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The Brauer–Siegel theorem concerns the size of the product of the class number and the regulator of a number field K. We derive bounds for this product in case K is a prime cyclotomic field, distinguishing between whether there is a Siegel zero or not. In particular, we make a result of Tatuzawa (1953) more explicit. Our theoretical advancements are complemented by numerical illustrations that are consistent with our findings.

1. Introduction

Let K be a number field, $\mathcal{O}$ its ring of integers and s a complex variable. For $\Re(s) > 1$ the *Dedekind zeta function* is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}},$$

where $\mathfrak{a}$ ranges over the nonzero ideals in $\mathcal{O}$, $\mathfrak{p}$ over the prime ideals in $\mathcal{O}$, and $N\mathfrak{a}$ denotes the *absolute norm* of $\mathfrak{a}$, that is, the index of $\mathfrak{a}$ in $\mathcal{O}$. It is known that $\zeta_K(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, and that it has a simple pole at $s = 1$. It has residue

$$\mathcal{R}(K) = \frac{2^{r_1}(2\pi)^{r_2} h(K)\,\mathrm{Reg}(K)}{\omega_K \sqrt{d_K}}, \tag{1}$$

where r_1 and r_2 denote the number of real, respectively complex, embeddings of K, d_K its absolute value of the discriminant, ω_K its roots of unity, $\mathrm{Reg}(K)$ its regulator and $h(K)$ its class number. Formula (1) is called the *analytic class number formula*. In it the only mysterious quantity is $h(K)\,\mathrm{Reg}(K)$ and one could hope to get bounds on it via estimates of $\mathcal{R}(K)$. For example, under the generalized Riemann hypothesis and the strong Artin conjecture for $\zeta_K(s)/\zeta(s)$, one has

$$\Big(\frac{1}{2} + o(1)\Big)\frac{\zeta(n)}{e^{\gamma}\log\log d_K} \le \mathcal{R}(K) \le (2 + o(1))^{n-1}(e^{\gamma}\log\log d_K)^{n-1}, \tag{2}$$

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where n denotes the degree of K, see [4, Section 3], and γ is Euler's constant. In 2015, Louboutin [13] (see also [12]) gave a weaker, but unconditional, bound for $\mathcal{R}(K)$. More precisely, he demonstrated that for any given $\varepsilon > 0$ and $n_0 \geq 5$, there exists a number ρ_0 such that for each number field K of degree $n \geq n_0$ and $d_K \geq \rho_0^n$, we have

$$\mathcal{R}(K) \leq \frac{3}{\sqrt{n}}\left(\frac{c(1+\varepsilon)e^{\gamma+\sqrt{6/n}}\log d_K}{n}\right)^{n-1}, \tag{3}$$

where $c = \frac{1}{2}\bigl(1 - \frac{1}{\sqrt{5}}\bigr)$.

From now on we focus exclusively on cyclotomic fields $K = \mathbb{Q}(\zeta_q)$ of prime conductor $q \geq 3$. We have

$$\zeta_K(s) = \zeta(s) \prod_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} L(s, \chi), \tag{4}$$

where χ runs over the $q-2$ nonprincipal characters in the group X_q of order $q-1$ of the Dirichlet characters modulo q. We write $h(q)$ for $h(K)$ and $\mathrm{Reg}(q)$ for $\mathrm{Reg}(K)$ and put $H(q) = 2\sqrt{q}\bigl(\frac{q}{2\pi}\bigr)^{(q-1)/2}$. Identity (4) in combination with (1) and using that $d_K = q^{q-2}$ gives

$$\mathcal{R}(q) = \prod_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} L(1, \chi) = \frac{h(q)\,\mathrm{Reg}(q)}{H(q)}.$$

In analogy with the terminology used for the relative class number, we will call $\mathcal{R}(q)$ the *Brauer–Siegel ratio* for a prime cyclotomic field, a term that seems to have been introduced by Ulmer [23] in the context of abelian varieties over function fields.

Tatuzawa [22, Theorem 3] proved that for every $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that

$$\frac{c(\varepsilon)}{q^{\varepsilon}} < \mathcal{R}(q) < (\log q)^c, \tag{5}$$

where $c > 0$ is an absolute constant. Here, adapting the technique used by Kandhil, Languasco, Moree, Saad Eddin, and Sedunova [8] to study the order of magnitude of the Kummer ratio[1] for the relative class number of a prime cyclotomic field, we bound the value of c in (5). We show that c is essentially at most 2 in the most general case; otherwise it is less than 1. Moreover, we explicitly show the role of the *Siegel zero* in both the bounds appearing in (5). Stark [21, Lemma 3][2] showed that $\zeta_K(s)$ (with $K \neq \mathbb{Q}$) has at most one zero in the region in the complex plane

[1] This analogy also speaks in favor of the terminology Brauer–Siegel ratio.

[2] In fact Stark's region is not the largest known; see, e.g., Louboutin [14].

determined by

$$\Re(s) \geq 1 - \frac{1}{4\log d_K}, \quad |\Im(s)| \leq \frac{1}{4\log d_K}.$$

If such a zero exists, it is real, simple and usually called a *Siegel zero*. In case $K = \mathbb{Q}(\zeta_q)$, we denote this zero by β_q. If nonempty, the set of Dirichlet characters modulo q such that $L(\beta_q, \chi) = 0$ contains only one primitive character (the *exceptional character* (mod q)). Exceptional characters are known to be real and quadratic. Siegel proved in 1936 that for every $\varepsilon > 0$ there exists a constant $c_1(\varepsilon)$ such that

$$\beta_q < 1 - c_1(\varepsilon)\, q^{-\varepsilon}, \tag{6}$$

where $c_1(\varepsilon)$ is ineffective; see, for example, Davenport [5, p. 127, equation (5)]. If an effective constant $c_2 > 0$ is needed, the best known estimate for β_q was established by Page in 1935:

$$\beta_q < 1 - c_2\, q^{-1/2}(\log q)^{-2}; \tag{7}$$

see, for example, Davenport [5, p. 96, equation (12)].

In our result the Siegel-zero contribution will be expressed using the *exponential integral function* defined for $x > 0$ as

$$E_1(x) := \int_x^\infty \frac{dt}{te^t}.$$

Since the derivative $(1 - e^{-x})/x$ of $E_1(x) + \log x$ is positive for $x > 0$, it follows that $E_1(x) + \log x = E_1(1) + \int_x^1 (e^{-t} - 1)\,\frac{dt}{t}$ is an increasing function of $x > 0$. Therefore,

$$E_1(x) = -\log x + O(1) \quad (0 < x \leq 1). \tag{8}$$

We are now ready to state our main theorem that makes (5) more explicit.

Theorem 1. *Let $\ell(q)$ be a function that tends arbitrarily slowly and monotonically to infinity as q tends to infinity. There is an effectively computable prime q_0 (possibly depending on ℓ) such that the following statements are true*:

(1) *If for some $q \geq q_0$ the family of Dirichlet L-series $L(s, \chi)$ has no Siegel zero for $\chi \in X_q$, $\chi \neq \chi_0$, then*

$$\frac{e^{-1.87}}{(\log q)^{1-\xi}} < \prod_{\substack{\chi \in X_q \\ \chi \neq \chi_0}} L(1, \chi) < e^{0.51}\,(\log q)^{1-\xi}, \tag{9}$$

for some absolute constant $\xi > 0$.

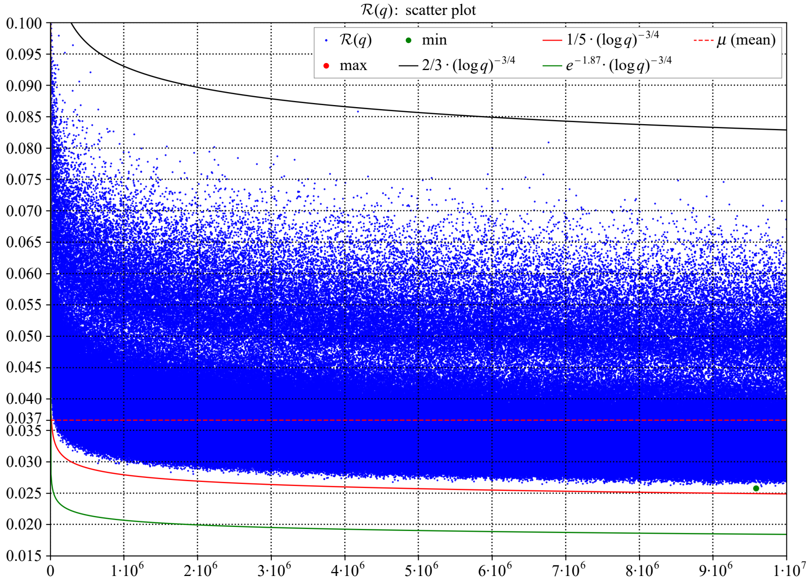

Figure 1. The values of $\mathcal{R}(q)$, q prime, $3 \le q \le 10^7$. The maximal value (red dot) is attained at $q = 3$ and its value is $0.604599\ldots$; much larger than the other plotted values. The red dashed line represents the mean value.

(2) *If for some $q \ge q_0$ the family of Dirichlet L-series $L(s, \chi)$ has a Siegel zero β_q for the only quadratic character χ in X_q, then*

$$\frac{e^{-1.87} e^{-E_1(1-\beta_q)}}{(\log q)^2\, \ell(q)} < \prod_{\substack{\chi \in X_q \\ \chi \ne \chi_0}} L(1, \chi) < e^{0.51} e^{-E_1(1-\beta_q)} (\log q)^2\, \ell(q).$$

We were able to perform extensive computations of $\mathcal{R}(q) = \prod_{\chi \in X_q,\, \chi \ne \chi_0} L(1, \chi)$ for the odd primes up to 10^7 using the fast Fourier transform method already presented in [9; 10]; see also [11]. They show a remarkable fit between $\mathcal{R}(q)$ and $c/(\log q)^{3/4}$, with $c \in \left(\frac{1}{5}, \frac{2}{3}\right)$; see Figure 1. In this respect, the scatter plot of the normalized values $\mathcal{R}(q)(\log q)^{3/4}$ presented in Figure 2 is particularly relevant. We think it is possible that the "true" order of magnitude for $\mathcal{R}(q)$ in Theorem 1 might be the one on the left-hand side of (9) with $\xi = \frac{1}{4}$. In Figure 3 we show the histograms obtained using the values presented into the first two figures.

Remark 2. All constants in Theorem 1 can be further sharpened by arguing as in Remark 7 below.

Remark 3. It is a consequence of Theorem 1 that asymptotically the upper bounds (2) and (3) are quite weak for prime cyclotomic fields. However, the *lower* bound in (2) seems reasonably sharp in this case.

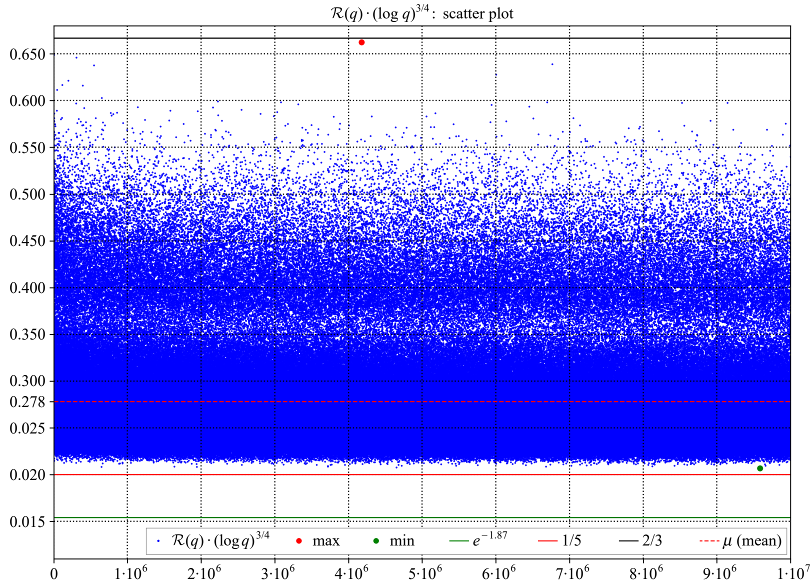

Figure 2. The values of $\mathcal{R}(q)(\log q)^{3/4}$, q prime, $3 \le q \le 10^7$. The red dashed line represents the mean value.

Remark 4. Using (6), the bound for $E_1(x)$ in (8) leads to

$$1 \ll E_1(1-\beta_q) = -\log(1-\beta_q) + O(1) < \varepsilon \log q + c_3(\varepsilon),$$

where $c_3(\varepsilon)$ is ineffective. Using the weaker, but with an effective constant, estimate (7) we obtain that

$$1 \ll E_1(1-\beta_q) < \tfrac{1}{2}\log q + 2\log\log q + c_4,$$

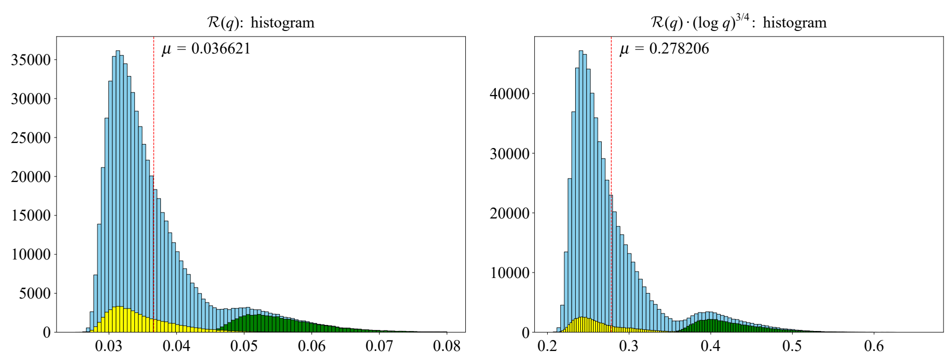

Figure 3. On the left: the values of $\mathcal{R}(q)$ (cerulean bars), q prime, $3 \le q \le 10^7$, but the contributions of the primes $q \ge 5$ such that $2q+1$ is prime (green bars) or $2q-1$ is prime (yellow bars) are superimposed. On the right: idem, but for the normalized values $\mathcal{R}(q)(\log q)^{3/4}$. The red dashed lines represent the mean values.

where $c_4 > 0$ is an effective constant. We also recall that Bessassi [2, Theorem 17] proved that $\beta_q < 1-6/(\pi\sqrt{q})$ for $q \equiv 3 \pmod 4$ and hence in this case one obtains $1 \ll E_1(1-\beta_q) < \frac{1}{2}\log q + \log(\pi/6)$.

Clearly Theorem 1 has implications for the asymptotic estimates of $h(q)\,\mathrm{Reg}(q)$; for example, (2) and the estimates of Remark 4 yield

$$\log(h(q)\,\mathrm{Reg}(q)) = \frac{q}{2}\log\Big(\frac{q}{2\pi}\Big) + O(\log\log q) \quad (q \to \infty),$$

improving significantly on the Brauer–Siegel implication

$$\log(h(q)\,\mathrm{Reg}(q)) \sim \log\sqrt{d_q} \sim \frac{q}{2}\log q \quad (q \to \infty).$$

The paper is organized as follows: In Section 2 we recall results we need (mainly from prime number theory) and in Section 3 we prove a useful lemma about a sum over prime powers in an arithmetic progression modulo a prime $q \geq 3$. Section 4 is devoted to the proof of Theorem 1.

An extended version of this paper also dealing with the computation of relevant prime sums and connections and analogies with the Mertens' constants in arithmetic progressions is available on the arXiv.

2. Preliminaries

2.1. ***Notation.*** We will use the standard notation

$$\pi(t) = \sum_{p\le t} 1, \qquad \pi(t;d,b) = \sum_{\substack{p\le t\\ p\equiv b \ (\mathrm{mod}\ d)}} 1,$$

$$\theta(t;d,b) = \sum_{\substack{p\le t\\ p\equiv b \ (\mathrm{mod}\ d)}} \log p, \quad \psi(t;d,b) = \sum_{\substack{n\le t\\ n\equiv b \ (\mathrm{mod}\ d)}} \Lambda(n),$$

where Λ denotes the von Mangoldt function and b and d are coprime.

2.2. ***Siegel zeros.*** The presence of a Siegel zero strongly influences the distribution of the primes in the progressions modulo q. We present two classical results in this direction we will make use of.

Classical Theorem (Brun–Titchmarsh[3]). *Let $x, y > 0$ and a, q be positive integers such that $(a,q) = 1$. Then, for all $y > q$, we have*

$$\pi(x+y;q,a) - \pi(x;q,a) < \frac{2y}{\varphi(q)\log(y/q)}. \tag{10}$$

[3] For a proof, see, e.g., the work of Montgomery and Vaughan [17, Theorem 2].

In particular, a key role is played by the constant 2 present in (10). From the works of Motohashi [18], Friedlander and Iwaniec [7], Ramaré [20, Theorems 6.5 and 6.6] and Maynard [16], it is well known that replacing the constant 2 with any value less than 2 is equivalent with assuming that there does not exist a Siegel zero for $\prod_{\chi \in X_q,\, \chi \neq \chi_0} L(s, \chi)$.

In Theorem 1(1) we will in fact assume that $\prod_{\chi \in X_q,\, \chi \neq \chi_0} L(s, \chi)$ has no Siegel zero and we will make use of the following result by Maynard [16, Proposition 3.5, second part].

Theorem (Maynard). *There is a fixed constant $\varepsilon > 0$ such that there exists an effectively computable constant q_1, such that if the set of the nonprincipal Dirichlet L-functions* $(\mathrm{mod}\, q)$, *for $q > q_1$, does not have a Siegel zero then for $x \geq q^{7.999}$ and for any b coprime with q we have that*

$$\left| \psi(x; q, b) - \frac{x}{\varphi(q)} \right| < \frac{(1-\varepsilon)x}{\varphi(q)}. \tag{11}$$

From now on $\log_2 x$ denotes $\log \log x$. The following theorem of Dusart [6, Theorem 5.5] will play a crucial role in the proof of our main result. It is an effective version of a sharpened form of the second theorem of Mertens (1872).

Theorem (Dusart). *For $x \geq 2278383$ we have*

$$\left| \sum_{p \leq x} \frac{1}{p} - \log_2 x - \mathcal{M} \right| \leq \frac{0.2}{(\log x)^3}, \tag{12}$$

where $\mathcal{M}$, the Meissel–Mertens constant, is given by the infinite sum

$$\mathcal{M} := \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) = \gamma - \sum_{p \geq 2} \sum_{m \geq 2} \frac{1}{mp^m}.$$

One has $\mathcal{M} \approx 0.261497212847643$; for more decimals see [19].

3. A useful lemma

For q a prime and b an integer, let

$$S_q(b) := \sum_{\substack{m \geq 2 \\ p^m \equiv b \ (\mathrm{mod}\ q)}} \frac{1}{mp^m}, \tag{13}$$

where the sum is over all pure prime powers that are congruent to b $(\mathrm{mod}\ q)$. This quantity will play a role in the proof of Theorem 1.

We will need the following lemma, the proof of which is similar to a well-known result by Ankeny and Chowla; see the estimate of C_4 in [1].

Lemma 5. *Put* $\alpha(m) := \frac{1}{2}(m^2-m)$, $\beta(m) := \frac{1}{2}(m^2+m)-1$ *and*

$$\mathcal{A} := \sum_{m\geq 2}\frac{1}{m}\sum_{k=\alpha(m)}^{\beta(m)}\frac{1}{k}. \tag{14}$$

For any odd prime number q *and for every* b *coprime to* q, *we have*

$$R(q,b) := (q-1)S_q(b) \leq \mathcal{A} + \left(\frac{\pi^2}{6}-\mathcal{A}\right)\frac{1}{q}, \tag{15}$$

where $S_q(b)$ *is defined in* (13). *In particular,* $R(q,b) \leq 1.608$ *for* $q \geq 7$ *and every* b *coprime to* q.

Proof. Note that without loss of generalization we may assume that $1 \leq b \leq q-1$. The contribution of the terms to $S_q(b)$ with $2 \leq p \leq q+1$ and $p \geq q+2$ are denoted by H (head) and T (tail), respectively. We now proceed to bound the tail T. For a given $m \geq 2$, let $x_{m,j}$, $1 \leq j \leq f(m)$, denote the $f(m)$ integral solutions in $\{2,\ldots,q+1\}$ of $x^m \equiv b \pmod q$. Note that $f(m) \leq m$. Since p must be equal to one of the $x_{m,j} \pmod q$, $p \geq q+2$ is greater than any $x_{m,j}$, we have

$$T \leq \sum_{m\geq 2}\sum_{j=1}^{f(m)}\sum_{k\geq 1}\frac{q-1}{m(x_{m,j}+kq)^m} \leq \sum_{m\geq 2}\sum_{k\geq 1}\frac{q-1}{(kq)^m} \leq \zeta(2)\sum_{m\geq 2}\frac{q-1}{q^m} = \frac{\zeta(2)}{q}. \tag{16}$$

We now bound the head H. For a given $m \geq 2$, let $g(m)$ denote the number of solutions in primes contained in $\{2,\ldots,q+1\}$ of $x^m \equiv b \pmod q$. Clearly $g(m) \leq f(m) \leq m$. Due to the weight $1/m$ in the definition of $S_q(b)$, it is more unfavorable to have a square, say, followed by a cube, than the other way around in the progression $b+q, b+2q, \ldots$. Thus we may assume we have $g(2)$ squares, followed by $g(3)$ cubes and so on. Again due to the weight $1/m$, the most unfavorable situation arises if $g(m) = m$, that is, two squares followed by three cubes and so on. Since $b+kq \leq kq$, we then find

$$H \leq \frac{q-1}{q}\sum_{m\geq 2}\frac{1}{m}\sum_{k=\alpha(m)}^{\beta(m)}\frac{1}{k} = \frac{q-1}{q}\mathcal{A}.$$

The result follows from adding H and T and doing some simple numerics. □

Remark 6 (precise numerical approximation of $\mathcal{A}$). Since $\beta(m) = \alpha(m+1)-1$, we obtain

$$\mathcal{A} = \sum_{m\geq 2}\frac{1}{m}(H_{\alpha(m+1)-1} - H_{\alpha(m)-1}) = \sum_{m\geq 2}\frac{H_{\alpha(m+1)-1}}{m^2+m} = \frac{\gamma}{2} + \frac{1}{2}\sum_{j\geq 3}\frac{\psi(\alpha(j))}{\alpha(j)},$$

where $\psi(x)$ is the *digamma function*, H_n denotes the n-th harmonic number, $H_0 = 0$ and we also used that $\psi(n) = H_{n-1} - \gamma$ for every $n \geq 1$. Recalling that $\psi(x) < \log x$, the third formula for $\mathcal{A}$ shows that the series converges, although not very quickly. However, it can be used to evaluate $\mathcal{A}$, since there exist very fast and accurate algorithms to compute $\psi(x)$ for positive x. For example, truncating the final sum in the expression for $\mathcal{A}$ at 10^{10} gives

$$\mathcal{A} \approx 1.6000883438\dots. \tag{17}$$

Remark 7. The first estimate in (16) together with $f(m) \le m$ and $x_{m,j} \ge 2$ leads to

$$\begin{aligned} T &\le (q-1)\sum_{m\ge 2}\sum_{k\ge 1}\frac{1}{(2+kq)^m} = (q-1)\biggl(\sum_{m\ge 2}\frac{\zeta(m,2/q)}{q^m} - \frac{1}{2}\biggr) \\ &= \frac{q-1}{q}\biggl(\psi\Bigl(\frac{2}{q}\Bigr) - \psi\Bigl(\frac{1}{q}\Bigr)\biggr) - \frac{q-1}{2}, \end{aligned}$$

where $\zeta(s,x)$ denotes the *Hurwitz zeta-function* and we used equations (4.1)–(4.3) from [3] to obtain a closed formula for the series involving the Hurwitz zeta-function values. Inserting this into the body of Lemma 5 we can replace (15) with the sharper (but less elegant) estimate

$$R(q,b) \le \frac{q-1}{q}\biggl(\mathcal{A} + \psi\Bigl(\frac{2}{q}\Bigr) - \psi\Bigl(\frac{1}{q}\Bigr)\biggr) - \frac{q-1}{2}, \tag{18}$$

from which one can infer that $R(q,b) < 1.600177$ for every prime $q \ge 3$ and b coprime to q, with the maximum of the right-hand side in (18) attained at $q = 229$.

4. Proof of Theorem 1

Using the Euler product for $L(1,\chi)$ with $\chi \ne \chi_0$, and Taylor's formula for $\log(1-u)$, we obtain

$$\log \mathcal{R}(q) = -\sum_{\chi\ne\chi_0}\sum_p \log\biggl(1 - \frac{\chi(p)}{p}\biggr) = \sum_{\chi\ne\chi_0}\sum_p\sum_{m\ge 1}\frac{\chi(p^m)}{mp^m} = \Sigma_1 + \Sigma_2, \tag{19}$$

say, where Σ_1 is the contribution of the primes ($m = 1$) and Σ_2 that of the prime powers ($m \ge 2$).

We first estimate Σ_2. Suppose that $(a,q) = (b,q) = 1$ and $b \equiv a \pmod q$. Then, using

$$\frac{1}{q-1}\sum_{\chi \,(\mathrm{mod}\, q)}\chi(a) = \begin{cases} 1, & a \equiv 1 \pmod q, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

we obtain

$$\Sigma_2 = (q-1)\sum_{\substack{m\ge 2 \\ p^m \equiv 1 \,(\mathrm{mod}\, q)}}\frac{1}{mp^m} - \sum_{\substack{m\ge 2 \\ p\ne q}}\frac{1}{mp^m}.$$

Recalling (13), it is easy to see that

$$\sum_{\substack{m\ge 2\\ p\ne q}} \frac{1}{mp^m} = \sum_{b=1}^{q-1} S_q(b),$$

and hence

$$-\frac{1}{q-1}\sum_{b=1}^{q-1} R(q,b) = -\sum_{b=1}^{q-1} S_q(b) < \Sigma_2 < (q-1)S_q(1) = R(q,1).$$

On invoking Lemma 5 we then obtain

$$|\Sigma_2| < \mathcal{A} + \frac{\zeta(2)-\mathcal{A}}{q}, \tag{21}$$

where $\mathcal{A}$ is defined in (14) and evaluated in (17).

We now proceed to define some quantities that will be useful later to estimate Σ_1. For any $b \in \{1,\dots,q-1\}$ and $x > 0$ let

$$S_q(b,x) := \sum_{\substack{p\le x\\ p\equiv b \ (\mathrm{mod}\ q)}} \frac{1}{p} \quad \text{and} \quad S(x) := \sum_{\substack{p\le x\\ p\ne q}} \frac{1}{p}. \tag{22}$$

Using (20) again, for any $x > 0$ we have

$$\sum_{\chi\ne\chi_0}\sum_{p\le x} \frac{\chi(p)}{p} = (q-1)S_q(1,x) - S(x). \tag{23}$$

As a consequence we obtain

$$\Sigma_1 = \lim_{x\to\infty} ((q-1)S_q(1,x) - S(x)).$$

We begin by estimating $S(x)$, followed by estimating $S_q(1,x)$ (which will bring the possible Siegel zero into play). From now on we will assume that q is a sufficiently large prime. Substituting $x = x_1 = q^{\ell(q)}$ in (12), we obtain

$$S(x_1) \ge \log_2 q + \log \ell(q) + 0.261497 + \frac{1}{\ell(q)\log q} - \frac{1}{q} \tag{24}$$

and

$$S(x_1) \le \log_2 q + \log \ell(q) + 0.261498 + \frac{1}{\log q}. \tag{25}$$

We will use (21) and (24)–(25) in the proofs of both parts of Theorem 1.

4.1. ***Proof of Theorem 1*(2).** The starting point is (19). We split the prime sum Σ_1 into three subsums S_1, S_2, S_3 defined according to whether $p \le x_1$, $x_1 < p \le x_2$ or $p \ge x_2$, with $x_1 = q^{\ell(q)}$ and $x_2 = e^q$.

We start by estimating S_1: Recalling [8, equations (26)–(27)] and using (22), we obtain

$$(q-1)S_q(1,x) < 2\left(\log_2\left(\frac{x}{q}\right) + C_1 + \frac{1}{\log q}\right), \tag{26}$$

where $C_1 = -0.4152617906$ and $x \ge q^2$. Combining (23), (24)–(25) and (26) we have

$$\begin{aligned} S_1 := \sum_{\chi \ne \chi_0} \sum_{p \le x_1} \frac{\chi(p)}{p} &\le \log_2 q + \log \ell(q) + 2C_1 - 0.261497 + \frac{2}{\log q} \\ &< \log_2 q + \log \ell(q) - 1.09202 + \frac{2}{\log q} \end{aligned} \tag{27}$$

and

$$S_1 > -\log_2 q - \log \ell(q) - 0.261498 - \frac{1}{\log q}. \tag{28}$$

We will now proceed to estimate S_3. By (20) and the partial summation formula, we have

$$\begin{aligned} S_3 = \sum_{\chi \ne \chi_0} \sum_{p \ge x_2} \frac{\chi(p)}{p} &= (q-1) \sum_{\substack{p \ge x_2 \\ p \equiv 1 \pmod q}} \frac{1}{p} - \sum_{\substack{p \ge x_2 \\ p \ne q}} \frac{1}{p} \\ &= \frac{1}{q} + \lim_{y \to \infty} \frac{((q-1)\pi(y;q,1) - \pi(y))}{y} - \frac{(q-1)\pi(x_2;q,1) - \pi(x_2)}{x_2} \\ &\qquad + \int_{x_2}^{\infty} \frac{(q-1)\pi(u;q,1) - \pi(u)}{u^2}\,du \\ &\ll q^2 e^{-c_1\sqrt{q}}, \end{aligned} \tag{29}$$

where $c_1 > 0$ is an absolute constant. In the final estimate, we have used both the prime number theorem and the Siegel–Walfisz theorem.

It remains to estimate S_2. Recall now (see, e.g., [5, Chapter 19]) that if χ is a nonprincipal character modulo q and $2 \le T \le x$, then

$$\theta(x,\chi) := \sum_{p \le x} \chi(p) \log p = -\delta_{\beta_q} \frac{x^{\beta_q}}{\beta_q} - \sum_{|\gamma| \le T}{}' \frac{x^{\rho}}{\rho} + O\left(\frac{x(\log qx)^2}{T} + \sqrt{x}\right), \tag{30}$$

where $\delta_{\beta_q} = 1$ if the Siegel zero β_q exists and is zero otherwise, and $\sum'$ is the sum over all nontrivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$, with the exception of β_q and its symmetric zero $1 - \beta_q$.

By the partial summation formula and (30) with $T = q^4$, we have

$$(31)\qquad S_2 := \sum_{\chi \neq \chi_0} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p}$$
$$= \sum_{\chi \neq \chi_0} \left(\frac{\theta(x_2, \chi)}{x_2 \log x_2} - \frac{\theta(x_1, \chi)}{x_1 \log x_1} + \int_{x_1}^{x_2} \theta(u, \chi) \frac{1 + \log u}{(u \log u)^2}\, du \right)$$
$$= -\delta_{\beta_q} \int_{x_1}^{x_2} \frac{u^{\beta_q - 2}}{\log u}\, du - \int_{x_1}^{x_2} \Bigl(\sum_{\chi \neq \chi_0} \sideset{}{'}\sum_{|\gamma| \leq q^4} u^{\rho - 2} \Bigr) \frac{du}{\log u} + (q-1) E_q,$$

and

$$E_q \ll \int_{x_1}^{x_2} \left(\frac{(\log qu)^2}{q^4 u} + \frac{1}{u^{3/2}} \right) \frac{du}{\log u} \ll \frac{1}{q^2}.$$

By using [15, Lemmas 7 and 8],[4] we obtain

$$(32)\qquad \int_{x_1}^{x_2} \Bigl(\sum_{\chi \neq \chi_0} \sideset{}{'}\sum_{|\gamma| \leq q^4} u^{\rho - 2} \Bigr) \frac{du}{\log u} \ll \frac{1}{\ell(q)}.$$

In this case we have that $\delta_{\beta_q} = 1$ in (31); we now proceed to evaluate the term depending on β_q. A direct computation using that $\log x_2 = q$ gives

$$\int_{x_1}^{x_2} \frac{u^{\beta_q - 2}}{\log u}\, du = \int_{\log x_1}^{\log x_2} \frac{dt}{t e^{(1-\beta_q)t}} = E_1(1 - \beta_q) - \int_{1-\beta_q}^{(1-\beta_q)\log x_1} \frac{dt}{t e^t} - E_1(q(1-\beta_q)),$$

where $E_1(u)$ denotes the exponential integral function. Recalling that $x_1 = q^{\ell(q)}$, where $\ell(q)$ tends to infinity arbitrarily slowly and monotonically as q tends to infinity, we have

$$(33)\qquad \int_{1-\beta_q}^{(1-\beta_q)\log x_1} \frac{dt}{t e^t} \leq \log_2 x_1 = \log_2 q + \log \ell(q) \quad \text{and} \quad E_1(q(1-\beta_q)) \ll \frac{1}{q}.$$

Inserting (32)–(33) into (31), we finally get

$$(34)\qquad |S_2 + E_1(1 - \beta_q)| \leq \log_2 q + \log \ell(q) + o(1).$$

Combining (27)–(29) and (34), in this case we obtain

$$(35)\qquad \Sigma_1 + E_1(1 - \beta_q) < 2 \log_2 q + 2 \log \ell(q) - 1.0920$$

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[4]Note that [15, Lemma 7] holds for every T and x_1 for which $\lim_{q \to \infty} \log(qT)/\log x_1 = 0$. This allows us to choose $T = q^4$ and $x_1 = q^{\ell(q)}$, where $\ell(q)$ tends to infinity arbitrarily slowly and monotonically as q tends to infinity. The final error term in [15, Lemma 8] is then $\ll 1/\ell(q) = o(1)$, as q tends to infinity.

and

$$\Sigma_1 + E_1(1-\beta_q) > -2\log_2 q - 2\log \ell(q) - 0.2615. \tag{36}$$

The proof is concluded by combining (19), (21) and (35)–(36) (recall that Σ_2 is bounded in (21)).

4.2. ***Proof of Theorem 1*(1).** The quantities S_1, S_2, S_3 are the same ones defined in Section 4.1. We first remark that for S_3 we can reuse (29). We now estimate S_2. In this case $\delta_{\beta_q} = 0$ and, arguing as in (31)–(32), we have

$$S_2 \ll \frac{1}{\ell(q)}. \tag{37}$$

We now estimate S_1: Since $\delta_{\beta_q} = 0$, we can use a sharper version of the Brun–Titchmarsh theorem. In particular, we can use (11) with $\varepsilon = 2\xi$. Since

$$\theta(x;q,b) = \psi(x;q,b) + O\big(\sqrt{x}\,\big),$$

we conclude that, for $x \ge q^{7.999}$ and b coprime with q,

$$\theta(x;q,b) < 2(1-\xi)\frac{x}{\varphi(q)} + C\sqrt{x}, \tag{38}$$

where $C > 0$ is a suitable constant. Using (38) we can replace (26) with

$$(q-1)S_q(1,x) < 2(1-\xi)\log_2 x + 2C_1 + \frac{c}{\log q}, \tag{39}$$

for $x > q^8$, where $c > 0$ is an effective constant.

A way to prove (39) for $x > q^8$ is the following. By the partial summation formula and using (38) we find

$$\begin{aligned}(q-1)&\sum_{\substack{kq<p\le x\\ p\equiv 1 \ (\mathrm{mod}\ q)}} \frac{1}{p} \\ &= (q-1)\left(\frac{\theta(x;q,1)}{x\log x} - \frac{\theta(kq;q,1)}{kq\log(kq)} + \int_{kq}^{x} \theta(u;q,1)\frac{1+\log u}{(u\log u)^2}\,du\right) \\ &< 2(1-\xi)\left(\frac{1}{\log x} + \int_{kq}^{x}\left(1+\frac{1}{\log u}\right)\frac{du}{u\log u}\right) + \frac{C}{q} \\ &\le 2(1-\xi)(\log_2 x - \log_2(kq)) + \frac{c}{\log q} \\ &\le 2(1-\xi)\log_2 x - 2\log_2 k + \frac{c}{\log q},\end{aligned} \tag{40}$$

where $c > 0$ is an effective constant. In deriving (40) we also used that $x > q^8$ is equivalent to $\sqrt{x} < x^{3/4}/q^2$. From equation (25) of [8] we also have

$$(q-1)\sum_{\substack{p\le kq\\ p\equiv 1 \pmod q}}\frac{1}{p}\le\sum_{j=1}^{(k-1)/2}\frac{q-1}{2jq-1}<\frac{1}{2}\sum_{j=1}^{(k-1)/2}\frac{1}{j}=\frac{1}{2}H_{(k-1)/2},\tag{41}$$

where $H_n := \sum_{j=1}^{n}\frac{1}{j}$ is the n-th harmonic number. Letting

$$c_1(k) := \tfrac{1}{4}H_{(k-1)/2}-\log_2 k,$$

we now choose k such that $c_1(k)$ is minimal. It is not hard to see that $k = 55$ and that

$$c_1(55) < C_1 = -0.4152617906.$$

Inequality (39) then follows from combining (40)–(41).

Using (37), (39), (24)–(25) and arguing as in (27)–(28), we can replace (35)–(36) with

$$\Sigma_1 < (1-2\xi)\log_2 q+\log\ell(q)-1.0920 < (1-\xi)\log_2 q-1.0920\tag{42}$$

and

$$\Sigma_1 > -(1-2\xi)\log_2 q-\log\ell(q)-0.2615 > -(1-\xi)\log_2 q-0.2615,\tag{43}$$

respectively.

The proof is concluded by combining (19), (21) and (42)–(43). □

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