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MINIMAL FREE RESOLUTIONS OF NUMERICAL SEMIGROUP ALGEBRAS VIA APÉRY SPECIALIZATION

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Numerical semigroups with multiplicity m are parametrized by integer points in a polyhedral cone C_m, according to Kunz. For the toric ideal of any such semigroup, the main result here constructs a free resolution whose overall structure is identical for all semigroups parametrized by the relative interior of a fixed face of C_m. The matrix entries of this resolution are monomials whose exponents are parametrized by the coordinates of the corresponding point in C_m, and minimality of the resolution is achieved when the semigroup is of maximal embedding dimension, which is the case when it is parametrized by the interior of C_m itself.

1. Introduction

Given a numerical semigroup S, the corresponding semigroup algebra has a defining toric ideal I_S. While the study of algebraic invariants of this numerical semigroup ideal I_S falls within the broader study of toric ideals, the family of numerical semigroup ideals forms a rich and interesting area of study that often affords more refined general results than those known or possible for the general toric setting. Our aim is to uniformly construct explicit free resolutions that are minimal for numerical semigroups with maximal embedding dimension, are parametrized by Apéry data, and therefore specialize to minimal free resolutions for numerical semigroups with arbitrary embedding dimension.

For general toric ideals, there is a substantial literature on their resolutions. In 1998, Peeva and Sturmfels described minimal free resolutions for generic lattice ideals [23]. More recently, Tchernev gave an explicit recursive algorithm for canonical minimal resolutions of toric rings [28]. Further, Li, Miller, and Ordog construct a canonical minimal free resolution of an arbitrary positively graded lattice ideal with a closed-form combinatorial description of the differential in

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characteristic 0 and all but finitely many positive characteristics [21]. However, these constructions are all quite general, and one would hope that in the special case of numerical semigroups, explicit resolutions of I_S more directly tied to the combinatorics of S are possible.

Free resolutions of I_S for a numerical semigroup S are known in some special cases. The surveys [12; 27] include most results concerning special families. We outline a few here. If S is maximal embedding dimension (MED) and I_S is determinantal (that is, generated by the minors of a matrix), then I_S is resolved by the Eagon–Northcott complex [15]. This accounts for some, but not all, MED numerical semigroups; a characterization of determinantal MED numerical semigroups is given in [19]. If S is generated by an arithmetic sequence, then I_S is minimally resolved by a variant of the Eagon–Northcott complex [13]. If S is obtained as a gluing of two numerical semigroups T and T', then a minimal free resolution of I_S can be obtained from the minimal free resolutions of I_T and $I_{T'}$ via a mapping cone construction [11]; this includes the case where I_S is complete intersection. If S has at most 3 generators, then a minimal free resolution is known. Numerous families of 4-generated numerical semigroups have also been investigated; see the survey [27] for more detail.

The present work is motivated by recent papers that examine a family of convex rational polyhedra C_m called *Kunz cones*, one for each integer $m \geq 2$, for which each numerical semigroup S with multiplicity m — that is, with $m = \min(S \setminus \{0\})$ — corresponds to an integer point of C_m. These were first introduced in [20], and most subsequent papers on the topic have employed lattice point techniques to examine enumerative questions [1; 17; 26]. However, seemingly overlooked for decades was another result in [20] that proved two numerical semigroups S and T correspond to points (relative) interior to the same face of C_m if and only if certain artinian quotients of I_S and I_T coincide. Indeed, a corollary of this result, namely that MED numerical semigroups are precisely those lying in the interior of C_m, was the only reference to the faces of C_m in the literature until a series of recent papers [4; 18] unknowingly rederived a combinatorial version of Kunz's result: S and T lie in the same face of C_m if and only if certain subsets of their divisibility posets coincide.

Kunz also observed in [20] that, as a consequence of his result, $\beta_d(I_S) = \beta_d(I_T)$ for every d, though his approach did not allow for explicit construction of syzygies for I_S and I_T. The combinatorial viewpoint of [4; 18] was recently used in [14] to make Kunz's enumerative result algebraic in the case $d = 1$: if S and T lie interior to the same face of C_m, then minimal binomial generating sets are explicitly constructed for I_S and I_T that coincide in all terms except the exponents of a single variable.

Our main result is to similarly make Kunz's result algebraic for all positive d: we construct explicit free resolutions of all numerical semigroup ideals. These resolutions are minimal when the semigroup has maximal embedding dimension

$$0 \leftarrow R \xleftarrow{\varnothing\begin{array}{cccccc} \mathbf{1,1} & \mathbf{2,2} & \mathbf{3,3} & \mathbf{2,1} & \mathbf{3,1} & \mathbf{3,2} \\ {}[x_1^2 - x_2 y^{b_{11}} & x_2^2 - y^{b_{22}} & x_3^2 - x_2 y^{b_{33}} & x_1x_2 - x_3 y^{b_{12}} & x_1x_3 - y^{b_{13}} & x_2x_3 - x_1 y^{b_{23}}] \end{array}}$$

$$R^6 \xleftarrow{\begin{array}{c|cccccccc} & \mathbf{1,12} & \mathbf{1,13} & \mathbf{2,12} & \mathbf{2,23} & \mathbf{3,13} & \mathbf{3,23} & \mathbf{2,13} & \mathbf{3,12} \\ \hline \mathbf{1,1} & -x_2 & -x_3 & & & & & y^{b_{23}} & y^{b_{23}} \\ \mathbf{2,2} & -y^{b_{11}} & & x_1 & -x_3 & & y^{b_{33}} & & \\ \mathbf{3,3} & & & & & x_1 & x_2 & -y^{b_{12}} & \\ \mathbf{2,1} & x_1 & & -x_2 & y^{b_{23}} & y^{b_{33}} & & -x_3 & \\ \mathbf{3,1} & y^{b_{12}} & x_1 & & & -x_3 & -y^{b_{23}} & & -x_2 \\ \mathbf{3,2} & & -y^{b_{11}} & -y^{b_{12}} & x_2 & & -x_3 & x_1 & x_1 \end{array}} R^8 \xleftarrow{\begin{array}{c|ccc} & \mathbf{1,[3]} & \mathbf{2,[3]} & \mathbf{3,[3]} \\ \hline \mathbf{1,12} & x_3 & -y^{b_{23}} & \\ \mathbf{1,13} & -x_2 & & y^{b_{23}} \\ \mathbf{2,12} & & x_3 & -y^{b_{33}} \\ \mathbf{2,23} & -y^{b_{11}} & x_1 & \\ \mathbf{3,13} & y^{b_{12}} & & -x_2 \\ \mathbf{3,23} & & -y^{b_{12}} & x_1 \\ \mathbf{2,13} & x_1 & -x_2 & \\ \mathbf{3,12} & -x_1 & & x_3 \end{array}} R^3 \leftarrow 0$$

Figure 1. The Apéry resolution for $m = 4$. The exponents $b_{i,j}$ are constants depending on the particular numerical semigroup S.

(equivalently, if S lies in the interior of C_m) and are minimalized uniformly for all numerical semigroups lying interior to the same face of the Kunz cone C_m. More precisely, our contributions are as follows:

(1) We introduce the Apéry toric ideal J_S of S, an analogue of I_S that lies in a ring with m variables instead of a ring with one variable per minimal generator of S. A generating set for J_S can be obtained by concatenating any generating set for I_S and a regular sequence with one element for each additional variable.

(2) For any positive integer $m \geq 2$, we construct a free resolution of J_S called the *Apéry resolution*. The rank of the d-th free module depends only on m and d, and the positions of the nonzero entries of the matrices representing the boundary maps depend only on m. An example of this for $m = 4$ is given in Figure 1, where the values $b_{i,j}$ depend on S.

(3) When S corresponds to a point interior to C_m, i.e., when S is MED and thus $J_S = I_S$, the Apéry resolution is a minimal free resolution of I_S.

(4) For any numerical semigroups S and T corresponding to points interior to the same face F of C_m, we prove there exists a uniform method for modifying the Apéry resolutions of J_S and J_T to minimal free resolutions in such a way that the resulting ranks of the free modules and the positions of the nonzero entries of the matrices representing the boundary maps depend only on m, F, and d.

The term "specialization" in the title and Section 4 refers to passage from the interior of the Kunz cone to a face, which entails some facet inequalities becoming equalities. Consequently, some exponents on y variables (as in Figure 1) pass from positive to 0, which results in the specialization that sets $y = 1$. Further substitutions among the x variables — extraneous ones are set equal to monomials in the others — combine in Step (4) with row and column operations to produce minimal free resolutions from the original Apéry resolution.

The remainder of this paper is structured as follows. Section 2 reviews basic properties of numerical semigroups and Kunz cones and defines the modules and maps used in the Apéry resolution. Section 3 proves that the Apéry resolution is indeed a resolution and establishes the minimality of this resolution when S is MED. Section 4 describes how to modify the Apéry resolution in a uniform way for all numerical semigroups in the interior of a fixed face of C_m to obtain a minimal resolution. Further research directions are outlined in Section 5.

2. Kunz polyhedra and Apéry resolutions

2.1. ***Semigroups and toric ideals.*** A *numerical semigroup* is a subsemigroup of $(\mathbb{Z}_{\geq 0}, +)$ that contains 0 and has finite complement. Throughout this work, fix a numerical semigroup $S \subset \mathbb{Z}_{\geq 0}$ with *multiplicity*

$$\mathsf{m}(S) = \min(S \setminus \{0\}) = m$$

and write

$$\begin{aligned} \operatorname{Ap}(S) &= \{n \in S : n - m \notin S\} \\ &= \{0, a_1, \dots, a_{m-1}\} \end{aligned}$$

for the *Apéry set* consisting of the minimal element of S from each equivalence class modulo m, where each a_i satisfies $a_i \equiv i \mod m$. For convenience, define $a_0 = m$; this convention plays an important role in our later formulas. In particular,

$$S = \langle m, a_1, \dots, a_{m-1} \rangle = \langle a_0, a_1, \dots, a_{m-1} \rangle,$$

though this generating set need not be the unique minimal generating set $\mathcal{A}(S)$ of S, such as when $a_i + a_j = a_{i+j}$ for some i, j, where indices are summed modulo m. The semigroup S has *maximal embedding dimension* (MED) if $\mathcal{A}(S) = \{a_0, \dots, a_{m-1}\}$.

Example 2.1. The semigroup $S = \langle 4, 9, 11, 14 \rangle$ has multiplicity $\mathsf{m}(S) = 4$ and Apéry set $\operatorname{Ap}(S) = \{0, 9, 14, 11\}$. The semigroup $T = \langle 4, 13, 23 \rangle$ has multiplicity $\mathsf{m}(T) = 4$ and $\operatorname{Ap}(T) = \{0, 13, 26, 23\}$. Note that $a_1 + a_1 = a_2$ in T, and thus the Apéry set is not a minimal generating set.

Let $R = \Bbbk[x_0, x_1, \dots, x_{m-1}]$ with the natural grading by $\mathbb{Z}$ via $\deg(x_i) = a_i$ and set $y = x_0$. The *Apéry toric ideal* of S is the kernel $J_S = \ker(\varphi)$ of the homomorphism

$$\begin{aligned} \varphi : R &\to \Bbbk[t], \\ x_i &\mapsto t^{a_i}, \end{aligned}$$

and the *defining toric ideal* of S is

$$I_S = J_S \cap \Bbbk[x_i : a_i \in \mathcal{A}(S)].$$

For every $1 \leq i, j \leq m-1$ define

$$c_{i,j} = \frac{1}{m}(a_i + a_j - a_{i+j}) \geq 0 \tag{2-1}$$

and

$$b_{i,j} = \begin{cases} c_{i,j} & \text{if } i+j \neq m, \\ c_{i,j}+1 & \text{if } i+j = m. \end{cases}$$

In particular, $c_{i,j} = 0$ if and only if $a_i + a_j = a_{i+j}$; this is impossible if $i + j = m$, since m is the multiplicity, so $b_{i,j} = 0$ if and only if $c_{i,j} = 0$. It is known that

$$J_S = \langle x_i x_j - y^{c_{i,j}} x_{i+j} : 1 \leq i \leq j \leq m-1 \rangle, \tag{2-2}$$

(see, e.g., [25, Section 8.4]), though it also follows from Lemma 3.2 here.

Example 2.2. The semigroup $S = \langle 4, 9, 11, 14\rangle$ has $(a_1, a_2, a_3) = (9, 14, 11)$ and

$$J_S = I_S = \langle x_1^2 - yx_2, x_1x_2 - y^3x_3, x_1x_3 - y^4y, x_2^2 - y^6y, x_2x_3 - x_1y^4, x_3^2 - x_2y^2\rangle.$$

The terms here are written in a way that emphasizes the convention $a_0 = m$ and $x_0 = y$, such as to produce the binomial $x_1x_3 - y^4x_0 = x_1x_3 - y^4y = x_1x_3 - y^5$.

The semigroup $T = \langle 4, 13, 23\rangle$ has $(a_1, a_2, a_3) = (13, 26, 23)$ and Apéry ideal

$$\begin{aligned} J_T &= \langle x_1^2 - x_2, x_1x_2 - y^4x_3, x_1x_3 - y^9, x_2^2 - y^{13}, x_2x_3 - x_1y^9, x_3^2 - x_2y^5\rangle \\ &= \langle x_1^2 - x_2, x_1^3 - y^4x_3, x_1x_3 - y^9, x_3^2 - x_1^2y^5\rangle \end{aligned}$$

and defining toric ideal

$$I_T = \langle x_1^3 - y^4x_3, x_1x_3 - y^9, x_3^2 - x_1^2y^5\rangle = J_T \cap \Bbbk[y, x_1, x_3].$$

2.2. ***Kunz cone.*** We describe the Kunz cone and its relationship to the values $b_{i,j}$. Letting $\mathrm{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$ with $a_i \equiv i \bmod m$ for each i as in Section 2.1, the *Apéry coordinate vector* of S with respect to m is the tuple $(a_1, \ldots, a_{m-1})$. The following set of linear inequalities exactly characterizes the set of Apéry coordinate vectors for numerical semigroups of multiplicity m [18; 20].

Definition 2.3. For each $m \geq 2$, the *Kunz cone* $C_m \subseteq \mathbb{R}^{m-1}_{\geq 0}$ has facet inequalities

$$z_i + z_j \geq z_{i+j} \quad \text{for } 1 \leq i \leq j \leq m-1 \text{ with } i+j \neq m,$$

where addition of subscripts is modulo m.

Lemma 2.4. *If S is a numerical semigroup of multiplicity m, then $b_{i,j} = 0$ if and only if $a_i + a_j = a_{i+j}$. Hence the Apéry coordinate vector of S lies on the boundary of C_m if and only if $b_{i,j} = 0$ for some i, j.*

Proof. This follows from the definitions, using (2-1) for the claim about $b_{i,j}$. □

The lemma has the following consequence [20].

Proposition 2.5. *A vector $z = (z_1, \ldots, z_{m-1}) \in \mathbb{Z}^{m-1}_{\geq 1}$ with $z_i \equiv i \bmod m$ for all i lies in C_m if and only if z is the Apéry coordinate vector of a numerical semigroup S. Moreover, z is in the interior of C_m if and only if S has maximal embedding dimension.*

Example 2.6. The cone $C_4 \subseteq \mathbb{R}^3_{\geq 0}$ is defined by the inequalities

$$z_2 + z_3 \geq z_1, \quad z_1 + z_2 \geq z_3, \quad 2z_1 \geq z_2, \quad \text{and} \quad 2z_3 \geq z_2,$$

and has extremal rays generated by $(1, 0, 1)$, $(1, 2, 3)$, $(1, 2, 1)$, and $(3, 2, 1)$. All positive-dimensional faces of C_4 contain numerical semigroups (in the sense of Proposition 2.5) except the rays through $(1, 0, 1)$ and $(1, 2, 1)$. Numerical semigroups on the rays through $(1, 2, 3)$ and $(3, 2, 1)$ have embedding dimension 2, and numerical semigroups in the relative interior of the facets $z_1 + z_2 = z_3$ and $z_2 + z_3 = z_1$ are complete intersections [2]. In particular, a minimal free resolution for the defining toric ideal of any semigroup in these 4 faces is known.

The numerical semigroup S from Example 2.2 corresponds to the point $(9, 14, 11)$ in the relative interior of C_4, while T corresponds to the point $(13, 26, 23)$ in the relative interior of the facet $2z_1 = z_2$. A minimal free resolution for J_S is obtained by substituting the appropriate values for $b_{i,j}$ in the free resolution in Figure 1, while a minimal free resolution for J_T is obtained via analogous substitution into Figure 4 (at the beginning of Section 4). This leaves the facet $2z_3 = z_2$, and, courtesy of the action of $\mathbb{Z}_4^*$ on C_4, free resolutions for semigroups in this face can be obtained from the ones exhibited in Figure 4 by interchanging 1s and 3s in every subscript.

We record here the result of Kunz that seems to be overlooked in the literature concerning when numerical semigroups reside in the interior of a given face of C_m.

Theorem 2.7 [20, Propositions 2.3 and 2.6]. *Two numerical semigroups S and T with multiplicity m lie in the interior of the same face of C_m if and only if*

$$R/(J_S + \langle y \rangle) \cong R/(J_T + \langle y \rangle).$$

Moreover, in this case, $\beta_d(I_S) = \beta_d(I_T)$ for every d.

2.3. ***Modules and maps for the Apéry resolution.*** For any numerical semigroup S of multiplicity m, this subsection defines the free modules and linear maps between them that form the *Apéry resolution*

$$\mathcal{F}_\bullet : 0 \leftarrow R \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

of J_S. Theorem 3.4 shows that it is a resolution. Of particular note is that the ranks of its modules and the locations of the nonzero coefficients in the matrices representing its linear maps depend only on m, not on the actual values of $\mathrm{Ap}(S)$.

Theorem 3.4 and Corollary 3.5 show that this resolution is minimal if and only if S has maximal embedding dimension, i.e., corresponds to a point interior to C_m. Theorem 4.4 shows that when S lies on the boundary of C_m, a minimal resolution of J_S can be obtained from the Apéry resolution in a manner that is uniform for all semigroups in the interior of a fixed face of C_m, parametrized by the $b_{i,j}$.

2.3.1. *Modules.* For $d = 1, \dots, m-1$, define F_d to be the free module over R with formal basis elements

$$\{e_{i,A} : i \in [m-1],\ A \subset [m-1],\ |A| = d,\ i \geq \min(A)\},$$

where $\deg(e_{i,A}) = a_i + \sum_{j \in A} a_j$ and $[m-1] = \{1, 2, \dots, m-1\}$. Since every pair (i, A) with $|A| = d$ and $i < \min(A)$ corresponds to a $(d{+}1)$-element subset $\{i\} \cup A$ of $[m-1]$, it is immediate that

$$\operatorname{rank} F_d = (m-1)\binom{m-1}{d} - \binom{m-1}{d+1} = d\binom{m}{d+1}.$$

Example 2.8. For $m = 3$, $F_0 = Re_\varnothing$, where $Re_\varnothing = \{re_\varnothing : r \in R\}$. Similarly,

$$\begin{aligned} F_1 &= Re_{1,\{1\}} + Re_{2,\{2\}} + Re_{2,\{1\}} \\ &= \{\alpha e_{1,\{1\}} + \beta e_{2,\{2\}} + \gamma e_{2,\{1\}} : \alpha, \beta, \gamma \in R\} \end{aligned}$$

with $\deg(e_{1,\{1\}}) = a_1 + a_1$, $\deg(e_{2,\{2\}}) = a_2 + a_2$, and $\deg(e_{2,\{1\}}) = a_2 + a_1$. Note that $\operatorname{rank} F_1 = 1 \cdot \binom{3}{1+1}$. Finally,

$$F_2 = Re_{1,12} + Re_{2,12}$$

with $\deg(e_{1,12}) = a_1 + a_1 + a_2$ and $\deg(e_{2,12}) = a_2 + a_1 + a_2$.

2.3.2. *Maps.* A few notational conventions help to define the boundary maps between the F_d. For $A \subseteq [m-1]$, set

$$\operatorname{sign}(j, A) = (-1)^t \quad \text{for } j \in A = \{\ell_0 < \ell_1 < \cdots < \ell_t = j < \cdots < \ell_r\}.$$

For convenience, set $e_{0,A} = 0$, and for $i \in [m-1]$ with $i < \min(A)$, define

$$e_{i,A} = \sum_{j \in A} \operatorname{sign}(j, A) e_{j, A \cup i \setminus j}\,. \tag{2-3}$$

As a consequence of this definition of $e_{i,A}$, for each $B \subseteq [m-1]$,

$$\sum_{i \in B} \operatorname{sign}(i, B) e_{i, B \setminus i} = 0. \tag{2-4}$$

With these conventions in hand, and considering $i + j$ modulo m in subscripts as usual, define the map $\partial_d : F_d \to F_{d-1}$ by

$$e_{i,A} \mapsto \sum_{j \in A} \operatorname{sign}(j, A)(x_j e_{i, A \setminus j} - y^{b_{i,j}} e_{i+j, A \setminus j}) \tag{2-5}$$

with the exception of $d = 1$, in which case

$$\partial_1(e_{i,j}) = x_i x_j - y^{c_{i,j}} x_{i+j}\,.$$

$$0 \leftarrow R \xleftarrow{\varnothing\begin{bmatrix} \overset{\mathbf{1,1}}{x_1^2 - x_2 y^{b_{11}}} & \overset{\mathbf{2,2}}{x_2^2 - x_1 y^{b_{22}}} & \overset{\mathbf{2,1}}{x_1 x_2 - y^{b_{12}}} \end{bmatrix}} R^3 \xleftarrow{\begin{array}{c|cc} & \mathbf{1,12} & \mathbf{2,12} \\ \hline \mathbf{1,1} & -x_2 & y^{b_{22}} \\ \mathbf{2,2} & -y^{b_{11}} & x_1 \\ \mathbf{2,1} & x_1 & -x_2 \end{array}} R^2 \leftarrow 0$$

Figure 2. The Apéry resolution for $m = 3$.

Example 2.9. Figure 2 shows the modules and maps for the case $m = 3$. Note that by definition the bases for the modules are indexed by (i, A) pairs, and these are used to label the rows and columns of the matrices representing the maps. Consider the term $\partial_2(e_{1,12})$, which by definition is

$$\begin{aligned} \partial_2(e_{1,12}) &= (x_1 e_{1,2} - y^{b_{1,1}} e_{2,2}) - (x_2 e_{1,1} - y^{b_{1,2}} e_{0,1}) \\ &= x_1 e_{2,1} - y^{b_{1,1}} e_{2,2} - x_2 e_{1,1}, \end{aligned}$$

where the relation $e_{1,2} - e_{2,1} = 0$ is used. This illustrates how (2-4) ensures that ∂_d is well defined.

Example 2.10. Figure 1 shows the modules and maps for the case $m = 4$. The ranks for the modules and the general structure of the maps are independent of the values of $\mathrm{Ap}(S)$; indeed, the only variance is found in the values of the exponents on the y-variables in the matrices. The Apéry resolution of I_S for S introduced in Example 2.1 is given in the upper portion of Figure 3 (toward the end of Section 3).

3. Maximal embedding dimension numerical semigroups

This section contains a proof that the Apéry resolution is indeed a resolution of J_S. We begin with a brief review of Schreyer's theorem, which identifies a Gröbner basis (under a carefully chosen term order) for the syzygy module of a Gröbner basis, and which is the core tool in our proof. Then, we prove two lemmas that verify important subtleties about generating sets of J_S and the boundary maps ∂_d; specifically, these maps are consistent with the definition of $e_{i,A}$ in (2-3) and (2-4) when $i < \min A$, and substituting (2-3) into the definition of ∂_d still yields matrix entries that are monomials. Finally, we prove our main result.

We begin with a statement of Schreyer's theorem. Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a Gröbner basis for a submodule $M \subseteq R^d$ with respect to a fixed term order $\preceq$. Let $e_1, \dots, e_s$ denote the standard basis of R^s, and let $\mathrm{In}_{\preceq}(v)$ denote the initial term of v with respect to the term order $\preceq$.

Theorem 3.1 (Schreyer's theorem). *There exist explicitly defined elements $s_{i,j} \in R^s$ that form a Gröbner basis for the syzygy module of $\mathcal{G}$ with respect to the monomial order $>_{\mathcal{G}}$ on R^s defined as follows: $x^\alpha e_i >_{\mathcal{G}} x^\beta e_j$ if $\mathrm{In}_{\preceq}(x^\alpha e_i) \succ \mathrm{In}_{\preceq}(x^\beta e_j)$ in R^m, or if $\mathrm{In}_{\preceq}(x^\alpha e_i) = \mathrm{In}_{\preceq}(x^\beta e_j)$ and $i < j$.*

For a textbook treatment of this theorem, including detailed definitions of $s_{i,j}$ and a proof, see [5, Chapter 5, (3.3)].

There is an interesting connection between the resolution we study and the origins of Schreyer's theorem. The authors of [9] produce a resolution that is isomorphic to the one defined in Section 2. In fact, Frank-Olaf Schreyer informed us (in personal communication) that resolution of the ideals in (2-2) was the initial motivation for both [9] and what is now known as Schreyer's theorem; see [5, Chapter 5, (3.3)]. However, the particular form taken by our explicit matrices clarifies the sense in which our resolution is compatible with specialization in the sense of Theorem 4.4. This point has some subtlety: obtaining a resolution isomorphic to ours need not be sufficient for the purpose of specialization (see, for instance, the resolutions and discussion in Remark 3.6). As such, we include in this section a full proof of our main result, Theorem 3.4.

Lemma 3.2. *The generating set* (2-2) *is a Gröbner basis for J_S under any term order $\preceq$ on R for which $x^a y^r \succ x^b y^s$ whenever $a_1 + \cdots + a_{m-1} > b_1 + \cdots + b_{m-1}$, where x^a and x^b are monomials in $x_1, \ldots, x_{m-1}$.*

Proof. Since J_S is generated by binomials, it suffices to consider binomials when computing initial ideals. The key observation is that in any graded degree, exactly one monomial in R has the form $x_i y^a$ with $a \in \mathbb{Z}_{\geq 0}$, since the graded degrees of the variables x_i are distinct modulo m. Hence the larger term under $\preceq$ in any nonzero binomial from J_S is divisible by $x_i x_j = \mathrm{In}_{\preceq}(x_i x_j - y^{c_{i,j}} x_{i+j})$ for some $i, j \in [m-1]$. As such,

$$\mathrm{In}_{\preceq}(J_S) = \langle x_i x_j : 1 \leq i, j \leq m-1 \rangle,$$

and thus the generating set in (2-2) is a Gröbner basis for J_S. □

The next aim is to establish that applying ∂ to $e_{i,A}$ when $i < \min(A)$ using the expression given in (2-5) is consistent with (2-4); this is needed when considering the result of applying ∂ repeatedly. Further, careful analysis of the use of (2-4) is required when $i + j < \min(A \setminus j)$ in (2-5). These issues are addressed in the following lemma.

Lemma 3.3. *The maps ∂_d respect* (2-4), *that is, applying the definition of ∂_d to the left-hand side of* (2-3) *yields the image of the right-hand side under ∂_d. Furthermore, $\mathcal{F}_\bullet$ is a complex, and for $d > 1$, the entries of each ∂_d are monomials.*

Proof. If $d = 1$, then (2-3) yields $e_{i,j} = e_{j,i}$, and the first claim is immediate. If $d > 1$, then for each $B \subseteq [m-1]$ with $|B| = d + 1$,

$$\sum_{i \in B} \mathrm{sign}(i, B) \partial e_{i,B \setminus i} = \sum_{i \in B} \mathrm{sign}(i, B) \sum_{j \in B \setminus i} \mathrm{sign}(j, B \setminus i)(x_j e_{i,B \setminus ij} - y^{b_{i,j}} e_{i+j,B \setminus ij}),$$

wherein the coefficient of $y^{b_{i,j}} e_{i+j,B\setminus ij}$ for distinct $i, j \in B$ equals

$$\operatorname{sign}(i, B) \operatorname{sign}(j, B \setminus i) + \operatorname{sign}(j, B) \operatorname{sign}(i, B \setminus j) = 0$$

and the remaining terms yield

$$\begin{aligned}
\sum_{i \in B} \operatorname{sign}(i, B) \partial e_{i,B\setminus i} &= \sum_{i \in B} \operatorname{sign}(i, B) \sum_{j \in B\setminus i} \operatorname{sign}(j, B \setminus i) x_j e_{i,B\setminus ij} \\
&= \sum_{j \in B} x_j \sum_{i \in B\setminus j} \operatorname{sign}(i, B) \operatorname{sign}(j, B \setminus i) e_{i,B\setminus ij} \\
&= -\sum_{j \in B} \operatorname{sign}(j, B) x_j \sum_{i \in B\setminus j} \operatorname{sign}(i, B \setminus j) e_{i,B\setminus ij} \\
&= -\sum_{j \in B} \operatorname{sign}(j, B) x_j \cdot 0 \\
&= 0.
\end{aligned}$$

Now proceed to the claim that each ∂_d is a matrix whose entries are monomials. Call $e_{i,A}$ *squarefree* if $i \notin A$. Note that every term in (2-4) is squarefree, and no two equalities of the form (2-4) share any terms. As such, to ensure substituting (2-3) into the definition of ∂_d does not produce nonmonomial matrix entries (i.e., that doing so does not contribute a term with a free generator of F_d already appearing in the sum), it suffices to prove that no two squarefree terms in $\partial e_{i,A}$ lie in the same equality in (2-4). To this end, fix $j, k \in A$. If $e_{i,A\setminus j}$ and $e_{i,A\setminus k}$ lie in the same equality in (2-4), then $(A \setminus j) \cup i = (A \setminus k) \cup i$ and thus $j = k$. If $e_{i+j,A\setminus j}$ and $e_{i+k,A\setminus k}$ lie in the same equality in (2-4), then $i + j \notin A$ but $i + j \in (A \cup \{i + k\}) \setminus k$, so necessarily $i + j = i + k$ and thus $j = k$. Lastly, if $e_{i,A\setminus j}$ and $e_{i+k,A\setminus k}$ lie in the same equality in (2-4), then $i + k \notin A$ but $i + k \in (A \setminus j) \cup i$, so $i + k = i$, which is impossible.

It remains to prove that $\mathcal{F}_\bullet$ is a complex. First, suppose $A = \{j, k\}$ with $j < k$ and $i \geq j$. If $i + j$, $i + k$, and $i + j + k$ are all nonzero, then

$$\begin{aligned}
\partial^2 e_{i,A} &= x_j \partial e_{i,k} - x_k \partial e_{i,j} - y^{b_{i,j}} \partial e_{i+j,k} + y^{b_{i,k}} \partial e_{i+k,j} \\
&= x_j (x_i x_k - y^{c_{i,k}} x_{i+k}) - x_k (x_i x_j - y^{c_{i,j}} x_{i+j}) \\
&\qquad - y^{b_{i,j}} (x_{i+j} x_k - y^{c_{i+j,k}} x_{i+j+k}) + y^{b_{i,k}} (x_{i+k} x_j - y^{c_{i+k,j}} x_{i+j+k}) \\
&= x_{i+j} x_k (y^{c_{i,j}} - y^{b_{i,j}}) + x_{i+k} x_j (y^{c_{i,k}} - y^{b_{i,k}}) + x_{i+j+k} (y^{b_{i,j}} y^{c_{i+j,k}} - y^{b_{i,k}} y^{c_{i+k,j}}) \\
&= 0
\end{aligned}$$

by homogeneity of ∂. In the event $i + j = 0$, or $i + k = 0$, or $i + j + k = 0$, replacing x_0 with zero as appropriate in the above algebra yields the desired equality. For all remaining cases, $|A| > 2$, and in the expansion of

$$\begin{aligned}\partial^2 e_{i,A} &= \sum_{j\in A} \operatorname{sign}(j, A) x_j \partial e_{i,A\setminus j} - \sum_{j\in A} \operatorname{sign}(j, A) y^{b_{i,j}} \partial e_{i+j,A\setminus j} \\ &= \sum_{j\in A} \operatorname{sign}(j, A) x_j \Big(\sum_{k\in A\setminus j} \operatorname{sign}(k, A\setminus j)(x_k e_{i,A\setminus jk} - y^{b_{i,k}} e_{i+k,A\setminus jk}) \Big) \\ &\quad - \sum_{j\in A} \operatorname{sign}(j, A) y^{b_{i,j}} \Big(\sum_{k\in A\setminus j} \operatorname{sign}(k, A\setminus j)(x_k e_{i+j,A\setminus jk} - y^{b_{i+j,k}} e_{i+j+k,A\setminus jk}) \Big),\end{aligned}$$

the terms $x_j x_k e_{i,A\setminus jk}$, $x_j y^{b_{i,k}} e_{i+k,A\setminus jk}$, and $y^{b_{i,j}+b_{i+j,k}} e_{i+j+k,A\setminus jk}$ each have coefficient

$$\operatorname{sign}(j, A) \operatorname{sign}(k, A \setminus j) + \operatorname{sign}(k, A) \operatorname{sign}(j, A \setminus k) = 0$$

for any distinct $j, k \in A$. □

Theorem 3.4. *The complex $\mathcal{F}_\bullet$ is a resolution.*

Proof. Proceed by induction on d to show that the columns of the matrices for ∂_d form a Gröbner basis for $\ker(\partial_{d-1})$. The case $d = 1$ is handled by Lemma 3.2, so suppose $d = 2$. Let $\preceq'$ denote the partial order on F_1 given by $x^\beta e_{k,\ell} \preceq' x^\alpha e_{i,j}$ whenever

$$\operatorname{In}_\preceq(\partial_1(x^\beta e_{k,b})) \prec \operatorname{In}_\preceq(\partial_1(x^\alpha e_{i,a})),$$

or when equality holds above and $a < b$, or if equality holds above, $a = b$, and $i < k$. Note $x^\beta e_{j,\ell} \preceq' x^\alpha e_{i,k}$ whenever x^α has higher total degree in $x_1, \ldots, x_{m-1}$ than x^β, so

$$\operatorname{In}_{\preceq'}(\partial_2(e_{i,jk})) = x_k e_{i,j} \quad \text{where } j < k \text{ and } i \ge j.$$

Theorem 3.1 implies the elements

$$s_{i,a;\,k,b} = \frac{L}{x_i x_a} e_{i,a} - \frac{L}{x_k x_b} e_{k,b} - \sum_{\ell\ge c\ge 1} f_{\ell,c} e_{\ell,c} \quad \text{for } i \ge a,\ k \ge b \tag{3-1}$$

form a Gröbner basis for $\ker(\partial_1)$ under $\preceq'$, where L equals $\operatorname{lcm}(x_i x_a, x_k x_b)$ and the $f_{\ell,c}$ are coefficients obtained from polynomial long division when dividing $S(\partial_1(e_{i,a}), \partial_1(e_{k,b}))$ by (2-2). In particular, we claim

$$\operatorname{In}_{\preceq'}(\ker(\partial_1)) = \langle x_k e_{i,j} \mid j < k \text{ and } i \ge j \rangle$$

is generated by initial terms of the columns of ∂_2: by construction $\operatorname{In}_{\preceq'}(s_{i,a;\,k,b})$ must be one of the first two terms in (3-1) and $\partial_1\big(\frac{L}{x_i x_a} e_{i,a}\big) = \partial_1\big(\frac{L}{x_k x_b} e_{k,b}\big)$, so without loss of generality say $e_{k,b} \prec' e_{i,a}$. Then either $a < b \le k$ and x_k or x_b appear as a coefficient of $e_{i,a}$, or $a = b$, $a \le i < k$ and x_k appears as a coefficient of $e_{i,a}$, so $\operatorname{In}_{\preceq'}(s_{i,a;\,k,b})$ is divisible by the initial term of some column of ∂_2. This implies that $\operatorname{In}_{\preceq'}(\operatorname{im}(\partial_2)) = \operatorname{In}_{\preceq'}(\ker(\partial_1))$, which, together with $\operatorname{im}(\partial_2) \subseteq \ker(\partial_1)$, implies $\operatorname{im}(\partial_2) = \ker(\partial_1)$ and the columns of ∂_2 form a Gröbner basis under $\preceq'$.

Lastly, suppose $d > 2$, let $\preceq$ denote the term order on F_{d-2} obtained inductively, and let $\preceq'$ denote the term order on F_{d-1} so that $x^\beta e_{j,B} \preceq' x^\alpha e_{i,A}$ whenever

$$\mathrm{In}_\preceq(x^\beta \partial_{d-1}(e_{j,B})) \prec \mathrm{In}_\preceq(x^\alpha \partial_{d-1}(e_{i,A})),$$

or if equality holds above and A precedes B lexicographically, or if equality holds above, $A = B$, and $i < j$. One readily obtains

$$\mathrm{In}_{\preceq'}(\partial_d(e_{i,A})) = x_j e_{i,A\setminus j} \quad \text{with } \ j = \max(A)$$

after checking that

- $x^\alpha e_{k,B} \preceq' x^\beta e_{\ell,C}$ whenever x^β has higher total degree in $x_1, \ldots, x_{m-1}$ than x^α;
- $x_k\, \mathrm{In}_\preceq(\partial_{d-1}(e_{i,A\setminus k})) = x_\ell\, \mathrm{In}_\preceq(\partial_{d-1}(e_{i,A\setminus \ell}))$ for all $k, \ell \in A$; and
- the substitution (2-3) need only be made if $\min(A) \le i < \min(A \setminus \min(A))$, in which case $A \setminus j$ lexicographically precedes the second subscript of every summand in (2-4).

The equality $S(\partial_{d-1}(e_{i,A}), \partial_{d-1}(e_{j,B})) = 0$ holds due to initial terms having distinct basis vectors unless $i = j$, $A = C \cup \{\gamma\}$, and $B = C \cup \{\delta\}$ for some $\delta, \gamma \in [m-1]$ and some nonempty $C \subseteq [m-1]$ with $\delta, \gamma > \max(C)$. As such, Schreyer's theorem yields

$$\mathrm{In}_{\preceq'}(\ker(\partial_{d-1})) = \langle x_\delta e_{i,C} : i, \delta \in [m-1],\ C \subseteq [m-1],\ \delta > \max(C)\rangle,$$

and since $x_\delta e_{i,C} = \mathrm{In}_{\preceq'}(\partial_d(e_{i,C\cup\delta}))$ for each i, δ, and C, the columns of ∂_d form a Gröbner basis for $\ker(\partial_{d-1})$. The proof is completed by observing that induction also ensures none of the initial terms in question involve $e_{i,A}$ with $i < \min(A)$. □

Corollary 3.5. *The resolution $\mathcal{F}_\bullet$ is minimal if and only if S is MED.*

Proof. A resolution is minimal if and only if the matrices for ∂_d contain no nonzero constant entries. The only entries that depend on $a_1, \ldots, a_{m-1}$ are powers of y, and their exponents $b_{i,j}$ are all strictly positive precisely when S is MED. □

Remark 3.6. MED semigroups whose associated toric ideal is determinantal are exactly those semigroups where $a_1, a_2, \ldots, a_{m-1}$ form an arithmetic sequence (not necessarily in that order) [15; 19]. In this case, I_S is resolved by the Eagon–Northcott complex [6]; a detailed treatment on the Eagon–Northcott resolution can be found in [8, Appendix A2H]. The strict requirements on an MED semigroup to make its associated toric ideal determinantal mean that such semigroups form only a small proportion of all numerical semigroups: in the Kunz cone, these semigroups lie in the union of a finite set of affine 2-planes, whose union cannot be the whole cone. Although relatively few toric ideals of MED semigroup ideals are minimally resolved by Eagon–Northcott complexes, the occasional overlap does mean that all toric ideals for MED numerical semigroups share Betti numbers with

$$0 \leftarrow R \xleftarrow{\begin{bmatrix} x_1^2 - x_2y^2 & x_2^2 - y^5 & x_3^2 - x_2y^3 & x_1x_2 - x_3y^2 & x_1x_3 - y^5 & x_2x_3 - x_1y^3 \end{bmatrix}}$$

$$R^6 \xleftarrow{\begin{bmatrix} -x_2 & -x_3 & & & & & y^3 & y^3 \\ -y^2 & & x_1 & -x_3 & & y^3 & & \\ & & & & x_1 & x_2 & -y^2 & \\ x_1 & & -x_2 & y^3 & y^3 & & -x_3 & \\ y^2 & x_1 & & & -x_3 & -y^3 & & -x_2 \\ & -y^2 & -y^2 & x_2 & & -x_3 & x_1 & x_1 \end{bmatrix}} R^8 \xleftarrow{\begin{bmatrix} x_3 & -y^3 & \\ -x_2 & & y^3 \\ & x_3 & -y^3 \\ -y^2 & x_1 & \\ y^2 & & -x_2 \\ & -y^2 & x_1 \\ x_1 & -x_2 & \\ -x_1 & & x_3 \end{bmatrix}} R^3 \leftarrow 0$$

$$0 \leftarrow R \xleftarrow{\begin{bmatrix} x_1^2 - x_2y^2 & x_2^2 - x_1x_3 & x_3^2 - x_2y^3 & x_1x_2 - x_3y^2 & x_1x_3 - y^5 & x_2x_3 - x_1y^3 \end{bmatrix}}$$

$$R^6 \xleftarrow{\begin{bmatrix} x_2 & x_3 & x_3 & & & & & y^3 \\ y^2 & & x_1 & x_3 & & y^3 & & \\ & & & x_1 & x_1 & x_2 & y^2 & \\ -x_1 & & -x_2 & & y^3 & & x_3 & \\ & -x_1 & & & -x_3 & & -x_2 & -x_2 \\ & y^2 & & -x_2 & & -x_3 & & x_1 \end{bmatrix}} R^8 \xleftarrow{\begin{bmatrix} -x_3 & -y^3 & \\ x_2 & x_3 & \\ & -x_3 & -y^3 \\ y^2 & x_1 & \\ & -x_1 & -x_2 \\ & y^2 & x_1 \\ -x_1 & -x_2 & \\ & x_2 & x_3 \end{bmatrix}} R^3 \leftarrow 0$$

Figure 3. The Apéry resolution (above) and Eagon–Northcott resolution (below) for I_S where $S = \langle 4, 9, 10, 11 \rangle$.

the Eagon–Northcott resolution of a $2\times m$ matrix, despite the impossibility of using the Eagon–Northcott construction to resolve most such toric ideals.

Even in the case where the ideal is determinantal, the Apéry resolution differs from the Eagon–Northcott resolution. As an example, consider the numerical semigroup $S = \langle 4, 9, 10, 11 \rangle$, whose defining toric ideal I_S is generated by the 2×2 minors of

$$\begin{bmatrix} x_1 & x_2 & x_3 & y^3 \\ y^2 & x_1 & x_2 & x_3 \end{bmatrix}.$$

The key difference is the presentation of the generators of I_S. Namely, the generators as provided in (2-2) are of the form $x_ix_j - x_{i+j}y^{b_{i,j}}$, while those given by determinants may have the form $x_ix_j - x_{i+1}x_{j-1}$. Figure 3 shows the Apéry resolution and the Eagon–Northcott resolution of I_S, with basis elements in the Eagon–Northcott resolution ordered to mimic the Apéry resolution. It is worth noting that in the $m = 3$ case, a_1 and a_2 trivially form an arithmetic sequence, and in fact the Apéry resolution and the Eagon–Northcott resolution coincide.

Remark 3.7. When S is MED, quotienting the Apéry resolution of I_S by the ideal $\langle y \rangle$, as Kunz does in Theorem 2.7 with the ring R/I_S, yields a minimal resolution of the ideal $\langle x_1, \dots, x_{m-1} \rangle^2$ over the ambient polynomial ring $\Bbbk[x_1, \dots, x_{m-1}]$. This ideal is known to be resolved by the Eagon–Northcott complex on the $2 \times m$ matrix

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & 0 \\ 0 & x_1 & x_2 & x_3 & \cdots & x_{m-1} \end{bmatrix}.$$

Thus, in the MED case, the Eagon–Northcott complex "sits inside" the Apéry resolution; indeed, it is the result of an artinian reduction of R/I_S.

4. Specialization for arbitrary numerical semigroups

The Apéry resolution can be thought of as a family of free resolutions, one for the Apéry ideal J_S of each numerical semigroup S with multiplicity m, that is parametrized by the values $b_{i,j}$. Given a numerical semigroup S, a free resolution of J_S is obtained by simply computing the values $b_{i,j}$ from the Apéry set of S and substituting them into the Apéry resolution. By Corollary 3.5, restricting to semigroups S in the interior of C_m, the Apéry resolutions form a parametrized family of minimal free resolutions.

The main result of this section is Theorem 4.4, which implies that for each face F of C_m, there exists a family of minimal free resolutions, one for the Apéry ideal J_S of each numerical semigroup S indexed by the interior of F, that is analogously parametrized by the positive $b_{i,j}$. Figure 4 depicts one such resolution for the $z_2 = 2z_1$ facet of C_4. Our proof of Theorem 4.4 is nonconstructive: it argues that there exists a change of basis for the Apéry resolution, depending only on F, that yields the desired minimal free resolution of J_S as a summand. With Proposition 4.1, which gives the algebraic relationship between minimal resolutions of J_S and I_S, the Betti numbers of J_S and I_S can be recovered from F (Corollaries 4.3 and 4.5).

$$0 \leftarrow R \xleftarrow{\ \begin{array}{c|cccc} & \mathbf{1,1} & \mathbf{3,3} & \mathbf{2,1} & \mathbf{3,1} \\ \mathbf{000} & x_1^2 - x_2 & x_3^2 - x_1^2 y^{b_{33}} & x_1^3 - x_3 y^{b_{12}} & x_1 x_3 - y^{b_{13}} \end{array}\ } R^4$$

$$R^4 \xleftarrow{\ \begin{array}{c|ccccc} & \mathbf{3,23} & \mathbf{2,12} & \mathbf{2,13} & \mathbf{3,13} & \mathbf{3,12} \\ \hline \mathbf{1,1} & x_3^2 - x_1^2 y^{b_{33}} & x_1^3 - x_3 y^{b_{12}} & x_1 x_3 - y^{b_{13}} & & \\ \mathbf{3,3} & -(x_1^2 - x_2) & & & x_1 & -y^{b_{12}} \\ \mathbf{2,1} & & -(x_1^2 - x_2) & & y^{b_{33}} & -x_3 \\ \mathbf{3,1} & & & -(x_1^2 - x_2) & -x_3 & x_1^2 \end{array}\ } R^5 \xleftarrow{\ \begin{array}{c|cc} & \mathbf{3,[3]} & \mathbf{2,[3]} \\ \hline \mathbf{3,23} & x_1 & -y^{b_{12}} \\ \mathbf{2,12} & y^{b_{33}} & -x_3 \\ \mathbf{2,13} & -x_3 & x_1^2 \\ \mathbf{3,13} & x_1^2 - x_2 & \\ \mathbf{3,12} & & x_1^2 - x_2 \end{array}\ } R^2 \leftarrow 0$$

Figure 4. A specialization of the $m = 4$ Apéry resolution where $b_{11} = 0$, so $a_2 = 2a_1$. Note this forces $b_{13} = b_{23}$.

Proposition 4.1. *A minimal free resolution of J_S can be obtained as the tensor product of a minimal free resolution of I_S with a Koszul complex.*

Proof. Nonminimality of $m, a_1, \dots, a_{m-1}$ as generators for S is reflected in J_S by binomial generators without y. More specifically, if $a_i + a_j = a_{i+j}$, then $b_{i,j} = 0$ and $x_i x_j - x_{i+j}$ appears in J_S. Let $\mathcal{A}(S) = \{m, a_{i_1}, a_{i_2}, \dots, a_{i_r}\}$ be the elements a_i that minimally generate S. Though I_S naturally lives in $\Bbbk[y, x_{i_1}, x_{i_2}, \dots, x_{i_r}]$, consider it as an ideal in R via the natural inclusion map. For each nonzero $w \in \mathrm{Ap}(S) \setminus \mathcal{A}(S)$, pick one of the binomials $f_w = x_w - x_u x_v$. These binomials form a regular sequence on R, so the ideal I_W generated by the f_w is resolved by a Koszul complex $\mathcal{K}_\bullet$. Writing $\mathcal{G}_\bullet$ for a minimal free resolution of I_S, the only nontrivial homology of $\mathcal{G}_\bullet \otimes_R \mathcal{K}_\bullet$ occurs in homological degree 0 and is isomorphic to $H_0(\mathcal{G}_\bullet) \otimes H_0(\mathcal{K}_\bullet) = R/I_S \otimes_R R/I_W = R/J_S$, where the last equality is because the f_w form a regular sequence over R/I_S. Therefore $\mathcal{G}_\bullet \otimes \mathcal{K}_\bullet$ is a minimal free resolution of R/J_S. □

Example 4.2. The underlying structure as a tensor of two resolutions is readily seen in Figure 4, which resolves J_S for $\mathrm{Ap}(S) = \{4, a_1, 2a_1, a_3\}$. This example was obtained by computing the Apéry resolution for J_S and then trimming away any constant entries using row and column operations as described in Theorem 4.4.

We include the proof of the following, despite its appearance in Theorem 2.7 as recovered from [20], to demonstrate how the Apéry resolution maps in Theorem 3.4 lend themselves to specialization to the faces of C_m, as well as to contrast its content with that of Theorem 4.4.

Corollary 4.3. *Let S and T be numerical semigroups corresponding to points interior to the same face F of the Kunz cone $\mathcal{C}_m$. The Apéry ideals of S and T share the same Betti numbers, as do the defining toric ideals of S and T. In particular, $\beta_d(J_S) = \beta_d(J_T)$ and $\beta_d(I_S) = \beta_d(I_T)$ for all $d \geq 0$.*

Proof. Let $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ be the Apéry resolutions of J_S and J_T, respectively. In the case that S and T are both MED, so the face F is the entirety of C_m, both resolutions are minimal by Corollary 3.5 and have the same modules at each homological degree, so $\beta_d(J_S) = \beta_d(J_T)$ holds immediately.

If $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ are not minimal, then the resolutions have ± 1 entries in identical places in their resolutions, once again because S and T lie interior to the same face F and thus have the same $b_{i,j} = 0$, meaning that the same entries $\pm y^{b_{i,j}}$ become ± 1. Because the Betti numbers of any positively graded ideal I equal the dimensions of the graded vector spaces $\mathrm{Tor}_\bullet(I, \Bbbk)$, consider $\mathcal{F}_\bullet \otimes_\Bbbk \Bbbk$ and $\mathcal{F}'_\bullet \otimes_\Bbbk \Bbbk$. The differentials in these complexes are identical: they are matrices of 0s and ± 1s with units in matching places. Therefore, their kernels and images are the same at each homological degree, so

$$\beta_d(J_S) = \dim \mathrm{Tor}_d^{\Bbbk}(J_S, \Bbbk) = \dim \mathrm{Tor}_d^{\Bbbk}(J_T, \Bbbk) = \beta_d(J_T).$$

Next consider I_S and I_T. By Proposition 4.1,

$$\mathcal{F}_\bullet = (\mathcal{G}_\bullet \otimes \mathcal{K}_\bullet) \quad \text{and} \quad \mathcal{F}'_\bullet = (\mathcal{G}'_\bullet \otimes \mathcal{K}_\bullet),$$

where $\mathcal{G}_\bullet$ and $\mathcal{G}'_\bullet$ are minimal free resolutions of I_S and I_T, respectively, and $\mathcal{K}_\bullet$ is the Koszul resolution on the extraneous binomials. Tensoring with $\mathcal{K}_\bullet$ exerts the same invertible change on the Betti numbers of $\mathcal{G}_\bullet$ and $\mathcal{G}'_\bullet$. More specifically, let

$$g_S(t) = \sum_{i=0}^{p} \beta_i(I_S)t^i \quad \text{and} \quad f_S(t) = \sum_{i=0}^{q} \beta_i(J_S)t^i$$

be the generating functions for the Betti numbers of $\mathcal{G}_\bullet$ and $\mathcal{F}_\bullet$, respectively. Since $\mathcal{K}_\bullet$ is a Koszul resolution of $r = m - |\mathcal{A}(S)|$ elements,

$$f_S(t) = (1+t)^r g_S(t).$$

Thus,

$$(1+t)^r g_S(t) = f_S(t) = f_T(t) = (1+t)^r g_T(t),$$

and therefore $g_S(t) = g_T(t)$, meaning $\beta_i(I_S) = \beta_i(I_T)$ for all $i \geq 0$. □

Theorem 4.4. *Consider the set*

$$\mathcal{M} = \{x_i : 1 \leq i \leq m-1\} \cup \{y^{b_{i,j}} : 1 \leq i,\, j \leq m-1\}$$

of formal symbols appearing as matrix entries in Apéry resolutions. (*Lemma 3.3 ensures every nonzero matrix entry is accounted for in* $\mathcal{M}$). *Fix a face F of C_m. There is a sequence of matrices, whose entries are* $\Bbbk$*-linear combinations of formal products of elements of* $\mathcal{M}$, *with the following property*: *for each numerical semigroup S indexed by the relative interior of F, substituting R-variables and the values $b_{i,j}$ for S into the entries of each matrix yields boundary maps for a graded minimal free resolution of J_S.*

Proof. Fix a numerical semigroup S with multiplicity m. Let

$$\mathcal{N}_S = \{x_i : 1 \leq i \leq m-1\} \cup \{y^{b_{i,j}} : 1 \leq i,\, j \leq m-1 \text{ and } b_{i,j} > 0\} \subseteq \mathcal{M}$$

denote the set of elements of $\mathcal{M}$ corresponding to positive-degree monomials in R under the grading by S. If S is MED, then $\mathcal{M} = \mathcal{N}_S$; otherwise they are distinct.

By [7, Theorem 20.2] (see also [22, Exercises 1.10 and 1.11]), the matrices in any free resolution for J_S can, via a sequence of row and column operations that preserve homogeneity, be turned into block diagonal matrices with 2 blocks:

(i) a matrix with no nonzero constant entries and at least one nonzero entry in each row and column; and

(ii) a matrix with no nonconstant entries and at most one nonzero entry in each row and column.

After doing this, restricting to each block described in (i) yields a minimal free resolution for J_S.

One way to select the aforementioned row and column operations is as follows. Begin with the matrices M_i for the maps ∂_i for the Apéry resolution, and perform the following for each $i = 1, 2, \dots, m-1$, assuming that, as a result of prior operations, any column of M_i with a nonzero constant entry has no other nonzero entries:

- First use nonzero constant entries of M_i to clear all other entries in their respective rows. If $i = 1$, then no such rows exist. Fix a row R of M_i with a nonzero constant entry c, say in column C_1 with corresponding row R_1 in the matrix M_{i+1}. For each nonzero entry f in R, say in a column $C_2 \neq C_1$ with corresponding row R_2 in M_{i+1}, subtract $c^{-1} f \cdot C_1$ from C_2 and add $c^{-1} f \cdot R_2$ to R_1. Once this is done, c will be the only nonzero entry in R, and in fact c will be the only nonzero entry in row R and column C_1. Moreover, since $M_i M_{i+1} = 0$, the row R_1 in M_{i+1} only has entries 0.

- Next use nonzero constant entries of M_{i+1} to clear all other entries in their respective columns. Fix a column C of M_{i+1} with a nonzero constant entry c, say in row R_1 with corresponding column C_1 in the matrix M_i. For each nonzero entry f in C, say in a row $R_2 \neq R_1$ with corresponding column C_2 in M_i, subtract $c^{-1} f \cdot R_1$ from R_2 and add $c^{-1} f \cdot C_2$ to C_1. Once this is done, c will be the only nonzero entry in column C, so since $M_i M_{i+1} = 0$, the column C_1 now only has entries 0. Moreover, all changes to M_i only affect the (now all 0) column C_1, so M_i still has the property that every nonzero constant entry is the only nonzero entry in its row and column.

Once the above operations are completed for each i, the rows and columns may be permuted to obtain the desired blocks.

The key observation is that in the above sequence of row and column operations, the entry f is an existing matrix entry. As such, after each row or column operation, every matrix entry g can be written as a $\Bbbk$-linear combination of products of (possibly constant) elements of $\mathcal{M}$. Moreover, if g is a nonzero constant, then g has degree 0 under the grading by S, so by homogeneity, the aforementioned expression for g cannot contain any monomials in $\mathcal{N}_S$, since it must be a $\Bbbk$-linear combination of products of degree-0 elements of $\mathcal{M}$.

Now, fix a numerical semigroup T in the same face F of the Kunz cone C_m as S. The sets $\mathcal{M} \setminus \mathcal{N}_S$ and $\mathcal{M} \setminus \mathcal{N}_T$ each contain $y^{b_{i,j}}$ whenever F is contained in the facet $z_i + z_j = z_{i+j}$, and thus $\mathcal{N}_S = \mathcal{N}_T$. As a consequence of the preceding paragraph, applying an identical sequence of row and column operations to the Apéry resolution for J_T yields nonzero constant entries in precisely the same locations at each step of the process. This completes the proof. □

Theorem 4.4 yields the following graded refinement of Corollary 4.3.

Corollary 4.5. *For each* $i \in \{0, \dots, m-1\}$, *writing* $[i]_m = i + m\mathbb{Z}$,

$$\sum_{b \in [i]_m} \beta_{d,b}(J_S) = \sum_{b \in [i]_m} \beta_{d,b}(J_{S'}).$$

The same relationship holds between the Betti numbers of I_S *and* $I_{S'}$.

Proof. Apply Theorem 4.4 for the first claim, and subsequently apply Proposition 4.1 for the final claim. □

5. Open questions

Several of the open questions presented here relate to the defining toric ideal I_S. One of the main results of [18] identifies a finite poset corresponding to each face F of C_m, called the *Kunz poset* of F. If a point interior to F indexes a numerical semigroup S, then this poset coincides with the divisibility poset of $\mathrm{Ap}(S)$. In [14], the Kunz poset of F is used to obtain a parametrized family of minimal binomial generating sets, one for the defining toric ideal I_S of each numerical semigroup S in F. The last three binomials in the first matrix in Figure 4 constitute one such example for the relevant facet F of C_4. This provides a natural candidate for the first matrix in the resolution conjectured as follows.

Conjecture 5.1. *For each face* F *of* C_m, *there exists a parametrized family of minimal resolutions, one for the defining toric ideal* I_S *of each numerical semigroup* S *indexed by the interior of* F, *akin to those obtained in Theorem 4.4 for Apéry ideals.*

Unlike the proof of Theorem 4.4, the proofs in [14] are constructive, utilizing the Kunz poset structure of the face F containing S. Intuitively, the set of factorizations of elements of $\mathrm{Ap}(S)$ (a set which depends only on F) forms the staircase of a monomial ideal M. If each element of $\mathrm{Ap}(S)$ factors uniquely, then I_S has exactly one binomial generator for each minimal monomial generator of M. If some elements of $\mathrm{Ap}(S)$ have multiple factorizations, then graph-theoretic methods can be used to partition some of the minimal generators of M into blocks (called *outer Betti elements*) and construct one minimal binomial generator of I_S for each block. We refer the reader to [14, Section 5] for the full construction; additional examples can be found in [10].

The nonconstructive nature of the proof of Theorem 4.4, along with the constructive nature of the proofs in [14], motivates the following.

Problem 5.2. Find an explicit combinatorial (e.g., poset-theoretic) construction of the matrices obtained in Theorem 4.4 and conjectured in Conjecture 5.1.

There is a long history of topological formulas for Betti numbers of graded ideals (for instance, Hochster's formula for squarefree monomial ideals [16] (see

[22, Chapter 1]), squarefree divisor complexes for toric ideals [3], or the use of poset homology for computing Poincaré series of semigroup algebras [24]). The following is thus a natural problem.

Problem 5.3. Given a face F of C_m, find a topological formula for extracting the value in the equation in Corollary 4.5 from the Kunz poset of F.

As mentioned in Example 2.6, the ray $(1, 2, 1)$ of C_4 contains positive integer points, but none correspond to a numerical semigroup under Proposition 2.5. Indeed, the first and third coordinates of any such point must be equal, but any point $(a_1, a_2, a_3) \in C_4$ corresponding to a numerical semigroup must have $a_i \equiv i \bmod 4$ for each i. However, the construction in [18] still associates a poset to this ray, and naively following the construction in [14] for a binomial generating set with $y = 0$ yields the artinian binomial ideal $\langle x_1^2 - x_3^2, x_1^3, x_1x_3, x_3^3\rangle$. This motivates the following.

Problem 5.4. Extend the correspondence in Proposition 2.5 to a family of lattice ideals that includes the defining toric ideals of numerical semigroups but reaches points in faces of C_m that do not contain numerical semigroups.

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