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Suppose $b=\{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than 1 and $D=\{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets. Let $\mu_{b,D}$ be the Cantor–Moran measure defined by

$$\mu_{b,D}=\delta_{\frac{1}{b_1}\mathscr{D}_1}*\delta_{\frac{1}{b_1b_2}\mathscr{D}_2}*\delta_{\frac{1}{b_1b_2b_3}\mathscr{D}_3}*\cdots.$$

We first prove that $L^2(\mu_{b,D})$ possesses an exponential orthonormal basis if and only if N_n divides b_n for each $n\geq 2$. Subsequently, we show that the generalized Fuglede's conjecture holds for such Cantor–Moran measures. An immediate consequence of this result is the equivalence between the existence of an exponential orthonormal basis and the integer-tiling of $D_n=\mathscr{D}_n+b_n\mathscr{D}_{n-1}+b_2\cdots b_n\mathscr{D}_1$ for $n\geq 1$.

1. Introduction

A Borel probability measure μ on $\mathbb{R}^d$ is called a *spectral measure* if there exists a countable set $\Lambda\subset\mathbb{R}^d$ (called a *spectrum*) such that the set of exponential functions $E(\Lambda):=\{e^{2\pi i\lambda\cdot x}:\lambda\in\Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. If $\Omega\subset\mathbb{R}^d$ is a measurable set with finite positive Lebesgue measure and $\mathscr{L}_\Omega$ is a spectral measure, then we say that Ω is a *spectral set*. Here $\mathscr{L}_K$ denotes the normalized Lebesgue measure restricted to a measurable set K of finite positive Lebesgue measure.

It is well known from classical Fourier analysis that the unit cube $[0,1]^d$ is a spectral set with a spectrum $\mathbb{Z}^d$. What other sets Ω can be spectral? The research on this problem has been influenced for many years by a famous paper due to Fuglede [1974], who suggested that there should be a concrete, geometric way to characterize the spectral sets.

Conjecture (Fuglede's conjecture). A set $\Omega\subset\mathbb{R}^d$ is spectral if and only if it can tile the space by translations.

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We say Ω is a translational tile if there exists a discrete set $\mathcal{J}$ such that the translated copies $\{\Omega + t : t \in \mathcal{J}\}$ constitute a partition of $\mathbb{R}^d$ up to measure zero. Although the conjecture was eventually disproved in dimension 3 or higher in its full generality [Kolountzakis and Matolcsi 2006b; 2006a; Tao 2004], most of the known examples of spectral sets are constructed from translational tiles [Łaba 2001; Fu et al. 2015; Dai et al. 2013; Dai et al. 2014; Dai 2016; 2012]. An important result proved by Lev and Matolcsi [2022] is that all spectral sets must admit a "weak tiling" which is a generalization of translational tiling in its measure-theoretic form. Let $K \subset \mathbb{R}^d$ be a bounded, measurable set. We say that another measurable, but possibly unbounded, set $\Sigma \subset \mathbb{R}^d$ admits a weak tiling by translates of K if there exists a positive, locally finite (Borel) measure ν on $\mathbb{R}^d$ such that $\mathbf{1}_K * \nu = \mathbf{1}_\Sigma$, where $\mathbf{1}_A$ denotes the indicator function of a set A.

Jorgensen and Pedersen [1998] widened the scope of Fuglede's conjecture and discovered that the standard middle-fourth Cantor measure $\mu_{1/(2k),\{0,2\}}$ is a spectral measure. It is the first spectral measure that is nonatomic and singular. Strichartz [2000; 2006] discovered a surprising and interesting phenomenon: the Fourier series corresponding to certain spectra of $\mu_{4^{-1},\{0,2\}}$ can exhibit significantly better convergence properties than their classical counterparts on the unit interval. Specifically, the Fourier series of continuous functions converge uniformly, and Fourier series of L^p-functions converge in the L^p-norm for $1 \le p < \infty$. Following these discoveries, there has been considerable research on such measures [An and Wang 2021; Dutkay and Haussermann 2016; Łaba and Wang 2002; Fu et al. 2015; Dutkay et al. 2009; Dai 2012; Dai et al. 2013; Dai et al. 2014; Dai 2016; Hu and Lau 2008; Dutkay et al. 2019], and a celebrated open problem was to characterize the spectral property of the Cantor measures $\mu_{\rho,\mathcal{D}}$, $0 < \rho < 1$ among the N-Bernoulli convolutions

$$\mu_{\rho,\mathcal{D}}(\cdot) = \frac{1}{N}\sum_{d\in\mathcal{D}} \mu_{\rho,\mathcal{D}}(\rho^{-1}(\cdot)\cdot - d),$$

where $\mathcal{D} = \{0, 1, \ldots, N-1\}$.

N-Bernoulli convolutions have been studied extensively in many areas of mathematics, including Fourier analysis, dynamical systems, integer tiles, wavelet theory, algebraic number theory and fractal geometry, since the 1930s (e.g., see [Wintner 1935]). Hu and Lau [2008], as well as Dai [2012], showed that the above Cantor measures $\mu_{1/(2k),\{0,2\}}$ are the only class of spectral measures among the $\mu_{\rho,\{0,2\}}$. Dai, He and Lau [Dai et al. 2014] demonstrated that a similar result holds for N-Bernoulli measures $\mu_{\rho,\mathcal{D}}$.

Later on, Strichartz [2000] formulated the most general fractal spectral measures one can possibly generate. Let $\boldsymbol{b} = \{b_n\}_{n=1}^\infty$ be a sequence of integers bigger than 1, and let $\boldsymbol{D} = \{\mathcal{D}_n\}_{n=1}^\infty$ be a sequence of integer digit sets. Let δ_a be the Dirac measure, and write

$$\delta_E = \frac{1}{\#E} \sum_{e \in E} \delta_e$$

for a finite set E. Write

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \delta_{\frac{1}{b_1 b_2 b_3}\mathscr{D}_3} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n},$$

where $*$ denotes convolution. If the sequence of convolutions $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ with compact support, then we call $\mu_{\boldsymbol{b},\boldsymbol{D}}$ a *Cantor–Moran measure*, as a generalization of the standard Cantor measure studied first by Moran [1946]. This opens up a research field for orthogonal harmonic analysis of Cantor–Moran measures (e.g., [An et al. 2019; An and He 2014; Deng and Li 2022; Li et al. 2022]).

In this paper, we study the spectrality of the Cantor–Moran measures generated by an integer sequence $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ with $b_n \geq 2$ and a sequence of consecutive digit sets $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$, i.e., $\mathscr{D}_n = \{0, 1, \ldots, N_n - 1\}$. We first provide a sufficient and necessary condition for the existence of such Cantor–Moran measures.

Theorem 1.1. *Let $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than* 1, *and let $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ be a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \ldots, N_n - 1\}$, where $N_n \geq 2$. Then the sequence of discrete measures*

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n}$$

converges weakly to a Borel probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ if and only if

$$\sum_{n=1}^{\infty} \frac{N_n}{b_1 b_2 \cdots b_n} < \infty. \tag{1-1}$$

In this case, $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is supported on a compact set

$$T(\boldsymbol{b}, \boldsymbol{D}) = \left\{ \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n} : d_n \in \mathscr{D}_n \right\} := \sum_{n=1}^{\infty} \frac{\mathscr{D}_n}{b_1 b_2 \cdots b_n}.$$

The spectral properties of such measures were first studied by An and He [2014] as a generalization of the N-Bernoulli convolutions ($b_n = b$ and $D_n = \{0, 1, \ldots, N-1\}$ for all n). The first-named author and He showed that $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral when N_n divides b_n for each $n \geq 1$. Under the condition that $\{N_n\}_{n=1}^{\infty}$ is bounded, it has been proved in [Deng and Li 2022] that the condition N_n divides b_n for each $n \geq 1$ is also necessary for $\mu_{\boldsymbol{b},\boldsymbol{D}}$ to be spectral. These Cantor–Moran measures also show that spectral measures can have support of any Hausdorff dimension [Dai and Sun 2015]. Furthermore, Cantor–Moran measures offer new examples of fractal measures that admit a Fourier frame but not a Fourier orthonormal basis [Gabardo and Lai 2014], which lead to a new avenue to study a long-standing problem: whether a middle-third Cantor measure has a Fourier frame.

Our motivation to extend the N-Bernoulli convolutions to this class of measures stems from the conjecture by Gabardo and Lai [2014], and we aim to answer a question on the relationship between Cantor–Moran spectral measures and integer tiles. To describe a unifying framework bridging the gap between singular spectral measures and spectral sets, Gabardo and Lai [2014] extended the classical Fuglede's conjecture to a more generalized form.

Conjecture (generalized Fuglede's conjecture). A compactly supported Borel probability measure μ on $\mathbb{R}$ is spectral if and only if there exists a Borel probability ν on $\mathbb{R}$ and a fundamental domain Q of some lattice on $\mathbb{R}$ such that $\mu * \nu = \mathscr{L}_Q$.

Deterministic positive results about Cantor measure have appeared in many papers (e.g., [An and Wang 2021; Dutkay et al. 2009; Dai 2012; Dai et al. 2013; Dai et al. 2014; Dai 2016; Hu and Lau 2008; Dutkay et al. 2019; Łaba and Wang 2002]). However, there are relatively few results regarding Cantor–Moran measures. Gabardo and Lai [2014] showed that if both μ and ν are two singular probability measures with $\mu * \nu = \mathscr{L}_{[0,1]}$, then they are both Cantor–Moran measures. In this paper, we will demonstrate that the generalized Fuglede's conjecture holds for our target measure.

Theorem 1.2. *Suppose $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than* 1, *and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \dots, N_n - 1\}$, where $N_n \geq 2$. Then the following are equivalent*:

(i) *The Cantor–Moran measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral.*

(ii) *There exists a Borel probability ν such that $\mu_{\boldsymbol{b},\boldsymbol{D}} * \nu = \mathscr{L}_{[0,N_1/b_1]}$.*

(iii) *N_n divides b_n for each $n \geq 2$.*

Remark. Lai and Wang conjectured that if $D_n = \{0, 1, \dots, N_n - 1\}$ is a continuous digit set for each $n \geq 1$ and the associated measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral, then N_n divides b_n for all n [Lai and Wang 2017, Conjecture 4.3]. Theorem 1.2 confirms this conjecture and resolves their conjecture from the perspective of tiling (convolution is now regarded as the tiling operation).

The implication (ii) $\Rightarrow$ (i) stems from [Gabardo and Lai 2014, Theorem 1.1], and the proof of (iii) $\Rightarrow$ (i) is provided in [An and He 2014, Theorem 1.4], while (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are apparently new in this generality. We now outline the strategy of the proof. Let us set up the notation:

$$\mu_{\boldsymbol{b},\boldsymbol{D}} = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \delta_{\frac{1}{b_1 b_2 b_3}\mathscr{D}_3} * \cdots = \mu_n * \mu_{>n},$$

where μ_n is the convolution of the first n discrete measures, and $\mu_{>n}$ is the remaining part. Based on the aforementioned decomposition of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, one seeks to observe the behavior of μ_n and $\mu_{>n}$ under the condition that $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure. Actually, if $0 \in \Lambda$ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, then, for any $n \geq 1$, we can construct spectra of μ_n and $\mu_{>n}$ relying on a maximal decomposition (see Definition 3.1) of Λ with respect to $(\mu_n, \mu_{>n})$.

Theorem 1.3. *Suppose $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than* 1, *and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets with $\mathscr{D}_n = \{0, 1, \dots, N_n - 1\}$, where $N_n \geq 2$. If $0 \in \Lambda$ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, then, for each $n \geq 1$, we have a maximal decomposition of Λ with respect to $(\mu_n, \mu_{>n})$, denoted by $\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$, such that $\mathscr{A}$ is a spectrum of μ_n and each Λ_α is a spectrum of $\mu_{>n}$.*

Applying Theorem 1.3, we can reduce (i) $\Rightarrow$ (iii) to the following theorem, considering μ_n as a spectral measure.

Theorem 1.4. *The discrete measure $\mu_n = *_{j=1}^{n} \delta_{\frac{1}{b_1 \cdots b_j}\mathscr{D}_j}$ is spectral if and only if N_j divides b_j for each $2 \leq j \leq n$.*

Then (i) $\Rightarrow$ (iii) follows from Theorems 1.3 and 1.4. We observe that if $b_n = r_n N_n$ for some integer r_n, then

$$\mathscr{D}_n \oplus N_n\{0, 1, \dots, r_n - 1\} = \{0, 1, \dots, b_n - 1\}.$$

Here the direct sum $A \oplus B$ means that $a + b$ are all distinct elements for all $a \in A$ and $b \in B$. This can yield (iii) $\Rightarrow$ (ii) in Theorem 1.2.

A digit set $\mathscr{D}$ is called an *integer tile* if $\mathscr{D}$ tiles some cyclic group $\mathbb{Z}_n$, i.e., there exists $\mathscr{B}$ such that $\mathscr{D} \oplus \mathscr{B} \equiv \mathbb{Z}_n \pmod{n}$. The study of integer tiles has a long history related to the geometry of numbers [Coven and Meyerowitz 1999; Łaba and Londner 2023; Newman 1977; Tijdeman 1995; Sands 1979]. In Theorem 1.5, we present an intriguing result regarding the relationship between the Cantor–Moran spectral measure and integer tiles. Write

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n} = \delta_{\frac{1}{b_1 b_2 \cdots b_n}\boldsymbol{D}_n},$$

where $\boldsymbol{D}_n = \mathscr{D}_n + b_n\mathscr{D}_{n-1} + \cdots + b_2 \cdots b_n \mathscr{D}_1$ is the first n terms iterated digit set of $\{\mathscr{D}_n\}_{n=1}^{\infty}$.

Theorem 1.5. *Suppose $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ is a sequence of integers bigger than* 1, *and $\boldsymbol{D} = \{\mathscr{D}_n\}_{n=1}^{\infty}$ is a sequence of consecutive digit sets. Then the Cantor–Moran measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is spectral if and only if, for each $n \geq 1$, $\boldsymbol{D}_n = \mathscr{D}_n \oplus b_n \mathscr{D}_{n-1} \oplus \cdots \oplus b_2 \cdots b_n \mathscr{D}_1$ is an integer tile.*

We organize this paper as follows. In Section 2, we will study the weak convergence of infinite convolutions and give the proof of Theorem 1.1. Also, we will introduce some basic definitions and properties of spectral measures. In Section 3, we will discuss the distribution of any bizero set of the spectral measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$, and prove Theorem 1.3. We will devote Section 4 to proving Theorems 1.2 and 1.4. In Section 5, we will prove Theorem 1.5 and propose an open question on the relationship between the spectral Cantor–Moran measure and the tiling of integers.

2. Notation and preliminaries

2.1. ***Weak convergence of convolutions.*** Using Kolmogorov's three-series theorem, Li, Miao and Wang [Li et al. 2022] provided sufficient and necessary conditions for the existence of infinite convolutions.

Theorem 2.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of nonnegative finite subsets of $\mathbb{R}$ satisfying that $\#A_n \geq 2$ for each $n \geq 1$. Let $\nu_n = \delta_{A_1} * \cdots * \delta_{A_n}$. Then the sequence of convolutions $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\#A_n} \sum_{a \in A_n} \frac{a}{1+a} < \infty. \tag{2-1}$$

Proof of Theorem 1.1. The weak convergence of $\{\mu_n\}_{n=1}^{\infty}$ is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{N_n} \sum_{d \in \mathscr{D}_n} \frac{d}{b_1 \cdots b_n + d} < \infty, \tag{2-2}$$

by Theorem 2.1. Note that

$$\sum_{n=1}^{\infty} \frac{1}{N_n} \sum_{d \in \mathscr{D}_n} \frac{d}{b_1 \cdots b_n + d} = \sum_{\{n : N_n - 1 > b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d} + \sum_{\{n : N_n - 1 \leq b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d}. \tag{2-3}$$

To prove sufficiency, suppose (1-1) holds. We first claim $\{n : N_n - 1 > b_1 \cdots b_n\}$ is a finite set. Indeed, if $\{n : N_n - 1 > b_1 \cdots b_n\}$ is an infinite set, then

$$\sum_{n=1}^{\infty} \frac{N_n}{b_1 \cdots b_n} \geq \sum_{\{n : N_n - 1 > b_1 \cdots b_n\}} \frac{N_n}{b_1 \cdots b_n} = \infty.$$

We get a contradiction. Then the claim follows. Hence

$$\sum_{\{n : N_n - 1 > b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d} < \infty. \tag{2-4}$$

Notice that

$$\begin{aligned} \sum_{\{n : N_n - 1 \leq b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n + d} &\leq \sum_{\{n : N_n - 1 \leq b_1 \cdots b_n\}} \frac{1}{N_n} \sum_{d=0}^{N_n - 1} \frac{d}{b_1 \cdots b_n} \\ &= \sum_{\{n : N_n - 1 \leq b_1 \cdots b_n\}} \frac{N_n - 1}{2 b_1 \cdots b_n} \\ &< \sum_{n=1}^{\infty} \frac{N_n}{b_1 \cdots b_n} < \infty. \end{aligned} \tag{2-5}$$

Then (2-2) follows from (2-3), (2-4) and (2-5).

Next we prove necessity. Suppose (2-2) holds. Then it follows from (2-3) that

$$\sum_{\{n:N_n-1>b_1\cdots b_n\}} \frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d}<\infty, \tag{2-6}$$
$$\sum_{\{n:N_n-1\le b_1\cdots b_n\}} \frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d}<\infty.$$

We assert that $\{n : N_n-1>b_1\cdots b_n\}$ is a finite set. Otherwise,

$$\begin{aligned}\infty &> 4\sum_{n=1}^{\infty}\frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d}\\ &\ge 4\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{1}{N_n}\frac{1}{b_1\cdots b_n+N_n-1}\sum_{d=0}^{N_n-1}d\\ &=4\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{2(N_n-1)}{b_1\cdots b_n+N_n-1}=\infty.\end{aligned}$$

This gives a contradiction, and the assertion follows. Consequently,

$$\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}<\infty. \tag{2-7}$$

Note that

$$\begin{aligned}\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{1}{N_n}\sum_{d=0}^{N_n-1}\frac{d}{b_1\cdots b_n+d} &\ge \sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{1}{N_n}\frac{1}{b_1\cdots b_n+N_n-1}\sum_{d=0}^{N_n-1}d\\ &=\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n-1}{2(b_1\cdots b_n+N_n-1)}\\ &\ge\frac{1}{2}\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n-1}{2b_1\cdots b_n}.\end{aligned}$$

It follows that

$$\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}<\infty. \tag{2-8}$$

This together with (2-7) yields that

$$\sum_{n=1}^{\infty}\frac{N_n}{b_1\cdots b_n}=\sum_{\{n:N_n-1>b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}+\sum_{\{n:N_n-1\le b_1\cdots b_n\}}\frac{N_n}{b_1\cdots b_n}<\infty,$$

completing the proof. □

Corollary 2.2. *Let* $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ *be a sequence of integers bigger than* 1, *and* $\boldsymbol{D} = \{\mathcal{D}_n\}_{n=1}^{\infty}$ *be a sequence of consecutive digit sets with* $\mathcal{D}_n = \{0, 1, \dots, N_n - 1\}$, *where* $N_n \geq 2$. *If* $N_n \leq b_n$ *for each* $n \geq 2$, *then* μ_n *converges weakly to a Borel probability measure* $\mu_{\boldsymbol{b},\boldsymbol{D}}$.

Proof. As $2 \leq N_n \leq b_n$ for each $n \geq 2$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{N_n}{b_1 \cdots b_n} &= \frac{N_1}{b_1} + \sum_{n=2}^{\infty} \frac{N_n}{b_1 \cdots b_n} \\ &\leq \frac{N_1}{b_1} + \sum_{n=2}^{\infty} \frac{1}{b_1 \cdots b_{n-1}} \leq \frac{N_1}{b_1} + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = \frac{N_1}{b_1} + 1 < \infty.\end{aligned}$$

Applying Theorem 1.1, the assertion follows. □

2.2. ***Spectral measure theoretic preliminaries.*** Let μ be a Borel probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined as usual:

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} \, \mathrm{d}\mu(x) \quad \text{for any } \xi \in \mathbb{R}.$$

We will denote by $\mathcal{Z}(\hat{\mu}) = \{\xi \in \mathbb{R} : \hat{\mu}(\xi) = 0\}$ the zero set of $\hat{\mu}$, and by e_λ the exponential function $e^{-2\pi i \lambda x}$. Then for a discrete set $\Lambda \subset \mathbb{R}$, $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set of $L^2(\mu)$ if and only if $\hat{\mu}(\lambda - \lambda') = 0$ for $\lambda \neq \lambda' \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}). \tag{2-9}$$

In this case, we say that Λ is a *bizero set* of μ. Moreover, Λ is called a *maximal bizero set* if it is maximal in $\mathcal{Z}(\hat{\mu})$ to have the set difference property. Since bizero sets (or spectra) are invariant under translation, without loss of generality, we always assume that $0 \in \Lambda$ in this paper. For $\xi \in \mathbb{R}$, write

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2.$$

The following criterion is a universal test to decide whether a countable set $\Lambda \subset \mathbb{R}$ is a bizero set (a spectrum) of μ or not.

Theorem 2.3 [Jorgensen and Pedersen 1998; Li et al. 2024]. *Let* μ *be a Borel probability measure*, *and let* $\Lambda \subset \mathbb{R}$ *be a countable set. Then*:

(i) Λ *is a bizero set of* μ *if and only if* $Q_\Lambda(\xi) \leq 1$ *for* $\xi \in \mathbb{R}$.

(ii) Λ *is a spectrum of* μ *if and only if* $Q_\Lambda(\xi) \equiv 1$ *for* $\xi \in \mathbb{R}$.

(iii) $Q_\Lambda(\xi)$ *has an entire analytic extension to* $\mathbb{C}$ *if* Λ *is a bizero set of* μ.

As a simple consequence of Theorem 2.3, the following useful theorem was proved in [Dai et al. 2014] and will be used to prove our main result.

Theorem 2.4. *Let $\mu = \nu * \omega$ be the convolution of two probability measures ν and ω that are not Dirac measures. Suppose that Λ is a bizero set of ν. Then Λ is also a bizero set of μ, but it cannot be a spectrum of μ.*

3. Proof of Theorem 1.3

In this section, we intend to complete the proof of Theorem 1.3. For simplicity, we write $\boldsymbol{b}_{m,n} = b_m \cdots b_n$ if $m \le n$. If $m = n$, we simply denote $\boldsymbol{b}_{m,n}$ by $\boldsymbol{b}_n$. Recall that $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a convolution, i.e., $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_n * \mu_{>n}$ for any integer $n \ge 1$, where

$$\mu_n := \mathop{\ast}_{k=1}^{n} \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k} \quad \text{and} \quad \mu_{>n} := \mathop{\ast}_{k=n+1}^{\infty} \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}. \tag{3-1}$$

So for any bizero set $0 \in \Lambda$ of μ, we must have

$$\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\} \subset \mathscr{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathscr{Z}(\hat{\mu}_n) \cup \mathscr{Z}(\hat{\mu}_{>n}).$$

We now introduce the definition of a *maximal decomposition of* Λ, which will be frequently used in the sequel.

Definition 3.1. Let $\mu = \nu * \omega$ be the convolution of two probability measures ν and ω. Suppose $0 \in \Lambda$ is a spectrum of μ and $0 \in \mathscr{A} \subset \Lambda$ is a maximal bizero set of ν. For any $\alpha \in \mathscr{A}$, let

$$\Lambda_\alpha := \big\{\lambda \in \Lambda : \lambda - \alpha \in \mathscr{Z}(\hat{\omega}) \setminus \mathscr{Z}(\hat{\nu})\big\} \cup \{\alpha\}.$$

We call the union

$$\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$$

a *maximal decomposition* with respect to (ν, ω).

Remark. In the definition, the maximality of $0 \in \mathscr{A} \subset \Lambda$ means that for any $\lambda \in \Lambda \setminus \mathscr{A}$, there is an $\alpha \in \mathscr{A}$ such that $\lambda - \alpha \in \mathscr{Z}(\hat{\omega}) \setminus \mathscr{Z}(\hat{\nu})$.

The maximal decomposition was first introduced by An and Wang [2021]. By the Jorgensen–Pedersen lemma [1998], Λ is a spectrum for a probability measure $\mu_{\boldsymbol{b},\boldsymbol{D}}$ if and only if

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi + \lambda)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}.$$

Substituting the maximal decomposition $\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$ with respect to $(\mu_n, \mu_{>n})$ into the above equation, we have

$$1 \le \sum_{a \in \mathscr{A}} \sum_{\lambda \in \Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi + \lambda)|^2 = \sum_{a \in \mathscr{A}} \sum_{\lambda \in \Lambda_a} |\hat{\mu}_n(\xi + \lambda)|^2 \, |\hat{\mu}_{>n}(\xi + \lambda)|^2.$$

In this paper, we are going to find a maximal decomposition $\Lambda = \bigcup_{\alpha \in \mathscr{A}} \Lambda_\alpha$ with respect to $(\mu_n, \mu_{>n})$ such that $\mathscr{A}$ is a spectrum of μ_n and Λ_a is a spectrum of $\mu_{>n}$ for each $a \in \mathscr{A}$.

With a direct calculation, we have $\mathcal{Z}(\hat{\delta}_{\mathcal{D}_k}) = (1/N_k)(\mathbb{Z} \setminus N_k\mathbb{Z})$. So

$$\mathcal{Z}(\hat{\mu}_n) = \bigcup_{k=1}^{n} \frac{\boldsymbol{b}_k}{N_k}(\mathbb{Z} \setminus N_k\mathbb{Z}), \quad \mathcal{Z}(\hat{\mu}_{>n}) = \bigcup_{k=n+1}^{\infty} \frac{\boldsymbol{b}_k}{N_k}(\mathbb{Z} \setminus N_k\mathbb{Z}), \tag{3-2}$$

and

$$\mathcal{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathcal{Z}(\hat{\mu}_n) \cup \mathcal{Z}(\hat{\mu}_{>n}) = \bigcup_{k=1}^{\infty} \frac{\boldsymbol{b}_k}{N_k}(\mathbb{Z} \setminus N_k\mathbb{Z}). \tag{3-3}$$

Now we make an important observation that will be used to guarantee a strong link between the spectral measures μ_n, $\mu_{>n}$ and $\mu_{\boldsymbol{b},\boldsymbol{D}}$.

Lemma 3.2. *Suppose $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure and $\{\lambda, \gamma\}$ is a bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. If $\lambda \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n})$ and $\gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathcal{D}_k}) \setminus \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n})$ with $k > n$, then we must have*

$$\lambda - \gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n}).$$

Proof. The bizero property of $\{\lambda, \gamma\}$ implies that

$$\lambda - \gamma \in \mathcal{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}).$$

Suppose to the contrary that

$$\lambda - \gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_j}\mathcal{D}_j}) \setminus \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathcal{D}_n})$$

for some $j \neq n$. By (3-2), we can write them as

$$\lambda = \frac{\boldsymbol{b}_n}{N_n}a_n, \quad \gamma = \frac{\boldsymbol{b}_k}{N_k}a_k \quad \text{and} \quad \lambda - \gamma = \frac{\boldsymbol{b}_j}{N_j}a_j,$$

where $a_i \in \mathbb{Z} \setminus N_i\mathbb{Z},\ i \in \{n, k, j\}$. Without loss of generality, we assume $j > n$. After some rearrangement, we have

$$\frac{a_n}{N_n} = \frac{\boldsymbol{b}_{n+1,k}}{N_k}a_k + \frac{\boldsymbol{b}_{n+1,j}}{N_j}a_j. \tag{3-4}$$

Reduce all fractions in the above equation to their simplest form, i.e.,

$$\frac{a_n'}{N_n'} = \frac{\boldsymbol{b}_{n+1,k}'}{N_k'}a_k + \frac{\boldsymbol{b}_{n+1,j}'}{N_j'}a_j = \frac{\boldsymbol{b}_{n+1,k}'a_kN_j' + \boldsymbol{b}_{n+1,j}'a_jN_k'}{N_k'N_j'},$$

where

$$\gcd(a_n', N_n') = 1, \quad \gcd(\boldsymbol{b}_{n+1,k}', N_k') = 1 \quad \text{and} \quad \gcd(\boldsymbol{b}_{n+1,j}', N_j') = 1. \tag{3-5}$$

This implies N_n' divides $N_k'N_j'$. Since $a_n \in \mathbb{Z} \setminus N_n\mathbb{Z}$, we have $N_n' > 1$. Let s_n be a prime factor of N_n', and of course it is also a prime factor of N_n. Then we must have

$$s_n \mid N_k' \quad \text{or} \quad s_n \mid N_j'.$$

Without loss of generality, we assume $N'_k = s_n t'_k$ for some integer t'_k. It follows from the second equation in (3-5) that

$$\gcd(s_n, \boldsymbol{b}'_{n+1,k}) = 1. \tag{3-6}$$

Write $t_k = \gcd(N_k, \boldsymbol{b}_{n+1,k})$. Then $N_k = N'_k t_k = s_n t_k t'_k$ and $\boldsymbol{b}_{n+1,k} = \boldsymbol{b}'_{n+1,k} t_k$. Let $\mathscr{E}_{t_k} = \{0, 1, \ldots, t_k - 1\}$, $\mathscr{E}_{t'_k} = \{0, 1, \ldots, t'_k - 1\}$ and $\mathscr{E}_{s_n} = \{0, 1, \ldots, s_n - 1\}$. We can factorize $\mathscr{D}_k$ as

$$\mathscr{D}_k = \mathscr{E}_{t_k} \oplus t_k \mathscr{E}_{s_n} \oplus s_n t_k \mathscr{E}_{t'_k}.$$

Write

$$\nu = \mathop{\ast}_{i \neq k} \delta_{\frac{1}{\boldsymbol{b}_i}\mathscr{D}_i} \ast \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{E}_{t_k}} \ast \delta_{\frac{1}{\boldsymbol{b}_k} s_n t_k \mathscr{E}_{t'_k}}.$$

Then $\mu_{\boldsymbol{b},\boldsymbol{D}} = \nu \ast \delta_{\frac{1}{\boldsymbol{b}_k} t_k \mathscr{E}_{s_n}}$. Note that

$$\begin{aligned}
\mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} t_k \mathscr{E}_{s_n}}) &= \frac{\boldsymbol{b}_n \boldsymbol{b}_{n+1,k}}{t_k s_n}(\mathbb{Z} \setminus s_n\mathbb{Z}) \\
&= \frac{\boldsymbol{b}_n \boldsymbol{b}'_{n+1,k}}{s_n}(\mathbb{Z} \setminus s_n\mathbb{Z}) \\
&\subset \frac{\boldsymbol{b}_n}{s_n}(\mathbb{Z} \setminus s_n\mathbb{Z}) && (\text{because } \gcd(\boldsymbol{b}'_{n+1,k}, s_n) = 1) \\
&\subset \frac{\boldsymbol{b}_n}{N_n}(\mathbb{Z} \setminus N_n\mathbb{Z}) && (\text{because } s_n \mid N_n) \\
&= \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathscr{D}_n}) \subset \mathfrak{Z}(\hat{\nu}).
\end{aligned}$$

So

$$\mathfrak{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathfrak{Z}(\hat{\nu}) \cup \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} t_k \mathscr{E}_{s_n}}) = \mathfrak{Z}(\hat{\nu}).$$

Let Λ be an arbitrary bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. The above equation implies that it also a bizero set of ν. It follows from Theorem 2.4 that Λ cannot be a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. Therefore, $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is not a spectral measure, a contradiction. □

Lemma 3.3. *Suppose $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure and $\{\lambda, \gamma\}$ is a bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$. Let $n \geq 1$ be an integer.*

(i) *If $\lambda \in \mathfrak{Z}(\hat{\mu}_n)$ and $\gamma \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)$, then*

$$\lambda - \gamma \in \mathfrak{Z}(\hat{\mu}_n).$$

(ii) *If $\lambda, \gamma \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)$, then*

$$\lambda - \gamma \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

Proof. (i) The assumption and (3-2) imply that

$$\lambda \in \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}) \quad \text{and} \quad \gamma \in \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_m}\mathscr{D}_m}) \setminus \mathfrak{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}) \qquad \text{for some } k \leq n < m.$$

We know from Lemma 3.2 that

$$\lambda - \gamma \in \mathcal{Z}(\hat{\delta}_{\frac{1}{b_k}\mathcal{D}_k}) \subset \mathcal{Z}(\hat{\mu}_n).$$

(ii) It should be noticed that $\mathcal{Z}(\hat{\nu}) = -\mathcal{Z}(\hat{\nu})$ for any measure ν. Suppose to the contrary that

$$\lambda' := \lambda - \gamma \in \mathcal{Z}(\hat{\mu}_n).$$

Since

$$\lambda = \lambda' - (-\gamma) \in \mathcal{Z}(\hat{\mu}_{>n}) \setminus \mathcal{Z}(\hat{\mu}_n) \subset \mathcal{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}),$$

$\{\lambda', -\gamma\}$ is a bizero set of $\mu_{\boldsymbol{b},\boldsymbol{D}}$ with $\lambda' \in \mathcal{Z}(\hat{\mu}_n)$ and $-\gamma \in \mathcal{Z}(\hat{\mu}_{>n}) \setminus \mathcal{Z}(\hat{\mu}_n)$. It follows from (i) that $\lambda = \lambda' - (-\gamma) \in \mathcal{Z}(\hat{\mu}_n)$, a contradiction. □

As a consequence of Lemma 3.3, we have:

Corollary 3.4. *Suppose that* $0 \in \Lambda$ *is a spectrum of* $\mu_{\boldsymbol{b},\boldsymbol{D}}$*. Then* $\Lambda \cap \mathcal{Z}(\hat{\mu}_n) \neq \varnothing$ *for any* $n \geq 1$.

Proof. The bizero property of Λ implies that

$$\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}) = \mathcal{Z}(\hat{\mu}_n) \cup \mathcal{Z}(\hat{\mu}_{>n}).$$

If the assertion is not true for some $n \geq 1$, then we have

$$\Lambda \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{>n}) \setminus \mathcal{Z}(\hat{\mu}_n).$$

From Lemma 3.3(ii), we have

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{>n}) \setminus \mathcal{Z}(\hat{\mu}_n),$$

that is, Λ is a bizero set of $\mu_{>n}$. It follows from Theorem 2.4 that Λ cannot be a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, which is a contradiction. □

Lemma 3.5. *Let* $0 \in \Lambda$ *be a spectrum of* $\mu_{\boldsymbol{b},\boldsymbol{D}}$*. Then* $\Lambda \subset (1/N_1)\mathbb{Z}$.

Proof. We write $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_1 * \mu_{>1}$ and decompose $\Lambda = \bigcup_{\alpha \in \mathcal{A}} \Lambda_\alpha$ into a maximal form with respect to $(\mu_1, \mu_{>1})$. That is, $0 \in \mathcal{A} \subset \Lambda$ is a maximal bizero set of μ_1, and

$$\Lambda_\alpha := \left\{\lambda \in \Lambda : \lambda - \alpha \in \mathcal{Z}(\hat{\mu}_{>1}) \setminus \mathcal{Z}(\hat{\mu}_1)\right\} \cup \{\alpha\} \quad \text{for all } \alpha \in \mathcal{A}.$$

It follows from Corollary 3.4 that $\mathcal{A} \setminus \{0\}$ is not empty. Recall that

$$\mathcal{Z}(\hat{\mu}_1) = \mathcal{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_1}\mathcal{D}_1}) = \frac{\boldsymbol{b}_1}{N_1}(\mathbb{Z} \setminus N_1\mathbb{Z}) \subset \frac{1}{N_1}\mathbb{Z}.$$

For any element $\lambda \in \Lambda$, if $\lambda \in \mathcal{A}$, then

$$\lambda \in \mathcal{A} \subset \mathcal{Z}(\hat{\mu}_1) \cup \{0\} \subset \frac{1}{N_1}\mathbb{Z}.$$

Otherwise, the maximality of $\mathscr{A}$ implies that there is a $\alpha \in \mathscr{A}$ such that

$$\lambda - \alpha \in \mathfrak{Z}(\hat{\mu}_{>1}) \setminus \mathfrak{Z}(\hat{\mu}_1).$$

Take an element $\alpha' \in \mathscr{A} \setminus \{\alpha\}$. Then $\alpha' - \alpha \in \mathfrak{Z}(\hat{\mu}_1)$. From Lemma 3.3(i), we have

$$\alpha' - \lambda = (\alpha' - \alpha) - (\lambda - \alpha) \in \mathfrak{Z}(\hat{\mu}_1).$$

Hence

$$\lambda = \alpha' - (\alpha' - \lambda) \in \mathscr{A} - \mathfrak{Z}(\hat{\mu}_1) \subset \frac{1}{N_1}\mathbb{Z}.$$

So $\Lambda \subset (1/N_1)\mathbb{Z}$. □

Theorem 1.3 shows if $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure, then any "truncation" of it is still a spectral measure. The proof is inspired by [An and Wang 2021, Proposition 4.3].

Proof of Theorem 1.3. For any $n \geq 1$, we write $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_n * \mu_{>n}$ and decompose Λ into a maximal form $\bigcup_{\alpha \in \mathscr{A}_0} \Lambda_\alpha$ with respect to $(\mu_n, \mu_{>n})$. That is, $0 \in \mathscr{A}_0 \subset \Lambda$ is a maximal bizero set of μ_n, and

$$\Lambda_\alpha := \big\{\lambda \in \Lambda : \lambda - \alpha \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)\big\} \cup \{\alpha\} \quad \text{for all } \alpha \in \mathscr{A}_0.$$

We first claim that the $\{\Lambda_\alpha\}_{\alpha \in \mathscr{A}_0}$ are disjoint pairwise. Otherwise, suppose there is an element λ such that

$$\lambda \in \Lambda_\alpha \cap \Lambda_{\alpha'}$$

for some $\alpha \neq \alpha' \in \mathscr{A}_0$. Since $\alpha - \alpha' \in \mathfrak{Z}(\hat{\mu}_n)$, we have $\alpha \in \Lambda_\alpha \setminus \Lambda_{\alpha'}$ and $\alpha' \in \Lambda_{\alpha'} \setminus \Lambda_\alpha$. So $\lambda \neq \alpha$ and $\lambda \neq \alpha'$. From the definition of Λ_α, we have

$$\lambda - \alpha, \lambda - \alpha' \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

Taking $\lambda_1 = \lambda - \alpha'$ and $\lambda_2 = \lambda - \alpha$ in Lemma 3.3, we have

$$\alpha - \alpha' = (\lambda - \alpha') - (\lambda - \alpha) \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

This contradicts the fact that $\mathscr{A}_0$ is a bizero set of μ_n. The claim holds.

For any $\alpha \in \mathscr{A}_0$ and $\lambda_\alpha \neq \lambda'_\alpha \in \Lambda_\alpha$, we have

$$\lambda_\alpha - \alpha,\ \lambda'_\alpha - \alpha \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n).$$

From Lemma 3.3, we have

$$\lambda_\alpha - \lambda'_\alpha = (\lambda_\alpha - \alpha) - (\lambda'_\alpha - \alpha) \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n). \tag{3-7}$$

For any $\alpha' \neq \alpha \in \mathscr{A}_0$, as $\lambda_\alpha \notin \Lambda_{\alpha'}$, we have

$$\lambda_\alpha - \alpha' \in \mathfrak{Z}(\hat{\mu}_n).$$

Because $\lambda_{\alpha'} - \alpha' \in \mathfrak{Z}(\hat{\mu}_{>n}) \setminus \mathfrak{Z}(\hat{\mu}_n)$ for any $\lambda_{\alpha'} \in \Lambda_{\alpha'}$, Lemma 3.2 implies that

$$\lambda_\alpha - \lambda_{\alpha'} = (\lambda_\alpha - \alpha') - (\lambda_{\alpha'} - \alpha') \in \mathfrak{Z}(\hat{\mu}_n). \tag{3-8}$$

Summarizing (3-7) and (3-8), we have

$$(\Lambda_\alpha-\Lambda_\alpha)\setminus\{0\}\subset\mathscr{Z}(\hat{\mu}_{>n})\setminus\mathscr{Z}(\hat{\mu}_n)\quad\text{and}\quad\Lambda_\alpha-\Lambda_{\alpha'}\subset\mathscr{Z}(\hat{\mu}_n). \tag{3-9}$$

For any $\xi\in[0,1]$, we have

$$Q_\Lambda(\xi)=\sum_{\lambda\in\Lambda}|\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi+\lambda)|^2=\sum_{\alpha\in\mathscr{A}_0}\sum_{\lambda_\alpha\in\Lambda_\alpha}|\hat{\mu}_n(\xi+\lambda_\alpha)|^2\,|\hat{\mu}_{>n}(\xi+\lambda_\alpha)|^2. \tag{3-10}$$

We define an equivalence relationship $\sim$ such that $\lambda\sim\lambda'$ whenever $\lambda'-\lambda\in b_1b_2\cdots b_n\mathbb{Z}$. Then the quotient group $\Lambda_\alpha/\sim=\{[\lambda]:\lambda\in\Lambda_\alpha\}$ is a partition of Λ_α, where

$$[\lambda]:=\{\lambda'\in\Lambda_\alpha:\lambda'\sim\lambda\}.$$

Since $\Lambda\subset(1/N_1)\mathbb{Z}$ (from Lemma 3.5), $\Lambda_\alpha/\sim$ is a finite set, which we will denote by $\{[\lambda_{\alpha,1}],\dots,[\lambda_{\alpha,n_\alpha}]\}$. Note that $\hat{\mu}_n$ is $(b_1b_2\cdots b_n)$-periodic. For any $\lambda\in[\lambda_{\alpha,i}]$,

$$|\hat{\mu}_n(\xi+\lambda)|=|\hat{\mu}_n(\xi+\lambda_{\alpha,i})|\quad\text{for all }\xi\in\mathbb{R}.$$

For any $\xi\in[0,1]$ and $\alpha\in\mathscr{A}_0$, there is a unique $\lambda_{\alpha,i(\xi)}$ with $i(\xi)\in\{1,2,\dots,n_\alpha\}$ such that

$$|\hat{\mu}_n(\xi+\lambda_{\alpha,i(\xi)})|=\max\{|\hat{\mu}_n(\xi+\lambda_{\alpha,i})|:1\le i\le n_\alpha\}=\max\{|\hat{\mu}_n(\xi+\lambda_\alpha)|:\lambda_\alpha\in\Lambda_\alpha\}.$$

As $\mathscr{A}_0$ and $\Lambda_\alpha/\sim$ are finite sets, we can find a finite set $\{\lambda_{\alpha,i_\alpha}\}_{\alpha\in\mathscr{A}_0}$ such that $\lambda_{\alpha,i(\xi_j)}=\lambda_{\alpha,i_\alpha}$ for infinitely many $\{\xi_j\}_{j=1}^\infty$. Combined with (3-10), we have

$$\begin{aligned}Q_\Lambda(\xi_j)&\le\sum_{\alpha\in\mathscr{A}_0}|\hat{\mu}_n(\xi_j+\lambda_{\alpha,i_\alpha})|^2\Big(\sum_{\lambda_\alpha\in\Lambda_\alpha}|\hat{\mu}_{>n}(\xi_j+\lambda_\alpha)|^2\Big)\\&\le\sum_{\alpha\in\mathscr{A}_0}|\hat{\mu}_n(\xi_j+\lambda_{\alpha,i_\alpha})|^2\\&\le 1,\end{aligned}$$

where the last two inequalities follow from (3-9) and Theorem 2.3(i). On the other hand, as Λ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$, we have $Q_\Lambda(\xi)\equiv 1$. This forces

$$\sum_{\lambda_\alpha\in\Lambda_\alpha}|\hat{\mu}_{>n}(\xi_j+\lambda_\alpha)|^2\equiv 1\quad\text{and}\quad\sum_{\alpha\in\mathscr{A}_0}|\hat{\mu}_n(\xi_j+\lambda_{\alpha,i_\alpha})|^2\equiv 1\qquad\text{for all }j\ge 1.$$

As all $\xi_j\in[0,1]$, the entire function property implies that

$$\sum_{\lambda_\alpha\in\Lambda_\alpha}|\hat{\mu}_{>n}(\xi+\lambda_\alpha)|^2\equiv 1\quad\text{and}\quad\sum_{\alpha\in\mathscr{A}_0}|\hat{\mu}_n(\xi+\lambda_{\alpha,i_\alpha})|^2\equiv 1\qquad\text{for all }\xi\in\mathbb{R}.$$

Hence $\mathscr{A}=\{\lambda_{\alpha,i_\alpha}\}_{\alpha\in\mathscr{A}_0}\subset\Lambda$ is a spectrum of μ_n and each Λ_α is a spectrum of $\mu_{>n}$. We now decompose $\Lambda=\bigcup_{\alpha\in\mathscr{A}}\Lambda_\alpha$ into a maximal form with respect to $(\mu_n,\mu_{>n})$. Then the desired result follows. □

4. Proofs of Theorems 1.2 and 1.4

Theorem 1.3 gives a necessary condition for $\mu_{\boldsymbol{b},\boldsymbol{D}}$ to be spectral. In this section, we will focus on analyzing the spectrality of μ_n, $n \geq 2$. From (3-2), for any $1 \leq k < n$, we have

$$\mu_n = \mu_k * \mu_{k+1,n},$$

where

$$\mu_k = \mathop{\ast}_{j=1}^{k} \delta_{\frac{1}{b_j}\mathcal{D}_j} \quad \text{and} \quad \mu_{k+1,n} = \mathop{\ast}_{j=k+1}^{n} \delta_{\frac{1}{b_j}\mathcal{D}_j}.$$

With the same proof, the conclusions in Lemma 3.3 and Theorem 1.3 are also true for μ_n.

Lemma 4.1. *Let n and k be two integers such that $n \geq 2$ and $1 \leq k < n$. Suppose μ_n is a spectral measure and $\{\lambda, \gamma\}$ is a bizero set of μ_n.*

(i) *If $\lambda \in \mathfrak{Z}(\hat{\mu}_k)$ and $\gamma \in \mathfrak{Z}(\hat{\mu}_{k+1,n}) \setminus \mathfrak{Z}(\hat{\mu}_k)$, then $\lambda - \gamma \in \mathfrak{Z}(\hat{\mu}_k)$.*

(ii) *If $\lambda, \gamma \in \mathfrak{Z}(\hat{\mu}_{k+1,n}) \setminus \mathfrak{Z}(\hat{\mu}_k)$, then $\lambda - \gamma \in \mathfrak{Z}(\hat{\mu}_{k+1,n}) \setminus \mathfrak{Z}(\hat{\mu}_k)$.*

Proof. The proof is the same as that of Lemma 3.3. □

Theorem 4.2. *If $0 \in \Lambda$ is a spectrum of μ_n, then, for each $1 \leq k < n$, we can perform a maximal decomposition on it, denoted by $\Lambda = \bigcup_{\alpha \in \mathcal{A}} \Lambda_\alpha$, such that $\mathcal{A}$ is a spectrum of μ_k and each Λ_α is a spectrum of $\mu_{k+1,n}$.*

Proof. The proof is the same as that of Theorem 1.3. □

Lemma 4.3. *Suppose that $\mathcal{E}_N = \{0, 1, \ldots, N-1\}$ is a consecutive digit set with cardinality N. Then $\delta_{\mathcal{E}_N}$ is a spectral measure. Moreover, $0 \in \mathcal{C}$ is a spectrum of $\delta_{\mathcal{E}_N}$ if and only if $\#\mathcal{C} = N$ and $\mathcal{C} \equiv \{0, 1/N, \ldots, (N-1)/N\} \pmod{\mathbb{Z}}$.*

Proof. It has been proved in [An and He 2014] that $\delta_{\mathcal{E}_N}$ is a spectral measure and admits a spectrum $\{0, 1/N, \ldots, (N-1)/N\}$. Since $\hat{\delta}_{\mathcal{E}_N}$ is 1-periodic, for any digit set $0 \in \mathcal{C}$ with $\#\mathcal{C} = N$ and $\mathcal{C} \equiv \{0, 1/N, \ldots, (N-1)/N\} \pmod{\mathbb{Z}}$, we have

$$\sum_{c \in \mathcal{C}} |\hat{\delta}_{\mathcal{E}_N}(\xi + c)|^2 = \sum_{j=0}^{N-1} \left| \hat{\delta}_{\mathcal{E}_N}\left(\xi + \frac{j}{N}\right) \right|^2.$$

It follows from Theorem 2.3(ii) that $\mathcal{C}$ is a spectrum of $\delta_{\mathcal{E}_N}$.

Next we will prove sufficiency. Suppose $0 \in \mathcal{C}$ is a spectrum of $\delta_{\mathcal{E}_N}$. The bizero property implies that

$$\mathcal{C} \setminus \{0\} \subset (\mathcal{C} - \mathcal{C}) \setminus \{0\} \subset \mathfrak{Z}(\hat{\delta}_{\mathcal{E}_N}) = \frac{1}{N}(\mathbb{Z} \setminus N\mathbb{Z}).$$

Therefore $\mathcal{C} \pmod{\mathbb{Z}} \subset \{0, 1/N, \ldots, (N-1)/N\}$. Completeness implies $\#\mathcal{C} = \dim L^2(\delta_{\mathcal{E}_N}) = N$, so

$$\mathcal{C} \equiv \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right\} \pmod{\mathbb{Z}}.$$

□

Proof of Theorem 1.4. The sufficiency can be found in [An and He 2014]. For the necessity, suppose on the contrary that there is $2 \le k \le n$ such that N_k does not divide b_k. Write $d = \gcd(b_k, N_k)$, and write $b_k = db'_k$, $N_k = dN'_k$. Then N'_k has a prime factor s_k which does not divide b'_k. Let $N'_k = s_k t_k$. Then $N_k = dt_k s_k$.

According to Theorem 4.2, the spectrality of $\mu_n = \mu_{k-2} * \mu_{k-1,n}$ implies that $\mu_{k-1,n}$ is also a spectral measure. If $n = k$, then $\mu_{k-1,n} = \mu_{k-1,n}$. If $n > k$, then $\mu_{k-1,n} = \mu_{k-1,k} * \mu_{k+1,n}$. From Theorem 4.2 again, $\mu_{k-1,k}$ is a spectral measure too. Suppose $0 \in \Lambda$ is a spectrum of $\mu_{k-1,k} = \delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}} * \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}$. We can decompose it into a maximal form

$$\Lambda = \bigcup_{a \in \mathscr{A}} \Lambda_a$$

with respect to $(\delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}}, \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k})$; that is, $0 \in \mathscr{A}$ is a maximal set of $\delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}}$, and each

$$\Lambda_a = \left\{\lambda \in \Lambda : \lambda - a \in \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}) \setminus \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}})\right\} \cup \{a\}$$

is a spectrum of $\delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{D}_k}$. Moreover, one can check that $\{\Lambda_a\}_{a\in\mathscr{A}}$ are disjoint pairwise. As $0 \in \Lambda_0$, it follows from Lemma 4.3 that

$$\Lambda_0 = \boldsymbol{b}_k\left\{0, \frac{1}{N_k} + m_1, \dots, \frac{N_k - 1}{N_k} + m_{N_k-1}\right\}.$$

Fix an element $a \in \mathscr{A} \setminus \{0\} \subset \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}})$. Then $a = (\boldsymbol{b}_{k-1}/N_{k-1})m$ for some $m \in \mathbb{Z}\setminus N_{k-1}\mathbb{Z}$. Recall $N_k = dt_k s_k$, where s_k is a prime integer and $1 \le t_k \le N_k - 1$. So

$$\boldsymbol{b}_k\left(\frac{t_k}{N_k} + m_{t_k}\right) \in \Lambda_0.$$

This, together with $\Lambda_0 \cap \Lambda_a = \varnothing$, implies that $\boldsymbol{b}_k(t_k/N_k + m_{t_k}) \notin \Lambda_a$ and so

$$\frac{\boldsymbol{b}_{k-1}}{N_{k-1}}m - \boldsymbol{b}_k\left(\frac{t_k}{N_k} + m_{t_k}\right) = \frac{\boldsymbol{b}_{k-1}}{N_{k-1}}m' \in \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}})$$

for some $m' \in \mathbb{Z} \setminus N_{k-1}\mathbb{Z}$. After some rearrangement, we have

$$\frac{m - m' - N_{k-1}b_k m_{t_k}}{N_{k-1}} = \frac{b_k t_k}{N_k} = \frac{b'_k}{s_k}.$$

This implies that s_k divides N_{k-1} since $\gcd(s_k, b'_k) = 1$. Let $\mathscr{E}_d = \{0, 1, \dots, d-1\}$, $\mathscr{E}_{s_k} = \{0, 1, \dots, s_k - 1\}$ and $\mathscr{E}_{t_k} = \{0, 1, \dots, t_k - 1\}$. We factorize $\mathscr{D}_k$ as

$$\mathscr{D}_k = \mathscr{E}_d \oplus d\mathscr{E}_{s_k} \oplus ds_k\mathscr{E}_{t_k}.$$

Write

$$\nu = \delta_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}} * \delta_{\frac{1}{\boldsymbol{b}_k}\mathscr{E}_d} * \delta_{\frac{1}{\boldsymbol{b}_k}s_k d\mathscr{E}_{t_k}}.$$

Then $\mu_{k-1,k} = \nu * \delta_{\frac{1}{\boldsymbol{b}_k} d\mathscr{E}_{s_k}}$. Note that

$$\begin{aligned}\mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} d\mathscr{E}_{s_k}}) &= \frac{\boldsymbol{b}_{k-1} b_k}{d s_k}(\mathbb{Z} \setminus s_k\mathbb{Z}) \\ &= \frac{\boldsymbol{b}_{k-1} b_k'}{s_k}(\mathbb{Z} \setminus s_k\mathbb{Z}) \\ &\subset \frac{\boldsymbol{b}_{k-1}}{s_k}(\mathbb{Z} \setminus s_k\mathbb{Z}) && (\text{because } \gcd(b_k', s_k) = 1) \\ &\subset \frac{\boldsymbol{b}_{k-1}}{N_{k-1}}(\mathbb{Z} \setminus N_{k-1}\mathbb{Z}) && (\text{because } s_k \mid N_{k-1}) \\ &= \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_{k-1}}\mathscr{D}_{k-1}}) \subset \mathscr{Z}(\hat{\nu}).\end{aligned}$$

So

$$\mathscr{Z}(\hat{\mu}_{k-1,k}) = \mathscr{Z}(\hat{\nu}) \cup \mathscr{Z}(\hat{\delta}_{\frac{1}{\boldsymbol{b}_k} d\mathscr{E}_{s_k}}) = \mathscr{Z}(\hat{\nu}).$$

This implies that Λ is also a bizero set of ν. It follows from Theorem 2.4 that Λ cannot be a spectrum of $\mu_{k-1,k}$, which is a contradiction. Hence N_k divides b_k for each $2 \le k \le n$. □

Before proving Theorem 1.2, we need some useful lemmas. The following lemma is well known. For the reader's convenience, we provide a proof here.

Lemma 4.4. *Λ is a spectrum of $\mu_{\boldsymbol{b},\boldsymbol{D}}$ if and only if $(1/a)\Lambda$ is a spectrum of $\mu_{\boldsymbol{b},a\boldsymbol{D}}$ for any $a \in \mathbb{R} \setminus \{0\}$.*

Proof. Note that

$$\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{\frac{1}{\boldsymbol{b}_n} a\mathscr{D}_n}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{\frac{1}{\boldsymbol{b}_n}\mathscr{D}_n}(a\xi) = \hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(a\xi).$$

Hence

$$\sum_{\gamma\in(1/a)\Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(\xi+\gamma)|^2 = \sum_{\gamma\in(1/a)\Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(a(\xi+\gamma))|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{\boldsymbol{b},\boldsymbol{D}}(a\xi+\lambda)|^2.$$

The assertion follows from Theorem 2.3. □

The following theorem has been proved by Gabardo and Lai [2014].

Theorem 4.5 [Gabardo and Lai 2014]. *Any positive Borel measures μ and ν such that $\mu * \nu = \mathscr{L}_{[0,1]}$ are spectral measures.*

Now we have all ingredients for the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) ⇒ (iii): This follows from Theorems 1.3 and 1.4.

(iii) ⇒ (ii): Suppose N_n divides b_n for each $n \ge 2$. Set $r_n = b_n/N_n$ for $n \ge 2$. So

$$\mathscr{D}_n \oplus N_n\{0, 1, \dots, r_n - 1\} = \{0, 1, \dots, b_n - 1\}$$

for $n \geq 2$. We write $C_1 = \{0\}$ and $C_n = N_n\{0, 1, \ldots, r_n - 1\}$ for $n \geq 2$ and let $\mu_{\boldsymbol{b},\boldsymbol{C}}$ be the Cantor–Moran measure generated by $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty}$ and $\boldsymbol{C} = \{C_n\}_{n=1}^{\infty}$. Then

$$\mu_{\boldsymbol{b},\boldsymbol{D}} * \mu_{\boldsymbol{b},\boldsymbol{C}} = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \mathscr{L}_{[0,1/b_1]} = \mathscr{L}_{[0,N_1/b_1]}.$$

This implies that the generalized Fuglede's conjecture holds for $\mu_{\boldsymbol{b},\boldsymbol{D}}$.

(ii) $\Rightarrow$ (i): Suppose there exists a Borel probability measure ν such that $\mu_{\boldsymbol{b},\boldsymbol{D}} * \nu = \mathscr{L}_{[0,N_1/b_1]}$. Let $f(x) = (N_1/b_1)x$ for $x \in \mathbb{R}$. Then $\mu_{\boldsymbol{b},(b_1/N_1)\boldsymbol{D}} * (\nu \circ f) = \mathscr{L}_{[0,1]}$. From Theorem 4.5, $\mu_{\boldsymbol{b},(b_1/N_1)\boldsymbol{D}}$ is a spectral measure and so is $\mu_{\boldsymbol{b},\boldsymbol{D}}$ according to Lemma 4.4. □

5. Proof of Theorem 1.5 and an open question

In this section, we first provide the proof of Theorem 1.5. Then, we conclude this paper with a conjecture on the relationship between the spectral Cantor–Moran measure and the tiling of integers.

We first introduce a theorem from Tijdeman [1995].

Theorem 5.1 [Tijdeman 1995]. *Suppose that A is finite, $0 \in A \cap B$ and $A \oplus B = \mathbb{Z}$. If r and $\#A$ are relatively prime, then $rA \oplus B = \mathbb{Z}$.*

Proof of Theorem 1.5. It would be easier to prove that $\boldsymbol{D}_n = \mathscr{D}_n \oplus b_n\mathscr{D}_{n-1} \oplus \cdots \oplus b_2 \cdots b_n\mathscr{D}_1$ is an integer tile if and only if N_n divides b_n for each $n \geq 2$. Suppose for any $n \geq 2$, $b_n = r_nN_n$ for some integer r_n. Then it is clear that $\boldsymbol{D}_n = \mathscr{D}_n + b_n\mathscr{D}_{n-1} + \cdots + b_2 \cdots b_n\mathscr{D}_1$ is a direct sum. Write $C_1 = \{0\}$, $C_n = N_n\{0, 1, \ldots, r_n - 1\}$ and

$$\boldsymbol{C}_n = C_n \oplus b_nC_{n-1} \oplus \cdots \oplus b_2 \cdots b_nC_1$$

for $n \geq 2$. Then $\mathscr{D}_n \oplus C_n = \{0, 1, \ldots, b_n - 1\}$ and so

$$\boldsymbol{D}_n \oplus \boldsymbol{C}_n = \{0, 1, \ldots, N_1b_2 \cdots b_n - 1\}$$

for each $n \geq 1$. This implies that $\boldsymbol{D}_n$ is an integer tile.

Suppose that the converse conclusion is false; that is, suppose that $\boldsymbol{D}_n = \mathscr{D}_n + b_n\mathscr{D}_{n-1} + \cdots + b_2 \cdots b_n\mathscr{D}_1$ is a direct sum and an integer tile, but there is an N_n that does not divide b_n. Then there must be a prime factor p_n of N_n which is coprime with b_n. Note that

$$\mathscr{D}_n = \{0, 1, \ldots, p_n - 1\} \oplus p_n\{0, 1, \ldots, N_n/p_n - 1\}. \tag{5-1}$$

The integer-tiling property of $\boldsymbol{D}_n$ implies that we can find $0 \in C \subset \mathbb{Z}$ such that

$$\boldsymbol{D}_n \oplus C = \{0, 1, \ldots, p_n - 1\} \oplus b_n\boldsymbol{D}_{n-1} \oplus p_n\{0, 1, \ldots, N_n/p_n - 1\} \oplus C = \mathbb{Z}.$$

Take $A = \{0, 1, \ldots, p_n - 1\}$ and $B = b_n \boldsymbol{D}_{n-1} \oplus p_n\{0, 1, \ldots, N_n/p_n - 1\} \oplus C$. As $\gcd(p_n, b_n) = 1$ and $\#A = p_n$, from Theorem 5.1 we have

$$b_n A \oplus b_n \boldsymbol{D}_{n-1} \oplus p_n\{0, 1, \ldots, N_n/p_n - 1\} \oplus C = \mathbb{Z}.$$

The above direct sum cannot happen as $\{0, 1\} \subset A \cap \boldsymbol{D}_{n-1}$. We have a contradiction and this shows our desired statement holds. □

Theorem 1.5 demonstrates a strong connection between the spectral Cantor–Moran measure and integer tiles. For the self-similar case $((b_n, \mathscr{D}_n) \equiv (b, \mathscr{D}))$, Łaba and Wang [2002] conjectured that if $\mu_{b,\mathscr{D}}$ is a spectral measure, then $\mathscr{D}$ is an integer tile. However, being an integer tile is not a sufficient condition. Here is a simple example observed by An, He and Lau [An et al. 2015]: Let $b = 8$ and $\mathscr{D} = \{0, 1, 8, 9\}$. Then $\mathscr{D}$ tiles $\mathbb{Z}_{16}$, but the self-similar measure $\mu_{8,\mathscr{D}}$ is not spectral. This is mainly because $\mathscr{D} + 8\mathscr{D}$ is not a direct sum.

Recall that $\mu_{\boldsymbol{b},\boldsymbol{D}} = \mu_n * \mu_{>n}$ and

$$\mu_n = \delta_{\frac{1}{b_1}\mathscr{D}_1} * \delta_{\frac{1}{b_1 b_2}\mathscr{D}_2} * \cdots * \delta_{\frac{1}{b_1 b_2 \cdots b_n}\mathscr{D}_n} = \delta_{\frac{1}{b_1 b_2 \cdots b_n}\boldsymbol{D}_n},$$

where $\boldsymbol{D}_n = \mathscr{D}_n + b_n \mathscr{D}_{n-1} + b_2 \cdots b_n \mathscr{D}_1$. According to Fuglede's conjecture on cyclic groups, one may ask:

Question. If $\mu_{\boldsymbol{b},\boldsymbol{D}}$ is a spectral measure, is each of the iterated digit sets $\boldsymbol{D}_n = \mathscr{D}_n + b_n \mathscr{D}_{n-1} + b_2 \cdots b_n \mathscr{D}_1$ (for $n \geq 1$) an integer tile?

Our Theorem 1.5 settles the case where $\mathscr{D}_n = \{0, 1, \ldots, N_n - 1\}$. And we eagerly anticipate further positive outcomes regarding this inquiry.

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References

[An and He 2014] L. X. An and X. G. He, "A class of spectral Moran measures", *J. Funct. Anal.* **266**:1 (2014), 343–354. MR Zbl

[An and Wang 2021] L. X. An and C. Wang, "On self-similar spectral measures", *J. Funct. Anal.* **280**:3 (2021), art. id. 108821. MR Zbl

[An et al. 2015] L. X. An, X. G. He, and K. S. Lau, "Spectrality of a class of infinite convolutions", *Adv. Math.* **283** (2015), 362–376. MR Zbl

[An et al. 2019] L. X. An, X. Y. Fu, and C. K. Lai, "On spectral Cantor–Moran measures and a variant of Bourgain's sum of sine problem", *Adv. Math.* **349** (2019), 84–124. MR Zbl

[Coven and Meyerowitz 1999] E. M. Coven and A. Meyerowitz, "Tiling the integers with translates of one finite set", *J. Algebra* **212**:1 (1999), 161–174. MR Zbl

[Dai 2012] X. R. Dai, "When does a Bernoulli convolution admit a spectrum?", *Adv. Math.* **231**:3-4 (2012), 1681–1693. MR Zbl

[Dai 2016] X. R. Dai, "Spectra of Cantor measures", *Math. Ann.* **366**:3-4 (2016), 1621–1647. MR Zbl

[Dai and Sun 2015] X. R. Dai and Q. Y. Sun, "Spectral measures with arbitrary Hausdorff dimensions", *J. Funct. Anal.* **268**:8 (2015), 2464–2477. MR Zbl

[Dai et al. 2013] X. R. Dai, X. G. He, and C. K. Lai, "Spectral property of Cantor measures with consecutive digits", *Adv. Math.* **242** (2013), 187–208. MR Zbl

[Dai et al. 2014] X. R. Dai, X. G. He, and K. S. Lau, "On spectral N-Bernoulli measures", *Adv. Math.* **259** (2014), 511–531. MR Zbl

[Deng and Li 2022] Q. R. Deng and M. T. Li, "Spectrality of Moran-type self-similar measures on $\mathbb{R}$", *J. Math. Anal. Appl.* **506**:1 (2022), art. id. 125547. MR Zbl

[Dutkay and Haussermann 2016] D. E. Dutkay and J. Haussermann, "Number theory problems from the harmonic analysis of a fractal", *J. Number Theory* **159** (2016), 7–26. MR Zbl

[Dutkay et al. 2009] D. E. Dutkay, D. G. Han, and Q. Y. Sun, "On the spectra of a Cantor measure", *Adv. Math.* **221**:1 (2009), 251–276. MR Zbl

[Dutkay et al. 2019] D. E. Dutkay, J. Haussermann, and C. K. Lai, "Hadamard triples generate self-affine spectral measures", *Trans. Amer. Math. Soc.* **371**:2 (2019), 1439–1481. MR Zbl

[Fu et al. 2015] X. Y. Fu, X. G. He, and K. S. Lau, "Spectrality of self-similar tiles", *Constr. Approx.* **42**:3 (2015), 519–541. MR Zbl

[Fuglede 1974] B. Fuglede, "Commuting self-adjoint partial differential operators and a group theoretic problem", *J. Functional Analysis* **16** (1974), 101–121. MR Zbl

[Gabardo and Lai 2014] J.-P. Gabardo and C. K. Lai, "Spectral measures associated with the factorization of the Lebesgue measure on a set via convolution", *J. Fourier Anal. Appl.* **20**:3 (2014), 453–475. MR Zbl

[Hu and Lau 2008] T. Y. Hu and K. S. Lau, "Spectral property of the Bernoulli convolutions", *Adv. Math.* **219**:2 (2008), 554–567. MR Zbl

[Jorgensen and Pedersen 1998] P. E. T. Jorgensen and S. Pedersen, "Dense analytic subspaces in fractal L^2-spaces", *J. Anal. Math.* **75** (1998), 185–228. MR Zbl

[Kolountzakis and Matolcsi 2006a] M. N. Kolountzakis and M. Matolcsi, "Complex Hadamard matrices and the spectral set conjecture", *Collect. Math.* Extra Volume (2006), 281–291. MR Zbl

[Kolountzakis and Matolcsi 2006b] M. N. Kolountzakis and M. Matolcsi, "Tiles with no spectra", *Forum Math.* **18**:3 (2006), 519–528. MR Zbl

[Łaba 2001] I. Łaba, "Fuglede's conjecture for a union of two intervals", *Proc. Amer. Math. Soc.* **129**:10 (2001), 2965–2972. MR Zbl

[Łaba and Londner 2023] I. Łaba and I. Londner, "The Coven–Meyerowitz tiling conditions for 3 odd prime factors", *Invent. Math.* **232**:1 (2023), 365–470. MR Zbl

[Łaba and Wang 2002] I. Łaba and Y. Wang, "On spectral Cantor measures", *J. Funct. Anal.* **193**:2 (2002), 409–420. MR Zbl

[Lai and Wang 2017] C. K. Lai and Y. Wang, "Non-spectral fractal measures with Fourier frames", *J. Fractal Geom.* **4**:3 (2017), 305–327. MR Zbl

[Lev and Matolcsi 2022] N. Lev and M. Matolcsi, "The Fuglede conjecture for convex domains is true in all dimensions", *Acta Math.* **228**:2 (2022), 385–420. MR Zbl

[Li et al. 2022] W. X. Li, J. J. Miao, and Z. Q. Wang, "Weak convergence and spectrality of infinite convolutions", *Adv. Math.* **404** (2022), art. id. 108425. MR Zbl

[Li et al. 2024] W. X. Li, J. J. Miao, and Z. Q. Wang, "Spectrality of random convolutions generated by finitely many Hadamard triples", *Nonlinearity* **37**:1 (2024), art. id. 015003. MR Zbl

[Moran 1946] P. A. P. Moran, "Additive functions of intervals and Hausdorff measure", *Proc. Cambridge Philos. Soc.* **42** (1946), 15–23. MR Zbl

[Newman 1977] D. J. Newman, "Tesselation of integers", *J. Number Theory* **9**:1 (1977), 107–111. MR Zbl

[Sands 1979] A. D. Sands, "On Keller's conjecture for certain cyclic groups", *Proc. Edinburgh Math. Soc.* (2) **22**:1 (1979), 17–21. MR Zbl

[Strichartz 2000] R. S. Strichartz, "Mock Fourier series and transforms associated with certain Cantor measures", *J. Anal. Math.* **81** (2000), 209–238. MR Zbl

[Strichartz 2006] R. S. Strichartz, "Convergence of mock Fourier series", *J. Anal. Math.* **99** (2006), 333–353. MR Zbl

[Tao 2004] T. Tao, "Fuglede's conjecture is false in 5 and higher dimensions", *Math. Res. Lett.* **11**:2-3 (2004), 251–258. MR Zbl

[Tijdeman 1995] R. Tijdeman, "Decomposition of the integers as a direct sum of two subsets", pp. 261–276 in *Number theory* (Paris, 1992–1993), edited by S. David, London Math. Soc. Lecture Note Ser. **215**, Cambridge Univ. Press, 1995. MR Zbl

[Wintner 1935] A. Wintner, "On symmetric Bernoulli convolutions", *Bull. Amer. Math. Soc.* **41**:2 (1935), 137–138. MR Zbl

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