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We provide examples of pairs of closed, connected Legendrian nonisotopic Legendrian submanifolds (Λ_-, Λ_+) of the $(4n+1)$-dimensional contact vector space, $n > 1$, such that there exist Lagrangian concordances from Λ_- to Λ_+ and from Λ_+ to Λ_-. This contradicts antisymmetry of the Lagrangian concordance relation, and, in particular, implies that Lagrangian concordances with connected Legendrian ends do not define a partial order in high dimensions. In addition, we explain how to get the same result for the relation given by exact Lagrangian cobordisms with connected Legendrian ends in the $(2n+1)$-dimensional contact vector space, $n > 1$.

1. Introduction and main result

Lagrangian cobordisms between Legendrian submanifolds naturally arise in the symplectic field theory of Eliashberg, Givental and Hofer [11]. They have been actively studied for more than two decades; for the basics of this theory we refer to [1; 2; 4; 9]. We concentrate on the most basic Lagrangian cobordisms, namely Lagrangian concordances with connected Legendrian ends, in the symplectization of the contact vector space $\mathbb{R}^{2n+1}_{\mathrm{st}} := (\mathbb{R}^{2n+1}, dz - y\,dx)$. Lagrangian concordances with connected Legendrian ends define a relation on closed, connected Legendrian submanifolds, i.e., we say that Λ_- is related to Λ_+ and write $\Lambda_- <_L \Lambda_+$ if there is a Lagrangian concordance L from Λ_- to Λ_+. This relation was studied in [2; 5]. It is reflexive and transitive. By the result of Chantraine it follows that this relation is not symmetric [3]; see also [5]. For a while there was an open question of whether this relation defines a partial order; see [2; 16; 15]. Recently, Dimitroglou Rizell and the author in [7] constructed a series of examples of Lagrangian concordances between Legendrian knots that contradict the antisymmetry property. In other words, in [7], Dimitroglou Rizell and the author constructed pairs (Λ_-, Λ_+) of Legendrian nonisotopic Legendrian knots such that there are Lagrangian concordances from Λ_- to Λ_+ and from Λ_+ to Λ_-.

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We construct examples of Lagrangian concordances with connected Legendrian ends that contradict antisymmetry in high dimensions. More precisely, we prove the following:

Theorem 1. *There is a pair of closed, connected Legendrian nonisotopic Legendrian submanifolds* (Λ_-, Λ_+) *of* $\mathbb{R}^{4n+1}_{\mathrm{st}}$, $n > 1$, *such that there exist two Lagrangian concordances* $L_\pm$ *from* Λ_- *to* Λ_+ *and* $L_\mp$ *from* Λ_+ *to* Λ_- *in the symplectization of* $\mathbb{R}^{4n+1}_{\mathrm{st}}$, $n > 1$.

Theorem 1 leads to the following corollary:

Corollary 2. *Lagrangian concordances with connected Legendrian ends do not define a partial order on closed, connected Legendrian submanifolds of* $\mathbb{R}^{4n+1}_{\mathrm{st}}$ *for* $n > 1$.

In Section 3 we explain how to get the same result for the relation given by exact Lagrangian cobordisms with connected Legendrian ends in $\mathbb{R}^{2n+1}_{\mathrm{st}}$, $n > 1$.

2. Proof of Theorem 1

First we recall that from the result of Murphy [14, Theorem 1.2] and the discussion right after it follows that every formal Legendrian isotopy class in $\mathbb{R}^{2k+1}_{\mathrm{st}}$, $k > 1$, contains loose Legendrian embeddings, and every two formally isotopic loose Legendrian embeddings are Legendrian isotopic.

Then we assume that $k = 2n$, $n > 1$. Following the result of Ekholm, Etnyre and Sullivan [8, Proposition 3.2] observe that for a Legendrian submanifold Λ of $\mathbb{R}^{4n+1}_{\mathrm{st}}$, the Thurston–Bennequin number $\mathrm{tb}(\Lambda)$ is a topological invariant given by

$$\mathrm{tb}(\Lambda) = (-1)^{n+1}\tfrac{1}{2}\chi(\Lambda).$$

Now we take a closed, stably parallelizable, simply connected Λ and two Legendrian embeddings $i_- : \Lambda \to \mathbb{R}^{4n+1}_{\mathrm{st}}$ and $i_+ : \Lambda \to \mathbb{R}^{4n+1}_{\mathrm{st}}$ such that $r(i_-(\Lambda)) = r(i_+(\Lambda))$ and $\Lambda_- := i_-(\Lambda)$, $\Lambda_+ := i_+(\Lambda)$ are loose Legendrian nonisotopic Legendrian submanifolds. From [14, Proposition A.4(c)] it follows that for a fixed rotation class, there is exactly one such couple Λ_-, Λ_+ up to Legendrian isotopy.

Remark 3. In particular, for our construction we can take $\Lambda = S^{2n}$.

Remark 4. Following [8, Proposition 3.2] we see that $\mathrm{tb}(\Lambda_-) = \mathrm{tb}(\Lambda_+)$. By the original assumption, the rotation classes of Λ_- and of Λ_+ coincide. Since Λ_-, Λ_+ are loose, they have acyclic Legendrian contact homology DGAs. Hence, we conclude that Λ_- and Λ_+ have the same Legendrian invariants.

From the classical result of Wu [17] it follows that every two embeddings of a connected $2n$-dimensional manifold into $\mathbb{R}^{4n+1}$, $n \geq 1$, are smoothly isotopic. Hence, Λ_- is smoothly isotopic to Λ_+. Thus, in particular, there is a smooth concordance from Λ_- to Λ_+ and we denote the corresponding smooth concordance by $L_{\pm}^{C^\infty}$, and there is a smooth concordance from Λ_+ to Λ_- and we denote it by $L_{\mp}^{C^\infty}$.

For simplicity, from now on let us assume that $\Lambda = S^{2n}$. Clearly $\Lambda_-, \Lambda_+ \simeq S^{2n}$ are closed, connected, loose, and the complexified tangent bundle of a concordance over S^{2n} is trivial. Now recall that in high dimensions Eliashberg and Murphy established an h-principle for exact Lagrangian embeddings with loose concave Legendrian ends [10, Theorem 2.2].

Remark 5. Even though [10, Theorem 2.2] is written for exact Lagrangian caps with loose concave Legendrian ends, the proof of [10, Theorem 2.2] can be adapted to hold for exact Lagrangian cobordisms with loose concave Legendrian ends and possibly nontrivial convex Legendrian ends. Observe that this adaptation is possible, since all the homotopies and isotopies from Theorems 2.2 and 2.3 in [10], where the former is deduced from the latter, are compactly supported.

By applying the h-principle of Eliashberg and Murphy that we discussed in Remark 5 to concordances $L_{\pm}^{C^\infty}$, $L_{\mp}^{C^\infty}$ with loose ends over Λ_-, Λ_+ we get Lagrangian concordances $L_\pm$ from Λ_- to Λ_+ and $L_\mp$ from Λ_+ to Λ_-. Since Λ_- is not Legendrian isotopic to Λ_+, this leads to the fact that Lagrangian concordances with connected Legendrian ends do not define an antisymmetric relation. In particular, it implies that Lagrangian concordances with connected Legendrian ends do not define a partial order for closed, connected Legendrian submanifolds of $\mathbb{R}^{4n+1}_{\text{st}}$, where $n > 1$. This finishes the proof.

Remark 6. The proof of Theorem 1 heavily relies on the description of Legendrian isotopy classes of loose Legendrian submanifolds in $\mathbb{R}^{2k+1}_{\text{st}}$ written by Murphy in [14, Proposition A.4]. For $k > 2$ even, a given stably parallelizable, simply connected k-manifold Λ, and a given rotation class r, this classification provides two Legendrian nonisotopic Legendrian embeddings of Λ realizing r. The fact that there are two Legendrian nonisotopic loose Legendrian embeddings of Λ realizing the same pair of classical invariants (r, tb) is crucial for the proof. Now note that for odd $k > 1$, Murphy in [14, Proposition A.4] proved that if two formal Legendrian embeddings have the same Thurston–Bennequin number and rotation class, then they are formally Legendrian isotopic. This, together with [14, Theorem 1.2] and the discussion right after it, implies that for a given stably parallelizable k-manifold Λ and a pair of classical Legendrian invariants (r, tb), there is exactly one loose Legendrian embedding realizing this pair. This, in particular, implies that the method we used for the proof of Theorem 1 for even k cannot be easily adjusted to the case when k is odd.

3. Exact Lagrangian cobordisms

If, instead of the class of Lagrangian concordances with connected Legendrian ends, we consider the class of exact Lagrangian cobordisms with connected Legendrian

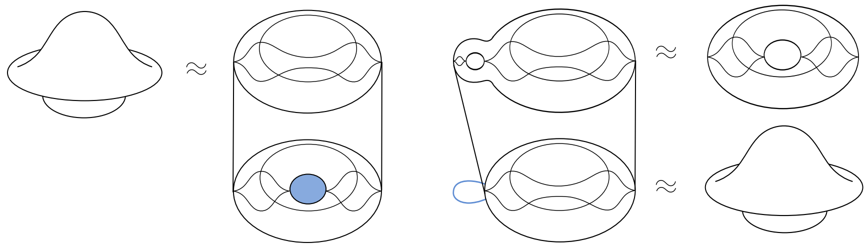

Figure 1. The pair of exact Lagrangian cobordisms $L^{S^2}_{T^2}$ (left) and $L^{T^2}_{S^2}$ (right) from [6, Section 2.3].

ends in $\mathbb{R}^{2n+1}_{\mathrm{st}}$, $n > 1$, and define the corresponding relation, then we can construct examples of Lagrangian cobordisms with nontrivial topology that contradict the antisymmetry.

For that, we consider the example from [6, Section 2.3], where Dimitroglou Rizell and the author constructed the noncylindrical exact Lagrangian endocobordism which consists of the following concatenated pieces (these appear in Figure 1):

- exact Lagrangian cobordism $L^{T^2}_{S^2}$ from the standard Legendrian 2-sphere to the Legendrian torus which is a front spun of $\mathrm{tb} = -1$ Legendrian unknot, and
- exact Lagrangian cobordism $L^{S^2}_{T^2}$ from the Legendrian torus which is a front spun of $\mathrm{tb} = -1$ Legendrian unknot to the standard Legendrian 2-sphere.

This pair of cobordisms $L^{T^2}_{S^2}$, $L^{S^2}_{T^2}$ contradicts the antisymmetry property by the obvious topological reason, i.e., by the fact that S^2 is not diffeomorphic to T^2. This example can easily be extended to high dimensions by using the front spinning construction (or the spherical spinning construction) applied to $L^{S^2}_{T^2}$ and $L^{T^2}_{S^2}$ as explained in [12; 13]. This way we obtain the exact Lagrangian cobordisms from the Legendrian Λ diffeomorphic to $S^{i_1} \times \cdots \times S^{i_k} \times S^2$ to the Legendrian Λ' diffeomorphic to $S^{i_1} \times \cdots \times S^{i_k} \times T^2$ and from Λ' to Λ, which implies that the relation defined by exact Lagrangian cobordisms with connected Legendrian ends for closed, connected Legendrian submanifolds of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ is not antisymmetric for all $n > 1$, and hence it is not a partial order for all $n > 1$.

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