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THE GENERALIZED FUGLEDE'S CONJECTURE HOLDS FOR A CLASS OF CANTOR–MORAN MEASURES

LI-XIANG AN, QIAN LI AND MIN-MIN ZHANG

Suppose $b = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than 1 and $D = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets. Let $\mu_{b,D}$ be the Cantor–Moran measure defined by

$$\mu_{b,D} = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \delta_{\frac{1}{b_1 b_2 b_3}\mathscr{D}_3} * \cdots .$$

We first prove that $L^2(\mu_{b,D})$ possesses an exponential orthonormal basis if and only if N_n divides b_n for each $n \geq 2$. Subsequently, we show that the generalized Fuglede's conjecture holds for such Cantor–Moran measures. An immediate consequence of this result is the equivalence between the existence of an exponential orthonormal basis and the integer-tiling of $D_n = \mathscr{D}_n + b_n \mathscr{D}_{n-1} + b_2 \cdots b_n \mathscr{D}_1$ for $n \geq 1$.

1. Introduction

A Borel probability measure μ on $\mathbb{R}^d$ is called a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ (called a *spectrum*) such that the set of exponential functions $E(\Lambda) := \{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. If $\Omega \subset \mathbb{R}^d$ is a measurable set with finite positive Lebesgue measure and $\mathscr{L}_\Omega$ is a spectral measure, then we say that Ω is a *spectral set*. Here $\mathscr{L}_K$ denotes the normalized Lebesgue measure restricted to a measurable set K of finite positive Lebesgue measure.

It is well known from classical Fourier analysis that the unit cube $[0, 1]^d$ is a spectral set with a spectrum $\mathbb{Z}^d$. What other sets Ω can be spectral? The research on this problem has been influenced for many years by a famous paper due to Fuglede [1974], who suggested that there should be a concrete, geometric way to characterize the spectral sets.

Conjecture (Fuglede's conjecture). A set $\Omega \subset \mathbb{R}^d$ is spectral if and only if it can tile the space by translations.

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We say Ω is a translational tile if there exists a discrete set $\mathcal{J}$ such that the translated copies $\{\Omega + t : t \in \mathcal{J}\}$ constitute a partition of $\mathbb{R}^d$ up to measure zero. Although the conjecture was eventually disproved in dimension 3 or higher in its full generality [Kolountzakis and Matolcsi 2006b; 2006a; Tao 2004], most of the known examples of spectral sets are constructed from translational tiles [Łaba 2001; Fu et al. 2015; Dai et al. 2013; Dai et al. 2014; Dai 2016; 2012]. An important result proved by Lev and Matolcsi [2022] is that all spectral sets must admit a "weak tiling" which is a generalization of translational tiling in its measure-theoretic form. Let $K \subset \mathbb{R}^d$ be a bounded, measurable set. We say that another measurable, but possibly unbounded, set $\Sigma \subset \mathbb{R}^d$ admits a weak tiling by translates of K if there exists a positive, locally finite (Borel) measure ν on $\mathbb{R}^d$ such that $\mathbf{1}_K * \nu = \mathbf{1}_\Sigma$, where $\mathbf{1}_A$ denotes the indicator function of a set A.

Jorgensen and Pedersen [1998] widened the scope of Fuglede's conjecture and discovered that the standard middle-fourth Cantor measure $\mu_{1/(2k),\{0,2\}}$ is a spectral measure. It is the first spectral measure that is nonatomic and singular. Strichartz [2000; 2006] discovered a surprising and interesting phenomenon: the Fourier series corresponding to certain spectra of $\mu_{4^{-1},\{0,2\}}$ can exhibit significantly better convergence properties than their classical counterparts on the unit interval. Specifically, the Fourier series of continuous functions converge uniformly, and Fourier series of L^p-functions converge in the L^p-norm for $1 \le p < \infty$. Following these discoveries, there has been considerable research on such measures [An and Wang 2021; Dutkay and Haussermann 2016; Łaba and Wang 2002; Fu et al. 2015; Dutkay et al. 2009; Dai 2012; Dai et al. 2013; Dai et al. 2014; Dai 2016; Hu and Lau 2008; Dutkay et al. 2019], and a celebrated open problem was to characterize the spectral property of the Cantor measures $\mu_{\rho,\mathcal{D}}$, $0 < \rho < 1$ among the N-Bernoulli convolutions

$$\mu_{\rho,\mathcal{D}}(\,\cdot\,) = \frac{1}{N} \sum_{d \in \mathcal{D}} \mu_{\rho,\mathcal{D}}(\rho^{-1}(\,\cdot\,) \cdot - d),$$

where $\mathcal{D} = \{0, 1, \dots, N-1\}$.

N-Bernoulli convolutions have been studied extensively in many areas of mathematics, including Fourier analysis, dynamical systems, integer tiles, wavelet theory, algebraic number theory and fractal geometry, since the 1930s (e.g., see [Wintner 1935]). Hu and Lau [2008], as well as Dai [2012], showed that the above Cantor measures $\mu_{1/(2k),\{0,2\}}$ are the only class of spectral measures among the $\mu_{\rho,\{0,2\}}$. Dai, He and Lau [Dai et al. 2014] demonstrated that a similar result holds for N-Bernoulli measures $\mu_{\rho,\mathcal{D}}$.

Later on, Strichartz [2000] formulated the most general fractal spectral measures one can possibly generate. Let $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than 1, and let $\boldsymbol{D} = \{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of integer digit sets. Let δ_a be the Dirac measure, and write

$$\delta_E = \frac{1}{\#E} \sum_{e \in E} \delta_e$$

for a finite set E. Write

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \delta_{\frac{1}{b_1 b_2 b_3}\mathscr{D}_3} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n},$$

where $*$ denotes convolution. If the sequence of convolutions $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ with compact support, then we call $\mu_{\boldsymbol{b},\boldsymbol{D}}$ a *Cantor–Moran measure*, as a generalization of the standard Cantor measure studied first by Moran [1946]. This opens up a research field for orthogonal harmonic analysis of Cantor–Moran measures (e.g., [An et al. 2019; An and He 2014; Deng and Li 2022; Li et al. 2022]).

In this paper, we study the spectrality of the Cantor–Moran measures generated by an integer sequence $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ with $b_n \geq 2$ and a sequence of consecutive digit sets $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$, i.e., $\mathscr{D}_n = \{0, 1, \ldots, N_n - 1\}$. We first provide a sufficient and necessary condition for the existence of such Cantor–Moran measures.

Theorem 1.1. *Let $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than* 1, *and let $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ be a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \ldots, N_n - 1\}$, where $N_n \geq 2$. Then the sequence of discrete measures*

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n}$$

converges weakly to a Borel probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ if and only if

$$\sum_{n=1}^{\infty} \frac{N_n}{b_1 b_2 \cdots b_n} < \infty. \tag{1-1}$$

In this case, $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is supported on a compact set

$$T(\boldsymbol{b}, \boldsymbol{D}) = \left\{ \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n} : d_n \in \mathscr{D}_n \right\} := \sum_{n=1}^{\infty} \frac{\mathscr{D}_n}{b_1 b_2 \cdots b_n}.$$

The spectral properties of such measures were first studied by An and He [2014] as a generalization of the N-Bernoulli convolutions ($b_n = b$ and $D_n = \{0, 1, \ldots, N-1\}$ for all n). The first-named author and He showed that $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral when N_n divides b_n for each $n \geq 1$. Under the condition that $\{N_n\}_{n=1}^{\infty}$ is bounded, it has been proved in [Deng and Li 2022] that the condition N_n divides b_n for each $n \geq 1$ is also necessary for $\mu_{\boldsymbol{b},\boldsymbol{D}}$ to be spectral. These Cantor–Moran measures also show that spectral measures can have support of any Hausdorff dimension [Dai and Sun 2015]. Furthermore, Cantor–Moran measures offer new examples of fractal measures that admit a Fourier frame but not a Fourier orthonormal basis [Gabardo and Lai 2014], which lead to a new avenue to study a long-standing problem: whether a middle-third Cantor measure has a Fourier frame.

Our motivation to extend the N-Bernoulli convolutions to this class of measures stems from the conjecture by Gabardo and Lai [2014], and we aim to answer a question on the relationship between Cantor–Moran spectral measures and integer tiles. To describe a unifying framework bridging the gap between singular spectral measures and spectral sets, Gabardo and Lai [2014] extended the classical Fuglede's conjecture to a more generalized form.

Conjecture (generalized Fuglede's conjecture). A compactly supported Borel probability measure μ on $\mathbb{R}$ is spectral if and only if there exists a Borel probability ν on $\mathbb{R}$ and a fundamental domain Q of some lattice on $\mathbb{R}$ such that $\mu * \nu = \mathscr{L}_Q$.

Deterministic positive results about Cantor measure have appeared in many papers (e.g., [An and Wang 2021; Dutkay et al. 2009; Dai 2012; Dai et al. 2013; Dai et al. 2014; Dai 2016; Hu and Lau 2008; Dutkay et al. 2019; Łaba and Wang 2002]). However, there are relatively few results regarding Cantor–Moran measures. Gabardo and Lai [2014] showed that if both μ and ν are two singular probability measures with $\mu * \nu = \mathscr{L}_{[0,1]}$, then they are both Cantor–Moran measures. In this paper, we will demonstrate that the generalized Fuglede's conjecture holds for our target measure.

Theorem 1.2. *Suppose $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than* 1, *and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \dots, N_n - 1\}$, where $N_n \geq 2$. Then the following are equivalent*:

(i) *The Cantor–Moran measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral.*

(ii) *There exists a Borel probability ν such that $\mu_{\boldsymbol{b},\boldsymbol{D}} * \nu = \mathscr{L}_{[0,N_1/b_1]}$.*

(iii) *N_n divides b_n for each $n \geq 2$.*

Remark. Lai and Wang conjectured that if $D_n = \{0, 1, \dots, N_n - 1\}$ is a continuous digit set for each $n \geq 1$ and the associated measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral, then N_n divides b_n for all n [Lai and Wang 2017, Conjecture 4.3]. Theorem 1.2 confirms this conjecture and resolves their conjecture from the perspective of tiling (convolution is now regarded as the tiling operation).

The implication (ii) $\Rightarrow$ (i) stems from [Gabardo and Lai 2014, Theorem 1.1], and the proof of (iii) $\Rightarrow$ (i) is provided in [An and He 2014, Theorem 1.4], while (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are apparently new in this generality. We now outline the strategy of the proof. Let us set up the notation:

$$\mu_{\boldsymbol{b},\boldsymbol{D}} = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \delta_{\frac{1}{b_1 b_2 b_3}\mathscr{D}_3} * \cdots = \mu_n * \mu_{>n},$$

where μ_n is the convolution of the first n discrete measures, and $\mu_{>n}$ is the remaining part. Based on the aforementioned decomposition of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, one seeks to observe the behavior of μ_n and $\mu_{>n}$ under the condition that $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure. Actually, if $0 \in \Lambda$ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, then, for any $n \geq 1$, we can construct spectra of μ_n and $\mu_{>n}$ relying on a maximal decomposition (see Definition 3.1) of Λ with respect to $(\mu_n, \mu_{>n})$.

Theorem 1.3. *Suppose $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than 1, and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \dots, N_n - 1\}$, where $N_n \geq 2$. If $0 \in \Lambda$ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, then, for each $n \geq 1$, we have a maximal decomposition of Λ with respect to $(\mu_n, \mu_{>n})$, denoted by $\Lambda = \bigcup_{\alpha\in\mathscr{A}} \Lambda_\alpha$, such that $\mathscr{A}$ is a spectrum of μ_n and each Λ_α is a spectrum of $\mu_{>n}$.*

Applying Theorem 1.3, we can reduce (i) $\Rightarrow$ (iii) to the following theorem, considering μ_n as a spectral measure.

Theorem 1.4. *The discrete measure $\mu_n = *_{j=1}^{n} \delta_{\frac{1}{b_1\cdots b_j}\mathscr{D}_j}$ is spectral if and only if N_j divides b_j for each $2 \leq j \leq n$.*

Then (i) $\Rightarrow$ (iii) follows from Theorems 1.3 and 1.4. We observe that if $b_n = r_n N_n$ for some integer r_n, then

$$\mathscr{D}_n \oplus N_n\{0, 1, \dots, r_n - 1\} = \{0, 1, \dots, b_n - 1\}.$$

Here the direct sum $A \oplus B$ means that $a + b$ are all distinct elements for all $a \in A$ and $b \in B$. This can yield (iii) $\Rightarrow$ (ii) in Theorem 1.2.

A digit set $\mathscr{D}$ is called an *integer tile* if $\mathscr{D}$ tiles some cyclic group $\mathbb{Z}_n$, i.e., there exists $\mathscr{B}$ such that $\mathscr{D} \oplus \mathscr{B} \equiv \mathbb{Z}_n \pmod{n}$. The study of integer tiles has a long history related to the geometry of numbers [Coven and Meyerowitz 1999; Łaba and Londner 2023; Newman 1977; Tijdeman 1995; Sands 1979]. In Theorem 1.5, we present an intriguing result regarding the relationship between the Cantor–Moran spectral measure and integer tiles. Write

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \cdots * \delta_{\frac{1}{b_1 b_2\cdots b_n}\mathscr{D}_n} = \delta_{\frac{1}{b_1 b_2\cdots b_n}\boldsymbol{D}_n},$$

where $\boldsymbol{D}_n = \mathscr{D}_n + b_n\mathscr{D}_{n-1} + \cdots + b_2\cdots b_n\mathscr{D}_1$ is the first n terms iterated digit set of $\{\mathscr{D}_n\}_{n=1}^{\infty}$.

Theorem 1.5. *Suppose $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than 1, and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets. Then the Cantor–Moran measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral if and only if, for each $n \geq 1$, $\boldsymbol{D}_n = \mathscr{D}_n \oplus b_n\mathscr{D}_{n-1} \oplus \cdots \oplus b_2\cdots b_n\mathscr{D}_1$ is an integer tile.*

We organize this paper as follows. In Section 2, we will study the weak convergence of infinite convolutions and give the proof of Theorem 1.1. Also, we will introduce some basic definitions and properties of spectral measures. In Section 3, we will discuss the distribution of any bizero set of the spectral measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$, and prove Theorem 1.3. We will devote Section 4 to proving Theorems 1.2 and 1.4. In Section 5, we will prove Theorem 1.5 and propose an open question on the relationship between the spectral Cantor–Moran measure and the tiling of integers.

2. Notation and preliminaries

2.1. ***Weak convergence of convolutions.*** Using Kolmogorov's three-series theorem, Li, Miao and Wang [Li et al. 2022] provided sufficient and necessary conditions for the existence of infinite convolutions.

Theorem 2.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of nonnegative finite subsets of $\mathbb{R}$ satisfying that $\#A_n \geq 2$ for each $n \geq 1$. Let $\nu_n = \delta_{A_1} * \cdots * \delta_{A_n}$. Then the sequence of convolutions $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\#A_n} \sum_{a \in A_n} \frac{a}{1+a} < \infty. \tag{2-1}$$

Proof of Theorem 1.1. The weak convergence of $\{\mu_n\}_{n=1}^{\infty}$ is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{N_n} \sum_{d \in \mathcal{D}_n} \frac{d}{b_1 \cdots b_n + d} < \infty, \tag{2-2}$$

by Theorem 2.1. Note that

$$\sum_{n=1}^{\infty} \frac{1}{N_n} \sum_{d \in \mathcal{D}_n} \frac{d}{b_1 \cdots b_n + d} = \sum_{\{n: N_n - 1 > b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d} + \sum_{\{n: N_n - 1 \leq b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d}. \tag{2-3}$$

To prove sufficiency, suppose (1-1) holds. We first claim $\{n : N_n - 1 > b_1 \cdots b_n\}$ is a finite set. Indeed, if $\{n : N_n - 1 > b_1 \cdots b_n\}$ is an infinite set, then

$$\sum_{n=1}^{\infty} \frac{N_n}{b_1 \cdots b_n} \geq \sum_{\{n: N_n - 1 > b_1 \cdots b_n\}} \frac{N_n}{b_1 \cdots b_n} = \infty.$$

We get a contradiction. Then the claim follows. Hence

$$\sum_{\{n: N_n - 1 > b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d} < \infty. \tag{2-4}$$

Notice that

$$\begin{aligned} \sum_{\{n: N_n - 1 \leq b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d} &\leq \sum_{\{n: N_n - 1 \leq b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n} \\ &= \sum_{\{n: N_n - 1 \leq b_1 \cdots b_n\}} \frac{N_n - 1}{2 b_1 \cdots b_n} \\ &< \sum_{n=1}^{\infty} \frac{N_n}{b_1 \cdots b_n} < \infty. \end{aligned} \tag{2-5}$$

Then (2-2) follows from (2-3), (2-4) and (2-5).

Next we prove necessity. Suppose (2-2) holds. Then it follows from (2-3) that

$$\sum_{\{n:N_n-1>b_1\cdots b_n\}} \frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d}<\infty, \qquad (2\text{-}6)$$
$$\sum_{\{n:N_n-1\le b_1\cdots b_n\}} \frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d}<\infty.$$

We assert that $\{n : N_n - 1 > b_1 \cdots b_n\}$ is a finite set. Otherwise,

$$\begin{aligned}\infty &> 4\sum_{n=1}^{\infty}\frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d}\\ &\ge 4\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{1}{N_n}\frac{1}{b_1\cdots b_n+N_n-1}\sum_{d=0}^{N_n-1}d\\ &=4\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{2(N_n-1)}{b_1\cdots b_n+N_n-1}=\infty.\end{aligned}$$

This gives a contradiction, and the assertion follows. Consequently,

$$\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}<\infty. \qquad (2\text{-}7)$$

Note that

$$\begin{aligned}\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d} &\ge \sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{1}{N_n}\frac{1}{b_1\cdots b_n+N_n-1}\sum_{d=0}^{N_n-1}d\\ &=\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n-1}{2(b_1\cdots b_n+N_n-1)}\\ &\ge\frac{1}{2}\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n-1}{2b_1\cdots b_n}.\end{aligned}$$

It follows that

$$\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}<\infty. \qquad (2\text{-}8)$$

This together with (2-7) yields that

$$\sum_{n=1}^{\infty}\frac{N_n}{b_1\cdots b_n}=\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}+\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}<\infty,$$

completing the proof. □

Corollary 2.2. *Let $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than* 1, *and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ be a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \dots, N_n - 1\}$, where $N_n \geq 2$. If $N_n \leq b_n$ for each $n \geq 2$, then μ_n converges weakly to a Borel probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$.*

Proof. As $2 \leq N_n \leq b_n$ for each $n \geq 2$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{N_n}{b_1 \cdots b_n} &= \frac{N_1}{b_1} + \sum_{n=2}^{\infty} \frac{N_n}{b_1 \cdots b_n} \\ &\leq \frac{N_1}{b_1} + \sum_{n=2}^{\infty} \frac{1}{b_1 \cdots b_{n-1}} \leq \frac{N_1}{b_1} + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = \frac{N_1}{b_1} + 1 < \infty.\end{aligned}$$

Applying Theorem 1.1, the assertion follows. □

2.2. ***Spectral measure theoretic preliminaries.*** Let μ be a Borel probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined as usual:

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x}\, \mathrm{d}\mu(x) \quad \text{for any } \xi \in \mathbb{R}.$$

We will denote by $\mathscr{Z}(\hat{\mu}) = \{\xi \in \mathbb{R} : \hat{\mu}(\xi) = 0\}$ the zero set of $\hat{\mu}$, and by e_λ the exponential function $e^{-2\pi i \lambda x}$. Then for a discrete set $\Lambda \subset \mathbb{R}$, $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set of $L^2(\mu)$ if and only if $\hat{\mu}(\lambda - \lambda') = 0$ for $\lambda \neq \lambda' \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathscr{Z}(\hat{\mu}). \tag{2-9}$$

In this case, we say that Λ is a *bizero set* of μ. Moreover, Λ is called a *maximal bizero set* if it is maximal in $\mathscr{Z}(\hat{\mu})$ to have the set difference property. Since bizero sets (or spectra) are invariant under translation, without loss of generality, we always assume that $0 \in \Lambda$ in this paper. For $\xi \in \mathbb{R}$, write

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2.$$

The following criterion is a universal test to decide whether a countable set $\Lambda \subset \mathbb{R}$ is a bizero set (a spectrum) of μ or not.

Theorem 2.3 [Jorgensen and Pedersen 1998; Li et al. 2024]. *Let μ be a Borel probability measure, and let $\Lambda \subset \mathbb{R}$ be a countable set. Then*:

(i) *Λ is a bizero set of μ if and only if $Q_\Lambda(\xi) \leq 1$ for $\xi \in \mathbb{R}$.*

(ii) *Λ is a spectrum of μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}$.*

(iii) *$Q_\Lambda(\xi)$ has an entire analytic extension to $\mathbb{C}$ if Λ is a bizero set of μ.*

As a simple consequence of Theorem 2.3, the following useful theorem was proved in [Dai et al. 2014] and will be used to prove our main result.

Theorem 2.4. *Let $\mu = \nu * \omega$ be the convolution of two probability measures ν and ω that are not Dirac measures. Suppose that Λ is a bizero set of ν. Then Λ is also a bizero set of μ, but it cannot be a spectrum of μ.*

3. Proof of Theorem 1.3

In this section, we intend to complete the proof of Theorem 1.3. For simplicity, we write $\boldsymbol{b}_{m,n} = b_m \cdots b_n$ if $m \le n$. If $m = n$, we simply denote $\boldsymbol{b}_{m,n}$ by $\boldsymbol{b}_n$. Recall that $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a convolution, i.e., $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_n * \mu_{>n}$ for any integer $n \ge 1$, where

$$\mu_n := \mathop{\ast}_{k=1}^{n} \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k} \quad \text{and} \quad \mu_{>n} := \mathop{\ast}_{k=n+1}^{\infty} \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}. \tag{3-1}$$

So for any bizero set $0 \in \Lambda$ of μ, we must have

$$\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\} \subset \mathscr{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathscr{Z}(\hat{\mu}_n) \cup \mathscr{Z}(\hat{\mu}_{>n}).$$

We now introduce the definition of a *maximal decomposition of* Λ, which will be frequently used in the sequel.

Definition 3.1. Let $\mu = \nu * \omega$ be the convolution of two probability measures ν and ω. Suppose $0 \in \Lambda$ is a spectrum of μ and $0 \in \mathscr{A} \subset \Lambda$ is a maximal bizero set of ν. For any $\alpha \in \mathscr{A}$, let

$$\Lambda_\alpha := \big\{\lambda \in \Lambda : \lambda - \alpha \in \mathscr{Z}(\hat{\omega}) \setminus \mathscr{Z}(\hat{\nu})\big\} \cup \{\alpha\}.$$

We call the union

$$\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$$

a *maximal decomposition* with respect to (ν, ω).

Remark. In the definition, the maximality of $0 \in \mathscr{A} \subset \Lambda$ means that for any $\lambda \in \Lambda \setminus \mathscr{A}$, there is an $\alpha \in \mathscr{A}$ such that $\lambda - \alpha \in \mathscr{Z}(\hat{\omega}) \setminus \mathscr{Z}(\hat{\nu})$.

The maximal decomposition was first introduced by An and Wang [2021]. By the Jorgensen–Pedersen lemma [1998], Λ is a spectrum for a probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ if and only if

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi + \lambda)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}.$$

Substituting the maximal decomposition $\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$ with respect to $(\mu_n, \mu_{>n})$ into the above equation, we have

$$1 \le \sum_{a \in \mathscr{A}} \sum_{\lambda \in \Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi + \lambda)|^2 = \sum_{a \in \mathscr{A}} \sum_{\lambda \in \Lambda_a} |\hat{\mu}_n(\xi + \lambda)|^2 \, |\hat{\mu}_{>n}(\xi + \lambda)|^2.$$

In this paper, we are going to find a maximal decomposition $\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$ with respect to $(\mu_n, \mu_{>n})$ such that $\mathscr{A}$ is a spectrum of μ_n and Λ_a is a spectrum of $\mu_{>n}$ for each $a \in \mathscr{A}$.

With a direct calculation, we have $\mathcal{Z}(\hat{\delta}_{\mathcal{D}_k}) = (1/N_k)(\mathbb{Z} \setminus N_k\mathbb{Z})$. So

$$\mathcal{Z}(\hat{\mu}_n) = \bigcup_{k=1}^{n} \frac{\boldsymbol{b}_k}{N_k}(\mathbb{Z} \setminus N_k\mathbb{Z}), \quad \mathcal{Z}(\hat{\mu}_{>n}) = \bigcup_{k=n+1}^{\infty} \frac{\boldsymbol{b}_k}{N_k}(\mathbb{Z} \setminus N_k\mathbb{Z}), \tag{3-2}$$

and

$$\mathcal{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathcal{Z}(\hat{\mu}_n) \cup \mathcal{Z}(\hat{\mu}_{>n}) = \bigcup_{k=1}^{\infty} \frac{\boldsymbol{b}_k}{N_k}(\mathbb{Z} \setminus N_k\mathbb{Z}). \tag{3-3}$$

Now we make an important observation that will be used to guarantee a strong link between the spectral measures μ_n, $\mu_{>n}$ and $\mu_{\boldsymbol{b},\boldsymbol{D}}$.

Lemma 3.2. *Suppose $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure and $\{\lambda, \gamma\}$ is a bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. If $\lambda \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n})$ and $\gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathcal{D}_k}) \setminus \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n})$ with $k > n$, then we must have*

$$\lambda - \gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n}).$$

Proof. The bizero property of $\{\lambda, \gamma\}$ implies that

$$\lambda - \gamma \in \mathcal{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}).$$

Suppose to the contrary that

$$\lambda - \gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_j}\mathcal{D}_j}) \setminus \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n})$$

for some $j \neq n$. By (3-2), we can write them as

$$\lambda = \frac{\boldsymbol{b}_n}{N_n}a_n, \quad \gamma = \frac{\boldsymbol{b}_k}{N_k}a_k \quad \text{and} \quad \lambda - \gamma = \frac{\boldsymbol{b}_j}{N_j}a_j,$$

where $a_i \in \mathbb{Z} \setminus N_i\mathbb{Z}$, $i \in \{n, k, j\}$. Without loss of generality, we assume $j > n$. After some rearrangement, we have

$$\frac{a_n}{N_n} = \frac{\boldsymbol{b}_{n+1,k}}{N_k}a_k + \frac{\boldsymbol{b}_{n+1,j}}{N_j}a_j. \tag{3-4}$$

Reduce all fractions in the above equation to their simplest form, i.e.,

$$\frac{a'_n}{N'_n} = \frac{\boldsymbol{b}'_{n+1,k}}{N'_k}a_k + \frac{\boldsymbol{b}'_{n+1,j}}{N'_j}a_j = \frac{\boldsymbol{b}'_{n+1,k}a_k N'_j + \boldsymbol{b}'_{n+1,j}a_j N'_k}{N'_k N'_j},$$

where

$$\gcd(a'_n, N'_n) = 1, \quad \gcd(\boldsymbol{b}'_{n+1,k}, N'_k) = 1 \quad \text{and} \quad \gcd(\boldsymbol{b}'_{n+1,j}, N'_j) = 1. \tag{3-5}$$

This implies N'_n divides $N'_k N'_j$. Since $a_n \in \mathbb{Z} \setminus N_n\mathbb{Z}$, we have $N'_n > 1$. Let s_n be a prime factor of N'_n, and of course it is also a prime factor of N_n. Then we must have

$$s_n \mid N'_k \quad \text{or} \quad s_n \mid N'_j.$$

Without loss of generality, we assume $N_k' = s_n t_k'$ for some integer t_k'. It follows from the second equation in (3-5) that

$$\gcd(s_n, \boldsymbol{b}_{n+1,k}') = 1. \tag{3-6}$$

Write $t_k = \gcd(N_k, \boldsymbol{b}_{n+1,k})$. Then $N_k = N_k' t_k = s_n t_k t_k'$ and $\boldsymbol{b}_{n+1,k} = \boldsymbol{b}_{n+1,k}' t_k$. Let $\mathscr{E}_{t_k} = \{0, 1, \ldots, t_k - 1\}$, $\mathscr{E}_{t_k'} = \{0, 1, \ldots, t_k' - 1\}$ and $\mathscr{E}_{s_n} = \{0, 1, \ldots, s_n - 1\}$. We can factorize $\mathscr{D}_k$ as

$$\mathscr{D}_k = \mathscr{E}_{t_k} \oplus t_k \mathscr{E}_{s_n} \oplus s_n t_k \mathscr{E}_{t_k'}.$$

Write

$$\nu = \mathop{\ast}_{i \neq k} \delta_{\frac{1}{\boldsymbol{b}_i}\mathscr{D}_i} \ast \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{E}_{t_k}} \ast \delta_{\frac{1}{\boldsymbol{b}_k} s_n t_k \mathscr{E}_{t_k'}}.$$

Then $\mu_{\boldsymbol{b},\boldsymbol{D}} = \nu \ast \delta_{\frac{1}{\boldsymbol{b}_k} t_k \mathscr{E}_{s_n}}$. Note that

$$\begin{aligned}
\mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} t_k \mathscr{E}_{s_n}}) &= \frac{\boldsymbol{b}_n \boldsymbol{b}_{n+1,k}}{t_k s_n}(\mathbb{Z} \setminus s_n\mathbb{Z}) \\
&= \frac{\boldsymbol{b}_n \boldsymbol{b}_{n+1,k}'}{s_n}(\mathbb{Z} \setminus s_n\mathbb{Z}) \\
&\subset \frac{\boldsymbol{b}_n}{s_n}(\mathbb{Z} \setminus s_n\mathbb{Z}) && \text{(because } \gcd(\boldsymbol{b}_{n+1,k}', s_n) = 1) \\
&\subset \frac{\boldsymbol{b}_n}{N_n}(\mathbb{Z} \setminus N_n\mathbb{Z}) && \text{(because } s_n \mid N_n) \\
&= \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathscr{D}_n}) \subset \mathfrak{Z}(\hat{\nu}).
\end{aligned}$$

So

$$\mathfrak{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathfrak{Z}(\hat{\nu}) \cup \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} t_k \mathscr{E}_{s_n}}) = \mathfrak{Z}(\hat{\nu}).$$

Let Λ be an arbitrary bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. The above equation implies that it also a bizero set of ν. It follows from Theorem 2.4 that Λ cannot be a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. Therefore, $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is not a spectral measure, a contradiction. □

Lemma 3.3. *Suppose $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure and $\{\lambda, \gamma\}$ is a bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. Let $n \geq 1$ be an integer.*

(i) *If $\lambda \in \mathfrak{Z}(\hat{\mu}_n)$ and $\gamma \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)$, then*

$$\lambda - \gamma \in \mathfrak{Z}(\hat{\mu}_n).$$

(ii) *If $\lambda, \gamma \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)$, then*

$$\lambda - \gamma \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

Proof. (i) The assumption and (3-2) imply that

$$\lambda \in \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}) \quad \text{and} \quad \gamma \in \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_m}\mathscr{D}_m}) \setminus \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}) \qquad \text{for some } k \leq n < m.$$

We know from Lemma 3.2 that

$$\lambda-\gamma\in\mathscr{Z}(\hat{\delta}_{\frac{1}{b_k}\mathscr{D}_k})\subset\mathscr{Z}(\hat{\mu}_n).$$

(ii) It should be noticed that $\mathscr{Z}(\hat{\nu})=-\mathscr{Z}(\hat{\nu})$ for any measure ν. Suppose to the contrary that

$$\lambda':=\lambda-\gamma\in\mathscr{Z}(\hat{\mu}_n).$$

Since

$$\lambda=\lambda'-(-\gamma)\in\mathscr{Z}(\hat{\mu}_{>n})\setminus\mathscr{Z}(\hat{\mu}_n)\subset\mathscr{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}),$$

$\{\lambda',-\gamma\}$ is a bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$ with $\lambda'\in\mathscr{Z}(\hat{\mu}_n)$ and $-\gamma\in\mathscr{Z}(\hat{\mu}_{>n})\setminus\mathscr{Z}(\hat{\mu}_n)$. It follows from (i) that $\lambda=\lambda'-(-\gamma)\in\mathscr{Z}(\hat{\mu}_n)$, a contradiction. □

As a consequence of Lemma 3.3, we have:

Corollary 3.4. *Suppose that* $0\in\Lambda$ *is a spectrum of* $\mu_{\boldsymbol{b},\boldsymbol{D}}$. *Then* $\Lambda\cap\mathscr{Z}(\hat{\mu}_n)\neq\varnothing$ *for any* $n\geq 1$.

Proof. The bizero property of Λ implies that

$$\Lambda\setminus\{0\}\subset(\Lambda-\Lambda)\setminus\{0\}\subset\mathscr{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}})=\mathscr{Z}(\hat{\mu}_n)\cup\mathscr{Z}(\hat{\mu}_{>n}).$$

If the assertion is not true for some $n\geq 1$, then we have

$$\Lambda\setminus\{0\}\subset\mathscr{Z}(\hat{\mu}_{>n})\setminus\mathscr{Z}(\hat{\mu}_n).$$

From Lemma 3.3(ii), we have

$$(\Lambda-\Lambda)\setminus\{0\}\subset\mathscr{Z}(\hat{\mu}_{>n})\setminus\mathscr{Z}(\hat{\mu}_n),$$

that is, Λ is a bizero set of $\mu_{>n}$. It follows from Theorem 2.4 that Λ cannot be a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, which is a contradiction. □

Lemma 3.5. *Let* $0\in\Lambda$ *be a spectrum of* $\mu_{\boldsymbol{b},\boldsymbol{D}}$. *Then* $\Lambda\subset(1/N_1)\mathbb{Z}$.

Proof. We write $\mu_{\boldsymbol{b},\boldsymbol{D}}=\mu_1*\mu_{>1}$ and decompose $\Lambda=\bigcup_{\alpha\in\mathscr{A}}\Lambda_\alpha$ into a maximal form with respect to $(\mu_1,\mu_{>1})$. That is, $0\in\mathscr{A}\subset\Lambda$ is a maximal bizero set of μ_1, and

$$\Lambda_\alpha:=\left\{\lambda\in\Lambda:\lambda-\alpha\in\mathscr{Z}(\hat{\mu}_{>1})\setminus\mathscr{Z}(\hat{\mu}_1)\right\}\cup\{\alpha\}\quad\text{for all }\alpha\in\mathscr{A}.$$

It follows from Corollary 3.4 that $\mathscr{A}\setminus\{0\}$ is not empty. Recall that

$$\mathscr{Z}(\hat{\mu}_1)=\mathscr{Z}(\hat{\delta}_{\frac{1}{b_1}\mathscr{D}_1})=\frac{\boldsymbol{b}_1}{N_1}(\mathbb{Z}\setminus N_1\mathbb{Z})\subset\frac{1}{N_1}\mathbb{Z}.$$

For any element $\lambda\in\Lambda$, if $\lambda\in\mathscr{A}$, then

$$\lambda\in\mathscr{A}\subset\mathscr{Z}(\hat{\mu}_1)\cup\{0\}\subset\frac{1}{N_1}\mathbb{Z}.$$

Otherwise, the maximality of $\mathscr{A}$ implies that there is a $\alpha \in \mathscr{A}$ such that

$$\lambda - \alpha \in \mathfrak{Z}(\hat{\mu}_{>1}) \setminus \mathfrak{Z}(\hat{\mu}_1).$$

Take an element $\alpha' \in \mathscr{A} \setminus \{\alpha\}$. Then $\alpha' - \alpha \in \mathfrak{Z}(\hat{\mu}_1)$. From Lemma 3.3(i), we have

$$\alpha' - \lambda = (\alpha' - \alpha) - (\lambda - \alpha) \in \mathfrak{Z}(\hat{\mu}_1).$$

Hence

$$\lambda = \alpha' - (\alpha' - \lambda) \in \mathscr{A} - \mathfrak{Z}(\hat{\mu}_1) \subset \frac{1}{N_1}\mathbb{Z}.$$

So $\Lambda \subset (1/N_1)\mathbb{Z}$. □

Theorem 1.3 shows if $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure, then any "truncation" of it is still a spectral measure. The proof is inspired by [An and Wang 2021, Proposition 4.3].

Proof of Theorem 1.3. For any $n \geq 1$, we write $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_n * \mu_{>n}$ and decompose Λ into a maximal form $\bigcup_{\alpha \in \mathscr{A}_0} \Lambda_\alpha$ with respect to $(\mu_n, \mu_{>n})$. That is, $0 \in \mathscr{A}_0 \subset \Lambda$ is a maximal bizero set of μ_n, and

$$\Lambda_\alpha := \big\{\lambda \in \Lambda : \lambda - \alpha \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)\big\} \cup \{\alpha\} \quad \text{for all } \alpha \in \mathscr{A}_0.$$

We first claim that the $\{\Lambda_\alpha\}_{\alpha \in \mathscr{A}_0}$ are disjoint pairwise. Otherwise, suppose there is an element λ such that

$$\lambda \in \Lambda_\alpha \cap \Lambda_{\alpha'}$$

for some $\alpha \neq \alpha' \in \mathscr{A}_0$. Since $\alpha - \alpha' \in \mathfrak{Z}(\hat{\mu}_n)$, we have $\alpha \in \Lambda_\alpha \setminus \Lambda_{\alpha'}$ and $\alpha' \in \Lambda_{\alpha'} \setminus \Lambda_\alpha$. So $\lambda \neq \alpha$ and $\lambda \neq \alpha'$. From the definition of Λ_α, we have

$$\lambda - \alpha, \lambda - \alpha' \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

Taking $\lambda_1 = \lambda - \alpha'$ and $\lambda_2 = \lambda - \alpha$ in Lemma 3.3, we have

$$\alpha - \alpha' = (\lambda - \alpha') - (\lambda - \alpha) \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

This contradicts the fact that $\mathscr{A}_0$ is a bizero set of μ_n. The claim holds.

For any $\alpha \in \mathscr{A}_0$ and $\lambda_\alpha \neq \lambda'_\alpha \in \Lambda_\alpha$, we have

$$\lambda_\alpha - \alpha,\ \lambda'_\alpha - \alpha \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

From Lemma 3.3, we have

$$\lambda_\alpha - \lambda'_\alpha = (\lambda_\alpha - \alpha) - (\lambda'_\alpha - \alpha) \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n). \tag{3-7}$$

For any $\alpha' \neq \alpha \in \mathscr{A}_0$, as $\lambda_\alpha \notin \Lambda_{\alpha'}$, we have

$$\lambda_\alpha - \alpha' \in \mathfrak{Z}(\hat{\mu}_n).$$

Because $\lambda_{\alpha'} - \alpha' \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)$ for any $\lambda_{\alpha'} \in \Lambda_{\alpha'}$, Lemma 3.2 implies that

$$\lambda_\alpha - \lambda_{\alpha'} = (\lambda_\alpha - \alpha') - (\lambda_{\alpha'} - \alpha') \in \mathfrak{Z}(\hat{\mu}_n). \tag{3-8}$$

Summarizing (3-7) and (3-8), we have

$$(\Lambda_\alpha - \Lambda_\alpha)\setminus\{0\} \subset \mathscr{Z}(\hat{\mu}_{>n})\setminus\mathscr{Z}(\hat{\mu}_n) \quad \text{and} \quad \Lambda_\alpha - \Lambda_{\alpha'} \subset \mathscr{Z}(\hat{\mu}_n). \tag{3-9}$$

For any $\xi \in [0, 1]$, we have

$$Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi+\lambda)|^2 = \sum_{\alpha\in\mathscr{A}_0}\sum_{\lambda_\alpha\in\Lambda_\alpha} |\hat{\mu}_n(\xi+\lambda_\alpha)|^2\, |\hat{\mu}_{>n}(\xi+\lambda_\alpha)|^2. \tag{3-10}$$

We define an equivalence relationship $\sim$ such that $\lambda\sim\lambda'$ whenever $\lambda'-\lambda \in b_1b_2\cdots b_n\mathbb{Z}$. Then the quotient group $\Lambda_\alpha/\sim = \{[\lambda] : \lambda\in\Lambda_\alpha\}$ is a partition of Λ_α, where

$$[\lambda] := \{\lambda' \in \Lambda_\alpha : \lambda' \sim \lambda\}.$$

Since $\Lambda \subset (1/N_1)\mathbb{Z}$ (from Lemma 3.5), $\Lambda_\alpha/\sim$ is a finite set, which we will denote by $\{[\lambda_{\alpha,1}], \dots, [\lambda_{\alpha,n_\alpha}]\}$. Note that $\hat{\mu}_n$ is $(b_1b_2\cdots b_n)$-periodic. For any $\lambda \in [\lambda_{\alpha,i}]$,

$$|\hat{\mu}_n(\xi+\lambda)| = |\hat{\mu}_n(\xi+\lambda_{\alpha,i})| \quad \text{for all } \xi\in\mathbb{R}.$$

For any $\xi\in[0,1]$ and $\alpha\in\mathscr{A}_0$, there is a unique $\lambda_{\alpha,i(\xi)}$ with $i(\xi)\in\{1,2,\dots,n_\alpha\}$ such that

$$|\hat{\mu}_n(\xi+\lambda_{\alpha,i(\xi)})| = \max\{|\hat{\mu}_n(\xi+\lambda_{\alpha,i})| : 1\le i\le n_\alpha\} = \max\{|\hat{\mu}_n(\xi+\lambda_\alpha)| : \lambda_\alpha\in\Lambda_\alpha\}.$$

As $\mathscr{A}_0$ and $\Lambda_\alpha/\sim$ are finite sets, we can find a finite set $\{\lambda_{\alpha,i_\alpha}\}_{\alpha\in\mathscr{A}_0}$ such that $\lambda_{\alpha,i(\xi_j)} = \lambda_{\alpha,i_\alpha}$ for infinitely many $\{\xi_j\}_{j=1}^\infty$. Combined with (3-10), we have

$$\begin{aligned} Q_\Lambda(\xi_j) &\le \sum_{\alpha\in\mathscr{A}_0} |\hat{\mu}_n(\xi_j+\lambda_{\alpha,i_\alpha})|^2 \Big(\sum_{\lambda_\alpha\in\Lambda_\alpha} |\hat{\mu}_{>n}(\xi_j+\lambda_\alpha)|^2\Big) \\ &\le \sum_{\alpha\in\mathscr{A}_0} |\hat{\mu}_n(\xi_j+\lambda_{\alpha,i_\alpha})|^2 \\ &\le 1, \end{aligned}$$

where the last two inequalities follow from (3-9) and Theorem 2.3(i). On the other hand, as Λ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, we have $Q_\Lambda(\xi)\equiv 1$. This forces

$$\sum_{\lambda_\alpha\in\Lambda_\alpha} |\hat{\mu}_{>n}(\xi_j+\lambda_\alpha)|^2 \equiv 1 \quad \text{and} \quad \sum_{\alpha\in\mathscr{A}_0} |\hat{\mu}_n(\xi_j+\lambda_{\alpha,i_\alpha})|^2 \equiv 1 \qquad \text{for all } j\ge 1.$$

As all $\xi_j\in[0,1]$, the entire function property implies that

$$\sum_{\lambda_\alpha\in\Lambda_\alpha} |\hat{\mu}_{>n}(\xi+\lambda_\alpha)|^2 \equiv 1 \quad \text{and} \quad \sum_{\alpha\in\mathscr{A}_0} |\hat{\mu}_n(\xi+\lambda_{\alpha,i_\alpha})|^2 \equiv 1 \qquad \text{for all } \xi\in\mathbb{R}.$$

Hence $\mathscr{A} = \{\lambda_{\alpha,i_\alpha}\}_{\alpha\in\mathscr{A}_0} \subset \Lambda$ is a spectrum of μ_n and each Λ_α is a spectrum of $\mu_{>n}$. We now decompose $\Lambda = \bigcup_{\alpha\in\mathscr{A}}\Lambda_\alpha$ into a maximal form with respect to $(\mu_n, \mu_{>n})$. Then the desired result follows. □

4. Proofs of Theorems 1.2 and 1.4

Theorem 1.3 gives a necessary condition for $\mu_{\boldsymbol{b},\boldsymbol{D}}$ to be spectral. In this section, we will focus on analyzing the spectrality of μ_n, $n \geq 2$. From (3-2), for any $1 \leq k < n$, we have

$$\mu_n = \mu_k * \mu_{k+1,n},$$

where

$$\mu_k = \mathop{\ast}_{j=1}^{k} \delta_{\frac{1}{b_j}\mathscr{D}_j} \quad \text{and} \quad \mu_{k+1,n} = \mathop{\ast}_{j=k+1}^{n} \delta_{\frac{1}{b_j}\mathscr{D}_j}.$$

With the same proof, the conclusions in Lemma 3.3 and Theorem 1.3 are also true for μ_n.

Lemma 4.1. *Let n and k be two integers such that $n \geq 2$ and $1 \leq k < n$. Suppose μ_n is a spectral measure and $\{\lambda, \gamma\}$ is a bizero set of μ_n.*

(i) *If $\lambda \in \mathscr{Z}(\hat{\mu}_k)$ and $\gamma \in \mathscr{Z}(\hat{\mu}_{k+1,n}) \setminus \mathscr{Z}(\hat{\mu}_k)$, then $\lambda - \gamma \in \mathscr{Z}(\hat{\mu}_k)$.*

(ii) *If $\lambda, \gamma \in \mathscr{Z}(\hat{\mu}_{k+1,n}) \setminus \mathscr{Z}(\hat{\mu}_k)$, then $\lambda - \gamma \in \mathscr{Z}(\hat{\mu}_{k+1,n}) \setminus \mathscr{Z}(\hat{\mu}_k)$.*

Proof. The proof is the same as that of Lemma 3.3. □

Theorem 4.2. *If $0 \in \Lambda$ is a spectrum of μ_n, then, for each $1 \leq k < n$, we can perform a maximal decomposition on it, denoted by $\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$, such that $\mathscr{A}$ is a spectrum of μ_k and each Λ_α is a spectrum of $\mu_{k+1,n}$.*

Proof. The proof is the same as that of Theorem 1.3. □

Lemma 4.3. *Suppose that $\mathscr{E}_N = \{0, 1, \dots, N-1\}$ is a consecutive digit set with cardinality N. Then $\delta_{\mathscr{E}_N}$ is a spectral measure. Moreover, $0 \in \mathscr{C}$ is a spectrum of $\delta_{\mathscr{E}_N}$ if and only if $\#\mathscr{C} = N$ and $\mathscr{C} \equiv \{0, 1/N, \dots, (N-1)/N\} \pmod{\mathbb{Z}}$.*

Proof. It has been proved in [An and He 2014] that $\delta_{\mathscr{E}_N}$ is a spectral measure and admits a spectrum $\{0, 1/N, \dots, (N-1)/N\}$. Since $\hat{\delta}_{\mathscr{E}_N}$ is 1-periodic, for any digit set $0 \in \mathscr{C}$ with $\#\mathscr{C} = N$ and $\mathscr{C} \equiv \{0, 1/N, \dots, (N-1)/N\} \pmod{\mathbb{Z}}$, we have

$$\sum_{c \in \mathscr{C}} |\hat{\delta}_{\mathscr{E}_N}(\xi + c)|^2 = \sum_{j=0}^{N-1} \left| \hat{\delta}_{\mathscr{E}_N}\left(\xi + \frac{j}{N}\right) \right|^2.$$

It follows from Theorem 2.3(ii) that $\mathscr{C}$ is a spectrum of $\delta_{\mathscr{E}_N}$.

Next we will prove sufficiency. Suppose $0 \in \mathscr{C}$ is a spectrum of $\delta_{\mathscr{E}_N}$. The bizero property implies that

$$\mathscr{C} \setminus \{0\} \subset (\mathscr{C} - \mathscr{C}) \setminus \{0\} \subset \mathscr{Z}(\hat{\delta}_{\mathscr{E}_N}) = \frac{1}{N}(\mathbb{Z} \setminus N\mathbb{Z}).$$

Therefore $\mathscr{C} \pmod{\mathbb{Z}} \subset \{0, 1/N, \dots, (N-1)/N\}$. Completeness implies $\#\mathscr{C} = \dim L^2(\delta_{\mathscr{E}_N}) = N$, so

$$\mathscr{C} \equiv \left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}\right\} \pmod{\mathbb{Z}}.$$ □

Proof of Theorem 1.4. The sufficiency can be found in [An and He 2014]. For the necessity, suppose on the contrary that there is $2 \le k \le n$ such that N_k does not divide b_k. Write $d = \gcd(b_k, N_k)$, and write $b_k = db'_k$, $N_k = dN'_k$. Then N'_k has a prime factor s_k which does not divide b'_k. Let $N'_k = s_k t_k$. Then $N_k = dt_k s_k$.

According to Theorem 4.2, the spectrality of $\mu_n = \mu_{k-2} * \mu_{k-1,n}$ implies that $\mu_{k-1,n}$ is also a spectral measure. If $n = k$, then $\mu_{k-1,n} = \mu_{k-1,n}$. If $n > k$, then $\mu_{k-1,n} = \mu_{k-1,k} * \mu_{k+1,n}$. From Theorem 4.2 again, $\mu_{k-1,k}$ is a spectral measure too. Suppose $0 \in \Lambda$ is a spectrum of $\mu_{k-1,k} = \delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}} * \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}$. We can decompose it into a maximal form

$$\Lambda = \bigcup_{a \in \mathscr{A}} \Lambda_a$$

with respect to $(\delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}}, \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k})$; that is, $0 \in \mathscr{A}$ is a maximal set of $\delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}}$, and each

$$\Lambda_a = \left\{\lambda \in \Lambda : \lambda - a \in \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}) \setminus \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}})\right\} \cup \{a\}$$

is a spectrum of $\delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}$. Moreover, one can check that $\{\Lambda_a\}_{a \in \mathscr{A}}$ are disjoint pairwise. As $0 \in \Lambda_0$, it follows from Lemma 4.3 that

$$\Lambda_0 = \boldsymbol{b}_k \left\{0, \frac{1}{N_k} + m_1, \ldots, \frac{N_k - 1}{N_k} + m_{N_k - 1}\right\}.$$

Fix an element $a \in \mathscr{A} \setminus \{0\} \subset \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}})$. Then $a = (\boldsymbol{b}_{k-1}/N_{k-1})m$ for some $m \in \mathbb{Z} \setminus N_{k-1}\mathbb{Z}$. Recall $N_k = dt_k s_k$, where s_k is a prime integer and $1 \le t_k \le N_k - 1$. So

$$\boldsymbol{b}_k \left(\frac{t_k}{N_k} + m_{t_k}\right) \in \Lambda_0.$$

This, together with $\Lambda_0 \cap \Lambda_a = \varnothing$, implies that $\boldsymbol{b}_k(t_k/N_k + m_{t_k}) \notin \Lambda_a$ and so

$$\frac{\boldsymbol{b}_{k-1}}{N_{k-1}} m - \boldsymbol{b}_k \left(\frac{t_k}{N_k} + m_{t_k}\right) = \frac{\boldsymbol{b}_{k-1}}{N_{k-1}} m' \in \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}})$$

for some $m' \in \mathbb{Z} \setminus N_{k-1}\mathbb{Z}$. After some rearrangement, we have

$$\frac{m - m' - N_{k-1} b_k m_{t_k}}{N_{k-1}} = \frac{b_k t_k}{N_k} = \frac{b'_k}{s_k}.$$

This implies that s_k divides N_{k-1} since $\gcd(s_k, b'_k) = 1$. Let $\mathscr{E}_d = \{0, 1, \ldots, d-1\}$, $\mathscr{E}_{s_k} = \{0, 1, \ldots, s_k - 1\}$ and $\mathscr{E}_{t_k} = \{0, 1, \ldots, t_k - 1\}$. We factorize $\mathscr{D}_k$ as

$$\mathscr{D}_k = \mathscr{E}_d \oplus d\mathscr{E}_{s_k} \oplus ds_k \mathscr{E}_{t_k}.$$

Write

$$\nu = \delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}} * \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{E}_d} * \delta_{\frac{1}{\boldsymbol{b}_k} s_k d \mathscr{E}_{t_k}}.$$

Then $\mu_{k-1,k} = \nu * \delta_{\frac{1}{\boldsymbol{b}_k} d\mathscr{E}_{s_k}}$. Note that

$$\begin{aligned}
\mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} d\mathscr{E}_{s_k}}) &= \frac{\boldsymbol{b}_{k-1} b_k}{d s_k}(\mathbb{Z} \setminus s_k \mathbb{Z}) \\
&= \frac{\boldsymbol{b}_{k-1} b'_k}{s_k}(\mathbb{Z} \setminus s_k \mathbb{Z}) \\
&\subset \frac{\boldsymbol{b}_{k-1}}{s_k}(\mathbb{Z} \setminus s_k \mathbb{Z}) \qquad \text{(because } \gcd(b'_k, s_k) = 1) \\
&\subset \frac{\boldsymbol{b}_{k-1}}{N_{k-1}}(\mathbb{Z} \setminus N_{k-1} \mathbb{Z}) \qquad \text{(because } s_k \mid N_{k-1}) \\
&= \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}} \mathscr{D}_{k-1}}) \subset \mathfrak{Z}(\hat{\nu}).
\end{aligned}$$

So

$$\mathfrak{Z}(\hat{\mu}_{k-1,k}) = \mathfrak{Z}(\hat{\nu}) \cup \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} d\mathscr{E}_{s_k}}) = \mathfrak{Z}(\hat{\nu}).$$

This implies that Λ is also a bizero set of ν. It follows from Theorem 2.4 that Λ cannot be a spectrum of $\mu_{k-1,k}$, which is a contradiction. Hence N_k divides b_k for each $2 \le k \le n$. □

Before proving Theorem 1.2, we need some useful lemmas. The following lemma is well known. For the reader's convenience, we provide a proof here.

Lemma 4.4. *Λ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$ if and only if $(1/a)\Lambda$ is a spectrum of $\mu_{\boldsymbol{b},a\boldsymbol{D}}$ for any $a \in \mathbb{R} \setminus \{0\}$.*

Proof. Note that

$$\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{\frac{1}{\boldsymbol{b}_n} a\mathscr{D}_n}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{\frac{1}{\boldsymbol{b}_n} \mathscr{D}_n}(a\xi) = \hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(a\xi).$$

Hence

$$\sum_{\gamma \in (1/a)\Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi + \gamma)|^2 = \sum_{\gamma \in (1/a)\Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(a(\xi + \gamma))|^2 = \sum_{\lambda \in \Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(a\xi + \lambda)|^2.$$

The assertion follows from Theorem 2.3. □

The following theorem has been proved by Gabardo and Lai [2014].

Theorem 4.5 [Gabardo and Lai 2014]. *Any positive Borel measures μ and ν such that $\mu * \nu = \mathscr{L}_{[0,1]}$ are spectral measures.*

Now we have all ingredients for the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) ⇒ (iii): This follows from Theorems 1.3 and 1.4.

(iii) ⇒ (ii): Suppose N_n divides b_n for each $n \ge 2$. Set $r_n = b_n/N_n$ for $n \ge 2$. So

$$\mathscr{D}_n \oplus N_n\{0, 1, \ldots, r_n - 1\} = \{0, 1, \ldots, b_n - 1\}$$

for $n \geq 2$. We write $C_1 = \{0\}$ and $C_n = N_n\{0, 1, \ldots, r_n - 1\}$ for $n \geq 2$ and let $\mu_{\boldsymbol{b},\boldsymbol{C}}$ be the Cantor–Moran measure generated by $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ and $\boldsymbol{C} = \{C_n\}_{n=1}^{\infty}$. Then

$$\mu_{\boldsymbol{b},\boldsymbol{D}} * \mu_{\boldsymbol{b},\boldsymbol{C}} = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \mathscr{L}_{[0,1/b_1]} = \mathscr{L}_{[0,N_1/b_1]}.$$

This implies that the generalized Fuglede's conjecture holds for $\mu_{\boldsymbol{b},\boldsymbol{D}}$.

(ii) ⇒ (i): Suppose there exists a Borel probability measure ν such that $\mu_{\boldsymbol{b},\boldsymbol{D}} * \nu = \mathscr{L}_{[0,N_1/b_1]}$. Let $f(x) = (N_1/b_1)x$ for $x \in \mathbb{R}$. Then $\mu_{\boldsymbol{b},(b_1/N_1)\boldsymbol{D}} * (\nu \circ f) = \mathscr{L}_{[0,1]}$. From Theorem 4.5, $\mu_{\boldsymbol{b},(b_1/N_1)\boldsymbol{D}}$ is a spectral measure and so is $\mu_{\boldsymbol{b},\boldsymbol{D}}$ according to Lemma 4.4. □

5. Proof of Theorem 1.5 and an open question

In this section, we first provide the proof of Theorem 1.5. Then, we conclude this paper with a conjecture on the relationship between the spectral Cantor–Moran measure and the tiling of integers.

We first introduce a theorem from Tijdeman [1995].

Theorem 5.1 [Tijdeman 1995]. *Suppose that A is finite*, $0 \in A \cap B$ *and* $A \oplus B = \mathbb{Z}$. *If r and $\#A$ are relatively prime*, *then* $rA \oplus B = \mathbb{Z}$.

Proof of Theorem 1.5. It would be easier to prove that $\boldsymbol{D}_n = \mathscr{D}_n \oplus b_n\mathscr{D}_{n-1} \oplus \cdots \oplus b_2 \cdots b_n\mathscr{D}_1$ is an integer tile if and only if N_n divides b_n for each $n \geq 2$. Suppose for any $n \geq 2$, $b_n = r_n N_n$ for some integer r_n. Then it is clear that $\boldsymbol{D}_n = \mathscr{D}_n + b_n\mathscr{D}_{n-1} + \cdots + b_2 \cdots b_n\mathscr{D}_1$ is a direct sum. Write $C_1 = \{0\}$, $C_n = N_n\{0, 1, \ldots, r_n - 1\}$ and

$$\boldsymbol{C}_n = C_n \oplus b_n C_{n-1} \oplus \cdots \oplus b_2 \cdots b_n C_1$$

for $n \geq 2$. Then $\mathscr{D}_n \oplus C_n = \{0, 1, \ldots, b_n - 1\}$ and so

$$\boldsymbol{D}_n \oplus \boldsymbol{C}_n = \{0, 1, \ldots, N_1 b_2 \cdots b_n - 1\}$$

for each $n \geq 1$. This implies that $\boldsymbol{D}_n$ is an integer tile.

Suppose that the converse conclusion is false; that is, suppose that $\boldsymbol{D}_n = \mathscr{D}_n + b_n\mathscr{D}_{n-1} + \cdots + b_2 \cdots b_n\mathscr{D}_1$ is a direct sum and an integer tile, but there is an N_n that does not divide b_n. Then there must be a prime factor p_n of N_n which is coprime with b_n. Note that

$$\mathscr{D}_n = \{0, 1, \ldots, p_n - 1\} \oplus p_n\{0, 1, \ldots, N_n/p_n - 1\}. \tag{5-1}$$

The integer-tiling property of $\boldsymbol{D}_n$ implies that we can find $0 \in C \subset \mathbb{Z}$ such that

$$\boldsymbol{D}_n \oplus C = \{0, 1, \ldots, p_n - 1\} \oplus b_n\boldsymbol{D}_{n-1} \oplus p_n\{0, 1, \ldots, N_n/p_n - 1\} \oplus C = \mathbb{Z}.$$

Take $A = \{0, 1, \dots, p_n - 1\}$ and $B = b_n \boldsymbol{D}_{n-1} \oplus p_n\{0, 1, \dots, N_n/p_n - 1\} \oplus C$. As $\gcd(p_n, b_n) = 1$ and $\#A = p_n$, from Theorem 5.1 we have

$$b_n A \oplus b_n \boldsymbol{D}_{n-1} \oplus p_n\{0, 1, \dots, N_n/p_n - 1\} \oplus C = \mathbb{Z}.$$

The above direct sum cannot happen as $\{0, 1\} \subset A \cap \boldsymbol{D}_{n-1}$. We have a contradiction and this shows our desired statement holds. □

Theorem 1.5 demonstrates a strong connection between the spectral Cantor–Moran measure and integer tiles. For the self-similar case $((b_n, \mathscr{D}_n) \equiv (b, \mathscr{D}))$, Łaba and Wang [2002] conjectured that if $\mu_{b,\mathscr{D}}$ is a spectral measure, then $\mathscr{D}$ is an integer tile. However, being an integer tile is not a sufficient condition. Here is a simple example observed by An, He and Lau [An et al. 2015]: Let $b = 8$ and $\mathscr{D} = \{0, 1, 8, 9\}$. Then $\mathscr{D}$ tiles $\mathbb{Z}_{16}$, but the self-similar measure $\mu_{8,\mathscr{D}}$ is not spectral. This is mainly because $\mathscr{D} + 8\mathscr{D}$ is not a direct sum.

Recall that $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_n * \mu_{>n}$ and

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n} = \delta_{\frac{1}{b_1 b_2 \cdots b_n}\boldsymbol{D}_n},$$

where $\boldsymbol{D}_n = \mathscr{D}_n + b_n \mathscr{D}_{n-1} + b_2 \cdots b_n \mathscr{D}_1$. According to Fuglede's conjecture on cyclic groups, one may ask:

Question. If $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure, is each of the iterated digit sets $\boldsymbol{D}_n = \mathscr{D}_n + b_n \mathscr{D}_{n-1} + b_2 \cdots b_n \mathscr{D}_1$ (for $n \ge 1$) an integer tile?

Our Theorem 1.5 settles the case where $\mathscr{D}_n = \{0, 1, \dots, N_n - 1\}$. And we eagerly anticipate further positive outcomes regarding this inquiry.

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LI-XIANG AN
SCHOOL OF MATHEMATICS AND STATISTICS
CENTRAL CHINA NORMAL UNIVERSITY
WUHAN
CHINA
anlixiang@163.com

QIAN LI
SCHOOL OF MATHEMATICS AND STATISTICS
HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
WUHAN
CHINA
liqian303606@163.com

MIN-MIN ZHANG
DEPARTMENT OF APPLIED MATHEMATICS
ANHUI UNIVERSITY OF TECHNOLOGY
MA'ANSHAN
CHINA
zhangminmin0907@163.com

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MINIMAL FREE RESOLUTIONS OF NUMERICAL SEMIGROUP ALGEBRAS VIA APÉRY SPECIALIZATION

BENJAMIN BRAUN, TARA GOMES, EZRA MILLER,
CHRISTOPHER O'NEILL AND ALEKSANDRA SOBIESKA

Numerical semigroups with multiplicity m are parametrized by integer points in a polyhedral cone C_m, according to Kunz. For the toric ideal of any such semigroup, the main result here constructs a free resolution whose overall structure is identical for all semigroups parametrized by the relative interior of a fixed face of C_m. The matrix entries of this resolution are monomials whose exponents are parametrized by the coordinates of the corresponding point in C_m, and minimality of the resolution is achieved when the semigroup is of maximal embedding dimension, which is the case when it is parametrized by the interior of C_m itself.

1. Introduction

Given a numerical semigroup S, the corresponding semigroup algebra has a defining toric ideal I_S. While the study of algebraic invariants of this numerical semigroup ideal I_S falls within the broader study of toric ideals, the family of numerical semigroup ideals forms a rich and interesting area of study that often affords more refined general results than those known or possible for the general toric setting. Our aim is to uniformly construct explicit free resolutions that are minimal for numerical semigroups with maximal embedding dimension, are parametrized by Apéry data, and therefore specialize to minimal free resolutions for numerical semigroups with arbitrary embedding dimension.

For general toric ideals, there is a substantial literature on their resolutions. In 1998, Peeva and Sturmfels described minimal free resolutions for generic lattice ideals [23]. More recently, Tchernev gave an explicit recursive algorithm for canonical minimal resolutions of toric rings [28]. Further, Li, Miller, and Ordog construct a canonical minimal free resolution of an arbitrary positively graded lattice ideal with a closed-form combinatorial description of the differential in

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characteristic 0 and all but finitely many positive characteristics [21]. However, these constructions are all quite general, and one would hope that in the special case of numerical semigroups, explicit resolutions of I_S more directly tied to the combinatorics of S are possible.

Free resolutions of I_S for a numerical semigroup S are known in some special cases. The surveys [12; 27] include most results concerning special families. We outline a few here. If S is maximal embedding dimension (MED) and I_S is determinantal (that is, generated by the minors of a matrix), then I_S is resolved by the Eagon–Northcott complex [15]. This accounts for some, but not all, MED numerical semigroups; a characterization of determinantal MED numerical semigroups is given in [19]. If S is generated by an arithmetic sequence, then I_S is minimally resolved by a variant of the Eagon–Northcott complex [13]. If S is obtained as a gluing of two numerical semigroups T and T', then a minimal free resolution of I_S can be obtained from the minimal free resolutions of I_T and $I_{T'}$ via a mapping cone construction [11]; this includes the case where I_S is complete intersection. If S has at most 3 generators, then a minimal free resolution is known. Numerous families of 4-generated numerical semigroups have also been investigated; see the survey [27] for more detail.

The present work is motivated by recent papers that examine a family of convex rational polyhedra C_m called *Kunz cones*, one for each integer $m \geq 2$, for which each numerical semigroup S with multiplicity m — that is, with $m = \min(S \setminus \{0\})$ — corresponds to an integer point of C_m. These were first introduced in [20], and most subsequent papers on the topic have employed lattice point techniques to examine enumerative questions [1; 17; 26]. However, seemingly overlooked for decades was another result in [20] that proved two numerical semigroups S and T correspond to points (relative) interior to the same face of C_m if and only if certain artinian quotients of I_S and I_T coincide. Indeed, a corollary of this result, namely that MED numerical semigroups are precisely those lying in the interior of C_m, was the only reference to the faces of C_m in the literature until a series of recent papers [4; 18] unknowingly rederived a combinatorial version of Kunz's result: S and T lie in the same face of C_m if and only if certain subsets of their divisibility posets coincide.

Kunz also observed in [20] that, as a consequence of his result, $\beta_d(I_S) = \beta_d(I_T)$ for every d, though his approach did not allow for explicit construction of syzygies for I_S and I_T. The combinatorial viewpoint of [4; 18] was recently used in [14] to make Kunz's enumerative result algebraic in the case $d = 1$: if S and T lie interior to the same face of C_m, then minimal binomial generating sets are explicitly constructed for I_S and I_T that coincide in all terms except the exponents of a single variable.

Our main result is to similarly make Kunz's result algebraic for all positive d: we construct explicit free resolutions of all numerical semigroup ideals. These resolutions are minimal when the semigroup has maximal embedding dimension

$$0 \leftarrow R \xleftarrow{\;\begin{array}{c|cccccc} & \mathbf{1,1} & \mathbf{2,2} & \mathbf{3,3} & \mathbf{2,1} & \mathbf{3,1} & \mathbf{3,2} \\ \hline \varnothing & x_1^2 - x_2y^{b_{11}} & x_2^2 - y^{b_{22}} & x_3^2 - x_2y^{b_{33}} & x_1x_2 - x_3y^{b_{12}} & x_1x_3 - y^{b_{13}} & x_2x_3 - x_1y^{b_{23}} \end{array}\;} R^6$$

$$R^6 \xleftarrow{\;\begin{array}{c|cccccccc} & \mathbf{1,12} & \mathbf{1,13} & \mathbf{2,12} & \mathbf{2,23} & \mathbf{3,13} & \mathbf{3,23} & \mathbf{2,13} & \mathbf{3,12} \\ \hline \mathbf{1,1} & -x_2 & -x_3 & & & & & y^{b_{23}} & y^{b_{23}} \\ \mathbf{2,2} & -y^{b_{11}} & & x_1 & -x_3 & & y^{b_{33}} & & \\ \mathbf{3,3} & & & & & x_1 & x_2 & -y^{b_{12}} & \\ \mathbf{2,1} & x_1 & & -x_2 & y^{b_{23}} & y^{b_{33}} & & -x_3 & \\ \mathbf{3,1} & y^{b_{12}} & x_1 & & & -x_3 & -y^{b_{23}} & & -x_2 \\ \mathbf{3,2} & & -y^{b_{11}} & -y^{b_{12}} & x_2 & & -x_3 & x_1 & x_1 \end{array}\;} R^8$$

$$R^8 \xleftarrow{\;\begin{array}{c|ccc} & \mathbf{1,[3]} & \mathbf{2,[3]} & \mathbf{3,[3]} \\ \hline \mathbf{1,12} & x_3 & -y^{b_{23}} & \\ \mathbf{1,13} & -x_2 & & y^{b_{23}} \\ \mathbf{2,12} & & x_3 & -y^{b_{33}} \\ \mathbf{2,23} & -y^{b_{11}} & x_1 & \\ \mathbf{3,13} & y^{b_{12}} & & -x_2 \\ \mathbf{3,23} & & -y^{b_{12}} & x_1 \\ \mathbf{2,13} & x_1 & -x_2 & \\ \mathbf{3,12} & -x_1 & & x_3 \end{array}\;} R^3 \leftarrow 0$$

Figure 1. The Apéry resolution for $m = 4$. The exponents $b_{i,j}$ are constants depending on the particular numerical semigroup S.

(equivalently, if S lies in the interior of C_m) and are minimalized uniformly for all numerical semigroups lying interior to the same face of the Kunz cone C_m. More precisely, our contributions are as follows:

(1) We introduce the Apéry toric ideal J_S of S, an analogue of I_S that lies in a ring with m variables instead of a ring with one variable per minimal generator of S. A generating set for J_S can be obtained by concatenating any generating set for I_S and a regular sequence with one element for each additional variable.

(2) For any positive integer $m \geq 2$, we construct a free resolution of J_S called the *Apéry resolution*. The rank of the d-th free module depends only on m and d, and the positions of the nonzero entries of the matrices representing the boundary maps depend only on m. An example of this for $m = 4$ is given in Figure 1, where the values $b_{i,j}$ depend on S.

(3) When S corresponds to a point interior to C_m, i.e., when S is MED and thus $J_S = I_S$, the Apéry resolution is a minimal free resolution of I_S.

(4) For any numerical semigroups S and T corresponding to points interior to the same face F of C_m, we prove there exists a uniform method for modifying the Apéry resolutions of J_S and J_T to minimal free resolutions in such a way that the resulting ranks of the free modules and the positions of the nonzero entries of the matrices representing the boundary maps depend only on m, F, and d.

The term “specialization” in the title and Section 4 refers to passage from the interior of the Kunz cone to a face, which entails some facet inequalities becoming equalities. Consequently, some exponents on y variables (as in Figure 1) pass from positive to 0, which results in the specialization that sets $y = 1$. Further substitutions among the x variables — extraneous ones are set equal to monomials in the others — combine in Step (4) with row and column operations to produce minimal free resolutions from the original Apéry resolution.

The remainder of this paper is structured as follows. Section 2 reviews basic properties of numerical semigroups and Kunz cones and defines the modules and maps used in the Apéry resolution. Section 3 proves that the Apéry resolution is indeed a resolution and establishes the minimality of this resolution when S is MED. Section 4 describes how to modify the Apéry resolution in a uniform way for all numerical semigroups in the interior of a fixed face of C_m to obtain a minimal resolution. Further research directions are outlined in Section 5.

2. Kunz polyhedra and Apéry resolutions

2.1. ***Semigroups and toric ideals.*** A *numerical semigroup* is a subsemigroup of $(\mathbb{Z}_{\geq 0}, +)$ that contains 0 and has finite complement. Throughout this work, fix a numerical semigroup $S \subset \mathbb{Z}_{\geq 0}$ with *multiplicity*

$$\mathsf{m}(S) = \min(S \setminus \{0\}) = m$$

and write

$$\begin{aligned}\mathrm{Ap}(S) &= \{n \in S : n - m \notin S\} \\ &= \{0, a_1, \ldots, a_{m-1}\}\end{aligned}$$

for the *Apéry set* consisting of the minimal element of S from each equivalence class modulo m, where each a_i satisfies $a_i \equiv i \mod m$. For convenience, define $a_0 = m$; this convention plays an important role in our later formulas. In particular,

$$S = \langle m, a_1, \ldots, a_{m-1} \rangle = \langle a_0, a_1, \ldots, a_{m-1} \rangle,$$

though this generating set need not be the unique minimal generating set $\mathcal{A}(S)$ of S, such as when $a_i + a_j = a_{i+j}$ for some i, j, where indices are summed modulo m. The semigroup S has *maximal embedding dimension* (MED) if $\mathcal{A}(S) = \{a_0, \ldots, a_{m-1}\}$.

Example 2.1. The semigroup $S = \langle 4, 9, 11, 14 \rangle$ has multiplicity $\mathsf{m}(S) = 4$ and Apéry set $\mathrm{Ap}(S) = \{0, 9, 14, 11\}$. The semigroup $T = \langle 4, 13, 23 \rangle$ has multiplicity $\mathsf{m}(T) = 4$ and $\mathrm{Ap}(T) = \{0, 13, 26, 23\}$. Note that $a_1 + a_1 = a_2$ in T, and thus the Apéry set is not a minimal generating set.

Let $R = \Bbbk[x_0, x_1, \ldots, x_{m-1}]$ with the natural grading by $\mathbb{Z}$ via $\deg(x_i) = a_i$ and set $y = x_0$. The *Apéry toric ideal* of S is the kernel $J_S = \ker(\varphi)$ of the homomorphism

$$\begin{aligned}\varphi : R &\to \Bbbk[t], \\ x_i &\mapsto t^{a_i},\end{aligned}$$

and the *defining toric ideal* of S is

$$I_S = J_S \cap \Bbbk[x_i : a_i \in \mathcal{A}(S)].$$

For every $1 \leq i, j \leq m - 1$ define

$$c_{i,j} = \frac{1}{m}(a_i + a_j - a_{i+j}) \geq 0 \tag{2-1}$$

and

$$b_{i,j} = \begin{cases} c_{i,j} & \text{if } i+j \neq m, \\ c_{i,j}+1 & \text{if } i+j = m. \end{cases}$$

In particular, $c_{i,j} = 0$ if and only if $a_i + a_j = a_{i+j}$; this is impossible if $i + j = m$, since m is the multiplicity, so $b_{i,j} = 0$ if and only if $c_{i,j} = 0$. It is known that

$$J_S = \langle x_i x_j - y^{c_{i,j}} x_{i+j} : 1 \le i \le j \le m-1 \rangle, \tag{2-2}$$

(see, e.g., [25, Section 8.4]), though it also follows from Lemma 3.2 here.

Example 2.2. The semigroup $S = \langle 4, 9, 11, 14\rangle$ has $(a_1, a_2, a_3) = (9, 14, 11)$ and

$$J_S = I_S = \langle x_1^2 - yx_2,\, x_1x_2 - y^3x_3,\, x_1x_3 - y^4y,\, x_2^2 - y^6y,\, x_2x_3 - x_1y^4,\, x_3^2 - x_2y^2\rangle.$$

The terms here are written in a way that emphasizes the convention $a_0 = m$ and $x_0 = y$, such as to produce the binomial $x_1x_3 - y^4x_0 = x_1x_3 - y^4y = x_1x_3 - y^5$.

The semigroup $T = \langle 4, 13, 23\rangle$ has $(a_1, a_2, a_3) = (13, 26, 23)$ and Apéry ideal

$$\begin{aligned} J_T &= \langle x_1^2 - x_2,\, x_1x_2 - y^4x_3,\, x_1x_3 - y^9,\, x_2^2 - y^{13},\, x_2x_3 - x_1y^9,\, x_3^2 - x_2y^5\rangle \\ &= \langle x_1^2 - x_2,\, x_1^3 - y^4x_3,\, x_1x_3 - y^9,\, x_3^2 - x_1^2y^5\rangle \end{aligned}$$

and defining toric ideal

$$I_T = \langle x_1^3 - y^4x_3,\, x_1x_3 - y^9,\, x_3^2 - x_1^2y^5\rangle = J_T \cap \Bbbk[y, x_1, x_3].$$

2.2. *Kunz cone.* We describe the Kunz cone and its relationship to the values $b_{i,j}$. Letting $\mathrm{Ap}(S) = \{0, a_1, \dots, a_{m-1}\}$ with $a_i \equiv i \bmod m$ for each i as in Section 2.1, the *Apéry coordinate vector* of S with respect to m is the tuple $(a_1, \dots, a_{m-1})$. The following set of linear inequalities exactly characterizes the set of Apéry coordinate vectors for numerical semigroups of multiplicity m [18; 20].

Definition 2.3. For each $m \ge 2$, the *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}_{\ge 0}$ has facet inequalities

$$z_i + z_j \ge z_{i+j} \quad \text{for } 1 \le i \le j \le m-1 \text{ with } i+j \neq m,$$

where addition of subscripts is modulo m.

Lemma 2.4. *If S is a numerical semigroup of multiplicity m, then $b_{i,j} = 0$ if and only if $a_i + a_j = a_{i+j}$. Hence the Apéry coordinate vector of S lies on the boundary of C_m if and only if $b_{i,j} = 0$ for some i, j.*

Proof. This follows from the definitions, using (2-1) for the claim about $b_{i,j}$. □

The lemma has the following consequence [20].

Proposition 2.5. *A vector $z = (z_1, \dots, z_{m-1}) \in \mathbb{Z}^{m-1}_{\ge 1}$ with $z_i \equiv i \bmod m$ for all i lies in C_m if and only if z is the Apéry coordinate vector of a numerical semigroup S. Moreover, z is in the interior of C_m if and only if S has maximal embedding dimension.*

Example 2.6. The cone $C_4 \subseteq \mathbb{R}^3_{\geq 0}$ is defined by the inequalities

$$z_2 + z_3 \geq z_1, \quad z_1 + z_2 \geq z_3, \quad 2z_1 \geq z_2, \quad \text{and} \quad 2z_3 \geq z_2,$$

and has extremal rays generated by $(1, 0, 1)$, $(1, 2, 3)$, $(1, 2, 1)$, and $(3, 2, 1)$. All positive-dimensional faces of C_4 contain numerical semigroups (in the sense of Proposition 2.5) except the rays through $(1, 0, 1)$ and $(1, 2, 1)$. Numerical semigroups on the rays through $(1, 2, 3)$ and $(3, 2, 1)$ have embedding dimension 2, and numerical semigroups in the relative interior of the facets $z_1 + z_2 = z_3$ and $z_2 + z_3 = z_1$ are complete intersections [2]. In particular, a minimal free resolution for the defining toric ideal of any semigroup in these 4 faces is known.

The numerical semigroup S from Example 2.2 corresponds to the point $(9, 14, 11)$ in the relative interior of C_4, while T corresponds to the point $(13, 26, 23)$ in the relative interior of the facet $2z_1 = z_2$. A minimal free resolution for J_S is obtained by substituting the appropriate values for $b_{i,j}$ in the free resolution in Figure 1, while a minimal free resolution for J_T is obtained via analogous substitution into Figure 4 (at the beginning of Section 4). This leaves the facet $2z_3 = z_2$, and, courtesy of the action of $\mathbb{Z}_4^*$ on C_4, free resolutions for semigroups in this face can be obtained from the ones exhibited in Figure 4 by interchanging 1s and 3s in every subscript.

We record here the result of Kunz that seems to be overlooked in the literature concerning when numerical semigroups reside in the interior of a given face of C_m.

Theorem 2.7 [20, Propositions 2.3 and 2.6]. *Two numerical semigroups S and T with multiplicity m lie in the interior of the same face of C_m if and only if*

$$R/(J_S + \langle y \rangle) \cong R/(J_T + \langle y \rangle).$$

Moreover, in this case, $\beta_d(I_S) = \beta_d(I_T)$ for every d.

2.3. ***Modules and maps for the Apéry resolution.*** For any numerical semigroup S of multiplicity m, this subsection defines the free modules and linear maps between them that form the *Apéry resolution*

$$\mathcal{F}_\bullet : 0 \leftarrow R \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

of J_S. Theorem 3.4 shows that it is a resolution. Of particular note is that the ranks of its modules and the locations of the nonzero coefficients in the matrices representing its linear maps depend only on m, not on the actual values of $\mathrm{Ap}(S)$.

Theorem 3.4 and Corollary 3.5 show that this resolution is minimal if and only if S has maximal embedding dimension, i.e., corresponds to a point interior to C_m. Theorem 4.4 shows that when S lies on the boundary of C_m, a minimal resolution of J_S can be obtained from the Apéry resolution in a manner that is uniform for all semigroups in the interior of a fixed face of C_m, parametrized by the $b_{i,j}$.

2.3.1. *Modules.* For $d = 1, \dots, m-1$, define F_d to be the free module over R with formal basis elements

$$\{e_{i,A} : i \in [m-1],\ A \subset [m-1],\ |A| = d,\ i \geq \min(A)\},$$

where $\deg(e_{i,A}) = a_i + \sum_{j \in A} a_j$ and $[m-1] = \{1, 2, \dots, m-1\}$. Since every pair (i, A) with $|A| = d$ and $i < \min(A)$ corresponds to a $(d+1)$-element subset $\{i\} \cup A$ of $[m-1]$, it is immediate that

$$\operatorname{rank} F_d = (m-1)\binom{m-1}{d} - \binom{m-1}{d+1} = d\binom{m}{d+1}.$$

Example 2.8. For $m = 3$, $F_0 = Re_\varnothing$, where $Re_\varnothing = \{re_\varnothing : r \in R\}$. Similarly,

$$\begin{aligned} F_1 &= Re_{1,\{1\}} + Re_{2,\{2\}} + Re_{2,\{1\}} \\ &= \{\alpha e_{1,\{1\}} + \beta e_{2,\{2\}} + \gamma e_{2,\{1\}} : \alpha, \beta, \gamma \in R\} \end{aligned}$$

with $\deg(e_{1,\{1\}}) = a_1 + a_1$, $\deg(e_{2,\{2\}}) = a_2 + a_2$, and $\deg(e_{2,\{1\}}) = a_2 + a_1$. Note that $\operatorname{rank} F_1 = 1 \cdot \binom{3}{1+1}$. Finally,

$$F_2 = Re_{1,12} + Re_{2,12}$$

with $\deg(e_{1,12}) = a_1 + a_1 + a_2$ and $\deg(e_{2,12}) = a_2 + a_1 + a_2$.

2.3.2. *Maps.* A few notational conventions help to define the boundary maps between the F_d. For $A \subseteq [m-1]$, set

$$\operatorname{sign}(j, A) = (-1)^t \quad \text{for } j \in A = \{\ell_0 < \ell_1 < \cdots < \ell_t = j < \cdots < \ell_r\}.$$

For convenience, set $e_{0,A} = 0$, and for $i \in [m-1]$ with $i < \min(A)$, define

$$e_{i,A} = \sum_{j \in A} \operatorname{sign}(j, A) e_{j, A \cup i \setminus j}\,. \tag{2-3}$$

As a consequence of this definition of $e_{i,A}$, for each $B \subseteq [m-1]$,

$$\sum_{i \in B} \operatorname{sign}(i, B) e_{i, B \setminus i} = 0. \tag{2-4}$$

With these conventions in hand, and considering $i + j$ modulo m in subscripts as usual, define the map $\partial_d : F_d \to F_{d-1}$ by

$$e_{i,A} \mapsto \sum_{j \in A} \operatorname{sign}(j, A)(x_j e_{i, A \setminus j} - y^{b_{i,j}} e_{i+j, A \setminus j}) \tag{2-5}$$

with the exception of $d = 1$, in which case

$$\partial_1(e_{i,j}) = x_i x_j - y^{c_{i,j}} x_{i+j}\,.$$

$$0 \leftarrow R \xleftarrow{\begin{array}{c} \\ \varnothing \end{array}\begin{array}{ccc} \mathbf{1,1} & \mathbf{2,2} & \mathbf{2,1} \\ \left[x_1^2 - x_2 y^{b_{11}}\right. & x_2^2 - x_1 y^{b_{22}} & \left. x_1 x_2 - y^{b_{12}}\right] \end{array}} R^3 \xleftarrow{\begin{array}{c} \\ \mathbf{1,1} \\ \mathbf{2,2} \\ \mathbf{2,1} \end{array}\begin{array}{cc} \mathbf{1,12} & \mathbf{2,12} \\ \multicolumn{2}{c}{\left[\begin{array}{cc} -x_2 & y^{b_{22}} \\ -y^{b_{11}} & x_1 \\ x_1 & -x_2 \end{array}\right]} \end{array}} R^2 \leftarrow 0$$

Figure 2. The Apéry resolution for $m = 3$.

Example 2.9. Figure 2 shows the modules and maps for the case $m = 3$. Note that by definition the bases for the modules are indexed by (i, A) pairs, and these are used to label the rows and columns of the matrices representing the maps. Consider the term $\partial_2(e_{1,12})$, which by definition is

$$\begin{aligned} \partial_2(e_{1,12}) &= (x_1 e_{1,2} - y^{b_{1,1}} e_{2,2}) - (x_2 e_{1,1} - y^{b_{1,2}} e_{0,1}) \\ &= x_1 e_{2,1} - y^{b_{1,1}} e_{2,2} - x_2 e_{1,1}, \end{aligned}$$

where the relation $e_{1,2} - e_{2,1} = 0$ is used. This illustrates how (2-4) ensures that ∂_d is well defined.

Example 2.10. Figure 1 shows the modules and maps for the case $m = 4$. The ranks for the modules and the general structure of the maps are independent of the values of $\mathrm{Ap}(S)$; indeed, the only variance is found in the values of the exponents on the y-variables in the matrices. The Apéry resolution of I_S for S introduced in Example 2.1 is given in the upper portion of Figure 3 (toward the end of Section 3).

3. Maximal embedding dimension numerical semigroups

This section contains a proof that the Apéry resolution is indeed a resolution of J_S. We begin with a brief review of Schreyer's theorem, which identifies a Gröbner basis (under a carefully chosen term order) for the syzygy module of a Gröbner basis, and which is the core tool in our proof. Then, we prove two lemmas that verify important subtleties about generating sets of J_S and the boundary maps ∂_d; specifically, these maps are consistent with the definition of $e_{i,A}$ in (2-3) and (2-4) when $i < \min A$, and substituting (2-3) into the definition of ∂_d still yields matrix entries that are monomials. Finally, we prove our main result.

We begin with a statement of Schreyer's theorem. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a Gröbner basis for a submodule $M \subseteq R^d$ with respect to a fixed term order $\preceq$. Let $e_1, \dots, e_s$ denote the standard basis of R^s, and let $\mathrm{In}_\preceq(v)$ denote the initial term of v with respect to the term order $\preceq$.

Theorem 3.1 (Schreyer's theorem). *There exist explicitly defined elements $s_{i,j} \in R^s$ that form a Gröbner basis for the syzygy module of $\mathcal{G}$ with respect to the monomial order $>_\mathcal{G}$ on R^s defined as follows: $x^\alpha e_i >_\mathcal{G} x^\beta e_j$ if $\mathrm{In}_\preceq(x^\alpha e_i) \succ \mathrm{In}_\preceq(x^\beta e_j)$ in R^m, or if $\mathrm{In}_\preceq(x^\alpha e_i) = \mathrm{In}_\preceq(x^\beta e_j)$ and $i < j$.*

For a textbook treatment of this theorem, including detailed definitions of $s_{i,j}$ and a proof, see [5, Chapter 5, (3.3)].

There is an interesting connection between the resolution we study and the origins of Schreyer's theorem. The authors of [9] produce a resolution that is isomorphic to the one defined in Section 2. In fact, Frank-Olaf Schreyer informed us (in personal communication) that resolution of the ideals in (2-2) was the initial motivation for both [9] and what is now known as Schreyer's theorem; see [5, Chapter 5, (3.3)]. However, the particular form taken by our explicit matrices clarifies the sense in which our resolution is compatible with specialization in the sense of Theorem 4.4. This point has some subtlety: obtaining a resolution isomorphic to ours need not be sufficient for the purpose of specialization (see, for instance, the resolutions and discussion in Remark 3.6). As such, we include in this section a full proof of our main result, Theorem 3.4.

Lemma 3.2. *The generating set* (2-2) *is a Gröbner basis for J_S under any term order $\preceq$ on R for which $x^a y^r \succ x^b y^s$ whenever $a_1 + \cdots + a_{m-1} > b_1 + \cdots + b_{m-1}$, where x^a and x^b are monomials in $x_1, \ldots, x_{m-1}$.*

Proof. Since J_S is generated by binomials, it suffices to consider binomials when computing initial ideals. The key observation is that in any graded degree, exactly one monomial in R has the form $x_i y^a$ with $a \in \mathbb{Z}_{\geq 0}$, since the graded degrees of the variables x_i are distinct modulo m. Hence the larger term under $\preceq$ in any nonzero binomial from J_S is divisible by $x_i x_j = \mathrm{In}_{\preceq}(x_i x_j - y^{c_{i,j}} x_{i+j})$ for some $i, j \in [m-1]$. As such,

$$\mathrm{In}_{\preceq}(J_S) = \langle x_i x_j : 1 \leq i, j \leq m-1 \rangle,$$

and thus the generating set in (2-2) is a Gröbner basis for J_S. □

The next aim is to establish that applying ∂ to $e_{i,A}$ when $i < \min(A)$ using the expression given in (2-5) is consistent with (2-4); this is needed when considering the result of applying ∂ repeatedly. Further, careful analysis of the use of (2-4) is required when $i + j < \min(A \setminus j)$ in (2-5). These issues are addressed in the following lemma.

Lemma 3.3. *The maps ∂_d respect* (2-4), *that is, applying the definition of ∂_d to the left-hand side of* (2-3) *yields the image of the right-hand side under ∂_d. Furthermore, $\mathcal{F}_\bullet$ is a complex, and for $d > 1$, the entries of each ∂_d are monomials.*

Proof. If $d = 1$, then (2-3) yields $e_{i,j} = e_{j,i}$, and the first claim is immediate. If $d > 1$, then for each $B \subseteq [m-1]$ with $|B| = d + 1$,

$$\sum_{i \in B} \mathrm{sign}(i, B) \partial e_{i,B \setminus i} = \sum_{i \in B} \mathrm{sign}(i, B) \sum_{j \in B \setminus i} \mathrm{sign}(j, B \setminus i)(x_j e_{i,B\setminus ij} - y^{b_{i,j}} e_{i+j,B\setminus ij}),$$

wherein the coefficient of $y^{b_{i,j}} e_{i+j,B\setminus ij}$ for distinct $i, j \in B$ equals

$$\operatorname{sign}(i, B)\operatorname{sign}(j, B \setminus i) + \operatorname{sign}(j, B)\operatorname{sign}(i, B \setminus j) = 0$$

and the remaining terms yield

$$\begin{aligned}
\sum_{i\in B} \operatorname{sign}(i, B)\partial e_{i,B\setminus i} &= \sum_{i\in B} \operatorname{sign}(i, B) \sum_{j\in B\setminus i} \operatorname{sign}(j, B \setminus i) x_j e_{i,B\setminus ij} \\
&= \sum_{j\in B} x_j \sum_{i\in B\setminus j} \operatorname{sign}(i, B)\operatorname{sign}(j, B \setminus i) e_{i,B\setminus ij} \\
&= -\sum_{j\in B} \operatorname{sign}(j, B) x_j \sum_{i\in B\setminus j} \operatorname{sign}(i, B \setminus j) e_{i,B\setminus ij} \\
&= -\sum_{j\in B} \operatorname{sign}(j, B) x_j \cdot 0 \\
&= 0.
\end{aligned}$$

Now proceed to the claim that each ∂_d is a matrix whose entries are monomials. Call $e_{i,A}$ *squarefree* if $i \notin A$. Note that every term in (2-4) is squarefree, and no two equalities of the form (2-4) share any terms. As such, to ensure substituting (2-3) into the definition of ∂_d does not produce nonmonomial matrix entries (i.e., that doing so does not contribute a term with a free generator of F_d already appearing in the sum), it suffices to prove that no two squarefree terms in $\partial e_{i,A}$ lie in the same equality in (2-4). To this end, fix $j, k \in A$. If $e_{i,A\setminus j}$ and $e_{i,A\setminus k}$ lie in the same equality in (2-4), then $(A \setminus j) \cup i = (A \setminus k) \cup i$ and thus $j = k$. If $e_{i+j,A\setminus j}$ and $e_{i+k,A\setminus k}$ lie in the same equality in (2-4), then $i + j \notin A$ but $i + j \in (A \cup \{i + k\}) \setminus k$, so necessarily $i + j = i + k$ and thus $j = k$. Lastly, if $e_{i,A\setminus j}$ and $e_{i+k,A\setminus k}$ lie in the same equality in (2-4), then $i + k \notin A$ but $i + k \in (A \setminus j) \cup i$, so $i + k = i$, which is impossible.

It remains to prove that $\mathcal{F}_\bullet$ is a complex. First, suppose $A = \{j, k\}$ with $j < k$ and $i \geq j$. If $i + j$, $i + k$, and $i + j + k$ are all nonzero, then

$$\begin{aligned}
\partial^2 e_{i,A} &= x_j \partial e_{i,k} - x_k \partial e_{i,j} - y^{b_{i,j}} \partial e_{i+j,k} + y^{b_{i,k}} \partial e_{i+k,j} \\
&= x_j (x_i x_k - y^{c_{i,k}} x_{i+k}) - x_k (x_i x_j - y^{c_{i,j}} x_{i+j}) \\
&\qquad - y^{b_{i,j}} (x_{i+j} x_k - y^{c_{i+j,k}} x_{i+j+k}) + y^{b_{i,k}} (x_{i+k} x_j - y^{c_{i+k,j}} x_{i+j+k}) \\
&= x_{i+j} x_k (y^{c_{i,j}} - y^{b_{i,j}}) + x_{i+k} x_j (y^{c_{i,k}} - y^{b_{i,k}}) + x_{i+j+k} (y^{b_{i,j}} y^{c_{i+j,k}} - y^{b_{i,k}} y^{c_{i+k,j}}) \\
&= 0
\end{aligned}$$

by homogeneity of ∂. In the event $i + j = 0$, or $i + k = 0$, or $i + j + k = 0$, replacing x_0 with zero as appropriate in the above algebra yields the desired equality. For all remaining cases, $|A| > 2$, and in the expansion of

$$\begin{aligned}\partial^2 e_{i,A} &= \sum_{j\in A} \operatorname{sign}(j, A) x_j \partial e_{i,A\setminus j} - \sum_{j\in A} \operatorname{sign}(j, A) y^{b_{i,j}} \partial e_{i+j,A\setminus j} \\ &= \sum_{j\in A} \operatorname{sign}(j, A) x_j \Big(\sum_{k\in A\setminus j} \operatorname{sign}(k, A\setminus j)(x_k e_{i,A\setminus jk} - y^{b_{i,k}} e_{i+k,A\setminus jk}) \Big) \\ &\quad - \sum_{j\in A} \operatorname{sign}(j, A) y^{b_{i,j}} \Big(\sum_{k\in A\setminus j} \operatorname{sign}(k, A\setminus j)(x_k e_{i+j,A\setminus jk} - y^{b_{i+j,k}} e_{i+j+k,A\setminus jk}) \Big),\end{aligned}$$

the terms $x_j x_k e_{i,A\setminus jk}$, $x_j y^{b_{i,k}} e_{i+k,A\setminus jk}$, and $y^{b_{i,j}+b_{i+j,k}} e_{i+j+k,A\setminus jk}$ each have coefficient

$$\operatorname{sign}(j, A) \operatorname{sign}(k, A \setminus j) + \operatorname{sign}(k, A) \operatorname{sign}(j, A \setminus k) = 0$$

for any distinct $j, k \in A$. □

Theorem 3.4. *The complex $\mathcal{F}_\bullet$ is a resolution.*

Proof. Proceed by induction on d to show that the columns of the matrices for ∂_d form a Gröbner basis for $\ker(\partial_{d-1})$. The case $d = 1$ is handled by Lemma 3.2, so suppose $d = 2$. Let $\preceq'$ denote the partial order on F_1 given by $x^\beta e_{k,\ell} \preceq' x^\alpha e_{i,j}$ whenever

$$\operatorname{In}_\preceq(\partial_1(x^\beta e_{k,b})) \prec \operatorname{In}_\preceq(\partial_1(x^\alpha e_{i,a})),$$

or when equality holds above and $a < b$, or if equality holds above, $a = b$, and $i < k$. Note $x^\beta e_{j,\ell} \preceq' x^\alpha e_{i,k}$ whenever x^α has higher total degree in $x_1, \dots, x_{m-1}$ than x^β, so

$$\operatorname{In}_{\preceq'}(\partial_2(e_{i,jk})) = x_k e_{i,j} \quad \text{where } j < k \text{ and } i \geq j.$$

Theorem 3.1 implies the elements

$$s_{i,a;\,k,b} = \frac{L}{x_i x_a} e_{i,a} - \frac{L}{x_k x_b} e_{k,b} - \sum_{\ell \geq c \geq 1} f_{\ell,c} e_{\ell,c} \quad \text{for } i \geq a,\ k \geq b \tag{3-1}$$

form a Gröbner basis for $\ker(\partial_1)$ under $\preceq'$, where L equals $\operatorname{lcm}(x_i x_a, x_k x_b)$ and the $f_{\ell,c}$ are coefficients obtained from polynomial long division when dividing $S(\partial_1(e_{i,a}), \partial_1(e_{k,b}))$ by (2-2). In particular, we claim

$$\operatorname{In}_{\preceq'}(\ker(\partial_1)) = \langle x_k e_{i,j} \mid j < k \text{ and } i \geq j \rangle$$

is generated by initial terms of the columns of ∂_2: by construction $\operatorname{In}_{\preceq'}(s_{i,a;\,k,b})$ must be one of the first two terms in (3-1) and $\partial_1\big(\frac{L}{x_i x_a} e_{i,a}\big) = \partial_1\big(\frac{L}{x_k x_b} e_{k,b}\big)$, so without loss of generality say $e_{k,b} \prec' e_{i,a}$. Then either $a < b \leq k$ and x_k or x_b appear as a coefficient of $e_{i,a}$, or $a = b$, $a \leq i < k$ and x_k appears as a coefficient of $e_{i,a}$, so $\operatorname{In}_{\preceq'}(s_{i,a;\,k,b})$ is divisible by the initial term of some column of ∂_2. This implies that $\operatorname{In}_{\preceq'}(\operatorname{im}(\partial_2)) = \operatorname{In}_{\preceq'}(\ker(\partial_1))$, which, together with $\operatorname{im}(\partial_2) \subseteq \ker(\partial_1)$, implies $\operatorname{im}(\partial_2) = \ker(\partial_1)$ and the columns of ∂_2 form a Gröbner basis under $\preceq'$.

Lastly, suppose $d > 2$, let $\preceq$ denote the term order on F_{d-2} obtained inductively, and let $\preceq'$ denote the term order on F_{d-1} so that $x^\beta e_{j,B} \preceq' x^\alpha e_{i,A}$ whenever

$$\mathrm{In}_\preceq(x^\beta \partial_{d-1}(e_{j,B})) \prec \mathrm{In}_\preceq(x^\alpha \partial_{d-1}(e_{i,A})),$$

or if equality holds above and A precedes B lexicographically, or if equality holds above, $A = B$, and $i < j$. One readily obtains

$$\mathrm{In}_{\preceq'}(\partial_d(e_{i,A})) = x_j e_{i,A\setminus j} \quad \text{with } \ j = \max(A)$$

after checking that

- $x^\alpha e_{k,B} \preceq' x^\beta e_{\ell,C}$ whenever x^β has higher total degree in $x_1, \ldots, x_{m-1}$ than x^α;
- $x_k \,\mathrm{In}_\preceq(\partial_{d-1}(e_{i,A\setminus k})) = x_\ell \,\mathrm{In}_\preceq(\partial_{d-1}(e_{i,A\setminus \ell}))$ for all $k, \ell \in A$; and
- the substitution (2-3) need only be made if $\min(A) \le i < \min(A \setminus \min(A))$, in which case $A \setminus j$ lexicographically precedes the second subscript of every summand in (2-4).

The equality $S(\partial_{d-1}(e_{i,A}), \partial_{d-1}(e_{j,B})) = 0$ holds due to initial terms having distinct basis vectors unless $i = j$, $A = C \cup \{\gamma\}$, and $B = C \cup \{\delta\}$ for some $\delta, \gamma \in [m-1]$ and some nonempty $C \subseteq [m-1]$ with $\delta, \gamma > \max(C)$. As such, Schreyer's theorem yields

$$\mathrm{In}_{\preceq'}(\ker(\partial_{d-1})) = \langle x_\delta e_{i,C} : i, \delta \in [m-1],\ C \subseteq [m-1],\ \delta > \max(C)\rangle,$$

and since $x_\delta e_{i,C} = \mathrm{In}_{\preceq'}(\partial_d(e_{i,C\cup\delta}))$ for each i, δ, and C, the columns of ∂_d form a Gröbner basis for $\ker(\partial_{d-1})$. The proof is completed by observing that induction also ensures none of the initial terms in question involve $e_{i,A}$ with $i < \min(A)$. □

Corollary 3.5. *The resolution $\mathcal{F}_\bullet$ is minimal if and only if S is MED.*

Proof. A resolution is minimal if and only if the matrices for ∂_d contain no nonzero constant entries. The only entries that depend on $a_1, \ldots, a_{m-1}$ are powers of y, and their exponents $b_{i,j}$ are all strictly positive precisely when S is MED. □

Remark 3.6. MED semigroups whose associated toric ideal is determinantal are exactly those semigroups where $a_1, a_2, \ldots, a_{m-1}$ form an arithmetic sequence (not necessarily in that order) [15; 19]. In this case, I_S is resolved by the Eagon–Northcott complex [6]; a detailed treatment on the Eagon–Northcott resolution can be found in [8, Appendix A2H]. The strict requirements on an MED semigroup to make its associated toric ideal determinantal mean that such semigroups form only a small proportion of all numerical semigroups: in the Kunz cone, these semigroups lie in the union of a finite set of affine 2-planes, whose union cannot be the whole cone. Although relatively few toric ideals of MED semigroup ideals are minimally resolved by Eagon–Northcott complexes, the occasional overlap does mean that all toric ideals for MED numerical semigroups share Betti numbers with

$$0 \leftarrow R \xleftarrow{\begin{bmatrix} x_1^2 - x_2y^2 & x_2^2 - y^5 & x_3^2 - x_2y^3 & x_1x_2 - x_3y^2 & x_1x_3 - y^5 & x_2x_3 - x_1y^3 \end{bmatrix}}$$

$$R^6 \xleftarrow{\begin{bmatrix} -x_2 & -x_3 & & & & & y^3 & y^3 \\ -y^2 & & x_1 & -x_3 & & y^3 & & \\ & & & & x_1 & x_2 & -y^2 & \\ x_1 & & -x_2 & y^3 & y^3 & & -x_3 & \\ y^2 & x_1 & & & -x_3 & -y^3 & & -x_2 \\ & -y^2 & -y^2 & x_2 & & -x_3 & x_1 & x_1 \end{bmatrix}} R^8 \xleftarrow{\begin{bmatrix} x_3 & -y^3 & \\ -x_2 & & y^3 \\ & x_3 & -y^3 \\ -y^2 & x_1 & \\ y^2 & & -x_2 \\ & -y^2 & x_1 \\ x_1 & -x_2 & \\ -x_1 & & x_3 \end{bmatrix}} R^3 \leftarrow 0$$

$$0 \leftarrow R \xleftarrow{\begin{bmatrix} x_1^2 - x_2y^2 & x_2^2 - x_1x_3 & x_3^2 - x_2y^3 & x_1x_2 - x_3y^2 & x_1x_3 - y^5 & x_2x_3 - x_1y^3 \end{bmatrix}}$$

$$R^6 \xleftarrow{\begin{bmatrix} x_2 & x_3 & x_3 & & & & & y^3 \\ y^2 & & x_1 & x_3 & & y^3 & & \\ & & & x_1 & x_1 & x_2 & y^2 & \\ -x_1 & & -x_2 & & y^3 & & x_3 & \\ & -x_1 & & & -x_3 & & -x_2 & -x_2 \\ & y^2 & & -x_2 & & -x_3 & & x_1 \end{bmatrix}} R^8 \xleftarrow{\begin{bmatrix} -x_3 & -y^3 & \\ x_2 & x_3 & \\ & -x_3 & -y^3 \\ y^2 & x_1 & \\ & -x_1 & -x_2 \\ & y^2 & x_1 \\ -x_1 & -x_2 & \\ & x_2 & x_3 \end{bmatrix}} R^3 \leftarrow 0$$

Figure 3. The Apéry resolution (above) and Eagon–Northcott resolution (below) for I_S where $S = \langle 4, 9, 10, 11 \rangle$.

the Eagon–Northcott resolution of a $2 \times m$ matrix, despite the impossibility of using the Eagon–Northcott construction to resolve most such toric ideals.

Even in the case where the ideal is determinantal, the Apéry resolution differs from the Eagon–Northcott resolution. As an example, consider the numerical semigroup $S = \langle 4, 9, 10, 11 \rangle$, whose defining toric ideal I_S is generated by the 2×2 minors of

$$\begin{bmatrix} x_1 & x_2 & x_3 & y^3 \\ y^2 & x_1 & x_2 & x_3 \end{bmatrix}.$$

The key difference is the presentation of the generators of I_S. Namely, the generators as provided in (2-2) are of the form $x_i x_j - x_{i+j} y^{b_{i,j}}$, while those given by determinants may have the form $x_i x_j - x_{i+1} x_{j-1}$. Figure 3 shows the Apéry resolution and the Eagon–Northcott resolution of I_S, with basis elements in the Eagon–Northcott resolution ordered to mimic the Apéry resolution. It is worth noting that in the $m = 3$ case, a_1 and a_2 trivially form an arithmetic sequence, and in fact the Apéry resolution and the Eagon–Northcott resolution coincide.

Remark 3.7. When S is MED, quotienting the Apéry resolution of I_S by the ideal $\langle y \rangle$, as Kunz does in Theorem 2.7 with the ring R/I_S, yields a minimal resolution of the ideal $\langle x_1, \ldots, x_{m-1} \rangle^2$ over the ambient polynomial ring $\Bbbk[x_1, \ldots, x_{m-1}]$. This ideal is known to be resolved by the Eagon–Northcott complex on the $2\times m$ matrix

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & 0 \\ 0 & x_1 & x_2 & x_3 & \cdots & x_{m-1} \end{bmatrix}.$$

Thus, in the MED case, the Eagon–Northcott complex "sits inside" the Apéry resolution; indeed, it is the result of an artinian reduction of R/I_S.

4. Specialization for arbitrary numerical semigroups

The Apéry resolution can be thought of as a family of free resolutions, one for the Apéry ideal J_S of each numerical semigroup S with multiplicity m, that is parametrized by the values $b_{i,j}$. Given a numerical semigroup S, a free resolution of J_S is obtained by simply computing the values $b_{i,j}$ from the Apéry set of S and substituting them into the Apéry resolution. By Corollary 3.5, restricting to semigroups S in the interior of C_m, the Apéry resolutions form a parametrized family of minimal free resolutions.

The main result of this section is Theorem 4.4, which implies that for each face F of C_m, there exists a family of minimal free resolutions, one for the Apéry ideal J_S of each numerical semigroup S indexed by the interior of F, that is analogously parametrized by the positive $b_{i,j}$. Figure 4 depicts one such resolution for the $z_2 = 2z_1$ facet of C_4. Our proof of Theorem 4.4 is nonconstructive: it argues that there exists a change of basis for the Apéry resolution, depending only on F, that yields the desired minimal free resolution of J_S as a summand. With Proposition 4.1, which gives the algebraic relationship between minimal resolutions of J_S and I_S, the Betti numbers of J_S and I_S can be recovered from F (Corollaries 4.3 and 4.5).

$$0 \leftarrow R \xleftarrow{\;\mathbf{000}\;\left[\begin{array}{c|ccc} \mathbf{1,1} & \mathbf{3,3} & \mathbf{2,1} & \mathbf{3,1} \\ x_1^2 - x_2 & x_3^2 - x_1^2 y^{b_{33}} & x_1^3 - x_3 y^{b_{12}} & x_1 x_3 - y^{b_{13}} \end{array}\right]\;} R^4$$

$$R^4 \xleftarrow{\;\begin{array}{c} \\ \mathbf{1,1} \\ \mathbf{3,3} \\ \mathbf{2,1} \\ \mathbf{3,1} \end{array} \left[\begin{array}{ccc|cc} \mathbf{3,23} & \mathbf{2,12} & \mathbf{2,13} & \mathbf{3,13} & \mathbf{3,12} \\ x_3^2 - x_1^2 y^{b_{33}} & x_1^3 - x_3 y^{b_{12}} & x_1 x_3 - y^{b_{13}} & & \\ \hline -(x_1^2 - x_2) & & & x_1 & -y^{b_{12}} \\ & -(x_1^2 - x_2) & & y^{b_{33}} & -x_3 \\ & & -(x_1^2 - x_2) & -x_3 & x_1^2 \end{array}\right]\;} R^5 \xleftarrow{\;\begin{array}{c} \\ \mathbf{3,23} \\ \mathbf{2,12} \\ \mathbf{2,13} \\ \mathbf{3,13} \\ \mathbf{3,12} \end{array} \left[\begin{array}{cc} \mathbf{3,[3]} & \mathbf{2,[3]} \\ x_1 & -y^{b_{12}} \\ y^{b_{33}} & -x_3 \\ -x_3 & x_1^2 \\ \hline x_1^2 - x_2 & \\ & x_1^2 - x_2 \end{array}\right]\;} R^2 \leftarrow 0$$

Figure 4. A specialization of the $m = 4$ Apéry resolution where $b_{11} = 0$, so $a_2 = 2a_1$. Note this forces $b_{13} = b_{23}$.

Proposition 4.1. *A minimal free resolution of J_S can be obtained as the tensor product of a minimal free resolution of I_S with a Koszul complex.*

Proof. Nonminimality of $m, a_1, \ldots, a_{m-1}$ as generators for S is reflected in J_S by binomial generators without y. More specifically, if $a_i + a_j = a_{i+j}$, then $b_{i,j} = 0$ and $x_i x_j - x_{i+j}$ appears in J_S. Let $\mathcal{A}(S) = \{m, a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ be the elements a_i that minimally generate S. Though I_S naturally lives in $\Bbbk[y, x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$, consider it as an ideal in R via the natural inclusion map. For each nonzero $w \in \mathrm{Ap}(S) \setminus \mathcal{A}(S)$, pick one of the binomials $f_w = x_w - x_u x_v$. These binomials form a regular sequence on R, so the ideal I_W generated by the f_w is resolved by a Koszul complex $\mathcal{K}_\bullet$. Writing $\mathcal{G}_\bullet$ for a minimal free resolution of I_S, the only nontrivial homology of $\mathcal{G}_\bullet \otimes_R \mathcal{K}_\bullet$ occurs in homological degree 0 and is isomorphic to $H_0(\mathcal{G}_\bullet) \otimes H_0(\mathcal{K}_\bullet) = R/I_S \otimes_R R/I_W = R/J_S$, where the last equality is because the f_w form a regular sequence over R/I_S. Therefore $\mathcal{G}_\bullet \otimes \mathcal{K}_\bullet$ is a minimal free resolution of R/J_S. □

Example 4.2. The underlying structure as a tensor of two resolutions is readily seen in Figure 4, which resolves J_S for $\mathrm{Ap}(S) = \{4, a_1, 2a_1, a_3\}$. This example was obtained by computing the Apéry resolution for J_S and then trimming away any constant entries using row and column operations as described in Theorem 4.4.

We include the proof of the following, despite its appearance in Theorem 2.7 as recovered from [20], to demonstrate how the Apéry resolution maps in Theorem 3.4 lend themselves to specialization to the faces of C_m, as well as to contrast its content with that of Theorem 4.4.

Corollary 4.3. *Let S and T be numerical semigroups corresponding to points interior to the same face F of the Kunz cone $\mathcal{C}_m$. The Apéry ideals of S and T share the same Betti numbers, as do the defining toric ideals of S and T. In particular, $\beta_d(J_S) = \beta_d(J_T)$ and $\beta_d(I_S) = \beta_d(I_T)$ for all $d \geq 0$.*

Proof. Let $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ be the Apéry resolutions of J_S and J_T, respectively. In the case that S and T are both MED, so the face F is the entirety of C_m, both resolutions are minimal by Corollary 3.5 and have the same modules at each homological degree, so $\beta_d(J_S) = \beta_d(J_T)$ holds immediately.

If $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ are not minimal, then the resolutions have ± 1 entries in identical places in their resolutions, once again because S and T lie interior to the same face F and thus have the same $b_{i,j} = 0$, meaning that the same entries $\pm y^{b_{i,j}}$ become ± 1. Because the Betti numbers of any positively graded ideal I equal the dimensions of the graded vector spaces $\mathrm{Tor}_\bullet(I, \Bbbk)$, consider $\mathcal{F}_\bullet \otimes_\Bbbk \Bbbk$ and $\mathcal{F}'_\bullet \otimes_\Bbbk \Bbbk$. The differentials in these complexes are identical: they are matrices of 0s and ± 1s with units in matching places. Therefore, their kernels and images are the same at each homological degree, so

$$\beta_d(J_S) = \dim \mathrm{Tor}_d^\Bbbk(J_S, \Bbbk) = \dim \mathrm{Tor}_d^\Bbbk(J_T, \Bbbk) = \beta_d(J_T).$$

Next consider I_S and I_T. By Proposition 4.1,

$$\mathcal{F}_\bullet = (\mathcal{G}_\bullet \otimes \mathcal{K}_\bullet) \quad \text{and} \quad \mathcal{F}'_\bullet = (\mathcal{G}'_\bullet \otimes \mathcal{K}_\bullet),$$

where $\mathcal{G}_\bullet$ and $\mathcal{G}'_\bullet$ are minimal free resolutions of I_S and I_T, respectively, and $\mathcal{K}_\bullet$ is the Koszul resolution on the extraneous binomials. Tensoring with $\mathcal{K}_\bullet$ exerts the same invertible change on the Betti numbers of $\mathcal{G}_\bullet$ and $\mathcal{G}'_\bullet$. More specifically, let

$$g_S(t) = \sum_{i=0}^{p} \beta_i(I_S)t^i \quad \text{and} \quad f_S(t) = \sum_{i=0}^{q} \beta_i(J_S)t^i$$

be the generating functions for the Betti numbers of $\mathcal{G}_\bullet$ and $\mathcal{F}_\bullet$, respectively. Since $\mathcal{K}_\bullet$ is a Koszul resolution of $r = m - |\mathcal{A}(S)|$ elements,

$$f_S(t) = (1+t)^r g_S(t).$$

Thus,

$$(1+t)^r g_S(t) = f_S(t) = f_T(t) = (1+t)^r g_T(t),$$

and therefore $g_S(t) = g_T(t)$, meaning $\beta_i(I_S) = \beta_i(I_T)$ for all $i \geq 0$. □

Theorem 4.4. *Consider the set*

$$\mathcal{M} = \{x_i : 1 \leq i \leq m-1\} \cup \{y^{b_{i,j}} : 1 \leq i,\, j \leq m-1\}$$

of formal symbols appearing as matrix entries in Apéry resolutions. (*Lemma 3.3 ensures every nonzero matrix entry is accounted for in* $\mathcal{M}$). *Fix a face F of C_m. There is a sequence of matrices, whose entries are* $\Bbbk$*-linear combinations of formal products of elements of* $\mathcal{M}$, *with the following property*: *for each numerical semigroup S indexed by the relative interior of F, substituting R-variables and the values $b_{i,j}$ for S into the entries of each matrix yields boundary maps for a graded minimal free resolution of J_S.*

Proof. Fix a numerical semigroup S with multiplicity m. Let

$$\mathcal{N}_S = \{x_i : 1 \leq i \leq m-1\} \cup \{y^{b_{i,j}} : 1 \leq i,\, j \leq m-1 \text{ and } b_{i,j} > 0\} \subseteq \mathcal{M}$$

denote the set of elements of $\mathcal{M}$ corresponding to positive-degree monomials in R under the grading by S. If S is MED, then $\mathcal{M} = \mathcal{N}_S$; otherwise they are distinct.

By [7, Theorem 20.2] (see also [22, Exercises 1.10 and 1.11]), the matrices in any free resolution for J_S can, via a sequence of row and column operations that preserve homogeneity, be turned into block diagonal matrices with 2 blocks:

(i) a matrix with no nonzero constant entries and at least one nonzero entry in each row and column; and

(ii) a matrix with no nonconstant entries and at most one nonzero entry in each row and column.

After doing this, restricting to each block described in (i) yields a minimal free resolution for J_S.

One way to select the aforementioned row and column operations is as follows. Begin with the matrices M_i for the maps ∂_i for the Apéry resolution, and perform the following for each $i = 1, 2, \ldots, m-1$, assuming that, as a result of prior operations, any column of M_i with a nonzero constant entry has no other nonzero entries:

- First use nonzero constant entries of M_i to clear all other entries in their respective rows. If $i = 1$, then no such rows exist. Fix a row R of M_i with a nonzero constant entry c, say in column C_1 with corresponding row R_1 in the matrix M_{i+1}. For each nonzero entry f in R, say in a column $C_2 \neq C_1$ with corresponding row R_2 in M_{i+1}, subtract $c^{-1} f \cdot C_1$ from C_2 and add $c^{-1} f \cdot R_2$ to R_1. Once this is done, c will be the only nonzero entry in R, and in fact c will be the only nonzero entry in row R and column C_1. Moreover, since $M_i M_{i+1} = 0$, the row R_1 in M_{i+1} only has entries 0.

- Next use nonzero constant entries of M_{i+1} to clear all other entries in their respective columns. Fix a column C of M_{i+1} with a nonzero constant entry c, say in row R_1 with corresponding column C_1 in the matrix M_i. For each nonzero entry f in C, say in a row $R_2 \neq R_1$ with corresponding column C_2 in M_i, subtract $c^{-1} f \cdot R_1$ from R_2 and add $c^{-1} f \cdot C_2$ to C_1. Once this is done, c will be the only nonzero entry in column C, so since $M_i M_{i+1} = 0$, the column C_1 now only has entries 0. Moreover, all changes to M_i only affect the (now all 0) column C_1, so M_i still has the property that every nonzero constant entry is the only nonzero entry in its row and column.

Once the above operations are completed for each i, the rows and columns may be permuted to obtain the desired blocks.

The key observation is that in the above sequence of row and column operations, the entry f is an existing matrix entry. As such, after each row or column operation, every matrix entry g can be written as a $\Bbbk$-linear combination of products of (possibly constant) elements of $\mathcal{M}$. Moreover, if g is a nonzero constant, then g has degree 0 under the grading by S, so by homogeneity, the aforementioned expression for g cannot contain any monomials in $\mathcal{N}_S$, since it must be a $\Bbbk$-linear combination of products of degree-0 elements of $\mathcal{M}$.

Now, fix a numerical semigroup T in the same face F of the Kunz cone C_m as S. The sets $\mathcal{M} \setminus \mathcal{N}_S$ and $\mathcal{M} \setminus \mathcal{N}_T$ each contain $y^{b_{i,j}}$ whenever F is contained in the facet $z_i + z_j = z_{i+j}$, and thus $\mathcal{N}_S = \mathcal{N}_T$. As a consequence of the preceding paragraph, applying an identical sequence of row and column operations to the Apéry resolution for J_T yields nonzero constant entries in precisely the same locations at each step of the process. This completes the proof. □

Theorem 4.4 yields the following graded refinement of Corollary 4.3.

Corollary 4.5. *For each* $i \in \{0, \ldots, m-1\}$, *writing* $[i]_m = i + m\mathbb{Z}$,

$$\sum_{b \in [i]_m} \beta_{d,b}(J_S) = \sum_{b \in [i]_m} \beta_{d,b}(J_{S'}).$$

The same relationship holds between the Betti numbers of I_S *and* $I_{S'}$.

Proof. Apply Theorem 4.4 for the first claim, and subsequently apply Proposition 4.1 for the final claim. □

5. Open questions

Several of the open questions presented here relate to the defining toric ideal I_S. One of the main results of [18] identifies a finite poset corresponding to each face F of C_m, called the *Kunz poset* of F. If a point interior to F indexes a numerical semigroup S, then this poset coincides with the divisibility poset of $\mathrm{Ap}(S)$. In [14], the Kunz poset of F is used to obtain a parametrized family of minimal binomial generating sets, one for the defining toric ideal I_S of each numerical semigroup S in F. The last three binomials in the first matrix in Figure 4 constitute one such example for the relevant facet F of C_4. This provides a natural candidate for the first matrix in the resolution conjectured as follows.

Conjecture 5.1. *For each face* F *of* C_m, *there exists a parametrized family of minimal resolutions*, *one for the defining toric ideal* I_S *of each numerical semigroup* S *indexed by the interior of* F, *akin to those obtained in Theorem 4.4 for Apéry ideals.*

Unlike the proof of Theorem 4.4, the proofs in [14] are constructive, utilizing the Kunz poset structure of the face F containing S. Intuitively, the set of factorizations of elements of $\mathrm{Ap}(S)$ (a set which depends only on F) forms the staircase of a monomial ideal M. If each element of $\mathrm{Ap}(S)$ factors uniquely, then I_S has exactly one binomial generator for each minimal monomial generator of M. If some elements of $\mathrm{Ap}(S)$ have multiple factorizations, then graph-theoretic methods can be used to partition some of the minimal generators of M into blocks (called *outer Betti elements*) and construct one minimal binomial generator of I_S for each block. We refer the reader to [14, Section 5] for the full construction; additional examples can be found in [10].

The nonconstructive nature of the proof of Theorem 4.4, along with the constructive nature of the proofs in [14], motivates the following.

Problem 5.2. Find an explicit combinatorial (e.g., poset-theoretic) construction of the matrices obtained in Theorem 4.4 and conjectured in Conjecture 5.1.

There is a long history of topological formulas for Betti numbers of graded ideals (for instance, Hochster's formula for squarefree monomial ideals [16] (see

[22, Chapter 1]), squarefree divisor complexes for toric ideals [3], or the use of poset homology for computing Poincaré series of semigroup algebras [24]). The following is thus a natural problem.

Problem 5.3. Given a face F of C_m, find a topological formula for extracting the value in the equation in Corollary 4.5 from the Kunz poset of F.

As mentioned in Example 2.6, the ray $(1, 2, 1)$ of C_4 contains positive integer points, but none correspond to a numerical semigroup under Proposition 2.5. Indeed, the first and third coordinates of any such point must be equal, but any point $(a_1, a_2, a_3) \in C_4$ corresponding to a numerical semigroup must have $a_i \equiv i \mod 4$ for each i. However, the construction in [18] still associates a poset to this ray, and naively following the construction in [14] for a binomial generating set with $y = 0$ yields the artinian binomial ideal $\langle x_1^2 - x_3^2, x_1^3, x_1x_3, x_3^3\rangle$. This motivates the following.

Problem 5.4. Extend the correspondence in Proposition 2.5 to a family of lattice ideals that includes the defining toric ideals of numerical semigroups but reaches points in faces of C_m that do not contain numerical semigroups.

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BENJAMIN BRAUN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KENTUCKY
LEXINGTON, KY
UNITED STATES
benjamin.braun@uky.edu

TARA GOMES
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN
UNITED STATES
gomes072@umn.edu

EZRA MILLER
DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NC
UNITED STATES
ezra@math.duke.edu

CHRISTOPHER O'NEILL
DEPARTMENT OF MATHEMATICS AND STATISTICS
SAN DIEGO STATE UNIVERSITY
SAN DIEGO, CA
UNITED STATES
cdoneill@sdsu.edu

ALEKSANDRA SOBIESKA
DEPARTMENT OF MATHEMATICS
MARSHALL UNIVERSITY
HUNTINGTON, WV
UNITED STATES
sobieskasnyd@marshall.edu

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BAR-NATAN HOMOLOGY FOR NULL HOMOLOGOUS LINKS IN $\mathbb{RP}^3$

DAREN CHEN

We introduce Bar-Natan homology for null homologous links in $\mathbb{RP}^3$ over the field of two elements. It is a deformation of the Khovanov homology in $\mathbb{RP}^3$ defined by Asaeda, Przytycki and Sikora. We also define an s-invariant from this deformation using the same recipe as for links in S^3, and prove some genus bound using it. The key ingredient is the notion of "twisted orientation" for null homologous links and cobordisms in $\mathbb{RP}^3$.

1. Introduction

Recently, Manolescu and Willis introduced Lee homology and the associated s-invariant for links in $\mathbb{RP}^3$ in [13]. This is a deformation of the Khovanov homology in $\mathbb{RP}^3$ over the base ring $\mathbb{Z}$, first introduced by Asaeda, Przytycki and Sikora in [1] for fields of characteristic 2, and later extended by Gabrovšek in [6] to the base ring $\mathbb{Z}$ by fixing certain sign conventions.

As with the usual Lee homology, the construction in [13] works as long as the characteristic of the base ring is not 2. For links in S^3, the solution for rings of characteristic 2 is to use the Bar-Natan deformation of the Khovanov homology, introduced by Bar-Natan in [2], instead of the Lee deformation. Here, we adapt the same approach for null homologous links in $\mathbb{RP}^3$. Similar to the S^3 case, the homology itself admits a simple description in terms of the number of components of the links. (Note that $H_1(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}_2$, the property that $[L] = 0 \in H_1(\mathbb{RP}^3, \mathbb{Z})$ does not depend on the choice of orientation on components of L, so we can discuss null homologous links without specifying the orientations of L.)

Theorem 1.1. *For a twisted oriented null homologous link $L \subset \mathbb{RP}^3$, one can associate a Bar-Natan chain complex* $\mathrm{CBN}(L)$ *over the base field $\mathbb{F} = \mathbb{F}_2$, whose homology* $\mathrm{HBN}(L)$ *is an invariant of L as a bigraded vector space. More specifically,*

$$\dim(\mathrm{HBN}(L)) = \begin{cases} 0 & \text{if } L \text{ has a component which is nonzero in } H_1(\mathbb{RP}^3, \mathbb{Z}); \\ 2^{|L|} & \text{otherwise (all components of } L \text{ are null homologous)}, \end{cases}$$

and there is a basis of $\mathrm{HBN}(L)$ *given by* $\{s_o \mid o \text{ is a twisted orientation of } L\}$.

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See Definitions 2.8 and 2.9, as well as Lemma 2.10, for the notion of twisted orientation on null homologous links in $\mathbb{RP}^3$. In short, a twisted orientation is an assignment of arrows on each segment of the link projection in $\mathbb{RP}^2$, which reverses direction each time it crosses a fixed essential unknot. More canonically, it is an orientation on the double cover $\widetilde{L}$ of L in S^3 that is reversed by the deck transformation on S^3 of the covering map $S^3 \to \mathbb{RP}^3$. In particular, if L has any homologically essential components, then such a twisted orientation does not exist, and $\mathrm{HBN}(L)$ vanishes in this case.

The reason for the appearance of the twisted orientation is that when we perform the Bar-Natan deformation, we are forced to assign the nontrivial map id_V to the 1-1 bifurcation in the definition of the Bar-Natan chain complex. This differs from the Lee deformation and the usual Khovanov homology in $\mathbb{RP}^3$, where the map assigned to the 1-1 bifurcation is the zero map. Therefore, we introduce the extra twisting to make the 1-1 bifurcation behave in a manner more consistent with the 1-2 and 2-1 bifurcations. By using the notion of "twisted orientation" instead of "orientation", essentially all the proofs for the usual Bar-Natan and Lee homology in S^3 as in [8; 15; 17] carry over in our setting with minor modifications.

As expected, since $\dim(\mathrm{HBN}(K)) = 2$ for a null homologous knot in $\mathbb{RP}^3$, one can define an s-invariant, denoted $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$, from the quantum filtration on $\mathrm{CBN}(K)$ and use it to establish a genus bound in the usual way. See Definitions 4.1 and 4.3 for the precise definition of $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$. It is worth noting that, unlike the usual case where the genus bound applies to the class of orientable slice surfaces, here we obtain a genus bound for twisted orientable slice surfaces. See Definitions 3.1 and 4.8 for precise definitions of twisted orientable cobordisms and slice surfaces. Roughly speaking, this means that the double cover of the surface in $S^3 \times I$ is orientable, and the fiberwise deck transformation τ in the S^3-direction reverses the orientation. We establish the following relationship between $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ and the Euler characteristic of the twisted orientable slice surface, which is analogous to the usual statement for the s-invariant in S^3.

Theorem 1.2. *Suppose $\Sigma : L \to L'$ is a twisted orientable cobordism between the null homologous links L and L' in $\mathbb{RP}^3 \times I$. Then one can define a filtered chain map of filtration degree $\chi(\Sigma)$,*

$$F_\Sigma : \mathrm{CBN}_{*,*}(L) \to \mathrm{CBN}_{*,*}(L'),$$

such that

$$F_\Sigma([s_o]) = \sum_{\{o_i\}} [s_{o_{i_{|L'}}}].$$

Here, o is a twisted orientation on L, $\{o_i\}$ is the set of twisted orientations on Σ that restrict to o on L, $o_{i_{|L'}}$ represents the restriction of such orientations on L',

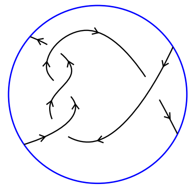

Figure 1. An example of difference in s-invariants defined using Lee deformation and Bar-Natan deformation.

and s_o, respectively $s_{o_{i_{|L'}}}$, are the corresponding canonical generators of $\mathrm{HBN}(L)$, *respectively* $\mathrm{HBN}(L')$, *defined in Definition 2.14.*

In particular, when both L and L' are null homologous knots and Σ is connected, F_Σ is a quasi-isomorphism of filtration degree $\chi(\Sigma)$. Further specifying to the case where L' is the trivial unknot in $\mathbb{RP}^3$, one gets

$$-\chi(\Sigma) \geq |s^{\mathrm{BN}}_{\mathbb{RP}^3}(L)|,$$

for any twisted orientable slice surface Σ of the knot L.

A twisted orientable slice surface can be either orientable or unorientable, and not every (un)orientable slice surface is necessarily twisted orientable. See Example 3.2 for different possibilities of the combinations of (non)twisted orientable/(un)orientable cobordisms.

The s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ shares similar formal properties with the usual s-invariant in S^3. See Proposition 4.5 for mirroring, Proposition 4.6 for taking a local connected sum with a local knot and Proposition 4.7 for “positive knots” in the sense of twisted orientations. All of these follow from the same arguments as in the case of S^3. We also provide a simple example to demonstrate the difference between the s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ defined in this paper and the s-invariant defined as in [13]. Consider the knot shown in Figure 1, which is “positive” with respect to the twisted orientation, so we have

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = 3.$$

However, one can construct an orientable slice surface of this knot with genus 1, so the s-invariant of it defined in [13] satisfies

$$|s(K)| \leq 2.$$

In addition, we provide a family of knots in which their two s-invariants differ by an arbitrarily large amount, obtained by inserting more and more full twists in this example. See Example 4.11 for a more detailed discussion.

Organization of the paper. In Section 2, we define the Bar-Natan deformation of the Khovanov complex for null homologous links in $\mathbb{RP}^3$ over the field $\mathbb{F}_2$. We also introduce the notion of twisted orientation and prove Theorem 1.1. In Section 3, we define cobordism maps on the Bar-Natan chain complex for twisted orientable cobordisms and establish the first statement of Theorem 1.2. In Section 4, we define the Bar-Natan s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$, discuss its formal properties, and complete the proof of Theorem 1.2. In Section 5, we discuss some further directions based on this work.

2. Bar-Natan homology for null homologous links in $\mathbb{RP}^3$

We work over the field $\mathbb{F} = \mathbb{F}_2$ of two elements, unless otherwise specified.

Let $V = \mathbb{F}\langle 1, x\rangle$ denote the graded vector space generated by $1, x$, where the quantum grading is defined as

$$\mathrm{qdeg}(1) = 1, \quad \mathrm{qdeg}(x) = -1.$$

Definition 2.1. The *Bar-Natan Frobenius algebra structure on V* is a deformation of the usual Frobenius structure on $V = H^*(S^2)$. It consists of the tuple $(V, m, \Delta, \iota, \eta)$, where the multiplication $m : V \otimes V \to V$ is given by

$$1\otimes 1 \to 1, \quad 1\otimes x \to x, \quad x\otimes 1 \to x, \quad x\otimes x \to x,$$

the comultiplication $\Delta : V \to V \otimes V$ is given by

$$1 \to 1\otimes x + x\otimes 1 + 1\otimes 1, \quad x \to x\otimes x,$$

the unit $\iota : \mathbb{F} \to V$ is given by

$$1 \to 1,$$

and the counit $\eta : V \to \mathbb{F}$ is given by

$$1 \to 0, \quad x \to 1.$$

Remark 2.2. Some signs in the comultiplication Δ may differ from the usual convention, but this makes no difference as we are working over the field $\mathbb{F}$ of characteristic 2. We also use the version in which the extra deformation variable of the Bar-Natan deformation is set to 1, so all the structure maps are filtered rather than grading preserving, after a suitable shift in the quantum grading.

Definition 2.3. A link L in $\mathbb{RP}^3$ is *null homologous* if

$$[L] = 0 \in H_1(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}/2.$$

If particular, we do not require L to be oriented, as different orientations do not affect $[L]$ in $H_1(\mathbb{RP}^3, \mathbb{Z})$, which is $\mathbb{Z}/2$.

Let L be a null homologous link in $\mathbb{RP}^3$. Suppose L is disjoint from a fixed point $* \in \mathbb{RP}^3$. Then, we obtain a link projection diagram D of L in $\mathbb{RP}^2$ using the

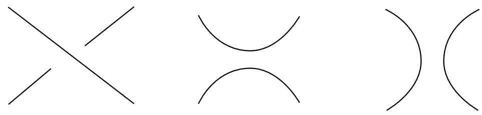

Figure 2. The middle and right illustrate a 0- and 1-smoothing, respectively.

twisted I-bundle structure $\mathbb{RP}^3 \setminus \{*\} \cong \mathbb{RP}^2 \tilde{\times} I$. Suppose D has n crossings. By choosing an ordering of the crossings, we form the usual cube

$$\underline{2}^n := (0 \to 1)^n$$

of smoothings D_v of D for each vertex $v \in \underline{2}^n$, following the convention of 0- and 1-smoothings as indicated in Figure 2.

Since we start with a null homologous link and the smoothing process does not change the homology class, it is straightforward to see that each component in the smoothing D_v is a local unknot in $\mathbb{RP}^2$ for every vertex $v \in \underline{2}^n$. We will apply the Bar-Natan Frobenius algebra to this cube of smoothings, i.e., we associate the vector space V to each local unknot and then take the tensor product for each smoothing D_v.

As for the edge maps, they correspond to change a 0-smoothing to a 1-smoothing at one crossing. Unlike the usual case for link projections in $\mathbb{R}^2$, there are three possibilities as indicated in Figure 3:

- 1-2 bifurcation, which splits one circle into two circles;
- 2-1 bifurcation, which merges two circles into one circle;
- 1-1 bifurcation, which twists one circle into another circle.

We will represent $\mathbb{RP}^2$ as a disk with half of its boundary (the blue circle) identified with the other half. We assign the multiplication map

$$m : V \otimes V \to V$$

to the 2-1 bifurcation and the comultiplication map

$$\Delta : V \to V \otimes V$$

to the 1-2 bifurcation map, as usual. For the 1-1 bifurcation, this corresponds to a map

$$f : V \to V,$$

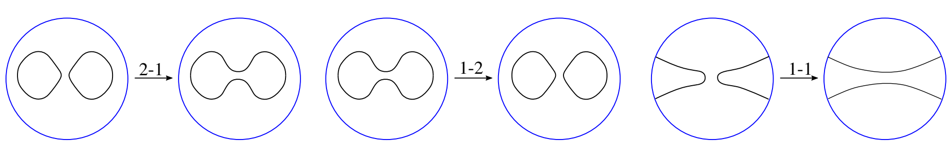

Figure 3. 2-1, 1-2 and 1-1 bifurcations.

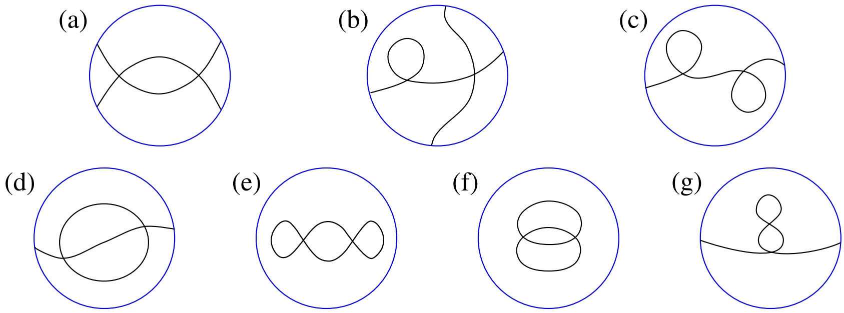

Figure 4. Singular graphs in $\mathbb{RP}^2$ with 2 singular points. See [6].

which is not part of the Bar-Natan Frobenius algebra structure of V as defined above. It turns out that there is a unique choice of f over the base field $\mathbb{F}_2$ that gives a filtered chain complex if we feed the cube of smoothings by the Frobenius algebra V, and use m, Δ, f for the edge maps.

Lemma 2.4. *The only possible choice of $f : V \to V$ for the* 1-1 *bifurcation map over $\mathbb{F}_2$ that gives a chain complex, with $f : V \to V[1]$ being filtered, is*

$$f = \mathrm{id}_V.$$

Here $V[1]$ denotes the graded vector space obtained from V by shifting the quantum grading up by 1.

Proof. To obtain a chain complex, we require that each face of the cube of smoothings gives a commutative diagram (recall that we are working over the field $\mathbb{F}_2$). This amounts to checking that each link diagram in $\mathbb{RP}^2$ with 2 crossings gives a commutative square. A list of such link diagrams is obtained from the singular diagrams in Figure 4 by replacing each singular point by an over or under crossing. Since we are considering null homologous links, only (a), (b), (e), and (f) are possible. A typical square of smoothings and the corresponding maps are shown in Figure 5.

The condition that these squares are commutative gives the following restrictions on the map $f : V \to V$:

(1) $f^2 = m \circ \Delta = \mathrm{id}_V$,

(2) $m \circ (f \otimes \mathrm{id}_V) = m \circ (\mathrm{id}_V \otimes f) = f \circ m$,

(3) $(f \otimes \mathrm{id}_V) \circ \Delta = (\mathrm{id}_V \otimes f) \circ \Delta = \Delta \circ f$.

Also, when defining the chain complex, we will apply a shift in the quantum grading of degree $|v| = \#$ of 1s in $v \in \underline{2}^n$ to the vector space corresponding to the smoothing D_v. Therefore, to ensure the edge maps are filtered, i.e., nondecreasing in the quantum grading, we also require

$$f : V \to V[1] \text{ to be filtered.}$$

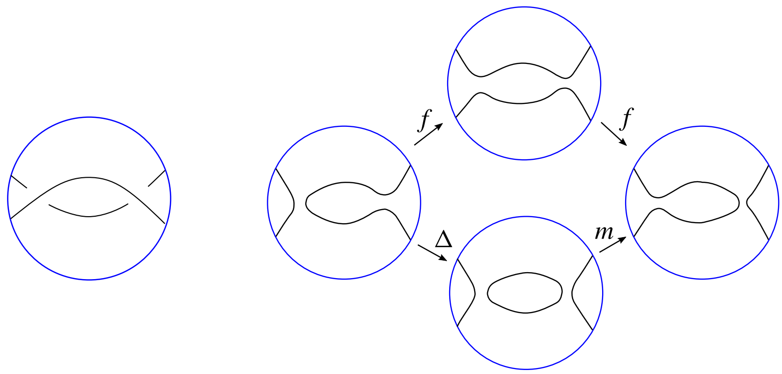

Figure 5. One example of a square in the cube of resolution.

Now, it is easy to see that the only possible choice of f satisfying the above conditions is

$$f = \mathrm{id}_V. \qquad \square$$

Remark 2.5. In [1], Asaeda, Przytycki and Sikora extended the Khovanov homology to links in $\mathbb{RP}^3$ over the field $\mathbb{F}_2$, where they assigned the 0 map to the 1-1 bifurcation map. Because they use the usual Frobenius algebra for Khovanov homology, the corresponding condition for f becomes $f^2 = m \circ \Delta = 0$, so $f = 0$ is a natural choice. In [6], Gabrovšek introduced some sign conventions to make the extension of Khovanov homology work over rings of characteristic 0, where the 1-1 bifurcation map is also assigned 0. More recently, in [13], Manolescu and Willis extended the definition of Lee homology to links in $\mathbb{RP}^3$, where, again, the 1-1 bifurcation map is assigned 0. It is due to this difference that the s-invariant from the Bar-Natan deformation in this paper gives a genus bound on twisted orientable cobordisms, while the s-invariant from the Lee deformation in [13] gives a genus bound on orientable cobordisms.

In [4], the author defined some variation of the usual Khovanov homology in $\mathbb{RP}^3$. One can try applying the Bar-Natan Frobenius algebra structure on V instead of the usual Khovanov Frobenius algebra structure and see what happens. Unfortunately, it won't give a more interesting homology theory. The reason is that if one uses the Bar-Natan Frobenius algebra structure, then the requirement for f and g becomes $f \circ g = \mathrm{id}_{V_0}$, $g \circ f = \mathrm{id}_{V_1}$, which implies V_0 and V_1 are isomorphic, and the chain complex will just be a direct sum of n copies of the reduced version of the Bar-Natan chain complex defined in this paper, where $n = \dim(V_0) = \dim(V_1)$. In the Khovanov setting, the equations we need for f and g are $f \circ g = g \circ f = 0$, which leaves more room for the choice of V_0, V_1, f and g. Still, one could obtain different spectral sequences for different choices of V_0, V_1, f, and g. We have not attempted to explore the possibilities in this project.

By examining the commutativity of the remaining possible 2-crossing diagrams in $\mathbb{RP}^2$ that do not involve the 1-1 bifurcation (these are local diagrams, which means the check is the same as verifying whether the Bar-Natan Frobenius algebra structure on V gives a chain complex for links in S^3), we can define the following chain complex for the link diagram D in $\mathbb{RP}^2$.

Definition 2.6. Given an n-crossing link diagram D in $\mathbb{RP}^2$ of a null homologous link, the *unadjusted Bar-Natan chain complex* $\mathrm{CBN}^{\mathrm{un}}_{*,*}(D)$ is the bigraded chain complex

$$\mathrm{CBN}^{\mathrm{un}}_{i,*} = \bigoplus_{\substack{v\in\underline{2}^n \\ |v|=i}} C(D_v)[i], \quad \partial = \sum_{e:\text{edges in } \underline{2}^n} \partial_e,$$

where the first grading is the homological grading, and the second grading is the quantum grading. Denote its homology by $\mathrm{HBN}^{\mathrm{un}}(D)$. Here $C(D_v) = V^{\otimes k_v}$ if the smoothing D_v consists of k_v unknots, $[i]$ denotes shifting up by i in the quantum grading, and ∂_e applies m, Δ or f to the involved unknots, depending on whether the edge is a 2-1, 1-2, or 1-1 bifurcation.

It is called the "unadjusted" Bar-Natan complex because we will apply some global shifts in the homological and quantum gradings to make it a link invariant, similar to the usual definition for Khovanov/Bar-Natan homology in S^3. However, the way we apply the shifts is a bit unconventional, so we delay the discussion until later, when we introduce more terminology.

Note that the quantum grading-preserving part of the differential is exactly the same as the differential used in the definition of Khovanov homology for links in $\mathbb{RP}^3$ over $\mathbb{F}_2$ as in [1], where the map associated with the 1-1 bifurcation is the zero map. (Note that such a map needs to be grading preserving from V to $V[1]$, so id_V does not preserve the quantum grading.)

Now the natural next step is to check whether the homology depends only on the link rather than the link diagram. This is usually done by verifying the invariance of the homology under Reidemeister moves. As discussed in [5], for link projections in $\mathbb{RP}^2$, there is a similar list of Reidemeister moves that relate different projections of isotopic links in $\mathbb{RP}^3$, as shown in Figure 6. However, for the arguments in the later sections involving the s-invariant, we need stronger conditions on the induced map by Reidemeister moves, so we will again delay the discussion of invariance until later.

Recall that in the usual Bar-Natan homology for links in S^3, the dimension of the Bar-Natan homology is actually determined just by the number of components in the link, and there is a canonical basis of the homology corresponding to different choices of orientations on each component of the link. Furthermore, the maps induced by the Reidemeister moves will send this canonical basis to the

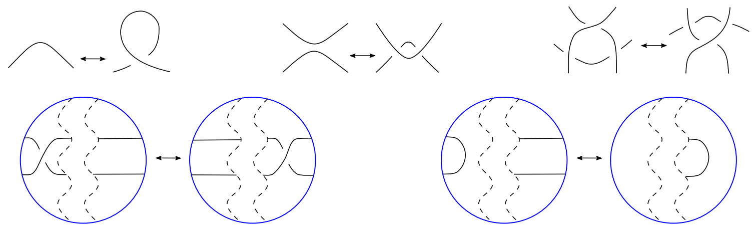

Figure 6. Reidemeister moves in $\mathbb{RP}^3$. Top: left, middle, and right correspond to R-I, R-II, and R-III, respectively. Bottom: left and right correspond to R-IV and R-V, respectively.

corresponding one after the Reidemeister moves. We will prove a similar result for the Bar-Natan homology for null homologous links in $\mathbb{RP}^3$: the dimension of the homology is determined by the number of null homologous components of the link, a canonical basis of the homology is given by "twisted orientations" on the link, and Reidemeister moves send the canonical basis to the canonical basis.

For that, we need to introduce the notion of a "twisted orientation". This is the central notion of the paper. Essentially, all the proofs for the usual Bar-Natan homology in S^3 apply in $\mathbb{RP}^3$ if we replace "orientation" in S^3 with "twisted orientation" in $\mathbb{RP}^3$.

Definition 2.7. Let U_1 denote the essential unknot in $\mathbb{RP}^2$, which is the quotient of the boundary of the disk. We represent $\mathbb{RP}^2$ by identifying the two halves of the boundary of a disk, and U_1 is represented by the quotient of the blue circles in all the figures.

Definition 2.8. Suppose D is a link diagram in $\mathbb{RP}^2$ such that each component of the link L represented by D in $\mathbb{RP}^3$ is null homologous. A *twisted orientation* on D is an assignment of arrows to each segment of D, such that it is reversed each time the link crosses U_1.

See Figure 7 for an illustration of the twisted orientation, along with a comparison to the usual orientation. Note that a twisted orientation only exists if we assume each component of D is null homologous since we cannot assign alternating arrows on a homologically essential component, which intersects U_1 an odd number of times.

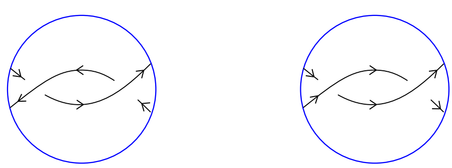

Figure 7. An example of a twisted orientation (left) compared with a usual orientation (right) on the same knot.

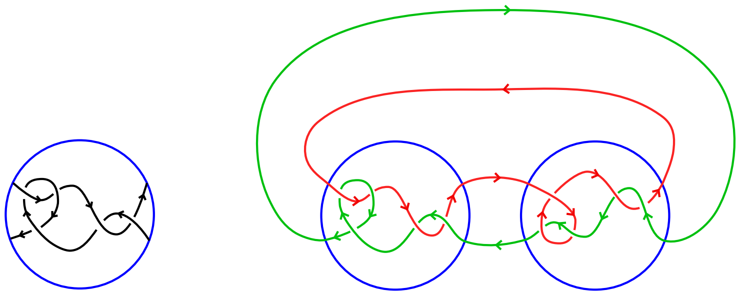

Figure 8. A twisted orientation on L (left) and the corresponding orientation on $\widetilde{L} = cl(T \circ F(T))$ (right) which is reversed by the action of τ.

More canonically, we have another way to view the twisted orientation that does not depend on the link diagram D.

Definition 2.9. Suppose L is a link in $\mathbb{RP}^3$ such that each component of L is null homologous. Let $\widetilde{L}$ denote the double cover of L in S^3, and let τ denote the deck transformation of the covering map $S^3 \to \mathbb{RP}^3$. A *twisted orientation* on L is an orientation on $\widetilde{L}$ that is reversed under the action of τ.

The next lemma proves that these two definitions agree, and when we refer to a twisted orientation on a link L in $\mathbb{RP}^3$, we mean either of the two, depending on the context.

Lemma 2.10. *Suppose D is a link diagram of a link L in $\mathbb{RP}^3$ such that each component of L is null homologous. Then a twisted orientation on D induces a twisted orientation on L, and vice versa. In particular, there are $2^{|L|}$ twisted orientations on L, where $|L|$ is the number of components of L.*

Proof. It is easy to see that a knot K in $\mathbb{RP}^3$ is null homologous if and only if its double cover $\widetilde{K}$ in S^3 is a 2-component link (rather that a knot). Therefore, due to the null homologous assumption, each component of L lifts to a two-component link in $\widetilde{L}$.

One way to draw a diagram of $\widetilde{L}$ is as follows. View the link diagram D in $\mathbb{RP}^2$ as an n-n tangle, denoted by T. Let $F(T)$ be the n-n tangle obtained by flipping T, i.e., rotating T by 180° about its middle horizontal axis in the plane. Then, $\widetilde{L}$ is the closure of the composition $T \circ F(T)$, where the deck transformation τ acts by swapping T and $F(T)$. See Figure 8 for an example.

Now consider a specific component K in L. Every time K hits the essential unknot U_1, it travels from one copy of $\mathbb{RP}^2$ to the other copy in the lifted picture of $\widetilde{K}$. Therefore, when we restrict to one copy of $\mathbb{RP}^2$, two adjacent segments (adjacent in the sense that they share common points on U_1) must come from different components of the lifted link $\widetilde{K}$ (labeled red and green in Figure 8, respectively).

The requirement that the arrow is reversed in the definition of D thus becomes the requirement that the deck transformation τ reverses the orientations on the two components of $\widetilde{K}$. □

Remark 2.11. If we are given an orientation on the lifted link $\widetilde{L}$ that is reversed by τ, and we present $\widetilde{L}$ as the closure of $T \circ F(T)$, there is an ambiguity in assigning the corresponding twisted orientation on the quotient link L. This ambiguity arises from whether we use T or $F(T)$ as a link diagram for L in $\mathbb{RP}^2$. After we fix a choice of the fundamental domain of the deck transformation τ (i.e., choosing T instead of $F(T)$), there is a canonical one-to-one correspondence between the set of twisted orientations of L and the set of orientations of $\widetilde{L}$ that are reversed by τ.

Now, we are going to define elements in the Bar-Natan homology corresponding to the twisted orientations. Later, we will show they form a basis of the Bar-Natan homology, as in the case of Bar-Natan homology in S^3.

Recall that we can diagonalize the Bar-Natan Frobenius algebra V using the basis $a = 1 + x$, $b = x$ (remember, we always work on the field $\mathbb{F}_2$), such that the multiplication and comultiplication become

$$m : V \otimes V \to V, \quad a \otimes a \to a, \quad b \otimes b \to b, \quad a \otimes b \to 0, \quad b \otimes a \to 0,$$
$$\Delta : V \to V \otimes V, \quad a \to a \otimes a, \quad b \to b \otimes b.$$

In the usual Bar-Natan homology in S^3, for each orientation of the link, one associates an element in the homology by forming the oriented resolutions and then assigning either a or b to each unknot component in the resolution, depending on their orientations and distance from infinity. Here, we will do something similar. Since a twisted orientation is just some assignment of arrows on each segment in the link diagram, and forming an oriented resolution only concerns the orientations near each crossing, we can form the "twisted orientation resolution" of a twisted oriented link diagram by performing the oriented resolution at each crossing in the usual sense. Then, we assign a or b to each unknot component in the twisted oriented resolution according to some rule. The rule requires some explanation, so let's go over it first.

Definition 2.12. Suppose D is a twisted oriented link diagram in $\mathbb{RP}^2$ of an unlink U in $\mathbb{RP}^3$ with no crossings, that is, it is a disjoint union of local unknots. Pick a point p on the essential unknot U_1, which is disjoint from the link diagram D. The choice of the point p gives a way to view the diagram D as an n-n tangle T. We form the lifted link $\widetilde{U}$ as the closure of $T \circ F(T)$ as before. As discussed in Lemma 2.10, a twisted orientation on D induces an orientation on $\widetilde{U}$ which is reversed under τ, in particular each component of $\widetilde{U}$ is oriented.

Define the following $\mathbb{Z}/2$-valued functions on components of $\widetilde{U}$: Let

$$d : \pi_0(\widetilde{U}) \to \mathbb{Z}/2$$

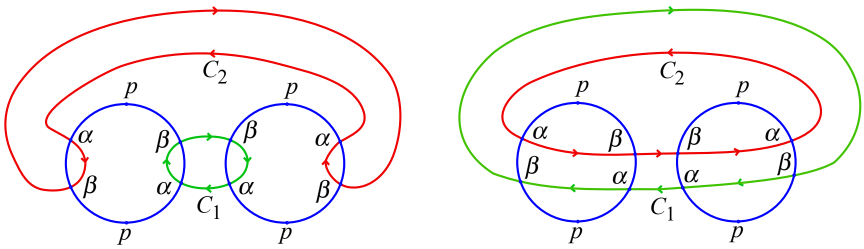

Figure 9. Two possibilities of the relative positions between C_1, C_2 and p.

be the number of circles in $\widetilde{L}$ separating the chosen component from infinity mod 2. Let

$$o : \pi_0(\widetilde{U}) \to \mathbb{Z}/2$$

equal 1 if the component is oriented counterclockwise, and 0 if it is oriented clockwise. Let

$$l : \pi_0(\widetilde{U}) \to \mathbb{Z}/2$$

be the sum of the above two functions mod 2:

$$l = d + o \text{ mod } 2.$$

Now we want to prove that l takes the same value on the two components in $\widetilde{U}$ that are mapped to the same component in U, so it descends to a map $l : \pi_0(U) \to \mathbb{Z}/2$.

Lemma 2.13. *Assume the same setting as in Definition 2.12. Suppose C_1 and C_2 are two components in $\widetilde{U}$ that are sent to the same component C in U under the quotient map $S^3 \to \mathbb{RP}^3$. Then*

$$l(C_1) = l(C_2).$$

Proof. By applying Reidemeister moves R-V away from the point p, we can assume that the local unknot C intersects the essential unknot U_1 at exactly two points, α and β. Then, there are two cases, depending on whether the point p lies inside the disk bounded by C, as shown in Figure 9. Later, when we refer to an arc in the proof, such as the arc $p\alpha$, we mean the open arc of U_1 traveling counterclockwise from p to α.

(1) In this case, the two components C_1 and C_2 are oriented in the same direction, so

$$o(C_1) = o(C_2).$$

The distance to infinity function $d(C_1)$ can be also be described by counting the number of intersections between $\widetilde{U}$ and the arc $p\alpha$ mod 2, and similarly, $d(C_2)$ counts the number of intersections between $\widetilde{U}$ and the arc βp mod 2. By the assumption that D is the diagram of an unlink, it is null homologous in particular.

Thus, the total number of intersections between $\widetilde{U}$ and the semicircle pp is even. Additionally, the link diagram D has no crossing, and the circle C_1 bounds a disk, so the number of intersections between $\widetilde{U}$ and the arc $\alpha\beta$ should also be even. Therefore, we conclude that

$$\#|\widetilde{U} \cap p\alpha| + \#|\widetilde{U} \cap \beta p| = 0 \bmod 2,$$

so

$$d(C_1) = \#|\widetilde{U} \cap p\alpha| = \#|\widetilde{U} \cap \beta p| = d(C_2) \bmod 2.$$

Hence, we have

$$l(C_1) = l(C_2) \bmod 2$$

in this case.

(2) This time, the orientations on C_1 and C_2 are opposite to each other, so

$$o(C_1) = o(C_2) + 1 \bmod 2.$$

Again, the distance function can be described in terms of the number of intersections between $\widetilde{U}$ and various arcs. In this case,

$$d(C_1) = \#|\widetilde{U} \cap p\alpha|, \quad d(C_2) = \#|\widetilde{U} \cap p\beta| = \#|\widetilde{U} \cap p\alpha| + 1 + \#|\widetilde{U} \cap \alpha\beta|.$$

By the assumptions on D that D has no crossings and each component of D is null homologous, we conclude that

$$\#|\widetilde{U} \cap \alpha\beta| = 0 \bmod 2,$$

so

$$d(C_1) = d(C_2) + 1 \bmod 2.$$

Therefore, we also have

$$l(C_1) = l(C_2) \bmod 2$$

in this case. □

With the help of the labeling function $l : \pi_0(L) \to \mathbb{Z}/2$, we can define our canonical generators associated with the twisted orientations.

Definition 2.14. Suppose D is a link diagram of a null homologous link L in $\mathbb{RP}^3$ with a twisted orientation o. Pick a point p on the essential unknot U_1 that is disjoint from the link diagram. Then, we define the *canonical element s_o associated with the twisted orientation o* in $\mathrm{CBN}^{\mathrm{un}}(D)$ as follows: First, form the twisted oriented resolution D_o according to o; then, apply the labeling function l to each component of the unlink D_o. Finally, assign the element $a = 1 + x$ to a component C in D_o if $l(C) = 0$, and $b = x$ to it if $l(C) = 1$.

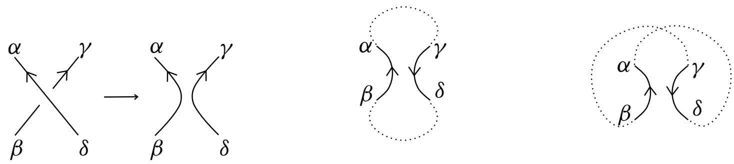

Figure 10. Two arcs sharing a crossing in the twisted oriented resolution won't belong to the same circle. Middle: each of the dotted arcs $\alpha\gamma$, $\beta\delta$ intersects U_1 an even number of times. Right: each of the dotted arcs $\alpha\gamma$, $\beta\delta$ intersects U_1 an odd number of times.

Remark 2.15. This definition depends on the choice of p in the following way: a different choice of p might result in a uniform change of $l(C)$ to $l(C)+1$ for each component C in D_o, leading to a switch of labels a and b. The point p serves as the point at infinity in the labeling function of link diagrams in $\mathbb{R}^2$.

As in the case of S^3, these canonical generators associated with the twisted orientations form a basis for the Bar-Natan homology. First, we will show that they actually lie in $\mathrm{HBN}^{\mathrm{un}}(L)$, and then we will prove that the dimensions match using an induction argument. The proof closely follows the original ones in [8] for Lee homology. See also [17] for a treatment of Bar-Natan homology (which is essentially the same as the one for Lee homology). The main difference in our case is that we need to take into account the 1-1 bifurcation and whether there is a homologically essential component in the link. These issues are resolved by the notion of twisted orientation.

Proposition 2.16. *For a null homologous link diagram D with a twisted orientation o, the canonical element s_o represents an element in the homology* $\mathrm{HBN}^{\mathrm{un}}(D)$.

Proof. Following Lee's argument, we aim to show that s_o lies in both $\ker(d)$ and $\ker(d^*)$, where d^* is the adjoint differential defined using the inner product on the Frobenius algebra V.

Let's take a closer look at what happens near a crossing. Without loss of generality, suppose the local picture near a crossing looks like the one in Figure 10. As in the usual proof for links in S^3, we want to show the two arcs $\alpha\beta$ and $\gamma\delta$ will not belong to the same circle in the resolution. In the usual proof, this possibility is ruled out because if these two arcs belong to the same circle, then the endpoint α would be adjacent to γ and β would be adjacent to δ in the circle due to the absence of crossings in the resolution. However, this arrangement is incompatible with the orientation on the arcs $\alpha\beta$ and $\gamma\delta$.

In our case, the situation is slightly different because the projection lies in $\mathbb{RP}^2$ instead of $\mathbb{R}^2$. We will again prove by contradiction that the arcs $\alpha\beta$ and $\gamma\delta$ will not belong to the same circle in the twisted oriented resolution. If they did, then there are two possibilities:

(1) If the endpoint α is adjacent to γ on the circle, then the arc $\alpha\gamma$ of the circle intersects the essential unknot U_1 an even number of times. Therefore, the twisted orientation on the circle will be from β to α and from γ to δ, or from α to β and from δ to γ, which is incompatible with the local orientation near the crossing.

(2) If the endpoint α is adjacent to δ on the circle, then the arc $\alpha\delta$ of the circle intersects the essential unknot U_1 an odd number of times. Therefore, the twisted orientation on the circle will be again from β to α and from γ to δ, or from α to β and from δ to γ, as the arrow is switched an odd number of times in the twisted orientation. But this is again incompatible with the local orientation near the crossing.

Therefore, we conclude that the arcs $\alpha\beta$ and $\gamma\delta$ near a crossing will not belong to the same circle in the twisted oriented resolution. This implies that if we change the smoothing at the crossing in the twisted orientation resolution, it will not result in a 1-1 bifurcation or a 1-2 bifurcation, which is exactly what we want to avoid, as the corresponding maps are nontrivial. The rest of the proof is exactly the same as in the usual case for links in S^3, as the rule we used to assign a and b to each circle in the resolution guarantees that the label assigned to the circle containing the arc $\alpha\beta$ is different from that assigned to the circle containing the arc $\gamma\delta$ in the resolution. Therefore, s_o lies in $\ker(d) \cap \ker(d^*)$. □

Now we prove a formula for the dimension of $\mathrm{HBN}^{\mathrm{un}}(D)$ using an induction argument on the number of crossings. There is an adjustment we need to make compared to the original argument in [8], as we need to discuss whether there is a homologically essential component in the link or not.

Proposition 2.17. *Suppose L is a null homologous link in $\mathbb{RP}^3$ with a link diagram D in $\mathbb{RP}^2$. Then*

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = \begin{cases} 0 & \text{if } L \text{ has some nonzero component in } H_1(\mathbb{RP}^3, \mathbb{Z}); \\ 2^{|L|} & \text{otherwise (if all components of } L \text{ are null homologous}). \end{cases}$$

Furthermore, a basis of $\mathrm{HBN}_{\mathrm{un}}(D)$ *is given by*

$$\{s_o \mid o \text{ is a twisted orientation of } L\}.$$

Remark 2.18. If L has a component which is nonzero in $H_1(\mathbb{RP}^3, \mathbb{Z})$, then there is no twisted orientation on L. Therefore, the set of basis elements given by twisted orientations is empty, and $\mathrm{HBN}^{\mathrm{un}}(L)$ is 0-dimensional in this case.

Proof. Following Lee's original proof, we first prove it for knots and 2-component links by induction on the number of crossings. It clearly holds for the unknot and the unlink with two local unknot components.

Now let D be a link diagram of a null homologous knot with n crossings. Pick one crossing, and let D_0, D_1 denote the 0- and 1-smoothing of D at this crossing,

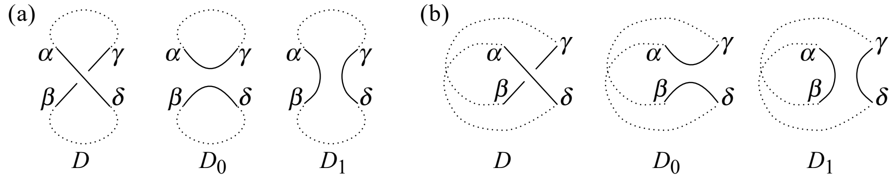

Figure 11. Resolve a null homologous knot/2-component link at a crossing.

respectively. Refer to (a) in Figure 11 as a local model near the crossing. Suppose, without loss of generality, that the endpoint α is adjacent to the endpoint γ on the knot K. (The case where α is adjacent to the endpoint β is the same, by changing the over-crossing to an under-crossing, which switches D_0 and D_1 but doesn't change the argument otherwise. The endpoint α won't be adjacent to δ, as that would give a 2-component link instead of a knot.) Then we divide into two cases depending on how many times the arc $\alpha\gamma$ of K intersects the essential unknot U_1.

(1) If the segment $\alpha\gamma$ intersects U_1 an odd number of times, so does the segment $\beta\delta$, since we assume K is null homologous. Then the link represented by D_0 consists of two homologically essential components, so by the induction hypothesis, it has trivial Bar-Natan homology. The diagram D_1 represents a null homologous knot with one fewer crossing than D, so by the induction hypothesis, we have

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) = 2.$$

Using the long exact sequence relating $\mathrm{HBN}^{\mathrm{un}}(D)$, $\mathrm{HBN}^{\mathrm{un}}(D_0)$, $\mathrm{HBN}^{\mathrm{un}}(D_1)$, we conclude that

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = 2.$$

(2) If the segment $\alpha\gamma$ intersects U_1 an even number of times, so does the segment $\beta\delta$. Then D_0 consists of two null homologous components and D_1 is a null homologous knot, so by the induction hypothesis,

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D_0) = 4, \quad \dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) = 2.$$

As in the usual case, out of the four twisted orientations on D_0, there are two that are compatible under the change of smoothings to twisted orientations on D_1, and the map in the long exact sequence will send these two twisted oriented generators in $\mathrm{HBN}^{\mathrm{un}}(D_0)$ to the corresponding ones in $\mathrm{HBN}^{\mathrm{un}}(D_1)$. Therefore,

$$2 \leq \dim(\mathrm{HBN}^{\mathrm{un}}(D)) \leq \dim(\mathrm{HBN}^{\mathrm{un}}(D_0)) + \dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) - 4 = 2.$$

Now suppose D is a diagram of a null homologous link L of two components, K_0 and K_1. Suppose first that there are no crossings between the two components in the link diagram D. Then K_0, K_1 must both be null homologous (otherwise, they would both be homologically essential, and there would have to be at least one crossing between these two components), and we can apply the Künneth formula to conclude

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = 4.$$

Suppose the two components share at least one crossing. Pick any crossing shared by them and form the 0- and 1-smoothing, D_0 and D_1, respectively. Refer to (b) in Figure 11 as a local model. We again divide into two cases depending on whether K_0 and K_1 are null homologous or not.

(1) If K_0, K_1 are both null homologous, then each of the segments $\alpha\delta$ and $\beta\gamma$ intersects the essential unknot U_1 an even number of times, and the two twisted orientations on D_0 are incompatible with the two twisted orientations on D_1 when changing the smoothing. Therefore, the map from $\mathrm{HBN}^{\mathrm{un}}(D_0)$ to $\mathrm{HBN}^{\mathrm{un}}(D_1)$ is 0 in the long exact sequence, and

$$4 \leq \dim(\mathrm{HBN}^{\mathrm{un}}(D)) \leq \dim(\mathrm{HBN}^{\mathrm{un}}(D_0)) + \dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) = 4.$$

(2) If K_0, K_1 are both homologically essential, then each of the segments $\alpha\delta$ and $\beta\gamma$ intersects the essential unknot U_1 an odd number of times, and the two twisted orientations on D_0 are compatible with the two twisted orientations on D_1 when changing the smoothing. Therefore, the map in the long exact sequence sends $\mathrm{HBN}^{\mathrm{un}}(D_0)$ isomorphically to $\mathrm{HBN}^{\mathrm{un}}(D_1)$, and hence

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = 0.$$

This finishes the discussion of knots and 2-component links. For links with more components, if there is a component that doesn't share a crossing with any other components, then we can apply the Künneth formula. Otherwise, there is at least one crossing shared by different components. Again, we divide into two cases depending on whether there exists a homological essential component or not.

(1) Suppose all components are null homologous. Then we apply the same argument as in the case (1) for 2-component links.
(2) Suppose there are some homological essential components. Then, there must be at least two such components, and they must share a crossing in the link diagram. Choose one such crossing, and then we apply the same argument as in case (2) for 2-component links. □

It is time to apply the global grading shift to ensure that the Bar-Natan homology is a link invariant in $\mathbb{RP}^3$ as a bigraded vector space. The usual convention is to apply a shift of $-n_-$ in the homological grading and a shift of $n_+ + 2n_-$ in the quantum grading, where n_+ and n_- are the numbers of positive and negative crossings for an oriented link, respectively. As expected, we will obtain an invariant for twisted oriented links in $\mathbb{RP}^3$, and we should use n_+ and n_- counted with respect to the twisted orientation.

Definition 2.19. Let D be a link diagram in $\mathbb{RP}^2$ of a null homologous link in $\mathbb{RP}^3$ with a twisted orientation. Let n_+ and n_- denote the numbers of positive and negative crossings, respectively, counted with respect to the twisted orientation on D. Then the *Bar-Natan chain complex* $\mathrm{CBN}_{*,*}(D)$ is defined as

$$\mathrm{CBN}_{*,*}(D) = \mathrm{CBN}^{\mathrm{un}}_{*,*}(D)\{-n_-\}[n_+ - 2n_-],$$

where $\mathrm{CBN}^{\mathrm{un}}_{*,*}(D)$ is the unadjusted Bar-Natan chain complex in Definition 2.6, $\{-n_-\}$ means shifting down by n_- in the homological grading, and $[n_+ - 2n_-]$ means shifting up by $n_+ - 2n_-$ in the quantum grading. The homology $\mathrm{HBN}_{*,*}(D)$ of $\mathrm{CBN}_{*,*}(D)$ is called the *Bar-Natan homology* of null homologous links in $\mathbb{RP}^3$ with a twisted orientation.

We are going to prove that $\mathrm{HBN}_{*,*}$ is an invariant of twisted oriented null homologous links in $\mathbb{RP}^3$ by exhibiting filtered maps between $\mathrm{CBN}_{*,*}$ induced by Reidemeister moves, which are isomorphisms on $\mathrm{HBN}_{*,*}$. Furthermore, we are going to prove that these filtered maps send the canonical generators associated with the twisted orientation to the corresponding canonical generators. This follows the same strategy as in Section 6 in [15]. See also Section 6 in [17] for the Reidemeister moves in the usual Bar-Natan homology. We need to check a few additional cases because the projection lies in $\mathbb{RP}^2$, but there is no extra difficulty.

Proposition 2.20. *For each Reidemeister move as drawn in Figure 6 relating link projections D_0 to D_1, we can define a filtered chain map*

$$\rho' : \mathrm{CBN}_{*,*}(D_0) \to \mathrm{CBN}_{*,*}(D_1),$$

such that the grading-preserving part ρ of ρ' gives an isomorphism on the Khovanov homology for links in $\mathbb{RP}^3$. Additionally,

$$\rho'([s_o]) = [s_{o'}],$$

where o is a twisted orientation on D_0 and o' is the induced twisted orientation on D_1.

Proof. There are three things to do in the proof: first, we need to give the definition of the filtered chain map ρ'; second, we need to check that the grading-preserving part ρ agrees with the one used in [6] for Reidemeister moves on Khovanov homology in $\mathbb{RP}^3$; third, we need to check whether ρ' sends canonical generators to canonical generators. As mentioned, the chain map ρ' we are going to use will be the same as the ones in [15], and the proof is purely a bookkeeping check. The situations for Reidemeister moves I, IV, and V are either trivial or identical to the case in S^3. However, for Reidemeister moves II and III, they will involve some

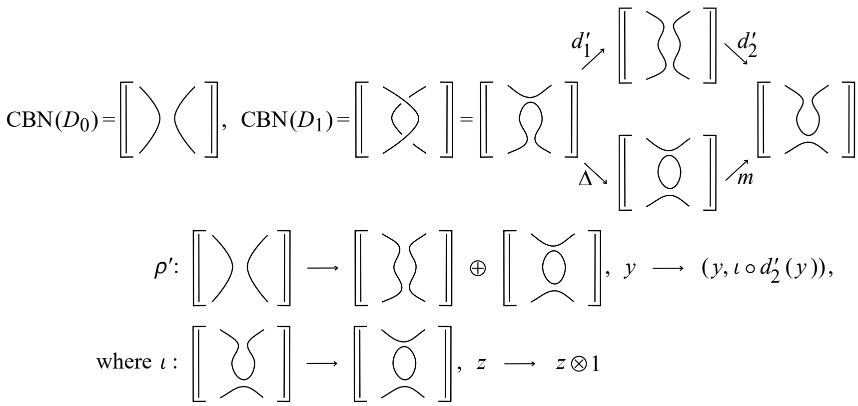

Figure 12. Definition of ρ' for R-II moves.

more case-by-case analysis than the usual situation in S^3. We will provide some examples and leave the rest to the reader.

For Reidemeister moves IV and V, they don't change the chain complex at all, and we can take ρ' to be the identity. It is worth mentioning that for the Reidemeister move V, as we move an arc through the essential unknot U_1, the orientations on this arc in o and o' will be reversed to each other.

For Reidemeister move I, since the operation of adding a curl is local, and the chain maps only involve 1-2 and 2-1 bifurcations, the same map and argument as in [15] work, with $a = 1 + x$, $b = x$. In terms of the global grading shift, if we add a positive curl, we also add 1 to n_+ for our definition of n_+. Therefore, the usual check of global grading shifts works in the same way.

For Reidemeister move II, refer to Figure 12 for the definition of the map ρ'. Here, ι is the map sending the state z to $z \otimes 1$, where 1 is assigned to the extra circle in the middle. In the chain complex CBN(D_1), we know the maps Δ and m because they correspond to a local splitting/merging of a circle. However, we don't know the maps d'_1 and d'_2: they could be any of Δ, m, or f depending on how the rest of the link diagram looks. But the point is we don't need to use their properties in the definition of ρ'. The grading-preserving part ρ of ρ' gives an isomorphism on Khovanov homology in $\mathbb{RP}^3$, using the usual proof of canceling acyclic complexes (note that we only use the property of Δ and m in the proof for invariance of Khovanov homology in $\mathbb{RP}^3$).

To check that ρ' sends canonical generators to canonical generators, we need to further divide into cases, depending on how the two arcs are connected and oriented in the link diagram.

(1) If the two arcs are oriented in the same direction, then, by the argument in Proposition 2.16, the two arcs will not belong to the same circle in the twisted

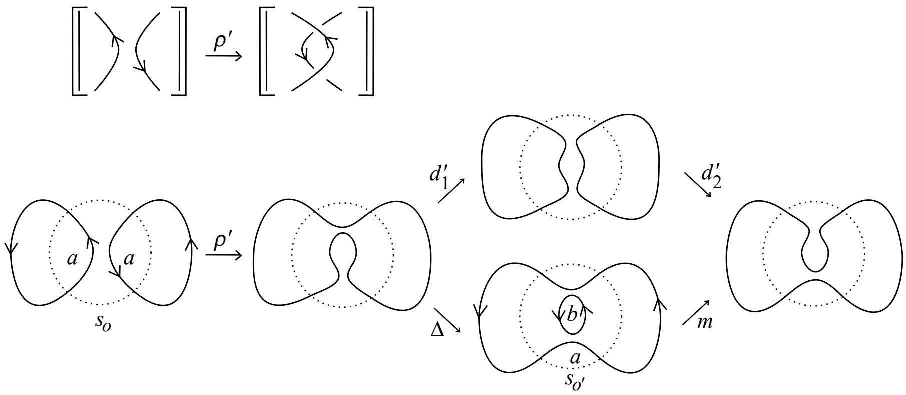

Figure 13. Schematic drawing for case (2)(a) of R-II moves.

oriented resolution, and the labels on these two circles in s_o are different. In this case, d_2' is a 2-1 bifurcation map, which is given by m, and $m(s_o) = 0$, since it merges two circles with different labels. Therefore,

$$\rho'(s_o) = (s_o, \iota \circ d_2'(s_o)) = (s_o, 0) = s_{o'}.$$

(2) If the two arcs are oriented in opposite directions, we divide into cases depending on whether these two arcs belong to the same circle or not in the twisted oriented resolution.

(a) If the two arcs belong to two different circles in the twisted oriented resolution, then the labels on the two circle will be the same in s_o; say, both of them are a, following the rules we assign labels in Definition 2.14. See Figure 13 for an illustration. Then, d_1' is a 1-2 bifurcation map, and d_2' is a 2-1 bifurcation map. Denote s_o by $a \otimes a$, which is the label of s_o on the two circle that are changing through the Reidemeister move II. Then

$$\iota \circ d_2'(a \otimes a) = \iota(a) = a \otimes 1 = a \otimes a + a \otimes b,$$

and

$$\rho'(a \otimes a) = (a \otimes a, a \otimes a + a \otimes b).$$

Note that

$$da = (d_1'(a), \Delta(a)) = (a \otimes a, a \otimes a),$$

so

$$[\rho'(s_o)] = [(0, a \otimes b)] = [s_{o'}] \text{ in } \mathrm{HBN}(D_1).$$

(b) If the two arcs belong to the same circle in the twisted oriented resolution, then again, we need to divide into two cases, depending on whether a 1-1 bifurcation is involved or not.

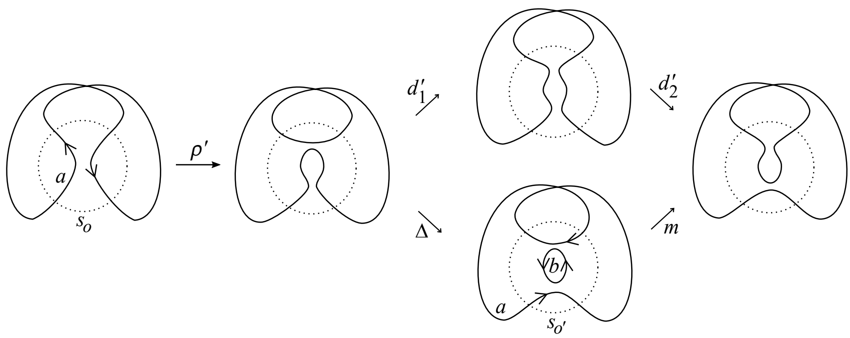

Figure 14. Schematic drawing for case (2)(b)(i) of R-II moves.

(i) Suppose both d_1' and d_2' are 1-1 bifurcations. Let's assume the label on the this circle in s_o is a. See Figure 14 for an illustration. Following the similar notation, we denote s_o by a, and we have

$$\iota \circ d_2'(a) = \iota(a) = a \otimes 1 = a \otimes a + a \otimes b,$$

so

$$\rho'(a) = (a, \iota \circ d_2'(a)) = (a, a \otimes a + a \otimes b).$$

Again, we can cancel $(a, a \otimes a)$ in homology, since

$$da = (d_1'(a), \Delta(a)) = (a, a \otimes a),$$

so

$$[\rho'(s_o)] = [(0, a \otimes b)] = [s_{o'}] \quad \text{in } \mathrm{HBN}(D_1).$$

(ii) Suppose that neither d_1' nor d_2' is a 1-1 bifurcation. (Note that d_1' is a 1-1 bifurcation if and only if d_2' is.) Then, it is the same as the usual check for the Reidemeister move II as in S^3, and we leave it to the reader.

For Reidemeister move III, the definition of the map ρ' is again similar to the corresponding one for Reidemeister move III in S^3, for the same reason as mentioned above: the edge maps, which we need some properties of to define ρ', are local, i.e., they only involve the splitting/merging of a local circle. The proof that the grading-preserving part of ρ' induces an isomorphism on Khovanov homology follows the same approach as in the case of S^3, as the proof of invariance of Khovanov homology under Reidemeister move III in $\mathbb{RP}^3$ is carried out similarly to that in S^3.

To verify that it sends canonical generators to canonical generators, we need to consider different cases depending on how each strand is oriented and how these strands are connected outside the local region in the twisted oriented resolution. This requires more case analysis than the proof in S^3, because the link diagram lies in $\mathbb{RP}^2$. We will illustrate one example and leave the rest to the reader.

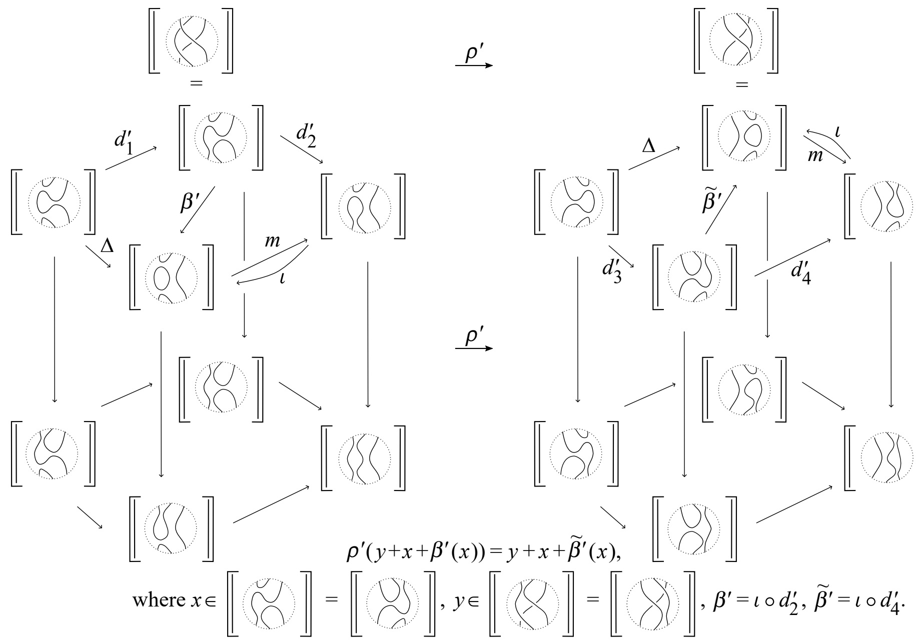

Figure 15. Definition of ρ' for R-III moves.

The definition of the map $\rho' : \mathrm{CBN}(D_0) \to \mathrm{CBN}(D_1)$ is the same as that in [15], which is recalled in Figure 15. The grading-preserving part of ρ' gives the induced map by Reidemeister move III on the Khovanov homology in $\mathbb{RP}^3$, as discussed above. Therefore, it remains to show

$$[\rho(s_o)] = [s_{o'}] \quad \text{in } \mathrm{HBN}(D_1).$$

As in the proof in [15], three out of the four situations regarding relative orientations lead to trivial checks. For the remaining relative orientation, there are several more cases to examine than in [15], as the diagram lies in $\mathbb{RP}^2$. In these additional cases, some of the d_1', d_2', d_3', and d_4' will be the 1-1 bifurcation map. As an example, we consider the following orientation on the strands and connectivity in the twisted oriented resolution, as shown in Figure 16. Note that the rule that we use to assign labels a or b to each circle depends on the choice of the point p. However, changing the position of p results in a uniform switch between a and b, so it does not affect the argument. Here, we demonstrate one possibility of the labeling for a specific choice of p lying in the indicated region in the diagram.

In this case, d_1' and d_2' are 1-1 bifurcations, d_3' is a 1-2 bifurcation, and d_4' is a 2-1 bifurcation. Let x be as shown in Figure 16. Then, by the computation in Figure 17,

$$[s_o] = [x + \beta'(x)] \quad \text{and} \quad [s_{o'}] = [x + \widetilde{\beta}'(x)].$$

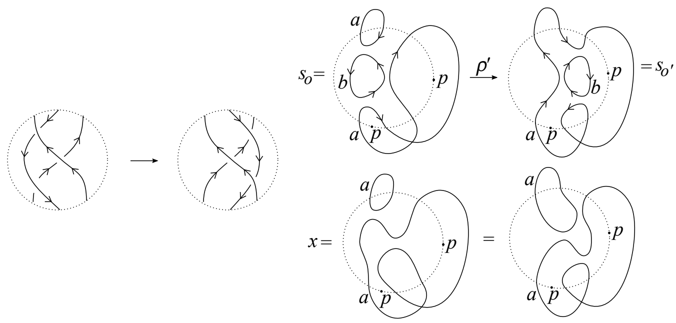

Figure 16. An example computation of ρ' for a R-III move (left), with given orientation and connectivity.

Therefore,

$$\rho'([s_o]) = \rho'([x + \beta(x)]) = [x + \widetilde{\beta}'(x)] = [s_{o'}] \quad \text{in HBN}(D_1). \qquad \square$$

Combining all the discussions in this section, we obtain a well-defined Bar-Natan homology for twisted oriented null homologous links in $\mathbb{RP}^3$.

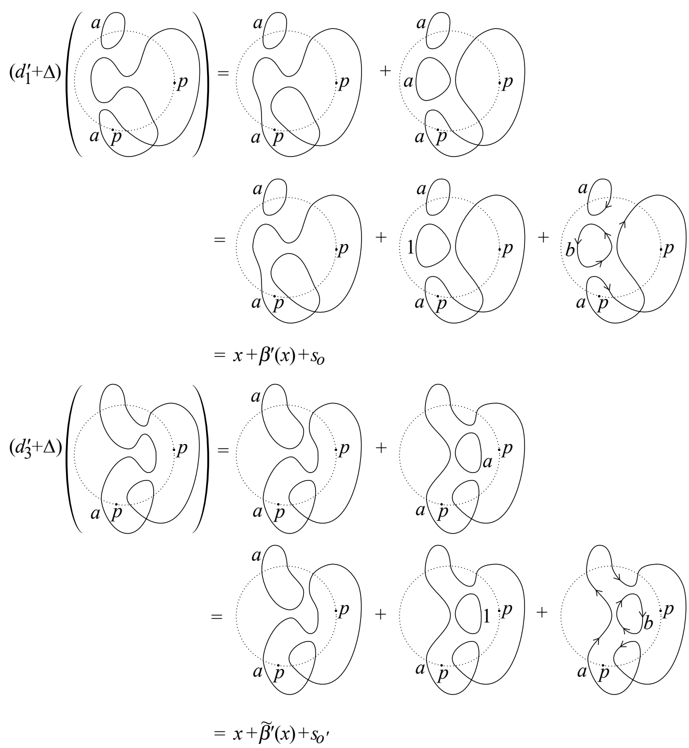

Figure 17. The homologous relation between $x + \beta'(x)$ with s_0, and between $x + \widetilde{\beta}'(x)$ with $s_{o'}$.

Theorem 2.21. *The Bar-Natan homology* HBN(L) *is a twisted oriented link invariant for null homologous links in* $\mathbb{RP}^3$ *as a bigraded vector space*, *with a canonical basis given by*

$$\{s_o \mid o \text{ is a twisted orientation of } L\}.$$

Remark 2.22. When L is a null homologous knot, the usual proof for knot in S^3, showing that s_o lies in homological grading 0, works here as well. This is because we use the twisted orientation to define n_+, n_-, and to form the resolution. For links, the homological grading of s_o can be determined from the linking number between components of the covering link $\widetilde{L}$ in S^3, with the orientation on $\widetilde{L}$ that lifts the twisted orientation.

3. Cobordism maps on Bar-Natan homology in $\mathbb{RP}^3$

In the usual setting for Bar-Natan or Lee homology in S^3, the reason one can get some genus bound from the homology is that one can associate some filtered chain maps to oriented cobordisms between links, which interact well with the canonical generators of the homology given by the orientations, and the filtration degree of the map is bounded by the Euler characteristic of the cobordism. The guiding principle of this paper is to replace "orientation" with "twisted orientation" throughout. Consequently, we need to develop a notion of twisted orientation on cobordisms. The definition of twisted orientable cobordism should ensure that it carries the twisted orientation from one end of the cobordism to the other end. Specifically, when the cobordism is formed by attaching bands, the process should be compatible with the twisted orientation. More formally, we provide the following definition from the perspective of double covers in S^3.

Definition 3.1. Let $\Sigma : L \to L'$ be a cobordism between null homologous links L and L' in $\mathbb{RP}^3 \times I$. A *twisted orientation* on Σ is defined as an orientation on its double cover $\widetilde{\Sigma} : \widetilde{L} \to \widetilde{L}'$ in $S^3 \times I$, which is reversed by the deck transformation $\tau : S^3 \times I \to S^3 \times I$ along the S^3-direction. The cobordism Σ is said to be *twisted orientable* if such a twisted orientation exists.

In particular, the double cover $\widetilde{\Sigma}$ must be orientable if Σ is twisted orientable. This definition is natural in light of the definition of twisted orientation on null homologous links in Definition 2.9.

Note that a twisted orientable cobordism in $\mathbb{RP}^3 \times I$ can be either orientable or unorientable in the usual sense, and there are orientable/unorientable cobordisms which are not twisted orientable, as demonstrated in the next example.

Example 3.2. Figure 18 illustrates examples of all four combinations of orientability and twisted orientability of surface cobordism in $\mathbb{RP}^3 \times I$.

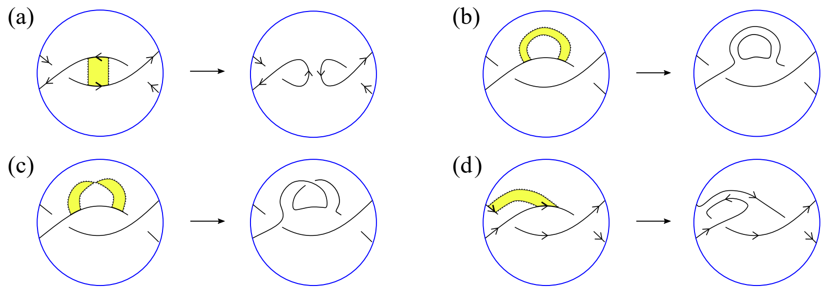

Figure 18. Different possibilities of twisted-orientability and usual orientability of a band attachment. Top left: unorientable and twisted orientable. Top right: orientable and twisted orientable. Bottom left: unorientable and not twisted orientable. Bottom right: orientable and not twisted orientable.

The cobordism in (a) is twisted orientable, with the arrows indicating twisted orientations on the boundary knots. The band is added in a manner compatible with the twisted orientations. However, this cobordism is unorientable in the usual sense, as it is topologically a Möbius band with a disk removed.

The cobordism in (b) is both twisted orientable and orientable in the usual sense.

The cobordism in (c) is neither twisted orientable nor orientable in the usual sense.

The cobordism in (d) is not twisted orientable. Here, the arrows represent usual orientations on the boundary knots, and the band is added in a way compatible with these orientations, making this cobordism orientable in the usual sense.

Now, we will prove an analogous statement about the interaction between twisted orientable cobordisms and the canonical generators of Bar-Natan homology generated by twisted orientations. The idea is to decompose the cobordism Σ into elementary pieces, corresponding to Reidemeister moves and a single handle attachment. To achieve this, we first prove that the twisted orientation on Σ descends to a twisted orientation on each elementary piece, as well as its input and output boundaries.

Lemma 3.3. *Suppose $\Sigma : L \to L'$ is a twisted orientable cobordism between null homologous links L and L' in $\mathbb{RP}^3 \times I$. We have the decomposition*

$$\Sigma = \Sigma_1 \circ \Sigma_2 \circ \cdots \circ \Sigma_n,$$

where each Σ_i corresponds to either a Reidemeister move or a single handle attachment. Let L_i denote the output boundary of the composition $\Sigma_1 \circ \Sigma_2 \circ \cdots \circ \Sigma_i$, with $L_0 = L$ and $L_n = L'$. Then, a twisted orientation on Σ restricts to a twisted orientation on each Σ_i and L_i.

Proof. Consider the double cover of the decomposition

$$\widetilde{\Sigma} = \widetilde{\Sigma_1} \circ \widetilde{\Sigma_2} \circ \cdots \circ \widetilde{\Sigma_n},$$

where each $\widetilde{\Sigma_i}$ corresponds to performing a pair of equivariant Reidemeister moves, or attaching a pair of equivariant handles. A twisted orientation on Σ is an orientation on $\widetilde{\Sigma}$ which is reversed under the action of the deck transformation τ in the S^3-direction. Since the action of τ is fiberwise in the S^3 direction of $S^3 \times I$, such an orientation on $\widetilde{\Sigma}$ restricts to an orientation on each $\widetilde{\Sigma_i}$ which is also reversed by τ. Consequently, the twisted orientation on Σ restricts to each Σ_i.

As for the links L_i, we again look at the double cover $\widetilde{L_i}$. Since these are regular fibers of the projection from $\widetilde{\Sigma}$ to I, a tubular neighborhood of $\widetilde{L_i}$ in $\widetilde{\Sigma}$ is homeomorphic to $\widetilde{L_i} \times [-\epsilon, \epsilon]$, where τ acts trivially in the $[-\epsilon, \epsilon]$-direction, as τ is a fiberwise action. However, τ acts in an orientation-reversing manner on the neighborhood $\widetilde{L_i} \times [-\epsilon, \epsilon]$, so it must reverse the orientation on $\widetilde{L_i}$. Hence, we obtain a twisted orientation on L_i. □

Remark 3.4. In particular, the above lemma implies that every component of L_i is null homologous, since L_i is twisted orientable. Hence, $\mathrm{HBN}(L_i)$ is nontrivial for each i. Note that not all closed loops in Σ are null homologous; for example, the fiber over the critical value of a 1-handle attachment could have a tubular neighborhood homeomorphic to a Möbius band, as seen in (a) of Figure 18. In this case, the action of τ on a tubular neighborhood $\widetilde{L} \times [-\epsilon, \epsilon]$ is the usual covering map from an annulus to a Möbius band, which is nontrivial in the $[-\epsilon, \epsilon]$-direction and orientation-preserving when restricted to $\widetilde{L}$.

Now, we can prove the analogous statement about the effect of the maps induced by twisted orientable cobordisms on Bar-Natan homology in $\mathbb{RP}^3$.

Proposition 3.5. *Suppose $\Sigma : L \to L'$ is a twisted orientable cobordism between null homologous links L and L' in $\mathbb{RP}^3 \times I$. Then, one can define a filtered chain map of degree $\chi(\Sigma)$,*

$$F_\Sigma : \mathrm{CBN}_{*,*}(L) \to \mathrm{CBN}_{*,*}(L'),$$

such that

$$(*) \qquad F_\Sigma([s_o]) = \sum_{\{o_i\}} [s_{o_{i_{|L'}}}],$$

where o is a twisted orientation on L, $\{o_i\}$ is the set of twisted orientations on Σ that restrict to o on L, $o_{i_{|L'}}$ is the restriction of such twisted orientations on L', and s_o (respectively $s_{o_{i_{|L'}}}$) are the corresponding canonical generators of $\mathrm{HBN}(L)$ *(respectively* $\mathrm{HBN}(L')$*) defined as in Definition 2.14.*

Proof. As in the proof of the analogous statement in [15], we divide the cobordism Σ into elementary pieces, define the map for each piece, and verify that the desired properties hold for each piece. The above lemma shows that each elementary piece of Σ is twisted orientable between twisted orientable links. See also Theorem 5.5 in [13] for a similar statement in the context of Lee homology in $\mathbb{RP}^3$.

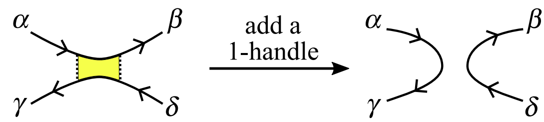

Figure 19. Local picture near a 1-handle attachment.

Recall Definition 2.12, where we describe the rules for assigning a and b to each circle in the twisted oriented resolution for the definition of s_o. In doing so, we need to choose a point p on the essential unknot U_1, which is away from the link diagram. Similarly, here we will choose a point p, such that $p \times I \subset \mathbb{RP}^3 \times I$ is away from the cobordism Σ. This chosen p will serve as the reference point when discussing the canonical generators in $\mathrm{HBN}(L_i)$ for each L_i.

The cobordism Σ can be expressed as the composition of a sequence of Reidemeister cobordisms and elementary Morse cobordisms. For the Reidemeister cobordisms, we will use the map ρ' defined in Proposition 2.20. Each of these Reidemeister cobordisms is topologically a cylinder with Euler characteristic 0, and there is a unique twisted orientation on it that extends the given twisted orientation on the input. Proposition 2.20 shows that these maps satisfy $(*)$.

Elementary Morse cobordisms are given by attaching 0-, 1-, or 2-handles. For the 0- and 2-handle attachments, which are local, we use the unit and counit maps:

$$\iota : \mathbb{F} \to V, \quad \eta : V \to \mathbb{F}$$

in the Frobenius algebra structure of V, as defined in Definition 2.1. The twisted orientability condition imposes no further restrictions on 0- and 2-handle attachments, as they correspond to attaching a pair of disjoint 0- and 2-handles in the double cover, where τ acts by switching the two copies. Such cobordisms are always twisted orientable, with a twisted orientation being a pair of opposite orientations on the two copies of the handles in the double cover. It is trivial to check that the desired properties hold for these cobordism maps, as in the usual case for cobordisms in $S^3 \times I$.

For the 1-handle attachments, we will use m, Δ, and f on each smoothing of the cube of resolutions, depending on whether it is a 2-1, 1-2, or 1-1 bifurcation in each resolution. Each of m, Δ, and f is a filtered map of degree -1, and the proof that such a definition gives a chain map is the same as checking $d^2 = 0$ in the Bar-Natan chain complex, which is covered in Lemma 2.4 and the discussion following it.

It remains to show that the canonical generators associated with twisted orientations behave properly under such maps. Again, the reason this works is that we attach 1-handles that are compatible with the twisted orientation. In particular, homologically essential knots are not created during the process of attaching 1-handles. More explicitly, the twisted orientations on the two arcs where we attach the 1-handle are as shown in Figure 19. Adding the 1-handle changes a twisted

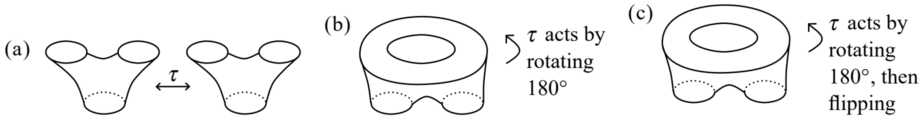

Figure 20. Action of τ on a pair of equivariant 1-handles.

oriented resolution into another twisted oriented resolution. There are three possible cases for how these two arcs are connected in the twisted orientation resolution:

(1) The two arcs $\alpha\beta$ and $\gamma\delta$ belong to the same circle in the twisted orientation resolution, and α is adjacent to γ on the circle. In this case, the induced map by the 1-handle attachment is Δ, such that

$$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b,$$

which proves that $F_\Sigma([s_o]) = [s_{o'}]$ in this case.

(2) The two arcs $\alpha\beta$ and $\gamma\delta$ belong to the same circle in the twisted orientation resolution, but α is adjacent to δ on the circle. In this case, the induced map by the 1-handle attachment is f, such that

$$f(a) = a, \quad f(b) = b,$$

so $F_\Sigma([s_o]) = [s_{o'}]$ as well.

(3) The two arcs $\alpha\beta$ and $\gamma\delta$ belong to different circles in the twisted oriented resolution. The labels on these two circles in s_o will be the same, by the rule in Definition 2.12. In this case, the induced map by the 1-handle attachment is m, such that

$$m(a \otimes a) = a, \quad m(b \otimes b) = b,$$

so $F_\Sigma([s_o]) = [s_{o'}]$.

In terms of the set of twisted orientations on Σ_i extending the given one on the boundary, we note the following. If Σ_i represents a 1-handle attachment that splits one circle into two circles, then it is topologically a pair-of-pants. A priori, there are two different possibilities for the covering $\widetilde{\Sigma_i}$ as shown in Figure 20(a) and (b). It is either a disjoint union of two pairs-of-pants where the action τ switches the two components, as shown in (a), or an annulus with two disks removed, as shown in (b), where the action of τ is a rotation by 180°. However, τ acts in an orientation-preserving way in case (b), so this possibility is ruled out by the assumption that Σ is twisted orientable. The situation for a 1-handle attachment that merges two circles into one circle is exactly the same, by turning the cobordism upside down. For a 1-handle attachment that twists one circle into another, the covering Σ_i is again an annulus with two disks removed, as shown in (c) of Figure 20, while the action τ is different: it is the usual covering map from an annulus to a Möbius band.

After this clarification, the rest of the proof follows exactly in the same way as in [15]. For Σ_i representing a 1-handle attachment that splits a circle into two or twists a circle, there is a unique twisted orientation on Σ_i that extends the twisted orientation on the input boundary. If Σ_i represents a 1-handle attachment merging two circles into one, then depending on which components these two circles belong to, either all twisted orientations on the input have a unique extension to Σ_i, or half of them have a unique extension. In the latter case,

$$F_{\Sigma_i}(s_o) = 0,$$

for twisted orientations o on the input which do not extend to Σ_i. □

Remark 3.6. We won't discuss the functoriality issue of cobordism maps in this paper, i.e., whether the map F_Σ is well defined up to filtered chain homotopy. The existence of such a map F_Σ with the stated property is enough to prove some genus bound.

If we restrict to connected cobordisms between null homologous knots, we obtain the following immediate corollary, which is what we need for the genus bound.

Corollary 3.7. *If L and L' are both null homologous knots and Σ is a connected, twisted orientable cobordism between them in $\mathbb{RP}^3 \times I$, then*

$$F_\Sigma([s_o]) = [s_{o'}],$$

where o' is the restriction of the unique twisted orientation on Σ that extends o, and

$$F_\Sigma : \mathrm{HBN}(L) \to \mathrm{HBN}(L')$$

is an isomorphism of filtration degree $\chi(\Sigma)$.

4. The s-invariant and genus bound

In the previous two sections, we have proven all the formal properties of the Bar-Natan homology required to define the s-invariant and bound slice genus. We summarize the results in this section, following exactly the same procedure as in the case of Bar-Natan or Lee homology in S^3. The only difference is that the class for which we obtain a genus bound is the class of twisted orientable slice surfaces of null homologous knot in $\mathbb{RP}^3 \times I$, rather than orientable slice surfaces. The treatment here closely follows the summary of the s-invariant for Bar-Natan homology in S^3 in Section 2 of [9].

Suppose L is a twisted oriented null homologous link in $\mathbb{RP}^3$. Denote its Bar-Natan chain complex by $C_{*,*}(L) = \mathrm{CBN}_{*,*}(L)$. Consider this finite-length filtration by the quantum grading on $C(L)$:

$$0 \subset \cdots \subset \mathcal{F}_{q+1}C(L) \subset \mathcal{F}_q C(L) \subset \mathcal{F}_{q-1}C(L) \subset \cdots \subset C(L),$$

where $\mathcal{F}_q C(L) = \bigoplus_{j\geq q} C_{*,j}(L)$. Note that due to the possible existence of 1-1 bifurcation in the edge map, which twists one circle into another circle, the parity of the quantum grading is no longer the same on the chain complex $C_{*,*}(L)$. Therefore, we increase the filtration degree q by 1 at each step, instead of by 2 as in the case of Bar-Natan chain complex for links in S^3. The differential map of the chain complex C respects the filtration in the quantum grading, as in the usual case for m, Δ, and we choose the map $f : V \to V[1]$ to be filtered in its definition, so each $\mathcal{F}_q C(L)$ is a subcomplex.

Definition 4.1. Let K be a twisted oriented null homologous knot in $\mathbb{RP}^3$. Define

$$s_{\min}(K) = \max\{q \in \mathbb{Z} \mid i_* : H(\mathcal{F}_q C(K)) \to H(C(K)) \cong \mathbb{F}^2 \text{ is surjective}\},$$
$$s_{\max}(K) = \max\{q \in \mathbb{Z} \mid i_* : H(\mathcal{F}_q C(K)) \to H(C(K)) \cong \mathbb{F}^2 \text{ is nonzero}\}.$$

Lemma 4.2. *We have* $s_{\max}(K) = s_{\min}(K) + 2$.

Proof. The same argument as in Proposition 2.6 of [9] applies here with a slight modification, so we provide only a sketch.

Consider the involution $I : C(K) \to C(K)$, which is induced by this involution on the Frobenius algebra V:

$$I(a) = b, \quad I(b) = a.$$

In terms of the basis $\{1, x\}$, this is

$$I(1) = 1, \quad I(x) = 1 + x.$$

Thus, I induces the identity map on the associated graded complex.

Choose a cycle $y \in \mathcal{F}_{s_{\min}(K)} C(K)$, such that $\{y, I(y)\}$ forms a basis of $H(C(K))$. Since the lowest grading parts of a and $I(a)$ are the same, and the grading non-preserving part of the map I on $C(K)$ raises the grading by at least 2, we have $y + I(y) \in \mathcal{F}_{s_{\min}(K)+2} C(K)$. Hence,

$$s_{\max}(K) \geq s_{\min}(K) + 2.$$

The proof of the inequality in the other direction follows exactly the same argument as for the s-invariant of the usual Bar-Natan homology, since taking the connected sum with a local unknot is a local operation, and the Bar-Natan chain complex behaves in the same way as in the case of S^3 with respect to this local operation. □

Hence, we can define the Bar-Natan s-invariant for null homologous knot in $\mathbb{RP}^3$ as usual.

Definition 4.3. Let K be a null homologous knot in $\mathbb{RP}^3$. The *Bar Natan s-invariant* of K, is defined as

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = \tfrac{1}{2}(s_{\min}(K) + s_{\max}(K)).$$

It is well defined, as we have verified that the maps induced by Reidemeister moves are filtered maps of filtration degree 0 in Proposition 2.20. Additionally, it does not depend on the twisted orientation on K, since the effect of reversing the twisted orientation is the same as switching a and b.

This s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ satisfies similar properties to those of the usual s-invariants. We summarize some of them here. Compare also with the corresponding statements about the s-invariant defined using the Lee deformation in [13].

Proposition 4.4. *If K is a local knot in $\mathbb{RP}^3$, that is, it is contained in some ball B^3 in $\mathbb{RP}^3$, then*

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = s^{\mathrm{BN}}(K),$$

where $s^{\mathrm{BN}}(K)$ denotes the s-invariants from the Bar-Natan homology for knots in S^3 over the field $\mathbb{F} = \mathbb{F}_2$.

Proof. A local knot is, in particular, null homologous, so we can define $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ for it. For a local knot, the notion of twisted orientation agrees with the usual notion of orientation, as we can draw a knot diagram for it that does not intersect the essential unknot U_1 at all, so no reversal of the arrow is needed. Therefore, the notions of n_+ and n_- with respect to the twisted orientation agree with the usual notions of n_+ and n_-. Additionally, there will be no 1-1 bifurcation appearing in the Bar-Natan chain complex for the knot diagram away from U_1. Thus, the notion $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ exactly matches with the usual notion of $s^{\mathrm{BN}}(K)$ when viewing K as a knot in S^3. □

Proposition 4.5. *Let $m(K)$ be the mirror of a null homologous knot in $\mathbb{RP}^3$, i.e., it is obtained by switching positive crossings with negative crossings in a knot diagram of K. Then,*

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(m(K)) = -s^{\mathrm{BN}}_{\mathbb{RP}^3}(K).$$

Proof. The usual argument, as in Proposition 3.9 of [15], using the dual chain complex works here as well, with the dual map of $f = \mathrm{id}_V$ being the identity $f^* = \mathrm{id}_{V^*}$ on V^*. □

Proposition 4.6. *If K is a null homologous knot in $\mathbb{RP}^3$ and K_l is a local knot in $\mathbb{RP}^3$, then we can form the connected sum*

$$K \# K_l \subset \mathbb{RP}^3 \# S^3 \cong \mathbb{RP}^3,$$

and we have

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K \# K_l) = s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) + s^{\mathrm{BN}}(K_l),$$

where again $s^{\mathrm{BN}}(K_l)$ is the s-invariant of K_l using the Bar-Natan homology in S^3 over the field $\mathbb{F} = \mathbb{F}_2$.

Proof. The same proof as in Proposition 3.11 of [15] works. Note that a twisted orientation on K and an orientation on K_l gives a twisted orientation on $K \# K_l$, provided they are compatible on the two arcs where the connected sum is performed. □

Proposition 4.7. *If K is positive with respect to the twisted orientation, that is, if $n_- = 0$ as counted using a twisted orientation on K, then we have the usual formula for its $s^{\mathrm{BN}}_{\mathbb{RP}^3}$-invariant:*

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = -k + n + 1,$$

where n is the number of crossings of K, and k is the number of circles in the twisted oriented resolution of K.

Proof. Exactly the same proof as in Section 5.2 of [15] works as well, with the word "orientation" replaced everywhere by "twisted orientation". □

Now, we discuss the genus bound that can be obtained using the s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$.

Definition 4.8. Let K be a null homologous knot in $\mathbb{RP}^3$. A surface Σ in $\mathbb{RP}^3 \times I$ is a *twisted orientable slice surface* of K if it is connected, twisted orientable, and satisfies

$$\partial \Sigma = \Sigma \cap (\mathbb{RP}^3 \times \{0\}) = K.$$

The most straightforward result that can be written down is in terms of the Euler characteristic.

Proposition 4.9. *If Σ is a twisted orientable slice surface of a null homologous knot K in $\mathbb{RP}^3$, then*

$$-\chi(\Sigma) \geq |s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)|.$$

Proof. It is a straightforward corollary of Corollary 3.7 applied to cobordisms from K to the trivial unknot U, and the fact $s^{\mathrm{BN}}_{\mathbb{RP}^3}(U) = 0$. Check Corollary 2.7 in [9] for a detailed explanation. □

The genus bound will depend on whether the twisted orientable slice surface is orientable in the usual sense, as the formulas for the Euler characteristic from the genus differ in these two case.

Corollary 4.10. *If Σ is a twisted orientable slice surface of a null homologous knot K in $\mathbb{RP}^3$, then*

$$g(\Sigma) \geq \begin{cases} |s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)| & \text{if } \Sigma \text{ is unorientable;} \\ \frac{1}{2}\,|s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)| & \text{if } \Sigma \text{ is orientable.} \end{cases}$$

As the class of twisted orientable slice surfaces is different from the usual notion of slice genus in $\mathbb{RP}^3 \times I$, it is natural to expect that this $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ provides different information than the s-invariant defined for knots in $\mathbb{RP}^3$ using the Lee deformation, as in [13]. We illustrate the difference in the following example.

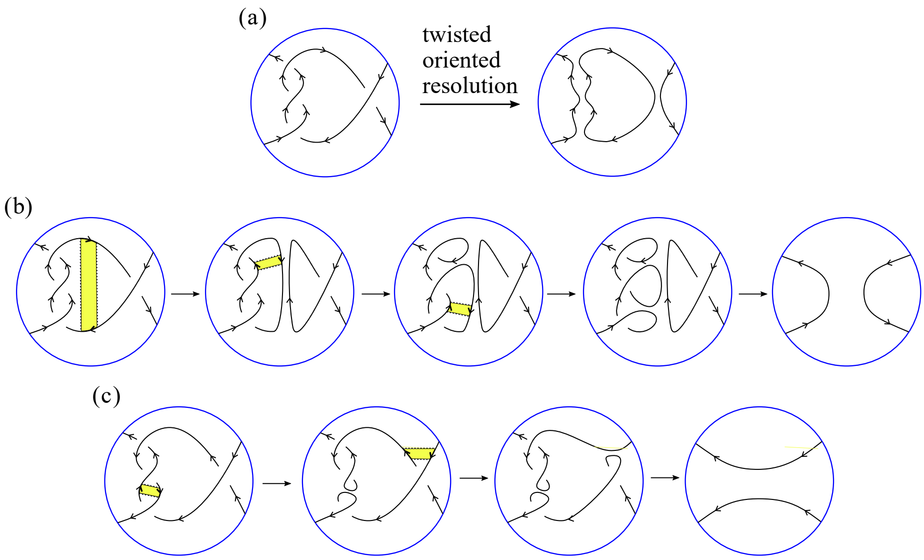

Figure 21. An example of difference in twisted orientable slice surface and usual orientable slice surface.

Example 4.11. Consider the following null homologous knot K as drawn in (a) of Figure 21. It is a positive knot with respect to the twisted orientation, with four crossings, and two circles in the twisted oriented resolution. By Proposition 4.7, we have

$$s^{\mathrm{BN}}_{\mathbb{RP}^3} = -2+4+1 = 3.$$

A twisted orientable slice surface of K with $\chi = -3$ is drawn in (b) of Figure 21, obtained by adding three bands compatible with the twisted orientation.

Therefore, a twisted orientable slice surface Σ of K that is actually orientable should have

$$g(\Sigma) \geq \tfrac{3}{2},$$

so it must have genus at least 2. However, there exists a orientable slice surface of K in the usual sense, which has genus 1, as shown in (c) of Figure 21. Note that the arrows in (b) represent a twisted orientation, while the arrows in (c) represent a usual orientation.

By inserting more and more full twists in this example, we obtain a family of null homologous knots K_n in $\mathbb{RP}^3$, whose two s-invariants, $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ and $s^{\mathrm{Lee}}_{\mathbb{RP}^3}(K)$, defined using the Bar-Natan and Lee deformation, respectively, can have arbitrarily large differences in their absolute values. Here, $s^{\mathrm{Lee}}_{\mathbb{RP}^3}(K)$ refers to the s-invariant defined in [13] using the Lee deformation. See Figure 22 for a diagram of K_n. It is a positive knot with respect to the twisted orientation and a negative knot with respect to the usual orientation. It has $2n+2$ crossings, and the twisted oriented resolution

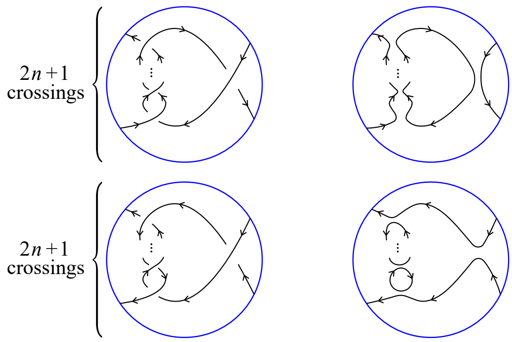

Figure 22. An example of knots with different $|s^{\mathrm{BN}}_{\mathbb{RP}^3}|$ and $|s^{\mathrm{Lee}}_{\mathbb{RP}^3}|$. Top left: K_n with a twisted orientation. Top right: a twisted oriented resolution of K_n. Bottom left: K_n with a usual orientation. Bottom right: an oriented resolution of K_n.

has two circles, while the oriented resolution has $2n+1$ circles. Therefore,

$$|s^{\mathrm{BN}}_{\mathbb{RP}^3}(K_n)| = |-2+(2n+2)+1| = 2n+1,$$
$$|s^{\mathrm{Lee}}_{\mathbb{RP}^3}(K_n)| = |-(-(2n+1)+(2n+2)+1)| = 2.$$

5. Further directions

We discuss several further directions to explore, listed in ascending order of scope.

• In [13], Manolescu and Willis defined the Lee homology and s-invariant for both null homologous and homologically essential links (class-0 and class-1 links in their notion), while we only defined the Bar-Natan homology and s-invariant for null homologous links. It is natural to ask what the counterpart for homologically essential links should be using Bar-Natan homology. Note that the algebraic structures of Khovanov homology (and Lee homology) for null homologous and homologically essential links are quite different. In any resolution of the link diagram of a homologically essential link, there will be a homologically essential unknot, and no 1-1 bifurcation will occur in the edge maps. Thus, instead of the extra map $f : V \to V$, what we require in the homologically essential case is a bimodule over the Frobenius algebra V, which will be assigned to the homologically essential unknot in each resolution.

• Naturally, one would like to ask how this Bar-Natan chain complex $\mathrm{CBN}(L)$ is related to an equivariant version of the Bar-Natan chain complex $\mathrm{CBN}(\widetilde{L})$ of the double cover $\widetilde{L}$ of L in S^3, and also the similar question for the Khovanov chain complex. When $\widetilde{L}$ is periodic or strongly invertible, there is a lot of work relating the Khovanov chain complex/stable homotopy type of the quotient link to the corresponding chain complex/stable homotopy type of the link itself with the action.

See, for example, [3; 16] for periodic links, and [10; 11] for strongly invertible knots. Since the Bar-Natan homology is much easier to describe than the Khovanov homology, we do obtain an inequality at the level of Bar-Natan homology:

$$\dim(\mathrm{HBN}(L)) \leq \dim(\mathrm{HBN}(\widetilde{L})),$$

where $\mathrm{HBN}(L)$ is the Bar-Natan homology of a null homologous link L in $\mathbb{RP}^3$ defined in this paper, and $\mathrm{HBN}(\widetilde{L})$ is the usual Bar-Natan homology for links in S^3. The reason is simple: as proved in Proposition 2.16, $\mathrm{HBN}(L)$ has a basis identified with the set of twisted orientations on L, which is by definition a subset of orientations on $\widetilde{L}$, and $\mathrm{HBN}(\widetilde{L})$ has a basis identified with the set of all orientations on $\widetilde{L}$. This suggests a possibility of a spectral sequence relating $\mathrm{HBN}(L)$ and $\mathrm{HBN}(\widetilde{L})$, and perhaps for the Khovanov homology as well. One possible starting point is to look at a link diagram of $\widetilde{L}$ as the closure of $T \circ F(T)$, as in Figure 8, on which one can formally define an action of the involution τ on the chain complex level for $\mathrm{CBN}(\widetilde{L})$ and $\mathrm{CKh}(\widetilde{L})$.

- For knots in S^3, the Lee and Bar-Natan deformation of the usual Khovanov homology are closely related to each other. For example, if $\mathbb{F}$ is a field of characteristic other than 2, then the s-invariant defined using the Bar-Natan deformation agrees with that defined using the Lee deformation. See Proposition 3.1 in [12]. However, in $\mathbb{RP}^3$, the behavior is quite different. For example, the class of slice surfaces whose genus is bounded by the s-invariant is not the same. The reason for this difference lies in the assignment of the map to the 1-1 bifurcation: it is the identity map in the Bar-Natan deformation, while it is 0 in the Lee deformation. It might be interesting to look at other extensions of the s-invariants to 3-manifolds using the Lee deformation, e.g., [7] for $S^1 \times D^2$, [14] for connected sums of $S^1 \times S^2$, and see what happens if one uses the Bar-Natan deformation instead of the Lee deformation in these cases.

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DAREN CHEN
DEPARTMENT OF MATHEMATICS
CALTECH
PASADENA, CA
UNITED STATES

darenc@caltech.edu

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ON PRINCIPAL SERIES REPRESENTATIONS OF QUASI-SPLIT REDUCTIVE p-ADIC GROUPS

MAARTEN SOLLEVELD

Let G be a quasi-split reductive group over a non-archimedean local field. We establish a local Langlands correspondence for all irreducible smooth complex G-representations in the principal series. The parametrization map is injective, and its image is an explicitly described set of enhanced L-parameters. Our correspondence is determined by the choice of a Whittaker datum for G, and it is canonical given that choice.

We show that our parametrization satisfies many expected properties, among others with respect to the enhanced L-parameters of generic representations, temperedness, cuspidal supports and central characters. Our correspondence lifts to a categorical level, where it makes the appropriate Bernstein blocks of G-representations naturally equivalent to module categories of Hecke algebras coming from Langlands parameters. Along the way we characterize genericity of G-representations in terms of representations of an affine Hecke algebra.

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Keywords: p-adic groups, principal series, local Langlands correspondence.

Introduction

Consider a quasi-split reductive group $G = \mathcal{G}(F)$ over a non-archimedean local field F. Let $\mathrm{Rep}(G)$ be the category of smooth G-representations on complex vector spaces and let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible representations in $\mathrm{Rep}(G)$. The conjectural local Langlands correspondence (LLC) asserts that $\mathrm{Irr}(G)$ is canonically partitioned into finite L-packets $\Pi_\phi(G)$, indexed by L-parameters ϕ. Some time after the initial formulation in [10], it was realized that $\Pi_\phi(G)$ should be parametrized by the set of irreducible representations of a finite component group R_ϕ. These conjectures have motivated a large part of the study of reductive groups over local fields in past decades; see the surveys [1; 10; 24; 26; 53].

This paper establishes a local Langlands correspondence for the most accessible class of G-representations, those in the principal series. To formulate the result precisely, we quickly recall some relevant notions.

Let $T \subset G$ be the centralizer of a maximal F-split torus in G, or equivalently a minimal Levi subgroup of G. Then T is itself a torus because G is quasi-split, and T is unique up to conjugation. Any representation of G that can be obtained from a smooth representation of T by parabolic induction and then taking a subquotient is called a principal series G-representation. These representations form a product of Bernstein blocks in $\mathrm{Rep}(G)$. We denote the set of irreducible principal series G-representations by $\mathrm{Irr}(G, T)$. We warn that some L-packets contain elements of $\mathrm{Irr}(G, T)$ and also other elements of $\mathrm{Irr}(G)$.

It has turned out that the representation ρ_π of R_ϕ associated to a given $\pi \in \mathrm{Irr}(G)$ is not canonically determined. To specify it uniquely one needs additional input, namely a Whittaker datum for G. Such a Whittaker datum can be used to normalize relevant intertwining operators, which then determine exactly how $\rho_\pi \in \mathrm{Irr}(R_\phi)$ is related to π. For non-quasi-split groups G such a normalization should also be possible [24, Conjecture 2.5], but it is much more involved.

We fix a Borel subgroup $B = TU$ and a nondegenerate character ξ of the unipotent radical U of B. Then (U, ξ), or rather its G-conjugacy class, is a Whittaker datum for G. Recall that $\pi \in \mathrm{Irr}(G)$ is called (U, ξ)-generic if $\mathrm{Hom}_U(\pi, \xi)$ is nonzero.

Let $\boldsymbol{W}_F$ be the Weil group of F, let $G^\vee$ be the complex dual group of G and let ${}^L G = G^\vee \rtimes \boldsymbol{W}_F$ be the Langlands dual group. In this introduction (but not in the body of the paper) we realize L-parameters for G as Weil–Deligne representations

$$\phi : \boldsymbol{W}_F \ltimes \mathbb{C} \to {}^L G.$$

Here $\boldsymbol{W}_F$ acts on $\mathbb{C}$ by $w \cdot z = \|w\| z$, where $\|w\| \in p^{\mathbb{Z}}$ is determined by $w(x) = x^{\|w\|}$ for all x in an algebraic closure of the residue field of F. The appropriate component group of such an L-parameter is

$$R_\phi = \pi_0\big(Z_{G^\vee}(\phi(\boldsymbol{W}_F \ltimes \mathbb{C}))/Z(G^\vee)^{\boldsymbol{W}_F}\big),$$

where $Z(G^\vee)$ denotes the center of $G^\vee$. An enhancement of ϕ is an irreducible R_ϕ-representation. Let $\Phi_e(G)$ be the set of enhanced L-parameters for G, considered up to $G^\vee$-conjugacy. An element $(\phi, \rho) \in \Phi_e(G)$ belongs to the principal series if its cuspidal support is an enhanced L-parameter for T. More explicitly, that means

- $\phi(\mathbf{W}_F)$ is contained in $T^\vee \rtimes \mathbf{W}_F$ (or some $G^\vee$-conjugate of $T^\vee \rtimes \mathbf{W}_F$, because ϕ is only given up to $G^\vee$-conjugacy),
- ρ appears in the homology of a certain variety of Borel subgroups.

We denote the subset of $\Phi_e(G)$ associated to the principal series by $\Phi_e(G, T)$. For a given ϕ it may happen that some enhancements yield elements of $\Phi_e(G, T)$, while other enhancements bring us outside $\Phi_e(G, T)$.

Our main result is a canonical LLC for principal series representations:

Theorem A (see Section 7). *The Whittaker datum (U, ξ) determines a canonical bijection*

$$\mathrm{Irr}(G, T) \leftrightarrow \Phi_e(G, T), \quad \pi(\phi, \rho) \leftarrow\!\shortmid (\phi, \rho), \quad \pi \mapsto (\phi_\pi, \rho_\pi),$$

with the following properties:

(a) $\pi(\phi, \rho)$ *is* (U, ξ)*-generic if and only if* ρ *is trivial and* $u_\phi = \phi(1, 1)$ *lies in the dense* $Z_{G^\vee}(\phi(\mathbf{W}_F))$*-orbit in*

$$\{v \in G^\vee : v \text{ is unipotent and } \phi(w) v \phi(w)^{-1} = v^{\|w\|} \text{ for all } w \in \mathbf{W}_F\}.$$

(b) $\pi(\phi, \rho)$ *is tempered* (*resp. essentially square integrable*) *if and only if* ϕ *is bounded* (*resp. discrete*).

(c) *The bijection is compatible with the cuspidal support maps on both sides.*

(d) *The bijection is equivariant for the canonical actions of* $H^1(\mathbf{W}_F, Z(G^\vee))$.

(e) *The bijection is compatible with the Langlands classification and* (*for tempered representations*) *with parabolic induction.*

All Borel's desiderata from [10, §10] *are satisfied. When* π *is given,* ϕ_π *is uniquely determined by* (a)–(e) *and the local Langlands correspondence for tori.*

For nonsplit quasi-split groups, the vast majority of the groups under consideration here, very little in this direction was previously known. On the other hand, for split groups many instances of Theorem A have been established before:

- Kazhdan and Lusztig [25] established a bijection with properties (b) and (e) for Iwahori-spherical representations, assuming that G is F-split and that $Z(G)$ is connected as an algebraic group. Their starting point is Borel's description [9] of those representations, in terms of Hecke algebras.

• Reeder [38] extended [25] to $\operatorname{Irr}(G,T)$ when G is F-split, $Z(G)$ is connected and the residual characteristic p of F is not "too small". This is based on work of Roche [40] and includes properties (a), (b) and (e). We note that here the Whittaker datum is unique up to G-conjugacy because $Z(G)$ is connected.

• In [2] a (noncanonical) bijection satisfying properties (b), (d) and (e) was established for $\operatorname{Irr}(G,T)$, when G is F-split and p is not too small.

• For quasi-split unitary groups with $p > 2$ a (noncanonical) bijection was constructed by the author's Ph.D. student Badea [7].

In all cases, a study of affine Hecke algebras constitutes the largest part of the argument. Thanks to [1; 50], that technique is now available in complete generality (even outside the principal series). The main novelties of this paper are:

- The construction of the LLC is canonical and uniform, over all non-archimedean local fields F and all quasi-split reductive F-groups.
- We can handle generic representations, even when not all Whittaker data for G are equivalent.
- Our LLC lifts to a categorical level, as follows. For each involved Bernstein block of G-representations, the LLC comes from a canonical equivalence between that block and the module category of a certain Hecke algebra defined entirely in terms of Langlands parameters.

We will now discuss the content of the paper in more detail, at the same time explaining parts of the proof of the main theorem.

We start with a Bernstein block $\operatorname{Rep}(G)^{\mathfrak{s}}$ in the principal series, and a progenerator $\Pi_{\mathfrak{s}}$ thereof. Via [50] $\operatorname{Rep}(G)^{\mathfrak{s}}$ is equivalent to the module category of some Hecke algebra $\operatorname{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}$, which we analyze in Section 1. We show that $\operatorname{End}_G(\Pi_{\mathfrak{s}})$ is isomorphic to an affine Hecke algebra $\mathcal{H}(\mathfrak{s})$ (extended with a twisted group algebra), and we determine its q-parameters.

In Section 2 we involve the Whittaker datum, and that enables us to make the aforementioned isomorphism canonical. In the same way we show that the twist in the extension part of $\mathcal{H}(\mathfrak{s})$ is actually trivial, so that it is an extended affine Hecke algebra $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_{\mathfrak{s}}$. Here the Bernstein group $W_{\mathfrak{s}}$ associated to $\operatorname{Rep}(G)^{\mathfrak{s}}$ appears as $W(R_{\mathfrak{s}}^\vee) \rtimes \Gamma_{\mathfrak{s}}$ for a root system $R_{\mathfrak{s}}^\vee$.

A continuation of this analysis yields a useful criterion for genericity in terms of Hecke algebra modules. Let $\mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda) \subset \mathcal{H}(\mathfrak{s})^\circ$ be the finite-dimensional Iwahori–Hecke algebra from the Bernstein presentation of $\mathcal{H}(\mathfrak{s})^\circ$, where q_F is the cardinality of the residue field of F and $\lambda : R_{\mathfrak{s}}^\vee / W_{\mathfrak{s}} \to \mathbb{Z}_{\geq 0}$ is a label function. Recall that its Steinberg representation St is given by $T_{s_\alpha} \mapsto -1$ for every simple reflection $s_\alpha \in W(R_{\mathfrak{s}}^\vee)$. Let $\det : W_{\mathfrak{s}} \to \{\pm 1\}$ be the determinant of the action of $W_{\mathfrak{s}}$

on the lattice of F-rational cocharacters of T. We extend St to a one-dimensional representation (still denoted St) of $\mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak{s}}$ by making it det on $\Gamma_{\mathfrak{s}}$.

Theorem B (see Theorem 3.4). *Suppose that $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ has finite length. Then π is (U, ξ)-generic if and only if the associated $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$-module* $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$ *contains the Steinberg representation of $(\mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak{s}})^{\mathrm{op}}$.*

The notion of principal series enhanced L-parameters is worked out in Section 4. There we also recall the Hecke algebras on the Galois side of the LLC, from [6], and we compute their q-parameters. Via the LLC for tori we associate to $\mathrm{Rep}(G)^{\mathfrak{s}}$ a unique Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee}$ of $\Phi_e(G, T)$. That yields an extended affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$. The crucial step to pass from the p-adic side to the Galois side of the LLC is:

Theorem C (see Theorem 5.4). *There exists a canonical algebra isomorphism $\mathcal{H}(\mathfrak{s})^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$.*

The above steps make $\mathrm{Rep}(G)^{\mathfrak{s}}$ canonically equivalent to the module category of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$. In [6], $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))$ is parametrized by $\Phi_e(G)^{\mathfrak{s}^\vee}$. We want to use that, but it does not quite suffice because we also need to keep track of genericity of representations. Therefore we revisit several constructions from [6], in our setting of the principal series. The main point of Section 6 is to show that through all those steps the one-dimensional representation det of $\big(\mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak{s}}\big)^{\mathrm{op}}$ is transformed into an analogous representation det for an extended graded Hecke algebra. That enables us to normalize the parametrization of $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))$, so that it matches generic representations with the desired kind of enhanced L-parameters.

With that settled the preparations are complete, and the bijection in Theorem A is obtained as

$$\mathrm{Irr}(G)^{\mathfrak{s}} \leftrightarrow \mathrm{Irr}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}) \leftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}) \leftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})) \leftrightarrow \Phi_e(G)^{\mathfrak{s}^\vee}.$$

The properties of the bijection $\mathrm{Irr}(G, T) \leftrightarrow \Phi_e(G, T)$, actually a few more than mentioned already, are checked in the remainder of Section 7.

Several further research topics are suggested by the above theorems.

- Like in [2, §17], one would like to show that the LLC is functorial with respect to those homomorphisms of reductive p-adic groups that have commutative kernel and commutative cokernel. That should be doable with the methods from [48]. In particular that can be applied to automorphisms of G from conjugation with elements of $\mathcal{G}_{\mathrm{ad}}(F)$, and this will show how the LLC changes if one modifies the Whittaker datum.

- Suppose that ϕ is discrete and $Z(G)$ is compact. It is conjectured in [23] that the formal degree of the square-integrable representation $\pi(\phi, \rho)$ equals $\dim(\rho)$ times the adjoint γ-factor of ϕ (with suitable normalizations on both sides). While this adjoint γ-factor can be computed as in [18, Appendix A], it may be difficult to determine this formal degree. The reason is that one would like to use a type, but sometimes it is not known whether a type for the involved Bernstein block exists.

• Every L-packet conjecturally supports a stable distribution on G. For L-packets that are entirely contained in $\mathrm{Irr}(G, T)$, one could try to prove that the distribution $\sum_{\rho \in \mathrm{Irr}(R_\phi)} \dim(\rho) \operatorname{tr} \pi(\phi, \rho)$ is stable.

• A modern geometric approach to the Langlands correspondence [16; 22; 55] predicts that the derived category of $\mathrm{Rep}(G)$ embeds in a derived category of coherent sheaves on a stack of Langlands parameters. It would be interesting to transfer the obtained natural equivalence

$$\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))$$

to a setting with such coherent sheaves; this would establish a part of the conjectures in [16; 22; 55]. It is reasonable to expect that can be done, because $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is constructed from $\Phi_e(G)^{\mathfrak{s}^\vee}$ and because on the underlying cuspidal level the local Langlands correspondence for tori achieves it already.

1. Hecke algebras for principal series representations

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}_F$. Let q_F be the cardinality of the residue field. Let $|\cdot|_F : F \to \mathbb{R}_{\geq 0}$ be the norm and fix an element $\varpi_F \in \mathfrak{o}_F$ with norm q_F^{-1}. We also fix a separable closure F_s of F and we let $\boldsymbol{W}_F \subset \mathrm{Gal}(F_s/F)$ be the Weil group. Field extensions of F will by default be contained in F_s.

Let $\mathcal{G}$ be a quasi-split reductive F-group, where we include connectedness in the definition of quasi-split. Let $\mathcal{S}$ be a maximal F-split torus in $\mathcal{G}$. We write $G = \mathcal{G}(F)$, $S = \mathcal{S}(F)$, etcetera. Since $\mathcal{G}$ is F-quasi-split, the centralizer $\mathcal{T}$ of $\mathcal{S}$ in $\mathcal{G}$ is a maximal F-torus. It is also a minimal F-Levi subgroup, a Levi factor of a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$. The Weyl group of $(\mathcal{G}, \mathcal{S})$ and (G, S) is

$$W(\mathcal{G}, \mathcal{S}) = N_{\mathcal{G}}(\mathcal{S})/Z_{\mathcal{G}}(\mathcal{S}) = N_{\mathcal{G}}(S)/\mathcal{T} \cong N_G(S)/T = N_G(T)/T.$$

This is also the Weyl group of the root system $R(\mathcal{G}, \mathcal{S})$.

Let T_{cpt} be the unique maximal compact subgroup of T and let $X_{\mathrm{nr}}(T)$ be the group of unramified characters of T, that is, the characters that are trivial on T_{cpt}. Pick any smooth character $\chi_0 : T \to \mathbb{C}^\times$ and write $\chi_c = \chi_0|_{T_{\mathrm{cpt}}}$. Then χ_c determines $X_{\mathrm{nr}}(T)\chi_0$ and conversely $X_{\mathrm{nr}}(T)\chi_0$ determines χ_c.

We denote the category of smooth G-representations on complex vector spaces by $\mathrm{Rep}(G)$, and the set of equivalence classes of irreducible objects therein by $\mathrm{Irr}(G)$. The set $X_{\mathrm{nr}}(T)\chi_0 = \mathrm{Irr}(T)^{\mathfrak{s}_T}$ (with $\mathfrak{s}_T = [T, \chi_0]_T$) is also known as a Bernstein component of $\mathrm{Irr}(T)$. From $\mathrm{Irr}(T)^{\mathfrak{s}_T}$ one derives a Bernstein component $\mathrm{Irr}(G)^{\mathfrak{s}}$, where $\mathfrak{s} = [T, \chi_0]_G$. It consists of the irreducible subquotients of the normalized parabolic inductions

$$I_B^G(\chi) = \mathrm{ind}_B^G(\chi \otimes \delta_B^{1/2}) \quad \text{with } \chi \in X_{\mathrm{nr}}(T)\chi_0.$$

We recall that the Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ is the full subcategory of $\mathrm{Rep}(G)$ made up by the representations π such that every irreducible subquotient of π belongs to $\mathrm{Irr}(G)^{\mathfrak{s}}$. The standard way to classify $\mathrm{Irr}(G)^{\mathfrak{s}}$ is by describing $\mathrm{Rep}(G)^{\mathfrak{s}}$ as the module category of a Hecke algebra, and then using the representation theory of Hecke algebras. We do so with the method that provides maximal generality, from [21; 50].

We denote smooth induction with compact supports by ind. The T-representation $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c) \cong \chi_c \otimes \mathbb{C}[T/T_{\mathrm{cpt}}]$ is a progenerator of $\mathrm{Rep}(T)^{\mathfrak{s}_T}$. By the first and second adjointness theorems,

$$\Pi_{\mathfrak{s}} = I_B^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c))$$

is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$. Let $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ be the algebra of G-endomorphisms of $\Pi_{\mathfrak{s}}$, acting from the left on $\Pi_{\mathfrak{s}}$. Then the functors

(1-1) $$\mathrm{Rep}(G)^{\mathfrak{s}} \leftrightarrow \mathrm{End}_G(\Pi_{\mathfrak{s}})-\mathrm{Mod}, \quad \rho \mapsto \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \rho), \quad V \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}} \leftarrow\!\shortmid V,$$

are equivalences of categories [41, Theorem 1.8.2.1]. This is compatible with parabolic induction, in the following sense. Let $P = MR_u(P)$ be a parabolic subgroup of G, where $B \subset P$, M is a Levi factor of P and $T \subset M$. The diagram

(1-2) $$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longrightarrow & \mathrm{End}_G(\Pi_{\mathfrak{s}}) - \mathrm{Mod} \\ \uparrow I_P^G & & \uparrow \mathrm{ind}_{\mathrm{End}_M(\Pi_{\mathfrak{s}_M})}^{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \\ \mathrm{Rep}(M)^{\mathfrak{s}_M} & \longrightarrow & \mathrm{End}_M(\Pi_{\mathfrak{s}_M}) - \mathrm{Mod} \end{array}$$

commutes; see [47, Condition 4.1 and Lemma 5.1].

The algebra $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ was investigated in [50], in larger generality. We will make it more explicit in the current setting. Since $\dim \chi_0 = 1$, $\chi|_{T_{\mathrm{cpt}}}$ is irreducible and we may use [50, §10] with $E = E_1 = \mathbb{C}$ and $\sigma_1 = \sigma|_{T_{\mathrm{cpt}}} = \chi_c$. For comparison with [50] we also note that the group

$$X_{\mathrm{nr}}(T, \chi_0) = \{\chi \in X_{\mathrm{nr}}(T) : \chi \otimes \chi_0 \cong \chi_0\}$$

is trivial. We write

$$W_{\mathfrak{s}} = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(\mathfrak{s}_T) = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(X_{\mathrm{nr}}(T)\chi_0) = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(\chi_c).$$

This group acts naturally on the complex variety

$$T_{\mathfrak{s}} := \chi_0 X_{\mathrm{nr}}(T)$$

by $(w \cdot \chi)(t) = \chi(w^{-1}tw)$. The theory of the Bernstein center [8] says that

$$Z(\mathrm{Rep}(G)^{\mathfrak{s}}) \cong Z(\mathrm{End}_G(\Pi_{\mathfrak{s}})) \cong \mathcal{O}(X_{\mathrm{nr}}(T)\chi_0)^{W_{\mathfrak{s}}} = \mathcal{O}(X_{\mathrm{nr}}(T)\chi_0/W_{\mathfrak{s}}).$$

The algebra $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ contains $\mathcal{O}(X_{\mathrm{nr}}(T)\chi_0) = \mathcal{O}(T_{\mathfrak{s}})$ as maximal commutative subalgebra, and as a module over that subalgebra it is free with the basis $\{N_w : w \in W_{\mathfrak{s}}\}$ [50, Theorem 10.9].

We note that the inertial equivalence class $\mathfrak{s}$ for G can arise from different inertial equivalence classes for T. Namely, the possibilities are $w\mathfrak{s}_T = [T, w\chi_0]_T$ with $w \in W(\mathcal{G}, \mathcal{S})$. Thus $w\mathfrak{s} = \mathfrak{s}$ as inertial equivalence classes for G, but they are represented by different subsets of $\mathrm{Irr}(T)$. For any $w \in W(\mathcal{G}, \mathcal{S})$, the G-representations $\Pi_{\mathfrak{s}}$ and $\Pi_{w\mathfrak{s}} = I_B^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(w\chi_c))$ are isomorphic; see [39, §VI.10.1]. That yields an algebra isomorphism

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) \cong \mathrm{End}_G(\Pi_{w\mathfrak{s}}), \tag{1-3}$$

unique up to inner automorphisms of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$. In principle that suffices to compare the functors $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, ?)$ and $\mathrm{Hom}_G(\Pi_{w\mathfrak{s}}, ?)$ on the level of isomorphism classes of representations. Nevertheless, we will have to make (1-3) explicit later, and we prepare for that now.

Proposition 1.1. *Let $w \in W(\mathcal{G}, \mathcal{S})$ be of minimal length in $wW_{\mathfrak{s}}$. The isomorphism $\Pi_{\mathfrak{s}} \cong \Pi_{w\mathfrak{s}}$ can be chosen so that the induced algebra isomorphism* (1-3) *restricts to*

$$\mathcal{O}(T_{\mathfrak{s}}) \to \mathcal{O}(T_{w\mathfrak{s}}), \quad f \mapsto f \circ w^{-1}.$$

Proof. Let $w = s_r \cdots s_2 s_1$ be a reduced expression in the Weyl group $W(\mathcal{G}, \mathcal{S})$. Then each simple reflection s_j has minimal length in $s_j W_{s_{j-1}\cdots s_1\mathfrak{s}}$. In this way we reduce the proposition to the case $w = s_\alpha$ for a simple root $\alpha \in W(\mathcal{G}, \mathcal{S})$, with $s_\alpha \mathfrak{s}_T \neq \mathfrak{s}_T$.

Let $G_\alpha \subset G$ be the subgroup generated by $T \cup U_\alpha \cup U_{-\alpha}$. As

$$\Pi_{\mathfrak{s}} = I_{BG_\alpha}^G I_{B\cap G_\alpha}^{G_\alpha} \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$$

and similarly for $\Pi_{w\mathfrak{s}}$, it suffices to work with the reductive group G_α and its Borel subgroup $B \cap G_\alpha$. Equivalently, we may (and will) assume that $R(\mathcal{G}, \mathcal{S})$ has rank one. In this rank-one setting, an isomorphism

$$\Pi_{\mathfrak{s}} \cong \Pi_{s_\alpha \mathfrak{s}} \tag{1-4}$$

is exhibited in [39, Lemme VI.10.1]. We analyze that construction.

By [39, Corollaire VII.1.3],

$$I_B^G : \mathrm{Rep}(T)^{\mathfrak{s}_T} \to \mathrm{Rep}(G)^{\mathfrak{s}} \text{ is an equivalence of categories.} \tag{1-5}$$

Let $J_{\overline{B}}^G : \mathrm{Rep}(G) \to \mathrm{Rep}(T)$ be the normalized Jacquet restriction functor with respect to the opposite Borel subgroup $\overline{B}$. As $\mathfrak{s}_T \neq s_\alpha \mathfrak{s}_T$, Bernstein's geometric lemma [39, Théorème VI.5.1] entails that

$$J_{\overline{B}}^G I_B^G \pi \cong \pi \oplus s_\alpha^{-1} \cdot \pi \quad \text{for all } \pi \in \mathrm{Rep}(T)^{\mathfrak{s}_T}. \tag{1-6}$$

Let $\mathrm{pr}_{\mathfrak{s}_T} : \mathrm{Rep}(T) \to \mathrm{Rep}(T)^{\mathfrak{s}_T}$ be the projection provided by the Bernstein decomposition. From (1-6) we see that

$$\mathrm{pr}_{\mathfrak{s}_T} J_{\overline{B}}^G : \mathrm{Rep}(G)^{\mathfrak{s}} \to \mathrm{Rep}(T)^{\mathfrak{s}_T} \text{ is the inverse of (1-5).} \tag{1-7}$$

It follows (slightly varying from the proof of [39, Lemme VI.10.1] by using $\overline{B}$ instead of B) that (1-4) is determined by the choice of a T-isomorphism

$$\mathrm{pr}_{\mathfrak{s}_T} J_{\overline{B}}^G \Pi_{s_\alpha \mathfrak{s}} \cong \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c). \tag{1-8}$$

Pick a representative for s_α in $N_G(T)$. From (1-6) with $\pi = \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ we see that evaluation at s_α in $I_B^G\, \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ provides an isomorphism of T-representations

$$\mathrm{ev}_{s_\alpha} : \mathrm{pr}_{\mathfrak{s}} J_{\overline{B}}^G \Pi_{s_\alpha \mathfrak{s}} \to s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c). \tag{1-9}$$

Further we have a canonical T-isomorphism

$$s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c) \to \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \quad f \mapsto [t \mapsto f(s_\alpha t s_\alpha^{-1})]. \tag{1-10}$$

The composition of (1-9) and (1-10) gives us (1-8). Applying (1-5), we obtain (1-4).

The subalgebra $\mathcal{O}(T_{\mathfrak{s}})$ of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ arises as $I_B^G(\mathcal{O}(T_{\mathfrak{s}}))$, where $\mathcal{O}(T_{\mathfrak{s}})$ acts on $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c) \cong \mathcal{O}(T_{\mathfrak{s}})$ by multiplication; see [50]. From (1-10), $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ is naturally isomorphic to the regular representation of $\mathcal{O}(T_{\mathfrak{s}})$.

Similarly $\mathcal{O}(T_{w\mathfrak{s}})$ acts on $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c) \cong \mathcal{O}(T_{w\mathfrak{s}})$ by multiplication, and it becomes a subalgebra of $\mathrm{End}_G(\Pi_{s_\alpha \mathfrak{s}})$ via I_B^G. From (1-6) we see that its action on $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$, obtained via $J_{\overline{B}}^G I_B^G$, is

$$s_\alpha^{-1} : \mathcal{O}(T_{s_\alpha \mathfrak{s}}) \to \mathcal{O}(T_{\mathfrak{s}})$$

followed by the regular representation. In other words, the action of $f \in \mathcal{O}(T_{\mathfrak{s}})$ on $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ via (1-8) coincides with the action of $s_\alpha(f) = f \circ s_\alpha^{-1} \in \mathcal{O}(T_{\mathfrak{s}})$ on $s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha \chi_c)$ via (1-10). Applying the normalized parabolic induction functor I_B^G, we find that $I_B^G(f) \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ is transformed into $I_B^G(f \circ s_\alpha^{-1}) \in \mathrm{End}_G(\Pi_{s_\alpha \mathfrak{s}})$ by (1-4). □

We resume the analysis of $\mathrm{End}(\Pi_{\mathfrak{s}})$ with $\mathfrak{s} = [T, \chi_0]_G$. Let $R_{\mathfrak{s},\mu}$ be the set of roots $\alpha \in R(\mathcal{G}, \mathcal{S})$ for which Harish-Chandra's function μ_α is not constant on $X_{\mathrm{nr}}(T)\chi_0$. Then $R_{\mathfrak{s},\mu}$ is a root system and $W(R_{\mathfrak{s},\mu})$ is a normal subgroup of $W_{\mathfrak{s}}$ [21, Proposition 1.3]. As explained in [50, §3], we can modify χ_0 inside $X_{\mathrm{nr}}(T)\chi_0$

so that $W(R_{\mathfrak{s},\mu})$ fixes χ_0. Let $R^+_{\mathfrak{s},\mu}$ be the positive system determined by the chosen Borel subgroup $\mathcal{B}$ of $\mathcal{G}$. Then

$$W_{\mathfrak{s}} = W(R_{\mathfrak{s},\mu}) \rtimes \Gamma_{\mathfrak{s}},$$

where $\Gamma_{\mathfrak{s}}$ denotes the stabilizer of $R^+_{\mathfrak{s},\mu}$ in $W_{\mathfrak{s}}$. Following [50, §3], we use the lattice $T/T_{\mathrm{cpt}} \cong X^*(X_{\mathrm{nr}}(T))$, and the dual lattice $(T/T_{\mathrm{cpt}})^\vee \cong X_*(X_{\mathrm{nr}}(T))$. For $\alpha \in R_{\mathfrak{s},\mu}$ let $h^\vee_\alpha$ be the unique generator of $T/T_{\mathrm{cpt}} \cap \mathbb{Q}\alpha^\vee$ such that $|\alpha(h^\vee_\alpha)|_F > 1$. We put

$$R^\vee_{\mathfrak{s}} = \{h^\vee_\alpha : \alpha \in R_{\mathfrak{s},\mu}\} \subset T/T_{\mathrm{cpt}}$$

and we let $R_{\mathfrak{s}} \subset (T/T_{\mathrm{cpt}})^\vee$ be the dual root system. By [50, Proposition 3.1],

$$\mathcal{R}_{\mathfrak{s}} = (R^\vee_{\mathfrak{s}}, T/T_{\mathrm{cpt}}, R_{\mathfrak{s}}, (T/T_{\mathrm{cpt}})^\vee)$$

is a root datum with Weyl group $W(R^\vee_{\mathfrak{s}}) = W(R_{\mathfrak{s},\mu})$. Moreover $W_{\mathfrak{s}}$ acts naturally on $\mathcal{R}_{\mathfrak{s}}$ and $\Gamma_{\mathfrak{s}}$ is the $W_{\mathfrak{s}}$-stabilizer of the basis of $\mathcal{R}_{\mathfrak{s}}$ determined by $\mathcal{B}$.

The complex variety $T_{\mathfrak{s}}$ is isomorphic to $X_{\mathrm{nr}}(T)$ via multiplication with χ_0. Let $\mathcal{H}(\mathfrak{s})^\circ$ be the vector space $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathbb{C}[W(R^\vee_{\mathfrak{s}})]$, identified with the vector space $\mathcal{O}(X_{\mathrm{nr}}(T)) \otimes \mathbb{C}[W(R^\vee_{\mathfrak{s}})]$ via $X_{\mathrm{nr}}(T) \to T_{\mathfrak{s}}$. Given label functions λ, λ^* and $q \in \mathbb{C}^\times$, we build the affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q)$ (see, for instance, [3, Proposition 2.2] with z_j specialized to q). Via the above isomorphism of vector spaces we make $\mathcal{H}(\mathfrak{s})^\circ$ into an algebra which is isomorphic to $\mathcal{H}(\mathcal{R}_{\mathfrak{s}}, \lambda, \lambda^*, q)$. The group $\Gamma_{\mathfrak{s}}$ acts on $\mathcal{H}(\mathfrak{s})^\circ$ by algebra isomorphisms:

$$\gamma(f \otimes w) = f \circ \gamma^{-1} \otimes \gamma w \gamma^{-1}, \quad f \in \mathcal{O}(T_{\mathfrak{s}}),\ w \in W(R^\vee_{\mathfrak{s}}). \tag{1-11}$$

That gives rise to the crossed product algebra

$$\mathcal{H}(\mathfrak{s}) := \mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_{\mathfrak{s}},$$

which we would like to be isomorphic with $\mathrm{End}_G(\Pi_{\mathfrak{s}})$.

For s_α with $\alpha \in R_{\mathfrak{s},\mu}$ simple, and more generally for any $w \in W(R^\vee_{\mathfrak{s}})$, an element $N_w \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ is constructed in [50, Lemma 10.8 and remarks], it is called $q_F^{-\lambda(\alpha)/2} T'_w$ over there. It can be determined uniquely by the choice of a good maximal compact subgroup K of G, associated to a special vertex in apartment for $\mathcal{T}$ in the Bruhat–Tits building of $(\mathcal{G}, F)$.

For $\gamma \in \Gamma_{\mathfrak{s}}$ we have to be more careful, mainly because it need not fix χ_0. (The group $W_{\mathfrak{s}}$ fixes χ_0 when $\mathcal{G}$ is F-split, but the argument in that case does not generalize to $\mathcal{G}$ that only split over a ramified extension of F.) Since $X_{\mathrm{nr}}(T, \chi_0) = 1$, there exists a unique $\chi_\gamma \in X_{\mathrm{nr}}(T)$ such that $\gamma \cdot \chi_0 = \chi_0 \otimes \chi_\gamma$. Then χ_γ is fixed by $W(R_{\mathfrak{s}})$ [50, Lemma 3.5]. The element J_γ from [50, Theorem 10.9] comes from A_γ in [50, §5]. From [50, start of §5.1] we see that A_γ depends on χ_γ (which is unique) and on some

$$\rho_\gamma \in \mathrm{Hom}_T(\gamma\chi_0, \chi_0 \otimes \chi_\gamma).$$

For the latter we have a canonical choice, namely the identity on $\mathbb{C}$. Apart from that, A_γ depends only on the choice of K.

Theorem 1.2. *The above intertwining operators $N_w J_\gamma \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ give rise to an algebra isomorphism*

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) \cong \mathcal{H}(\mathfrak{s})^\circ \rtimes \mathbb{C}[\Gamma_{\mathfrak{s}}, \natural_{\mathfrak{s}}],$$

for a 2-cocycle $\natural_{\mathfrak{s}} : \Gamma_{\mathfrak{s}}^2 \to \mathbb{C}^\times$, suitable $W_{\mathfrak{s}}$-invariant label functions $\lambda : R_{\mathfrak{s}}^\vee \to \mathbb{Z}_{>0}$, $\lambda^* : R_{\mathfrak{s}}^\vee \to \mathbb{Z}_{\geq 0}$ and q-base $q_F^{1/2}$. This isomorphism is determined by the choice of a maximal compact subgroup K of G.

Proof. The isomorphism between $\mathcal{H}(\mathfrak{s})^\circ$ and the subalgebra of $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ generated by $\mathcal{O}(T_{\mathfrak{s}})$ and the N_w with $w \in W(R_{\mathfrak{s}}^\vee)$ is given in [50, Theorem 10.9]. The operators J_γ ($\gamma \in \Gamma_{\mathfrak{s}}$) in [50, Theorem 10.9] coincide with the $A_\gamma \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ from [50, §5.1]. The multiplication rules for the A_γ are given in [50, Proposition 5.2(a)]. As $X_{\mathrm{nr}}(T, \chi_0) = 1$, we get

$$A_\gamma A_{\gamma'} = \natural_{\mathfrak{s}}(\gamma, \gamma') A_{\gamma\gamma'}, \quad \gamma, \gamma' \in \Gamma_{\mathfrak{s}},$$

for some $\natural_{\mathfrak{s}}(\gamma, \gamma') \in \mathbb{C}^\times$. By the associativity of the multiplication, $\natural_{\mathfrak{s}}$ is a 2-cocycle. The other parts of [50, Proposition 5.2] also simplify, because χ_γ is fixed by $W(R_{\mathfrak{s}}^\vee)$. They show that

$$A_\gamma A_w = A_{\gamma w} \quad \text{and} \quad A_w A_\gamma = A_{w\gamma} \qquad \text{for } \gamma \in \Gamma_{\mathfrak{s}},\ w \in W(R_{\mathfrak{s}}^\vee).$$

This implies

$$A_\gamma^{-1} A_w A_\gamma = A_{\gamma^{-1} w \gamma} = A_{\gamma^{-1}} A_w A_\gamma. \tag{1-12}$$

In view of how N_w is constructed from A_w [50, §10], the relation (1-12) entails $A_\gamma^{-1} N_w A_\gamma = N_{\gamma^{-1} w \gamma}$. That and [50, (5.2)] show that $\Gamma_{\mathfrak{s}}$ acts on the image of $\mathcal{H}(\mathfrak{s})^\circ$ in $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ as in (1-11). Combining that with [50, Theorem 10.9] yields the required algebra isomorphism. □

We note that an important part of the structure of the algebra $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ consists of the labels $\lambda(h_\alpha^\vee)$, $\lambda^*(h_\alpha^\vee)$ with $\alpha \in R_{\mathfrak{s},\mu}$. Here the eigenvalues of N_{s_α} are $q_F^{\lambda(h_\alpha^\vee)/2}$ and $-q_F^{-\lambda(h_\alpha^\vee)/2}$. When we recall the known formulas for these labels, it will be convenient to consider all $\alpha \in R(\mathcal{G}, \mathcal{S})$ such that $s_\alpha \in W_{\mathfrak{s}}$.

Suppose first that $\mathcal{G}$ is F-split. By [52, Proposition 4.3], $\alpha \in R_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F^\times \to \mathbb{C}^\times$ is unramified. Further, by [52, Theorem 4.4],

$$\lambda(\alpha^\vee(\varpi_F^{-1})) = \lambda^*(\alpha^\vee(\varpi_F^{-1})) = 1. \tag{1-13}$$

Now we suppose that $\mathcal{G}$ quasi-split but not necessarily split. A special role is played by pairs of roots in type ${}^2A_{2n}$, such that the diagram automorphism permutes the pair. We settle the other cases before we turn to those exceptional roots.

Let $\alpha_{\mathcal{T}} \in R(\mathcal{G}, \mathcal{T})$ be a preimage of $\alpha \in R(\mathcal{G}, \mathcal{S})$ and let $\boldsymbol{W}_{F,\alpha_{\mathcal{T}}}$ be its stabilizer. The splitting field $F_\alpha = F_s^{\boldsymbol{W}_{F,\alpha_{\mathcal{T}}}}$ of α is unique up to isomorphism. Let $f(F_\alpha/F)$ be the residual degree of F_α/F.

Assuming that α is not exceptional, the issue can be reduced to (1-13). Indeed, by [52, §4.2], $\alpha \in R_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times$ is unramified. Moreover, by [52, Corollary 4.5],

$$\lambda(\alpha^\vee(\varpi_{F_\alpha}^{-1})) = \lambda^*(\alpha^\vee(\varpi_{F_\alpha}^{-1})) = f(F_\alpha/F). \tag{1-14}$$

In most cases $h_\alpha^\vee = \alpha^\vee(\varpi_{F_\alpha}^{-1})$ in T/T_{cpt}, and sometimes $\alpha^\vee(\varpi_{F_\alpha}^{-1}) = (h_\alpha^\vee)^2$ in T/T_{cpt}. In the latter cases, for instance, $\mathrm{PGL}_2(F)$,

$$\lambda(h_\alpha^\vee) = f(F_\alpha/F) \quad \text{and} \quad \lambda^*(h_\alpha^\vee) = 0. \tag{1-15}$$

The exceptional roots occur only when $R_{\mathfrak{s},\mu}$ has a component of type BC_n which comes from a component of type ${}^2A_{2n}$ in $R(\mathcal{G}, \mathcal{S})$. Consider an indivisible root $\alpha \in R_{\mathfrak{s},\mu}$ which comes from two adjacent roots in ${}^2A_{2n}$. As explained in [52, §4.2], the computation of the parameters for this α can be reduced to the quasi-split group $\mathrm{SU}_3(F_\alpha/F_{2\alpha})$. Moreover, since the groups of unramified characters of $\mathrm{SU}_3(F_\alpha/F_{2\alpha})$, $U_3(F_\alpha/F_{2\alpha})$ and $\mathrm{PU}_3(F_\alpha/F_{2\alpha})$ are naturally identified, the reductions from [52, §2] apply to these groups in the strong sense that in these instances of [52, Proposition 2.4] no doubling or halving of roots can occur. Consequently the labels for $\alpha \in R(\mathcal{G}, \mathcal{S})$ are precisely $f(F_{2\alpha}/F)$ times the labels for α as root for $U_3(F_\alpha/F_{2\alpha})$.

For $U_3(F_\alpha/F_{2\alpha})$ all q-parameters for principal series representations were computed via types by Badea [7]. The outcome can be summarized as follows.

- If $F_\alpha/F_{2\alpha}$ is unramified and χ_c is trivial on $T_{\mathrm{cpt}} \cap \mathrm{SU}_3(F_\alpha/F_{2\alpha})$, then $\alpha \in R_{\mathfrak{s},\mu}$ and $\lambda(h_\alpha^\vee) = 3$, $\lambda^*(h_\alpha^\vee) = 1$.
- If $F_\alpha/F_{2\alpha}$ is unramified and χ_c is nontrivial on $T_{\mathrm{cpt}} \cap \mathrm{SU}_3(F_\alpha/F_{2\alpha})$, then $\alpha \in R_{\mathfrak{s},\mu}$ and $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee) = 1$.
- If $F_\alpha/F_{2\alpha}$ is ramified, then $\alpha \in R_{\mathfrak{s},\mu}$ if and only if $\chi \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times$ is nontrivial on $\mathfrak{o}_{F_{2\alpha}}^\times$. (We note that $\chi^2 \circ \alpha^\vee|_{\mathfrak{o}_{F_{2\alpha}}^\times} = 1$ because $s_\alpha \chi_c = \chi_c$.) When this condition is fulfilled, we have
$$\lambda(\alpha^\vee(\varpi_{F_{2\alpha}}^{-1})) = \lambda^*(\alpha^\vee(\varpi_{F_{2\alpha}}^{-1})) = 1$$
and $\lambda(h_\alpha^\vee) = 1$, $\lambda^*(h_\alpha^\vee) = 0$.

We warn that in [7] it is assumed throughout that the residual characteristic of F is not 2. For unramified characters χ this restriction is not necessary, because in those cases the Hecke algebras and the parameters were already known from [9]. However, for other χ the tricky calculations in [7, §2.7 and §5.2.1] do not work in residual characteristic 2.

For F of arbitrary characteristic, the Hecke algebra parameters for $U_3(F_\alpha/F_{2\alpha})$ can also be determined via the endoscopic methods from [36]; see [52, Theorem 4.9]. That shows that the above formulas also apply when the residual characteristic of F is 2.

2. Whittaker normalization

Unfortunately the isomorphism from Theorem 1.2 is not entirely canonical, because it depends on a good maximal compact subgroup K of G, and often G has more than one conjugacy class of such subgroups. Further, it may be expected that the 2-cocycle $\natural_{\mathfrak{s}}$ of $\Gamma_{\mathfrak{s}}$ is trivial, because G is quasi-split. We will fix both issues by using a Whittaker datum. Let U be the unipotent radical of B (since all Borel subgroups of G are conjugate, the choice of B is inessential.) Let $\xi : U \to \mathbb{C}^\times$ be a nondegenerate smooth character, which means that it is nontrivial on every root subgroup U_α with $\alpha \in R(\mathcal{G}, \mathcal{S})$ simple. Then the G-conjugacy class of (U, ξ) is a Whittaker datum for G.

Recall that a Whittaker functional for $\pi \in \mathrm{Rep}(G)$ is an element of

$$\mathrm{Hom}_U(\pi, \xi) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\xi)),$$

where Ind denotes smooth induction. We say that π is generic, or more precisely (U, ξ)-generic, if it admits a nonzero Whittaker functional. It is well known [42; 44] that every representation $I_B^G(\chi)$ with $\chi \in \mathrm{Irr}(T)$ is generic, and that its space of Whittaker functionals has dimension one. For the upcoming arguments we need a larger but modest supply of generic representations.

Proposition 2.1. *Suppose that $R(\mathcal{G}, \mathcal{S})$ and $R_{\mathfrak{s},\mu}$ have rank one. Then $|W_{\mathfrak{s}}| = 2$ and by Theorem 1.2 $\mathcal{H}(\mathfrak{s})$ is an affine Hecke algebra with a unique positive root $h_\alpha^\vee$. Let $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$ be the Steinberg representation of $\mathcal{H}(\mathfrak{s})$, the unique essentially discrete series representation with an $\mathcal{O}(T_{\mathfrak{s}})$-weight of the form $\chi_0|\alpha|_F^s$ with $s \in \mathbb{R}$.*

(a) *The G-representation $\mathrm{St}_{\mathfrak{s}} := \mathrm{St}_{\mathcal{H}(\mathfrak{s})} \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}}$ is generic.*

(b) *Suppose that $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$. In that case $\mathcal{H}(\mathfrak{s})$ has a unique essentially discrete series representation $\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}$ with an $\mathcal{O}(T_{\mathfrak{s}})$-weight of the form $\chi_0|\alpha|_F^{ia+s}$ where $s, a \in \mathbb{R}$ and $|\alpha(h_\alpha^\vee)|_F^{ia} = -1$; see* [49, §2.2]. *Then the G-representation $\mathrm{St}_{\mathfrak{s}-} := \mathrm{St}_{\mathcal{H}(\mathfrak{s})-} \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}}$ is generic.*

(c) *Suppose that the coroot of $h_\alpha^\vee$ lies in $2(T/T_{\mathrm{cpt}})^\vee$ and that $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$. Choose $a \in \mathbb{R}$ as in* (b). *Then $I_B^G(\chi_0|\alpha|_F^{ia})$ is a direct sum of two irreducible* *subrepresentations. One of them, say* $\pi^g_{\mathfrak{s}-}$, *is* (U, ξ)*-generic and the other, say* $\pi^n_{\mathfrak{s}-}$, *is not.*

(d) *The irreducible* G*-representations in* (a)–(c) *are unitary.*

Proof. (a) As $R(\mathcal{G}, \mathcal{S})$ and $R_{\mathfrak{s},\mu}$ have the same rank, the equivalence of categories (1-1) translates "essentially square integrable" into "essentially discrete series" [50, Theorem 9.6.c]. In particular $\mathrm{St}_{\mathfrak{s}}$ is an essentially square-integrable G-representation. The assumptions of the proposition amount to the assumptions for [43, Theorem 8.1]. Part (b) of that result provides the desired conclusion, at least when $\mathrm{char}(F) = 0$. The version of [43, Theorem 8.1] with $\mathrm{char}(F) > 0$ was established in [29, Theorem 5.5].

(b) This is analogous to (a).

(c) It is well known (see, for instance, [49, §2.2]) that $\mathrm{ind}^{\mathcal{H}(\mathfrak{s})}_{\mathcal{O}(T_{\mathfrak{s}})}(\chi_0|\alpha|^{ia}_F)$ is a direct sum of two one-dimensional representations, say $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ and $\pi^n_{\mathcal{H}(\mathfrak{s})-}$. Writing $\pi^{g/n}_{\mathfrak{s}-} = \pi^{g/n}_{\mathcal{H}(\mathfrak{s})-} \otimes_{\mathrm{End}_G(\Pi_{\mathfrak{s}})} \Pi_{\mathfrak{s}}$, we obtain

$$I^G_B(\chi_0|\alpha|^{ia}_F) = \pi^g_{\mathfrak{s}-} \oplus \pi^n_{\mathfrak{s}-}.$$

Since $\dim \mathrm{Hom}_U\big(I^G_B(\chi_0|\alpha|^{ia}_F), \xi\big) = 1$, exactly one of these direct summands is generic (which one depends on ξ). By renaming if necessary, we can make $\pi^g_{\mathfrak{s}-}$ generic.

(d) This holds because these representations are tempered and irreducible [39, Corollaire VII.2.6]. □

2.1. ***Modules of Whittaker functionals.*** For our purposes it is more convenient to analyze a perspective on generic representations which is dual to the traditional view. For $(\pi, V) \in \mathrm{Rep}(G)$ let $V^\dagger$ be the smooth Hermitian dual space, that is, the vector space of all conjugate-linear maps $\ell : V \to \mathbb{C}$ which factor through the projection $V \to V^K$ for some compact open subgroup K of G. The Hermitian dual representation $\pi^\dagger$ on $V^\dagger$ is defined by

$$(\pi^\dagger(g)\ell)(v) = \ell(\pi(g^{-1})v) \quad \text{for all } v \in V.$$

Equivalently, $\pi^\dagger$ is the smooth contragredient of the complex conjugate of π. If π is unitary and admissible, then $\pi^\dagger$ is isomorphic to π via the G-invariant inner product.

Lemma 2.2. *If* $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$, *then also* $\pi^\dagger \in \mathrm{Rep}(G)^{\mathfrak{s}}$.

Proof. Let $\mathfrak{s}' = [M, \sigma]_G$ be any inertial equivalence class different from $\mathfrak{s}$. We may assume that σ is unitary, so $\sigma^\dagger \cong \sigma$. Let $P \subset G$ be a parabolic subgroup with Levi factor M and let $M_1 \subset M$ be the subgroup generated by all compact subgroups of M.

Then $I_P^G(\mathrm{ind}_{M_1}^M(\sigma))$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}'}$; see [39, Théorème VI.10.1]. With Bernstein's second adjointness we compute

$$\begin{aligned}(2\text{-}1)\quad \mathrm{Hom}_G\big(I_P^G(\mathrm{ind}_{M_1}^M(\sigma)), \pi^\dagger\big) &\cong \mathrm{Hom}_M(\mathrm{ind}_{M_1}^M(\sigma), J_{\bar{P}}^G(\pi^\dagger)) \cong \mathrm{Hom}_M(\mathrm{ind}_{M_1}^M(\sigma), (J_P^G\pi)^\dagger) \\ &\cong \mathrm{Hom}_{M_1}(\sigma, (J_P^G\pi)^\dagger) \cong \mathrm{Hom}_{M_1}(J_P^G(\pi), \sigma).\end{aligned}$$

Since $[M, \sigma]_G \neq \mathfrak{s}$, $J_P^G(\pi)$ does not have any irreducible subquotient isomorphic with σ or an unramified twist of σ. Hence (2-1) is zero. This means that the component of $\pi^\dagger$ in $\mathrm{Rep}(G)^{\mathfrak{s}'}$ is zero for any $\mathfrak{s}' \neq \mathfrak{s}$. □

From [12, (2.1.1)] one sees that the Hermitian dual of $\mathrm{ind}_U^G(\xi)$ is $\mathrm{Ind}_U^G(\xi)$, with respect to the pairing

$$\mathrm{Ind}_U^G(\xi) \times \mathrm{ind}_U^G(\xi) \to \mathbb{C}, \quad \langle f_1, f_2\rangle = \int_{U\backslash G} f_1(g)\overline{f_2(g)}\,\mathrm{d}g.$$

Hence there is a natural isomorphism

$$(2\text{-}2)\qquad \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\xi)) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \pi^\dagger).$$

By Lemma 2.2 and (1-1), the right-hand side is isomorphic with

$$(2\text{-}3)\qquad \mathrm{Hom}_{\mathrm{End}_G(\Pi_\mathfrak{s})}\big(\mathrm{Hom}_G(\Pi_\mathfrak{s}, \mathrm{ind}_U^G(\xi)), \mathrm{Hom}_G(\Pi_\mathfrak{s}, \pi^\dagger)\big).$$

Thus any nonzero Whittaker functional for π yields a nonzero element of (2-3). This prompts us to analyze $\mathrm{Hom}_G(\Pi_\mathfrak{s}, \mathrm{ind}_U^G(\xi))$ as an $\mathrm{End}_G(\Pi_\mathfrak{s})^{\mathrm{op}}$-module. By [12, Theorem 2.2] there are these canonical isomorphisms of T-representations:

$$(2\text{-}4)\qquad J_{\bar{B}}^G\,\mathrm{ind}_U^G(\xi) \cong \mathrm{ind}_{U\cap T}^T(\xi) = \mathrm{ind}_{\{e\}}^T(\mathrm{triv}).$$

From that we compute

$$\begin{aligned}(2\text{-}5)\qquad \mathrm{Hom}_G(\Pi_\mathfrak{s}, \mathrm{ind}_U^G(\xi)) &= \mathrm{Hom}_G\big(I_B^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)), \mathrm{ind}_U^G(\xi)\big) \\ &\cong \mathrm{Hom}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), J_{\bar{B}}^G\,\mathrm{ind}_U^G(\xi)) \\ &\cong \mathrm{Hom}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \mathrm{ind}_{\{e\}}^T(\mathrm{triv})).\end{aligned}$$

The Bernstein decomposition of $\mathrm{Rep}(T)$ entails that only the part of $\mathrm{ind}_{\{e\}}^T(\mathrm{triv})$ on which T_{cpt} acts according to χ_c contributes to the right-hand side. Hence (2-5) is naturally isomorphic with

$$(2\text{-}6)\quad \mathrm{Hom}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)) \cong \mathrm{Hom}_{T_{\mathrm{cpt}}}(\chi_c, \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)) \cong \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c).$$

This vector space contains a canonical unit vector, namely $\chi_c \in \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ or equivalently $\mathbb{1} \in \mathcal{O}(T_\mathfrak{s})$. We use blackboard bold to indicate that it is an element of (2-6), not of $\mathrm{End}_G(\Pi_\mathfrak{s})$.

We want to normalize our intertwining operators N_w so that they act on $\mathbb{1}$ in an easy way. Any $f \in \mathcal{O}(T_\mathfrak{s}) \cong \mathrm{ind}_{T_{\mathrm{cpt}}}^{T}(\chi_c)$ can be regarded as an element of $\mathrm{End}_G(\Pi_\mathfrak{s})$, namely I_B^G applied to multiplication by f. The action of that on (2-6) is again multiplication by f. Thus (2-23) is free of rank one as an $\mathcal{O}(T_\mathfrak{s})$-module, and $\mathbb{1}$ forms a canonical basis.

Let $\mathbb{C}(T_\mathfrak{s})$ be the field of rational functions on $T_\mathfrak{s}$, the quotient field of $\mathcal{O}(T_\mathfrak{s})$. It follows from Bernstein's geometric lemma [39, Théorème VI.5.1] that

$$\mathrm{End}_G(I_B^G \mathbb{C}(T_\mathfrak{s})) \cong \mathrm{End}_G(I_B^G \mathcal{O}(T_\mathfrak{s})) \otimes_{\mathcal{O}(T_\mathfrak{s})} \mathbb{C}(T_\mathfrak{s}); \tag{2-7}$$

see [50, Lemma 5.3]. The natural isomorphisms (2-5) and (2-6) extend to

$$\mathrm{Hom}_G(I_B^G \mathcal{O}(T_\mathfrak{s}), \mathrm{ind}_U^G(\xi)) \otimes_{\mathcal{O}(T_\mathfrak{s})} \mathbb{C}(T_\mathfrak{s}) \cong \mathbb{C}(T_\mathfrak{s}), \tag{2-8}$$

and as a module over (2-7) this is an extension of scalars of (2-6). The advantage of this setup is:

Proposition 2.3. *Theorem 1.2 extends to an algebra isomorphism*

$$\mathrm{End}_G(I_B^G \mathbb{C}(T_\mathfrak{s})) \cong (\mathbb{C}(T_\mathfrak{s}) \rtimes W(R_\mathfrak{s}^\vee)) \rtimes \mathbb{C}[\Gamma_\mathfrak{s}, \natural_\mathfrak{s}].$$

Proof. This is a direct consequence of Theorem 1.2 and §5.1 (in particular Corollary 5.8) of [50]. □

In Proposition 2.3 the basis elements of $\mathbb{C}[\Gamma_\mathfrak{s}, \natural_\mathfrak{s}]$ are the same $J_\gamma = A_\gamma$ as in Theorem 1.2. The basis elements of

$$\mathbb{C}[W(R_\mathfrak{s}^\vee)] \subset \mathrm{End}_G(I_B^G \mathbb{C}(T_\mathfrak{s}))$$

are the $\mathcal{T}_w$ from [50, Proposition 5.5], which are expressed in terms of the N_w in Lemma 10.8 and the preceding remarks of [50]. Proposition 2.3 enables us to analyze the actions on (2-6) and on (2-8) more explicitly.

For $w \in W(R_\mathfrak{s}^\vee)$, $\gamma \in \Gamma_\mathfrak{s}$ and $f \in \mathcal{O}(T_\mathfrak{s})$,

$$\begin{aligned} f \cdot \mathcal{T}_w J_\gamma &= \mathbb{1} \cdot f \mathcal{T}_w J_\gamma = (\mathbb{1} \cdot \mathcal{T}_w J_\gamma) \cdot (J_\gamma^{-1} \mathcal{T}_w^{-1} f \mathcal{T}_w J_\gamma) \\ &= (\mathbb{1} \cdot \mathcal{T}_w J_\gamma) \cdot (f \circ w\gamma) = (f \circ w\gamma)(\mathbb{1} \cdot \mathcal{T}_w J_\gamma) \end{aligned} \tag{2-9}$$

in (2-8). Notice that $\mathbb{1} \cdot J_\gamma$ must be invertible in $\mathcal{O}(T_\mathfrak{s})$, because J_γ is invertible in $\mathrm{End}_G(\Pi_\mathfrak{s})$.

We write $\theta_{n\alpha}$ for $\theta_{nh_\alpha^\vee}$, where $n \in \mathbb{Z}$ and $\alpha \in R_\mathfrak{s}^\vee$. We also write

$$q_\alpha = q_F^{(\lambda(h_\alpha^\vee) + \lambda^*(h_\alpha^\vee))/2} \quad \text{and} \quad q_{\alpha*} = q_F^{(\lambda(h_\alpha^\vee) - \lambda^*(h_\alpha^\vee))/2}.$$

Proposition 2.4. *For each simple root $h_\alpha^\vee \in R_\mathfrak{s}^\vee$ there exists $n_\alpha \in \mathbb{Z}$ such that $\mathbb{1} \cdot \mathcal{T}_{s_\alpha} = -\theta_{n_\alpha \alpha}$ in* (2-8).

Proof. The operators $N_{s_\alpha} \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ and $\mathcal{T}_{s_\alpha}$ arise by parabolic induction from the analogous elements for the Levi subgroup G_α of G generated by $T \cup U_\alpha \cup U_{-\alpha}$. Hence it suffices to work in G_α, which means that we may assume that $R(\mathcal{G}, \mathcal{S})$ and $R_{\mathfrak{s},\mu}$ have rank one.

First we consider the cases where $q_{\alpha *} \neq 1$, or equivalently $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$. From [50, (5.19)] we know that $\mathcal{T}_{s_\alpha}(q_\alpha - \theta_{-\alpha})(q_{\alpha *} + \theta_{-\alpha}) \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$. Hence we can write

$$\mathbb{1} \cdot \mathcal{T}_{s_\alpha} = f_1 (q_\alpha - \theta_{-\alpha})^{-1} (q_{\alpha *} + \theta_{-\alpha})^{-1} \quad \text{with } f_1 \in \mathcal{O}(T_{\mathfrak{s}}).$$

The relations $\mathcal{T}_{s_\alpha}^2 = 1$ and (2-9) imply that

$$1 = (\mathbb{1} \cdot \mathcal{T}_{s_\alpha})\, s_\alpha(\mathbb{1} \cdot \mathcal{T}_{s_\alpha}) = \frac{f_1\, s_\alpha(f_1)}{(q_\alpha - \theta_\alpha)(q_{\alpha *} + \theta_\alpha)(q_\alpha - \theta_{-\alpha})(q_{\alpha *} + \theta_{-\alpha})}.$$

It follows that there exist $\epsilon \in \{\pm 1\}$ and $n_\alpha \in \mathbb{Z}$ such that

$$f_1 = \epsilon \theta_{n_\alpha \alpha} (q_\alpha - \theta_{\pm \alpha})(q_{\alpha *} + \theta_{\pm' \alpha}),$$

for suitable signs $\pm$, $\pm'$. Equivalently

$$\mathbb{1} \cdot \mathcal{T}_{s_\alpha} = \epsilon \theta_{n_\alpha \alpha} \left(\frac{q_\alpha - \theta_\alpha}{q_\alpha - \theta_{-\alpha}} \right)^{\eta} \left(\frac{q_{\alpha *} + \theta_\alpha}{q_{\alpha *} + \theta_{-\alpha}} \right)^{\eta'} =: \epsilon \theta_{n_\alpha \alpha} f_2, \tag{2-10}$$

where $\eta, \eta' \in \{0, 1\}$.

Under our assumption, $\alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee$ and s_α fixes any $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(h_\alpha^\vee) = -1$. Notice that $f_2(\chi) = 1$ whenever $\theta_\alpha(\chi) \in \{\pm 1\}$. As in Lemma 10.7(b) of [50], define

$$\epsilon_\alpha = \begin{cases} 1 & \text{if } I_B^G(\mathrm{ev}_\chi)\mathcal{T}_{s_\alpha} = -I_B^G(\mathrm{ev}_\chi), \\ 0 & \text{otherwise.} \end{cases} \tag{2-11}$$

By [50, Lemma 10.8],

$$q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1 = (\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1} \tag{2-12}$$

belongs to $\mathrm{End}_G(\Pi_{\mathfrak{s}})$. In particular

$$\mathbb{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) = \frac{(\epsilon \theta_{(n_\alpha - \epsilon_\alpha)\alpha} f_2 + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)}{\theta_{2\alpha} - 1} \tag{2-13}$$

lies in $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}})$. Specializing the numerator of (2-13) at χ' with $\theta_\alpha(\chi') = 1$ gives $(\epsilon + 1)(q_\alpha - 1)(q_{\alpha *} + 1)$. Since $q_\alpha > 1$ and (2-13) has no poles, this implies $\epsilon = -1$.

Let $\mathcal{G}_{\mathrm{der}}$ be the derived group of $\mathcal{G}$ and write $\mathfrak{s}_{\mathrm{der}} = [\chi|_{T \cap G_{\mathrm{der}}}, T \cap G_{\mathrm{der}}]_{G_{\mathrm{der}}}$. By construction $\mathcal{H}(\mathfrak{s}_{\mathrm{der}})$ is the subalgebra of $\mathcal{H}(\mathfrak{s})$ generated by $\mathbb{C}[T \cap G_{\mathrm{der}} / T_{\mathrm{cpt}} \cap G_{\mathrm{der}}]$

and N_{s_α}. From [49, §2.2] we recall that $\mathrm{St}_{\mathcal{H}(\mathfrak{s})} : \mathcal{H}(\mathfrak{s}) \to \mathbb{C}$ is given on $\mathcal{H}(\mathfrak{s}_{\mathrm{der}})$ by

$$\mathrm{St}_{\mathcal{H}(\mathfrak{s})}(N_{s_\alpha}) = -q_F^{-\lambda(h_\alpha^\vee)/2}, \quad \mathrm{St}_{\mathcal{H}(\mathfrak{s})}(\theta_{n\alpha}) = q_\alpha^{-n}.$$

From Proposition 2.1(a) and (d) we know that

$$\mathrm{Hom}_G(\mathrm{St}_{\mathfrak{s}}, \mathrm{Ind}_U^G(\xi)) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \mathrm{St}_{\mathfrak{s}}) \text{ has dimension } 1.$$

As in (2-3), any nonzero Whittaker functional yields a surjection

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}}) \to \mathrm{St}_{\mathcal{H}(\mathfrak{s})}.$$

This is an $\mathcal{O}(T_{\mathfrak{s}})$-module homomorphism, so up to rescaling it must be evaluation at χ_{St}, the unique $\mathcal{O}(T_{\mathfrak{s}})$-weight of $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$. Since $\mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda)$ acts on $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$ via the sign representation,

$$\theta_x \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) \in \ker(\mathrm{ev}_{\chi_{\mathrm{St}}}) \quad \text{for all } x \in T/T_{\mathrm{cpt}}. \tag{2-14}$$

For $x = 0$ we can make that more explicit with (2-12):

$$\begin{aligned} &\mathbb{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1) \\ &= (\theta_0 - \theta_{(n_\alpha - \epsilon_\alpha)\alpha} f_2)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1} \\ &= \frac{(q_\alpha - \theta_{-\alpha})^\eta (q_{\alpha *} + \theta_{-\alpha})^{\eta'} - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_\alpha - \theta_\alpha)^\eta (q_{\alpha *} + \theta_\alpha)^{\eta'}}{(q_\alpha - \theta_{-\alpha})^\eta (q_{\alpha *} + \theta_{-\alpha})^{\eta'}} \frac{(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)}{(\theta_{2\alpha} - 1)} \end{aligned}$$

When $\eta = 1$, this reduces to

$$\frac{(q_\alpha - \theta_{-\alpha})(q_{\alpha *} + \theta_{-\alpha})^{\eta'} - \theta_{(n_\alpha - \epsilon_\alpha)\alpha}(q_\alpha - \theta_\alpha)(q_{\alpha *} + \theta_\alpha)^{\eta'}}{(q_{\alpha *} + \theta_{-\alpha})^{\eta'}} \frac{\theta_\alpha(\theta_\alpha q_{\alpha *} + 1)}{(\theta_{2\alpha} - 1)}. \tag{2-15}$$

Evaluation at χ_{St} sends this element to

$$\frac{-q_\alpha^{\epsilon_\alpha - n_\alpha}(q_\alpha - q_\alpha^{-1})(q_{\alpha *} + q_\alpha^{-1})^{\eta'} q_\alpha^{-1}(q_\alpha^{-1} q_{\alpha *} + 1)}{(q_{\alpha *} + q_\alpha)(q_\alpha^{-2} - 1)} \neq 0.$$

That contradicts (2-14), so that η must be 0.

We recall from [49, proof of Theorem 2.4(c)] that $\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}$ is given on $\mathcal{H}(\mathfrak{s}_{\mathrm{der}})$ by

$$\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}(N_{s_\alpha}) = -q_F^{-\lambda(h_\alpha^\vee)/2}, \quad \mathrm{St}_{\mathcal{H}(\mathfrak{s})-}(\theta_{n\alpha}) = (-q_{\alpha *}^{-1})^n.$$

By Proposition 2.1(b) and (d),

$$\mathrm{Hom}_G(\mathrm{St}_{\mathfrak{s}-}, \mathrm{Ind}_U^G(\xi)) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \mathrm{St}_{\mathfrak{s}-}) \text{ has dimension } 1.$$

As above, this gives a surjection

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}}) \to \mathrm{St}_{\mathcal{H}(\mathfrak{s})-},$$

which (up to rescaling) is evaluation at the $\mathcal{O}(T_{\mathfrak{s}})$-weight $\chi_{\mathrm{St}-}$ of $\mathrm{St}_{\mathcal{H}(\mathfrak{s})-}$. Then $\ker(\mathrm{ev}_{\chi_{\mathrm{St}-}})$ contains

$$\begin{aligned} &\mathbb{1}\cdot(q_F^{\lambda(h_\alpha^\vee)/2}N_{s_\alpha}+1) \\ &= \mathbb{1}\cdot(\mathcal{T}_{s_\alpha}\theta_{-\epsilon_\alpha\alpha}+1)(\theta_\alpha q_\alpha-1)(\theta_\alpha q_{\alpha*}+1)(\theta_{2\alpha}-1)^{-1} \\ &= (\theta_0-\theta_{(n_\alpha-\epsilon_\alpha)\alpha}(q_{\alpha*}+\theta_\alpha/q_{\alpha*}+\theta_{-\alpha})^{\eta'})(\theta_\alpha q_\alpha-1)(\theta_\alpha q_{\alpha*}+1)(\theta_{2\alpha}-1)^{-1} \\ &= \frac{(q_{\alpha*}+\theta_{-\alpha})^{\eta'}-\theta_{(n_\alpha-\epsilon_\alpha)\alpha}(q_{\alpha*}+\theta_\alpha)^{\eta'}}{(q_{\alpha*}+\theta_{-\alpha})^{\eta'}}\frac{(\theta_\alpha q_\alpha-1)(\theta_\alpha q_{\alpha*}+1)}{(\theta_{2\alpha}-1)}. \end{aligned} \tag{2-16}$$

When $\eta'=1$, (2-16) simplifies to

$$(q_{\alpha*}+\theta_{-\alpha}-\theta_{(n_\alpha-\epsilon_\alpha)\alpha}(q_{\alpha*}+\theta_\alpha))\frac{(\theta_\alpha q_\alpha-1)\theta_\alpha}{(\theta_{2\alpha}-1)}.$$

Evaluation at $\chi_{\mathrm{St}-}$ results in

$$\frac{(-q_{\alpha*}^{-1})^{n_\alpha-\epsilon_\alpha}(q_\alpha-q_{\alpha*}^{-1})(-q_{\alpha*}^{-1}q_\alpha-1)q_{\alpha*}^{-1}}{q_{\alpha*}^{-2}-1}.$$

This is nonzero because $q_\alpha\geq q_{\alpha*}>1$. But then (2-16) does not lie in the kernel of $\mathrm{ev}_{\chi_{\mathrm{St}-}}$, a contradiction. Therefore η' must be 0.

Now we consider the cases with $q_{\alpha*}=1$, or equivalently $\lambda(h_\alpha^\vee)=\lambda^*(h_\alpha^\vee)$. Then we can omit all factors $q_{\alpha*}+\theta_{\pm\alpha}$, and we can replace $\theta_{2\alpha}-1$ by $\theta_\alpha-1$. The above argument with $\mathrm{St}_{\mathcal{H}(\mathfrak{s})}$ still applies, and shows that $\eta=0$. □

For the moment we continue to work in G_α. Assume that $\alpha^\sharp\in 2(T/T_{\mathrm{cpt}})^\vee$ and $\lambda(h_\alpha^\vee)=\lambda^*(h_\alpha^\vee)$. The one-dimensional $\mathcal{H}(\mathfrak{s})$-representation $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ from Proposition 2.1(c) extends canonically to a representation of $\mathcal{H}(\mathfrak{s})+\mathcal{T}_{s_\alpha}\mathcal{H}(\mathfrak{s})$, because $\mathcal{T}_{s_\alpha}$ does not have a pole at $|\alpha|_F^{ia}$. In particular $\pi^g_{\mathcal{H}(\mathfrak{s})-}$ determines a character of the order-two group $\langle\mathcal{T}_{s_\alpha}\rangle$. We define

$$\epsilon_\alpha=\begin{cases}1 & \text{if } \pi^g_{\mathcal{H}(\mathfrak{s})-}|_{\langle\mathcal{T}_{s_\alpha}\rangle}=\text{triv},\\ 0 & \text{if } \pi^g_{\mathcal{H}(\mathfrak{s})-}|_{\langle\mathcal{T}_{s_\alpha}\rangle}=\text{sign}.\end{cases} \tag{2-17}$$

This complements the definition of ϵ_α when $\alpha^\sharp\in 2(T/T_{\mathrm{cpt}})^\vee$ and $\lambda(h_\alpha^\vee)\neq\lambda^*(h_\alpha^\vee)$; see (2-11). Together these provide a function

$$\epsilon_?:\{h_\alpha^\vee\in R_{\mathfrak{s}}^\vee \text{ simple}:\alpha^\sharp\in 2(T/T_{\mathrm{cpt}})^\vee\}\to\{0,1\}. \tag{2-18}$$

Lemma 2.5. (a) *The function* (2-18) *is* $\Gamma_{\mathfrak{s}}$*-invariant.*

(b) *Take* α *in the domain of* $\epsilon_?$ *and let* n_α *be as in Proposition 2.4. Then* $n_\alpha-\epsilon_\alpha$ *is even.*

Proof. (a) For $\gamma \in \Gamma_{\mathfrak{s}}$, represented in $N_G(T)$, we have $\gamma G_\alpha \gamma^{-1} = G_{\gamma(\alpha)}$ and $J_\gamma \mathcal{T}_{s_\alpha} J_\gamma^{-1} = \mathcal{T}_{s_{\gamma(\alpha)}}$. Further

$$\mathrm{Ad}(\gamma) I_{B\cap G_\alpha}^{G_\alpha}(\chi_0 |\alpha|_F^z) \cong I_{B\cap G_{\gamma(\alpha)}}^{G_{\gamma(\alpha)}}(\chi_0|\alpha|_F^z) \quad \text{for any } z \in \mathbb{C}.$$

When $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$, we apply this with $z = ia$. We note that $I_{G_\alpha B}^G(\pi_{\mathfrak{s}-}^g)$ is generic while $I_{G_\alpha B}^G(\pi_{\mathfrak{s}-}^n)$ is not. Because $I_{G_\alpha B}^G(\pi_{\mathfrak{s}-}^g) = I_{G_{\gamma(\alpha)}B}^G \mathrm{Ad}(\gamma)(\pi_{\mathfrak{s}-}^g)$, we conclude that $\pi_{\mathfrak{s}-}^g$ for $G_{\gamma(\alpha)}$ is obtained from $\pi_{\mathfrak{s}-}^g$ for G_α by $\mathrm{Ad}(\gamma)$. Hence $\epsilon_{\gamma(\alpha)} = \epsilon_\alpha$.

When $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$, the same argument works with the irreducible representation $I_{B\cap G_\alpha}^{G_\alpha}(\chi_0|\alpha|_F^{ia})$.

(b) Suppose that $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$. Recall from the proof of Proposition 2.4 that $\epsilon f_2 = -1$. Specializing the numerator of (2-13) at χ with $\theta_\alpha(\chi) = -1$ gives

$$(-(-1)^{n_\alpha - \epsilon_\alpha} + 1)(-q_\alpha - 1)(-q_{\alpha*} + 1) = ((-1)^{n_\alpha - \epsilon_\alpha} - 1)(q_\alpha + 1)(1 - q_{\alpha*}).$$

Again this must be 0 by (2-13). Using $q_{\alpha*} \neq 1$ we find that $n_\alpha - \epsilon_\alpha$ is even.

Suppose that $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$ and $\pi_{\mathcal{H}(\mathfrak{s})-}^g|_{\langle \mathcal{T}_{s_\alpha}\rangle} = \mathrm{triv}$. By Proposition 2.1(d), any Whittaker functional for $\pi_{\mathcal{H}(\mathfrak{s})-}^g$ gives a surjection

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}}) \to \pi_{\mathcal{H}(\mathfrak{s})-}^g.$$

As an $\mathcal{O}(T_{\mathfrak{s}})$-module homomorphism it is (up to scaling) evaluation at $\chi_- := \chi_0|\alpha|_F^{ia}$, a character such that $\theta_\alpha(\chi_-) = -1$. Then $\ker(\mathrm{ev}_{\chi_-})$ contains

$$\mathbb{1} \cdot (\mathcal{T}_{s_\alpha} - 1) = -\theta_{n_\alpha \alpha} - \theta_0,$$

so n_α is odd. Recall that $\epsilon_\alpha = 1$ in this case.

Suppose that $\lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee)$ and $\pi_{\mathcal{H}(\mathfrak{s})-}^g|_{\langle \mathcal{T}_{s_\alpha}\rangle} = \mathrm{sign}$. Then $\ker(\mathrm{ev}_{\chi_-})$ contains

$$\mathbb{1} \cdot (\mathcal{T}_{s_\alpha} + 1) = -\theta_{n_\alpha \alpha} + \theta_0,$$

so n_α is even. Here $\epsilon_\alpha = 0$, so again $n_\alpha - \epsilon_\alpha$ is even. □

2.2. ***Normalization of intertwining operators.*** With Lemma 2.5(a), we can extend $\epsilon_?$ to a $W_{\mathfrak{s}}$-invariant function on $\{h_\alpha^\vee \in R_{\mathfrak{s}}^\vee : \alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee\}$. In [50], ϵ_α was only defined when $\lambda(h_\alpha^\vee) \neq \lambda^*(h_\alpha^\vee)$, implicitly saying that it is 0 otherwise. We can just as well use ϵ_α for any simple $h_\alpha^\vee$ with $\alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee$, Lemma 2.5(a) ensures that all the computations from [50] remain valid. In particular we can now (re)define $N_{s_\alpha} \in \mathrm{End}_G(\Pi_{\mathfrak{s}})$ by

$$q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1 = (\mathcal{T}_{s_\alpha}\theta_{-\epsilon_\alpha \alpha} + 1)\ (\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha*} + 1)(\theta_{2\alpha} - 1)^{-1} \tag{2-19}$$

for any simple $h_\alpha^\vee$ with $\alpha^\sharp \in 2(T/T_{\mathrm{cpt}})^\vee$. The analogous formula when $\alpha^\sharp \notin 2(T/T_{\mathrm{cpt}})^\vee$ is slightly simpler:

$$q_F^{\lambda(h_\alpha^\vee)/2} N_{s_\alpha} + 1 = (\mathcal{T}_{s_\alpha} + 1)\ (\theta_\alpha q_\alpha - 1)(\theta_\alpha - 1)^{-1}. \tag{2-20}$$

Recall that the isomorphism in Theorem 1.2 was determined by the choice of a good maximal compact subgroup K of G, associated to a special vertex in the apartment for $\mathcal{T}$ in the Bruhat–Tits building of $(\mathcal{G}, F)$.

Lemma 2.6. *The good maximal compact subgroup K can be replaced by a G-conjugate, such that the isomorphism in Theorem 1.2 satisfies, for all simple roots $h_\alpha^\vee \in R_{\mathfrak{s}}^\vee$,*

$$\mathbb{1} \cdot \mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} = -\mathbb{1} \quad \textit{and} \quad \mathbb{1} \cdot N_{s_\alpha} = -q_F^{-\lambda(h_\alpha^\vee)/2} \mathbb{1}.$$

Proof. Recall the integers n_α from Proposition 2.4. We will tacitly put $\epsilon_\alpha = 0$ when $\alpha^\sharp \notin 2(T/T_{\mathrm{cpt}})^\vee$. Select y in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R_{\mathfrak{s}}, \mathbb{Z})$ so that $\langle y, \alpha^\sharp \rangle = n_\alpha - \epsilon_\alpha$ for every simple root $\alpha^\sharp \in R_{\mathfrak{s}}$. By Lemma 2.5(b), y can be extended to an element of $\mathrm{Hom}_{\mathbb{Z}}((T/T_{\mathrm{cpt}})^\vee, \mathbb{Z}) = T/T_{\mathrm{cpt}}$, which we still denote by y. The automorphism $\mathrm{Ad}(\theta_y)$ of $\mathcal{H}(\mathfrak{s})^\circ$ extends uniquely to an automorphism of $\mathcal{H}(\mathfrak{s})^\circ \otimes_{\mathcal{O}(T_{\mathfrak{s}})} \mathbb{C}(T_{\mathfrak{s}})$, which satisfies

$$\mathrm{Ad}(\theta_y)(\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha}) = \mathcal{T}_{s_\alpha} \theta_{s_\alpha(y) - y} \theta_{-\epsilon_\alpha} = \mathcal{T}_{s_\alpha} \theta_{(-n_\alpha)\alpha}. \tag{2-21}$$

By Proposition 2.4,

$$\mathbb{1} \cdot \mathrm{Ad}(\theta_y)(\mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha}) = -\mathbb{1}. \tag{2-22}$$

For any representative y_G of y in T, $K' = \mathrm{Ad}(y_G)K$ is another good maximal compact subgroup of G. If we replace K by K', we must replace the representatives $\tilde{w} \in K$ for $w \in W(\mathcal{G}, \mathcal{S})$, which are used in the constructions behind Theorem 1.2, by representatives in K'. Which choice in K' does not matter, we take

$$\mathrm{Ad}(y_G^{-1})\tilde{w} = \tilde{w} w^{-1}(y_G^{-1}) y_G \in K'.$$

For a simple root, that means

$$\mathrm{Ad}(y_G)\tilde{s}_\alpha T_{\mathrm{cpt}} = \tilde{s}_\alpha (h_\alpha^\vee)^{\langle y, \alpha^\sharp \rangle} \in N_G(T)/T_{\mathrm{cpt}}.$$

According to Proposition 3.1 of [21], the effect of this replacement on $\mathcal{T}_{s_\alpha}$ is left composition with $s_\alpha(\theta_\alpha^{-\langle y, \alpha^\sharp \rangle}) = \theta_\alpha^{\langle y, \alpha^\sharp \rangle}$ or equivalently right multiplication with $\theta_\alpha^{-\langle y, \alpha^\sharp \rangle}$. In view of (2-21), the effect of $\mathrm{Ad}(y_G^{-1})$ on $\mathcal{H}(\mathfrak{s})^\circ$ is precisely $\mathrm{Ad}(\theta_y)$.

By (2-22), the new element $\mathcal{T}'_{s_\alpha} = \mathcal{T}_{s_\alpha} \theta_{\langle y, \alpha^\sharp \rangle \alpha}$ has the same ϵ_α as before, and n_α has become ϵ_α. Now (2-22) says that $\mathbb{1} \cdot \mathcal{T}'_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} = -\mathbb{1}$. The equations (2-19) and (2-20) for the new elements N'_{s_α} become

$$q_F^{\lambda(h_\alpha^\vee)/2} N'_{s_\alpha} + 1 = (\mathcal{T}'_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} + 1)(\theta_\alpha q_\alpha - 1)(\theta_\alpha q_{\alpha *} + 1)(\theta_{2\alpha} - 1)^{-1}.$$

In view of (2-22), this implies

$$\mathbb{1} \cdot (q_F^{\lambda(h_\alpha^\vee)/2} N'_{s_\alpha} + 1) = 0 \in \mathbb{C}(T_{\mathfrak{s}}).$$

Equivalently, we obtain $\mathbb{1} \cdot N_{s_\alpha} = -q_F^{-\lambda(h_\alpha^\vee)/2} \mathbb{1}$. □

From now on we choose K as in the statement of Lemma 2.6. For $w \in W_{\mathfrak{s}}$ let $\det(w)$ be the determinant of the action of w on $(T/T_{\mathrm{cpt}}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\det : W_{\mathfrak{s}} \to \mathbb{R}^\times$ is a quadratic character extending the sign character of $W(R_{\mathfrak{s}}^\vee)$.

For $\gamma \in \Gamma_{\mathfrak{s}}$ we write $\mathbb{1} \cdot J_\gamma = z_\gamma \theta_{x_\gamma}$ with $z_\gamma \in \mathbb{C}^\times$ and $x_\gamma \in T/T_{\mathrm{cpt}}$. Consider the operators

$$N_\gamma = \det(\gamma) z_\gamma^{-1} \theta_{-x_\gamma} J_\gamma \in \mathrm{End}_G(\Pi_{\mathfrak{s}}).$$

From (2-9) we see that

$$\mathbb{1} \cdot N_\gamma = \det(\gamma) \mathbb{1}, \quad \gamma \in \Gamma_{\mathfrak{s}}. \tag{2-23}$$

Theorem 2.7. *The operators $N_w N_\gamma$, for $w \in W(R_{\mathfrak{s}}^\vee)$ with K as in Lemma 2.6 and N_γ with $\gamma \in \Gamma_{\mathfrak{s}}$ as above, provide an algebra isomorphism*

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) \cong \mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_{\mathfrak{s}} = \mathcal{H}(\mathfrak{s}).$$

Given the Whittaker datum (U, ξ), this isomorphism is canonical.

Proof. By direct computation, using Lemma 2.6,

$$\mathbb{1} \cdot \mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha} = -\det(\gamma) z_\gamma \theta_{s_\alpha(x_\gamma)}. \tag{2-24}$$

A similar computation shows that

$$\mathbb{1} \cdot \mathcal{T}_{s_{\gamma(\alpha)}} \theta_{\epsilon_\alpha \gamma(\alpha)} J_\gamma = -\mathbb{1} \cdot J_\gamma = -\det(\gamma) z_\gamma \theta_{x_\gamma}. \tag{2-25}$$

As $J_\gamma \mathcal{T}_{s_\alpha} \theta_{-\epsilon_\alpha \alpha}$ equals $\mathcal{T}_{s_{\gamma(\alpha)}} \theta_{\epsilon_\alpha \gamma(\alpha)} J_\gamma$, (2-24) and (2-25) are equal, and we deduce that $s_\alpha(x_\gamma) = x_\gamma$. Hence x_γ is fixed by each such s_α, and by the entire group $W(R_{\mathfrak{s}}^\vee)$. Now we can easily check that the N_γ satisfy these desired relations:

$$\begin{aligned}
\mathbb{1} \cdot N_{\gamma \tilde{\gamma}} &= \det(\gamma \tilde{\gamma}) \mathbb{1} = \det(\gamma) \det(\tilde{\gamma}) \mathbb{1} = \mathbb{1} \cdot N_\gamma N_{\tilde{\gamma}}, \\
N_\gamma f N_\gamma^{-1} &= J_\gamma f J_\gamma^{-1} = f \circ \gamma^{-1}, \quad f \in \mathcal{O}(T_{\mathfrak{s}}), \\
N_\gamma \mathcal{T}_{s_\alpha} N_\gamma^{-1} &= z_\gamma^{-1} \theta_{-x_\gamma} J_\gamma \mathcal{T}_{s_\alpha} J_\gamma^{-1} \theta_{x_\gamma} z_\gamma = \theta_{-x_\gamma} \mathcal{T}_{s_{\gamma\alpha}} \theta_{x_\gamma} = \mathcal{T}_{s_{\gamma\alpha}}.
\end{aligned}$$

The first two of these relations imply that

$$N_{\gamma \tilde{\gamma}} = N_\gamma N_{\tilde{\gamma}} \quad \text{for all } \gamma, \tilde{\gamma} \in \Gamma_{\mathfrak{s}}.$$

Thus with respect to the given $\mathcal{O}(T_{\mathfrak{s}})$-basis, $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ becomes $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma$.

Any two isomorphisms of this kind differ by an automorphism ψ of $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_{\mathfrak{s}}$. Since the subalgebra $\mathcal{O}(T_{\mathfrak{s}})$ is mapped naturally to $\mathrm{End}_G(\Pi_{\mathfrak{s}})$, ψ is the identity on that subalgebra. Hence ψ extends to an automorphism of the version of $\mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_{\mathfrak{s}}$ with $\mathbb{C}(T_{\mathfrak{s}})$. Then (2-9) entails that ψ multiplies each basis element $\mathcal{T}_w N_\gamma$ by an element of $\mathcal{O}(T_{\mathfrak{s}})$. Combining that with Lemma 2.6 and (2-23), we find that ψ is the identity. □

Theorem 2.7 shows in particular that the 2-cocycle $\natural_{\mathfrak{s}}$ from Theorem 1.2 becomes trivial in $H^2(\Gamma_{\mathfrak{s}}, \mathbb{C}^\times)$.

Recall from page 276 that $\mathfrak{s}$ can also arise from $w\mathfrak{s}_T = [T, w\chi_0]_T$ for any $w \in W(\mathcal{G}, \mathcal{S})$. To compare all these cases, it suffices to consider one w from every left coset of $W_{\mathfrak{s}} = \mathrm{Stab}_{W(\mathcal{G},\mathcal{S})}(\mathfrak{s}_T)$.

Proposition 2.8. *Let $w \in W(\mathcal{G}, \mathcal{S})$ be of minimal length in $wW_{\mathfrak{s}}$.*

(a) *The isomorphism $\Pi_{\mathfrak{s}} \cong \Pi_{w\mathfrak{s}}$ from* [39, §VI.10.1] *can be normalized so that it sends $\mathbb{1} \in \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi))$ to $\mathbb{1} \in \mathrm{Hom}_G(\Pi_{w\mathfrak{s}}, \mathrm{ind}_U^G(\xi))$.*

(b) *In that situation the induced algebra isomorphism $\mathrm{End}_G(\Pi_{\mathfrak{s}}) \cong \mathrm{End}_G(\Pi_{w\mathfrak{s}})$ is given by $f \mapsto f \circ w^{-1}$ for $f \in \mathcal{O}(T_{\mathfrak{s}})$ and $N_v \mapsto N_{wvw^{-1}}$ for $v \in W_{\mathfrak{s}}$.*

Proof. (a) The isomorphism of G-representations

$$\phi_w : \Pi_{w\mathfrak{s}} \xrightarrow{\sim} \Pi_{\mathfrak{s}}$$

from Proposition 1.1 induces a map

$$\mathcal{O}(T_{\mathfrak{s}}) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \to \mathrm{Hom}_G(\Pi_{w\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{w\mathfrak{s}}) \tag{2-26}$$

and a compatible algebra isomorphism

$$\mathrm{Ad}(\phi_w^{-1}) : \mathrm{End}_G(\Pi_{\mathfrak{s}}) \to \mathrm{End}_G(\Pi_{w\mathfrak{s}}).$$

In view of (2-4)–(2-6) and Proposition 1.1, (2-26) must be $f \mapsto f \circ w^{-1}$ followed by multiplication with some element of $\mathcal{O}(T_{w\mathfrak{s}})$.

Like in the proof of Proposition 1.1, we reduce to the case where $R(\mathcal{G}, \mathcal{S})$ has rank one, $w = s_\alpha$ is a simple reflection and $s_\alpha\mathfrak{s}_T \neq \mathfrak{s}_T$. We represent s_α in the maximal compact subgroup K from Lemma 2.6. Consider $\chi_c \in \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ and

$$\mathbb{1} \in \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathrm{Hom}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c), \mathrm{ind}_{\{e\}}^T(\mathrm{triv})).$$

Recall from (2-5) that here the isomorphism is given by $J_{\overline{B}}^G$ and the natural transformation $\mathrm{id} \to J_{\overline{B}}^G I_B^G$. By definition $J_{\overline{B}}^G(\mathbb{1})\chi_c = \chi_c$. We want to determine

$$J_{\overline{B}}^G(\mathbb{1}) J_{\overline{B}}^G(\phi_{s_\alpha})(s_\alpha\chi_c) \in J_{\overline{B}}^G\, \mathrm{ind}_U^G(\xi), \tag{2-27}$$

where $s_\alpha\chi$ is considered as an element of $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha\chi_c) \subset J_{\overline{B}}^G \Pi_{s_\alpha\mathfrak{s}}$. From (1-6) we see that $J_{\overline{B}}^G(\phi_{s_\alpha})$ on $\mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha\chi_c)$ equals

$$s_\alpha \circ [J_{\overline{B}}^G(\phi_{s_\alpha}) \text{ on } s_\alpha^{-1} \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(s_\alpha\chi_c)] \circ s_\alpha^{-1}.$$

Similarly $J_{\overline{B}}^G(\mathbb{1})$ on $s_\alpha \cdot \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$ equals

$$s_\alpha \circ [J_{\overline{B}}^G(\mathbb{1}) \text{ on } \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)] \circ s_\alpha^{-1}.$$

Now we can compute (2-27). First $s_\alpha \chi$ is mapped to χ_c by s_α^{-1}, then (1-8)–(1-10) show that $J_{\overline{B}}^G(\phi_{s_\alpha})$ sends that to $\chi_c \in \mathrm{ind}_{T_{\mathrm{cpt}}}^T(\chi_c)$. Applying $J_{\overline{B}}^G(\mathbb{1})$ returns $\chi_c \in \mathrm{ind}_{\{e\}}^T(\mathrm{triv})$ and finally the action of s_α yields $s_\alpha \chi_c$. Hence

$$J_{\overline{B}}^G(\mathbb{1}) J_{\overline{B}}^G(\phi_{s_\alpha})(s_\alpha \chi_c) = s_\alpha \chi_c = J_{\overline{B}}^G(\mathbb{1})(s_\alpha \chi_c),$$

where the second $\mathbb{1}$ comes from $\Pi_{s_\alpha \mathfrak{s}}$. In view of (2-5) and (2-6), that implies

$$J_{\overline{B}}^G(\mathbb{1}) J_{\overline{B}}^G(\phi_{s_\alpha}) = J_{\overline{B}}^G(\mathbb{1}) \quad \text{and} \quad \mathbb{1} \circ \phi_{s_\alpha} = \mathbb{1}.$$

(b) Since w has minimal length in $w W_{\mathfrak{s}}$, it also has minimal length in $w W(R_{\mathfrak{s},\mu})$. According to [6, Lemma 2.4(a)], $w(R^+_{\mathfrak{s},\mu}) \subset R(\mathcal{B}, \mathcal{S})$. By the $W(\mathcal{G}, \mathcal{S})$-equivariance of Harish-Chandra μ-functions also $w(R^+_{\mathfrak{s},\mu}) \subset R_{w\mathfrak{s},\mu}$, and therefore $w(R^+_{\mathfrak{s},\mu}) = R^+_{w\mathfrak{s},\mu}$.

The proof of (a) shows that (2-26) equals $f \mapsto f \circ w^{-1}$. With the rough description of the $\mathcal{H}(\mathfrak{s})$-action on $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi))$ from (2-9) we deduce that $\mathrm{Ad}(\phi_w^{-1})$ sends each $N_v \in \mathcal{H}(\mathfrak{s})$ to $N_{wvw^{-1}}$ times an element of $\mathcal{O}(T_{\mathfrak{s}^\vee})$. The more precise descriptions from Lemma 2.6 and (2-23) show that in fact $\mathrm{Ad}(\phi_w^{-1}) N_v$ equals $N_{wvw^{-1}}$ for any $v \in W_{\mathfrak{s}}$. □

3. Characterization of generic representations

We want to parametrize $\mathrm{Irr}(G, T)$ so that the generic representations correspond to the expected kind of enhanced L-parameters. To that end a simple characterization of genericity in terms of Hecke algebras will be indispensable.

We start with a complete description of $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi))$ as a right $\mathcal{H}(\mathfrak{s})$-module. Let $\mathcal{H}(W(R_{\mathfrak{s}}^\vee, q_F^\lambda) \subset \mathcal{H}(\mathfrak{s})^\circ$ be the finite-dimensional Iwahori–Hecke algebra spanned by the N_w with $w \in W(R_{\mathfrak{s}}^\vee)$. The Steinberg representation of $\mathcal{H}(W(R_{\mathfrak{s}}^\vee, q_F^\lambda))$ is defined by

$$\mathrm{St}(N_{s_\alpha}) = -q_F^{\lambda(h_\alpha^\vee)/2} \quad \text{for simple } h_\alpha^\vee \in R_{\mathfrak{s}}^\vee. \tag{3-1}$$

We extend this to a representation St of

$$\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda) := \mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak{s}}$$

by $\mathrm{St}(N_w N_\gamma) = \mathrm{St}(N_w)\det(\gamma)$. Note that this formula also defines a representation of the opposite algebra $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}$.

Special cases of the next result were established in [13] (for the Iwahori-spherical Bernstein component of a split group) and in [35] (for principal series representations of split reductive p-adic groups, with some extra conditions).

Lemma 3.1. *We have the following isomorphism of $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$-representations*:

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathrm{ind}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(\mathrm{St}).$$

Proof. Let $w \in W(R_{\mathfrak{s}}^\vee)$ and $\gamma \in \Gamma_{\mathfrak{s}}$. By Lemma 2.6 and (2-23),

$$\mathbb{1} \cdot N_w N_\gamma = \mathbb{1} \cdot \mathrm{St}(N_w N_\gamma) \in \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)).$$

As vector spaces,

$$\mathcal{H}(\mathfrak{s}) = \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda).$$

Further $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathcal{O}(T_{\mathfrak{s}})$ as $\mathcal{O}(T_{\mathfrak{s}})$-modules, with basis vector $\mathbb{1}$. Hence

$$\mathrm{ind}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(\mathrm{St}) \to \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)), \quad h \otimes 1 \mapsto h \cdot \mathbb{1}, \tag{3-2}$$

is an isomorphism of $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$-modules. □

The criterium for genericity, namely that (2-2) is nonzero, can be put in a more manageable form with Lemma 3.1. For any $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$,

$$\begin{aligned}
\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\xi)) &\cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \pi^\dagger) \\
&\cong \mathrm{Hom}_{\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}}\big(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)), \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)\big) \\
&\cong \mathrm{Hom}_{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}\big((\mathrm{ind}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(\mathrm{St})), \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)\big) \\
&\cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}(\mathrm{St}, \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)).
\end{aligned}$$

Corollary 3.2. *A representation $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ is (U, ξ)-generic if and only if the $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}$-module $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)$ contains* St.

In this corollary the effect of $\pi \mapsto \pi^\dagger$ on $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$-modules is not obvious, we analyze that in several steps. With the *-structure and the trace from [46, §3.1],

$$\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda) = \mathcal{H}(W(R_{\mathfrak{s}}^\vee), q_F^\lambda) \rtimes \Gamma_{\mathfrak{s}} \tag{3-3}$$

is a finite-dimensional Hilbert algebra, so it is, in particular, semisimple.

Recall that a standard G-representation is of the form $I_P^G(\tau \otimes \chi)$, where $P = MN$ is a parabolic subgroup of G, τ is an irreducible tempered M-representation and $\chi : M \to \mathbb{R}_{>0}$ is an unramified character in positive position with respect to P. By conjugating P and M, we may assume that $T \subset M$.

Lemma 3.3. *Suppose that $\tau \in \mathrm{Rep}(M)^{[T, \chi_0]_M}$. Then $I_P^G(\tau \otimes \chi) \in \mathrm{Rep}(G)^{\mathfrak{s}}$ and*

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, I_P^G(\tau \otimes \chi)^\dagger) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, I_P^G(\tau \otimes \chi)) \quad \textit{as } \mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}\textit{-modules.}$$

Proof. The representation $I_P^G(\tau \otimes \chi)$ has cuspidal support $\mathrm{Sc}(\tau) \otimes \chi|_T \in [T, \chi_0]_T$, so it belongs to $\mathrm{Rep}(G)^{[T, \chi_0]_G}$. By [39, IV.2.1.2],

$$I_P^G(\tau \otimes \chi)^\dagger \cong I_P^G((\tau \otimes \chi)^\dagger) \cong I_P^G(\tau^\dagger \otimes \chi^\dagger).$$

Since χ is real valued and τ is unitary [39, Corollaire VII.2.6], the right-hand side is isomorphic with $I_P^G(\tau \otimes \chi^{-1})$. Consider the continuous path

$$[-1, 1] \to \mathrm{Rep}(G)^{\mathfrak{s}}, \quad t \mapsto I_P^G(\tau \otimes \chi^t).$$

Via the equivalence of categories (1-1), we obtain a continuous path in $\mathrm{Mod}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}})$. Modules of a finite-dimensional semisimple algebra are stable under continuous deformations, so

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, I_P^G(\tau \otimes \chi)^\dagger) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, I_P^G(\tau \otimes \chi^{-1})) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, I_P^G(\tau \otimes \chi))$$

as $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}$-modules. □

We are ready to establish a useful characterization of genericity, without Hermitian duals. The next result is formulated for finite-length representations, but we believe it is also valid without that condition. To study it for arbitrary representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$ one likely needs Hermitian duals of modules over affine Hecke algebras.

Theorem 3.4. *Suppose that $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ has finite length. Then π is (U, ξ)-generic if and only if* $\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi), \mathrm{St})$ *is nonzero.*

Proof. Since π has finite length, we can form its semisimplification π_{ss}. Then $\pi_{ss}^\dagger$ is the semisimplification of $\pi^\dagger$. By (3-3) the module category of $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}$ is semisimple. In particular

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}(\mathrm{St}, \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)) \tag{3-4}$$

does not change if we replace $\pi^\dagger$ by $\pi_{ss}^\dagger$. Since we only need semisimplifications of modules here, we may pass to the Grothendieck group of finite-length representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$. The standard modules in $\mathrm{Rep}(G)^{\mathfrak{s}}$ form a $\mathbb{Z}$-basis of that Grothendieck group. Indeed, that is a consequence of the Langlands classification [39, Théorème VII.4.2] and the property that the irreducible quotient of a standard module is the unique maximal constituent in a certain sense [11, §XI.2].

For each such standard module we have Lemma 3.3, and hence the conclusion of Lemma 3.3 extends to the entire Grothendieck group of the finite-length part of $\mathrm{Rep}(G)^{\mathfrak{s}}$. In particular

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_{ss}^\dagger) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_{ss}) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$$

as $\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}$-modules. Hence the vector space (3-4) is isomorphic with

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}(\mathrm{St}, \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)).$$

By the semisimplicity of the involved algebra, this has the same dimension as

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^\lambda)^{\mathrm{op}}}(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi), \mathrm{St}). \tag{3-5}$$

We conclude by applying Corollary 3.2 to (3-4) and (3-5). □

From Theorem 3.4 it is easy to prove an analogue of the uniqueness (up to scalars) of Whittaker functionals [42; 44] in the context of Hecke algebras. Let M be a standard Levi subgroup of G and write $\mathfrak{s}_M = [T, \chi_0]_M$. Via parabolic induction $\mathcal{H}(\mathfrak{s}_M) \cong \mathrm{End}_M(\Pi_{\mathfrak{s}_M})$ becomes a subalgebra of $\mathcal{H}(\mathfrak{s}) \cong \mathrm{End}_G(\Pi_{\mathfrak{s}})$. In fact the constructions in [50, §10.2] show that $\mathcal{H}(\mathfrak{s}_M)$ is a parabolic subalgebra of $\mathcal{H}(\mathfrak{s})$, in the sense of [47, p. 216]. The functor $\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}$ corresponds to parabolic induction from M to G; see [41, Proposition 1.8.5.1].

Lemma 3.5. *Let V be an irreducible $\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}$-module. Then*

$$\dim \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}}(\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}} V, \mathrm{St}) \leq 1.$$

Proof. By the Bernstein presentation of $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$ we can simplify the module:

$$\mathrm{Res}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}} V) = \mathrm{ind}_{\mathcal{H}(W_{\mathfrak{s}_M}, q_F^{\lambda})^{\mathrm{op}}}^{\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}}(\mathrm{Res}_{\mathcal{H}(W_{\mathfrak{s}_M}, q_F^{\lambda})^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}} V).$$

With Frobenius reciprocity it follows that

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}}(\mathrm{ind}_{\mathcal{H}(\mathfrak{s}_M)^{\mathrm{op}}}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}} V, \mathrm{St}) \cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}_M}, q_F^{\lambda})^{\mathrm{op}}}(V, \mathrm{St}). \tag{3-6}$$

This reduces the lemma to the case $M = G$, which we investigate next.

As $\mathcal{H}(\mathfrak{s})$ has finite rank as a module over its center, V has finite dimension. Hence V contains an eigenvector for $\mathcal{O}(T_{\mathfrak{s}})$, say with character t. Then

$$0 \neq \mathrm{Hom}_{\mathcal{O}(T_{\mathfrak{s}})}(t, V) \cong \mathrm{Hom}_{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(t), V),$$

so V is a quotient of $\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(t)$. For multiplicities upon restriction to the finite-dimensional semisimple subalgebra $\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}$, that means

$$\dim \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}}(V, \mathrm{St}) \leq \dim \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}}(\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(t), \mathrm{St}). \tag{3-7}$$

By the presentation of $\mathcal{H}(\mathfrak{s})$, $\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(\mathfrak{s})^{\mathrm{op}}}(t) \cong \mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}$ as $\mathcal{H}(W_{\mathfrak{s}}, q_F^{\lambda})^{\mathrm{op}}$-modules. Hence the right-hand side of (3-7) is 1. □

4. Hecke algebras for principal series L-parameters

Recall that we fixed a separable closure F_s of F. Let $\boldsymbol{I}_F \subset \boldsymbol{W}_F \subset \mathrm{Gal}(F_s/F)$ be the inertia subgroup of the Weil group and pick a geometric Frobenius element Frob_F of $\boldsymbol{W}_F$. Let $G^\vee$ be the complex dual group of G and let ${}^L G = G^\vee \rtimes \boldsymbol{W}_F$ be the Langlands dual group. Let $\Phi(G)$ be the set of L-parameters $\phi : \boldsymbol{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$, considered modulo $G^\vee$-conjugacy.

For an L-parameter ϕ we have the component group

$$R_\phi = \pi_0(Z_{G^\vee}(\phi)/Z(G^\vee)^{\boldsymbol{W}_F}),$$

this is the appropriate version because G is quasi-split. We define a (G-relevant) enhancement of ϕ to be an irreducible representation of the finite group R_ϕ. Compared to [3], the quasi-splitness of G allows us to focus on the enhancements whose $Z(G^\vee_{\mathrm{sc}})$-character is trivial, and that eliminates the need to consider the centralizer of ϕ in $G^\vee_{\mathrm{sc}}$. We denote the set of $G^\vee$-conjugacy classes of enhanced L-parameters for G by $\Phi_e(G)$.

Recall [3] that there exists a notion of cuspidality and a cuspidal support map Sc for enhanced L-parameters. The map Sc associates to each $(\phi, \rho) \in \Phi_e(G)$ a F-Levi subgroup L of G and a cuspidal enhanced L-parameter for L (unique up to $G^\vee$-conjugation). We say that (ϕ, ρ) is a principal series L-parameter if $\mathrm{Sc}(\phi, \rho)$ is an enhanced L-parameter for T (or a G-conjugate of T). In that case $\mathrm{Sc}(\phi, \rho)$ is unique up to $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)$-conjugacy. In other words, $\mathrm{Sc}(\phi, \rho)$ as an element of $\Phi_e(T)$ is unique up to conjugacy by $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)/T^\vee$.

For the maximal torus T, the dual group $T^\vee$ is a complex torus. In particular any L-parameter for T is trivial on $\mathrm{SL}_2(\mathbb{C})$ and has trivial component group. Hence an element of $\Phi_e(T)$ is just the $T^\vee$-conjugacy class of a homomorphism $\hat{\chi} : \boldsymbol{W}_F \to {}^L T$. Every element of $\Phi_e(T)$ is cuspidal, because T has no proper Levi subgroups.

To describe principal series (enhanced) L-parameters more explicitly, we consider an arbitrary $(\phi, \rho) \in \Phi_e(G)$. We want to determine $\mathrm{Sc}(\phi, \rho) = (L, \psi, \epsilon)$. By construction

$$\psi|_{\boldsymbol{I}_F} = \phi|_{\boldsymbol{I}_F} \quad \text{and} \quad \psi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big) = \phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big). \tag{4-1}$$

In order that $L = T$, it is necessary that $\phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big) \in T^\vee \mathrm{Frob}_F$ and $\phi(i) \in T^\vee i$ for all $i \in \boldsymbol{I}_F$. The group

$$H^\vee := Z_{G^\vee}(\phi(\boldsymbol{W}_F))$$

is reductive [53, (4.4.f)] and

$$R_\phi = \pi_0(Z_{G^\vee}(\phi)/Z(G^\vee)^{\boldsymbol{W}_F}) \quad \text{is equal to} \quad \pi_0\big(Z_{H^\vee}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big)/Z(G^\vee)^{\boldsymbol{W}_F}\big).$$

This group is a quotient of

$$\pi_0\big(Z_{H^\vee}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big)\big) \cong \pi_0(Z_{H^\vee}(u_\phi)),$$

where $u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$. Thus we can regard ρ as an irreducible representation of $\pi_0(Z_{H^\vee}(u_\phi))$. Let $(M^\vee, u_\psi, \epsilon)$ be the cuspidal quasisupport of (u_ϕ, ρ) for $H^\vee$, as in [3, §5]. Then ψ is the L-parameter determined (up to conjugacy) by (4-1) and u_ψ, while ϵ is as above and $L^\vee = Z_{H^\vee}(Z(M^\vee)^\circ)$.

For $L^\vee = T^\vee$ we need $M^\vee = T^\vee$, which implies that $u_\psi = 1$. There is an explicit criterium for $\mathrm{Sc}(u_\phi, \rho) = (T^\vee, 1, \epsilon)$ with arbitrary ϵ, as follows. Let $\mathcal{B}^{u_\phi}_{H^\vee}$ be the variety of Borel subgroups of $H^{\vee,\circ}$ that contain u_ϕ, it carries a natural action of $Z_{H^\vee}(u_\phi)$. As $Z_{H^{\vee,\circ}}(u_\phi)$ is a union of connected components of $Z_{H^\vee}(u_\phi)$, its component group $\pi_0(Z_{H^{\vee,\circ}}(u_\phi))$ is a subgroup of $\pi_0(Z_{H^\vee}(u_\phi))$. Let ρ° be any irreducible constituent of $\rho|_{\pi_0(Z_{H^{\vee,\circ}}(u_\phi))}$. Then the criterium says: ρ° appears in the action of $\pi_0(Z_{H^{\vee,\circ}}(u_\phi))$ on the (top-degree) homology of $\mathcal{B}^{u_\phi}_{H^\vee}$.

Summarizing, we found the following necessary and sufficient conditions for $(\phi, \rho) \in \Phi_e(G)$ to be a principal series enhanced L-parameter:

(i) $\phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big), \phi(i) \in T^\vee \rtimes \boldsymbol{W}_F$ for any $i \in \boldsymbol{I}_F$;

(ii) ρ° appears in $H_*(\mathcal{B}^{u_\phi}_{H^\vee})$ where $H^\vee = Z_{G^\vee}(\phi(\boldsymbol{W}_F))$.

We note that under these conditions $\mathrm{Sc}(\phi, \rho)$ does not depend on u_ϕ or ρ. Moreover it equals $\mathrm{Sc}(\phi, \mathrm{triv})$, because $H^{\mathrm{top}}(\mathcal{B}^{u_\phi}_{H^\vee})$ is a permutation representation of R_ϕ (with as permuted objects the irreducible components of $\mathcal{B}^{u_\phi}_{H^\vee}$), and that always contains the trivial representation. With this in mind, we call $\phi \in \Phi(G)$ a principal series L-parameter if (i) holds.

Recall from [20, §3.3.1] that there is a natural isomorphism

$$X_{\mathrm{nr}}(G) \cong (Z(G^\vee)^{\boldsymbol{I}_F})^\circ_{\mathrm{Frob}}, \tag{4-2}$$

and that the group of unramified characters $X_{\mathrm{nr}}(T)$ is naturally isomorphic to the group $(T^{\vee,\boldsymbol{I}_F})_{\boldsymbol{W}_F}{}^\circ$. We will sometimes identify these groups and write simply $X_{\mathrm{nr}}(T)$. We note that $(T^{\vee,\boldsymbol{I}_F})_{\boldsymbol{W}_F}$ acts on $\Phi(T)$ by

$$(z\hat{\chi})|_{\boldsymbol{I}_F} = \hat{\chi}|_{\boldsymbol{I}_F}, \quad (z\hat{\chi})(\mathrm{Frob}_F) = z(\hat{\chi}(\mathrm{Frob}_F))$$

for $z \in T^{\vee,\boldsymbol{I}_F}$ and $\hat{\chi} \in \Phi(T)$. A Bernstein component of $\Phi_e(T) = \Phi(T)$ is by definition one $X_{\mathrm{nr}}(T)$-orbit in $\Phi(T)$. We will usually write this as $\mathfrak{s}^\vee_T = X_{\mathrm{nr}}(T)\hat{\chi}$ for one $\hat{\chi} \in \Phi(T)$. It gives rise to a Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee} := \mathrm{Sc}^{-1}(T, \mathfrak{s}^\vee_T)$ in the principal series part of $\Phi_e(G)$.

Next we make the extended affine Hecke algebra $\mathcal{H}(\mathfrak{s}^\vee, z)$ from [6] explicit. The maximal commutative subalgebra of $\mathcal{H}(\mathfrak{s}^\vee, z)$ is $\mathcal{O}(\mathfrak{s}^\vee) \otimes \mathbb{C}[z, z^{-1}]$, where z is a formal variable. In this context we prefer to write $T_{\mathfrak{s}^\vee}$ for $\mathfrak{s}^\vee$, to emphasize that it is a complex torus (as a variety, in general it does not have a canonical multiplication).

The group $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)/T^\vee$ acts naturally on $\Phi(T)$, by conjugation. Let $W_{\mathfrak{s}^\vee}$ denote the stabilizer of $\mathfrak{s}^\vee$ in $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)/T^\vee$. Although $W_{\mathfrak{s}^\vee}$ need not be a Weyl group, it always contains the Weyl group of a root system. Namely, consider the group $J = Z_{G^\vee}(\hat{\chi}(\boldsymbol{I}_F))$, with the torus $(T^{\vee,\boldsymbol{W}_F})^\circ$ and the maximal torus $T^\vee$. According to [6, Proposition 3.9(a) and (3.22)], $R(J^\circ, (T^{\vee,\boldsymbol{W}_F})^\circ)$ is a root system and $W_{\mathfrak{s}^\vee}$ acts naturally on it. Moreover [6, Proposition 3.9(b)] says that for a suitable choice of $\hat{\chi}$ in $T_{\mathfrak{s}^\vee}$ the set of indivisible roots

$$R(J^\circ, (T^{\vee,\boldsymbol{W}_F})^\circ)_{\mathrm{red}} \text{ equals } R\big(Z_{G^\vee}(\hat{\chi}(\boldsymbol{W}_F))^\circ, (T^{\vee,\boldsymbol{W}_F})^\circ\big)_{\mathrm{red}}.$$

For such a choice of a basepoint $\hat{\chi}$ of $T_{\mathfrak{s}^\vee}$,

$$W_{\mathfrak{s}^\vee}^\circ := W\big(R(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ)\big)$$

is a normal subgroup of $W_{\mathfrak{s}}$. Let $R^+(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ)$ be the positive root system determined by the Borel subgroup $B^\vee$ of $G^\vee$. Then $W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee}^\circ \rtimes \Gamma_{\mathfrak{s}^\vee}$, where $\Gamma_{\mathfrak{s}^\vee}$ denotes the stabilizer of $R^+(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ)$ in $W_{\mathfrak{s}^\vee}$.

The root system for $\mathcal{H}(\mathfrak{s}^\vee, z)$ will essentially be $R(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ)$, but we still need to rescale the elements [6, §3.2]. We note that the inclusion $(T^{\vee, \boldsymbol{W}_F})^\circ \to T^\vee$ induces a surjection

$$\mathrm{pr} : R(J^\circ, T^\vee) \cup \{0\} \to R(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ) \cup \{0\}.$$

In [6, Definition 3.11], positive integers m_α for $\alpha^\vee \in R(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ)_{\mathrm{red}}$ are defined as follows:

- Suppose that $\mathrm{pr}^{-1}(\{\alpha^\vee\})$ meets $k > 1$ connected components of $R(J^\circ, T^\vee)$. These k components are permuted transitively by Frob_F. Then m_α equals k times the analogous number m'_α obtained by replacing F by its degree-k unramified extension (or equivalently replacing Frob_F by Frob_F^k).
- Suppose that $\mathrm{pr}^{-1}(\{\alpha^\vee\})$ lies in a single connected component of $R(J^\circ, T^\vee)$. Then m_α is the smallest natural number such that $\ker(m_\alpha \alpha^\vee)$ contains the kernel of the canonical surjection

$$(T^{\vee, \boldsymbol{W}_F})^\circ \to (T^{\vee, \boldsymbol{I}_F})_{\boldsymbol{W}_F}{}^\circ \cong X_{\mathrm{nr}}(T). \tag{4-3}$$

In fact it is easy to identify the kernel of (4-3) as

$$(T^{\vee, \boldsymbol{W}_F})^\circ_{\mathrm{Frob}_F} := (T^{\vee, \boldsymbol{W}_F})^\circ \cap (1 - \mathrm{Frob}_F) T^{\vee, \boldsymbol{I}_F}.$$

Lemma 4.1. *The number m_α equals $f(F_\alpha / F)$, where $\boldsymbol{W}_{F_\alpha}$ is the $\boldsymbol{W}_F$-stabilizer of a lift of $\alpha^\vee$ to $R(G^\vee, T^\vee)$.*

Proof. The number m_α can be related to the structure of the F-group $\mathcal{G}$. Let $\mathcal{G}_\alpha$ be the F-simple almost direct factor of $\mathcal{G}$ such that $\mathrm{pr}^{-1}(\{\alpha^\vee\})$ consists of roots coming from $G_\alpha^\vee$. Write $\mathcal{G}_\alpha = \mathrm{Res}_{E_\alpha/F} \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha$ is absolutely simple. The injection $\mathcal{G}_\alpha \to \mathcal{G}$ induces a surjection ${}^L G \to {}^L G_\alpha$ which does not change $\alpha^\vee((T^{\vee, \boldsymbol{W}_F})^\circ_{\mathrm{Frob}_F})$. Knowing that, the first bullet above says that m_α equals $f(E_\alpha / F)$ times the number m'_α for $\mathcal{H}_\alpha(E_\alpha)$.

Let $\mathcal{T}_\alpha$ be the maximal torus of $\mathcal{H}_\alpha$ with $\mathrm{Res}_{E_\alpha/F} \mathcal{T}_\alpha = \mathcal{T} \cap \mathcal{G}_\alpha$. The Weil group $\boldsymbol{W}_{E_\alpha}$ acts on the irreducible root system $R(\mathcal{H}_\alpha^\vee, \mathcal{T}_\alpha^\vee)$, and the set of orbits is in bijection with the irreducible component of $R(J^\circ, (T^{\vee, \boldsymbol{W}_F})^\circ)$ containing $\alpha^\vee$. Let $\boldsymbol{W}_{F_\alpha}$ be the $\boldsymbol{W}_{E_\alpha}$-stabilizer of an element $\alpha'^\vee \in R(\mathcal{H}_\alpha^\vee, \mathcal{T}_\alpha^\vee)$ that corresponds to $\alpha^\vee$. Then $\alpha^\vee = \alpha'^\vee|_{T_\alpha^{\boldsymbol{W}_{F_\alpha}}}$.

Suppose that the elements of $\boldsymbol{W}_{E_\alpha}\alpha'^\vee$ are mutually orthogonal, which happens in most cases. From the definitions we see that

$$|\alpha^\vee((T^{\vee,\boldsymbol{W}_F})^\circ_{\mathrm{Frob}_F})| = f(F_\alpha/E_\alpha).$$

Here the relevant elements of $(T^{\vee,\boldsymbol{W}_F})^\circ_{\mathrm{Frob}_F}$ are the powers of

$$(1-\mathrm{Frob}_F)t, \quad \text{where } (\mathrm{Frob}^n\alpha'^\vee)(t) = \exp\Big(\frac{2\pi i n}{f(F_\alpha/E_\alpha)}\Big).$$

We find $m'_\alpha = f(F_\alpha/E_\alpha)$ and $m_\alpha = f(F_\alpha/F)$.

Next we consider the cases where the elements of $\boldsymbol{W}_{E_\alpha}\alpha'^\vee$ are not mutually orthogonal. Classification shows that $R(\mathcal{H}^\vee, \mathcal{T}_\alpha^\vee)$ has type ${}^2A_{2n}$ and

$$|\boldsymbol{W}_{E_\alpha}\alpha'^\vee| = [F_\alpha : E_\alpha] = 2,$$

so that $H_{\alpha,\mathrm{ad}} \cong \mathrm{PU}_{2n+1}(F_\alpha/E_\alpha)$. Direct computations show that, when F_α/E_α is ramified, $m'_\alpha = 1$ and $m_\alpha = f(E_\alpha/F) = f(F_\alpha/F)$. Similarly, when F_α/E_α is unramified, $m'_\alpha = 2$ and $m_\alpha = 2f(E_\alpha/F) = f(F_\alpha/F)$. □

Lemma 4.1 and [6, Lemma 3.12] yield the precise definition of the root system for $\mathcal{H}(\mathfrak{s}^\vee, z)$:

$$R_{\mathfrak{s}^\vee} = \{m_\alpha\alpha^\vee : \alpha^\vee \in R(J^\circ, (T^{\vee,\boldsymbol{W}_F})^\circ)_{\mathrm{red}}\}.$$

This root system is endowed with an action of $W_{\mathfrak{s}^\vee}$. Hence $W_{\mathfrak{s}^\vee}$ also acts on the resulting root datum from [6, §3.2]:

$$\begin{aligned}\mathcal{R}_{\mathfrak{s}^\vee} &= \big(R_{\mathfrak{s}^\vee}, X^*((T^{\vee,\boldsymbol{I}_F})^\circ_{\boldsymbol{W}_F}), R^\vee_{\mathfrak{s}^\vee}, X_*((T^{\vee,\boldsymbol{I}_F})^\circ_{\boldsymbol{W}_F})\big)\\ &= (R_{\mathfrak{s}^\vee}, T/T_{\mathrm{cpt}}, R^\vee_{\mathfrak{s}^\vee}, (T/T_{\mathrm{cpt}})^\vee).\end{aligned}$$

The label functions λ, λ^* for $\mathcal{H}(\mathfrak{s}^\vee, z)$ are determined in [6, Proposition 3.14]. Suppose first that the elements of $\boldsymbol{W}_{E_\alpha}\alpha'^\vee$ are mutually orthogonal (in the notation from the proof of Lemma 4.1), and that the same holds for $\alpha^\vee/2$ whenever $\alpha^\vee/2$ can be lifted to $R(G^\vee, (T^{\vee,\boldsymbol{W}_F})^\circ)$. In these nonexceptional cases

$$\lambda(m_\alpha\alpha^\vee) = \lambda^*(m_\alpha\alpha^\vee) = m_\alpha = f(F_\alpha/F). \tag{4-4}$$

If in addition $m_\alpha\alpha^\vee \in 2X^*((T^{\vee,\boldsymbol{I}_F})^\circ_{\boldsymbol{W}_F}) = 2(T/T_{\mathrm{cpt}})$, then we can get the same Hecke algebras with $m_\alpha\alpha^\vee/2$ instead of $m_\alpha\alpha^\vee$, and

$$\lambda(m_\alpha\alpha^\vee/2) = m_\alpha = f(F_\alpha/F),\ \lambda^*(m_\alpha\alpha^\vee/2) = 0. \tag{4-5}$$

We call the remaining cases exceptional, these occur only when $R(G^\vee, T^\vee)$ has a component of type ${}^2A_{2n}$ and $\alpha^\vee$ or $\alpha^\vee/2$ comes from two nonorthogonal roots that are exchanged by the diagram automorphism. As noted in the proof of Lemma 4.1,

$$H_{\alpha,\mathrm{ad}} \cong \mathrm{PU}_{2n+1}(F_\alpha/E_\alpha).$$

The groups $\mathrm{PU}_{2n+1}(F_\alpha/E_\alpha)$, $\mathrm{SU}_{2n+1}(F_\alpha/E_\alpha)$ and $U_{2n+1}(F_\alpha/E_\alpha)$ give the same root system, the same unramified characters and the same groups $(T^{\vee,\boldsymbol{W}_F})^\circ$. Hence the relevant data for H_α can be reduced (via its derived group) to those for $U_{2n+1}(F_\alpha/E_\alpha)$, and it suffices to continue the analysis in the latter group.

For $U_{2n+1}(F_\alpha/E_\alpha)$ all the labels were computed in [5, §5]. For convenience we provide an overview, where we remark that the labels from [5] still have to be multiplied by $f(E_\alpha/F)$ to account for the restriction of scalars $\mathcal{G}_\alpha(F) = \mathcal{H}_\alpha(E_\alpha)$, as in the proof of Lemma 4.1. We write $\alpha^\vee = \alpha'^\vee + \alpha''^\vee$, where $\alpha'^\vee$ and $\alpha''^\vee$ are nonorthogonal roots in A_{2n} exchanged by the diagram automorphism.

- Suppose F_α/E_α is unramified and $\hat{\chi}(\boldsymbol{W}_{E_\alpha}) \subset Z(\mathrm{GL}_{2n+1}(\mathbb{C})) \rtimes \boldsymbol{W}_{E_\alpha}$. Then $\lambda(\alpha^\vee) = 3$ and $\lambda^*(\alpha^\vee) = 1$.
- Suppose F_α/E_α is unramified and $\hat{\chi}(\boldsymbol{W}_{E_\alpha}) \not\subset Z(\mathrm{GL}_{2n+1}(\mathbb{C})) \rtimes \boldsymbol{W}_{E_\alpha}$. Here we need $\hat{\chi}(\boldsymbol{W}_{E_\alpha})$ to fix $U_{\alpha^\vee}$ pointwise for $\alpha^\vee \in R_{\mathfrak{s}^\vee}$. Under that condition $\lambda(\alpha^\vee) = \lambda^*(\alpha^\vee) = 1$.
- Suppose F_α/E_α is ramified. When $\hat{\chi} \circ \alpha^\vee : F_\alpha \to \mathbb{C}^\times$ is conjugate orthogonal, $\alpha^\vee \notin R_{\mathfrak{s}^\vee}$. Otherwise $\hat{\chi} \circ \alpha^\vee$ is conjugate symplectic, in which case $\alpha^\vee \in R_{\mathfrak{s}^\vee}$ and $\lambda(\alpha^\vee) = \lambda^*(\alpha^\vee) = 1$. Equivalently, using $\alpha^\vee/2$ as root,

$$\lambda(\alpha^\vee/2) = 1, \quad \lambda^*(\alpha^\vee/2) = 0. \tag{4-6}$$

The algebra $\mathcal{H}(\mathfrak{s}^\vee, z)$ has a subalgebra $\mathcal{H}(\mathfrak{s}^\vee, z)^\circ$, whose underlying vector space is

$$\mathcal{O}(T_{\mathfrak{s}^\vee}) \otimes \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[W^\circ_{\mathfrak{s}^\vee}].$$

It is isomorphic to the affine Hecke algebra $\mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, z)$, for suitable label functions λ, λ^*. The identification of the vector spaces comes from the elements $N_w \in \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, z)$ and the bijection

$$(T^{\boldsymbol{I}_F})_{\boldsymbol{W}_F}{}^\circ \to T_{\mathfrak{s}^\vee}, \quad t \mapsto t\hat{\chi}. \tag{4-7}$$

Theorem 4.2. *There is a canonical algebra isomorphism*

$$\mathcal{H}(\mathfrak{s}^\vee, z) \cong \mathcal{H}(\mathfrak{s}^\vee, z)^\circ \rtimes \Gamma_{\mathfrak{s}^\vee}.$$

Proof. By design $\mathcal{H}(\mathfrak{s}^\vee, z)$ is free as an $\mathcal{H}(\mathfrak{s}^\vee, z)^\circ$-module, with a basis indexed by $\Gamma_{\mathfrak{s}^\vee}$. More precisely, by [6, Proposition 3.15(a)] the actions of $\Gamma_{\mathfrak{s}^\vee}$ on $R_{\mathfrak{s}^\vee}$, $T_{\mathfrak{s}^\vee}$ and $\mathcal{O}(T_{\mathfrak{s}^\vee})$ naturally induce an action of $\Gamma_{\mathfrak{s}^\vee}$ on $\mathcal{H}(\mathfrak{s}^\vee, z)^\circ$. For every $\gamma \in \Gamma_{\mathfrak{s}^\vee}$, that yields an element of $\mathcal{H}(\mathfrak{s}^\vee, z)$, unique up to scaling.

For the Langlands parameters under consideration, the sheaf $q\mathcal{E}$ from [3; 6] is just the constant sheaf with stalk $\mathbb{C}$ on the point $1 \in T^\vee$. It follows that there is a canonical choice for the map qb_γ from [6, (3.39)], namely the identity. Then $\gamma \mapsto qb_\gamma$ is multiplicative, the scalars $\lambda_{\gamma,\gamma'}$ in the proof of [6, Proposition 3.15(b)] reduce to 1 and $\mathbb{C}[\Gamma_{\mathfrak{s}^\vee}]$ embeds in $\mathcal{H}(\mathfrak{s}^\vee, z)$ as the span of these qb_γ. With this in place, [6, Proposition 3.15(a)] provides the desired statement. □

Next we specialize z to $q_F^{1/2}$, which yields the algebra

$$\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) = \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^\circ \rtimes \Gamma_{\mathfrak{s}^\vee} \cong \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}^\vee}. \tag{4-8}$$

We note that here the isomorphism depends on the choice of the basepoint $\hat{\chi}$ of $T_{\mathfrak{s}^\vee}$. From (4-8) we see that the center of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is

$$Z(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})) = \mathcal{O}(T_{\mathfrak{s}^\vee})^{W_{\mathfrak{s}}} = \mathcal{O}(T_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee}).$$

The main use of the algebras (4-8) lies in the following result.

Theorem 4.3 [6, Theorem 3.18]. *There exists a canonical bijection*

$$\Phi_e(G)^{\mathfrak{s}^\vee} \to \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})), \quad (\phi, \rho) \mapsto \overline{M}(\phi, \rho, q_F^{1/2}),$$

such that:

(a) *$\overline{M}(\phi, \rho, q_F^{1/2})$ admits the central character $W_{\mathfrak{s}^\vee}\tilde{\phi} \in T_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee}$, where $\tilde{\phi}|_{\boldsymbol{I}_F} = \phi|_{\boldsymbol{I}_F}$ and*

$$\tilde{\phi}(\mathrm{Frob}_F) = \phi\Big(\mathrm{Frob}_F, \Big(\begin{smallmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{smallmatrix}\Big)\Big).$$

(b) *ϕ is bounded if and only if $\overline{M}(\phi, \rho, q_F^{1/2})$ is tempered.*

(c) *ϕ is discrete if and only if $\overline{M}(\phi, \rho, q_F^{1/2})$ is essentially discrete series and the rank of $R_{\mathfrak{s}^\vee}$ equals the F-split rank of $\mathcal{T}/Z(\mathcal{G})$.*

(d) *The bijection is equivariant for the canonical actions of $Z(G^\vee)^{\boldsymbol{I}_F} \cap (T^{\vee,\boldsymbol{I}_F})^\circ$.*

We note that in [6] the canonicity is obtained in a slightly weaker sense, by interpreting the subalgebra of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ spanned by the N_γ with $\gamma \in \Gamma_{\mathfrak{s}^\vee}$ as the endomorphism algebra of a certain perverse sheaf [6, (2.5)]. We got rid of that subtlety in the proof of Theorem 4.2.

For (d) we recall that any element $t \in Z(G^\vee)^{\boldsymbol{I}_F}$ determines a weakly unramified character of G [20, §3.3.1], and that character is trivial on T_{cpt} if and only if $t \in (T^{\vee,\boldsymbol{I}_F})^\circ$. To $t \in Z(G^\vee)^{\boldsymbol{I}_F} \cap (T^{\vee,\boldsymbol{I}_F})^\circ$ we associate the automorphism

$$xN_w \mapsto x(t)xN_w, \quad x \in T/T_{\mathrm{cpt}},\ w \in W_{\mathfrak{s}^\vee},$$

of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, where x is regarded as function on $T_{\mathfrak{s}^\vee}$ via (4-7). The action of t on $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))$ is composition with the above automorphism.

5. Comparison of Hecke algebras

We start with a Bernstein component $\mathfrak{s}_T$ for T. Recall that this is just a $X_{\mathrm{nr}}(T)$-coset in $\mathrm{Irr}(T)$. The local Langlands correspondence for tori [28; 54] associates to $\mathfrak{s}_T$ a $X_{\mathrm{nr}}(T)$-orbit in $\Phi(T)$, that is, one Bernstein component $\mathfrak{s}_T^\vee$ in $\Phi_e(T)$. Let $W_{\mathfrak{s}}$ be the stabilizer of $\mathfrak{s}_T$ in $N_G(T)/T$ and let $W_{\mathfrak{s}^\vee}$ be the stabilizer of $\mathfrak{s}_T^\vee$ in $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)/T^\vee$.

Lemma 5.1. *There is a natural isomorphism $W_{\mathfrak{s}} \cong W_{\mathfrak{s}^\vee}$.*

Proof. The root datum

$$(R(\mathcal{G}, \mathcal{T}), X^*(\mathcal{T}), R(G^\vee, T^\vee), X^*(T^\vee))$$

comes with a natural group isomorphism

$$W(\mathcal{G}, \mathcal{T}) \cong W(G^\vee, T^\vee). \tag{5-1}$$

From [1, Proposition 3.1 and its proof] we know that (5-1) restricts to an isomorphism

$$N_G(T)/T \cong N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)/T^\vee = W(G^\vee, T^\vee)^{\boldsymbol{W}_F}. \tag{5-2}$$

The LLC for tori is natural, so compatible with isomorphisms of

$$(X^*(T), X_*(T)) = (X_*(T^\vee), X^*(T^\vee)).$$

In particular it is equivariant for the action of (5-1), and hence for the action of (5-2). Thus the action of $N_G(T)/T$ on $\mathrm{Irr}(T)$ is turned into the conjugation action of $W(G^\vee, T^\vee)^{\boldsymbol{W}_F}$ by the LLC. In particular (5-2) matches $\mathrm{Stab}_{N_G(T)/T}(\mathfrak{s}_T)$ with the $W(G^\vee, T^\vee)^{\boldsymbol{W}_F}$-stabilizer of the image $\mathfrak{s}_T^\vee$ of $\mathfrak{s}_T$ in $\Phi_e(T)$. □

Similarly we can compare root systems on both sides of the LLC.

Lemma 5.2. *There exists a natural bijection between $R_{\mathfrak{s}}^\vee$ and $R_{\mathfrak{s}^\vee}$, which preserves positivity.*

Proof. Pick any $\chi \in \mathfrak{s}_T$.

By construction $R_{\mathfrak{s}}^\vee$ consists of positive multiples of the $\alpha^\vee \in R(\mathcal{G}, \mathcal{S})^\vee$ for which $\alpha \in R_{\mathfrak{s},\mu}$. Similarly $R_{\mathfrak{s}^\vee}$ consists of positive multiples of the $\alpha^\vee$ in

$$R\big(Z_{G^\vee}(\hat{\chi}(\boldsymbol{I}_F)), T^{\vee,\boldsymbol{W}_F,\circ}\big)_{\mathrm{red}} \subset R(G^\vee, T^{\vee,\boldsymbol{W}_F,\circ})_{\mathrm{red}} \cong R(G^\vee, S^\vee)_{\mathrm{red}} \cong R(\mathcal{G}, \mathcal{S})^\vee_{\mathrm{red}}$$

for which $\hat{\chi}(\boldsymbol{I}_F)$ fixes $U_{\alpha^\vee}$ or $U_{2\alpha^\vee}$ in $Z_{G^\vee}(\hat{\chi}(\boldsymbol{I}_F))$. Since both $R_{\mathfrak{s}}^\vee$ and $R_{\mathfrak{s}^\vee}$ are reduced root systems, this means that there exists at most one bijection $R_{\mathfrak{s}}^\vee \to R_{\mathfrak{s}^\vee}$ which scales each root by a positive real number. Positivity in $R_{\mathfrak{s}}^\vee$ is determined by $\mathcal{B}$ and positivity in $R_{\mathfrak{s}^\vee}$ is determined by $B^\vee$, so such a bijection would automatically preserve positivity of roots.

It remains to check that for $R_{\mathfrak s}^\vee$ and $R_{\mathfrak s^\vee}$ the same elements of $R(\mathcal G, \mathcal S)_{\mathrm{red}}^\vee$ are relevant. For the nonexceptional roots we know from (1-13)–(1-14) that $\alpha \in \Sigma_{\mathfrak s,\mu}$ if and only if $\chi \circ \alpha^\vee : F_\alpha^\times \to \mathbb C^\times$ is unramified. Via the LLC for tori that becomes

$$\alpha^\vee \circ \hat\chi : \boldsymbol W_{F_\alpha} \to \mathbb C \rtimes \boldsymbol W_{F_\alpha} \text{ restricts to the identity on } \boldsymbol I_{F_\alpha}.$$

In this setting the roots in the associated $\boldsymbol W_F$-orbit in $R(G^\vee, T^\vee)$ are mutually orthogonal, permuted by $\boldsymbol W_F$ and fixed by $\boldsymbol W_{F_\alpha}$. Hence $\alpha^\vee \circ \hat\chi(\boldsymbol I_{F_\alpha})$ fixes $U_{\alpha^\vee}$ pointwise, which means that α belongs to $R\big(Z_{G^\vee}(\hat\chi(\boldsymbol I_F)), T^{\vee,\boldsymbol W_F,\circ}\big)_{\mathrm{red}}$. This argument also works in the opposite direction, so $\alpha^\vee \in R_{\mathfrak s^\vee}$ if and only if $\alpha \in R_{\mathfrak s,\mu}$.

For the exceptional roots $\alpha^\vee$ with $s_\alpha \in W_{\mathfrak s} \cong W_{\mathfrak s^\vee}$, we saw on page 280 and after (4-5) that on both sides the issue can be reduced to a unitary group U_{2n+1}. From the list of cases at the end of Section 1 it is clear that if U_{2n+1} is unramified, $\alpha^\vee$ is relevant for $R_{\mathfrak s}^\vee$ if and only if it is relevant for $R_{\mathfrak s^\vee}$.

When the involved group U_{2n+1} only splits over a ramified extension, we need to check one more detail to arrive at the same conclusion. Namely, if $\alpha^\vee \circ \hat\chi : \boldsymbol W_{E_\alpha} \to \mathbb C^\times$ is conjugate orthogonal (respectively conjugate symplectic) then $\chi \circ \alpha^\vee : \mathfrak o_{E_\alpha}^\vee \to \mathbb C^\times$ must be trivial (respectively of order two). This is exactly [19, Lemma 3.4]. □

Lemma 5.2 implies that the isomorphism (5-2) restricts to $W_{\mathfrak s}^\circ \cong W_{\mathfrak s^\vee}^\circ$. We choose a $W_{\mathfrak s^\vee}^\circ$-invariant base point $\hat\chi_0$ of $\mathfrak s_T^\vee$ as in Section 4. We use the image χ_0 of $\hat\chi_0$ under the LLC as a basepoint of $\mathfrak s_T$. By the aforementioned equivariance of the LLC for tori, χ_0 is invariant under $W_{\mathfrak s}^\circ$.

Recall that $h_\alpha^\vee \in R_{\mathfrak s}^\vee$ generates $\mathbb Q\alpha^\vee \cap T/T_{\mathrm{cpt}}$. The element $m_\alpha\alpha^\vee$ does not necessarily generate $\mathbb Q\alpha^\vee \cap T/T_{\mathrm{cpt}}$. However, since $R_{\mathfrak s^\vee}$ is part of the root datum $\mathcal R_{\mathfrak s^\vee}$, $m_\alpha\alpha^\vee$ is at most divisible by 2 in T/T_{cpt} (namely when it is a long root in a type-C root system). For a better comparison, we replace $m_\alpha\alpha^\vee$ by $m_\alpha\alpha^\vee/2$ whenever that is possible. That option was already taken into account in Section 4. We denote the new multiple of $\alpha^\vee$ by $\tilde m_\alpha\alpha^\vee$ and we write

$$\tilde R_{\mathfrak s^\vee} = \{\tilde m_\alpha\alpha^\vee : \alpha^\vee \in R(J^\circ, T^{\vee,\boldsymbol W_F,\circ})\}.$$

Now Lemma 5.2 entails that the isomorphism

$$X^*(T_{\mathfrak s}) \cong X^*(X_{\mathrm{nr}}(T)) \cong T/T_{\mathrm{cpt}} \cong X^*((T^{\vee,\boldsymbol I_F})^\circ_{\boldsymbol W_F}), \tag{5-3}$$

induced by the LLC for tori, sends $R_{\mathfrak s}^\vee$ bijectively to $R_{\mathfrak s^\vee}$.

Lemma 5.3. *For any $\alpha \in R_{\mathfrak s,\mu}$, $\lambda(h_\alpha^\vee) = \lambda(\tilde m_\alpha\alpha^\vee)$ and $\lambda^*(h_\alpha^\vee) = \lambda^*(\tilde m_\alpha\alpha^\vee)$.*

Proof. For the nonexceptional roots, this was checked in (1-14), (1-15), Lemma 4.1 and (4-4). For exceptional roots (i.e., those for which the issue can be reduced to a unitary group U_3), it is verified case-by-case in the lists at the end of Section 1 and just before (4-6). □

We are ready to establish the desired isomorphism between Hecke algebras on the two sides of the LLC.

Theorem 5.4. *There is a canonical algebra isomorphism* $\psi_{\mathfrak{s}} : \mathcal{H}(\mathfrak{s})^{\mathrm{op}} \to \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$, *given as follows*:

- *On* $\mathcal{O}(T_{\mathfrak{s}})$, $\psi_{\mathfrak{s}}$ *is induced by the bijection* $T_{\mathfrak{s}} \cong T_{\mathfrak{s}^\vee}$ *from the LLC for tori.*
- $\psi_{\mathfrak{s}}(N_w) = N_{w^{-1}}$ *for all* $w \in W_{\mathfrak{s}} \cong W_{\mathfrak{s}^\vee}$.

Proof. By Theorem 2.7 there is a unique isomorphism of $\mathcal{O}(T_{\mathfrak{s}^\vee})$-modules with these properties. By the $W_{\mathfrak{s}}$-equivariance of the LLC for tori via (5-2), $\mathcal{O}(T_{\mathfrak{s}}) \xrightarrow{\sim} \mathcal{O}(T_{\mathfrak{s}^\vee})$ is $W_{\mathfrak{s}}$-equivariant. Combine that with Lemma 5.3 and the multiplication rules in extended affine Hecke algebras [6, Proposition 2.2]. □

We note that Theorem 5.4 is compatible with parabolic induction from standard parabolic and standard Levi subgroups of G. Indeed, for a standard Levi subgroup M of G one obtains the same isomorphism as in Theorem 5.4, on the subalgebra generated by $\mathcal{O}(T_{\mathfrak{s}})$ and the N_w with $w \in N_M(T)/T$.

Remark 5.5. It is also possible to construct a canonical algebra isomorphism $\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$. To that end, we only have to change the condition $\psi_{\mathfrak{s}}(N_w) = N_{w^{-1}}$ in Theorem 5.4 to $\psi_{\mathfrak{s}}(N_w) = N_w$. The two options are related by the canonical isomorphism

$$\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) \xrightarrow{\sim} \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^{\mathrm{op}}, \qquad f N_w \to N_{w^{-1}} f, \quad f \in \mathcal{O}(T_{\mathfrak{s}^\vee}),\ w \in W_{\mathfrak{s}^\vee}.$$

6. Parameters of generic representations

With Theorem 5.4 and (5-3) we can reformulate Theorem 3.4 in terms of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-modules. Then it says: π is (U, ξ)-generic if and only if

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^{\lambda})}(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi), \mathrm{St}) \text{ is nonzero.} \tag{6-1}$$

We want to investigate which Langlands parameters should correspond to generic representations in Theorem 4.3. With the reduction theorems from [32, §8–9] we translate the study of (irreducible) representations of $\mathcal{H}(\mathfrak{s})^{\mathrm{op}} \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ to representations of graded Hecke algebras. Subsequently we take a closer look at the geometric construction of the representations of such algebras. We need to revisit the methods from [32] and [4; 6], because the aspects we are interested in were not considered previously and require some details.

6.1. ***Reduction to graded Hecke algebras.*** To ease the notation, from now on the elements of $R_{\mathfrak{s}^\vee}$ will be called just $\alpha^\vee$, instead of $m_\alpha \alpha^\vee$ as previously. For a $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-module V and $t \in T_{\mathfrak{s}^\vee}$ we write

$$V_t = \{v \in V : \text{there exists } n \in \mathbb{N} \text{ such that } (\theta_x - x(t))^n v = 0 \text{ for all } x \in X\}.$$

If V_t is nonzero, then we call t a weight of V. For a $W_{\mathfrak{s}^\vee}$-stable subset $U \subset T_{\mathfrak{s}^\vee}$, let $\mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee})_U$ be the category of finite-length $\mathcal{H}_{\mathfrak{s}^\vee}$-modules all whose $\mathcal{O}(T_{\mathfrak{s}^\vee})$-weights belong to U. There is this natural equivalence of categories:

$$\mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))_U \to \bigoplus_{t \in U/W_{\mathfrak{s}^\vee}} \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))_{W_{\mathfrak{s}^\vee} t}, \quad V \mapsto \bigoplus_{t \in U/W_{\mathfrak{s}^\vee}} \Big(\sum_{w \in W_{\mathfrak{s}^\vee}} V_{wt} \Big).$$

Let $T_{\mathfrak{s}^\vee,\mathrm{un}} \subset T_{\mathfrak{s}^\vee}$ be the maximal compact real subtorus. It is homeomorphic to the set of unitary characters in $T_{\mathfrak{s}} = X_{\mathrm{nr}}(T)\chi_0$. For $u \in T_{\mathfrak{s}^\vee,\mathrm{un}}$ we put

$$R_{\mathfrak{s}^\vee,u} = \{\alpha^\vee \in R_{\mathfrak{s}^\vee} : s_\alpha(u) = u\}.$$

This is a root system and its Weyl group is contained in $W_{\mathfrak{s}^\vee,u}$. Recall that we fixed a Borel subgroup $B^\vee \subset G^\vee$, which provides $R_{\mathfrak{s}^\vee,u}$ with a notion of positive roots. Let $\Gamma_{\mathfrak{s}^\vee,u}$ be the stabilizer of $R^+_{\mathfrak{s}^\vee,u} = R^+_{\mathfrak{s}^\vee} \cap R_{\mathfrak{s}^\vee,u}$ in $W_{\mathfrak{s}^\vee,u}$. Then

$$W_{\mathfrak{s}^\vee,u} = W(R_{\mathfrak{s}^\vee,u}) \rtimes \Gamma_{\mathfrak{s}^\vee,u}.$$

From these objects we build a new root datum

$$\mathcal{R}_{\mathfrak{s}^\vee,u} = (R_{\mathfrak{s}^\vee,u}, X^*(T_{\mathfrak{s}^\vee}), R^\vee_{\mathfrak{s}^\vee,u}, X_*(T_{\mathfrak{s}^\vee})),$$

which is endowed with an action of $\Gamma_{\mathfrak{s}^\vee,u}$. That gives rise to an extended affine Hecke algebra

$$\mathcal{H}_{\mathfrak{s}^\vee,u} = \mathcal{H}(\mathcal{R}_{\mathfrak{s}^\vee,u}, \lambda, \lambda^*, q_F^{1/2}) \rtimes \Gamma_{\mathfrak{s}^\vee,u}.$$

We denote the standard generators of this algebra (as an $\mathcal{O}(T_{\mathfrak{s}^\vee})$-module) by $N_{w,u}$, where $w \in W_{\mathfrak{s}^\vee,u}$.

The positive part of $X_{\mathrm{nr}}(T)$ is $X^+_{\mathrm{nr}}(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{R}_{>0})$. Via (4-2), $X^+_{\mathrm{nr}}(T)$ can be regarded as a subgroup of $(T^{\vee, \boldsymbol{I}_F})_{\boldsymbol{W}_F}{}^\circ$, and as such it acts on $T_{\mathfrak{s}^\vee}$. In particular that yields a subset $W_{\mathfrak{s}^\vee,u} X^+_{\mathrm{nr}}(T)u$ of $T_{\mathfrak{s}^\vee}$. Notice that every element of $T_{\mathfrak{s}^\vee}$ lies in a subset of the form $X^+_{\mathrm{nr}}(T)u$ with $u \in T_{\mathfrak{s}^\vee,\mathrm{un}}$.

Theorem 6.1. *There exists a canonical equivalence of categories*

$$\begin{aligned} \mathrm{ind}_u : \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{X^+_{\mathrm{nr}}(T)u} &\to \mathrm{Mod}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))_{W_{\mathfrak{s}^\vee} X^+_{\mathrm{nr}}(T)u}, \\ V_{X^+_{\mathrm{nr}}(T)u} := \sum_{t \in X^+_{\mathrm{nr}}(T)} V_{tu} &\leftarrow\!\shortmid V, \end{aligned}$$

such that:

(a) ind_u *is given by localization of the centers on both sides, followed by induction.*

(b) ind_u *and* ind_u^{-1} *preserve central characters.*

(c) *For* $V \in \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{X^+_{\mathrm{nr}}(T)u}$ *there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)}(\mathrm{ind}_u V, \mathrm{St}) \cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}(V, \mathrm{St}).$$

Proof. The original version of this equivalence is [32, Theorem 8.6], but the setup is slightly different there. The version we need, including the canonicity and the group $\Gamma_{\mathfrak{s}^\vee}$, is shown in [46, Theorem 2.1.2]. Strictly speaking $\Gamma_{\mathfrak{s}_\vee}$ must fix a point of $T_{\mathfrak{s}^\vee}$ in [46]. Fortunately, that does not play a role in the proof, it works in the generality of our setting because we consider u that need not be fixed by $W(R_{\mathfrak{s}^\vee})$. The properties (a) and (b) are checked in [6, Theorem 2.5].

By [6, Theorem 2.5] the effect of the thus obtained functor ind_u on $\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)$-modules is

$$V \mapsto \mathrm{ind}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}^{\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)} V. \tag{6-2}$$

In this expression $\mathcal{H}(W(R_{\mathfrak{s}^\vee}), q_F^\lambda)$ and $\mathbb{C}[\Gamma_{\mathfrak{s}^\vee} \cap \Gamma_{\mathfrak{s}^\vee,u}]$ are naturally subalgebras of $\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^\lambda)$, but we have to be careful with the $\tilde{N}_{w,u}$ for which $w \in \Gamma_{\mathfrak{s}^\vee,u}$ but $w \notin \Gamma_{\mathfrak{s}^\vee}$. From [32, §8] and [46, §2.1] one sees that $\tilde{N}_{w,u}$ is sent to

$$\widetilde{\mathcal{T}}_w 1_{X_{\mathrm{nr}}^+(T)u} = \mathcal{T}_w 1_{X_{\mathrm{nr}}^+(T)u}$$

in a suitable completion of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$. Here $\mathcal{T}_w$ is as in Section 2, transferred to completions of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ via Theorem 5.4. From (6-2) and Frobenius reciprocity (in a suitably completed algebra) we obtain (c). □

With Theorem 6.1 we can reduce the study of $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-modules that admit a central character to modules of another affine Hecke algebra, $\mathcal{H}_{\mathfrak{s}^\vee,u}$, such that for the new modules the compact part of the central character is fixed by the new extended Weyl group. In this process all relevant properties of modules are preserved.

Let $\mathfrak{T}_u(T_{\mathfrak{s}^\vee})$ be the tangent space of $T_{\mathfrak{s}^\vee}$ at u. It can be identified with the space $\mathbb{C} \otimes_{\mathbb{Z}} X_*(T_{\mathfrak{s}^\vee})$, so $R_{\mathfrak{s}^\vee,u}$ can be regarded as a subset of the cotangent space $\mathfrak{T}_u^*(T_{\mathfrak{s}^\vee})$. For $\alpha^\vee \in R_{\mathfrak{s}^\vee}$ we define a parameter

$$k_{\alpha^\vee}^u = \frac{(\lambda(h_\alpha^\vee) + \alpha(u)\lambda^*(h_\alpha^\vee)) \log(q_F)}{2} \in \mathbb{R}_{\geq 0}.$$

The graded Hecke algebra $\mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u)$ is defined to be the vector space $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})) \otimes \mathbb{C}[W(R_{\mathfrak{s}^\vee,u})]$ with multiplication given as follows:

- $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$ and $\mathbb{C}[W(R_{\mathfrak{s}^\vee,u})]$ are embedded as unital subalgebras.
- For $\alpha^\vee \in R_{\mathfrak{s}^\vee,u}$ simple and $f \in \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$,

$$s_\alpha f - s_\alpha(f) s_\alpha = \frac{k_{\alpha^\vee}^u (f - s_\alpha(f))}{\alpha^\vee}.$$

The group $\Gamma_{\mathfrak{s}^\vee,u}$ acts naturally on this algebra, by

$$\gamma(wf) = (\gamma w \gamma^{-1}) f \circ \gamma^{-1}, \quad w \in W(R_{\mathfrak{s}^\vee,u}),\ f \in \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})).$$

This action gives rise to the extended graded Hecke algebra

$$\mathbb{H}_{\mathfrak{s},u} = \mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes \Gamma_{\mathfrak{s}^\vee,u}.$$

Its center is $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))^{W_{\mathfrak{s}^\vee,u}}$ and weights of $\mathbb{H}_{\mathfrak{s}^\vee,u}$-modules are considered, by default, with respect to the maximal commutative subalgebra $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$. Like for affine Hecke algebras, for a $W_{\mathfrak{s}^\vee,u}$-stable subset $U \subset \mathfrak{T}_u(T_{\mathfrak{s}^\vee})$ we have the category $\mathrm{Mod}(\mathbb{H}_{\mathfrak{s}^\vee,u})_U$ of finite-length modules all whose $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$-weights belong to U.

Recall the exponential map for $T_{\mathfrak{s}^\vee}$ based at u:

$$\exp_u : \mathfrak{T}_u(T_{\mathfrak{s}^\vee}) \to T_{\mathfrak{s}^\vee}, \quad y \mapsto u \exp(y).$$

Theorem 6.2. *The map* $\exp_u$ *induces a canonical equivalence of categories*

$$\exp_{u*} : \mathrm{Mod}(\mathbb{H}_{\mathfrak{s}^\vee,u})_{\mathbb{R}\otimes X_*(T_{\mathfrak{s}^\vee})} \to \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{X^+_{\mathrm{nr}}(T)u},$$

such that:

(a) $\exp_{u*}$ *comes from an isomorphism* (*induced by* $\exp_u$) *between localized versions of* $\mathbb{H}_{\mathfrak{s}^\vee,u}$ *and of* $\mathcal{H}_{\mathfrak{s}^\vee,u}$.

(b) $\exp_{u*}$ *does not change the vector spaces underlying the modules.*

(c) *The effect of* $\exp_{u*}$ *on* $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$*-weights is* $\exp_u$.

(d) *For any* $V \in \mathrm{Mod}(\mathbb{H}_{\mathfrak{s}^\vee,u})_{\mathbb{R}\otimes X_*(T_{\mathfrak{s}^\vee})}$ *there is an isomorphism*

$$\mathrm{Hom}_{W_{\mathfrak{s}^\vee,u}}(V, \det) \cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}(\exp_{u*} V, \mathrm{St}).$$

Proof. The original version of this equivalence is [32, Theorem 9.3]. We use the version from [46, Theorem 2.1.4 and Corollary 2.1.5]. This includes the canonicity and the properties (a), (b) and (c).

One way to see (d) is via deformations of the parameters. We can scale the parameters k^u linearly to 0. That gives a family of extended graded Hecke algebras

$$\mathbb{H}_{\mathfrak{s}^\vee,u,\epsilon} = \mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), \epsilon k^u) \rtimes \Gamma_{\mathfrak{s}^\vee,u}, \quad \epsilon \in \mathbb{R}_{\geq 0}.$$

A module V can be "scaled" to modules V_ϵ, via the scaling homomorphisms $\mathbb{H}_{\mathfrak{s}^\vee,u,\epsilon} \to \mathbb{H}_{\mathfrak{s}^\vee}$ for $\epsilon \geq 0$ [46, (1.11)]. For $\epsilon = 0$ we obtain a module V_0 of

$$\mathbb{H}_{\mathfrak{s}^\vee,u,0} = \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})) \rtimes W_{\mathfrak{s}^\vee,u},$$

which equals V as a $\mathbb{C}[W_{\mathfrak{s}^\vee,u}]$-module and on which $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$ acts by evaluation at $0 \in \mathfrak{T}_u(T_{\mathfrak{s}^\vee})$.

For the affine Hecke algebra $\mathcal{H}_{\mathfrak{s}^\vee,u}$, the parameters can be scaled via $q_F \mapsto q_F^\epsilon$ with $\epsilon \in [0, 1]$. That yields a family of algebras

$$\mathcal{H}_{\mathfrak{s}^\vee,u,\epsilon} = \mathcal{H}(\mathcal{R}_u, \lambda, \lambda^*, q_F^\epsilon) \rtimes \Gamma_{\mathfrak{s}^\vee,u}, \quad \epsilon \in \mathbb{R}_{\geq 0}.$$

The module $\exp_{u*} V$ can be "scaled" accordingly via a functor

$$\tilde{\sigma}_\epsilon : \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u})_{\chi_+ u} \to \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}^\vee,u,\epsilon})_{\chi_+^\epsilon u}, \quad \epsilon \in [0,1];$$

see [46, Corollary 4.2.2]. In this process $\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)$ is replaced by the isomorphic semisimple algebra $\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^{\epsilon\lambda})$. The multiplicities

$$\dim \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^{\epsilon\lambda})}(\tilde{\sigma}_\epsilon(\exp_{u*} V), \mathrm{St})$$

depend continuously on $\epsilon \in [0,1]$ and they are integers, so in fact they are constant as functions of ϵ. It is known from [46, (4.6)–(4.7)] that

$$\exp_{u*}(V_\epsilon) = \tilde{\sigma}_\epsilon(\exp_{u*} V) \quad \text{for all } \epsilon \in [0,1].$$

We conclude that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)}(\exp_{u*} V), \mathrm{St}) &\cong \mathrm{Hom}_{\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^0)}(\tilde{\sigma}_0(\exp_{u*} V), \mathrm{St}) \\ &\cong \mathrm{Hom}_{W_{\mathfrak{s}^\vee,u}}(V_0, \det) = \mathrm{Hom}_{W_{\mathfrak{s}^\vee,u}}(V, \det). \quad \square \end{aligned}$$

In view of Theorems 3.4, 6.1 and 6.2, the role of genericity for $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is played by modules that the contain the character det of $\mathbb{C}[W_{\mathfrak{s}^\vee,u}]$. To analyze those, we bring the algebra in an easier form.

Let $R_{u>0}$ be the subset of $R_{\mathfrak{s}^\vee,u}$ consisting of the roots $\alpha^\vee$ for which $k^u_{\alpha^\vee} > 0$. Let $\Gamma_{u>0}$ be the stabilizer of $R^+_{u>0} = R_{u>0} \cap R^+_{\mathfrak{s}^\vee}$ in $W_{\mathfrak{s}^\vee,u}$.

Lemma 6.3. *The set $R_{u>0}$ is a root system and*

$$\mathbb{H}_{\mathfrak{s}^\vee,u} = \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes \Gamma_{u>0}.$$

Proof. The set $R_{u>0}$ is $W_{\mathfrak{s}^\vee,u}$-stable by the invariance properties of the labels. In particular it is stable under the reflections with respect to its roots, so it is a root system. In every irreducible component of $R_{\mathfrak{s}^\vee,u}$, $R_{u>0}$ is either everything or empty or the roots of one given length. By the simple transitivity of the action of $W(R_{u>0})$ on the collection of positive systems in $R_{u>0}$,

$$W_{\mathfrak{s}^\vee,u} = W(R_{u>0}) \rtimes \Gamma_{u>0}.$$

We note that

$$R_{\mathfrak{s}^\vee,u} \setminus R_{u>0} = \{\alpha^\vee \in R_{\mathfrak{s}^\vee,u} : \lambda(h_\alpha^\vee) = \lambda^*(h_\alpha^\vee),\ \alpha^\vee(u) = -1\}. \tag{6-3}$$

By reduction to irreducible root systems, and the classification thereof, one checks that $W(R_{\mathfrak{s}^\vee,u})$ is the semidirect product of $W(R_{u>0})$ and the subgroup generated by the reflections with respect to the simple roots in $R_{\mathfrak{s}^\vee,u} \setminus R_{u>0}$. For such reflections the multiplication relations in $\mathbb{H}_{\mathfrak{s}^\vee,u}$ simplify to $s_\alpha f = s_\alpha(f) s_\alpha$. That implies

$$\mathbb{H}(W(R_{\mathfrak{s}^\vee,u}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) = \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes (\Gamma_{u>0} \cap W(R_{\mathfrak{s}^\vee,u})),$$

which in turn implies the lemma. $\square$

The advantage of Lemma 6.3 is that via the new presentation the algebra becomes isomorphic to a graded Hecke algebra with equal parameters.

Lemma 6.4. *$\mathbb{H}_{\mathfrak{s}^\vee,u}$ is isomorphic to a graded Hecke algebra (extended by $\Gamma_{u>0}$) such that every root from $R_{u>0}$ has the parameter $\log(q_F)$.*

Proof. By [6, Proposition 3.14], $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is isomorphic to the graded Hecke algebra associated to a certain complex reductive group $\widetilde{G}$ and a cuspidal L-parameter with values in a quasi-Levi subgroup $\widetilde{M}$ of $\widetilde{G}$. In our specific setting $\widetilde{M}^\circ$ is a torus, because we only work with principal series L-parameters. In particular the cuspidal L-parameter is trivial on $\mathrm{SL}_2(\mathbb{C})$. Thus $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is a graded Hecke algebra associated to $\widetilde{G}$ and a cuspidal support whose unipotent (or nilpotent) element is trivial. By construction [6, §1] all the nonzero parameters are of the form $k^u_{\alpha^\vee} = c(\alpha^\vee)r_i$, where $r_i \in \mathbb{C}$ depends only on the connected component of the root system that contains $\alpha^\vee$. Further $c(\alpha^\vee) = 2$ by [31, §2] and our earlier specialization of z to $q_F^{1/2}$ entails $k_i = \log(q_F^{1/2}) = \log(q_F)/2$. Combine that with Lemma 6.3. □

6.2. ***Geometric representations of graded Hecke algebras.*** Since u corresponds to a unitary character of T, it is a bounded L-parameter for T. By [6, Proposition 3.14], the algebra $\mathbb{H}_{\mathfrak{s}^\vee,u}$ is of the form $\mathbb{H}(u, 0, \mathrm{triv}, \log(q_F)/2)$, where triv means the trivial local system on the trivial nilpotent orbit 0. The meaning of this statement is explained somewhat further in [6, (3.9) and below]. It can be formulated as

$$\mathbb{H}(u, 0, \mathrm{triv}, \log(q_F)/2) \cong \mathbb{H}(G_u^\vee, M^\vee, \mathrm{triv}, \log(q_F)/2).$$

In [6] the group $G_u^\vee$ is defined as $Z^1_{G^\vee_{\mathrm{sc}}}(u) \times X_{\mathrm{nr}}(G)$, but in our current setting we have just $G_u^\vee = Z_{G^\vee}(u)$. The reason is that at the start of Section 4 we refrained from involving the simply connected cover of $G^\vee_{\mathrm{der}}$, that would be superfluous for quasi-split groups. Similarly the group $M^\vee$, which is a quasi-Levi subgroup of $Z^1_{G^\vee_{\mathrm{sc}}}(u) \times X_{\mathrm{nr}}(G)$ in [6], becomes simply $T^\vee$ in our setup.

Notice that $G_u^\vee$ need not be connected. In fact the isomorphism

$$\mathbb{H}(G_u^\vee, T^\vee, \mathrm{triv}, \log(q_F)/2) \cong \mathbb{H}_{\mathfrak{s}^\vee,u} = \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u) \rtimes \Gamma_{u>0} \tag{6-4}$$

and Lemma 6.3 imply that $\pi_0(G_u^\vee) \cong \Gamma_{u>0}$. When we replace $G_u^\vee$ by its identity component, we obtain the subalgebra

$$\mathbb{H}^\circ_{\mathfrak{s}^\vee,u} := \mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2) \cong \mathbb{H}(W(R_{u>0}), \mathfrak{T}_u(T_{\mathfrak{s}^\vee}), k^u).$$

The irreducible representations of such graded Hecke algebras were parametrized and constructed geometrically in [31; 33]. The parameters are triples (σ, y, ρ°) where

(i) $\sigma \in \mathrm{Lie}(G_u^{\vee,\circ})$ is semisimple,

(ii) $y \in \mathrm{Lie}(G_u^{\vee,\circ})$ is nilpotent,

(iii) $[\sigma, y] = \log(q_F)y$,

(iv) ρ° is an irreducible representation of $\pi_0(Z_{G_u^{\vee,\circ}}/Z(G_u^{\vee,\circ}))$ satisfying the analogue of (ii) on page 129.

By [33] $G_u^{\vee,\circ}$-conjugacy classes of such triples are naturally in bijection with $\mathrm{Irr}\big(\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)\big)$. Let us write that as

$$(\sigma, y, \rho^\circ) \mapsto M^\circ_{y,\sigma,\rho^\circ}.$$

In [31; 33] there is an extra parameter $r \in \mathbb{C}$, but we suppress that because in this paper it will always be equal to $\log(q_F)/2$. From these parameters σ can always be chosen in $\mathrm{Lie}(T^\vee)$, and then $W(R_{u>0})\sigma$ is the central character of $M^\circ_{y,\sigma,\rho^\circ}$.

Lusztig's parametrization was slightly modified in [4, §3.5], essentially by composing it with the Iwahori–Matsumoto involution IM of $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$. To make that consistent, the above condition (iii) must be replaced by

(iii') $[\sigma, y] = -\log(q_F)y$.

We denote the resulting parametrization of

$$\mathrm{Irr}\big(\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)\big),$$

which is the one used in [6], by

$$(\sigma, y, \rho^\circ) \mapsto \overline{M}^\circ_{y,\sigma,\rho^\circ} := \mathrm{IM}^* M^\circ_{y,-\sigma,\rho^\circ}. \tag{6-5}$$

Proposition 6.5. *The irreducible* $\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \mathrm{triv}, \log(q_F)/2)$*-representation* $\overline{M}^\circ_{y,\sigma,\rho^\circ}$ *contains the sign representation of* $\mathbb{C}[W(R_{u>0})]$ *if and only if* ρ° *is the trivial representation and the* $Z_{G_u^{\vee,\circ}}(\sigma)$*-orbit of* y *is dense in*

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\}.$$

Proof. We may replace $G_u^{\vee,\circ}$ by any finite covering group, that does not change the associated graded Hecke algebra. In particular we may assume that the derived group of $G_u^{\vee,\circ}$ is simply connected.

Via [6, Theorems 2.5 and 2.11], analogous to Theorems 6.1 and 6.2, $\overline{M}^\circ_{y,\sigma,\rho^\circ}$ becomes an irreducible representation of the affine Hecke algebra associated to $(G_u^{\vee,\circ}, T^\vee, \mathrm{triv})$, with parameter q_F. By Proposition 2.18 in [6], $\overline{M}^\circ_{y,\sigma,\rho^\circ}$ is turned into the module $\widetilde{M}_{\exp(\sigma),\exp(y),\rho^\circ}$ associated by Kazhdan and Lusztig [25] to $(\exp(\sigma), \exp(y), \rho^\circ)$ and q_F. The paper [25] assumed that the derived group of the involved complex reductive group is simply connected but that condition was lifted in [38]. Furthermore, it was shown in [38, §7.2–7.3] that $\widetilde{M}_{\exp(\sigma),\exp(y),\rho^\circ}$ contains the Steinberg representation of $\mathcal{H}(W(R_{u>0}), q_F)$ if and only if ρ° is trivial and the $Z^\circ_{G_u^{\vee,\circ}}$-orbit of y is dense in

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : \mathrm{Ad}(\exp\sigma)Y = q_F^{-1}Y\}.$$

Now we go back to $\mathbb{H}(G_u^{\vee,\circ}, T^\vee, \text{triv}, \log(q_F)/2)$-modules, and we conclude with a version of Theorem 6.2(d). □

The parametrization of $\text{Irr}(\mathbb{H}^\circ_{\mathfrak{s}^\vee,u})$ from (6-5) has been generalized to $\mathbb{H}_{\mathfrak{s}^\vee,u}$ in [4, Theorem 3.20] and [6, Theorem 3.8]. The parameters are $G_u^\vee$-conjugacy classes of triples (σ, y, ρ) as above, with the only difference that ρ is now an irreducible representation of $\pi_0(Z_{G_u^\vee}(\sigma, y)/Z(G_u^{\vee,\circ}))$.

The two constructions are related as follows. To (σ, y) one associates [31] a $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u} \times \pi_0(Z_{G_u^{\vee,\circ}})$-representation $E^\circ_{y,-\sigma}$. Then

$$E^\circ_{y,-\sigma,\rho^\circ} = \text{Hom}_{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}(\rho^\circ, E^\circ_{y,-\sigma})$$

and $M^\circ_{y,-\sigma,\rho^\circ}$ is the unique irreducible quotient of that module.

Similarly an $\mathbb{H}_{\mathfrak{s}^\vee,u} \times \pi_0(Z_{G_u^\vee})$-representation $E_{y,-\sigma}$ can be constructed [4], and by [4, Lemma 3.3] there is a canonical isomorphism

$$E_{y,-\sigma} \cong \text{ind}_{\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}}^{\mathbb{H}_{\mathfrak{s}^\vee,u}} E^\circ_{y,\sigma}. \tag{6-6}$$

One defines

$$E_{y,-\sigma,\rho} = \text{Hom}_{\pi_0(Z_{G_u^\vee}(\sigma,y))}(\rho, E_{y,-\sigma}), \tag{6-7}$$

and then $M_{y,-\sigma,\rho}$ is the unique irreducible quotient of $E_{y,-\sigma,\rho}$.

Lemma 6.6. *Every semisimple $\sigma \in G_u^{\vee,\circ}$ can be extended to a triple as used in* (6-7) *such that $M_{y,-\sigma,\rho}$ contains the trivial representation of $\mathbb{C}[W(R_{u>0}) \rtimes \Gamma_{u>0}]$. Moreover (y, ρ) is unique up to $Z_{G_u^\vee}(\sigma)$-conjugacy, y lies in the dense $Z_{G_u^{\vee,\circ}}(\sigma)$-orbit in*

$$\{Y \in \text{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\}$$

and the restriction of ρ to $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y))$ is a multiple of the trivial representation.

Proof. Let $M_{y,-\sigma}$ be the maximal semisimple quotient $\mathbb{H}_{\mathfrak{s}^\vee,u}$-module of $E_{y,-\sigma}$. Then

$$M_{y,-\sigma,\rho} = \text{Hom}_{\pi_0(Z_{G_u^\vee}(y,\sigma))}(\rho, M_{y,-\sigma}), \tag{6-8}$$

for any eligible ρ. The same can be done for the analogous $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-modules. It follows from (6-6), (6-7) and (6-8) that

$$M_{y,-\sigma} \cong \text{ind}_{\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}}^{\mathbb{H}_{\mathfrak{s}^\vee,u}} M^\circ_{y,-\sigma}. \tag{6-9}$$

Recall that $\mathbb{H}_{\mathfrak{s}^\vee,u} = \mathbb{H}^\circ_{\mathfrak{s}^\vee,u} \rtimes \Gamma_{u>0}$. By Frobenius reciprocity and (6-9) the multiplicity of $\text{triv}_{W(R_{u>0})\rtimes\Gamma_{u>0}}$ in $M_{y,-\sigma}$ equals the multiplicity of $\text{triv}_{W(R_{u>0})}$ in $M^\circ_{y,-\sigma}$.

For any given σ, Proposition 6.5 and (6-8) for $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$ entail that $\text{triv}_{W(R_{u>0})}$ appears with multiplicity one in $M^\circ_{y,-\sigma}$ if y satisfies the density condition, and otherwise that multiplicity is zero. Hence $M_{y,-\sigma}$ contains $\mathrm{triv}_{W(R_{u>0})\rtimes\Gamma_{u>0}}$ if and only if y satisfies the condition from the statement, and then the multiplicity is one.

For such (σ, y), multiplicity one ensures that there exists a unique ρ such that $M_{y,\sigma,\rho}$ contains $\mathrm{triv}_{W(R_{u>0})\rtimes\Gamma_{u>0}}$. Let ρ° be an irreducible subrepresentation of ρ restricted to the normal subgroup $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y))$. By Clifford theory the restriction of ρ to $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y))$ is a multiple of $\bigoplus_g g\cdot\rho^\circ$, where g runs over $\pi_0(Z_{G_u^\vee}(\sigma, y))$ modulo the stabilizer of ρ°.

Suppose that ρ° is nontrivial. Then $g\cdot\rho^\circ$ is nontrivial for any $g \in \pi_0(Z_{G_u^\vee}(\sigma, y))$, and $M_{y,-\sigma,\rho}$ cannot contain any $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-submodule of the form $M^\circ_{y',\sigma',\mathrm{triv}}$. In this case $M_{y,-\sigma,\rho}$ does not contain $\mathrm{triv}_{W(R_{u>0})}$. □

To see that the parametrization of $\mathrm{Irr}(\mathbb{H}^{\mathfrak{s}^\vee,u})$ from [6, Theorem 3.8] has a property like Proposition 6.5, it remains to analyze the ρ determined by Lemma 6.6. To that end we have to delve more deeply into the underlying constructions.

By the naturality of the parametrization (6-5), the $\Gamma_{u>0}$-stabilizer of $\overline{M}^\circ_{y,\sigma,\rho^\circ}$ (or equivalently of $M^\circ_{y,-\sigma,\rho^\circ}$) equals the $\Gamma_{u>0}$-stabilizer of the $G_u^{\vee,\circ}$-orbit of (σ, y, ρ°). When $\rho^\circ = \mathrm{triv}$, that group depends only on (σ, y). When furthermore y satisfies the density condition from Proposition 6.5, the $\Gamma_{u>0}$-stabilizer of $\overline{M}^\circ_{y,\sigma,\mathrm{triv}}$ equals the $\Gamma_{u>0}$-stabilizer of the $G_u^{\vee,\circ}$-orbit of σ, which we denote by $\Gamma_{[\sigma]}$.

Lemma 6.7. *Let* $(\sigma, y, \mathrm{triv})$ *be as in Proposition 6.5. The action of* $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$ *on* $M^\circ_{y,-\sigma,\mathrm{triv}}$ *extends in a unique way to an action of* $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u} \rtimes \Gamma_{[\sigma]}$ *that contains the trivial representation of* $\mathbb{C}[W(R_{u>0}) \rtimes \Gamma_{[\sigma]}]$.

Proof. By Proposition 6.5, we have that $M^\circ_{y,-\sigma,\rho^\circ}$ contains the trivial representation of $\mathbb{C}[W(R_{u>0})]$, and by Lemma 3.5 it does so with multiplicity one. Any $\gamma \in \Gamma_{[\sigma]}$ stabilizes $M^\circ_{y,-\sigma,\mathrm{triv}}$, so there exists a linear bijection I_γ such that

$$I_\gamma \circ h = \gamma(h) \circ I_\gamma : M^\circ_{y,-\sigma,\rho^\circ} \to M^\circ_{y,-\sigma,\rho^\circ} \quad \text{for all } h \in \mathbb{H}^\circ_{\mathfrak{s}^\vee,u}.$$

Schur's lemma says that I_γ is unique up to scalars. Since $\mathrm{triv}_{W(R_{u>0})}$ is $\Gamma_{[\sigma]}$-stable and appears with multiplicity one in $M^\circ_{y,-\sigma,\rho^\circ}$, I_γ stabilizes the one-dimensional subspace which affords $\mathrm{triv}_{W(R_{u>0})}$. We normalize I_γ by requiring that it fixes $\mathrm{triv}_{W(R_{u>0})} \subset M^\circ_{y,-\sigma,\rho^\circ}$ pointwise, that is the only possibility if we want to end up with the trivial representation of $W(R_{u>0}) \rtimes \Gamma_{[\sigma]}$.

For any $\gamma, \gamma' \in \Gamma_{[\sigma]}$, $I_\gamma \circ I_{\gamma'}$ satisfies the same condition as $I_{\gamma\gamma'}$, so equals $I_{\gamma\gamma'}$. These I_γ provide the desired extension. □

The module $M^\circ_{y,\sigma,\rho^\circ}$ comes as the unique irreducible quotient of a standard module $E^\circ_{y,\sigma,\rho^\circ}$ [34, Theorem 1.15(a)]. The latter is a subspace of the homology of the variety $\mathcal{B}^y$ of Borel subgroups of $G_u^{\vee,\circ}$ that contain $\exp(y)$, with coefficients in a certain local system $\dot{\mathcal{L}}$. In our setting $\dot{\mathcal{L}}$ is trivial because it comes from the trivial local system on $\{0\}$. In terms of the $\Gamma_{u>0}$-stable Borel subgroup $B^\vee \cap G_u^{\vee,\circ}$

we have

$$E^{\circ}_{y,\sigma,\rho^{\circ}} \subset H_*(\mathcal{B}^y) = H_*\big(\{g \in G_u^{\vee,\circ}/B^{\vee} \cap G_u^{\vee,\circ} : \mathrm{Ad}(g^{-1})y \in \mathrm{Lie}(B^{\vee} \cap G_u^{\vee,\circ})\}\big).$$

From that and (6-7) we see that

$$E^{\circ}_{y,-\sigma,\mathrm{triv}} = H_*(\mathcal{B}^y)^{Z_{G_u^{\vee,\circ}}(y,\sigma)}. \tag{6-10}$$

Lemma 6.8. *Let* $(\sigma, y, \mathrm{triv})$ *be as in Proposition 6.5. Then*

$$M^{\circ}_{y,-\sigma,\mathrm{triv}} = E^{\circ}_{y,-\sigma,\mathrm{triv}} = H_*(\mathcal{B}^y)^{Z_{G_u^{\vee,\circ}}(y,\sigma)}$$

as vector spaces. The subspace $H_0(\mathcal{B}^y)^{Z_{G_u^{\vee,\circ}}(y,\sigma)}$ *has dimension equal to one and* $\mathbb{C}[W(R_{u>0})]$ *acts on it as the trivial representation.*

Proof. By [33, §10.4–10.8], every irreducible subquotient of $E^{\circ}_{y,-\sigma,\mathrm{triv}}$ different from $M^{\circ}_{y,-\sigma,\mathrm{triv}}$ is of the form $M^{\circ}_{y',-\sigma,\rho^{\circ}}$ with

$$\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y \;\subset\; \overline{\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y'}.$$

By the density condition on y, such a y' does not exist. Therefore $E^{\circ}_{y,-\sigma,\mathrm{triv}}$ is irreducible and equal to $M^{\circ}_{y,-\sigma,\mathrm{triv}}$.

Again by [33, §10.4–10.8], $M^{\circ}_{y,-\sigma,\mathrm{triv}}$ is a subquotient of $E^{\circ}_{0,-\sigma,\mathrm{triv}}$. As

$$\overline{\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y} = \{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\} \tag{6-11}$$

is a vector space, the intersection cohomology complex from the constant sheaf on $\mathrm{Ad}(Z_{G_u^{\vee,\circ}})y$ is the constant sheaf on (6-11). In view of [33, §10], restricting that sheaf to $\{0\}$ provides a natural nonzero $\mathbb{H}^{\circ}_{\mathfrak{s}^{\vee},u}$-module homomorphism

$$E^{\circ}_{y,-\sigma,\mathrm{triv}} \to E^{\circ}_{0,-\sigma,\mathrm{triv}}.$$

This realizes $M^{\circ}_{y,-\sigma,\mathrm{triv}}$ as subrepresentation of $E^{\circ}_{0,-\sigma,\mathrm{triv}}$.

Consider the algebra $A = \mathcal{O}(\mathrm{Lie}(G_u^{\vee,\circ})/\mathrm{Ad}(G) \times \mathbb{C})$ of conjugation invariant functions on the Lie algebra of $G_u^{\vee,\circ} \times \mathbb{C}^{\times}$. We recall from [31] that

$$E^{\circ}_{0,-\sigma,\rho^{\circ}} = \mathrm{Hom}(\rho^{\circ}, E^{\circ}_{y,\sigma}) = \mathrm{Hom}(\rho^{\circ}, \mathbb{C}_{-\sigma,\log(q_F)/2} \otimes_A H^A_*(\mathcal{B}^0)).$$

If we replace $\log(q_F)/2$ by an arbitrary $r \in \mathbb{C}$, we still obtain a module for a graded Hecke algebra, namely $\mathbb{H}(G_u^{\vee,\circ}, T^{\vee}, \mathrm{triv}, r)$. It is known from Proposition 7.2 of [31] that $H^A_*(\mathcal{B}^0)$ is a free A-module. That implies that the modules $\mathbb{C}_{-\sigma,r} \otimes_A H^A_*(\mathcal{B}^0)$ form an algebraic family parametrized by $r \in \mathbb{C}$ and a semisimple $\sigma \in \mathrm{Lie}(G_u^{\vee,\circ})$. In particular, as modules for the finite-dimensional semisimple subalgebra $\mathbb{C}[W(R_{u>0})]$ they do not depend on (σ, r).

For $r = 0$, $\sigma = 0$, the group $Z_{G_u^{\vee,\circ}}(\sigma, 0) = G_u^{\vee,\circ}$ is connected, and we obtain $E^{\circ}_{0,0} = H_*(\mathcal{B}^0)$. This is a $\mathbb{C}[W(R_{u>0})]$-representation with which the classical Springer correspondence can be constructed. Here we must use the version of the

Springer correspondence from [30], which by [30, Theorem 9.2] means that the trivial $W(R_{u>0})$-representation appears as

$$H_0(\mathrm{pt}) = H_0(\mathcal{B}^x) \cong H_0(\mathcal{B}^0)$$

for a regular unipotent element $x \in G_u^{\vee,\circ}$. As $\dim H_0(\mathcal{B}^0) = 1$, the parts of

$$M^\circ_{y,-\sigma,\mathrm{triv}} \subset E^\circ_{0,-\sigma,\mathrm{triv}} \subset \mathbb{C}_{-\sigma,\log(q_F)/2} \otimes_A H_*^A(\mathcal{B}^0)$$

in homological degree zero also have dimension one and carry the trivial representation of $W(R_{u>0})$. □

We are ready to prove the desired generalization of Proposition 6.5.

Theorem 6.9. *There exists a canonical bijection between* $\mathrm{Irr}(\mathbb{H}_{\mathfrak{s}^\vee,u})$ *and the set of* $G_u^\vee$*-conjugacy classes of triples* (σ, y, ρ), *where*

- $\sigma, y \in \mathrm{Lie}(G_u^\vee)$ *with* σ *semisimple*, y *nilpotent and* $[\sigma, y] = -\log(q_F)y$,
- ρ *is an irreducible representation of* $\pi_0(Z_{G_u^\vee}(\sigma, y)/Z(G_u^{\vee,\circ}))$, *such that any irreducible* $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y)/Z(G_u^{\vee,\circ}))$*-subrepresentation appears in the homology of the variety of Borel subgroups of* $G_u^{\vee,\circ}$ *that contain* $\exp(\sigma)$ *and* $\exp(y)$.

The module $\overline{M}_{y,\sigma,\rho}$ *associated to* (σ, y, ρ) *contains the character* det *of* $\mathbb{C}[W_{\mathfrak{s}^\vee,u}]$ *if and only if* ρ *is trivial and the* $\mathrm{Ad}(Z_{G_u^{\vee,\circ}}(\sigma))$*-orbit of* y *is dense in*

$$\{Y \in \mathrm{Lie}(G_u^{\vee,\circ}) : [\sigma, Y] = -\log(q_F)Y\}.$$

Proof. We start with the parametrization of $\mathrm{Irr}\big(\mathbb{H}(G_u^\vee, T^\vee, \mathrm{triv}, \log(q_F)/2)\big)$ provided by [4, §3.5] and [6, Theorem 3.8]. This has almost all the required properties, only the action of $\mathbb{C}[\Gamma_{u>0}]$ on the thus constructed modules can still be normalized in several ways.

Fix a nilpotent $y \in \mathrm{Lie}(G_u^{\vee,\circ})$ and consider the variety

$$\mathcal{P}_y := \{g \in G_u^\vee / B^\vee \cap G_u^{\vee,\circ} : \mathrm{Ad}(g^{-1})y \in \mathrm{Lie}(B^\vee \cap G_u^{\vee,\circ})\}.$$

The $\mathbb{H}_{\mathfrak{s}^\vee,u} \times \pi_0(Z_{G_u^\vee}(\sigma, y))$-representation $E_{y,-\sigma}$ equals $H_*(\mathcal{P}^y)$ as a vector space. The action of $\pi_0(Z_{G_u^\vee}(\sigma, y))$ on $H_*(\mathcal{P}^y)$ is induced by the natural left action of $Z_{G_u^\vee}(\sigma, y)$ on $\mathcal{P}_y$. An element $\gamma \in \Gamma_{u>0}$ acts on $\mathcal{P}^y$ by

$$r_\gamma^{-1} : g(B^\vee \cap G_u^{\vee,\circ}) \mapsto g\gamma^{-1} B^\vee \cap G_u^{\vee,\circ}, \tag{6-12}$$

which in fact makes $\mathcal{P}_y$ isomorphic to $\mathcal{B}^y \times \Gamma_{u>0}$. We normalize the action of $\mathbb{C}[\Gamma_{u>0}]$ on $E_{y,-\sigma}$, by defining it as $H_*(r_\gamma^{-1})$. (This normalization was not possible in [4], because there the homology of $\mathcal{P}_y$ had coefficients in a local system that could be nontrivial.)

From now on we assume that y satisfies the density condition from the statement. In view of Lemma 6.6, it remains to analyze the $\pi_0(Z_{G_u^\vee}(\sigma, y))$-invariants in $E_{y,-\sigma}$. We recall from [4, Lemma 3.12] that

$$\Gamma_{[\sigma]} \cong \pi_0(Z_{G_u^\vee}(\sigma, y))/\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y)).$$

Let $\gamma_{\sigma,y} \in Z_{G_u^\vee}(\sigma, y)$ be a representative of $\gamma \in \Gamma_{[\sigma]}$. Then $H_*(\mathcal{P}_y)^{\pi_0(Z_{G_u^\vee}(\sigma,y))}$ consists of the invariants for $\{\gamma_{\sigma,y} : \gamma \in \Gamma_{[\sigma]}\}$ in

$$H_*(\mathcal{P}_y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} = \bigoplus\nolimits_{w\in\Gamma_{u>0}} H_*(w\cdot\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}. \tag{6-13}$$

Fix $\gamma \in \Gamma_{[\sigma]}$ and consider the map

$$f_\gamma : \mathcal{B}^y \to \mathcal{B}^y, \quad g(B^\vee \cap G_u^{\vee,\circ}) \mapsto \gamma_{\sigma,y} g\gamma^{-1}(B^\vee \cap G_u^{\vee,\circ}). \tag{6-14}$$

It can be decomposed as

$$f_\gamma = l_{\gamma_{y,\sigma}} \circ \rho_\gamma^{-1} = r_\gamma^{-1} \circ l_{\gamma_{y,\sigma}}.$$

The induced map on $H_*(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}$ is the composition of the action of an $\mathbb{H}_{\mathfrak{s}^\vee,u}$-intertwiner $H_*(l_{\gamma_{y,\sigma}})$ from $\pi_0(Z_{G_u^{\vee,\circ}}(\sigma, y))$ and the action $H_*(\rho_\gamma^{-1})$ of $\gamma \in \mathbb{H}_{\mathfrak{s}^\vee,u}$, so it is an $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$-intertwiner

$$H_*(f_\gamma) : E^\circ_{y,-\sigma,\mathrm{triv}} \to \gamma \cdot E^\circ_{y,-\sigma,\mathrm{triv}}.$$

Let $\pi_{y,-\sigma}$ be the extension of $E^\circ_{y,-\sigma,\mathrm{triv}} = M^\circ_{y,-\sigma,\mathrm{triv}}$ to an $\mathbb{H}_{\mathfrak{s}^\vee,u} \rtimes \Gamma_{[\sigma]}$-representation from Lemma 6.8. Consider the composition

$$\pi_{y,-\sigma}(\gamma^{-1}) \circ H_*(f_\gamma) \in \mathrm{End}_{\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}}(E^\circ_{y,-\sigma,\mathrm{triv}}).$$

By Schur's lemma this is a scalar, say $\lambda \in \mathbb{C}$. We know from Lemma 6.8 that $H_0(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}$ has dimension one, so in terms of simplicial homology it is spanned by an element v that is the sum of one point from every connected component of $\mathcal{B}^y$. This v is fixed by $H_0(f_\gamma)$ because (6-14) is a homeomorphism. By Lemmas 6.7 and 6.8 also $\pi_{y,-\sigma}(\gamma^{-1})v = v$. Hence $\lambda = 1$ and $\pi_{y,-\sigma}(\gamma^{-1}) \circ H_*(f_\gamma)$ is the identity. Equivalently,

$$H_*(l_{\gamma_{y,\sigma}}) = H_*(r_\gamma) \circ \pi_{y,-\sigma}(\gamma) : H_*(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} \to H_*(\gamma\cdot\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}.$$

Specializing to $H_0(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} = \mathbb{C}v$ we obtain

$$H_0(l_{\gamma_{y,\sigma}}) = H_0(r_\gamma) : H_0(\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))} \to H_0(\gamma\cdot\mathcal{B}^y)^{\pi_0(Z_{G_u^{\vee,\circ}}(\sigma,y))}.$$

It follows that $H_0(\mathcal{P}^y)^{\pi_0(Z_{G_u^\vee}(\sigma,y))}$ contains the nonzero vector

$$\sum_{\gamma\in\Gamma_{u>0}} H_0(r_\gamma^{-1})v = \sum_{w\in\Gamma_{u>0}/\Gamma_{[\sigma]}} H_0(r_w^{-1}) \sum_{\gamma\in\Gamma_{[\sigma]}} H_0(l_{\gamma_{y,\sigma}})v_0.$$

Lemma 6.8 shows that this an element of $E_{y,-\sigma,\mathrm{triv}}$ fixed by $W(R_{u>0}) \rtimes \Gamma_{u>0}$. In other words,

(6-15) $\mathrm{IM}^* M_{y,-\sigma,\mathrm{triv}}$ contains the $\mathbb{C}[W(R_{u>0}) \rtimes \Gamma_{u>0}]$-representation sign $\rtimes$ triv.

Lemma 6.6 says that only the triples (y, σ, ρ) of the kind indicated in the statement have that property.

Finally, we slightly modify the construction from [4, §3.5]. Instead of extending the Iwahori–Matsumoto involution from $\mathbb{H}^\circ_{\mathfrak{s}^\vee,u}$ to $\mathbb{H}_{\mathfrak{s}^\vee,u}$ by making it the identity on $\mathbb{C}[\Gamma_{u>0}]$,

(6-16) we extend IM to $\mathbb{H}_{\mathfrak{s}^\vee,u}$ as multiplication by det on $\mathbb{C}[\Gamma_{u>0}]$.

Then (6-15) becomes: $\mathrm{IM}^* M_{y,-\sigma,\rho}$ contains $\det_{W_{\mathfrak{s}^\vee,u}}$ if and only if (σ, y, ρ) is as stated in the theorem. □

We note that (6-16) only differs from the usual Iwahori–Matsumoto involution on the extended graded Hecke algebra $\mathbb{H}_{\mathfrak{s}^\vee,u}$ by the automorphism

(6-17) $$\det_{\Gamma_{u>0}} : \gamma w f \mapsto \det(\gamma)\gamma w f, \quad \gamma \in \Gamma_{u>0},\ w \in W(R_{u>0}),\ f \in \mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee})).$$

Since $\det_{\Gamma_{u>0}}$ is the identity on $\mathcal{O}(\mathfrak{T}_u(T_{\mathfrak{s}^\vee}))$, it preserves all the properties (e.g., temperedness) that we need later on. The advantage of (6-16) over IM from [4] is that all reflections in $W(R_{u>0}) \rtimes \Gamma_{u>0}$ are treated in the same way, irrespective of their q-parameters in some Hecke algebra. This is needed to express Theorem 6.9 with $\det|_{W_{\mathfrak{s}^\vee,u}}$.

We wrap up this section by combining the main results.

Lemma 6.10. *We modify* [6, Theorem 3.18] (*see Theorem 4.3*) *by using* (6-16) *instead of the involution* IM *from* [4, §3.5]. *That yields a canonical bijection*

$$\Phi_e(G)^{\mathfrak{s}^\vee} \to \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})), \quad (\phi, \rho) \mapsto \overline{M}(\phi, \rho, q_F^{1/2}),$$

such that:

- *It has all the properties listed in* [6, Theorem 3.18].
- $\overline{M}(\phi, \rho, q_F^{1/2})$ *contains the Steinberg representation of* $\mathcal{H}(W_{\mathfrak{s}^\vee}, q_F^{\lambda})$ *if and only if* ρ *is trivial and* $\log \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$ *lies in the dense* $Z_{G^\vee}(\tilde{\phi}(\boldsymbol{W}_F))$*-orbit in*

$$\big\{ Y \in \mathrm{Lie}\big(Z_{G^\vee}(\phi(\boldsymbol{I}_F))\big) : \mathrm{Ad}(\tilde{\phi}(\mathrm{Frob}_F))Y = q_F^{-1} Y \big\}.$$

Here $\tilde{\phi}|_{\boldsymbol{I}_F} = \phi|_{\boldsymbol{I}_F}$ *and* $\tilde{\phi}(\mathrm{Frob}_F) = \phi\Big(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Big)$.

Proof. By design [6, Theorem 3.18] for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ is the composition of Theorems 6.1, 6.2 and 6.9, with (6-16) as only difference. Since each of the three involved bijections is canonical, so is our version of [6, Theorem 3.18]. As explained after (6-17), the automorphism $\det_{\Gamma_{u>0}}$ does not destroy any of the properties from [6, Theorem 3.18], so our bijection still satisfies all those properties.

In Theorem 6.9 we found a necessary and sufficient condition so that $\overline{M}_{y,\sigma,\rho} \in \mathrm{Irr}(\mathbb{H}_{\mathfrak{s}^\vee})$ contains $\det_{W_{\mathfrak{s}^\vee,u}}$. With Theorem 6.2 we can translate that to $\exp_{u*} \overline{M}_{y,\sigma,\rho}$. Thus the latter module contains the representation St of $\mathcal{H}(W_{\mathfrak{s}^\vee,u}, q_F^\lambda)$ if and only if ρ is trivial and the orbit of y is dense in

$$\{Y \in \mathrm{Lie}(Z_{G^\vee}(\phi(\boldsymbol{I}_F)) : \mathrm{Ad}(u \exp(\sigma))Y = q_F^{-1} Y\}.$$

With Theorem 6.1 we transfer that to a property of

$$\mathrm{ind}_u^{-1} \exp_{u*} \overline{M}_{y,\sigma,\rho} \in \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})).$$

The translation to L-parameters from [6] is such that $\mathrm{Sc}(\phi,\rho) = u \exp(\sigma)$ and $y = \log \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$. Thus we recover the characterization of genericity stated in the lemma. □

7. A canonical local Langlands correspondence

Recall that we fixed a quasi-split group $G = \mathcal{G}(F)$, a maximal split torus S of G, a Borel subgroup $B \subset G$ containing $T = Z_G(S)$ and a Whittaker datum (U, ξ). Given G, only ξ is really a choice, the other objects are unique up to G-conjugacy.

We denote the space of irreducible G-representations in the principal series by $\mathrm{Irr}(G,T)$, and we write $\Phi_e(G,T)$ for the set of principal series enhanced L-parameters in $\Phi_e(G)$.

Theorem 7.1. *The Whittaker datum (U,ξ) determines a canonical bijection*

$$\mathrm{Irr}(G,T) \leftrightarrow \Phi_e(G,T), \quad \pi \mapsto (\phi_\pi, \rho_\pi), \quad \pi(\phi,\rho) \leftarrow\!\shortmid (\phi,\rho).$$

Proof. Recall from (5-2) that the LLC for tori provides a $N_G(T)/T$-equivariant bijection between the Bernstein components of $\mathrm{Irr}(T)$ and the Bernstein components of $\Phi_e(T)$, say $\mathfrak{s}_T \mapsto \mathfrak{s}_T^\vee$.

Every principal series Bernstein component $\mathrm{Irr}(G)^{\mathfrak{s}}$ of $\mathrm{Irr}(G)$ determines a unique $N_G(T)/T$-orbit of Bernstein components $\mathrm{Irr}(T)^{\mathfrak{s}_T}$. Similarly every principal series Bernstein component $\Phi_e(G)^{\mathfrak{s}^\vee}$ determines a unique $N_G(T)/T$-orbit of Bernstein components $\Phi_e(G)^{\mathfrak{s}_T^\vee}$. Thus the LLC for tori induces a natural bijection between the Bernstein components of $\mathrm{Irr}(G,T)$ and those of $\Phi_e(G,T)$. We denote it by $\mathrm{Irr}(G)^{\mathfrak{s}} \mapsto \Phi_e(G)^{\mathfrak{s}^\vee}$, where typically $\mathfrak{s} = [T, \chi_0]_G$ and $\mathfrak{s}^\vee = (T, \hat{\chi}_0 X_{\mathrm{nr}}(T))$. From respectively (1-1) and Theorems 2.7, 5.4 and 4.3 (the latter in the form of Lemma 6.10), we obtain canonical bijections

$$\text{(7-1)}\quad \mathrm{Irr}(G)^{\mathfrak{s}} \leftrightarrow \mathrm{Irr}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}) \leftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}) \leftrightarrow \mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})) \leftrightarrow \Phi_e(G)^{\mathfrak{s}^\vee}.$$

Suppose we represent $\mathfrak{s}$ instead by $w\mathfrak{s}_T = [T, w\chi_0]_T$ with $w \in W(\mathcal{G}, \mathcal{S}) = N_G(T)/T$. Clearly we may assume that w has minimal length in $wW_\mathfrak{s}$. Start with any $\pi \in \mathrm{Irr}(G)^\mathfrak{s}$ and follow (7-1) to obtain

$$\pi_\mathfrak{s} \in \mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}}), \quad \pi_{\mathfrak{s}^\vee} \in \mathrm{Irr}(\mathcal{H}(\mathfrak{s}, q_F^{1/2})), \quad (\phi_\pi, \rho_\pi) \in \Phi_e(G, T).$$

We use the same notations with $w\mathfrak{s}$ instead of $\mathfrak{s}$. Proposition 2.8 implies that $\pi_{w\mathfrak{s}} = \pi_\mathfrak{s} \circ \mathrm{Ad}(\phi_w)$ where

$$\phi_w(f N_v) = (f \circ w) N_{w^{-1}vw} \quad \text{for } f \in \mathcal{O}(T_{w\mathfrak{s}}),\ v \in W_{w\mathfrak{s}}.$$

Now we consider w as element of $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)/T^\vee$ via (5-2), and we define an algebra isomorphism

$$\begin{aligned} \mathrm{Ad}(\phi_w)^\vee : \mathcal{H}(w\mathfrak{s}^\vee, q_F^{1/2}) &\to \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}), \\ f N_v &\mapsto (f \circ w) N_{w^{-1}vw}, \quad f \in \mathcal{O}(T_{w\mathfrak{s}^\vee}),\ v \in W_{w\mathfrak{s}^\vee}. \end{aligned}$$

With Theorem 5.4 we find that $\pi_{w\mathfrak{s}}$ is matched with $\pi_{w\mathfrak{s}^\vee} = \pi_{\mathfrak{s}^\vee} \circ \mathrm{Ad}(\phi_w)^\vee$. All the constructions behind Theorem 4.3 and Lemma 6.10 are equivariant for automorphisms of $(G^\vee \rtimes \boldsymbol{W}_F, T^\vee \rtimes \boldsymbol{W}_F)$ which preserve the projections to $\boldsymbol{W}_F$ and are algebraic on $G^\vee$. This means that $\pi_{\mathfrak{s}^\vee} \circ \mathrm{Ad}(\phi_w)^\vee$ is parametrized by $(w\phi_\pi w^{-1}, w \cdot \rho_\pi)$, for any representative of w in $N_{G^\vee}(T^\vee \rtimes \boldsymbol{W}_F)$. As $(w\phi_\pi w^{-1}, w \cdot \rho_\pi)$ equals (ϕ_π, ρ_π) in $\Phi_e(G)$, we deduce that the bijection between $\mathrm{Irr}(G)^\mathfrak{s}$ and $\Phi_e(G)^{\mathfrak{s}^\vee}$ from (7-1) does not depend on the choice of an inertial equivalence class for T underlying $\mathfrak{s}$.

Knowing that, we can unambiguously take the union of the bijections (7-1) over all Bernstein components of $\mathrm{Irr}(G, T)$. $\square$

Remark 7.2. If we had used the isomorphism $\mathcal{H}(\mathfrak{s}) \cong \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$ from Remark 5.5 instead of Theorem 5.4, then (7-1) would provide a canonical bijection between $\mathrm{Irr}(G)^\mathfrak{s}$ and $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})^{\mathrm{op}})$. That could be more natural, depending on the point of view.

In the remainder of this section we will show that the bijection from Theorem 7.1 has many desirable properties.

The definition of $\tilde\phi$ in Lemma 6.10 applies to any Langlands parameter $\phi \in \Phi(G)$. The group $Z_{G^\vee}(\tilde\phi(\boldsymbol{W}_F))$ acts by conjugation on the variety

$$V_{\tilde\phi} = \{v \in Z_{G^\vee}(\phi(\boldsymbol{I}_F)) : v \text{ is unipotent and } \tilde\phi(\mathrm{Frob}_F)^{-1} v \tilde\phi(\mathrm{Frob}_F) = v^{q_F}\}.$$

It is known from [15, Proposition 5.6.1] that $V_{\tilde\phi}$ is an affine space over $\mathbb{C}$ on which $Z_{G^\vee}(\tilde\phi(\boldsymbol{W}_F))$ acts with finitely many orbits, of which exactly one is open. Following [14, §0.6], we call $\phi \in \Phi(G)$ open if $u_\phi \in V_{\tilde\phi}$ lies in the open $Z_{G^\vee}(\tilde\phi(\boldsymbol{W}_F))$-orbit.

Lemma 7.3. *The representation $\pi(\phi, \rho) \in \mathrm{Irr}(G, T)$ is (U, ξ)-generic if and only if ϕ is open and ρ is trivial.*

Proof. By Theorem 3.4, $\pi(\phi, \rho)$ is (U, ξ)-generic if and only if the $\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}}$-module $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi(\phi, \rho))$ contains St. Via Theorems 5.4 and 4.3 that becomes the analogous statement for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-representations. In Lemma 6.10 we showed the equivalence with the stated conditions on ϕ and ρ, except unipotency. The conditions in Lemma 6.10 imply that $\log u_\phi$ must be nilpotent. Hence u_ϕ must be unipotent (as is any case required for Langlands parameters). □

We note that Lemma 7.3 agrees with the Reeder's findings [37; 38] for generic unipotent representations and generic principal series representations, in both cases for split reductive p-adic groups with connected center.

For the next properties of our LLC, the setup will be similar to [51, §5].

Lemma 7.4. *Theorem 7.1 is compatible with direct products of quasi-split F-groups.*

Proof. If $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, then all involved objects for $\mathcal{G}$ are naturally products of the analogous objects for G_1 and G_2. □

Recall that the group of (smooth) characters $\mathrm{Hom}(G, \mathbb{C}^\times)$ is naturally isomorphic to $H^1(\boldsymbol{W}_F, Z(G^\vee))$ [10; 27]. The former group acts on $\mathrm{Irr}(G)$ by tensoring, and that action commutes with the supercuspidal support map so stabilizes $\mathrm{Irr}(G, T)$.

On the other hand, $H^1(\boldsymbol{W}_F, Z(G^\vee))$ acts on $\Phi(G)$ by multiplication of maps $\boldsymbol{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$, where $H^1(\boldsymbol{W}_F, Z(G^\vee))$ gives maps that do not use $\mathrm{SL}_2(\mathbb{C})$). That action does not change R_ϕ, so it induces an action of $H^1(\boldsymbol{W}_F, Z(G^\vee))$ on $\Phi_e(G)$ which does not change the enhancements. This last action commutes with the cuspidal support maps, so it stabilizes $\Phi_e(G, T)$.

Lemma 7.5. *The bijection in Theorem 7.1 is $H^1(\boldsymbol{W}_F, Z(G^\vee))$-equivariant.*

Proof. For the fourth bijection in (7-1), such equivariance was shown in [51, Lemma 2.2(a)]. Here $z \in H^1(\boldsymbol{W}_F, Z(G^\vee))$ acts via the algebra isomorphism

$$\begin{aligned} \mathcal{H}(z) : \mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) &\to \mathcal{H}(z\mathfrak{s}^\vee, q_F^{1/2}), \\ f N_w &\mapsto (f \circ z^{-1}) N_w, \quad f \in \mathcal{O}(T_{\mathfrak{s}^\vee}),\ w \in W_{\mathfrak{s}^\vee}. \end{aligned}$$

In view of Theorem 5.4, the same formula also defines an algebra isomorphism

$$\mathcal{H}(z) : \mathcal{H}(\mathfrak{s})^{\mathrm{op}} \to \mathcal{H}(z\mathfrak{s})^{\mathrm{op}}.$$

We define an action of $H^1(\boldsymbol{W}_F, Z(G^\vee))$ on the union of the spaces $\mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}})$ by $z \cdot \tau = \tau \circ \mathcal{H}(z)^{-1}$. That renders the third bijection in (7-1) equivariant. Using Theorem 2.7 and the same argument we also make the second bijection in (7-1) equivariant for $H^1(\boldsymbol{W}_F, Z(G^\vee))$.

Finally, consider $\pi \in \mathrm{Irr}(G)^{\mathfrak{s}}$ and $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \in \mathrm{Irr}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{\mathrm{op}})$. Then $z \otimes \pi$ lies in $\mathrm{Irr}(G)^{z\mathfrak{s}}$ and

$$\begin{aligned}\mathrm{Hom}_G(\Pi_{z\mathfrak{s}}, z \otimes \pi) &= \mathrm{Hom}_G(I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T (z \otimes \chi), z \otimes \pi) \\ &\cong \mathrm{Hom}_G(z \otimes I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T (\chi), z \otimes \pi) \\ &= \mathrm{Hom}_G(I_B^G \mathrm{ind}_{T_{\mathrm{cpt}}}^T (\chi), \pi) \\ &= \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi).\end{aligned}$$

The isomorphism (from bottom to top) is given by translation by z on $\mathrm{Irr}(T)$. As modules over $\mathcal{H}(\mathfrak{s})$ and $\mathcal{H}(z\mathfrak{s})$, that isomorphism is implemented by composition with $\mathcal{H}(z)^{-1}$. Hence the first bijection in (7-1) is equivariant as well. □

It is clear that a principal series G-representation is supercuspidal if and only if G is a torus. Similarly, the discussion at the start of Section 4 entails that a principal series enhanced L-parameter for G is cuspidal if and only if G is a torus. The next result relates the cuspidal support maps on both sides, when G is not a torus.

Lemma 7.6. *Theorem 7.1 and the cuspidal support maps make the following diagram commute*:

$$\begin{array}{ccc} \mathrm{Irr}(G, T) & \longleftrightarrow & \Phi_e(G, T) \\ \downarrow \mathrm{Sc} & & \downarrow \mathrm{Sc} \\ \mathrm{Irr}(T)/N_G(T) & \xrightarrow{LLC} & \Phi(T)/N_{G^\vee}(T \rtimes \mathbf{W}_F) \end{array}$$

Proof. From the formula for the cuspidal support (4-1) and Theorem 4.2(a), we see that the central character of $\overline{M}(\phi, \rho, q_F^{1/2})$ is given by $\mathrm{Sc}(\phi, \rho)/W_{\mathfrak{s}^\vee} \in \Phi_e(T)/W_{\mathfrak{s}^\vee}$. Hence the central character of $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi(\phi, \rho))$ is the image $W_{\mathfrak{s}}\chi_\phi$ of $\mathrm{Sc}(\phi, \rho)/W_{\mathfrak{s}^\vee}$ in $\mathrm{Irr}(T)/W_{\mathfrak{s}}$.

More explicitly, $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$ acts on $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi(\phi, \rho))$ via $W_{\mathfrak{s}}\chi_\phi$. Then a glance at the construction of $\Pi_{\mathfrak{s}}$ reveals that $W_{\mathfrak{s}}\chi_\phi$ represents the supercuspidal support of $\pi(\phi, \rho)$. □

We turn to more analytic properties of G-representations.

Lemma 7.7. $\pi \in \mathrm{Irr}(G, T)$ *is tempered if and only if* $\phi_\pi \in \Phi(G)$ *is bounded.*

Proof. Theorem 4.2(b) says that the fourth bijection in (7-1) has the desired property. By Lemma 5.3 and Theorem 5.4, the third bijection in (7-1) preserves temperedness. By [50, Theorem 9.6(a)], so does the composition of the first and the second bijections in (7-1). □

Lemma 7.8. $\pi \in \mathrm{Irr}(G, T)$ *is essentially square integrable if and only if* ϕ *is discrete.*

Proof. Suppose first that $R_{\mathfrak{s},\mu}$ has smaller rank that $R(\mathcal{G},\mathcal{S})$. By [50, Theorem 9.6(b)], $\mathrm{Rep}(G)^{\mathfrak{s}}$ contains no essentially square-integrable representations. As $\mathrm{rk}(R(\mathcal{G},\mathcal{S}))$ equals the F-split rank of $\mathcal{G}$ and

$$\mathrm{rk}(R_{\mathfrak{s},\mu}) = \mathrm{rk}(R_{\mathfrak{s}}^{\vee}) = \mathrm{rk}(R_{\mathfrak{s}^{\vee}}),$$

Theorem 4.2(c) says that $\Phi_e(G)^{\mathfrak{s}^{\vee}}$ contains no discrete enhanced L-parameters.

Now we suppose that $\mathrm{rk}(R_{\mathfrak{s},\mu}) = \mathrm{rk}(R(\mathcal{G},\mathcal{S}))$. Then [50, Theorem 9.6(c)] says that (1-1) restricts to a bijection between essentially square-integrable representations in $\mathrm{Irr}(G)^{\mathfrak{s}}$ and essentially discrete series representations in $\mathrm{Irr}(\mathcal{H}(\mathfrak{s})^{\mathrm{op}})$. By Lemma 5.3 and Theorem 5.4, the latter set is naturally in bijection with the set of essentially discrete series representations in $\mathrm{Irr}(\mathcal{H}(\mathfrak{s}^{\vee}, q_F^{1/2}))$. Combine that with Theorem 4.2(c). □

Recall from [27, p. 20–23] and [10, §10] that every $\phi \in \Phi(G)$ determines in a canonical way a character χ_ϕ of $Z(G)$.

Lemma 7.9. *For any $(\phi,\rho) \in \Phi_e(G,T)$, the central character of $\pi(\phi,\rho)$ equals χ_ϕ.*

Proof. For any subquotient π of $I_B^G(\chi) = \mathrm{ind}_B^G(\chi \otimes \delta_B^{1/2})$, the central character of π equals $(\chi \otimes \delta_B^{1/2})|_{Z(G)} = \chi|_{Z(G)}$. In particular the central character of $\pi(\phi,\rho)$ equals $\mathrm{Sc}(\pi(\phi,\rho))|_{Z(G)}$. By Lemma 7.6 that is $\pi(\mathrm{Sc}(\phi,\rho))|_{Z(G)}$. With (4-1) we write it as

$$\chi|_{Z(G)}, \quad \text{where } \hat{\chi}|_{\mathbf{I}_F} = \phi|_{\mathbf{I}_F} \text{ and } \hat{\chi}(\mathrm{Frob}_F) = \phi\Bigl(\mathrm{Frob}_F, \begin{pmatrix} q_F^{-1/2} & 0 \\ 0 & q_F^{1/2} \end{pmatrix}\Bigr).$$

It remains to show that χ_ϕ equals $\chi|_{Z(G)}$, and to that end we revisit the construction from [10; 27]. Let $\overline{G}$ be a quasi-split reductive F-group with connected center, such that $\mathcal{G}_{\mathrm{der}} = \overline{\mathcal{G}}_{\mathrm{der}}$. Let $\bar{\phi} \in \Phi(\overline{G})$ be a lift of $\phi \in \Phi(G)$. With the canonical map $\bar{p} : {}^L\overline{G} \to {}^L Z(\overline{G})$ we obtain $\bar{p}(\bar{\phi}) \in \Phi(Z(\overline{G}))$. Via the LLC for tori that gives $\chi_{\bar{p}(\bar{\phi})} \in \mathrm{Irr}(Z(\overline{G}))$, and by definition $\chi_\phi = \chi_{\bar{p}(\bar{\phi})}|_{Z(G)}$.

Let $\overline{T} = Z_{\overline{G}}(S) = Z_{\overline{G}}(\overline{T})$. From (4-1) we see that, for any enhancement $\bar{\rho}$ of $\bar{\phi}$ such that $(\bar{\phi},\bar{\rho}) \in \Phi_e(\overline{G},\overline{T})$, we have $\mathrm{Sc}(\bar{\phi},\bar{\rho}) = (\bar{\psi},\bar{\epsilon})$, where $\bar{\psi} \in \Phi(\overline{T})$ is a lift of $\hat{\chi} \in \Phi(T)$. As $\bar{\phi}$ and $\bar{\psi}$ differ only by elements of ${\overline{G}^{\vee}}_{\mathrm{der}} \subset \ker(\bar{p})$, we have $\bar{p}\bar{\phi} = \bar{p}\bar{\psi}$. By the naturality of the LLC for tori, $\chi_{\bar{\psi}}$ extends both $\chi \in \mathrm{Irr}(T)$ and $\chi_{\bar{p}\bar{\psi}} = \chi_{\bar{p}\bar{\phi}} \in \mathrm{Irr}(Z(\overline{G}))$. Hence $\chi|_{Z(G)} = \chi_{\bar{p}\bar{\phi}}|_{Z(G)} = \chi_\phi$. □

Suppose that $P = MR_u(P)$ is a parabolic subgroup of G, where M is a Levi factor of P and $T \subset M$. We can use the normalized parabolic induction functor I_P^G to relate representations of M and of G.

The restriction of ξ to $U \cap M$ is a nondegenerate character ξ_M. We use the pair $(U \cap M, \xi_M)$ to define genericity of M-representations and to normalize the LLC for $\mathrm{Irr}(M,T)$.

Suppose furthermore that $\phi \in \Phi(G)$ factors via $\Phi(M)$. By Theorem 7.10(a) of [3] the group $R_\phi^M = \pi_0(Z_{M^\vee}(\phi)/Z(M^\vee))$ injects naturally into R_ϕ. Hence any enhancement of $\phi \in \Phi(G)$ can be considered as a (possibly reducible) representation of R_ϕ^M.

Lemma 7.10. *Let $(\phi, \rho^M) \in \Phi_e(M, T)$ be bounded. Then*

$$I_P^G \pi(\phi, \rho^M) \cong \bigoplus_\rho \mathrm{Hom}_{R_\phi^M}(\rho^M, \rho) \otimes \pi(\phi, \rho),$$

where the sum runs over all $\rho \in \mathrm{Irr}(R_\phi)$ with $\mathrm{Sc}(\phi, \rho) = \mathrm{Sc}(\phi, \rho^M)$.

Proof. By [6, Theorem 3.18(f) and Lemma 3.19(a)], the analogous statement holds for $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2})$-modules. Theorem 5.4 (which is compatible with parabolic induction) entails it also holds for $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$-modules. Then (1-2) enables us to transfer the desired statement from $\mathcal{H}(\mathfrak{s})^{\mathrm{op}}$ to $\mathrm{Rep}(G)^{\mathfrak{s}}$. □

Recall that the Langlands classification for irreducible G-representations [27; 39] associates to any $\pi \in \mathrm{Irr}(G)$ a unique standard parabolic subgroup $P = MR_u(P)$, a unique tempered $\tau \in \mathrm{Irr}(M)$ and a unique strictly positive $z \in \mathrm{Hom}(M, \mathbb{R}_{>0})$, such that π is the unique irreducible quotient of the standard module $I_P^G(\tau \otimes z)$. It has a counterpart for (enhanced) L-parameters [45]. Let $(\phi, \rho) \in \Phi_e(G, T)$ and let $(P = MR_u(P), \phi_b, z)$ be the triple associated to ϕ by [45, Theorem 4.6]. Here $\phi_b \in \Phi(M)$ is bounded and $z \in X_{\mathrm{nr}}(M) \cong (Z(M^\vee)^{\mathbf{I}_F})_{\mathbf{W}_F}{}^\circ$ is "strictly positive with respect to P". By [3, Theorem 7.10(b)] there are natural isomorphisms

$$R_{\phi_b}^M \cong R_{z\phi_b}^M = R_\phi^M \cong R_\phi.$$

Hence ρ can also be regarded as enhancement of $\phi \in \Phi(M)$ or $\phi_b \in \Phi(M)$.

Lemma 7.11. *In the above setting*:

(a) *$\pi(\phi, \rho)$ is the unique irreducible quotient of $I_P^G \pi^M(\phi, \rho)$.*

(b) *$\pi^M(\phi, \rho) = \pi^M(z\phi_b, \rho) = z \otimes \pi^M(\phi_b, \rho)$ with $\pi^M(\phi_b, \rho) \in \mathrm{Irr}(M)$ tempered.*

(c) *The triple associated to $\pi(\phi, \rho)$ by the Langlands classification for* $\mathrm{Irr}(G)$ *is $(P, \pi^M(\phi_b, \rho), z)$.*

Proof. (a) By [51, Proposition 2.3], the analogue in $\mathrm{Rep}(\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}))$ holds. As in the proof of Lemma 7.10, that can be transferred to $\mathrm{Rep}(G)^{\mathfrak{s}}$ via (1-2).

(b) This is a direct consequence of Lemmas 7.5 and 7.7.

(c) This follows from (a), (b) and the uniqueness in the Langlands classification. □

Suppose that F'/F is a finite extension inside the fixed separable closure F_s. Let $\mathcal{G}'$ be a quasi-split F'-group and put $\mathcal{G} = \mathrm{Res}_{F'/F}(\mathcal{G}')$. Then $\mathcal{G}(F) = \mathcal{G}'(F')$, so there is a natural bijection $\mathrm{Irr}(\mathcal{G}(F)) \to \mathrm{Irr}(\mathcal{G}'(F'))$. On the other hand, Shapiro's lemma provides a natural isomorphism

$$\mathrm{Sh} : \Phi_e(\mathcal{G}(F)) \to \Phi_e(\mathcal{G}'(F'));$$

see [17, Lemma A.3].

Let $\mathcal{T}'$ be the centralizer of a maximal F'-split torus of $\mathcal{G}'$ and put $\mathcal{T} = \mathrm{Res}_{F'/F}(\mathcal{T}')$.

Lemma 7.12. *The bijection in Theorem 7.1 is compatible with restriction of scalars, in the sense that the following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{G}(F), \mathcal{T}(F)) & \longrightarrow & \Phi_e(\mathcal{G}(F), \mathcal{T}(F)) \\ \downarrow & & \downarrow \mathrm{Sh} \\ \mathrm{Irr}(\mathcal{G}'(F'), \mathcal{T}'(F')) & \longrightarrow & \Phi_e(\mathcal{G}'(F'), \mathcal{T}'(F')) \end{array}$$

Proof. By [51, (26)], Sh induces a bijection from the set of Bernstein components of $\Phi_e(\mathcal{G}(F))$ to the analogous set for $\mathcal{G}'(F')$. This bijection commutes with the cuspidal support maps, so it also applies to $\Phi_e(\mathcal{G}(F), \mathcal{T}(F))$ and $\Phi_e(\mathcal{G}'(F'), \mathcal{T}'(F'))$. If $\mathfrak{s}^\vee$ corresponds to $\mathfrak{s}'^\vee$, there is a natural algebra isomorphism $\mathcal{H}(\mathfrak{s}^\vee, q_F^{1/2}) \cong \mathcal{H}(\mathfrak{s}'^\vee, q_{F'}^{1/2})$ [51, Lemma 2.4]. Combine that with (7-1). □

Finally we investigate in what sense our (enhanced) L-parameters are unique.

Lemma 7.13. *Let $\pi \in \mathrm{Irr}(G, T)$. Then the ϕ_π from Theorem 7.1 is uniquely determined by Lemmas 7.5, 7.6, 7.7 and 7.11.*

Proof. Suppose that π is tempered. Lemma 7.6 determines $\mathrm{Sc}(\phi_\pi, \rho_\pi) = \tilde{\phi}$ up to $N_{G^\vee}(T^\vee \rtimes \mathbf{W}_F)$. Lemma 7.8 says that ϕ_π must be bounded, so according to [14, §0.6] ϕ_π is an open Langlands parameter. In other words, u_{ϕ_π} is uniquely determined (up to $Z_{G^\vee}(\tilde{\phi_\pi}(\mathbf{W}_F))$-conjugacy) as an element of the open orbit in $V_{\tilde{\phi_\pi}}$. Thus ϕ_π is unique up to $G^\vee$-conjugacy.

Suppose now that π is not tempered. Let (P, τ, z) be the triple associated to π by the Langlands classification. Here τ is tempered, so the above determines $\phi_\tau \in \Phi(P/R_u(P), T)$ uniquely. Then Lemma 7.5 forces $\phi_{\tau\otimes z} = z \cdot \phi_\tau$ and Lemma 7.11 says that ϕ_π equals $z\phi_\tau$ up to $G^\vee$-conjugacy. □

It is less clear to what extent the enhancement ρ_π of ϕ_π is uniquely specified. Lemma 7.11 reduces this issue to tempered $\pi \in \mathrm{Irr}(G, T)$. Then ϕ_π is bounded, so open. By Lemma 7.3 the L-packet $\Pi_{\phi_\pi}(G)$ contains a unique generic member, namely $\pi(\phi_\pi, \mathrm{triv})$. That fixes the normalization of the intertwining operators from elements of R_{ϕ_π}, which then determines $\pi(\phi_\pi, \rho)$ for any $\rho \in \mathrm{Irr}(R_{\phi_\pi})$ such that $(\phi_\pi, \rho) \in \Phi_e(G, T)$. However, to make that precise one has to say on which module these intertwining operators acts. That involves the constructions with Hecke algebras in Sections 5 and 6. Those are canonical, but they may not be unique; see Remarks 5.5 and 7.2.

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MAARTEN SOLLEVELD
IMAPP
RADBOUD UNIVERSITEIT NIJMEGEN
NIJMEGEN
NETHERLANDS
m.solleveld@science.ru.nl

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CANONICAL METRIC CONNECTIONS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

SHUWEN CHEN AND FANGYANG ZHENG

We consider the conjecture of Chen and Nie concerning the space forms for canonical metric connections of compact Hermitian manifolds. We verify the conjecture for two special types of Hermitian manifolds: complex nilmanifolds with nilpotent J, and nonbalanced Bismut torsion-parallel manifolds.

1. Introduction and statement of results

The simplest kind of Riemannian manifolds are the so-called *space forms*, which means complete Riemannian manifolds with constant sectional curvature. Their universal covers are respectively the sphere S^n, the Euclidean space $\mathbb{R}^n$, or the hyperbolic space $\mathbb{H}^n$, equipped with (scaling of) the standard metrics.

In the complex case, the sectional curvature of Hermitian manifolds in general can no longer be constant (unless it is flat). Instead one requires the *holomorphic sectional curvature* to be constant. When the metric is Kähler, one gets the so-called *complex space forms*, namely complete Kähler manifolds with constant holomorphic sectional curvature. Analogous to the Riemannian case, their universal covers are the complex projective space $\mathbb{CP}^n$, the complex Euclidean space $\mathbb{C}^n$, or the complex hyperbolic space $\mathbb{CH}^n$, equipped with (scaling of) the standard metrics.

When a Hermitian metric is not Kähler, its curvature tensor does not obey all the Kähler symmetries in general. As a result, the holomorphic sectional curvature could no longer determine the entire curvature tensor. So one would naturally wonder about when can the holomorphic sectional curvature be constant. In this direction, a long-standing conjecture is the following:

Conjecture 1 (constant holomorphic sectional curvature conjecture). *Given any compact Hermitian manifold*, *if the holomorphic sectional curvature of its Chern*

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(*or Levi-Civita*) *connection is a constant* c, *then when* $c \neq 0$ *the metric must be Kähler* (*hence a complex space form*), *while when* $c = 0$ *the metric must be Chern* (*or Levi-Civita*) *flat.*

Note that when $n \geq 3$, there are compact Chern flat (or Levi-Civita flat) manifolds that are non-Kähler. Compact Chern flat manifolds are compact quotients of complex Lie groups by the classic theorem of Boothby [5], and compact Levi-Civita flat Hermitian threefolds were classified in [13], while for dimension 4 or higher it is still an open question. Note also that the compactness assumption is necessary for the conjecture.

For $n = 2$, Conjecture 1 was confirmed by Balas and Gauduchon [2; 3], Sato and Sekigawa [18], and Apostolov, Davidov, and Mushkarov [1] in the 1980s and 1990s. In higher dimensions, the first substantial result towards this conjecture is the result by Davidov, Grantcharov, and Mushkarov [11], in which they showed among other things that the only twistor space with constant holomorphic sectional curvature is the complex space form $\mathbb{CP}^3$. More recently, Chen, Chen, and Nie [9] showed that, for locally conformally Kähler manifolds, the conjecture holds provided that the holomorphic sectional curvature is a nonpositive constant. The conjecture is also known in some other special cases; see, for instance, [8; 15; 16; 20].

Given a Hermitian manifold (M^n, g), besides the Chern connection ∇^c and Levi-Civita connection ∇, there is another metric connection that is widely studied: the *Bismut connection* ∇^b. It is the connection compatible with both the metric g and the almost complex structure J, as well as having totally skew-symmetric torsion. Its existence and uniqueness was proved by Bismut in [4]. It was discovered independently by Strominger [19], so in some literature it was also called the Strominger connection.

Previously, we tried to extend Conjecture 1 to the Bismut connection case, and raised the following conjecture and question [7, Conjecture 2 and Question 1]:

Conjecture 2. *Given any compact Hermitian manifold*, *if the holomorphic sectional curvature of its Bismut connection is a nonzero constant*, *then the metric must be Kähler* (*hence a complex space form*).

Question 3. *What kind of compact Hermitian manifolds will have zero Bismut holomorphic sectional curvature but are not Bismut flat*?

The reason for the above splitting is due to the fact that there are examples of compact Hermitian manifolds with vanishing Bismut holomorphic sectional curvature but its Bismut curvature is not identically zero, e.g., the standard Hopf manifolds of dimension ≥ 3 (or standard Hopf surfaces but with a specially varied metric). In [7] we proved Conjecture 2 and answered Question 3 for the $n = 2$ case, and also answered them in the special case of complex nilmanifolds with nilpotent J and the Bismut Kähler-like manifolds (see [7, Theorems 3 and 4]).

The three canonical connections on a given Hermitian manifold (M^n, g), namely, ∇ (Levi-Civita), ∇^c (Chern) and ∇^b (Bismut), are all equal to each other when g is Kähler. While when g is not Kähler, they are linearly independent. The line of connections spanned by Chern and Bismut are called *Gauduchon connections*, discovered by Gauduchon [12]:

$$D^r = \tfrac{1}{2}(1+r)\nabla^c + \tfrac{1}{2}(1-r)\nabla^b, \quad r \in \mathbb{R}.$$

In the literature there are a number of different ways to parametrize these Gauduchon connections. Here we used Gauduchon's approach, so that $D^1 = \nabla^c$ is the Chern connection, $D^{-1} = \nabla^b$ is the Bismut connection. Note that in this parametrization, $D^0 = \frac{1}{2}(\nabla^c + \nabla^b)$ is the Hermitian projection of the Levi-Civita connection ∇, often called the *Lichnerowicz connection*. Let

$$D^r_s = (1-s)D^r + s\nabla, \quad (r, s) \in \Omega := \{s \neq 1\} \cup \{(0, 1)\} \subset \mathbb{R}^2.$$

In the rs-plane $\mathbb{R}^2$, the domain Ω is the cone over the r-axis with vertex $(0, 1)$, or equivalently, the *plane of canonical metric connections* D^r_s are the cone over the line of Gauduchon connections with vertex at the Levi-Civita connection. As is well known, when the metric g is Kähler, all D^r_s coincide, while when g is not Kähler, $D^r_s \neq D^{r'}_{s'}$ for any two distinct points (r, s), (r', s') in Ω.

In [6], H. Chen and X. Nie gave a beautiful characterization of the possible extension of Conjecture 1 to the 2-parameter family of canonical metric connections D^r_s. They discovered the particular subset (which will be called the *Chen–Nie curve* from now on)

$$\Gamma = \{(r, s) \in \mathbb{R}^2 \mid (1-r+rs)^2 + s^2 = 4\} \subset \Omega,$$

and proved the following theorem (see [6, Theorem 2.4]):

Theorem 4 (Chen–Nie). *Let (M^2, g) be a compact Hermitian surface with pointwise constant holomorphic sectional curvature with respect to its D^r_s connection. Then either g must be Kähler, or $(r, s) \in \Gamma$ and (M^2, g) is an isosceles Hopf surface equipped with an admissible metric.*

In other words, in order to extend Conjecture 1 to the D^r_s connections, one has to exclude the subset Γ and address its zero holomorphic sectional curvature case differently just like what we have seen in the Bismut connection case in [7]. Following their discovery, it is natural to propose the following:

Conjecture 5 (Chen–Nie). *Given any compact Hermitian manifold, if the holomorphic sectional curvature of its D^r_s connection is a nonzero constant, then the metric must be Kähler (hence a complex space form). If the holomorphic sectional curvature of its D^r_s connection is zero and $(r, s) \in \Omega \setminus \Gamma$, then g must be D^r_s flat.*

A companion question is the following:

Question 6. *For any $(r, s) \in \Gamma$, what kind of compact Hermitian manifolds will have its D^r_s connection being nonflat but with vanishing holomorphic sectional curvature*?

The aforementioned theorem of Chen and Nie says that Conjecture 5 is true when $n = 2$. Note that the Chen–Nie curve Γ lies between the two horizontal lines $s = -2$ and $s = 2$, passing through the point $\big(\frac{1}{3}, -2\big)$, $(-1, 2)$, $(-1, 0)$, $(3, 0)$, and with r approaching $\pm\infty$ when $s \to 1$. Note that D^{-1}_0 is just the Bismut connection ∇^b, while $D^3_0 = 2\nabla^c - \nabla^b$ is the reflection point of ∇^b with respect to ∇^c, sometimes called the *anti-Bismut connection*. In the notation of [25], $D^{1/3}_{-2} = \nabla^-$, $D^{-1}_2 = \nabla^+$ are also special connections in the sense that their curvature moves in sync with the curvature of ∇^b, for instance, the curvatures of ∇^+, ∇^- and ∇^b will obey all Kähler symmetries (namely, be Kähler-like) at the same time. Also, $\nabla^+ + \nabla^c = 2\nabla$, that is, ∇^+ is the reflection point of the Chern connection ∇^c with respect to the Levi-Civita connection ∇. Similarly, ∇^- is the reflection point of $D^{-1/3} = \frac{1}{3}\nabla^c + \frac{2}{3}\nabla^b$ with respect to $\nabla' = D^0_{-1}$, the so-called *anti-Levi-Civita connection*, while $D^{-1/3}$ is called the *minimal connection* since it has the smallest norm of torsion amongst all Gauduchon connections.

By a beautiful theorem of Lafuente and Stanfield [14, Theorem A], for any Gauduchon connection D^r other than Chern or Bismut, if a compact Hermitian manifold has flat (or more generally, Kähler-like) D^r, then the metric must be Kähler. For D^r_s with $s \neq 0$, it was proved in [25, Theorem 4] that if D^r_s is other than ∇, ∇', ∇^+ or ∇^-, and if a Hermitian manifold has flat (or more generally, Kähler-like) D^r_s, then the metric must be Kähler.

In other words, with the exception of the three canonical connections ∇^c, ∇^b, ∇ and ∇', ∇^+, ∇^-, for any other D^r_s, a compact Hermitian manifold with flat D^r_s must be Kähler.

The main purpose of this short article is to confirm Conjecture 5 for general dimensional Hermitian manifolds in the special case of either complex nilmanifolds (with nilpotent J in the sense of [10]) or nonbalanced Bismut torsion-parallel manifolds. Recall that by a *complex nilmanifold* here we mean a compact Hermitian manifold (M^n, g) such that its universal cover is a nilpotent Lie group equipped with a left-invariant complex structure and a compatible left-invariant metric. A compact Hermitian manifold (M^n, g) is said to be *Bismut torsion-parallel* (or BTP for brevity) if $\nabla^b T^b = 0$, where T^b is the torsion of the Bismut connection ∇^b. Also g is said to be *balanced* if $d(\omega^{n-1}) = 0$, where ω is the Kähler form of g. Examples and properties of BTP manifolds are discussed in [26; 27]. Note that any non-Kähler Bismut Kähler-like (BKL) manifold is always a nonbalanced BTP manifold.

Theorem 7. *Let (M^n, g) be a compact Hermitian manifold such that for some $(r, s) \in \Omega$, the canonical metric connection D^r_s of g has constant holomorphic sectional curvature c.*

(1) *Assume that* (M^n, g) *is a complex nilmanifold with nilpotent* J. *If* D^r_s *is not the Chern connection*, *then* $c = 0$, *the Lie group is abelian*, *and* (M^n, g) *is* (*a finite undercover of*) *a flat complex torus. If* D^r_s *is the Chern connection*, *then* $c = 0$, *the Lie group is a* (*nilpotent*) *complex Lie group*, *and* (M^n, g) *is Chern flat.*

(2) *If* (M^n, g) *is a nonbalanced BTP manifold*, *then* $c = 0$ *and* $(r, s) \in \Gamma$.

(3) *If* (M^3, g) *is balanced BTP of dimension* 3, *then either it is Kähler*, *or it is Chern flat and* $c = 0$, $(r, s) = (1, 0)$ (*namely*, *with* D^r_s *being the Chern connection*), *with* M^3 *being a compact quotient of the simple complex Lie group* $\mathrm{SO}(3, \mathbb{C})$ *equipped with the standard metric.*

The above theorem says that Conjecture 5 holds for all complex nilmanifolds with nilpotent J, all compact nonbalanced BTP manifolds, and all compact balanced BTP threefolds. In particular, since (non-Kähler) Bismut Kähler-like (BKL) manifolds or Vaisman manifolds are always nonbalanced BTP, we know that Conjecture 5 holds for BKL or Vaisman manifolds.

2. Preliminaries

First let us recall the definition of *sectional curvature* and *holomorphic sectional curvature*. Given a connection D on a differential manifold M^n, its torsion and curvature are respectively defined by

$$T^D(x, y) = D_x y - D_y x - [x, y], \qquad R^D_{xy} z = D_x D_y z - D_y D_x z - D_{[x,y]} z,$$

where x, y, z are vector fields on M. When M is equipped with a Riemannian metric $g = \langle \cdot\,, \cdot \rangle$, we could use g to lower the index and turn R^D into a (4, 0)-tensor (which we still denote by the same letter):

$$R^D(x, y, z, w) = \langle R^D_{xy} z, w \rangle,$$

where x, y, z, w are vector fields on M. Clearly, R^D is skew-symmetric with respect to its first two positions. If D is a *metric connection*, namely, $Dg = 0$, then R^D is skew-symmetric with respect to its last two positions as well, hence it becomes a bilinear form on $\Lambda^2 TM$. Note that the presence of torsion T^D usually will make the bilinear form R^D not symmetric in general. The *sectional curvature* of D is defined by

$$K^D(\pi) = -\frac{R^D(x \wedge y, x \wedge y)}{|x \wedge y|^2}, \quad \text{where } \pi = \mathrm{span}_{\mathbb{R}}\{x, y\} \subset T_p M.$$

Here as usual $|x \wedge y|^2 = |x|^2 |y|^2 - \langle x, y \rangle^2$. It is easy to see that the value $K^D(\pi)$ is independent of the choice of the basis of the 2-plane π in the tangent space $T_p M$. Since the bilinear form R^D may not be symmetric, the values of K in general will not determine the entire R^D (but only the symmetric part of R^D).

Now suppose (M^n, g) is a Hermitian manifold and D is a metric connection. Then besides the sectional curvature K^D, one also has the *holomorphic sectional curvature* H^D, which is the restriction of K^D on those 2-planes π that are J-invariant: $J\pi = \pi$. In this case, for any nonzero $x \in \pi$, $\{x, Jx\}$ is a basis of π, so we can rewrite H^D in complex coordinates:

$$H^D(X) = \frac{R^D(X, \overline{X}, X, \overline{X})}{|X|^4}, \quad X = x - \sqrt{-1}Jx.$$

Let $\{e_1, \dots, e_n\}$ be a local frame of type-(1, 0) complex tangent vector fields on M^n, and denote by $R^D_{i\bar{j}k\bar{\ell}} = R^D(e_i, \bar{e}_j, e_k, \bar{e}_\ell)$ the components of R^D under the frame e. We see that

$$H^D \equiv c \iff \widehat{R}^D_{i\bar{j}k\bar{\ell}} = \tfrac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}),$$

where

$$\widehat{R}^D_{i\bar{j}k\bar{\ell}} = \tfrac{1}{4}(R^D_{i\bar{j}k\bar{\ell}} + R^D_{k\bar{j}i\bar{\ell}} + R^D_{i\bar{\ell}k\bar{j}} + R^D_{k\bar{\ell}i\bar{j}})$$

is the symmetrization of R^D.

Next let us recall the structure equations of Hermitian manifolds. Let (M^n, g) be a Hermitian manifold and denote by ω the Kähler form associated with g. Denote by ∇, ∇^c, ∇^b the Levi-Civita, Chern, and Bismut connection, respectively. Denote by R the curvature of ∇, by $T^c = T$ and R^c the torsion and curvature of ∇^c, and by T^b and R^b the torsion and curvature of ∇^b. Under any local unitary frame e, let us write

$$T^c(e_i, e_k) = \sum_{j=1}^{n} T^j_{ik} e_j, \quad 1 \le i, k \le n.$$

Then T^j_{ik} are the Chern torsion components under e. Let φ be the coframe of local (1, 0)-forms dual to e, namely, $\varphi_i(e_j) = \delta_{ij}$. Denote by θ, Θ the matrices of connection and curvature of ∇^c under e. Let τ be the column vector under e of the Chern torsion, namely, $\tau_j = \frac{1}{2}\sum_{i,k} T^j_{ik}\varphi_i \wedge \varphi_k$. Then the structure equations and Bianchi identities are

$$d\varphi = -{}^t\theta \wedge \varphi + \tau, \qquad d\theta = \theta \wedge \theta + \Theta,$$
$$d\tau = -{}^t\theta \wedge \tau + {}^t\Theta \wedge \varphi, \quad d\Theta = \theta \wedge \Theta - \Theta \wedge \theta.$$

Similarly, denote by θ^b, Θ^b the matrices of connection and curvature of ∇^b under e. Then

$$\Theta^b = d\theta^b - \theta^b \wedge \theta^b.$$

Let $\gamma = \nabla^b - \nabla^c$ be the tensor, and for simplicity we will also write $\gamma = \theta^b - \theta$ under e. Then by [23] we have

$$\gamma e_i = \sum_j \gamma_{ij} e_j = \sum_{j,k} (T^j_{ik}\varphi_k - \overline{T^i_{jk}}\,\overline{\varphi}_k) e_j. \tag{1}$$

Also, following the notation of [22; 23], the Levi-Civita connection is given by

$$\nabla e_i = \sum_j \big((\theta_{ij} + \tfrac{1}{2}\gamma_{ij}) e_j + \overline{\beta_{ij}}\, \bar{e}_j \big), \quad \text{where } \beta_{ij} = \tfrac{1}{2} \sum_k \overline{T^k_{ij}} \varphi_k.$$

Therefore, if we write $e = {}^t(e_1, \dots, e_n)$ and $\varphi = {}^t(\varphi_1, \dots, \varphi_n)$ as column vectors, then under the frame ${}^t(e, \bar{e})$ the matrices of connection and curvature of ∇ are given by

$$\hat{\theta} = \begin{bmatrix} \theta_1 & \bar{\beta} \\ \beta & \overline{\theta_1} \end{bmatrix}, \quad \widehat{\Theta} = \begin{bmatrix} \Theta_1 & \overline{\Theta}_2 \\ \Theta_2 & \overline{\Theta}_1 \end{bmatrix}, \quad \theta_1 = \theta + \tfrac{1}{2}\gamma,$$

where

$$\begin{aligned} \Theta_1 &= d\theta_1 - \theta_1 \wedge \theta_1 - \bar{\beta} \wedge \beta, \\ \Theta_2 &= d\beta - \beta \wedge \theta_1 - \overline{\theta_1} \wedge \beta, \\ d\varphi &= -\,{}^t\theta_1 \wedge \varphi - {}^t\beta \wedge \overline{\varphi}. \end{aligned}$$

As is well known, the entries of the curvature matrix Θ are all (1, 1)-forms, while the entries of the column vector τ are all (2, 0)-forms, under any frame e. Since

$$\Theta_{ij} = \sum_{k,\ell=1}^n R^c_{k\bar{\ell}i\bar{j}}\, \varphi_k \wedge \overline{\varphi}_\ell, \quad \Theta^b_{ij} = \sum_{k,\ell=1}^n (R^b_{k\ell i\bar{j}}\, \varphi_k \wedge \varphi_\ell + R^b_{\bar{k}\bar{\ell}i\bar{j}}\, \overline{\varphi}_k \wedge \overline{\varphi}_\ell + R^b_{k\bar{\ell}i\bar{j}}\, \varphi_k \wedge \overline{\varphi}_\ell),$$

and

$$\begin{aligned} (\Theta_1)_{ij} &= \sum_{k,\ell=1}^n (R_{k\ell i\bar{j}}\, \varphi_k \wedge \varphi_\ell + R_{\bar{k}\bar{\ell}i\bar{j}}\, \overline{\varphi}_k \wedge \overline{\varphi}_\ell + R_{k\bar{\ell}i\bar{j}}\, \varphi_k \wedge \overline{\varphi}_\ell), \\ (\Theta_2)_{ij} &= \sum_{k,\ell=1}^n (R_{k\ell\bar{i}\bar{j}}\, \varphi_k \wedge \varphi_\ell + R_{k\bar{\ell}\bar{i}\bar{j}}\, \varphi_k \wedge \overline{\varphi}_\ell). \end{aligned}$$

From the structure equations and Bianchi identities, one gets this relationship between the three curvature tensors [7, Lemma 2]:

Lemma 8. *Let (M^n, g) be a Hermitian manifold. Under any local unitary frame e,*

$$R_{k\bar{\ell}i\bar{j}} - R^c_{k\bar{\ell}i\bar{j}} = -\tfrac{1}{2} T^j_{ik,\bar{\ell}} - \tfrac{1}{2}\overline{T^i_{j\ell,\bar{k}}} + \tfrac{1}{4} \sum_r (T^r_{ik} \overline{T^r_{j\ell}} - T^j_{kr} \overline{T^i_{\ell r}} - T^\ell_{ir} \overline{T^k_{jr}}),$$

$$R^b_{k\bar{\ell}i\bar{j}} - R^c_{k\bar{\ell}i\bar{j}} = -T^j_{ik,\bar{\ell}} - \overline{T^i_{j\ell,\bar{k}}} + \sum_r (T^r_{ik} \overline{T^r_{j\ell}} - T^j_{kr} \overline{T^i_{\ell r}}) \tag{2}$$

for any i, j, k, ℓ, where the indices after commas mean covariant derivatives with respect to ∇^c.

Note that the discrepancy in the coefficients here and [7, Lemma 2] is due to the fact that our T^j_{ik} is twice of that in [7]. For our later use, we will also need to express the covariant derivatives of torsion in terms of the Bismut connection. Let us use indices after semicolons to denote the covariant derivatives with respect to the Bismut connection. By the formula for γ (1) we have

$$
\begin{aligned}
T^j_{ik,\ell} - T^j_{ik;\ell} &= \sum_r (T^j_{rk}T^r_{i\ell} + T^j_{ir}T^r_{k\ell} - T^r_{ik}T^j_{r\ell}), \\
T^j_{ik,\bar\ell} - T^j_{ik;\bar\ell} &= \sum_r (-T^j_{rk}\overline{T^i_{r\ell}} - T^j_{ir}\overline{T^k_{r\ell}} + T^r_{ik}\overline{T^r_{j\ell}}).
\end{aligned} \tag{3}
$$

Next let us examine the curvature components of the canonical metric connections

$$
D^r_s = (1-s)D^r + s\nabla, \quad D^r = \tfrac{1}{2}(1+r)\nabla^c + \tfrac{1}{2}(1-r)\nabla^b, \qquad (r,s)\in\Omega.
$$

For convenience, in the following we will fix an arbitrary point $(r,s)\in\Omega$ and write D for D^r_s. Under the local unitary frame e, we have

$$
De_i = \sum_j (\theta^D_{ij}e_j + s\overline{\beta_{ij}}\,\bar e_j), \quad \theta^D = \theta + \tfrac{1}{2}(1-r+rs)\gamma = \theta + t\gamma,
$$

where we wrote $t = \frac{1}{2}(1-r+rs)$. So under the frame ${}^t(e,\bar e)$ the matrices of connection and curvature of D are given by

$$
\hat\theta^D = \begin{bmatrix} \theta^D & s\bar\beta \\ s\beta & \overline{\theta^D} \end{bmatrix}, \quad \widehat\Theta^D = \begin{bmatrix} \Theta^D_1 & \overline{\Theta^D_2} \\ \Theta^D_2 & \overline{\Theta^D_1} \end{bmatrix}, \quad \theta^D = \theta + t\gamma,
$$

where

$$
\begin{aligned}
\Theta^D_1 &= d\theta^D - \theta^D\wedge\theta^D - s^2\bar\beta\wedge\beta, \\
\Theta^D_2 &= s(d\beta - \beta\wedge\theta^D - \overline{\theta^D}\wedge\beta).
\end{aligned}
$$

For any fixed point $p\in M$, by the same proof of [23, Lemma 4], we may choose our local unitary frame e near p so that $\theta^D|_p = 0$. So at the point p we have $\theta|_p = -t\gamma|_p$. Let γ' be the (1, 0)-part of γ. Then we have $\gamma = \gamma' - \gamma'^*$ where γ'^* denotes the conjugate transpose of γ'. Note that we always have ${}^t\gamma'\wedge\varphi = -2\tau$, so by the structure equation we get

$$
\partial\varphi_r = \left(\tfrac{1}{2} - t\right)\sum_{i,k} T^r_{ik}\varphi_i\wedge\varphi_k, \quad \bar\partial\varphi_r = t\sum_{i,k}\overline{T^i_{rk}}\varphi_i\wedge\bar\varphi_k \qquad \text{at the point } p.
$$

Since $\theta^b|_p = -(t-1)\gamma|_p$, at p we have

$$
\begin{aligned}
\bar\partial\gamma'_{k\ell}|_p &= \bar\partial\sum_i T^\ell_{ki}\varphi_i = \sum_i (\bar\partial(T^\ell_{ki})\wedge\varphi_i + T^\ell_{ki}\bar\partial\varphi_i) \\
&= \sum_{i,j}\left(-\bar\partial_j(T^\ell_{ki}) + t\sum_r T^\ell_{kr}\overline{T^i_{rj}}\right)\varphi_i\wedge\bar\varphi_j \\
&= \sum_{i,j}\left(T^\ell_{ik;\bar j} + (t-1)\sum_r (T^r_{ik}\overline{T^r_{j\ell}} - T^\ell_{ir}\overline{T^k_{jr}}) - \sum_r T^\ell_{kr}\overline{T^i_{jr}}\right)\varphi_i\wedge\bar\varphi_j,
\end{aligned}
$$

where index after the semicolon stands for covariant derivative with respect to the Bismut connection ∇^b. At p, we have

$$\Theta_1^D = \Theta + \Phi, \quad \Phi = t d\gamma + t^2 \gamma \wedge \gamma - s^2 \bar{\beta} \wedge \beta.$$

We compute the (1, 1)-part of Φ at the point p:

$$\begin{aligned}(\Phi_{k\ell})^{1,1} &= t\bar{\partial}\gamma'_{k\ell} - t\partial\overline{\gamma'_{\ell k}} - t^2 \sum_r (\gamma'_{kr} \wedge \overline{\gamma'_{\ell r}} + \overline{\gamma'_{rk}} \wedge \gamma'_{r\ell}) - s^2 \sum_r \overline{\beta_{kr}} \wedge \beta_{r\ell} \\ &= \sum_{i,j} \big(t T^\ell_{ik;\bar{j}} + t\overline{T^k_{j\ell;\bar{i}}} + (t^2 - 2t)(w - v^\ell_i) - t(v^j_i + v^\ell_k) - \tfrac{1}{4}s^2 v^j_k \big)\varphi_i \wedge \overline{\varphi}_j.\end{aligned}$$

Here and from now on we will use these abbreviations:

$$w = \sum_r T^r_{ik}\overline{T^r_{j\ell}}, \quad v^j_i = \sum_r T^j_{ir}\overline{T^k_{\ell r}}, \quad v^\ell_i = \sum_r T^\ell_{ir}\overline{T^k_{jr}}, \tag{4}$$
$$v^j_k = \sum_r T^j_{kr}\overline{T^i_{\ell r}}, \quad v^\ell_k = \sum_r T^\ell_{kr}\overline{T^i_{jr}}.$$

We therefore conclude the following:

Lemma 9. *Let (M^n, g) be a Hermitian manifold. Under any local unitary frame e, the curvature of $D = D^r_s$ has components*

$$R^D_{i\bar{j}k\bar{\ell}} = R^c_{i\bar{j}k\bar{\ell}} + t(T^\ell_{ik;\bar{j}} + \overline{T^k_{j\ell;\bar{i}}}) + (t^2 - 2t)(w - v^\ell_i) - t(v^j_i + v^\ell_k) - \tfrac{1}{4}s^2 v^j_k,$$

for any $1 \le i, j, k, \ell \le n$, where $t = \frac{1}{2}(1 - r + rs)$, R^c is the Chern curvature, w and v^j_i etc. are given by (4), *and indices after the semicolon stand for covariant derivatives with respect to the Bismut connection ∇^b.*

Note that using this shorthand notation, by (3) and (2), we get:

Lemma 10. *Given a Hermitian manifold (M^n, g), under any local unitary frame e,*

$$R^b_{i\bar{j}k\bar{\ell}} - R^c_{i\bar{j}k\bar{\ell}} = T^\ell_{ik;\bar{j}} + \overline{T^k_{j\ell;\bar{i}}} + v^\ell_i - v^j_i - v^\ell_k - w \quad \textit{for all } 1 \le i, j, k, \ell \le n,$$

where w and v^j_i etc. are given by (4), *and the indices after the semicolon stand for covariant derivatives with respect to the Bismut connection ∇^b.*

Since $\widehat{T^j_{ik;\bar{\ell}}} = 0$, $\widehat{w} = 0$, and $\hat{v}^j_i = \hat{v}^j_k = \hat{v}^\ell_i = \hat{v}^\ell_k = \frac{1}{4}(v^j_i + v^j_k + v^\ell_i + v^\ell_k) := \hat{v}$, we finally end up with the following identity which holds for any Hermitian manifold:

$$\widehat{R}^D_{i\bar{j}k\bar{\ell}} = \widehat{R}^c_{i\bar{j}k\bar{\ell}} - \big(t^2 + \tfrac{1}{4}s^2\big)\hat{v} = \widehat{R}^b_{i\bar{j}k\bar{\ell}} + \big(1 - t^2 - \tfrac{1}{4}s^2\big)\hat{v}. \tag{5}$$

In the rest of this section, let us recall a basic formula for *Lie–Hermitian manifolds*, which means compact Hermitian manifolds with universal cover (G, J, g), where G is a Lie group equipped with a left-invariant complex structure J and a compatible left-invariant metric g. Denote by $\mathfrak{g}$ the Lie algebra of G. Then the left-invariant complex structure and metric on G correspond to a complex structure and a metric on $\mathfrak{g}$: the former means an almost complex structure J on the vector space $\mathfrak{g}$ satisfying the integrability condition

$$[x, y] - [Jx, Jy] + J[Jx, y] + J[x, Jy] = 0 \quad \text{for all } x, y \in \mathfrak{g},$$

while the latter means an inner product $g = \langle \cdot , \cdot \rangle$ on $\mathfrak{g}$ such that $\langle Jx, Jy\rangle = \langle x, y\rangle$ for any $x, y \in \mathfrak{g}$. Denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$, and by $\mathfrak{g}^{1,0}$ its (1, 0)-part, namely,

$$\mathfrak{g}^{1,0} = \{x - \sqrt{-1}Jx \mid x \in \mathfrak{g}\}.$$

By a *unitary frame* of $\mathfrak{g}$ we mean a basis $e = \{e_1, \dots, e_n\}$ of the complex vector space $\mathfrak{g}^{1,0} \cong \mathbb{C}^n$ so that $\langle e_i, \bar{e}_j\rangle = \delta_{ij}$ for any $1 \le i, j \le n$. Here we assume that G has real dimension $2n$ and we have extended $\langle \cdot , \cdot \rangle$ bilinearly over $\mathbb{C}$. Following the notation of [21; 22; 24] we will let

$$[e_i, e_j] = \sum_k C^k_{ij} e_k, \quad [e_i, \bar{e}_j] = \sum_k (\overline{D^i_{kj}} e_k - D^j_{ki} \bar{e}_k) \qquad \text{for all } 1 \le i, j \le n. \tag{6}$$

If we denote by φ the coframe dual to e, then the structure equation takes the form

$$d\varphi_i = -\sum_{j,k=1}^{n} \left(\tfrac{1}{2} C^i_{jk} \varphi_j \wedge \varphi_k + \overline{D^j_{ik}} \varphi_j \wedge \overline{\varphi}_k\right)$$

and the first Bianchi identity which coincides with the Jacobi identity becomes

$$\sum_{r=1}^{n} (C^r_{ij} C^\ell_{rk} + C^r_{jk} C^\ell_{ri} + C^r_{ki} C^\ell_{rj}) = 0,$$

$$\sum_{r=1}^{n} (C^r_{ik} D^\ell_{jr} + D^r_{ji} D^\ell_{rk} - D^r_{jk} D^\ell_{ri}) = 0,$$

$$\sum_{r=1}^{n} (C^r_{ik} \overline{D^r_{j\ell}} - C^j_{rk} \overline{D^i_{r\ell}} + C^j_{ri} \overline{D^k_{r\ell}} - D^\ell_{ri} \overline{D^k_{jr}} + D^\ell_{rk} \overline{D^i_{jr}}) = 0,$$

for any $1 \le i, j, k, \ell \le n$. Under the frame e, the Chern connection form and Chern torsion components are

$$\theta_{ij} = \sum_{k=1}^{n} (D^j_{ik} \varphi_k - \overline{D^i_{jk}} \overline{\varphi_k}), \quad T^j_{ik} = -C^j_{ik} - D^j_{ik} + D^j_{ki}.$$

From this, we get the expression for γ and the Bismut connection matrix

$$\theta^b_{ij} = \sum_{k=1}^{n} ((-C^j_{ik} + D^j_{ki}) \varphi_k + (\overline{C^i_{jk}} - \overline{D^i_{kj}})\, \overline{\varphi}_k).$$

Following [7], we can take the (1, 1)-part in $\Theta^b = d\theta^b - \theta^b \wedge \theta^b$ and obtain

$$R^b_{k\bar{\ell}i\bar{j}} = (C^r_{ik}\overline{C^r_{j\ell}} - C^j_{rk}\overline{C^i_{r\ell}}) - (C^r_{ik}\overline{D^r_{\ell j}} + \overline{C^r_{j\ell}}D^r_{ki}) + (C^j_{ir}\overline{D^k_{r\ell}} - C^j_{kr}\overline{D^i_{\ell r}}) \\ + (\overline{C^i_{jr}}D^\ell_{rk} - \overline{C^i_{\ell r}}D^j_{kr}) - (D^j_{ri}\overline{D^k_{r\ell}} + D^\ell_{rk}\overline{D^i_{rj}}) + (D^r_{ki}\overline{D^r_{\ell j}} - D^j_{kr}\overline{D^i_{\ell r}})$$

for any i, j, k, ℓ. Here r is summed up from 1 to n. For our later proofs, we will also need the following result for the symmetrization of R^b, which are (33) and (34) from [7]:

Lemma 11. *Let (G, J, g) be an even-dimensional Lie group equipped with a left-invariant complex structure and a compatible metric. Let e be a unitary frame of $\mathfrak{g}^{1,0}$ and C, D be defined by* (6)*. Then under e we have*

$$(7)\quad 4\widehat{R}^b_{k\bar{k}i\bar{i}} = -\big(|C^i_{rk}|^2+|C^k_{ri}|^2+2\,\mathrm{Re}(C^i_{ri}\overline{C^k_{rk}})\big) \\ +2\,\mathrm{Re}\big(\overline{C^i_{ir}}(D^k_{rk}-D^k_{kr})+\overline{C^k_{kr}}(D^i_{ri}-D^i_{ir})+\overline{C^i_{kr}}(D^i_{rk}-D^i_{kr})+\overline{C^k_{ir}}(D^k_{ri}-D^k_{ir})\big) \\ -2\big(|D^i_{rk}|^2+|D^k_{ri}|^2+2\,\mathrm{Re}(D^i_{ri}\overline{D^k_{rk}})\big)+\big(|D^r_{ki}|^2+|D^r_{ik}|^2+2\,\mathrm{Re}(D^r_{ik}\overline{D^r_{ki}})\big) \\ -\big(|D^i_{kr}|^2+|D^k_{ir}|^2+2\,\mathrm{Re}(D^i_{ir}\overline{D^k_{kr}})\big),$$

$$(8)\quad \widehat{R}^b_{i\bar{i}i\bar{i}} = -|C^i_{ri}|^2+2\,\mathrm{Re}(\overline{C^i_{ir}}(D^i_{ri}-D^i_{ir}))-2\,|D^i_{ri}|^2+|D^r_{ii}|^2-|D^i_{ir}|^2.$$

Finally let us recall this famous result of Salamon [17, Theorem 1.3]:

Theorem 12 (Salamon). *Let G be a nilpotent Lie group of dimension $2n$ equipped with a left-invariant complex structure. Then there exists a coframe $\varphi = \{\varphi_1, \dots, \varphi_n\}$ of left-invariant $(1, 0)$-forms on G such that*

$$d\varphi_1 = 0, \qquad d\varphi_i = \mathcal{I}\{\varphi_1, \dots, \varphi_{i-1}\} \quad \textit{for all } 2 \le i \le n,$$

where $\mathcal{I}$ stands for the ideal in exterior algebra of the complexified cotangent bundle generated by those $(1, 0)$-forms.

Note that when g is a compatible left-invariant metric, clearly one can choose the above coframe φ so that it is also unitary. In terms of the structure constants C and D given by (6), this means

$$C^j_{ik} = 0 \quad \text{unless } j > i \text{ or } j > k; \qquad D^j_{ik} = 0 \quad \text{unless } i > j.$$

If the complex structure J is nilpotent in the sense of Cordero, Fernández, Gray, and Ugarte [10], then there exists an invariant unitary coframe φ so that

$$(9)\quad C^j_{ik} = 0 \quad \text{unless } j > i \text{ and } j > k; \qquad D^j_{ik} = 0 \quad \text{unless } i > j \text{ and } i > k.$$

We do not know how to prove Conjecture 5 for all nilmanifolds at the present time, but for those with nilpotent J in the sense of [10], we will be able to confirm the conjecture with the help of (9) above.

3. The standard Hopf manifolds

Consider the standard (isosceles) Hopf manifold (M^n, g), $n \geq 2$, where the manifold and the Kähler form ω of the metric are given by

$$(10)\quad M^n = (\mathbb{C}^n \setminus \{0\})/\langle \phi \rangle, \quad \omega = \sqrt{-1}\frac{\partial\bar{\partial}|z|^2}{|z|^2}, \quad \phi(z_1, \ldots, z_n) = (a_1 z_1, \ldots, a_n z_n),$$

where a_i are constants satisfying $0 < |a_1| = \cdots = |a_n| < 1$. Here $(z_1, \ldots, z_n)$ denotes the standard Euclidean coordinate of $\mathbb{C}^n$ and $|z|^2$ is shorthand for $|z_1|^2 + \cdots + |z_n|^2$.

Near any given point $p \in M^n$, $(z_1, \ldots, z_n)$ gives a local holomorphic coordinate system, under which the metric g has components $g_{k\bar{\ell}} = \frac{1}{|z|^2}\delta_{k\ell}$. If we let $e_i = |z|\,\partial_i$, where $\partial_i = \frac{\partial}{\partial z_i}$, then e becomes a local unitary frame of (M^n, g). The Chern curvature components are

$$\begin{aligned} R^c_{i\bar{j}k\bar{\ell}} &= |z|^4 R^c(\partial_i, \bar{\partial}_j, \partial_k, \bar{\partial}_\ell) \\ &= |z|^4\Big(-\partial_i\bar{\partial}_j g_{k\bar{\ell}} + \sum_{r,s} \partial_i g_{k\bar{r}}\,\bar{\partial}_j g_{s\bar{\ell}}\, g^{\bar{r}s}\Big) = \delta_{ij}\delta_{k\ell} - \frac{\bar{z}_i z_j}{|z|^2}\delta_{k\ell}, \end{aligned}$$

for any $1 \leq i, j, k, \ell \leq n$. On the other hand, by the defining equation $\partial\omega^{n-1} = -\eta \wedge \omega^{n-1}$, we know that Gauduchon's torsion 1-form is $\eta = (n-1)\partial|z|^2$, and the Chern torsion components under the frame e are

$$T^j_{ik} = \frac{\bar{z}_k}{|z|}\delta_{ij} - \frac{\bar{z}_i}{|z|}\delta_{kj} \quad \text{for all } 1 \leq i, j, k \leq n.$$

From this, we compute

$$v^j_i = \sum_r T^j_{ir}\overline{T^k_{\ell r}} = \delta_{ij}\delta_{k\ell} + \frac{1}{|z|^2}(\bar{z}_i z_\ell \delta_{kj} - \bar{z}_i z_j \delta_{k\ell} - \bar{z}_k z_\ell \delta_{ij}).$$

Taking its symmetrization, we obtain

$$4\hat{v} = 2(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}) - \frac{1}{|z|^2}(\bar{z}_i z_j \delta_{k\ell} + \bar{z}_k z_\ell \delta_{ij} + \bar{z}_i z_\ell \delta_{kj} + \bar{z}_k z_j \delta_{i\ell}) = 4\widehat{R}^c.$$

So for any canonical metric connection D^r_s where $(r, s) \in \Omega$, by (5) we get

$$\widehat{R}^D = \widehat{R}^c - \big(t^2 + \tfrac{1}{4}s^2\big)\hat{v} = \big(1 - t^2 - \tfrac{1}{4}s^2\big)\widehat{R}^c.$$

In particular, whenever $t^2 + \frac{1}{4}s^2 = 1$, or equivalently, whenever (r, s) belongs to the Chen–Nie curve Γ, then one would have $\widehat{R}^D = 0$, that is, the canonical metric connection D^r_s for the standard Hopf manifold will have vanishing holomorphic sectional curvature. When $(r, s) \notin \Gamma$, on the other hand, we have

$$\widehat{R}^D_{1\bar{1}1\bar{1}} = \big(1 - t^2 - \tfrac{1}{4}s^2\big)\widehat{R}^c_{1\bar{1}1\bar{1}} = \big(1 - t^2 - \tfrac{1}{4}s^2\big)\Big(1 - \frac{|z_1|^2}{|z|^2}\Big).$$

Since $n \geq 2$, the right-hand side is not a constant function, thus the holomorphic sectional curvature of D^r_s cannot be a constant.

Next we want to check that, for $(r, s) \in \Omega$, when will the canonical metric connection D^r_s of the standard Hopf manifold be flat? To do this, let us fix any $1 \leq i, j, k, \ell \leq n$ and introduce the shorthand notation

$$b_{ij} = \frac{\bar{z}_i z_j}{|z|^2}\delta_{k\ell}, \quad b_{i\ell} = \frac{\bar{z}_i z_\ell}{|z|^2}\delta_{kj}, \quad b_{kj} = \frac{\bar{z}_k z_j}{|z|^2}\delta_{i\ell}, \quad b_{k\ell} = \frac{\bar{z}_k z_\ell}{|z|^2}\delta_{ij}.$$

We compute

$$w = \sum_r T^r_{ik}\overline{T^r_{j\ell}} = b_{ij} + b_{k\ell} - b_{i\ell} - b_{kj}.$$

Similarly,

$$\begin{aligned} v^j_i &= \delta_{ij}\delta_{k\ell} - b_{ij} - b_{k\ell} + b_{i\ell}, \quad & v^j_k &= \delta_{i\ell}\delta_{kj} + b_{k\ell} - b_{i\ell} - b_{kj}, \\ v^\ell_i &= \delta_{i\ell}\delta_{kj} + b_{ij} - b_{i\ell} - b_{kj}, \quad & v^\ell_k &= \delta_{ij}\delta_{k\ell} - b_{ij} - b_{k\ell} + b_{kj}. \end{aligned}$$

Also, $R^c_{i\bar{j}k\bar{\ell}}$ equals $\delta_{ij}\delta_{k\ell} - b_{ij}$. As is well known (see, for instance, the proof of [27, Lemma 4.6]), the standard Hopf manifolds given by (10) are Bismut torsion-parallel (or equivalently, Vaisman), so when we plug all of these expressions into the formula in Lemma 9 we end up with

$$\begin{aligned} R^D_{i\bar{j}k\bar{\ell}} = (1-2t)\delta_{ij}\delta_{k\ell} &+ \bigl(2t - t^2 - \tfrac{1}{4}s^2\bigr)\delta_{i\ell}\delta_{kj} \\ &+ +(2t-1)b_{ij}\bigl(t^2 - \tfrac{1}{4}s^2\bigr)b_{k\ell} + \bigl(\tfrac{1}{4}s^2 - t\bigr)(b_{i\ell} + b_{kj}). \end{aligned}$$

In particular, for any $i \neq k$, we have

$$\begin{aligned} R^D_{i\bar{i}k\bar{k}} &= (1-2t) + (2t-1)\frac{|z_i|^2}{|z|^2} + \bigl(t^2 - \tfrac{1}{4}s^2\bigr)\frac{|z_k|^2}{|z|^2}, \\ R^D_{i\bar{k}k\bar{i}} &= \bigl(2t - t^2 - \tfrac{1}{4}s^2\bigr) + \bigl(\tfrac{1}{4}s^2 - t\bigr)\left(\frac{|z_i|^2}{|z|^2} + \frac{|z_k|^2}{|z|^2}\right). \end{aligned}$$

Now assume that $R^D = 0$. When $n \geq 3$, each of the coefficients on the right-hand sides must be zero, so we get $1 - 2t = 0$, $\frac{1}{4}s^2 = t^2 = t$, which lead to a contradiction. This means that D^r_s can never be flat for any (r, s) when $n \geq 3$. When $n = 2$, however, we have $|z_i|^2 + |z_k|^2 = |z|^2$, so the vanishing of $R^D_{i\bar{i}k\bar{k}}$ and $R^D_{i\bar{k}k\bar{i}}$ in this case only give us

$$t^2 - \tfrac{1}{4}s^2 - 2t + 1 = 0, \quad t - t^2 = 0.$$

From this, we conclude that either $t = 1$ and $s = 0$, or $t = 0$ and $s = \pm 2$. Recall that $t = \frac{1}{2}(1 - r + rs)$, so we end up with three solutions:

$$(r, s) = (-1, 0), (-1, 2), \bigl(\tfrac{1}{3}, -2\bigr).$$

The corresponding connections are $D_0^{-1} = \nabla^b$, $D_2^{-1} = \nabla^+$, and $D_{-2}^{1/3} = \nabla^-$, namely, the Bismut connection, and the two vertices of the Chen–Nie curve Γ, which lies between the horizontal lines $s = -2$ and $s = 2$. Conversely, it is known that when $n = 2$, the isosceles Hopf surface is Bismut flat, hence is also ∇^+ and ∇^- flat ([25]).

In summary, we have proved the following:

Proposition 13. *Let* (M^n, g) *be a standard* (*isosceles*) *Hopf manifold given by* (10), *with* $n \geq 2$. *Then for any* $(r, s) \in \Omega \setminus \Gamma$, *the canonical metric connection* D_s^r *cannot have constant holomorphic sectional curvature. For any* $(r, s) \in \Gamma$, D_s^r *has vanishing holomorphic sectional curvature*, *but it is not flat except when* $n = 2$ *and* D_s^r *is* ∇^b, ∇^+ *or* ∇^-.

Recall that the Chen–Nie curve Γ is defined by $1 = t^2 + \frac{1}{4}s^2$ where $2t = 1 - r + rs$.

4. Proof of Theorem 7

In this section we will prove the main result, namely Theorem 7. Let us start with the nilmanifold case.

Proof of Theorem 7 for nilmanifolds. Let (M^n, g) be a complex nilmanifold, namely, a compact Hermitian manifold with universal cover (G, J, g), where G is a nilpotent Lie group, J a left-invariant complex structure on G, and g a left-invariant metric on G compatible with J. We assume that J is nilpotent in the sense of [10]. Now suppose that for some $(r, s) \in \Omega$, the holomorphic sectional curvature of the canonical metric connection $D = D_s^r$ is a constant c. This means that

$$\widehat{R}^D_{i\bar{j}k\bar{\ell}} = \tfrac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}) \quad \text{for all } 1 \leq i, j, k, \ell \leq n,$$

under any unitary frame e. By (5), we have

$$\widehat{R}^b_{i\bar{j}k\bar{\ell}} = \tfrac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}) + \big(t^2 + \tfrac{1}{4}s^2 - 1\big)\hat{v}.$$

Therefore,

$$\text{(11)} \quad \widehat{R}^b_{i\bar{i}k\bar{k}} = \tfrac{1}{2}c(1 + \delta_{ik}) + \big(t^2 + \tfrac{1}{4}s^2 - 1\big) \cdot \tfrac{1}{4}\sum_r \big(2\,\mathrm{Re}(T^i_{ir}\overline{T^k_{kr}}) + |T^k_{ir}|^2 + |T^i_{kr}|^2\big) \quad \text{for all } 1 \leq i, k \leq n.$$

Choose $i = k$, we get

$$\widehat{R}^b_{i\bar{i}i\bar{i}} = c + \big(t^2 + \tfrac{1}{4}s^2 - 1\big)\sum_r |T^i_{ir}|^2 = c + \big(t^2 + \tfrac{1}{4}s^2 - 1\big)\sum_{r>i} |D^i_{ri}|^2,$$

where in the last equality we used the fact that $T^j_{ik} = -C^j_{ik} - D^j_{ik} + D^j_{ki}$ and (9). Comparing the above identity with (8) and utilizing (9) again, we end up with

$$-c = \big(t^2 + \tfrac{1}{4}s^2 + 1\big)\sum_{r>i} |D^i_{ri}|^2 \quad \text{for all } 1 \leq i \leq n.$$

If we choose $i = n$, then the right-hand side is vacuum, so we know that $c = 0$. Hence $D^i_{*i} = 0$. From this, we get $T^i_{i*} = 0$. So (11) now takes the form

$$\widehat{R}^b_{i\bar{i}k\bar{k}} = \left(t^2 + \tfrac{1}{4}s^2 - 1\right)\tfrac{1}{4}\sum_r (|T^k_{ir}|^2 + |T^i_{kr}|^2) = \widehat{R}^b_{k\bar{k}i\bar{i}}.$$

Let us assume that $i < k$. Plugging the above into (7) and utilizing (9), we get

$$\begin{aligned} 4\widehat{R}^b_{k\bar{k}i\bar{i}} &= \left(t^2 + \tfrac{1}{4}s^2 - 1\right)\Big(\sum_{r<k}(|C^k_{ir}|^2 + |D^i_{kr}|^2) + \sum_{r>k}(|D^k_{ri}|^2 + |D^i_{rk}|^2)\Big) \\ &= \sum_{r<k}(-|C^k_{ri}|^2 + |D^r_{ki}|^2 - |D^i_{kr}|^2) - 2\sum_{r>k}(|D^i_{rk}|^2 + |D^k_{ri}|^2). \end{aligned}$$

That is,

$$\left(t^2 + \tfrac{1}{4}s^2 + 1\right)\sum_{r>k}(|D^k_{ri}|^2 + |D^i_{rk}|^2) + \left(t^2 + \tfrac{1}{4}s^2\right)\sum_{r<k}(|C^k_{ir}|^2 + |D^i_{kr}|^2) = \sum_{r<k}|D^r_{ki}|^2.$$

We already know that $D^j_{*j} = 0$ for any j. In particular $D^*_{2*} = 0$ by (9), so if we take $k = 2$ in the above identity, the right-hand side would be zero, thus we conclude that $D^2_{*1} = D^1_{*2} = 0$. Hence by (9) we have $D^*_{3*} = 0$. Take $k = 3$ in the above identity, again the right-hand side is zero which leads to $D^j_{*\ell} = 0$ whenever $j, \ell \leq 3$. Thus $D^*_{4*} = 0$ by (9). Repeating this process, we end up with $D = 0$. Then by the above identity again, we get $\left(t^2 + \frac{1}{4}s^2\right)C = 0$. Note that $(t, s) = (0, 0)$ means $(r, s) = (1, 0)$ or equivalently $D^r_s = \nabla^c$. So when D^r_s is not the Chern connection, we get $C = 0$, hence the Lie group G is abelian, and g is Kähler and flat. In this case (M^n, g) is a finite undercover of a flat complex torus. When $(t, s) = (0, 0)$, the connection D^r_s is the Chern connection. The vanishing of D means that the Lie group G is a complex Lie group, so g is Chern flat.

In summary, when D^r_s is not the Chern connection, the constancy of holomorphic sectional curvature for D^r_s would imply that the nilpotent group G must be abelian and g is Kähler and flat. When D^r_s is the Chern connection, the constancy of Chern holomorphic sectional curvature would imply that G is a (nilpotent) complex Lie group, and g is Chern flat. This completes the proof of Theorem 7 for the nilmanifold case. □

We remark that in the nilmanifold case we do not need to exclude any (r, s) values for the metric connection D^r_s. As one can see from the above proof, the technical assumption that J is nilpotent is crucial in the argument, and without which we do not know how to complete the proof. It would be an interesting question to answer though.

Next let us prove Theorem 7 in the BTP case.

Proof of Theorem 7 for nonbalanced BTP manifolds. Let (M^n, g) be a compact, nonbalanced BTP manifold. Assume that for some $(r, s) \in \Omega$, the canonical metric connection $D = D^r_s$ has constant holomorphic sectional curvature: $H^D = c$. Then under any local unitary frame e, we have $\widehat{R}^D_{i\bar{j}k\bar{\ell}} = \frac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj})$ for any $1 \leq i, j, k, \ell \leq n$. By (5), we get

$$\widehat{R}^b_{i\bar{j}k\bar{\ell}} = \tfrac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}) + \big(t^2 + \tfrac{1}{4}s^2 - 1\big)\hat{v} \quad \text{for all } 1 \leq i, j, k, \ell \leq n, \tag{12}$$

where $4\hat{v} = v^j_i + v^\ell_k + v^j_k + v^\ell_i$. Since g is nonbalanced BTP, by Definition 1.6 and Proposition 1.7 of [27], we know that locally on M^n there always exist the so-called *admissible frames*, which means a local unitary frame e such that the Chern torsion components under e enjoy the property $T^n_{ij} = 0$ and $T^j_{in} = \delta_{ij}a_i$ for any $1 \leq i, j \leq n$, where a_i are global constants on M^n, also, the Bismut curvature components satisfy $R^b_{n\bar{j}k\bar{\ell}} = R^b_{i\bar{j}n\bar{\ell}} = 0$ for any $1 \leq i, j, k, \ell \leq n$. Let us take $i = j$ and $k = \ell = n$ in (12). Then we get

$$0 = \tfrac{1}{2}c(1 + \delta_{in}) + \big(t^2 + \tfrac{1}{4}s^2 - 1\big)\tfrac{1}{4}\,|a_i|^2.$$

For $i = n$, since $a_n = 0$ we deduce $c = 0$. So the above equality becomes

$$\big(t^2 + \tfrac{1}{4}s^2 - 1\big)|a_i|^2 = 0$$

for each i. Since the metric is assumed to be nonbalanced, we have $a_1 + \cdots + a_{n-1} = \lambda > 0$, therefore those a_i cannot be all zero and we must have $t^2 + \frac{1}{4}s^2 - 1 = 0$. That is, the parameter (r, s) must belong to the Chen–Nie curve Γ. □

It remains to deal with the case of balanced BTP threefolds, which relies on the classification result for such threefolds in [26]. First we need the following:

Lemma 14. *Let (M^n, g) be a BTP manifold with its D^r_s connection having constant holomorphic sectional curvature c. Then under any local unitary frame e,*

$$R^b_{i\bar{j}k\bar{\ell}} = \tfrac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}) - \tfrac{1}{2}w + \tfrac{1}{4}\big(t^2 + \tfrac{1}{4}s^2 - 3\big)(v^j_i + v^\ell_k) + \tfrac{1}{4}\big(t^2 + \tfrac{1}{4}s^2 + 1\big)(v^\ell_i + v^j_k).$$

Proof. For BTP manifolds, by [26] we know that the Bismut curvature R^b always satisfies the symmetry condition $R^b_{i\bar{j}k\bar{\ell}} = R^b_{k\bar{\ell}i\bar{j}}$ and

$$Q_{i\bar{j}k\bar{\ell}} := R^b_{i\bar{j}k\bar{\ell}} - R^b_{k\bar{j}i\bar{\ell}} = -w - v^j_i - v^\ell_k + v^\ell_i + v^j_k,$$

under any local unitary frame. Therefore, for BTP manifolds,

$$\begin{aligned} \widehat{R}^b_{i\bar{j}k\bar{\ell}} &= \tfrac{1}{2}(R^b_{i\bar{j}k\bar{\ell}} + R^b_{k\bar{j}i\bar{\ell}}) = \tfrac{1}{2}(2R^b_{i\bar{j}k\bar{\ell}} - Q_{i\bar{j}k\bar{\ell}}) \\ &= R^b_{i\bar{j}k\bar{\ell}} + \tfrac{1}{2}(w + v^j_i + v^\ell_k - v^\ell_i - v^j_k). \end{aligned} \tag{13}$$

Under our assumption $H^D = c$, we have $\widehat{R}^D_{i\bar{j}k\bar{\ell}} = \frac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj})$. On the other hand, by (5) we get

$$\widehat{R}^D - \widehat{R}^b = \tfrac{1}{4}\big(1 - t^2 - \tfrac{1}{4}s^2\big)(v^j_i + v^\ell_k + v^\ell_i + v^j_k).$$

Plugging this into (13), we get the desired expression for R^b stated in the lemma. □

Lemma 14 implies that, for a BTP manifold with D^r_s holomorphic sectional curvature being a constant c, its Bismut curvature satisfies

$$R^b_{i\bar{i}k\bar{k}} = \tfrac{1}{2}c(1+\delta_{ik}) - \tfrac{1}{2}\sum_r |T^r_{ik}|^2 + \tfrac{1}{2}\big(t^2+\tfrac{1}{4}s^2-3\big)\,\mathrm{Re}\sum_r T^i_{ir}\overline{T^k_{kr}} + \tfrac{1}{4}\big(t^2+\tfrac{1}{4}s^2+1\big)\sum_r (|T^i_{kr}|^2+|T^k_{ir}|^2) \quad \text{for all } 1\le i,k\le n, \tag{14}$$

under any local unitary frame e.

Next let us recall the classification result from [26] for balanced BTP threefolds. Let (M^3, g) be a balanced, non-Kähler, compact BTP Hermitian threefold. By the observation in [26; 28], for any given point $p \in M$, there always exists a unitary frame e (which will be called *special frames* from now on) in a neighborhood of p such that under e the only possibly nonzero Chern torsion components are $a_i = T^i_{jk}$, where (ijk) is a cyclic permutation of (123). Furthermore, each a_i is a global constant on M^3, with $a_1 = \cdots = a_r > 0$, $a_{r+1} = \cdots = 0$, where $r = r_B \in \{1, 2, 3\}$ is the rank of the B tensor, which is the global 2-tensor on any Hermitian manifold defined under any unitary frame by $B_{i\bar{j}} = \sum_{k,\ell} T^j_{k\ell}\overline{T^i_{k\ell}}$. The conclusion in [26] indicates that any compact balanced (but non-Kähler) BTP threefold must be one of the following:

- $r_B = 3$, (M^3, g) is a compact quotient of the complex simple Lie group $\mathrm{SO}(3,\mathbb{C})$, in particular it is Chern flat.

- $r_B = 1$, (M^3, g) is the so-called Wallach threefold, namely, M^3 is biholomorphic to the flag variety $\mathbb{P}(T_{\mathbb{P}^2})$ while g is the Kähler–Einstein metric g_0 minus the square of the null-correlation section. Scale g by a positive constant if necessary, the Bismut curvature matrix under a special frame e is

$$\Theta^b = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ 0 & \alpha & \sigma \\ 0 & -\bar{\sigma} & \beta \end{bmatrix}, \quad \begin{cases} \alpha = \varphi_{1\bar{1}} + (1-b)\varphi_{2\bar{2}} + b\varphi_{3\bar{3}} + p\varphi_{2\bar{3}} + \bar{p}\varphi_{3\bar{2}}, \\ \beta = \varphi_{1\bar{1}} + b\varphi_{2\bar{2}} + (1-b)\varphi_{3\bar{3}} - p\varphi_{2\bar{3}} - \bar{p}\varphi_{3\bar{2}}, \\ \sigma = p\varphi_{2\bar{2}} - p\varphi_{3\bar{3}} + q\varphi_{2\bar{3}} + (1+b)\varphi_{3\bar{2}}, \end{cases} \tag{15}$$

where b is a real constant, p, q are complex constants, φ is the coframe dual to e, and we wrote $\varphi_{i\bar{j}}$ for $\varphi_i \wedge \overline{\varphi}_j$ for simplicity.

- $r_B = 2$, in this case (M^3, g) is said to be of *middle type*. Again under appropriate scaling of the metric, the Bismut curvature matrix under e becomes

$$\Theta^b = \begin{bmatrix} d\alpha & d\beta_0 & \\ -d\beta_0 & d\alpha & \\ & & 0 \end{bmatrix}, \quad \begin{cases} d\alpha = x(\varphi_{1\bar{1}} + \varphi_{2\bar{2}}) + iy(\varphi_{2\bar{1}} - \varphi_{1\bar{2}}), \\ d\beta_0 = -iy(\varphi_{1\bar{1}} + \varphi_{2\bar{2}}) + (x-2)(\varphi_{2\bar{1}} - \varphi_{1\bar{2}}), \end{cases} \tag{16}$$

where x, y are real-valued local smooth functions.

With this explicit information on Bismut curvature at hand, we are now ready to finish the proof of Theorem 7 for the balanced BTP threefold case.

Proof of Theorem 7 for balanced BTP threefolds. Let (M^3, g) be a compact balanced BTP Hermitian threefold. Assume that g is not Kähler (otherwise by the constancy of holomorphic sectional curvature we already know that the manifold is a complex space form). Suppose that for some $(r, s) \in \Omega$ the canonical metric connection D^r_s of g has constant holomorphic sectional curvature c.

First let us consider the $r_B = 3$ case. In this case g is Chern flat, so by (5) we have

$$\tfrac{1}{2}c(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}) = \widehat{R}^D_{i\bar{j}k\bar{\ell}} = -\big(t^2 + \tfrac{1}{4}s^2\big)\hat{v}. \tag{17}$$

Now let e be a special frame. Then the only nonzero Chern torsion components under e are $T^1_{23} = T^2_{31} = T^3_{12} = \lambda > 0$. Therefore $\hat{v}_{i\bar{i}i\bar{i}} = \sum_r |T^i_{ir}|^2 = 0$ for any $1 \le i \le 3$, while for any $1 \le i \ne k \le 3$ we have

$$4\hat{v}_{i\bar{i}k\bar{k}} = \sum_r \big(|T^i_{kr}|^2 + |T^k_{ir}|^2 + 2\,\mathrm{Re}(T^i_{ir}\overline{T^k_{kr}})\big) = \lambda^2 + \lambda^2 + 0 = 2\lambda^2.$$

If we let $i = j = k = \ell$ in (17), then we get $c = 0$. If we let $i = j \ne k = \ell$ in (17) instead, then we obtain $c = -\big(t^2 + \frac{1}{4}s^2\big)\lambda^2$. Thus $t = s = 0$. Since $t = \frac{1}{2}(1 - r + rs)$, this means that $r = 1$. Hence our connection D^r_s is D^1_0, which is the Chern connection ∇^c.

Next let us consider the $r_B = 1$ case. In this case (M^3, g) is the Wallach threefold. Under a special frame e, the only nonzero Chern torsion component is $T^1_{23} = \lambda > 0$. So as in the previous case $\hat{v}_{i\bar{i}i\bar{i}} = \sum_r |T^i_{ir}|^2 = 0$ for any $1 \le i \le 3$, and for any $1 \le i < k \le 3$ we have

$$4\hat{v}_{i\bar{i}k\bar{k}} = \sum_r \big(|T^i_{kr}|^2 + |T^k_{ir}|^2 + 2\,\mathrm{Re}(T^i_{ir}\overline{T^k_{kr}})\big) = \delta_{i1}\lambda^2.$$

Therefore by Lemma 14 we have $R^b_{i\bar{i}i\bar{i}} = c$ for any $1 \le i \le 3$ and

$$R^b_{i\bar{i}k\bar{k}} = \tfrac{1}{2}c - \tfrac{1}{2}\delta_{i2}\lambda^2 + \tfrac{1}{4}\big(t^2 + \tfrac{1}{4}s^2 + 1\big)\delta_{i1}\lambda^2 \quad \text{for all } 1 \le i < k \le 3. \tag{18}$$

On the other hand, by the formula (15) for the Bismut curvature matrix under e, we have

$$R^b_{1\bar{1}1\bar{1}} = 2, \quad R^b_{2\bar{2}2\bar{2}} = 1 - b, \quad R^b_{1\bar{1}2\bar{2}} = 1, \quad R^b_{2\bar{2}3\bar{3}} = b.$$

Therefore $2 = 1 - b = c$ which implies that $c = 2$ and $b = -1$, while by (18) we get

$$1 = \tfrac{1}{2}c + \tfrac{1}{4}\big(t^2 + \tfrac{1}{4}s^2 + 1\big)\lambda^2, \quad b = \tfrac{1}{2}c - \tfrac{1}{2}\lambda^2.$$

Note that the first equality in the above line gives a contradiction. So in this Fano case the holomorphic sectional curvature of D^r_s for any $(r, s) \in \Omega$ cannot be a constant.

Finally let us consider the $r_B = 2$ case. In this case, under a special frame e the only nonzero Chern torsion components are $T^1_{23} = T^2_{31} = \lambda > 0$. So by (14) we get $R^b_{i\bar{i}i\bar{i}} = c$ for each $1 \le i \le 3$, and, for any $1 \le i < k \le 3$,

$$R^b_{i\bar{i}k\bar{k}} = \tfrac{1}{2}c - \tfrac{1}{2}\lambda^2\delta_{k3} + \tfrac{1}{4}\big(t^2 + \tfrac{1}{4}s^2 + 1\big)\lambda^2(2 - \delta_{k3}).$$

From (16), we get $R^b_{1\bar{1}1\bar{1}} = x$, $R^b_{3\bar{3}3\bar{3}} = 0$, hence $x = c = 0$. Also by (16) we have $R^b_{1\bar{1}2\bar{2}} = x$, thus the above equality gives us

$$0 = x = R^b_{1\bar{1}2\bar{2}} = \tfrac{1}{2}c - 0 + \tfrac{1}{4}\big(t^2 + \tfrac{1}{4}s^2 + 1\big)\lambda^2(2-0) = \tfrac{1}{2}\big(t^2 + \tfrac{1}{4}s^2 + 1\big)\lambda^2,$$

which is clearly a contradiction. This shows that balanced BTP threefolds of middle type can never have constant holomorphic sectional curvature for D^r_s for any (r, s). This completes the proof the Theorem 7 for the case of balanced BTP threefolds. □

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SHUWEN CHEN
SCHOOL OF MATHEMATICAL SCIENCES
CHONGQING NORMAL UNIVERSITY
CHONGQING
CHINA
3153017458@qq.com

FANGYANG ZHENG
SCHOOL OF MATHEMATICAL SCIENCES
CHONGQING NORMAL UNIVERSITY
CHONGQQING
CHINA
20190045@cqnu.edu.cn

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THE WEIGHTS OF ISOLATED CURVE SINGULARITIES ARE DETERMINED BY HODGE IDEALS

YANG WANG, STEPHEN S.-T. YAU AND HUAIQING ZUO

We calculate Hodge ideals and Hodge moduli algebras for three types of isolated quasihomogeneous curve singularities. We show that Hodge ideals and Hodge moduli algebras of the singularities can determine the weights of the polynomials defining the singularities. We give some examples to explain why Hodge moduli algebras and the Hodge moduli sequence are better invariants than the characteristic polynomial (a topological invariant of the singularity) for nondegenerate quasihomogeneous singularities, in the sense that the characteristic polynomial cannot determine the weight type of the singularity.

1. Introduction

In [15; 16], the authors ask whether the topology of the singularity determines the weights of the polynomial defining the singularity. They showed that this is valid in the category of isolated singularities of Brieskorn–Pham type and isolated quasihomogeneous curve singularities.

Theorem 1.1 [15]. *The topology of a singularity of Brieskorn–Pham type determines the exponents* (*weight*) *of the polynomial defining the singularity.*

Theorem 1.2 [16]. *Let* $f_i(z_1, z_2)$, $i = 1, 2$, *be nondegenerate quasihomogeneous polynomials of weight* $(r_{i1}, r_{i2}; 1)$, $0 \le r_{i1} \le r_{i2} \le \frac{1}{2}$, *and let* V_i *be the germ of* $f_i(z_1, z_2) = 0$ *at the origin of* $\mathbb{C}^2$. *Then if* $(\mathbb{C}^2, V_1, 0) \simeq (\mathbb{C}^2, V_2, 0)$, *homeomorphically*, *we have* $(r_{11}, r_{12}) = (r_{21}, r_{22})$.

For quasihomogeneous surface singularities, there are some relevant results. Arnold [1, pages 91–131] and Orlik and Wagreich [9] showed that if $h(z_0, z_1, z_2)$ is a quasihomogeneous polynomial in $\mathbb{C}^3$ and $V = \{h(z) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the following seven classes below while keeping the differentiable structure of the link $K_V = S_\epsilon^{2n+1} \cap V$ constant:

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(I) $V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2}\}$, $a_0, a_1, a_2 > 1$,

(II) $V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}\}$, $a_0, a_1 > 1$, $a_2 > 0$,

(III) $V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2}\}$, $a_0 > 1$, $a_1, a_2 > 0$,

(IV) $V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2}\}$, $a_0 > 1$, $a_1, a_2 > 0$,

(V) $V(a_0, a_1, a_2; 5) = \{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}\}$, $a_0, a_1, a_2 > 0$,

(VI) $V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}\}$, $a_0 > 1$, $a_1, a_2, b_1, b_2 > 0$ satisfy $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$,

(VII) $V(a_0, a_1, a_2; 7) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}\}$, $a_0, a_1, a_2, b_1, b_2 > 0$ satisfy $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2(a_0 a_1 - 1)$.

Xu and Yau [14] proved that the above deformation is actually a topological trivial deformation as a pair (S^{2n+1}, K_V). Therefore any isolated quasihomogeneous surface singularity has the same topological type of one of the seven classes above. Let $\Delta_V(z)$ denote the characteristic polynomial of the *Milnor* fibration of $(V, 0)$.

Theorem 1.3 [14]. *If $(V, 0)$ and $(W, 0)$ are among the seven classes above, then $(\mathbb{C}^3, V, 0)$ is biholomorphic to $(\mathbb{C}^3, W, 0)$ if and only if $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$ with some exceptional cases. And $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$ if and only if $\pi_1(K_V) \simeq \pi_1(K_W)$ and $\Delta_V(z) = \Delta_W(z)$.*

The following are direct corollaries of the above theorem:

Corollary 1.4 [14]. *Let $(V, 0)$ and $(W, 0)$ be two isolated quasihomogeneous surface singularities in $\mathbb{C}^3$. Then $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$ if and only if $\pi_1(K_V) \simeq \pi_1(K_W)$ and $\Delta_V(z) = \Delta_W(z)$.*

Corollary 1.5 [14]. *Let $(V, 0)$ be an isolated quasihomogeneous surface singularity with weights (w_0, w_1, w_2). Then the topological type of $(V, 0)$ determines and is determined by its weights (w_0, w_1, w_2).*

Corollary 1.6 [14]. *Let $(V, 0)$ be an isolated singularity defined by a quasihomogeneous polynomial in $\mathbb{C}^3$ with weights (w_0, w_1, w_2). Then the fundamental group of the link $\pi_1(K_V)$ and the characteristic polynomial $\Delta_V(z)$ determine and are determined by the weights (w_0, w_1, w_2).*

The original motivation of [14] was to prove the *Zariski* conjecture (see [17]) for isolated quasihomogeneous surface singularities in $\mathbb{C}^3$: multiplicity is an invariant of topological type. As a corollary, they proved:

Corollary 1.7 [14]. *Let $(V, 0)$ and $(W, 0)$ be two isolated quasihomogeneous surface singularities in $\mathbb{C}^3$. If $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$, then V and W have the same multiplicity at the origin.*

Recall that in [3], we proved that a series of new invariants, Hodge moduli algebras and the Hodge moduli sequence, of the singularity are complete contact invariants for simple surface singularities. And our final aim is to extend this result to isolated quasihomogeneous surface singularities or even more general types of singularity. Note that in the proof of the above theorems, the characteristic polynomial of the singularity plays a fundamental role, since the characteristic polynomial is a topological invariant of the singularity. Motivated by these results and our former results, it is natural to ask whether we can replace the characteristic polynomial by Hodge ideals and Hodge moduli algebras of the singularity to determine the weights of the polynomials defining the singularities. That is, we want to prove if the i-th Hodge moduli algebras of two isolated quasihomogeneous curve singularities are isomorphic for all $i \geq 0$, then the weights of these two singularities are the same.

If $h(x, y)$ is a quasihomogeneous polynomial in $\mathbb{C}^2$ and $V = \{h(x, y) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the following three classes below while keeping the differentiable structure of the link $K_V = S_\epsilon^{2n+1} \cap V$ constant:

$$\begin{aligned}
\mathbb{F}_1(x, y) &= x^a + y^b, && a, b \geq 2,\\
\mathbb{F}_2(x, y) &= x^a + xy^b, && a \geq 2,\ b \geq 1,\\
\mathbb{F}_3(x, y) &= x^a y + xy^b, && a, b \geq 1.
\end{aligned}$$

After a tedious calculation for Hodge ideals and Hodge moduli algebras of isolated quasihomogeneous curve singularities of the above three types, we obtain the following.

Main Theorem A (0-th and 1-st Hodge moduli algebras determine weight type).
(1) *For isolated quasihomogeneous curve singularities*

$$D_1^{(a_1,b_1)} = \{x^{a_1} + y^{b_1} = 0\}, \quad 2 \leq a_1 \leq b_1,$$

and

$$D_2^{(a_2,b_2)} = \{x^{a_2} + xy^{b_2} = 0\}, \quad 1 \leq a_2 - 1 \leq b_2,$$

if their 0*-th and* 1*-st Hodge moduli algebras* (*taking* $\alpha = 1$ *in their Hodge ideals*) *are isomorphic*, *i.e.*,

$$M_0(D_1^{(a_1,b_1)}) \simeq M_0(D_2^{(a_2,b_2)}), \quad M_1(D_1^{(a_1,b_1)}) \simeq M_1(D_2^{(a_2,b_2)}),$$

then the weight types of $D_1^{(a_1,b_1)}$ *and* $D_2^{(a_2,b_2)}$ *are the same*, *i.e.*,

$$\mathrm{wt}(F_1^{(a_1,b_1)}) = \mathrm{wt}(F_2^{(a_2,b_2)}).$$

(2) *For isolated quasihomogeneous curve singularities*

$$D_2^{(a_2,b_2)} = \{x^{a_2} + xy^{b_2} = 0\}, \quad a_2 - 1 \geq b_2 \geq 1,$$

and

$$D_3^{(a_3,b_3)} = \{x^{a_3}y + xy^{b_3} = 0\}, \quad 1 \leq a_3 \leq b_3,$$

if their 0*-th and* 1*-st Hodge moduli algebras* (*taking* $\alpha = 1$ *in their Hodge ideals*) *are isomorphic*, *i.e.*,

$$M_0(D_2^{(a_2,b_2)}) \simeq M_0(D_3^{(a_3,b_3)}), \quad M_1(D_2^{(a_2,b_2)}) \simeq M_1(D_3^{(a_3,b_3)}),$$

then the weight types of $D_2^{(a_2,b_2)}$ *and* $D_3^{(a_3,b_3)}$ *are the same*, *i.e.*,

$$\mathrm{wt}(F_1^{(a_1,b_1)}) = \mathrm{wt}(F_3^{(a_3,b_3)}).$$

(3) *For isolated quasihomogeneous curve singularities*

$$D_1^{(a_1,b_1)} = \{x^{a_1} + y^{b_1} = 0\}, \quad a_1, b_1 \geq 2,$$

and

$$D_3^{(a_3,b_3)} = \{x^{a_3}y + xy^{b_3} = 0\}, \quad a_3, b_3 \geq 1,$$

their i*-th Hodge moduli algebras* (*taking* $\alpha = 1$ *in their Hodge ideals*) *are not isomorphic*, *for* $i = 0, 1$, *respectively.*

As a by-product, we obtain an inequality of the δ-invariant, 0-th Hodge moduli number and multiplicity for isolated quasihomogeneous curve singularities of the above three types:

Main Theorem B. (1) *For isolated quasihomogeneous curve singularities* $D_1^{(a,b)} = \{x^a + y^b = 0\}$, $a, b \geq 2$, *we have*

$$0 \leq \delta_1(a, b) - m_0(D_1^{(a,b)}) \leq \mathrm{mt}(D_1^{(a,b)}),$$

where $\delta_1(a, b)$ *is the* δ*-invariant of* $D_1^{(a,b)}$, $m_0(D_1^{(a,b)})$ *is the* 0*-th Hodge moduli number of the divisor* $D_1^{(a,b)}$ *for* $\alpha = 1$ *and* $\mathrm{mt}(D_1^{(a,b)})$ *is the multiplicity of* $D_1^{(a,b)}$.

(2) *For isolated quasihomogeneous curve singularities* $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, $a \geq 2$, $b \geq 1$, *we have*

$$1 \leq \delta_2(a, b) - m_0(D_2^{(a,b)}) \leq \mathrm{mt}(D_2^{(a,b)}),$$

where $\delta_2(a, b)$ *is the* δ*-invariant of* $D_2^{(a,b)}$, $m_0(D_2^{(a,b)})$ *is the* 0*-th Hodge moduli number of the divisor* $D_2^{(a,b)}$ *for* $\alpha = 1$ *and* $\mathrm{mt}(D_2^{(a,b)})$ *is the multiplicity of* $D_2^{(a,b)}$.

(3) *For isolated quasihomogeneous curve singularities* $D_3^{(a,b)} = \{x^a y + xy^b = 0\}$, $a, b \geq 1$, *we have*

$$2 \leq \delta_3(a, b) - m_0(D_3^{(a,b)}) \leq \mathrm{mt}(D_3^{(a,b)}),$$

where $\delta_3(a, b)$ *is the* δ*-invariant of* $D_3^{(a,b)}$, $m_0(D_3^{(a,b)})$ *is the* 0*-th Hodge moduli number of the divisor* $D_3^{(a,b)}$ *for* $\alpha = 1$ *and* $\mathrm{mt}(D_3^{(a,b)})$ *is the multiplicity of* $D_3^{(a,b)}$.

In Section 2, we recall a number of classical results on the Hodge ideals of effective $\mathbb{Q}$-divisors and the δ-invariants of curve singularities. We also collect some important lemmas and theorems that will be used in the following parts. In Section 3, we explicitly calculate Hodge ideals and Hodge moduli algebras of isolated quasihomogeneous curve singularities of three types. In Sections 4 and 5 we prove Main Theorems A and B by the results in Section 3. Finally, in Section 6 we give some examples to illustrate that Hodge moduli algebras and the Hodge moduli sequence are better invariants than the characteristic polynomial (a topological invariant of singularity) for nondegenerate quasihomogeneous singularities. Furthermore, from the observation of some examples, we raise a conjecture that the Hodge moduli numbers of isolated quasihomogeneous curve singularities remain constant under quasihomogeneous deformation. That is, Hodge moduli numbers of isolated quasihomogeneous curve singularities only depend on the weights of the singularities.

2. Preliminaries

2.1. ***Hodge ideals.*** In [7; 8], the authors extend the notion of Hodge ideals to the case when D is an arbitrary effective $\mathbb{Q}$-divisor on X, where X is a smooth complex variety. Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ are defined in terms of the Hodge filtration $F_\bullet$ on some $\mathscr{D}_X$-module associated with D (see [7, §2–§4] for more details). When D is an integral and reduced divisor, this recovers the definition of Hodge ideals $I_k(D)$ in [6].

Let X be a smooth complex variety, and $\mathscr{D}_X$ be the sheaf of differential operators on X. If H is an integral and reduced effective divisor on X, $D = \alpha H$, $\alpha \in \mathbb{Q}\cap(0, 1]$, let $\mathcal{O}_X(*D)$ be the sheaf of rational functions with poles along D. It is also a left $\mathscr{D}_X$-module underlying the mixed Hodge module $j_*\mathbb{Q}_U^H[n]$, where $U = X \setminus D$ and $j : U \hookrightarrow X$ is the inclusion map. Any $\mathscr{D}_X$-module associated with a mixed Hodge module has a good filtration $F_\bullet$, the Hodge filtration of the mixed Hodge module [12].

To study the Hodge filtration of $\mathcal{O}_X(*D)$, it seems easier to consider a series of ideal sheaves, defined by Mustaţă and Popa [6], which can be considered to be a generalization of multiplier ideals of divisors. The Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ of the divisor D are defined by

$$F_k\mathscr{O}_X(*D) = I_k(D) \otimes \mathscr{O}_X((k+1)D) \quad \text{for all } k \in \mathbb{N}.$$

These are coherent sheaves of ideals. See [6] for details and an extensive study of the ideals $I_k(D)$. Hodge ideals are indexed by the nonnegative integers; at the 0-th step, they essentially coincide with multiplier ideals. It turns out that $I_0(D) = \mathscr{J}((1-\epsilon)D)$, the multiplier ideal of the divisor $(1-\epsilon)D$, $0 < \epsilon \ll 1$. The multiplier ideal sheaves are ubiquitous objects in birational geometry, encoding local numerical invariants of singularities, and satisfying Kodaira-type vanishing theorems in the global setting. The Hodge ideals are interesting invariants of the singularities, they have similar properties as multiplier ideals.

We summarize the properties and results (see [7; 5]) of Hodge ideals as follows.

Given a reduced effective divisor H on a smooth complex variety X, $D = \alpha H$, $\alpha \in \mathbb{Q} \cap (0, 1]$, we also denote by Z the support of D. The sequence of Hodge ideals $I_k(D)$, with $k \geq 0$, satisfies these properties:

- $I_0(D)$ is the multiplier ideal $\mathcal{I}((1-\epsilon)D)$, so in particular $I_0(D) = \mathscr{O}_X$ if and only if the pair(X, D) is log canonically.
- When Z has simple normal crossings,

$$I_k(D) = I_k(Z) \otimes \mathscr{O}_X(Z - \lceil D \rceil),$$

where $I_k(Z)$ can be computed explicitly as in [6]. If Z is smooth, then $I_k(D) = \mathscr{O}_X(Z - \lceil D \rceil)$.

- The Hodge filtration is generated at level $n-1$, where $n = \dim X$, i.e.,

$$F_\ell \mathscr{D}_X \cdot (I_k(D) \otimes \mathscr{O}_X(kZ)h^{-\alpha}) = I_{k+\ell}(D) \otimes \mathscr{O}_X((k+\ell)Z)h^{-\alpha}$$

for all $k \geqslant n-1$ and $\ell \geqslant 0$.

- There are nontriviality criteria for $I_k(D)$ at a point $x \in D$ in terms of the multiplicity of D at x.
- If X is projective, $I_k(D)$ satisfy a vanishing theorem analogous to Nadel vanishing for multiplier ideals.
- If Y is a smooth divisor in X such that $Z|_Y$ is reduced, then $I_k(D)$ satisfy

$$I_k(D|_Y) \subseteq I_k(D) \cdot \mathscr{O}_Y,$$

with equality when Y is general.

- If $X \to T$ is a smooth family with a section $s : T \to X$, and D is a relative divisor on X that satisfies a suitable condition then

$$\{t \in T \mid I_k(D_t) \nsubseteq \mathfrak{m}^q_{s(t)}\}$$

is an open subset of T, for each $q \geqslant 1$.

- If D_1 and D_2 are $\mathbb{Q}$-divisors with supports Z_1 and Z_2, such that $Z_1 + Z_2$ is also reduced, then we have the subadditivity property

$$I_k(D_1 + D_2) \subseteq I_k(D_1) \cdot I_k(D_2)$$

For comparison, the list of properties of Hodge ideals in the case when D is reduced is summarized in [10]. The setting of $\mathbb{Q}$-divisors is more intricate. For instance, the bounds for the generation level of the Hodge filtration can become worse. Moreover, it is not known whether the inclusions $I_k(D) \subseteq I_{k-1}(D)$ continue to hold for arbitrary $\mathbb{Q}$-divisors. New phenomena appear as well: given two rational numbers $\alpha_1 < \alpha_2$, usually the ideals $I_k(\alpha_1 Z)$ and $I_k(\alpha_2 Z)$ cannot be compared for $k \geqslant 1$, unlike in the case of multiplier ideals.

We recall the following definition.

Definition 2.1. Let $f, g \in R = \mathbb{C}\{x_1, \dots, x_n\}$ which is the convergent power series ring. We say f and g are contact equivalent if the local $\mathbb{C}$-algebras $R/(f)$ and $R/(g)$ are isomorphic.

Definition 2.2. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $n \geq 2$, be an isolated hypersurface singularity. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f, $D^\alpha = \alpha H$, $\alpha \in \mathbb{Q} \cap (0, 1]$. We define the i-th Hodge moduli algebra of D^α to be the moduli algebra of the ideal $J_i(D^\alpha) := (f) + I_i(D^\alpha)$ (or J_i for short)

$$M_i(D^\alpha) := \mathbb{C}\{x_1, \dots, x_n\}/J_i(D^\alpha)$$

for $i \geq 0$ (or M_i for short), where $I_i(D^\alpha)$ is the i-th Hodge ideal (or I_i for short). The i-th Hodge moduli number of D^α is defined to be

$$m_i(D^\alpha) := \dim_{\mathbb{C}}(M_i(D^\alpha))$$

for $i \geq 0$ (or m_i for short). We define the Hodge moduli sequence of D to be the sequence

$$\{m_i\} := \{m_0, m_1, m_2, \dots\}.$$

Definition 2.3. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called weighted homogeneous if there exists positive rational numbers $w_1, \dots, w_n$ (that is, weights of $x_1, \dots, x_n$) and d such that $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with a nonzero coefficient. The number d is called the weighted homogeneous degree (w-deg) of f for weights w_j, $1 \leq j \leq n$. These w_j, $1 \leq j \leq n$, are called the weight type of f.

The Hodge filtration $F_\bullet$ of $\mathcal{O}_X(*D)$ is usually hard to describe. However, it does have an explicit formula in the case when D is defined by a reduced weighted homogeneous polynomial f which has an isolated singularity at the origin, which is proved by M. Saito [13]. To state Saito's result, we first clarify the notation as follows.

- Denote by $\mathcal{O} = \mathbb{C}\{x_1, \dots, x_n\}$ the ring of germs of holomorphic function for local coordinates $x_1, \dots, x_n$.
- Denote by $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of a holomorphic function that is quasihomogeneous, i.e., $f \in \mathcal{J}(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$, and with an isolated singularity at the origin. Kyoji Saito [11] showed that after a biholomorphic coordinate change, we can assume f is a weighted homogeneous polynomial with an isolated singularity at the origin. We will keep this assumption for f unless otherwise stated.

• Denote by $w = w(f) = (w_1, \dots, w_n)$ the weights of the weighted homogeneous polynomial f.

• Denote by $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of a holomorphic function, and we write

$$g = \sum_{A \in \mathbb{N}^n} g_A x^A,$$

where $A = (a_1, \dots, a_n)$, $g_A \in \mathbb{C}$ and $x^A = x_1^{a_1} \cdots x_n^{a_n}$.

• Denote by $\rho(g)$ the weight of an element $g \in \mathcal{O}$ defined by

$$\rho(g) = \left(\sum_{i=1}^{m} w_i \right) + \inf\{\langle w, A \rangle : g_A \neq 0\}.$$

The weight function ρ defines a filtration on $\mathcal{O}$ as

$$\mathcal{O}^{>k} = \{u \in \mathcal{O} : \rho(u) > k\},$$
$$\mathcal{O}^{\geq k} = \{u \in \mathcal{O} : \rho(u) \geq k\}.$$

Since we consider $\mathscr{D}_X$-modules locally around the isolated singularity, we can assume $X = \mathbb{C}^n$ and identify the stalk at the singularity to be that of $\mathscr{D}_X$-modules on $\mathbb{C}^n$. For example, we replace $F_k\mathcal{O}_{X,0}(*D)$ with $F_k\mathcal{O}_X(*D)$. Now we can state the formula proved by M. Saito (see [13, Theorem 0.7]):

$$F_k\mathcal{O}_X(*D) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \left(\frac{\mathcal{O}^{\geq i+1}}{f^{i+1}} \right) \quad \text{for all } k \in \mathbb{N}. \tag{1}$$

Since the Hodge filtration can be constructed on analogous $\mathscr{D}_X$-modules associated with any effective $\mathbb{Q}$-divisor D, so it satisfies a similar formula in the case when D is supported on a hypersurface defined by such a polynomial f.

Assume that the divisor is $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = (f = 0)$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin. In this case, the associated $\mathscr{D}_X$-module is the well-known twisted localization $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha}) := \mathcal{O}_X(*Z) f^{1-\alpha}$ (see more details in [7] about how to construct the Hodge filtration $F_\bullet\mathcal{M}(f^{1-\alpha})$). With new ingredients from Mustață and Popa [8], where this Hodge filtration is compared to the V-filtration on $\mathcal{M}(f^{1-\alpha})$, M. Zhang generalized Saito's formula and proved the following theorem:

Theorem 2.4 (Zhang, [18]). *If $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have*

$$F_k\mathcal{M}(f^{1-\alpha}) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \left(\frac{\mathcal{O}^{\geq \alpha+i}}{f^{i+1}} f^{1-\alpha} \right),$$

where the action $\cdot$ of $\mathscr{D}_X$ on the right-hand side is the action on the left $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha})$ defined by

$$D\cdot(wf^{1-\alpha}) := \left(D(w) + w\frac{(1-\alpha)D(f)}{f}\right)f^{1-\alpha} \quad \textit{for any } D\in \operatorname{Der}_{\mathbb{C}}\mathcal{O}_X.$$

Notice that if we set $\alpha = 1$, Theorem 2.4 recovers Saito's formula (1) mentioned above. For any polynomial f with an isolated singularity at the origin, it is well known that the *Milnor* algebra

$$\mathcal{A}_f := \mathbb{C}\{x_1,\dots,x_n\}/(\partial_1 f,\dots,\partial_n f)$$

is a finite-dimensional $\mathbb{C}$-vector space. Fix a monomial basis $\{v_1,\dots,v_\mu\}$ for this vector space, where μ is the dimension of $\mathcal{A}_f$ (i.e., *Milnor* number). The next theorem follows from Theorem 2.4.

Theorem 2.5 (Zhang, [18]). *If $D = \alpha Z$, where $0 < \alpha \le 1$ and $Z = \{f = 0\}$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have*

$$F_0\mathcal{M}(f^{1-\alpha}) = f^{-1}\cdot \mathcal{O}^{\ge\alpha} f^{1-\alpha}$$

and

$$F_k\mathcal{M}(f^{1-\alpha}) = \left(f^{-1}\cdot \sum_{v_j\in\mathcal{O}^{\ge k+1+\alpha}} \mathcal{O}_X\cdot v_j\right)f^{1-\alpha} + F_1\mathfrak{D}_X\cdot F_{k-1}\mathcal{M}(f^{1-\alpha}).$$

Alternatively, in terms of Hodge ideals, these formulas say that

$$I_0(D) = \mathcal{O}^{\ge\alpha}$$

and

$$I_{k+1}(D) = \sum_{v_j\in\mathcal{O}^{\ge k+1+\alpha}} \mathcal{O}_X\cdot v_j + \sum_{1\le i\le n,\, a\in I_k(D)} \mathcal{O}_X(f\partial_i a - (\alpha+k)a\partial_i f).$$

2.2. *Delta invariant of curve singularities.*

Definition 2.6 (δ-invariant). Let $f \in \mathbb{C}\{x, y\}$ be a reduced convergent power series, and let

$$\mathcal{O} = \mathbb{C}\{x,y\}/\langle f\rangle \hookrightarrow \overline{\mathcal{O}}$$

denote the normalization. Then we call

$$\delta(f) := \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O}$$

the δ-invariant of f.

Although we can explicitly calculate the δ-invariants of isolated quasihomogeneous singularities of three types F_1, F_2, F_3, by blowing up singularities and using the above theorem. We use Lemma 2.7, which is a very useful equality of the *Milnor* number, the δ-invariant and the number of irreducible factors of a curve singularity $\{f=0\}$, to show Lemmas 2.8, 2.9, and 2.10. And we only give proof for Lemma 2.8 for simplicity, since the proofs for Lemmas 2.9 and 2.10 are similar.

Lemma 2.7 [2, Proposition 3.35]. *Let $f \in \mathfrak{m} \subseteq \mathbb{C}\{x, y\}$ be reduced. Then*

$$\mu(f) = 2\delta(f) - r(f) + 1,$$

*where $\mu(f)$ is the **Milnor** number of f, $\delta(f)$ is the δ-invariant of f and $r(f)$ is the number of irreducible factors of f.*

Lemma 2.8. *For an isolated quasihomogeneous curve singularity of the form $D_1^{(a,b)} = \{x^a + y^b = 0\}$, defined by $F_1^{(a,b)} = x^a + y^b$, $a, b \geq 2$, its δ-invariant is*

$$\delta_1(a, b) = \frac{(a-1)(b-1) + \gcd(a, b) - 1}{2}.$$

In particular, $\delta_1(a, b) = \frac{(a-1)(b-1)}{2}$, if $\gcd(a, b) = 1$.

Proof. Since $\mu(f) = (a-1)(b-1)$ and $r(f) = \gcd(a, b)$,

$$\delta_1(a, b) = \frac{\mu(f) + r(f) - 1}{2} = \frac{(a-1)(b-1) + \gcd(a, b) - 1}{2}. \qquad \square$$

Lemma 2.9. *For an isolated quasihomogeneous curve singularity of the form $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, defined by $F_2^{(a,b)} = x^a + xy^b$, $a \geq 2$, $b \geq 1$, its δ-invariant is*

$$\delta_2(a, b) = \frac{a(b-1) + \gcd(a-1, b) + 1}{2}.$$

In particular, $\delta_2(a, b) = \frac{a(b-1)+2}{2}$, if $\gcd(a-1, b) = 1$.

Lemma 2.10. *For an isolated quasihomogeneous curve singularity of the form $D_3^{(a,b)} = \{x^a y + xy^b = 0\}$, defined by $F_3^{(a,b)} = x^a y + xy^b$, $a, b \geq 1$, its δ-invariant is*

$$\delta_3(a, b) = \frac{ab + \gcd(a-1, b-1) + 1}{2}.$$

In particular, $\delta_3(a, b) = \frac{ab+2}{2}$, if $\gcd(a-1, b-1) = 1$.

3. The first two Hodge ideals of three types of isolated quasihomogeneous curve singularities

In this section, we compute Hodge ideals of three types of isolated quasihomogeneous curve singularities for $\alpha = 1$ in Theorem 2.5. And $\mathcal{O}_X = \mathbb{C}\{x, y\}$ in the following computation. The following lemma is used in the computation of dimensions of Hodge moduli algebras.

Lemma 3.1. *For $n, m \in \mathbb{N}$, $n, m \geq 1$,*

$$\sum_{i=1}^{n-1} \left[\frac{mi}{n}\right] = \frac{(m-1)(n-1) + \gcd(m, n) - 1}{2},$$

Consider isolated quasihomogeneous curve singularities $D_1^{(a,b)} = \{x^a + y^b = 0\}$, defined by $F_1^{(a,b)} = x^a + y^b$. If $a \le b$, let $r = \frac{a}{\gcd(a,b)}$; then $1 \le r \le a$. Since $\frac{1+a-1}{a} + \frac{1}{b} \ge 1$, we have $x^{a-1} \in I_0(D_1)$. And we have

$$x^{k-1} y^{[\frac{b(a-k)}{a}]} \in I_0(D_1) \quad \text{for all } 1 \le k \le a-1,\ r \nmid k,$$

and

$$x^{ir-1} y^{b-\frac{ibr}{a}-1} \in I_0(D_1) \quad \text{for all } 1 \le i \le \gcd(a,b)-1.$$

So the 0-th Hodge ideal for D_1 is

$$\begin{aligned} J_0(D_1) = I_0(D_1) = \big(&x^{a-1}, x^{a-2} y^{[\frac{b}{a}]}, \dots, x^{a-r} y^{[\frac{b(r-1)}{a}]}, \\ &\dots, x^{ir-1} y^{b-\frac{ibr}{a}-1}, x^{ir-2} y^{[\frac{b(a-ir+1)}{a}]}, \dots, x^{(i-1)r} y^{[\frac{b(a-(i-1)r-1)}{a}]}, \\ &\dots, x^{r-1} y^{b-\frac{br}{a}-1}, x^{r-2} y^{[\frac{b(a-r+1)}{a}]}, \dots, y^{[\frac{b(a-1)}{a}]}\big), \end{aligned}$$

where $1 \le i \le \gcd(a,b)-1$. Its multiplicity $\mathrm{mt}(J_0(D_1))$ equals $a-1$. Using Lemma 3.1, we obtain the dimension of the 0-th Hodge moduli algebra $M_0(D_1) = \mathcal{O}_X / J_0(D_1)$:

$$\begin{aligned} m_0(D_1) &= \sum_{i=1}^{a-1} \left[\frac{bi}{a}\right] - (\gcd(a,b)-1) \\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} - (\gcd(a,b)-1) \\ &= \frac{(a-1)(b-1)-\gcd(a,b)+1}{2}. \end{aligned}$$

The 1-st Hodge ideal of D_1 is

$$\begin{aligned} J_1(D_1) &= (f) + I_0(D_1) \cdot (Jf) \\ &= \big(x^a + y^b, x^{a-2} y^b, x^{a-3} y^{[\frac{b}{a}]+b}, \dots, x^{a-r-1} y^{[\frac{b(r-1)}{a}]+b}, \\ &\qquad \dots, x^{ir-2} y^{2b-\frac{ibr}{a}-1}, \dots, x^{(i-1)r-1} y^{[\frac{b(a-(i-1)r-1)}{a}]+b}, \\ &\qquad \dots, x^{r-2} y^{2b-\frac{br}{a}-1}, \dots, y^{[\frac{b(a-2)}{a}]+b}, x^{a-1} y^{[\frac{b(a-1)}{a}]}\big), \end{aligned}$$

where $2 \le i \le \gcd(a,b)$. Its multiplicity $\mathrm{mt}(J_1(D_1))$ equals a. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_1) = \mathcal{O}_X / J_1(D_1)$ is

$$\begin{aligned} m_1(D_1) &= \sum_{i=1}^{a-2} \left(\left[\frac{bi}{a}\right] + b\right) - (\gcd(a,b)-1) + b + \left[\frac{b(a-1)}{a}\right] \\ &= \sum_{i=1}^{a-1} \left[\frac{bi}{a}\right] + (a-1)b - (\gcd(a,b)-1) \\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} + (a-1)b - (\gcd(a,b)-1) \\ &= \frac{(a-1)(3b-1)-\gcd(a,b)+1}{2}. \end{aligned}$$

If $a \geq b$, let $r = \frac{b}{\gcd(a,b)}$; then $1 \leq r \leq b$. By symmetry of a, b, we obtain the 0-th Hodge ideal for D_1:

$$\begin{aligned}J_0(D_1) &= I_0(D_1)\\ &= \big(y^{b-1}, y^{b-2}x^{[\frac{a}{b}]}, \dots, y^{b-r}x^{[\frac{a(r-1)}{b}]},\\ &\qquad \dots, y^{ir-1}x^{a-\frac{iar}{b}-1}, y^{ir-2}x^{[\frac{a(b-ir+1)}{b}]}, \dots, y^{(i-1)r}x^{[\frac{a(b-(i-1)r-1)}{b}]},\\ &\qquad\qquad \dots, y^{r-1}x^{a-\frac{ar}{b}-1}, y^{r-2}x^{[\frac{a(b-r+1)}{b}]}, \dots, x^{[\frac{a(b-1)}{b}]}\big),\end{aligned}$$

where $1 \leq i \leq \gcd(a,b)-1$. Its multiplicity $\mathrm{mt}(J_0(D_1))$ equals $b-1$. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_0(D_1) = \mathcal{O}_X/J_0(D_1)$ is

$$\begin{aligned}m_0(D_1) &= \sum_{i=1}^{b-1}\left[\frac{ai}{b}\right] - (\gcd(a,b)-1)\\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} - (\gcd(a,b)-1)\\ &= \frac{(a-1)(b-1)-\gcd(a,b)+1}{2}.\end{aligned}$$

And the 1-st Hodge ideal for D_1 is

$$\begin{aligned}J_1(D_1) &= (f) + I_0(D_1)\cdot(Jf)\\ &= \big(x^a+y^b, y^{b-2}x^a, y^{b-3}x^{[\frac{a}{b}]+a}, \dots, y^{b-r-1}x^{[\frac{a(r-1)}{b}]+a},\\ &\qquad \dots, y^{ir-2}x^{2a-\frac{iar}{b}-1}, \dots, y^{(i-1)r-1}x^{[\frac{a(b-(i-1)r-1)}{b}]+a},\\ &\qquad\qquad \dots, y^{r-2}x^{2a-\frac{ar}{b}-1}, \dots, x^{[\frac{a(b-2)}{b}]+a}, y^{b-1}x^{[\frac{a(b-1)}{b}]}\big),\end{aligned}$$

where $2 \leq i \leq \gcd(a,b)$. Its multiplicity $\mathrm{mt}(J_1(D_1))$ equals b. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_1) = \mathcal{O}_X/J_1(D_1)$ is

$$\begin{aligned}m_1(D_1) &= \sum_{i=1}^{b-2}\left(\left[\frac{ai}{b}\right]+a\right) - (\gcd(a,b)-1) + a + \left[\frac{a(b-1)}{b}\right]\\ &= \sum_{i=1}^{b-1}\left[\frac{ai}{b}\right] + (b-1)a - (\gcd(a,b)-1)\\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} + (b-1)a - (\gcd(a,b)-1)\\ &= \frac{(3a-1)(b-1)-\gcd(a,b)+1}{2}.\end{aligned}$$

Consider isolated quasihomogeneous curve singularities $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, defined by $F_2^{(a,b)} = x^a + xy^b$. If $a-1 \leq b$, let $r = \frac{a-1}{\gcd(a-1,b)}$; then $1 \leq r \leq a-1$.

Since $\frac{1+a-1}{a}+\frac{a-1}{ab}\geq 1$, we have $x^{a-1}\in I_0(D_2)$. And we have

$$x^k y^{[\frac{b(a-k-1)}{a-1}]}\in I_0(D_2)\quad \text{for all } 1\leq k\leq a-1,\ r\nmid k,$$

and

$$x^{ir}y^{b-\frac{ibr}{a-1}-1}\in I_0(D_2)\quad \text{for all } 1\leq i\leq \gcd(a-1,b)-1.$$

Since $\frac{1}{a}+\frac{(a-1)(1+b-1)}{ab}\geq 1$, we have $y^{b-1}\in I_0(D_2)$. So the 0-th Hodge ideal for D_2 is

$$\begin{aligned}J_0(D_2)&=I_0(D_2)\\&=\big(x^{a-1},x^{a-2}y^{[\frac{b}{a-1}]},\dots,x^{a-r}y^{[\frac{b(r-1)}{a-1}]},\\&\qquad\dots,x^{ir}y^{b-\frac{ibr}{a-1}-1},x^{ir-1}y^{[\frac{b(a-ir)}{a-1}]},\dots,x^{(i-1)r+1}y^{[\frac{b(a-((i-1)r+1)-1)}{a-1}]},\\&\qquad\dots,x^r y^{b-\frac{br}{a-1}-1},x^{r-1}y^{[\frac{b(a-r)}{a-1}]},\dots,xy^{[\frac{b(a-2)}{a-1}]},y^{b-1}\big),\end{aligned}$$

where $1\leq i\leq \gcd(a-1,b)-1$. Its multiplicity $\mathrm{mt}(J_0(D_2))$ equals $a-1$. The dimension of the 0-th Hodge moduli algebra $M_0(D_2)=\mathcal{O}_X/J_0(D_2)$, by Lemma 3.1, is

$$\begin{aligned}m_0(D_2)&=\sum_{i=1}^{a-2}\left[\frac{bi}{a-1}\right]-(\gcd(a-1,b)-1)+b-1\\&=\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}-(\gcd(a,b)-1)+b-1\\&=\frac{a(b-1)-\gcd(a-1,b)+1}{2}.\end{aligned}$$

And the 1-st Hodge ideal of D_2 is

$$\begin{aligned}J_1(D_2)&=(f)+I_0(D_2)\cdot(Jf)\\&=\big(x^a+xy^b,ax^{a-1}y^{b-1}+y^{2b-1}x^{a-1}y^b,x^{a-2}y^{[\frac{b}{a-1}]+b},\dots,x^{a-r}y^{[\frac{b(r-1)}{a-1}]+b},\\&\qquad\dots,x^{ir}y^{2b-\frac{ibr}{a-1}-1},\dots,x^{(i-1)r+1}y^{[\frac{b(a-((i-1)r+1)-1)}{a-1}]+b},\\&\qquad\dots,x^r y^{2b-\frac{br}{a-1}-1},\dots,xy^{[\frac{b(a-2)}{a-1}]+b}\big),\end{aligned}$$

where $1\leq i\leq \gcd(a-1,b)-1$. Its multiplicity $\mathrm{mt}(J_1(D_2))$ equals a. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_2)=\mathcal{O}_X/J_1(D_2)$ is

$$\begin{aligned}m_1(D_2)&=2b-1+\sum_{i=1}^{a-2}\left(\left[\frac{bi}{a-1}\right]+b\right)-(\gcd(a-1,b)-1)+b\\&=\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}+(a-2)b+3b-1-(\gcd(a-1,b)-1)\\&=\frac{a(3b-1)-\gcd(a-1,b)+1}{2}.\end{aligned}$$

If $a-1\ge b$, let $r=\frac{b}{\gcd(a-1,b)}$; then $1\le r\le b$. Since $\frac{1}{a}+\frac{(a-1)(1+b-1)}{ab}\ge 1$, we have $y^{b-1}\in I_0(D_2)$. And we have

$$x^{[\frac{(a-1)(b-k)}{b}]+1}y^{k-1}\in I_0(D_2)\quad\text{for all } 1\le k\le b,\ r\nmid k,$$

and

$$x^{a-1-\frac{i(a-1)r}{b}}\in I_0(D_2)\quad\text{for all } 1\le i\le \gcd(a-1,b).$$

So the 0-th Hodge ideal for D_2 is

$$\begin{aligned}J_0(D_2)&=I_0(D_2)\\&=\big(x^{[\frac{(a-1)(b-1)}{b}]+1},x^{[\frac{(a-1)(b-2)}{b}]+1}y,\dots,x^{a-1-\frac{(a-1)r}{b}}y^{r-1},\dots,\\&\qquad x^{[\frac{(a-1)(b-(i-1)r-1)}{b}]+1}y^{(i-1)r},x^{[\frac{(a-1)(b-(i-1)r-2)}{b}]+1}y^{(i-1)r+1},\dots,x^{a-1-\frac{i(a-1)r}{b}}y^{ir-1},\\&\qquad\dots,x^{[\frac{(a-1)(r-1)}{b}]+1}y^{b-r},x^{[\frac{(a-1)(r-2)}{b}]+1}y^{b-r+1},\dots,y^{b-1}\big),\end{aligned}$$

where $1\le i\le \gcd(a-1,b)$. Its multiplicity $\mathrm{mt}(J_0(D_2))$ equals $b-1$. By Lemma 3.1, the dimension of the 0-th Hodge moduli algebra $M_0(D_2)=\mathcal{O}_X/J_0(D_2)$ is

$$\begin{aligned}m_0(D_2)&=\sum_{i=1}^{b-1}\left(\left[\frac{(a-1)i}{b}\right]+1\right)-(\gcd(a-1,b)-1)\\&=\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}+b-1-(\gcd(a-1,b)-1)\\&=\frac{a(b-1)-\gcd(a-1,b)+1}{2}.\end{aligned}$$

And the 1-st Hodge ideal of D_2 is

$$\begin{aligned}J_1(D_2)&=(f)+I_0(D_2)\cdot(Jf)\\&=\big(x^a+xy^b,ax^{a-1}y^{b-1}+y^{2b-1},x^{[\frac{(a-1)(b-1)}{b}]+2}y^{b-1},x^{[\frac{(a-1)(b-2)}{b}]+2}y^{b},\dots,\\&\qquad x^{a-\frac{(a-1)r}{b}}y^{b+r-2},\dots,x^{[\frac{(a-1)(b-(i-1)r-1)}{b}]+2}y^{b+(i-1)r-1},\\&\qquad x^{[\frac{(a-1)(b-(i-1)r-2)}{b}]+2}y^{b+(i-1)r},\dots,x^{a-\frac{i(a-1)r}{b}}y^{b+ir-2},\dots,\\&\qquad x^{[\frac{(a-1)(r-1)}{b}]+2}y^{2b-r-1},x^{[\frac{(a-1)(r-2)}{b}]+2}y^{2b-r},\dots,xy^{2b-2}\big),\end{aligned}$$

where $1\le i\le \gcd(a-1,b)$. Its multiplicity $\mathrm{mt}(J_1(D_2))$ equals $b+1$. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_2)=\mathcal{O}_X/J_1(D_2)$ is

$$\begin{aligned}m_1(D_2)&=a(b-1)+\sum_{i=1}^{b-1}\left(\left[\frac{(a-1)i}{b}\right]+2\right)-(\gcd(a-1,b)-1)+1\\&=(a+2)(b-1)+\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}-(\gcd(a-1,b)-1)+1\\&=\frac{(3a+2)(b-1)-\gcd(a-1,b)+3}{2}.\end{aligned}$$

Consider isolated quasihomogeneous curve singularities $D_3^{(a,b)} = \{x^a y + xy^b = 0\}$, defined by $F_3^{(a,b)} = x^a y + xy^b$. If $a \le b$, let $r = \frac{a-1}{\gcd(a-1,b-1)}$; then $1 \le r \le a-1$. Since

$$\frac{(b-1)(1+a-1)}{ab-1} + \frac{a-1}{ab-1} \ge 1,$$

we have $x^{a-1} \in I_0(D_3)$. And we have

$$x^k y^{[\frac{(b-1)(a-1-k)}{a-1}]+1} \in I_0(D_3) \quad \text{for all } 1 \le k \le a-2,\ r \nmid k,$$

and

$$x^{ir} y^{b-1-\frac{i(b-1)r}{a-1}} \in I_0(D_3) \quad \text{for all } 1 \le i \le \gcd(a-1, b-1) - 1.$$

So the 0-th Hodge ideal for D_3 is

$$\begin{aligned} J_0(D_3) &= I_0(D_3) \\ &= \big(x^{a-1}, x^{a-2} y^{[\frac{b-1}{a-1}]+1}, \dots, x^{a-r} y^{[\frac{(b-1)(r-1)}{a-1}]+1}, \dots, \\ &\qquad x^{ir} y^{b-1-\frac{i(b-1)r}{a-1}}, x^{ir-1} y^{[\frac{(b-1)(a-ir)}{a-1}]+1}, \dots, x^{(i-1)r+1} y^{[\frac{(b-1)(a-1-(i-1)r-1)}{a-1}]+1}, \\ &\qquad \dots, x^r y^{b-1-\frac{(b-1)r}{a-1}}, x^{r-1} y^{[\frac{(b-1)(a-r)}{a-1}]+1}, \dots, xy^{[\frac{(b-1)(a-2)}{a-1}]+1}, y^{b-1}\big), \end{aligned}$$

where $1 \le i \le \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_0(D_3))$ equals $a-1$. The dimension of the 0-th Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X / J_0(D_3)$, by Lemma 3.1, is

$$\begin{aligned} m_0(D_3) &= \sum_{i=1}^{a-2} \left(\left[\frac{(b-1)i}{a-1} \right] + 1 \right) - (\gcd(a-1, b-1) - 1) + b - 1 \\ &= \frac{ab - \gcd(a-1, b-1) - 1}{2}. \end{aligned}$$

And the 1-st Hodge ideal of D_3 is

$$\begin{aligned} J_1(D_3) &= (f) + I_0(D_3) \cdot (Jf) \\ &= \big(x^a y + xy^b, x^{2a-1}, y^{2b-1} x^{2a-2} y, \dots, x^{2a-r-1} y^{[\frac{(b-1)(r-1)}{a-1}]+2}, \\ &\qquad \dots, x^{a-1+ir} y^{b-\frac{i(b-1)r}{a-1}}, \dots, x^{a+(i-1)r} y^{[\frac{(b-1)(a-1-(i-1)r-1)}{a-1}]+2}, \\ &\qquad \dots, x^{a-1+r} y^{b-\frac{(b-1)r}{a-1}}, \dots, x^a y^{[\frac{(b-1)(a-2)}{a-1}]+2}\big), \end{aligned}$$

where $1 \le i \le \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_1(D_3))$ equals $a+1$. The dimension of the 1-st Hodge moduli algebra $M_1(D_3) = \mathcal{O}_X / J_1(D_3)$, by Lemma 3.1, is

$$\begin{aligned} m_1(D_3) &= (2b-1) + (a-2)b + b + \sum_{i=1}^{a-2} \left(\left[\frac{(b-1)i}{a-1} \right] + 2 \right) - (\gcd(a-1, b-1) - 1) + 1 \\ &= \frac{a(3b+2) - \gcd(a-1, b-1) - 3}{2}. \end{aligned}$$

If $a \geq b$, let $r = \frac{b-1}{\gcd(a-1,b-1)}$; then $1 \leq r \leq b-1$. By symmetry of a, b, we obtain the 0-th Hodge ideal

$$\begin{aligned} J_0(D_3) &= I_0(D_3) \\ &= \big(y^{b-1}, y^{b-2}x^{[\frac{a-1}{b-1}]+1}, \dots, y^{b-r}x^{[\frac{(a-1)(r-1)}{b-1}]+1}, \dots, \\ &\qquad y^{ir}x^{a-1-\frac{i(a-1)r}{b-1}}, y^{ir-1}x^{[\frac{(a-1)(b-ir)}{b-1}]+1}, \dots, y^{(i-1)r+1}x^{[\frac{(a-1)(b-1-(i-1)r-1)}{b-1}]+1}, \\ &\qquad \dots, y^r x^{a-1-\frac{(a-1)r}{b-1}}, y^{r-1}x^{[\frac{(a-1)(b-r)}{b-1}]+1}, \dots, yx^{[\frac{(a-1)(b-2)}{b-1}]+1}, x^{a-1}\big), \end{aligned}$$

where $1 \leq i \leq \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_0(D_3))$ equals $b-1$. The dimension of the 0-th Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X/J_0(D_3)$, by Lemma 3.1, is

$$\begin{aligned} m_0(D_3) &= \sum_{i=1}^{b-2}\left(\left[\frac{(a-1)i}{b-1}\right]+1\right) - (\gcd(a-1, b-1)-1) + a - 1 \\ &= \frac{ab - \gcd(a-1, b-1) - 1}{2}. \end{aligned}$$

And the 1-st Hodge ideal of D_3 is

$$\begin{aligned} J_1(D_3) &= (f) + I_0(D_3)\cdot(Jf) \\ &= \big(x^a y + xy^b, x^{2a-1}, y^{2b-1}y^{2b-2}x, \dots, y^{2b-r-1}x^{[\frac{(a-1)(r-1)}{b-1}]+2}, \\ &\qquad \dots, y^{b-1+ir}x^{a-\frac{i(a-1)r}{b-1}}, \dots, y^{b+(i-1)r}x^{[\frac{(a-1)(b-1-(i-1)r-1)}{b-1}]+2}, \\ &\qquad \dots, y^{b-1+r}x^{a-\frac{(a-1)r}{b-1}}, \dots, y^a x^{[\frac{(a-1)(b-2)}{b-1}]+2}\big), \end{aligned}$$

where $1 \leq i \leq \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_1(D_3))$ equals $b+1$. The dimension of the 1-st Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X/J_1(D_3)$, by Lemma 3.1, is

$$\begin{aligned} m_1(D_3) &= (2a-1)+(b-2)a+a+\sum_{i=1}^{b-2}\left(\left[\frac{(a-1)i}{b-1}\right]+2\right)-(\gcd(a-1, b-1)-1)+1 \\ &= \frac{(3a+2)b-\gcd(a-1, b-1)-3}{2}. \end{aligned}$$

4. Proof of Main Theorem A

I. We compare singularities of types $\mathbb{F}_1$ and $\mathbb{F}_2$:

(1) Suppose for singularities

$$\begin{aligned} D_1 &= \{x^{a_1} + y^{b_1} = 0\}, \quad 2 \leq a_1 \leq b_1, \\ D_2 &= \{x^{a_2} + xy^{b_2} = 0\}, \quad 1 \leq a_2 - 1 \leq b_2, \end{aligned}$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_2), \quad M_1(D_1) \simeq M_1(D_2).$$

By our computation in Section 3 we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_1)) &= a_1 - 1,\\
\mathrm{mt}(J_1(D_1)) &= a_1,\\
m_0(D_1) &= \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\
m_1(D_1) &= \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_2)) &= a_2 - 1,\\
\mathrm{mt}(J_1(D_2)) &= a_2,\\
m_0(D_2) &= \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\
m_1(D_2) &= \frac{a_2(3b_2-1)-\gcd(a_2-1,b_2)+1}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_1 - 1 &= a_2 - 1,\\
a_1 &= a_2,\\
\frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2} &= \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\
\frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} &= \frac{a_2(3b_2-1)-\gcd(a_2-1,b_2)+1}{2},
\end{aligned}$$

that is,

$$\begin{aligned}
a_1 &= a_2,\\
(a_1-1)b_1 &= a_2b_2,\\
\gcd(a_1,b_1)-\gcd(a_2-1,b_2) &= a_2-(a_1-1).
\end{aligned}$$

Its solutions are $(a_1, b_1) = (a_2, a_2m)$, $(a_2, b_2) = (a_2, (a_2-1)m)$, where $a_2, m \in \mathbb{N}$, $a_2 \geq 2$, $m \geq 1$. And we have

$$\begin{aligned}
\mathrm{wt}(F_1) &= \left\{\frac{1}{a_1}, \frac{1}{b_1}\right\} = \left\{\frac{1}{a_2}, \frac{1}{a_2m}\right\},\\
\mathrm{wt}(F_2) &= \left\{\frac{1}{a_2}, \frac{a_2-1}{a_2b_2}\right\} = \left\{\frac{1}{a_2}, \frac{1}{a_2m}\right\}.
\end{aligned}$$

It follows that $\mathrm{wt}(F_1) = \mathrm{wt}(F_2)$. Under these conditions, we obtain

$$\begin{aligned}
J_0(D_1) &= (x^{a_2-1}, x^{a_2-2}y^{m-1}, \dots, y^{(a_2-1)m-1}),\\
J_0(D_2) &= (x^{a_2-1}, x^{a_2-2}y^{m-1}, \dots, y^{(a_2-1)m-1}),
\end{aligned}$$

i.e., $J_0(D_1) = J_0(D_2)$, which shows $M_0(D_1) \simeq M_0(D_2)$ directly.

(2) Suppose for singularities

$$D_1=\{x^{a_1}+y^{b_1}=0\},\qquad 2\le a_1\le b_1,$$
$$D_2=\{x^{a_2}+xy^{b_2}=0\},\quad a_2-1\ge b_2\ge 1,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1)\simeq M_0(D_2),\quad M_1(D_1)\simeq M_1(D_2).$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_1))&=a_1-1,\\
\mathrm{mt}(J_1(D_1))&=a_1,\\
m_0(D_1)&=\frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\
m_1(D_1)&=\frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_2))&=b_2-1,\\
\mathrm{mt}(J_1(D_2))&=b_2+1,\\
m_0(D_2)&=\frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\
m_1(D_2)&=\frac{(3a_2+2)(b_2-1)-\gcd(a_2-1,b_2)+3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_1-1&=b_2-1,\\
a_1&=b_2+1,\\
\frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2}&=\frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\
\frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}&=\frac{(3a_2+2)(b_2-1)-\gcd(a_2-1,b_2)+3}{2}.
\end{aligned}$$

It has no solution.

II. We compare singularities of types $\mathbb{F}_2$ and $\mathbb{F}_3$:

(1) Suppose for singularities

$$D_2=\{x^{a_2}+xy^{b_2}=0\},\quad a_2-1\ge b_2\ge 1,$$
$$D_3=\{x^{a_3}y+xy^{b_3}=0\},\qquad 1\le a_3\le b_3,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_2)\simeq M_0(D_3),\quad M_1(D_2)\simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}\operatorname{mt}(J_0(D_2)) &= b_2-1,\\ \operatorname{mt}(J_1(D_2)) &= b_2+1,\\ m_0(D_2) &= \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\ m_1(D_2) &= \frac{(3a_2+2)(b_2-1)-\gcd(a_2-1,b_2)+3}{2}.\end{aligned}$$

And

$$\begin{aligned}\operatorname{mt}(J_0(D_3)) &= a_3-1,\\ \operatorname{mt}(J_1(D_3)) &= a_3+1,\\ m_0(D_3) &= \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2},\\ m_1(D_3) &= \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2}.\end{aligned}$$

Hence we obtain the equations

$$\begin{gathered} b_2-1=a_3-1,\\ b_2+1=a_3+1,\\ \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2}=\frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2},\\ \frac{(3a_2+2)(b_2-1)-\gcd(a_2-1,b_2)+3}{2}=\frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2},\end{gathered}$$

that is,

$$\begin{aligned} b_2 &= a_3,\\ a_2b_2+b_2-a_2 &= a_3b_3+a_3-1,\\ \gcd(a_2-1,b_2)-\gcd(a_3-1,b_3-1) &= a_3-(b_2-1).\end{aligned}$$

Its solutions are $(a_2,b_2)=(mb_2+1,b_2)$, $(a_3,b_3)=(b_2,m(b_2-1)+1)$, where $b_2,m\in\mathbb{N}$, $b_2\geq 2$, $m\geq 1$. And we have

$$\begin{aligned}\operatorname{wt}(F_2) &= \left\{\frac{1}{a_2},\frac{a_2-1}{a_2b_2}\right\}=\left\{\frac{1}{mb_2+1},\frac{m}{mb_2+1}\right\},\\ \operatorname{wt}(F_3) &= \left\{\frac{b_3-1}{a_3b_3-1},\frac{a_3-1}{a_3b_3-1}\right\}=\left\{\frac{m}{mb_2+1},\frac{1}{mb_2+1}\right\}.\end{aligned}$$

It follows that $\operatorname{wt}(F_2)=\operatorname{wt}(F_3)$. Under these conditions, we obtain

$$\begin{aligned} J_0(D_2) &= (x^{m(b_2-1)},x^{m(b_2-2)}y,\ldots,y^{b_2-1}),\\ J_0(D_3) &= (y^{m(b_2-1)},y^{m(b_2-2)}x,\ldots,x^{b_2-1}),\end{aligned}$$

i.e., $J_0(D_2)\cong J_0(D_3)$, $x\mapsto y$, which shows $M_0(D_2)\cong M_0(D_3)$ directly.

(2) Suppose for singularities

$$D_2 = \{x^{a_2} + xy^{b_2} = 0\}, \quad 1 \le a_2 - 1 \le b_2,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \quad 1 \le a_3 \le b_3,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_2) \simeq M_0(D_3), \quad M_1(D_2) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_2)) &= a_2 - 1,\\
\mathrm{mt}(J_1(D_2)) &= a_2,\\
m_0(D_2) &= \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},\\
m_1(D_2) &= \frac{a_2(3b_2-1) - \gcd(a_2-1, b_2) + 1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_3)) &= a_3 - 1,\\
\mathrm{mt}(J_1(D_3)) &= a_3 + 1,\\
m_0(D_3) &= \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
m_1(D_3) &= \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{gathered}
a_2 - 1 = a_3 - 1,\\
a_2 = a_3 + 1,\\
\frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2} = \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
\frac{a_2(3b_2-1) - \gcd(a_2-1, b_2) + 1}{2} = \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{gathered}$$

It has no solution.

III. We compare singularities of types $\mathbb{F}_1$ and $\mathbb{F}_3$:

(1) Suppose for singularities

$$D_1 = \{x^{a_1} + y^{b_1} = 0\}, \quad 2 \le a_1 \le b_1,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \quad 1 \le a_3 \le b_3,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_3), \quad M_1(D_1) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_1)) &= a_1 - 1,\\
\mathrm{mt}(J_1(D_1)) &= a_1,\\
m_0(D_1) &= \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\
m_1(D_1) &= \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_3)) &= a_3 - 1,\\
\mathrm{mt}(J_1(D_3)) &= a_3 + 1,\\
m_0(D_3) &= \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2},\\
m_1(D_3) &= \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_1 - 1 &= a_3 - 1,\\
a_1 &= a_3 + 1,\\
\frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2} &= \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2},\\
\frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} &= \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2}.
\end{aligned}$$

It has no solutions.

(2) Suppose for singularities

$$\begin{aligned}
D_1 &= \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \le a_1 \le b_1,\\
D_3 &= \{x^{a_3}y + xy^{b_3} = 0\}, \quad a_3 \ge b_3 \ge 1,
\end{aligned}$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_3), \quad M_1(D_1) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_1)) &= a_1 - 1,\\
\mathrm{mt}(J_1(D_1)) &= a_1,\\
m_0(D_1) &= \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\
m_1(D_1) &= \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_3)) &= b_3 - 1,\\
\mathrm{mt}(J_1(D_3)) &= b_3 + 1,\\
m_0(D_3) &= \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
m_1(D_3) &= \frac{(3a_3+2)b_3 - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_1 - 1 &= b_3 - 1,\\
a_1 &= b_3 + 1,\\
\frac{(a_1-1)(b_1-1) - \gcd(a_1, b_1) + 1}{2} &= \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
\frac{(a_1-1)(3b_1-1) - \gcd(a_1, b_1) + 1}{2} &= \frac{(3a_3+2)b_3 - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

It has no solution.

5. Proof of Main Theorem B

(1) For isolated quasihomogeneous curve singularity

$$D_1^{(a,b)} = \{x^a + y^b = 0\}, \quad a, b \geq 2,$$

since the 0-th Hodge moduli number is

$$m_0(D_1^{(a,b)}) = \frac{(a-1)(b-1) - \gcd(a, b) + 1}{2},$$

we have

$$\delta_1(a, b) - m_0(D_1^{(a,b)}) = \gcd(a, b) - 1 \geq 0.$$

And we also have

$$\delta_1(a, b) - m_0(D_1^{(a,b)}) = \gcd(a, b) - 1 \leq \min\{a, b\} - 1 = \mathrm{mt}(D_1^{(a,b)}) - 1.$$

The equality holds if and only if $\min\{a, b\} = \gcd(a, b)$, i.e., $(a, b) = (a, am)$ or $(a, b) = (bm', b)$ for some $m, m' \in \mathbb{N}$.

(2) For isolated quasihomogeneous curve singularity

$$D_2^{(a,b)} = \{x^a + xy^b = 0\}, \quad a \geq 2, b \geq 1,$$

since the 0-th Hodge moduli number is

$$m_0(D_2^{(a,b)}) = \frac{a(b-1) - \gcd(a-1, b) + 1}{2},$$

we have

$$\delta_2(a, b) - m_0(D_2^{(a,b)}) = \gcd(a-1, b) \geq 1.$$

And we also have

$$\delta_2(a,b) - m_0(D_2^{(a,b)}) = \gcd(a-1,b) \le \min\{a-1,b\} = \mathrm{mt}(D_2^{(a,b)}) - 1.$$

The equality holds if and only if $\min\{a-1,b\} = \gcd(a-1,b)$, i.e., $(a,b) = (a,(a-1)m)$ or $(a,b) = (bm'+1,b)$ for some $m,m' \in \mathbb{N}$.

(3) For isolated quasihomogeneous curve singularity

$$D_3^{(a,b)} = \{x^a y + x y^b = 0\}, \quad a,b \ge 1,$$

since the 0-th Hodge moduli number is

$$m_0(D_3^{(a,b)}) = \frac{ab - \gcd(a-1,b-1)+1}{2},$$

we have

$$\delta_3(a,b) - m_0(D_3^{(a,b)}) = \gcd(a-1,b-1)+1 \ge 2.$$

And we also have

$$\begin{aligned}\delta_3(a,b) - m_0(D_3^{(a,b)}) &= \gcd(a-1,b-1)+1 \le \min\{a-1,b-1\}+1\\ &= \min\{a,b\} = \mathrm{mt}(D_3^{(a,b)}) - 1.\end{aligned}$$

The equality holds if and only if $\min\{a-1,b-1\} = \gcd(a-1,b-1)$, i.e., $(a,b) = (a,(a-1)m+1)$ or $(a,b) = ((b-1)m'+1,b)$ for some $m,m' \in \mathbb{N}$.

6. Some examples and conjectures

Example 6.1. Let curve singularities $H_1 = \{x^2y + xy^6 = 0\}$ be defined by a polynomial $f(x,y) = x^2y + xy^6$ and $H_2 = \{x^3y + xy^4 = 0\}$ be defined by a polynomial $g(x,y) = x^3y + xy^4$. Then f is quasihomogeneous of weight type $\left(\frac{5}{11}, \frac{1}{11}; 1\right)$ and g is quasihomogeneous of weight type $\left(\frac{3}{11}, \frac{2}{11}; 1\right)$. In [16], the characteristic polynomials of f and g coincide:

$$\Delta_f(t) = (t-1)(t^{11}-1) = \Delta_g(t).$$

So this tells us that the characteristic polynomial does not determine the weights of the nondegenerate quasihomogeneous polynomial defining the singularity.

However, their i-th Hodge moduli algebras $M_i(D^\alpha)$ are not isomorphic for $i \ge i_0(\alpha)$, for a big enough $i_0(\alpha)$. Precisely speaking,

$$M_i(D_1^\alpha) \not\cong M_i(D_2^\alpha) \quad \text{for all } i \ge 1,$$

where $D_j^\alpha = \alpha H_j$, $j = 1,2$, for $\alpha = 1$. In fact, we just simply observe this result by their Hodge moduli numbers are different for $i \ge 1$ as follows:

singularity	weight type	$m_0(D)$	$m_1(D)$	$m_2(D)$	$m_3(D)$	$m_4(D)$	$m_5(D)$
$x^2y + xy^6$	$\left(\frac{5}{11}, \frac{1}{11}; 1\right)$	5	18	32	46	60	74
$x^3y + xy^4$	$\left(\frac{3}{11}, \frac{2}{11}; 1\right)$	5	19	39	49	64	79

Example 6.2. Let $f(z_1,\dots,z_n,w_1,w_2)=z_1^2+\cdots+z_n^2+w_1^3+w_2^{2p}$ be a quasihomogeneous polynomial of weight type $\left(\frac{1}{2},\dots,\frac{1}{2},\frac{1}{3},\frac{1}{2p};1\right)$ with an isolated singularity at the origin for any $p\in\mathbb{N}$. Let $n\geq 0$, even and $\gcd(3,p)=1$. Then we know their characteristic polynomials are

$$\Delta_f(t)=\frac{t^{4p}+t^{2p}+1}{t^2+t+1}\quad\text{for all } n\geq 0\text{, even.}$$

Hence $\Delta_f(1)=1$. By Theorem 8.5 in [4], each of their links $K_f=S_\epsilon\cap\{f(z,w)=0\}$ is a topological sphere. Thus all K_f for all p, $(3,p)=1$, are homeomorphic to each other though $f(z_1,\dots,z_n,w_1,w_2)$ are of the different quasihomogeneous types for all p.

However, their i-th Hodge moduli algebras $M_i(D^\alpha)$ are not isomorphic for all $i\geq 1$ and $n\geq 2$, where $D^\alpha=\{f(z_1,\dots,z_n,w_1,w_2)=0\}$ for $\alpha=1$. In fact, we have their 0-th Hodge ideal

$$I_0(D)=\begin{cases}(1) & \text{if } n\geq 2\text{, or } n=0,\ p=1,2,\\ (w_1,w_2^{i_0}) & \text{if } n=0,\ p\geq 4,\end{cases}$$

where $i_0=\left\lceil\frac{4p}{3}\right\rceil-1$, is the smallest integer bigger than or equal to $\frac{4p}{3}-1$. Then we compute their 1-st Hodge ideal as follows. For example, if $n=2$, we have

$$\begin{aligned}I_1^{(2)}(D)&=\sum_{v_j\in\mathcal{O}^{\geq 2}}\mathcal{O}_X\cdot v_j+\sum_{\substack{1\leq i\leq 4\\ a\in I_0(D)}}\mathcal{O}_X(f\partial_i a-\alpha a\partial_i f)\\&=(w_2^{i_1},w_1w_2^{j_1})+(z_1,z_2,w_1^2,w_2^{2p-1})\\&=(z_1,z_2,w_1^2,w_1w_2^{j_1},w_2^{i_1}),\\ J_1^{(2)}(D)&=(f)+I_1^{(2)}(D)\\&=(z_1,z_2,w_1^2,w_1w_2^{j_1},w_2^{i_1}),\end{aligned}$$

where $i_1=\left\lceil\frac{4p}{3}\right\rceil-1$ and $j_1=\left\lceil\frac{2p}{3}\right\rceil-1$. And if $n=4$, we have

$$\begin{aligned}I_1^{(4)}(D)&=\sum_{v_j\in\mathcal{O}^{\geq 2}}\mathcal{O}_X\cdot v_j+\sum_{\substack{1\leq i\leq 6\\ a\in I_0(D)}}\mathcal{O}_X(f\partial_i a-\alpha a\partial_i f)\\&=(z_1,z_2,z_3,z_4,w_1^2,w_2^{2p-1}),\\ J_1^{(4)}(D)&=(f)+I_1^{(4)}(D)\\&=(z_1,z_2,z_3,z_4,w_1^2,w_2^{2p-1}).\end{aligned}$$

So we obtain their corresponding Hodge moduli algebras

$$\begin{aligned}M_1^{(2)}(D)&=\mathbb{C}\{z_1,z_2,w_1,w_2\}/I_1^{(2)}(D)=\mathbb{C}\{w_1,w_2\}/(w_1^2,w_1w_2^{j_1},w_2^{i_1}),\\ M_2^{(4)}(D)&=\mathbb{C}\{z_1,z_2,z_3,z_4,w_1,w_2\}/I_1^{(4)}(D)=\mathbb{C}\{w_1,w_2\}/(w_1^2,w_2^{2p-1}),\end{aligned}$$

which are not isomorphic obviously, since one can verify

$$\dim_{\mathbb{C}} M_1^{(2)}(D) = i_1 + j_1 < 2(2p-1) = \dim_{\mathbb{C}} M_1^{(4)}(D).$$

Thus these examples imply that Hodge moduli algebras and Hodge moduli numbers (or the Hodge moduli sequence) are better invariants than the characteristic polynomial (a topological invariant of the singularity) for nondegenerate quasihomogeneous singularities.

It is an interesting question interesting whether Hodge numbers, Hodge ideals and Hodge moduli algebras of singularities remain constant or isomorphic under some deformations, like quasihomogeneous or semiquasihomogeneous deformations (or, more generally, μ-constant deformations). We give an example to explain that the Hodge moduli numbers of isolated singularities may remain constant under quasihomogeneous deformation.

Example 6.3. For quasihomogeneous polynomial

$$f = x^2 + y^4$$

of weight $\mathrm{wt}(f) = \left(\frac{1}{2}, \frac{1}{4}; 1\right)$, let divisor $D_1^{\alpha} = \{f = 0\}$, where $\alpha = 1$. Then its 1-st Hodge ideal and Hodge moduli algebra are

$$J_1(D_1) = (x^2, xy, y^4),$$
$$M_1(D_1) = \mathbb{C}\{x, y\}/(x^2, xy, y^4).$$

And for quasihomogeneous polynomial $g = x^2 + y^4 + xy^2$ of weight $\mathrm{wt}(g) = \left(\frac{1}{2}, \frac{1}{4}; 1\right)$, which is a quasihomogeneous deformation of f, let divisor $D_2^{\alpha} = \{g = 0\}$, where $\alpha = 1$. Then its 1-st Hodge ideal and Hodge moduli algebra are

$$J_1(D_2) = (x^2, xy^2, 2xy + y^3, y^4),$$
$$M_1(D_2) = \mathbb{C}\{x, y\}/(x^2, xy^2, 2xy + y^3, y^4).$$

As $\mathbb{C}$-vector spaces, $M_1(D_1)$ and $M_1(D_2)$ have the same the $\mathbb{C}$-basis:

$$1, x, y, y^2, y^3.$$

Thus, the Hodge moduli numbers of D_1 and D_2 are the same.

So we raise a conjecture from the above example.

Conjecture 6.4. Suppose F_i is one of the three types[1] of quasihomogeneous polynomial in $\mathbb{C}^2$, $1 \le i \le 3$. Let $H_{i,t} = F_i + tG_i$ be a semiquasihomogeneous deformation of F_i, $t \in \mathbb{C}$, $1 \le i \le 3$. Then the k-th Hodge moduli algebras of the divisors $D_H^{\alpha} = \{H_{i,t} = 0\}$ for $\alpha = 1$, and $D_F^{\alpha} = \{F_i = 0\}$ for $\alpha = 1$, have the same basis

[1] See pages 351–352 in the introduction.

over $\mathbb{C}$ for all $k \geq 0$. Hence, their dimensions, i.e., their k-th Hodge moduli numbers, are the same,

$$m_k(D_H^\alpha) = m_k(D_F^\alpha) \quad \forall k \geq 0, \forall 1 \leq i \leq 3, \forall t \in \mathbb{C}.$$

And we can also ask whether the inequalities in Main Theorem B can be extended to more general singularities. Suppose F_i is one of the three types of quasihomogeneous curve singularities, $1 \leq i \leq 3$. Consider a μ-constant deformation $H_{i,t} = F_i + tG_i$ of F_i, $t \in \mathbb{C}$. If we furthermore assume $H_{i,t}$ is reduced, i.e., all distinct irreducible factors of $H_{i,t}$ have multiplicity 1, we have

$$\mu(H_{i,t}) = \mu(F_i),$$
$$r(H_{i,t}) \leq r(F_i).$$

By Lemma 2.7, we have

$$\delta(H_{i,t}) \leq \delta(F_i),$$

where μ, δ and r are the same notation as in Lemma 2.7. So we have a corollary:

Corollary 6.5. *Suppose the above Conjecture 6.4 is true. For a semiquasihomogeneous deformation $H_{i,t}$, $t \in \mathbb{C}$, of F_i, we have*

$$\delta(D_{i,t}) - m_0(D_{i,t}) \leq \mathrm{mt}(D_{i,t}),$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$, where $D_{i,t} = \{H_{i,t} = 0\}$ is the corresponding singularity, $\delta(D_{i,t})$ is the δ-invariant of $D_{i,t}$, $m_0(D_{i,t})$ is the 0-th Hodge moduli number of $D_{i,t}$, and $\mathrm{mt}(D_{i,t})$ is the multiplicity of $D_{i,t}$.

Proof. In fact, one can verify the multiplicity of F_i is not decreasing under the above semiquasihomogeneous deformation for all $1 \leq i \leq 3$, i.e.,

$$\mathrm{mt}(D_{i,t}) \geq \mathrm{mt}(D_{i,0})$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. And by Conjecture 6.4, we have

$$m_0(D_{i,t}) = m_0(D_{i,0})$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. Finally, by the discussion after Conjecture 6.4, we have

$$\delta(D_{i,t}) \leq \delta(D_{i,0})$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. So we have

$$\delta(D_{i,t}) - m_0(D_{i,t}) \leq \delta(D_{i,0}) - m_0(D_{i,0}) \leq \mathrm{mt}(D_{i,0}) \leq \mathrm{mt}(D_{i,t}),$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. □

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YANG WANG
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA
wangyang21@mails.tsinghua.edu.cn

STEPHEN S.-T. YAU
BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS (BIMSA)
BEIJING
CHINA
yau@uic.edu
and
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA

HUAIQING ZUO
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA
hqzuo@mail.tsinghua.edu.cn

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