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We introduce Bar-Natan homology for null homologous links in $\mathbb{RP}^3$ over the field of two elements. It is a deformation of the Khovanov homology in $\mathbb{RP}^3$ defined by Asaeda, Przytycki and Sikora. We also define an s-invariant from this deformation using the same recipe as for links in S^3, and prove some genus bound using it. The key ingredient is the notion of "twisted orientation" for null homologous links and cobordisms in $\mathbb{RP}^3$.

1. Introduction

Recently, Manolescu and Willis introduced Lee homology and the associated s-invariant for links in $\mathbb{RP}^3$ in [13]. This is a deformation of the Khovanov homology in $\mathbb{RP}^3$ over the base ring $\mathbb{Z}$, first introduced by Asaeda, Przytycki and Sikora in [1] for fields of characteristic 2, and later extended by Gabrovšek in [6] to the base ring $\mathbb{Z}$ by fixing certain sign conventions.

As with the usual Lee homology, the construction in [13] works as long as the characteristic of the base ring is not 2. For links in S^3, the solution for rings of characteristic 2 is to use the Bar-Natan deformation of the Khovanov homology, introduced by Bar-Natan in [2], instead of the Lee deformation. Here, we adapt the same approach for null homologous links in $\mathbb{RP}^3$. Similar to the S^3 case, the homology itself admits a simple description in terms of the number of components of the links. (Note that $H_1(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}_2$, the property that $[L] = 0 \in H_1(\mathbb{RP}^3, \mathbb{Z})$ does not depend on the choice of orientation on components of L, so we can discuss null homologous links without specifying the orientations of L.)

Theorem 1.1. *For a twisted oriented null homologous link $L \subset \mathbb{RP}^3$, one can associate a Bar-Natan chain complex* $\mathrm{CBN}(L)$ *over the base field $\mathbb{F} = \mathbb{F}_2$, whose homology* $\mathrm{HBN}(L)$ *is an invariant of L as a bigraded vector space. More specifically*,

$$\dim(\mathrm{HBN}(L)) = \begin{cases} 0 & \text{if } L \text{ has a component which is nonzero in } H_1(\mathbb{RP}^3, \mathbb{Z}); \\ 2^{|L|} & \text{otherwise (all components of } L \text{ are null homologous)}, \end{cases}$$

and there is a basis of $\mathrm{HBN}(L)$ *given by* $\{s_o \mid o \text{ is a twisted orientation of } L\}$.

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See Definitions 2.8 and 2.9, as well as Lemma 2.10, for the notion of twisted orientation on null homologous links in $\mathbb{RP}^3$. In short, a twisted orientation is an assignment of arrows on each segment of the link projection in $\mathbb{RP}^2$, which reverses direction each time it crosses a fixed essential unknot. More canonically, it is an orientation on the double cover $\widetilde{L}$ of L in S^3 that is reversed by the deck transformation on S^3 of the covering map $S^3 \to \mathbb{RP}^3$. In particular, if L has any homologically essential components, then such a twisted orientation does not exist, and $\mathrm{HBN}(L)$ vanishes in this case.

The reason for the appearance of the twisted orientation is that when we perform the Bar-Natan deformation, we are forced to assign the nontrivial map id_V to the 1-1 bifurcation in the definition of the Bar-Natan chain complex. This differs from the Lee deformation and the usual Khovanov homology in $\mathbb{RP}^3$, where the map assigned to the 1-1 bifurcation is the zero map. Therefore, we introduce the extra twisting to make the 1-1 bifurcation behave in a manner more consistent with the 1-2 and 2-1 bifurcations. By using the notion of "twisted orientation" instead of "orientation", essentially all the proofs for the usual Bar-Natan and Lee homology in S^3 as in [8; 15; 17] carry over in our setting with minor modifications.

As expected, since $\dim(\mathrm{HBN}(K)) = 2$ for a null homologous knot in $\mathbb{RP}^3$, one can define an s-invariant, denoted $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$, from the quantum filtration on $\mathrm{CBN}(K)$ and use it to establish a genus bound in the usual way. See Definitions 4.1 and 4.3 for the precise definition of $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$. It is worth noting that, unlike the usual case where the genus bound applies to the class of orientable slice surfaces, here we obtain a genus bound for twisted orientable slice surfaces. See Definitions 3.1 and 4.8 for precise definitions of twisted orientable cobordisms and slice surfaces. Roughly speaking, this means that the double cover of the surface in $S^3 \times I$ is orientable, and the fiberwise deck transformation τ in the S^3-direction reverses the orientation. We establish the following relationship between $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ and the Euler characteristic of the twisted orientable slice surface, which is analogous to the usual statement for the s-invariant in S^3.

Theorem 1.2. *Suppose $\Sigma : L \to L'$ is a twisted orientable cobordism between the null homologous links L and L' in $\mathbb{RP}^3 \times I$. Then one can define a filtered chain map of filtration degree $\chi(\Sigma)$,*

$$F_\Sigma : \mathrm{CBN}_{*,*}(L) \to \mathrm{CBN}_{*,*}(L'),$$

such that

$$F_\Sigma([s_o]) = \sum_{\{o_i\}} [s_{o_{i_{|L'}}}].$$

Here, o is a twisted orientation on L, $\{o_i\}$ is the set of twisted orientations on Σ that restrict to o on L, $o_{i_{|L'}}$ represents the restriction of such orientations on L',

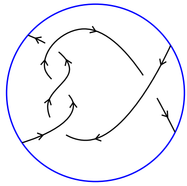

Figure 1. An example of difference in s-invariants defined using Lee deformation and Bar-Natan deformation.

and s_o, *respectively* $s_{o_{i_{|L'}}}$, *are the corresponding canonical generators of* $\mathrm{HBN}(L)$, *respectively* $\mathrm{HBN}(L')$, *defined in Definition 2.14.*

In particular, when both L *and* L' *are null homologous knots and* Σ *is connected,* F_Σ *is a quasi-isomorphism of filtration degree* $\chi(\Sigma)$. *Further specifying to the case where* L' *is the trivial unknot in* $\mathbb{RP}^3$, *one gets*

$$-\chi(\Sigma) \geq |s^{\mathrm{BN}}_{\mathbb{RP}^3}(L)|,$$

for any twisted orientable slice surface Σ *of the knot* L.

A twisted orientable slice surface can be either orientable or unorientable, and not every (un)orientable slice surface is necessarily twisted orientable. See Example 3.2 for different possibilities of the combinations of (non)twisted orientable/(un)orientable cobordisms.

The s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ shares similar formal properties with the usual s-invariant in S^3. See Proposition 4.5 for mirroring, Proposition 4.6 for taking a local connected sum with a local knot and Proposition 4.7 for "positive knots" in the sense of twisted orientations. All of these follow from the same arguments as in the case of S^3. We also provide a simple example to demonstrate the difference between the s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ defined in this paper and the s-invariant defined as in [13]. Consider the knot shown in Figure 1, which is "positive" with respect to the twisted orientation, so we have

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = 3.$$

However, one can construct an orientable slice surface of this knot with genus 1, so the s-invariant of it defined in [13] satisfies

$$|s(K)| \leq 2.$$

In addition, we provide a family of knots in which their two s-invariants differ by an arbitrarily large amount, obtained by inserting more and more full twists in this example. See Example 4.11 for a more detailed discussion.

Organization of the paper. In Section 2, we define the Bar-Natan deformation of the Khovanov complex for null homologous links in $\mathbb{RP}^3$ over the field $\mathbb{F}_2$. We also introduce the notion of twisted orientation and prove Theorem 1.1. In Section 3, we define cobordism maps on the Bar-Natan chain complex for twisted orientable cobordisms and establish the first statement of Theorem 1.2. In Section 4, we define the Bar-Natan s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$, discuss its formal properties, and complete the proof of Theorem 1.2. In Section 5, we discuss some further directions based on this work.

2. Bar-Natan homology for null homologous links in $\mathbb{RP}^3$

We work over the field $\mathbb{F} = \mathbb{F}_2$ of two elements, unless otherwise specified.

Let $V = \mathbb{F}\langle 1, x\rangle$ denote the graded vector space generated by $1, x$, where the quantum grading is defined as

$$\operatorname{qdeg}(1) = 1, \quad \operatorname{qdeg}(x) = -1.$$

Definition 2.1. The *Bar-Natan Frobenius algebra structure on V* is a deformation of the usual Frobenius structure on $V = H^*(S^2)$. It consists of the tuple $(V, m, \Delta, \iota, \eta)$, where the multiplication $m : V \otimes V \to V$ is given by

$$1 \otimes 1 \to 1, \quad 1 \otimes x \to x, \quad x \otimes 1 \to x, \quad x \otimes x \to x,$$

the comultiplication $\Delta : V \to V \otimes V$ is given by

$$1 \to 1 \otimes x + x \otimes 1 + 1 \otimes 1, \quad x \to x \otimes x,$$

the unit $\iota : \mathbb{F} \to V$ is given by

$$1 \to 1,$$

and the counit $\eta : V \to \mathbb{F}$ is given by

$$1 \to 0, \quad x \to 1.$$

Remark 2.2. Some signs in the comultiplication Δ may differ from the usual convention, but this makes no difference as we are working over the field $\mathbb{F}$ of characteristic 2. We also use the version in which the extra deformation variable of the Bar-Natan deformation is set to 1, so all the structure maps are filtered rather than grading preserving, after a suitable shift in the quantum grading.

Definition 2.3. A link L in $\mathbb{RP}^3$ is *null homologous* if

$$[L] = 0 \in H_1(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}/2.$$

If particular, we do not require L to be oriented, as different orientations do not affect $[L]$ in $H_1(\mathbb{RP}^3, \mathbb{Z})$, which is $\mathbb{Z}/2$.

Let L be a null homologous link in $\mathbb{RP}^3$. Suppose L is disjoint from a fixed point $* \in \mathbb{RP}^3$. Then, we obtain a link projection diagram D of L in $\mathbb{RP}^2$ using the

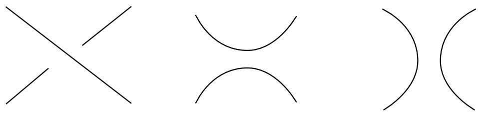

Figure 2. The middle and right illustrate a 0- and 1-smoothing, respectively.

twisted I-bundle structure $\mathbb{RP}^3 \setminus \{*\} \cong \mathbb{RP}^2 \tilde{\times} I$. Suppose D has n crossings. By choosing an ordering of the crossings, we form the usual cube

$$\underline{2}^n := (0 \to 1)^n$$

of smoothings D_v of D for each vertex $v \in \underline{2}^n$, following the convention of 0- and 1-smoothings as indicated in Figure 2.

Since we start with a null homologous link and the smoothing process does not change the homology class, it is straightforward to see that each component in the smoothing D_v is a local unknot in $\mathbb{RP}^2$ for every vertex $v \in \underline{2}^n$. We will apply the Bar-Natan Frobenius algebra to this cube of smoothings, i.e., we associate the vector space V to each local unknot and then take the tensor product for each smoothing D_v.

As for the edge maps, they correspond to change a 0-smoothing to a 1-smoothing at one crossing. Unlike the usual case for link projections in $\mathbb{R}^2$, there are three possibilities as indicated in Figure 3:

- 1-2 bifurcation, which splits one circle into two circles;
- 2-1 bifurcation, which merges two circles into one circle;
- 1-1 bifurcation, which twists one circle into another circle.

We will represent $\mathbb{RP}^2$ as a disk with half of its boundary (the blue circle) identified with the other half. We assign the multiplication map

$$m : V \otimes V \to V$$

to the 2-1 bifurcation and the comultiplication map

$$\Delta : V \to V \otimes V$$

to the 1-2 bifurcation map, as usual. For the 1-1 bifurcation, this corresponds to a map

$$f : V \to V,$$

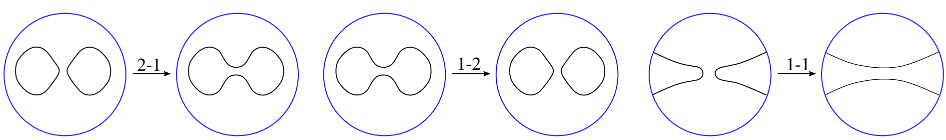

Figure 3. 2-1, 1-2 and 1-1 bifurcations.

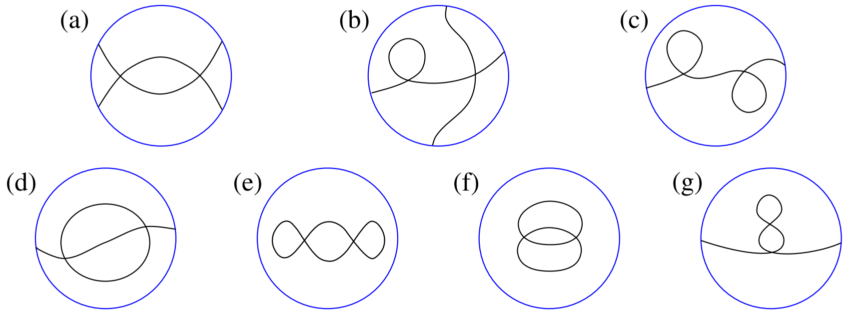

Figure 4. Singular graphs in $\mathbb{RP}^2$ with 2 singular points. See [6].

which is not part of the Bar-Natan Frobenius algebra structure of V as defined above. It turns out that there is a unique choice of f over the base field $\mathbb{F}_2$ that gives a filtered chain complex if we feed the cube of smoothings by the Frobenius algebra V, and use m, Δ, f for the edge maps.

Lemma 2.4. *The only possible choice of $f : V \to V$ for the* 1-1 *bifurcation map over $\mathbb{F}_2$ that gives a chain complex, with $f : V \to V[1]$ being filtered, is*

$$f = \mathrm{id}_V.$$

Here $V[1]$ denotes the graded vector space obtained from V by shifting the quantum grading up by 1.

Proof. To obtain a chain complex, we require that each face of the cube of smoothings gives a commutative diagram (recall that we are working over the field $\mathbb{F}_2$). This amounts to checking that each link diagram in $\mathbb{RP}^2$ with 2 crossings gives a commutative square. A list of such link diagrams is obtained from the singular diagrams in Figure 4 by replacing each singular point by an over or under crossing. Since we are considering null homologous links, only (a), (b), (e), and (f) are possible. A typical square of smoothings and the corresponding maps are shown in Figure 5.

The condition that these squares are commutative gives the following restrictions on the map $f : V \to V$:

(1) $f^2 = m \circ \Delta = \mathrm{id}_V$,

(2) $m \circ (f \otimes \mathrm{id}_V) = m \circ (\mathrm{id}_V \otimes f) = f \circ m$,

(3) $(f \otimes \mathrm{id}_V) \circ \Delta = (\mathrm{id}_V \otimes f) \circ \Delta = \Delta \circ f$.

Also, when defining the chain complex, we will apply a shift in the quantum grading of degree $|v| = \#$ of 1s in $v \in \underline{2}^n$ to the vector space corresponding to the smoothing D_v. Therefore, to ensure the edge maps are filtered, i.e., nondecreasing in the quantum grading, we also require

$$f : V \to V[1] \text{ to be filtered.}$$

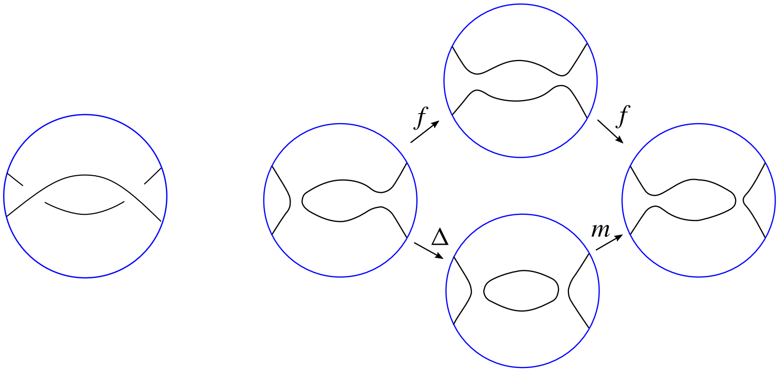

Figure 5. One example of a square in the cube of resolution.

Now, it is easy to see that the only possible choice of f satisfying the above conditions is

$$f = \mathrm{id}_V. \qquad \square$$

Remark 2.5. In [1], Asaeda, Przytycki and Sikora extended the Khovanov homology to links in $\mathbb{RP}^3$ over the field $\mathbb{F}_2$, where they assigned the 0 map to the 1-1 bifurcation map. Because they use the usual Frobenius algebra for Khovanov homology, the corresponding condition for f becomes $f^2 = m \circ \Delta = 0$, so $f = 0$ is a natural choice. In [6], Gabrovšek introduced some sign conventions to make the extension of Khovanov homology work over rings of characteristic 0, where the 1-1 bifurcation map is also assigned 0. More recently, in [13], Manolescu and Willis extended the definition of Lee homology to links in $\mathbb{RP}^3$, where, again, the 1-1 bifurcation map is assigned 0. It is due to this difference that the s-invariant from the Bar-Natan deformation in this paper gives a genus bound on twisted orientable cobordisms, while the s-invariant from the Lee deformation in [13] gives a genus bound on orientable cobordisms.

In [4], the author defined some variation of the usual Khovanov homology in $\mathbb{RP}^3$. One can try applying the Bar-Natan Frobenius algebra structure on V instead of the usual Khovanov Frobenius algebra structure and see what happens. Unfortunately, it won't give a more interesting homology theory. The reason is that if one uses the Bar-Natan Frobenius algebra structure, then the requirement for f and g becomes $f \circ g = \mathrm{id}_{V_0}$, $g \circ f = \mathrm{id}_{V_1}$, which implies V_0 and V_1 are isomorphic, and the chain complex will just be a direct sum of n copies of the reduced version of the Bar-Natan chain complex defined in this paper, where $n = \dim(V_0) = \dim(V_1)$. In the Khovanov setting, the equations we need for f and g are $f \circ g = g \circ f = 0$, which leaves more room for the choice of V_0, V_1, f and g. Still, one could obtain different spectral sequences for different choices of V_0, V_1, f, and g. We have not attempted to explore the possibilities in this project.

By examining the commutativity of the remaining possible 2-crossing diagrams in $\mathbb{RP}^2$ that do not involve the 1-1 bifurcation (these are local diagrams, which means the check is the same as verifying whether the Bar-Natan Frobenius algebra structure on V gives a chain complex for links in S^3), we can define the following chain complex for the link diagram D in $\mathbb{RP}^2$.

Definition 2.6. Given an n-crossing link diagram D in $\mathbb{RP}^2$ of a null homologous link, the *unadjusted Bar-Natan chain complex* $\mathrm{CBN}^{\mathrm{un}}_{*,*}(D)$ is the bigraded chain complex

$$\mathrm{CBN}^{\mathrm{un}}_{i,*} = \bigoplus_{\substack{v\in\underline{2}^n \\ |v|=i}} C(D_v)[i], \qquad \partial = \sum_{e:\text{edges in } \underline{2}^n} \partial_e,$$

where the first grading is the homological grading, and the second grading is the quantum grading. Denote its homology by $\mathrm{HBN}^{\mathrm{un}}(D)$. Here $C(D_v) = V^{\otimes k_v}$ if the smoothing D_v consists of k_v unknots, $[i]$ denotes shifting up by i in the quantum grading, and ∂_e applies m, Δ or f to the involved unknots, depending on whether the edge is a 2-1, 1-2, or 1-1 bifurcation.

It is called the "unadjusted" Bar-Natan complex because we will apply some global shifts in the homological and quantum gradings to make it a link invariant, similar to the usual definition for Khovanov/Bar-Natan homology in S^3. However, the way we apply the shifts is a bit unconventional, so we delay the discussion until later, when we introduce more terminology.

Note that the quantum grading-preserving part of the differential is exactly the same as the differential used in the definition of Khovanov homology for links in $\mathbb{RP}^3$ over $\mathbb{F}_2$ as in [1], where the map associated with the 1-1 bifurcation is the zero map. (Note that such a map needs to be grading preserving from V to $V[1]$, so id_V does not preserve the quantum grading.)

Now the natural next step is to check whether the homology depends only on the link rather than the link diagram. This is usually done by verifying the invariance of the homology under Reidemeister moves. As discussed in [5], for link projections in $\mathbb{RP}^2$, there is a similar list of Reidemeister moves that relate different projections of isotopic links in $\mathbb{RP}^3$, as shown in Figure 6. However, for the arguments in the later sections involving the s-invariant, we need stronger conditions on the induced map by Reidemeister moves, so we will again delay the discussion of invariance until later.

Recall that in the usual Bar-Natan homology for links in S^3, the dimension of the Bar-Natan homology is actually determined just by the number of components in the link, and there is a canonical basis of the homology corresponding to different choices of orientations on each component of the link. Furthermore, the maps induced by the Reidemeister moves will send this canonical basis to the

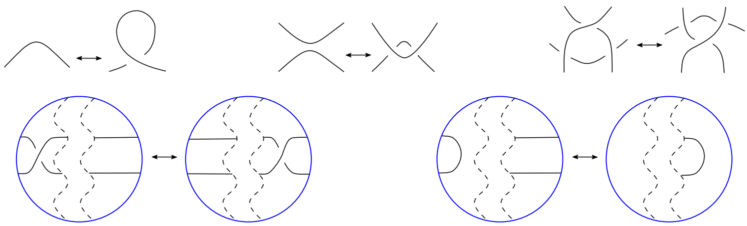

Figure 6. Reidemeister moves in $\mathbb{RP}^3$. Top: left, middle, and right correspond to R-I, R-II, and R-III, respectively. Bottom: left and right correspond to R-IV and R-V, respectively.

corresponding one after the Reidemeister moves. We will prove a similar result for the Bar-Natan homology for null homologous links in $\mathbb{RP}^3$: the dimension of the homology is determined by the number of null homologous components of the link, a canonical basis of the homology is given by "twisted orientations" on the link, and Reidemeister moves send the canonical basis to the canonical basis.

For that, we need to introduce the notion of a "twisted orientation". This is the central notion of the paper. Essentially, all the proofs for the usual Bar-Natan homology in S^3 apply in $\mathbb{RP}^3$ if we replace "orientation" in S^3 with "twisted orientation" in $\mathbb{RP}^3$.

Definition 2.7. Let U_1 denote the essential unknot in $\mathbb{RP}^2$, which is the quotient of the boundary of the disk. We represent $\mathbb{RP}^2$ by identifying the two halves of the boundary of a disk, and U_1 is represented by the quotient of the blue circles in all the figures.

Definition 2.8. Suppose D is a link diagram in $\mathbb{RP}^2$ such that each component of the link L represented by D in $\mathbb{RP}^3$ is null homologous. A *twisted orientation* on D is an assignment of arrows to each segment of D, such that it is reversed each time the link crosses U_1.

See Figure 7 for an illustration of the twisted orientation, along with a comparison to the usual orientation. Note that a twisted orientation only exists if we assume each component of D is null homologous since we cannot assign alternating arrows on a homologically essential component, which intersects U_1 an odd number of times.

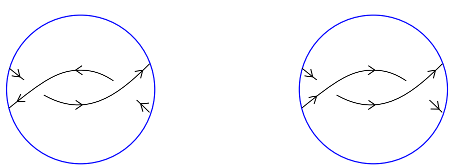

Figure 7. An example of a twisted orientation (left) compared with a usual orientation (right) on the same knot.

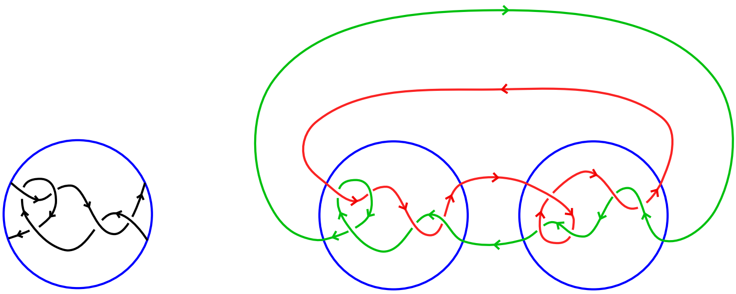

Figure 8. A twisted orientation on L (left) and the corresponding orientation on $\widetilde{L} = cl(T \circ F(T))$ (right) which is reversed by the action of τ.

More canonically, we have another way to view the twisted orientation that does not depend on the link diagram D.

Definition 2.9. Suppose L is a link in $\mathbb{RP}^3$ such that each component of L is null homologous. Let $\widetilde{L}$ denote the double cover of L in S^3, and let τ denote the deck transformation of the covering map $S^3 \to \mathbb{RP}^3$. A *twisted orientation* on L is an orientation on $\widetilde{L}$ that is reversed under the action of τ.

The next lemma proves that these two definitions agree, and when we refer to a twisted orientation on a link L in $\mathbb{RP}^3$, we mean either of the two, depending on the context.

Lemma 2.10. *Suppose D is a link diagram of a link L in $\mathbb{RP}^3$ such that each component of L is null homologous. Then a twisted orientation on D induces a twisted orientation on L, and vice versa. In particular, there are $2^{|L|}$ twisted orientations on L, where $|L|$ is the number of components of L.*

Proof. It is easy to see that a knot K in $\mathbb{RP}^3$ is null homologous if and only if its double cover $\widetilde{K}$ in S^3 is a 2-component link (rather that a knot). Therefore, due to the null homologous assumption, each component of L lifts to a two-component link in $\widetilde{L}$.

One way to draw a diagram of $\widetilde{L}$ is as follows. View the link diagram D in $\mathbb{RP}^2$ as an n-n tangle, denoted by T. Let $F(T)$ be the n-n tangle obtained by flipping T, i.e., rotating T by 180° about its middle horizontal axis in the plane. Then, $\widetilde{L}$ is the closure of the composition $T \circ F(T)$, where the deck transformation τ acts by swapping T and $F(T)$. See Figure 8 for an example.

Now consider a specific component K in L. Every time K hits the essential unknot U_1, it travels from one copy of $\mathbb{RP}^2$ to the other copy in the lifted picture of $\widetilde{K}$. Therefore, when we restrict to one copy of $\mathbb{RP}^2$, two adjacent segments (adjacent in the sense that they share common points on U_1) must come from different components of the lifted link $\widetilde{K}$ (labeled red and green in Figure 8, respectively).

The requirement that the arrow is reversed in the definition of D thus becomes the requirement that the deck transformation τ reverses the orientations on the two components of $\widetilde{K}$. □

Remark 2.11. If we are given an orientation on the lifted link $\widetilde{L}$ that is reversed by τ, and we present $\widetilde{L}$ as the closure of $T \circ F(T)$, there is an ambiguity in assigning the corresponding twisted orientation on the quotient link L. This ambiguity arises from whether we use T or $F(T)$ as a link diagram for L in $\mathbb{RP}^2$. After we fix a choice of the fundamental domain of the deck transformation τ (i.e., choosing T instead of $F(T)$), there is a canonical one-to-one correspondence between the set of twisted orientations of L and the set of orientations of $\widetilde{L}$ that are reversed by τ.

Now, we are going to define elements in the Bar-Natan homology corresponding to the twisted orientations. Later, we will show they form a basis of the Bar-Natan homology, as in the case of Bar-Natan homology in S^3.

Recall that we can diagonalize the Bar-Natan Frobenius algebra V using the basis $a = 1 + x, b = x$ (remember, we always work on the field $\mathbb{F}_2$), such that the multiplication and comultiplication become

$$m : V \otimes V \to V, \quad a \otimes a \to a, \quad b \otimes b \to b, \quad a \otimes b \to 0, \quad b \otimes a \to 0,$$
$$\Delta : V \to V \otimes V, \quad a \to a \otimes a, \quad b \to b \otimes b.$$

In the usual Bar-Natan homology in S^3, for each orientation of the link, one associates an element in the homology by forming the oriented resolutions and then assigning either a or b to each unknot component in the resolution, depending on their orientations and distance from infinity. Here, we will do something similar. Since a twisted orientation is just some assignment of arrows on each segment in the link diagram, and forming an oriented resolution only concerns the orientations near each crossing, we can form the "twisted orientation resolution" of a twisted oriented link diagram by performing the oriented resolution at each crossing in the usual sense. Then, we assign a or b to each unknot component in the twisted oriented resolution according to some rule. The rule requires some explanation, so let's go over it first.

Definition 2.12. Suppose D is a twisted oriented link diagram in $\mathbb{RP}^2$ of an unlink U in $\mathbb{RP}^3$ with no crossings, that is, it is a disjoint union of local unknots. Pick a point p on the essential unknot U_1, which is disjoint from the link diagram D. The choice of the point p gives a way to view the diagram D as an n-n tangle T. We form the lifted link $\widetilde{U}$ as the closure of $T \circ F(T)$ as before. As discussed in Lemma 2.10, a twisted orientation on D induces an orientation on $\widetilde{U}$ which is reversed under τ, in particular each component of $\widetilde{U}$ is oriented.

Define the following $\mathbb{Z}/2$-valued functions on components of $\widetilde{U}$: Let

$$d : \pi_0(\widetilde{U}) \to \mathbb{Z}/2$$

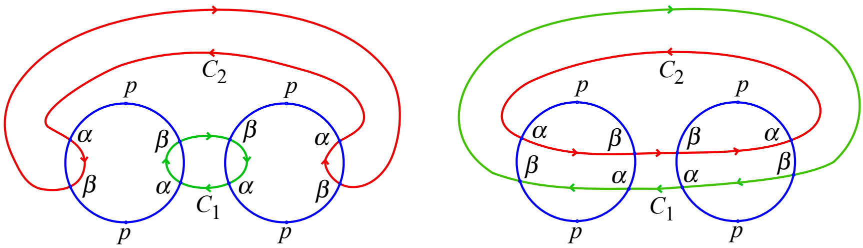

Figure 9. Two possibilities of the relative positions between C_1, C_2 and p.

be the number of circles in $\widetilde{L}$ separating the chosen component from infinity mod 2. Let

$$o : \pi_0(\widetilde{U}) \to \mathbb{Z}/2$$

equal 1 if the component is oriented counterclockwise, and 0 if it is oriented clockwise. Let

$$l : \pi_0(\widetilde{U}) \to \mathbb{Z}/2$$

be the sum of the above two functions mod 2:

$$l = d + o \text{ mod } 2.$$

Now we want to prove that l takes the same value on the two components in $\widetilde{U}$ that are mapped to the same component in U, so it descends to a map $l : \pi_0(U) \to \mathbb{Z}/2$.

Lemma 2.13. *Assume the same setting as in Definition 2.12. Suppose C_1 and C_2 are two components in $\widetilde{U}$ that are sent to the same component C in U under the quotient map $S^3 \to \mathbb{RP}^3$. Then*

$$l(C_1) = l(C_2).$$

Proof. By applying Reidemeister moves R-V away from the point p, we can assume that the local unknot C intersects the essential unknot U_1 at exactly two points, α and β. Then, there are two cases, depending on whether the point p lies inside the disk bounded by C, as shown in Figure 9. Later, when we refer to an arc in the proof, such as the arc $p\alpha$, we mean the open arc of U_1 traveling counterclockwise from p to α.

(1) In this case, the two components C_1 and C_2 are oriented in the same direction, so

$$o(C_1) = o(C_2).$$

The distance to infinity function $d(C_1)$ can be also be described by counting the number of intersections between $\widetilde{U}$ and the arc $p\alpha$ mod 2, and similarly, $d(C_2)$ counts the number of intersections between $\widetilde{U}$ and the arc βp mod 2. By the assumption that D is the diagram of an unlink, it is null homologous in particular.

Thus, the total number of intersections between $\widetilde{U}$ and the semicircle pp is even. Additionally, the link diagram D has no crossing, and the circle C_1 bounds a disk, so the number of intersections between $\widetilde{U}$ and the arc $\alpha\beta$ should also be even. Therefore, we conclude that

$$\#|\widetilde{U} \cap p\alpha| + \#|\widetilde{U} \cap \beta p| = 0 \bmod 2,$$

so

$$d(C_1) = \#|\widetilde{U} \cap p\alpha| = \#|\widetilde{U} \cap \beta p| = d(C_2) \bmod 2.$$

Hence, we have

$$l(C_1) = l(C_2) \bmod 2$$

in this case.

(2) This time, the orientations on C_1 and C_2 are opposite to each other, so

$$o(C_1) = o(C_2) + 1 \bmod 2.$$

Again, the distance function can be described in terms of the number of intersections between $\widetilde{U}$ and various arcs. In this case,

$$d(C_1) = \#|\widetilde{U} \cap p\alpha|, \quad d(C_2) = \#|\widetilde{U} \cap p\beta| = \#|\widetilde{U} \cap p\alpha| + 1 + \#|\widetilde{U} \cap \alpha\beta|.$$

By the assumptions on D that D has no crossings and each component of D is null homologous, we conclude that

$$\#|\widetilde{U} \cap \alpha\beta| = 0 \bmod 2,$$

so

$$d(C_1) = d(C_2) + 1 \bmod 2.$$

Therefore, we also have

$$l(C_1) = l(C_2) \bmod 2$$

in this case. □

With the help of the labeling function $l : \pi_0(L) \to \mathbb{Z}/2$, we can define our canonical generators associated with the twisted orientations.

Definition 2.14. Suppose D is a link diagram of a null homologous link L in $\mathbb{RP}^3$ with a twisted orientation o. Pick a point p on the essential unknot U_1 that is disjoint from the link diagram. Then, we define the *canonical element s_o associated with the twisted orientation o* in $\mathrm{CBN}^{\mathrm{un}}(D)$ as follows: First, form the twisted oriented resolution D_o according to o; then, apply the labeling function l to each component of the unlink D_o. Finally, assign the element $a = 1 + x$ to a component C in D_o if $l(C) = 0$, and $b = x$ to it if $l(C) = 1$.

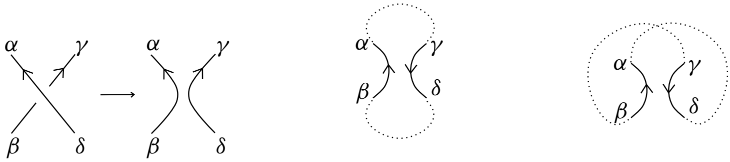

Figure 10. Two arcs sharing a crossing in the twisted oriented resolution won't belong to the same circle. Middle: each of the dotted arcs $\alpha\gamma$, $\beta\delta$ intersects U_1 an even number of times. Right: each of the dotted arcs $\alpha\gamma$, $\beta\delta$ intersects U_1 an odd number of times.

Remark 2.15. This definition depends on the choice of p in the following way: a different choice of p might result in a uniform change of $l(C)$ to $l(C)+1$ for each component C in D_o, leading to a switch of labels a and b. The point p serves as the point at infinity in the labeling function of link diagrams in $\mathbb{R}^2$.

As in the case of S^3, these canonical generators associated with the twisted orientations form a basis for the Bar-Natan homology. First, we will show that they actually lie in $\mathrm{HBN}^{\mathrm{un}}(L)$, and then we will prove that the dimensions match using an induction argument. The proof closely follows the original ones in [8] for Lee homology. See also [17] for a treatment of Bar-Natan homology (which is essentially the same as the one for Lee homology). The main difference in our case is that we need to take into account the 1-1 bifurcation and whether there is a homologically essential component in the link. These issues are resolved by the notion of twisted orientation.

Proposition 2.16. *For a null homologous link diagram D with a twisted orientation o, the canonical element s_o represents an element in the homology* $\mathrm{HBN}^{\mathrm{un}}(D)$.

Proof. Following Lee's argument, we aim to show that s_o lies in both $\ker(d)$ and $\ker(d^*)$, where d^* is the adjoint differential defined using the inner product on the Frobenius algebra V.

Let's take a closer look at what happens near a crossing. Without loss of generality, suppose the local picture near a crossing looks like the one in Figure 10. As in the usual proof for links in S^3, we want to show the two arcs $\alpha\beta$ and $\gamma\delta$ will not belong to the same circle in the resolution. In the usual proof, this possibility is ruled out because if these two arcs belong to the same circle, then the endpoint α would be adjacent to γ and β would be adjacent to δ in the circle due to the absence of crossings in the resolution. However, this arrangement is incompatible with the orientation on the arcs $\alpha\beta$ and $\gamma\delta$.

In our case, the situation is slightly different because the projection lies in $\mathbb{RP}^2$ instead of $\mathbb{R}^2$. We will again prove by contradiction that the arcs $\alpha\beta$ and $\gamma\delta$ will not belong to the same circle in the twisted oriented resolution. If they did, then there are two possibilities:

(1) If the endpoint α is adjacent to γ on the circle, then the arc $\alpha\gamma$ of the circle intersects the essential unknot U_1 an even number of times. Therefore, the twisted orientation on the circle will be from β to α and from γ to δ, or from α to β and from δ to γ, which is incompatible with the local orientation near the crossing.

(2) If the endpoint α is adjacent to δ on the circle, then the arc $\alpha\delta$ of the circle intersects the essential unknot U_1 an odd number of times. Therefore, the twisted orientation on the circle will be again from β to α and from γ to δ, or from α to β and from δ to γ, as the arrow is switched an odd number of times in the twisted orientation. But this is again incompatible with the local orientation near the crossing.

Therefore, we conclude that the arcs $\alpha\beta$ and $\gamma\delta$ near a crossing will not belong to the same circle in the twisted oriented resolution. This implies that if we change the smoothing at the crossing in the twisted orientation resolution, it will not result in a 1-1 bifurcation or a 1-2 bifurcation, which is exactly what we want to avoid, as the corresponding maps are nontrivial. The rest of the proof is exactly the same as in the usual case for links in S^3, as the rule we used to assign a and b to each circle in the resolution guarantees that the label assigned to the circle containing the arc $\alpha\beta$ is different from that assigned to the circle containing the arc $\gamma\delta$ in the resolution. Therefore, s_o lies in $\ker(d) \cap \ker(d^*)$. □

Now we prove a formula for the dimension of $\mathrm{HBN}^{\mathrm{un}}(D)$ using an induction argument on the number of crossings. There is an adjustment we need to make compared to the original argument in [8], as we need to discuss whether there is a homologically essential component in the link or not.

Proposition 2.17. *Suppose L is a null homologous link in $\mathbb{RP}^3$ with a link diagram D in $\mathbb{RP}^2$. Then*

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = \begin{cases} 0 & \text{if } L \text{ has some nonzero component in } H_1(\mathbb{RP}^3, \mathbb{Z}); \\ 2^{|L|} & \text{otherwise (if all components of } L \text{ are null homologous).} \end{cases}$$

Furthermore, a basis of $\mathrm{HBN}_{\mathrm{un}}(D)$ *is given by*

$$\{s_o \mid o \text{ is a twisted orientation of } L\}.$$

Remark 2.18. If L has a component which is nonzero in $H_1(\mathbb{RP}^3, \mathbb{Z})$, then there is no twisted orientation on L. Therefore, the set of basis elements given by twisted orientations is empty, and $\mathrm{HBN}^{\mathrm{un}}(L)$ is 0-dimensional in this case.

Proof. Following Lee's original proof, we first prove it for knots and 2-component links by induction on the number of crossings. It clearly holds for the unknot and the unlink with two local unknot components.

Now let D be a link diagram of a null homologous knot with n crossings. Pick one crossing, and let D_0, D_1 denote the 0- and 1-smoothing of D at this crossing,

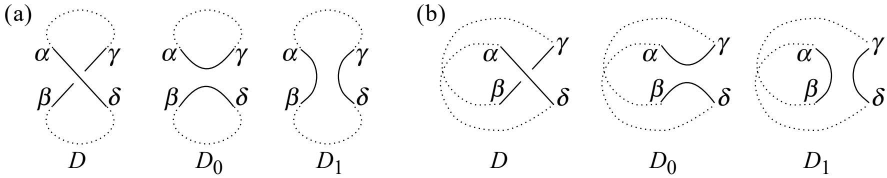

Figure 11. Resolve a null homologous knot/2-component link at a crossing.

respectively. Refer to (a) in Figure 11 as a local model near the crossing. Suppose, without loss of generality, that the endpoint α is adjacent to the endpoint γ on the knot K. (The case where α is adjacent to the endpoint β is the same, by changing the over-crossing to an under-crossing, which switches D_0 and D_1 but doesn't change the argument otherwise. The endpoint α won't be adjacent to δ, as that would give a 2-component link instead of a knot.) Then we divide into two cases depending on how many times the arc $\alpha\gamma$ of K intersects the essential unknot U_1.

(1) If the segment $\alpha\gamma$ intersects U_1 an odd number of times, so does the segment $\beta\delta$, since we assume K is null homologous. Then the link represented by D_0 consists of two homologically essential components, so by the induction hypothesis, it has trivial Bar-Natan homology. The diagram D_1 represents a null homologous knot with one fewer crossing than D, so by the induction hypothesis, we have

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) = 2.$$

Using the long exact sequence relating $\mathrm{HBN}^{\mathrm{un}}(D)$, $\mathrm{HBN}^{\mathrm{un}}(D_0)$, $\mathrm{HBN}^{\mathrm{un}}(D_1)$, we conclude that

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = 2.$$

(2) If the segment $\alpha\gamma$ intersects U_1 an even number of times, so does the segment $\beta\delta$. Then D_0 consists of two null homologous components and D_1 is a null homologous knot, so by the induction hypothesis,

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D_0) = 4, \quad \dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) = 2.$$

As in the usual case, out of the four twisted orientations on D_0, there are two that are compatible under the change of smoothings to twisted orientations on D_1, and the map in the long exact sequence will send these two twisted oriented generators in $\mathrm{HBN}^{\mathrm{un}}(D_0)$ to the corresponding ones in $\mathrm{HBN}^{\mathrm{un}}(D_1)$. Therefore,

$$2 \le \dim(\mathrm{HBN}^{\mathrm{un}}(D)) \le \dim(\mathrm{HBN}^{\mathrm{un}}(D_0)) + \dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) - 4 = 2.$$

Now suppose D is a diagram of a null homologous link L of two components, K_0 and K_1. Suppose first that there are no crossings between the two components in the link diagram D. Then K_0, K_1 must both be null homologous (otherwise, they would both be homologically essential, and there would have to be at least one crossing between these two components), and we can apply the Künneth formula to conclude

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = 4.$$

Suppose the two components share at least one crossing. Pick any crossing shared by them and form the 0- and 1-smoothing, D_0 and D_1, respectively. Refer to (b) in Figure 11 as a local model. We again divide into two cases depending on whether K_0 and K_1 are null homologous or not.

(1) If K_0, K_1 are both null homologous, then each of the segments $\alpha\delta$ and $\beta\gamma$ intersects the essential unknot U_1 an even number of times, and the two twisted orientations on D_0 are incompatible with the two twisted orientations on D_1 when changing the smoothing. Therefore, the map from $\mathrm{HBN}^{\mathrm{un}}(D_0)$ to $\mathrm{HBN}^{\mathrm{un}}(D_1)$ is 0 in the long exact sequence, and

$$4 \leq \dim(\mathrm{HBN}^{\mathrm{un}}(D)) \leq \dim(\mathrm{HBN}^{\mathrm{un}}(D_0)) + \dim(\mathrm{HBN}^{\mathrm{un}}(D_1)) = 4.$$

(2) If K_0, K_1 are both homologically essential, then each of the segments $\alpha\delta$ and $\beta\gamma$ intersects the essential unknot U_1 an odd number of times, and the two twisted orientations on D_0 are compatible with the two twisted orientations on D_1 when changing the smoothing. Therefore, the map in the long exact sequence sends $\mathrm{HBN}^{\mathrm{un}}(D_0)$ isomorphically to $\mathrm{HBN}^{\mathrm{un}}(D_1)$, and hence

$$\dim(\mathrm{HBN}^{\mathrm{un}}(D)) = 0.$$

This finishes the discussion of knots and 2-component links. For links with more components, if there is a component that doesn't share a crossing with any other components, then we can apply the Künneth formula. Otherwise, there is at least one crossing shared by different components. Again, we divide into two cases depending on whether there exists a homological essential component or not.

(1) Suppose all components are null homologous. Then we apply the same argument as in the case (1) for 2-component links.

(2) Suppose there are some homological essential components. Then, there must be at least two such components, and they must share a crossing in the link diagram. Choose one such crossing, and then we apply the same argument as in case (2) for 2-component links. □

It is time to apply the global grading shift to ensure that the Bar-Natan homology is a link invariant in $\mathbb{RP}^3$ as a bigraded vector space. The usual convention is to apply a shift of $-n_-$ in the homological grading and a shift of $n_+ + 2n_-$ in the quantum grading, where n_+ and n_- are the numbers of positive and negative crossings for an oriented link, respectively. As expected, we will obtain an invariant for twisted oriented links in $\mathbb{RP}^3$, and we should use n_+ and n_- counted with respect to the twisted orientation.

Definition 2.19. Let D be a link diagram in $\mathbb{RP}^2$ of a null homologous link in $\mathbb{RP}^3$ with a twisted orientation. Let n_+ and n_- denote the numbers of positive and negative crossings, respectively, counted with respect to the twisted orientation on D. Then the *Bar-Natan chain complex* $\mathrm{CBN}_{*,*}(D)$ is defined as

$$\mathrm{CBN}_{*,*}(D) = \mathrm{CBN}^{\mathrm{un}}_{*,*}(D)\{-n_-\}[n_+ - 2n_-],$$

where $\mathrm{CBN}^{\mathrm{un}}_{*,*}(D)$ is the unadjusted Bar-Natan chain complex in Definition 2.6, $\{-n_-\}$ means shifting down by n_- in the homological grading, and $[n_+ - 2n_-]$ means shifting up by $n_+ - 2n_-$ in the quantum grading. The homology $\mathrm{HBN}_{*,*}(D)$ of $\mathrm{CBN}_{*,*}(D)$ is called the *Bar-Natan homology* of null homologous links in $\mathbb{RP}^3$ with a twisted orientation.

We are going to prove that $\mathrm{HBN}_{*,*}$ is an invariant of twisted oriented null homologous links in $\mathbb{RP}^3$ by exhibiting filtered maps between $\mathrm{CBN}_{*,*}$ induced by Reidemeister moves, which are isomorphisms on $\mathrm{HBN}_{*,*}$. Furthermore, we are going to prove that these filtered maps send the canonical generators associated with the twisted orientation to the corresponding canonical generators. This follows the same strategy as in Section 6 in [15]. See also Section 6 in [17] for the Reidemeister moves in the usual Bar-Natan homology. We need to check a few additional cases because the projection lies in $\mathbb{RP}^2$, but there is no extra difficulty.

Proposition 2.20. *For each Reidemeister move as drawn in Figure 6 relating link projections D_0 to D_1, we can define a filtered chain map*

$$\rho' : \mathrm{CBN}_{*,*}(D_0) \to \mathrm{CBN}_{*,*}(D_1),$$

such that the grading-preserving part ρ of ρ' gives an isomorphism on the Khovanov homology for links in $\mathbb{RP}^3$. Additionally,

$$\rho'([s_o]) = [s_{o'}],$$

where o is a twisted orientation on D_0 and o' is the induced twisted orientation on D_1.

Proof. There are three things to do in the proof: first, we need to give the definition of the filtered chain map ρ'; second, we need to check that the grading-preserving part ρ agrees with the one used in [6] for Reidemeister moves on Khovanov homology in $\mathbb{RP}^3$; third, we need to check whether ρ' sends canonical generators to canonical generators. As mentioned, the chain map ρ' we are going to use will be the same as the ones in [15], and the proof is purely a bookkeeping check. The situations for Reidemeister moves I, IV, and V are either trivial or identical to the case in S^3. However, for Reidemeister moves II and III, they will involve some

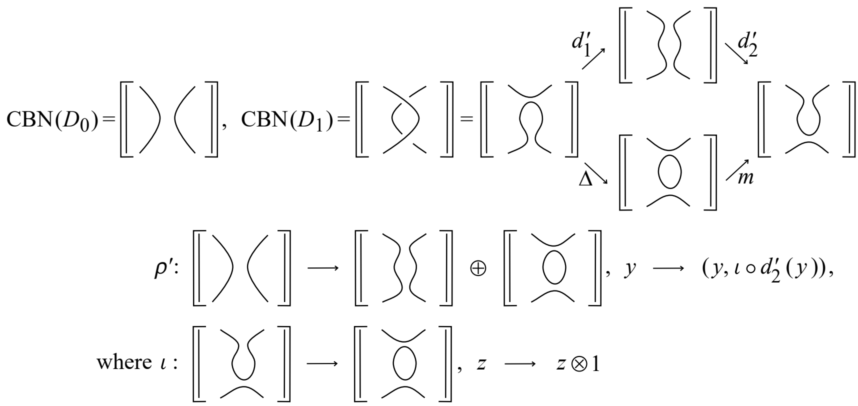

Figure 12. Definition of ρ' for R-II moves.

more case-by-case analysis than the usual situation in S^3. We will provide some examples and leave the rest to the reader.

For Reidemeister moves IV and V, they don't change the chain complex at all, and we can take ρ' to be the identity. It is worth mentioning that for the Reidemeister move V, as we move an arc through the essential unknot U_1, the orientations on this arc in o and o' will be reversed to each other.

For Reidemeister move I, since the operation of adding a curl is local, and the chain maps only involve 1-2 and 2-1 bifurcations, the same map and argument as in [15] work, with $a = 1 + x$, $b = x$. In terms of the global grading shift, if we add a positive curl, we also add 1 to n_+ for our definition of n_+. Therefore, the usual check of global grading shifts works in the same way.

For Reidemeister move II, refer to Figure 12 for the definition of the map ρ'. Here, ι is the map sending the state z to $z \otimes 1$, where 1 is assigned to the extra circle in the middle. In the chain complex CBN(D_1), we know the maps Δ and m because they correspond to a local splitting/merging of a circle. However, we don't know the maps d'_1 and d'_2: they could be any of Δ, m, or f depending on how the rest of the link diagram looks. But the point is we don't need to use their properties in the definition of ρ'. The grading-preserving part ρ of ρ' gives an isomorphism on Khovanov homology in $\mathbb{RP}^3$, using the usual proof of canceling acyclic complexes (note that we only use the property of Δ and m in the proof for invariance of Khovanov homology in $\mathbb{RP}^3$).

To check that ρ' sends canonical generators to canonical generators, we need to further divide into cases, depending on how the two arcs are connected and oriented in the link diagram.

(1) If the two arcs are oriented in the same direction, then, by the argument in Proposition 2.16, the two arcs will not belong to the same circle in the twisted

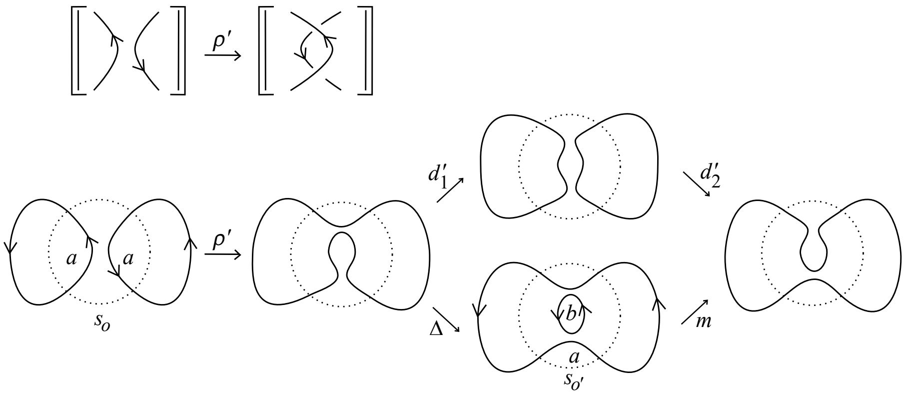

Figure 13. Schematic drawing for case (2)(a) of R-II moves.

oriented resolution, and the labels on these two circles in s_o are different. In this case, d'_2 is a 2-1 bifurcation map, which is given by m, and $m(s_o) = 0$, since it merges two circles with different labels. Therefore,

$$\rho'(s_o) = (s_o, \iota \circ d'_2(s_o)) = (s_o, 0) = s_{o'}.$$

(2) If the two arcs are oriented in opposite directions, we divide into cases depending on whether these two arcs belong to the same circle or not in the twisted oriented resolution.

(a) If the two arcs belong to two different circles in the twisted oriented resolution, then the labels on the two circle will be the same in s_o; say, both of them are a, following the rules we assign labels in Definition 2.14. See Figure 13 for an illustration. Then, d'_1 is a 1-2 bifurcation map, and d'_2 is a 2-1 bifurcation map. Denote s_o by $a \otimes a$, which is the label of s_o on the two circle that are changing through the Reidemeister move II. Then

$$\iota \circ d'_2(a \otimes a) = \iota(a) = a \otimes 1 = a \otimes a + a \otimes b,$$

and

$$\rho'(a \otimes a) = (a \otimes a, a \otimes a + a \otimes b).$$

Note that

$$da = (d'_1(a), \Delta(a)) = (a \otimes a, a \otimes a),$$

so

$$[\rho'(s_o)] = [(0, a \otimes b)] = [s_{o'}] \text{ in } \mathrm{HBN}(D_1).$$

(b) If the two arcs belong to the same circle in the twisted oriented resolution, then again, we need to divide into two cases, depending on whether a 1-1 bifurcation is involved or not.

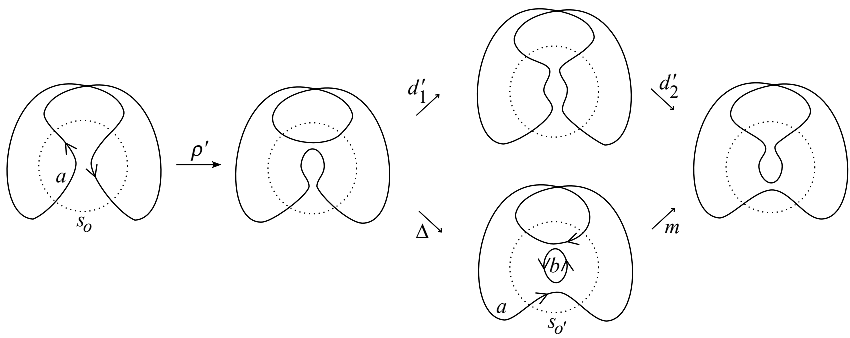

Figure 14. Schematic drawing for case (2)(b)(i) of R-II moves.

(i) Suppose both d_1' and d_2' are 1-1 bifurcations. Let's assume the label on the this circle in s_o is a. See Figure 14 for an illustration. Following the similar notation, we denote s_o by a, and we have

$$\iota \circ d_2'(a) = \iota(a) = a \otimes 1 = a \otimes a + a \otimes b,$$

so

$$\rho'(a) = (a, \iota \circ d_2'(a)) = (a, a \otimes a + a \otimes b).$$

Again, we can cancel $(a, a \otimes a)$ in homology, since

$$da = (d_1'(a), \Delta(a)) = (a, a \otimes a),$$

so

$$[\rho'(s_o)] = [(0, a \otimes b)] = [s_{o'}] \quad \text{in } \mathrm{HBN}(D_1).$$

(ii) Suppose that neither d_1' nor d_2' is a 1-1 bifurcation. (Note that d_1' is a 1-1 bifurcation if and only if d_2' is.) Then, it is the same as the usual check for the Reidemeister move II as in S^3, and we leave it to the reader.

For Reidemeister move III, the definition of the map ρ' is again similar to the corresponding one for Reidemeister move III in S^3, for the same reason as mentioned above: the edge maps, which we need some properties of to define ρ', are local, i.e., they only involve the splitting/merging of a local circle. The proof that the grading-preserving part of ρ' induces an isomorphism on Khovanov homology follows the same approach as in the case of S^3, as the proof of invariance of Khovanov homology under Reidemeister move III in $\mathbb{RP}^3$ is carried out similarly to that in S^3.

To verify that it sends canonical generators to canonical generators, we need to consider different cases depending on how each strand is oriented and how these strands are connected outside the local region in the twisted oriented resolution. This requires more case analysis than the proof in S^3, because the link diagram lies in $\mathbb{RP}^2$. We will illustrate one example and leave the rest to the reader.

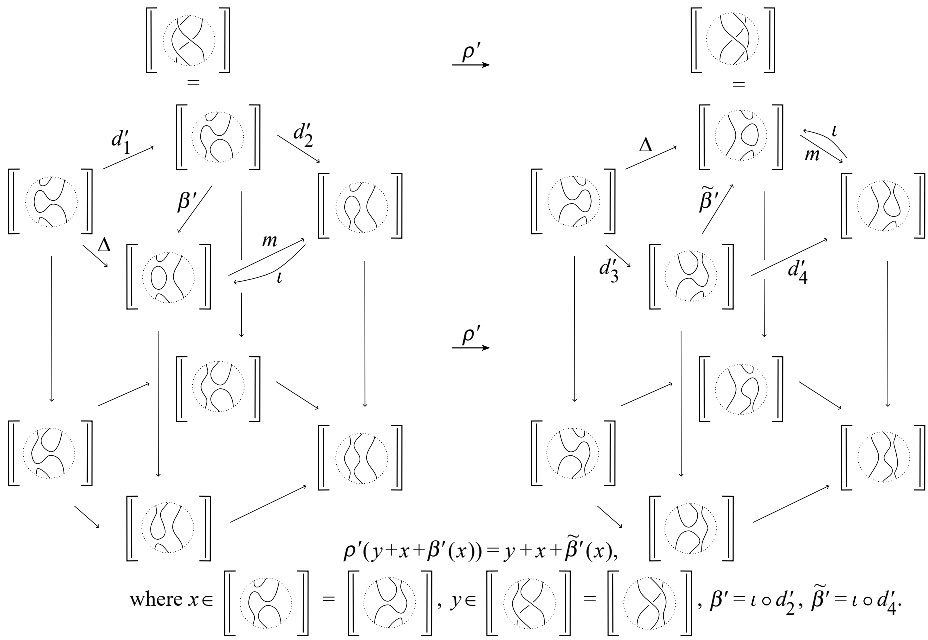

Figure 15. Definition of ρ' for R-III moves.

The definition of the map $\rho' : \mathrm{CBN}(D_0) \to \mathrm{CBN}(D_1)$ is the same as that in [15], which is recalled in Figure 15. The grading-preserving part of ρ' gives the induced map by Reidemeister move III on the Khovanov homology in $\mathbb{RP}^3$, as discussed above. Therefore, it remains to show

$$[\rho(s_o)] = [s_{o'}] \quad \text{in } \mathrm{HBN}(D_1).$$

As in the proof in [15], three out of the four situations regarding relative orientations lead to trivial checks. For the remaining relative orientation, there are several more cases to examine than in [15], as the diagram lies in $\mathbb{RP}^2$. In these additional cases, some of the d'_1, d'_2, d'_3, and d'_4 will be the 1-1 bifurcation map. As an example, we consider the following orientation on the strands and connectivity in the twisted oriented resolution, as shown in Figure 16. Note that the rule that we use to assign labels a or b to each circle depends on the choice of the point p. However, changing the position of p results in a uniform switch between a and b, so it does not affect the argument. Here, we demonstrate one possibility of the labeling for a specific choice of p lying in the indicated region in the diagram.

In this case, d'_1 and d'_2 are 1-1 bifurcations, d'_3 is a 1-2 bifurcation, and d'_4 is a 2-1 bifurcation. Let x be as shown in Figure 16. Then, by the computation in Figure 17,

$$[s_o] = [x + \beta'(x)] \quad \text{and} \quad [s_{o'}] = [x + \widetilde{\beta}'(x)].$$

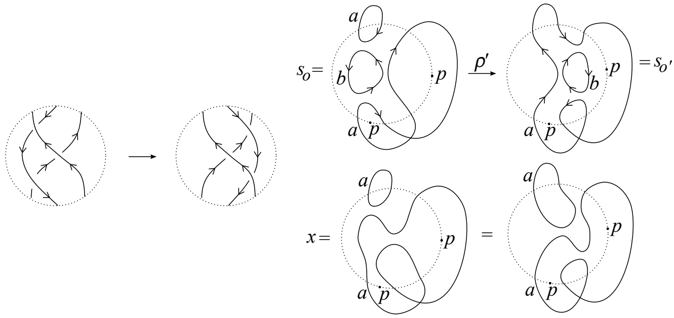

Figure 16. An example computation of ρ' for a R-III move (left), with given orientation and connectivity.

Therefore,

$$\rho'([s_o]) = \rho'([x+\beta(x)]) = [x+\widetilde{\beta}'(x)] = [s_{o'}] \quad \text{in } \mathrm{HBN}(D_1). \qquad \square$$

Combining all the discussions in this section, we obtain a well-defined Bar-Natan homology for twisted oriented null homologous links in $\mathbb{RP}^3$.

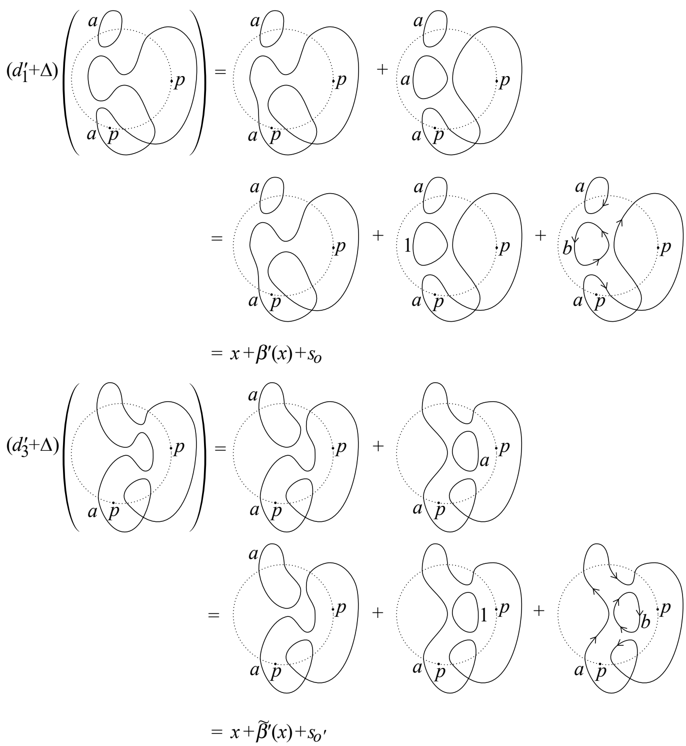

Figure 17. The homologous relation between $x+\beta'(x)$ with s_0, and between $x+\widetilde{\beta}'(x)$ with $s_{o'}$.

Theorem 2.21. *The Bar-Natan homology* $\mathrm{HBN}(L)$ *is a twisted oriented link invariant for null homologous links in* $\mathbb{RP}^3$ *as a bigraded vector space, with a canonical basis given by*

$$\{s_o \mid o \text{ is a twisted orientation of } L\}.$$

Remark 2.22. When L is a null homologous knot, the usual proof for knot in S^3, showing that s_o lies in homological grading 0, works here as well. This is because we use the twisted orientation to define n_+, n_-, and to form the resolution. For links, the homological grading of s_o can be determined from the linking number between components of the covering link $\widetilde{L}$ in S^3, with the orientation on $\widetilde{L}$ that lifts the twisted orientation.

3. Cobordism maps on Bar-Natan homology in $\mathbb{RP}^3$

In the usual setting for Bar-Natan or Lee homology in S^3, the reason one can get some genus bound from the homology is that one can associate some filtered chain maps to oriented cobordisms between links, which interact well with the canonical generators of the homology given by the orientations, and the filtration degree of the map is bounded by the Euler characteristic of the cobordism. The guiding principle of this paper is to replace "orientation" with "twisted orientation" throughout. Consequently, we need to develop a notion of twisted orientation on cobordisms. The definition of twisted orientable cobordism should ensure that it carries the twisted orientation from one end of the cobordism to the other end. Specifically, when the cobordism is formed by attaching bands, the process should be compatible with the twisted orientation. More formally, we provide the following definition from the perspective of double covers in S^3.

Definition 3.1. Let $\Sigma : L \to L'$ be a cobordism between null homologous links L and L' in $\mathbb{RP}^3 \times I$. A *twisted orientation* on Σ is defined as an orientation on its double cover $\widetilde{\Sigma} : \widetilde{L} \to \widetilde{L}'$ in $S^3 \times I$, which is reversed by the deck transformation $\tau : S^3 \times I \to S^3 \times I$ along the S^3-direction. The cobordism Σ is said to be *twisted orientable* if such a twisted orientation exists.

In particular, the double cover $\widetilde{\Sigma}$ must be orientable if Σ is twisted orientable. This definition is natural in light of the definition of twisted orientation on null homologous links in Definition 2.9.

Note that a twisted orientable cobordism in $\mathbb{RP}^3 \times I$ can be either orientable or unorientable in the usual sense, and there are orientable/unorientable cobordisms which are not twisted orientable, as demonstrated in the next example.

Example 3.2. Figure 18 illustrates examples of all four combinations of orientability and twisted orientability of surface cobordism in $\mathbb{RP}^3 \times I$.

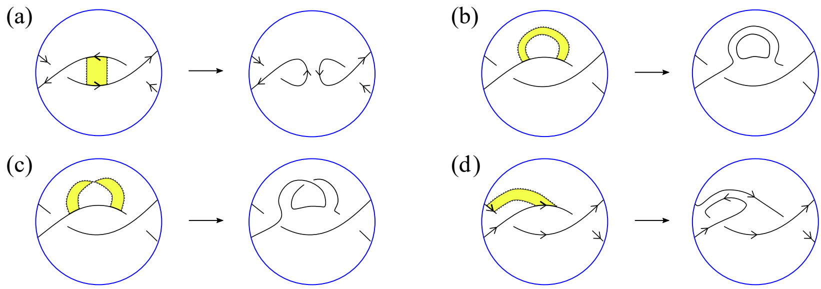

Figure 18. Different possibilities of twisted-orientability and usual orientability of a band attachment. Top left: unorientable and twisted orientable. Top right: orientable and twisted orientable. Bottom left: unorientable and not twisted orientable. Bottom right: orientable and not twisted orientable.

The cobordism in (a) is twisted orientable, with the arrows indicating twisted orientations on the boundary knots. The band is added in a manner compatible with the twisted orientations. However, this cobordism is unorientable in the usual sense, as it is topologically a Möbius band with a disk removed.

The cobordism in (b) is both twisted orientable and orientable in the usual sense.

The cobordism in (c) is neither twisted orientable nor orientable in the usual sense.

The cobordism in (d) is not twisted orientable. Here, the arrows represent usual orientations on the boundary knots, and the band is added in a way compatible with these orientations, making this cobordism orientable in the usual sense.

Now, we will prove an analogous statement about the interaction between twisted orientable cobordisms and the canonical generators of Bar-Natan homology generated by twisted orientations. The idea is to decompose the cobordism Σ into elementary pieces, corresponding to Reidemeister moves and a single handle attachment. To achieve this, we first prove that the twisted orientation on Σ descends to a twisted orientation on each elementary piece, as well as its input and output boundaries.

Lemma 3.3. *Suppose $\Sigma : L \to L'$ is a twisted orientable cobordism between null homologous links L and L' in $\mathbb{RP}^3 \times I$. We have the decomposition*

$$\Sigma = \Sigma_1 \circ \Sigma_2 \circ \cdots \circ \Sigma_n,$$

where each Σ_i corresponds to either a Reidemeister move or a single handle attachment. Let L_i denote the output boundary of the composition $\Sigma_1 \circ \Sigma_2 \circ \cdots \circ \Sigma_i$, with $L_0 = L$ and $L_n = L'$. Then, a twisted orientation on Σ restricts to a twisted orientation on each Σ_i and L_i.

Proof. Consider the double cover of the decomposition

$$\widetilde{\Sigma} = \widetilde{\Sigma_1} \circ \widetilde{\Sigma_2} \circ \cdots \circ \widetilde{\Sigma_n},$$

where each $\widetilde{\Sigma_i}$ corresponds to performing a pair of equivariant Reidemeister moves, or attaching a pair of equivariant handles. A twisted orientation on Σ is an orientation on $\widetilde{\Sigma}$ which is reversed under the action of the deck transformation τ in the S^3-direction. Since the action of τ is fiberwise in the S^3 direction of $S^3 \times I$, such an orientation on $\widetilde{\Sigma}$ restricts to an orientation on each $\widetilde{\Sigma_i}$ which is also reversed by τ. Consequently, the twisted orientation on Σ restricts to each Σ_i.

As for the links L_i, we again look at the double cover $\widetilde{L_i}$. Since these are regular fibers of the projection from $\widetilde{\Sigma}$ to I, a tubular neighborhood of $\widetilde{L_i}$ in $\widetilde{\Sigma}$ is homeomorphic to $\widetilde{L_i} \times [-\epsilon, \epsilon]$, where τ acts trivially in the $[-\epsilon, \epsilon]$-direction, as τ is a fiberwise action. However, τ acts in an orientation-reversing manner on the neighborhood $\widetilde{L_i} \times [-\epsilon, \epsilon]$, so it must reverse the orientation on $\widetilde{L_i}$. Hence, we obtain a twisted orientation on L_i. □

Remark 3.4. In particular, the above lemma implies that every component of L_i is null homologous, since L_i is twisted orientable. Hence, $\mathrm{HBN}(L_i)$ is nontrivial for each i. Note that not all closed loops in Σ are null homologous; for example, the fiber over the critical value of a 1-handle attachment could have a tubular neighborhood homeomorphic to a Möbius band, as seen in (a) of Figure 18. In this case, the action of τ on a tubular neighborhood $\widetilde{L} \times [-\epsilon, \epsilon]$ is the usual covering map from an annulus to a Möbius band, which is nontrivial in the $[-\epsilon, \epsilon]$-direction and orientation-preserving when restricted to $\widetilde{L}$.

Now, we can prove the analogous statement about the effect of the maps induced by twisted orientable cobordisms on Bar-Natan homology in $\mathbb{RP}^3$.

Proposition 3.5. *Suppose $\Sigma : L \to L'$ is a twisted orientable cobordism between null homologous links L and L' in $\mathbb{RP}^3 \times I$. Then, one can define a filtered chain map of degree $\chi(\Sigma)$,*

$$F_\Sigma : \mathrm{CBN}_{*,*}(L) \to \mathrm{CBN}_{*,*}(L'),$$

such that

$$F_\Sigma([s_o]) = \sum_{\{o_i\}} [s_{o_{i_{|L'}}}], \tag{$*$}$$

where o is a twisted orientation on L, $\{o_i\}$ is the set of twisted orientations on Σ that restrict to o on L, $o_{i_{|L'}}$ is the restriction of such twisted orientations on L', and s_o (respectively $s_{o_{i_{|L'}}}$) are the corresponding canonical generators of $\mathrm{HBN}(L)$ (*respectively* $\mathrm{HBN}(L')$) *defined as in Definition 2.14.*

Proof. As in the proof of the analogous statement in [15], we divide the cobordism Σ into elementary pieces, define the map for each piece, and verify that the desired properties hold for each piece. The above lemma shows that each elementary piece of Σ is twisted orientable between twisted orientable links. See also Theorem 5.5 in [13] for a similar statement in the context of Lee homology in $\mathbb{RP}^3$.

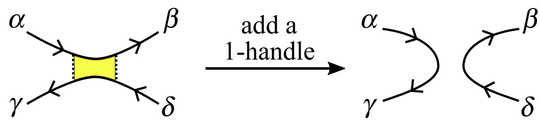

Figure 19. Local picture near a 1-handle attachment.

Recall Definition 2.12, where we describe the rules for assigning a and b to each circle in the twisted oriented resolution for the definition of s_o. In doing so, we need to choose a point p on the essential unknot U_1, which is away from the link diagram. Similarly, here we will choose a point p, such that $p \times I \subset \mathbb{RP}^3 \times I$ is away from the cobordism Σ. This chosen p will serve as the reference point when discussing the canonical generators in $\mathrm{HBN}(L_i)$ for each L_i.

The cobordism Σ can be expressed as the composition of a sequence of Reidemeister cobordisms and elementary Morse cobordisms. For the Reidemeister cobordisms, we will use the map ρ' defined in Proposition 2.20. Each of these Reidemeister cobordisms is topologically a cylinder with Euler characteristic 0, and there is a unique twisted orientation on it that extends the given twisted orientation on the input. Proposition 2.20 shows that these maps satisfy $(*)$.

Elementary Morse cobordisms are given by attaching 0-, 1-, or 2-handles. For the 0- and 2-handle attachments, which are local, we use the unit and counit maps:

$$\iota : \mathbb{F} \to V, \quad \eta : V \to \mathbb{F}$$

in the Frobenius algebra structure of V, as defined in Definition 2.1. The twisted orientability condition imposes no further restrictions on 0- and 2-handle attachments, as they correspond to attaching a pair of disjoint 0- and 2-handles in the double cover, where τ acts by switching the two copies. Such cobordisms are always twisted orientable, with a twisted orientation being a pair of opposite orientations on the two copies of the handles in the double cover. It is trivial to check that the desired properties hold for these cobordism maps, as in the usual case for cobordisms in $S^3 \times I$.

For the 1-handle attachments, we will use m, Δ, and f on each smoothing of the cube of resolutions, depending on whether it is a 2-1, 1-2, or 1-1 bifurcation in each resolution. Each of m, Δ, and f is a filtered map of degree -1, and the proof that such a definition gives a chain map is the same as checking $d^2 = 0$ in the Bar-Natan chain complex, which is covered in Lemma 2.4 and the discussion following it.

It remains to show that the canonical generators associated with twisted orientations behave properly under such maps. Again, the reason this works is that we attach 1-handles that are compatible with the twisted orientation. In particular, homologically essential knots are not created during the process of attaching 1-handles. More explicitly, the twisted orientations on the two arcs where we attach the 1-handle are as shown in Figure 19. Adding the 1-handle changes a twisted

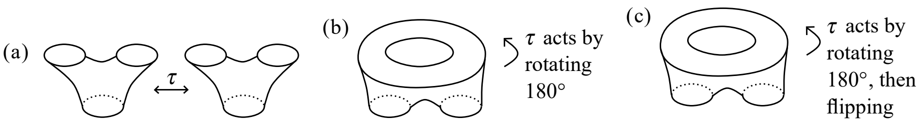

Figure 20. Action of τ on a pair of equivariant 1-handles.

oriented resolution into another twisted oriented resolution. There are three possible cases for how these two arcs are connected in the twisted orientation resolution:

(1) The two arcs $\alpha\beta$ and $\gamma\delta$ belong to the same circle in the twisted orientation resolution, and α is adjacent to γ on the circle. In this case, the induced map by the 1-handle attachment is Δ, such that

$$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b,$$

which proves that $F_\Sigma([s_o]) = [s_{o'}]$ in this case.

(2) The two arcs $\alpha\beta$ and $\gamma\delta$ belong to the same circle in the twisted orientation resolution, but α is adjacent to δ on the circle. In this case, the induced map by the 1-handle attachment is f, such that

$$f(a) = a, \quad f(b) = b,$$

so $F_\Sigma([s_o]) = [s_{o'}]$ as well.

(3) The two arcs $\alpha\beta$ and $\gamma\delta$ belong to different circles in the twisted oriented resolution. The labels on these two circles in s_o will be the same, by the rule in Definition 2.12. In this case, the induced map by the 1-handle attachment is m, such that

$$m(a \otimes a) = a, \quad m(b \otimes b) = b,$$

so $F_\Sigma([s_o]) = [s_{o'}]$.

In terms of the set of twisted orientations on Σ_i extending the given one on the boundary, we note the following. If Σ_i represents a 1-handle attachment that splits one circle into two circles, then it is topologically a pair-of-pants. A priori, there are two different possibilities for the covering $\widetilde{\Sigma_i}$ as shown in Figure 20(a) and (b). It is either a disjoint union of two pairs-of-pants where the action τ switches the two components, as shown in (a), or an annulus with two disks removed, as shown in (b), where the action of τ is a rotation by 180°. However, τ acts in an orientation-preserving way in case (b), so this possibility is ruled out by the assumption that Σ is twisted orientable. The situation for a 1-handle attachment that merges two circles into one circle is exactly the same, by turning the cobordism upside down. For a 1-handle attachment that twists one circle into another, the covering Σ_i is again an annulus with two disks removed, as shown in (c) of Figure 20, while the action τ is different: it is the usual covering map from an annulus to a Möbius band.

After this clarification, the rest of the proof follows exactly in the same way as in [15]. For Σ_i representing a 1-handle attachment that splits a circle into two or twists a circle, there is a unique twisted orientation on Σ_i that extends the twisted orientation on the input boundary. If Σ_i represents a 1-handle attachment merging two circles into one, then depending on which components these two circles belong to, either all twisted orientations on the input have a unique extension to Σ_i, or half of them have a unique extension. In the latter case,

$$F_{\Sigma_i}(s_o) = 0,$$

for twisted orientations o on the input which do not extend to Σ_i. □

Remark 3.6. We won't discuss the functoriality issue of cobordism maps in this paper, i.e., whether the map F_Σ is well defined up to filtered chain homotopy. The existence of such a map F_Σ with the stated property is enough to prove some genus bound.

If we restrict to connected cobordisms between null homologous knots, we obtain the following immediate corollary, which is what we need for the genus bound.

Corollary 3.7. *If L and L' are both null homologous knots and Σ is a connected, twisted orientable cobordism between them in $\mathbb{RP}^3 \times I$, then*

$$F_\Sigma([s_o]) = [s_{o'}],$$

where o' is the restriction of the unique twisted orientation on Σ that extends o, and

$$F_\Sigma : \mathrm{HBN}(L) \to \mathrm{HBN}(L')$$

is an isomorphism of filtration degree $\chi(\Sigma)$.

4. The s-invariant and genus bound

In the previous two sections, we have proven all the formal properties of the Bar-Natan homology required to define the s-invariant and bound slice genus. We summarize the results in this section, following exactly the same procedure as in the case of Bar-Natan or Lee homology in S^3. The only difference is that the class for which we obtain a genus bound is the class of twisted orientable slice surfaces of null homologous knot in $\mathbb{RP}^3 \times I$, rather than orientable slice surfaces. The treatment here closely follows the summary of the s-invariant for Bar-Natan homology in S^3 in Section 2 of [9].

Suppose L is a twisted oriented null homologous link in $\mathbb{RP}^3$. Denote its Bar-Natan chain complex by $C_{*,*}(L) = \mathrm{CBN}_{*,*}(L)$. Consider this finite-length filtration by the quantum grading on $C(L)$:

$$0 \subset \cdots \subset \mathcal{F}_{q+1}C(L) \subset \mathcal{F}_q C(L) \subset \mathcal{F}_{q-1}C(L) \subset \cdots \subset C(L),$$

where $\mathcal{F}_q C(L) = \bigoplus_{j \geq q} C_{*,j}(L)$. Note that due to the possible existence of 1-1 bifurcation in the edge map, which twists one circle into another circle, the parity of the quantum grading is no longer the same on the chain complex $C_{*,*}(L)$. Therefore, we increase the filtration degree q by 1 at each step, instead of by 2 as in the case of Bar-Natan chain complex for links in S^3. The differential map of the chain complex C respects the filtration in the quantum grading, as in the usual case for m, Δ, and we choose the map $f : V \to V[1]$ to be filtered in its definition, so each $\mathcal{F}_q C(L)$ is a subcomplex.

Definition 4.1. Let K be a twisted oriented null homologous knot in $\mathbb{RP}^3$. Define

$$s_{\min}(K) = \max\{q \in \mathbb{Z} \mid i_* : H(\mathcal{F}_q C(K)) \to H(C(K)) \cong \mathbb{F}^2 \text{ is surjective}\},$$
$$s_{\max}(K) = \max\{q \in \mathbb{Z} \mid i_* : H(\mathcal{F}_q C(K)) \to H(C(K)) \cong \mathbb{F}^2 \text{ is nonzero}\}.$$

Lemma 4.2. *We have* $s_{\max}(K) = s_{\min}(K) + 2$.

Proof. The same argument as in Proposition 2.6 of [9] applies here with a slight modification, so we provide only a sketch.

Consider the involution $I : C(K) \to C(K)$, which is induced by this involution on the Frobenius algebra V:

$$I(a) = b, \quad I(b) = a.$$

In terms of the basis $\{1, x\}$, this is

$$I(1) = 1, \quad I(x) = 1 + x.$$

Thus, I induces the identity map on the associated graded complex.

Choose a cycle $y \in \mathcal{F}_{s_{\min}(K)} C(K)$, such that $\{y, I(y)\}$ forms a basis of $H(C(K))$. Since the lowest grading parts of a and $I(a)$ are the same, and the grading non-preserving part of the map I on $C(K)$ raises the grading by at least 2, we have $y + I(y) \in \mathcal{F}_{s_{\min}(K)+2} C(K)$. Hence,

$$s_{\max}(K) \geq s_{\min}(K) + 2.$$

The proof of the inequality in the other direction follows exactly the same argument as for the s-invariant of the usual Bar-Natan homology, since taking the connected sum with a local unknot is a local operation, and the Bar-Natan chain complex behaves in the same way as in the case of S^3 with respect to this local operation. □

Hence, we can define the Bar-Natan s-invariant for null homologous knot in $\mathbb{RP}^3$ as usual.

Definition 4.3. Let K be a null homologous knot in $\mathbb{RP}^3$. The *Bar Natan s-invariant* of K, is defined as

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = \tfrac{1}{2}(s_{\min}(K) + s_{\max}(K)).$$

It is well defined, as we have verified that the maps induced by Reidemeister moves are filtered maps of filtration degree 0 in Proposition 2.20. Additionally, it does not depend on the twisted orientation on K, since the effect of reversing the twisted orientation is the same as switching a and b.

This s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ satisfies similar properties to those of the usual s-invariants. We summarize some of them here. Compare also with the corresponding statements about the s-invariant defined using the Lee deformation in [13].

Proposition 4.4. *If K is a local knot in $\mathbb{RP}^3$, that is, it is contained in some ball B^3 in $\mathbb{RP}^3$, then*

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = s^{\mathrm{BN}}(K),$$

where $s^{\mathrm{BN}}(K)$ denotes the s-invariants from the Bar-Natan homology for knots in S^3 over the field $\mathbb{F} = \mathbb{F}_2$.

Proof. A local knot is, in particular, null homologous, so we can define $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ for it. For a local knot, the notion of twisted orientation agrees with the usual notion of orientation, as we can draw a knot diagram for it that does not intersect the essential unknot U_1 at all, so no reversal of the arrow is needed. Therefore, the notions of n_+ and n_- with respect to the twisted orientation agree with the usual notions of n_+ and n_-. Additionally, there will be no 1-1 bifurcation appearing in the Bar-Natan chain complex for the knot diagram away from U_1. Thus, the notion $s^{\mathrm{BN}}_{\mathbb{RP}^3}$ exactly matches with the usual notion of $s^{\mathrm{BN}}(K)$ when viewing K as a knot in S^3. □

Proposition 4.5. *Let $m(K)$ be the mirror of a null homologous knot in $\mathbb{RP}^3$, i.e., it is obtained by switching positive crossings with negative crossings in a knot diagram of K. Then,*

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(m(K)) = -s^{\mathrm{BN}}_{\mathbb{RP}^3}(K).$$

Proof. The usual argument, as in Proposition 3.9 of [15], using the dual chain complex works here as well, with the dual map of $f = \mathrm{id}_V$ being the identity $f^* = \mathrm{id}_{V^*}$ on V^*. □

Proposition 4.6. *If K is a null homologous knot in $\mathbb{RP}^3$ and K_l is a local knot in $\mathbb{RP}^3$, then we can form the connected sum*

$$K \# K_l \subset \mathbb{RP}^3 \# S^3 \cong \mathbb{RP}^3,$$

and we have

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K \# K_l) = s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) + s^{\mathrm{BN}}(K_l),$$

where again $s^{\mathrm{BN}}(K_l)$ is the s-invariant of K_l using the Bar-Natan homology in S^3 over the field $\mathbb{F} = \mathbb{F}_2$.

Proof. The same proof as in Proposition 3.11 of [15] works. Note that a twisted orientation on K and an orientation on K_l gives a twisted orientation on $K \# K_l$, provided they are compatible on the two arcs where the connected sum is performed. □

Proposition 4.7. *If K is positive with respect to the twisted orientation, that is, if $n_- = 0$ as counted using a twisted orientation on K, then we have the usual formula for its $s^{\mathrm{BN}}_{\mathbb{RP}^3}$-invariant:*

$$s^{\mathrm{BN}}_{\mathbb{RP}^3}(K) = -k + n + 1,$$

where n is the number of crossings of K, and k is the number of circles in the twisted oriented resolution of K.

Proof. Exactly the same proof as in Section 5.2 of [15] works as well, with the word "orientation" replaced everywhere by "twisted orientation". □

Now, we discuss the genus bound that can be obtained using the s-invariant $s^{\mathrm{BN}}_{\mathbb{RP}^3}$.

Definition 4.8. Let K be a null homologous knot in $\mathbb{RP}^3$. A surface Σ in $\mathbb{RP}^3 \times I$ is a *twisted orientable slice surface* of K if it is connected, twisted orientable, and satisfies

$$\partial\Sigma = \Sigma \cap (\mathbb{RP}^3 \times \{0\}) = K.$$

The most straightforward result that can be written down is in terms of the Euler characteristic.

Proposition 4.9. *If Σ is a twisted orientable slice surface of a null homologous knot K in $\mathbb{RP}^3$, then*

$$-\chi(\Sigma) \geq |s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)|.$$

Proof. It is a straightforward corollary of Corollary 3.7 applied to cobordisms from K to the trivial unknot U, and the fact $s^{\mathrm{BN}}_{\mathbb{RP}^3}(U) = 0$. Check Corollary 2.7 in [9] for a detailed explanation. □

The genus bound will depend on whether the twisted orientable slice surface is orientable in the usual sense, as the formulas for the Euler characteristic from the genus differ in these two case.

Corollary 4.10. *If Σ is a twisted orientable slice surface of a null homologous knot K in $\mathbb{RP}^3$, then*

$$g(\Sigma) \geq \begin{cases} |s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)| & \textit{if } \Sigma \textit{ is unorientable;} \\ \frac{1}{2}\,|s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)| & \textit{if } \Sigma \textit{ is orientable.} \end{cases}$$

As the class of twisted orientable slice surfaces is different from the usual notion of slice genus in $\mathbb{RP}^3 \times I$, it is natural to expect that this $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ provides different information than the s-invariant defined for knots in $\mathbb{RP}^3$ using the Lee deformation, as in [13]. We illustrate the difference in the following example.

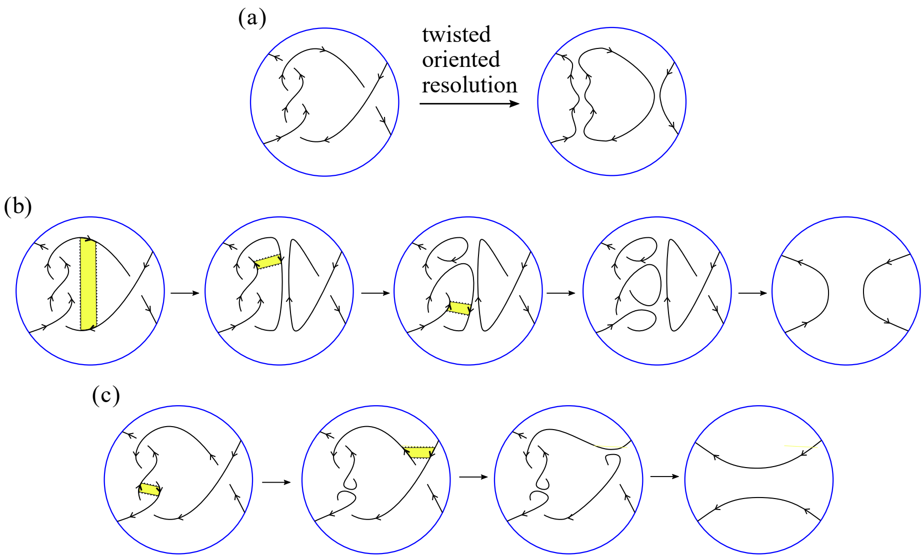

Figure 21. An example of difference in twisted orientable slice surface and usual orientable slice surface.

Example 4.11. Consider the following null homologous knot K as drawn in (a) of Figure 21. It is a positive knot with respect to the twisted orientation, with four crossings, and two circles in the twisted oriented resolution. By Proposition 4.7, we have

$$s^{\mathrm{BN}}_{\mathbb{RP}^3} = -2 + 4 + 1 = 3.$$

A twisted orientable slice surface of K with $\chi = -3$ is drawn in (b) of Figure 21, obtained by adding three bands compatible with the twisted orientation.

Therefore, a twisted orientable slice surface Σ of K that is actually orientable should have

$$g(\Sigma) \geq \tfrac{3}{2},$$

so it must have genus at least 2. However, there exists a orientable slice surface of K in the usual sense, which has genus 1, as shown in (c) of Figure 21. Note that the arrows in (b) represent a twisted orientation, while the arrows in (c) represent a usual orientation.

By inserting more and more full twists in this example, we obtain a family of null homologous knots K_n in $\mathbb{RP}^3$, whose two s-invariants, $s^{\mathrm{BN}}_{\mathbb{RP}^3}(K)$ and $s^{\mathrm{Lee}}_{\mathbb{RP}^3}(K)$, defined using the Bar-Natan and Lee deformation, respectively, can have arbitrarily large differences in their absolute values. Here, $s^{\mathrm{Lee}}_{\mathbb{RP}^3}(K)$ refers to the s-invariant defined in [13] using the Lee deformation. See Figure 22 for a diagram of K_n. It is a positive knot with respect to the twisted orientation and a negative knot with respect to the usual orientation. It has $2n + 2$ crossings, and the twisted oriented resolution

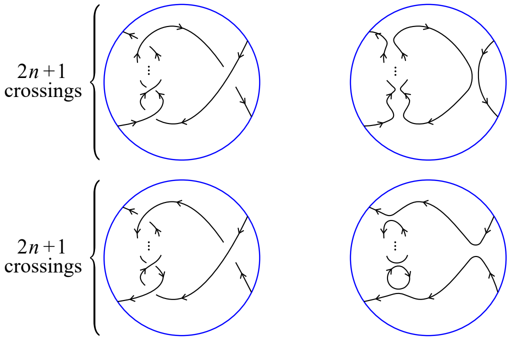

Figure 22. An example of knots with different $|s^{\mathrm{BN}}_{\mathbb{RP}^3}|$ and $|s^{\mathrm{Lee}}_{\mathbb{RP}^3}|$. Top left: K_n with a twisted orientation. Top right: a twisted oriented resolution of K_n. Bottom left: K_n with a usual orientation. Bottom right: an oriented resolution of K_n.

has two circles, while the oriented resolution has $2n+1$ circles. Therefore,

$$|s^{\mathrm{BN}}_{\mathbb{RP}^3}(K_n)| = |-2+(2n+2)+1| = 2n+1,$$
$$|s^{\mathrm{Lee}}_{\mathbb{RP}^3}(K_n)| = |-(-(2n+1)+(2n+2)+1)| = 2.$$

5. Further directions

We discuss several further directions to explore, listed in ascending order of scope.

- In [13], Manolescu and Willis defined the Lee homology and s-invariant for both null homologous and homologically essential links (class-0 and class-1 links in their notion), while we only defined the Bar-Natan homology and s-invariant for null homologous links. It is natural to ask what the counterpart for homologically essential links should be using Bar-Natan homology. Note that the algebraic structures of Khovanov homology (and Lee homology) for null homologous and homologically essential links are quite different. In any resolution of the link diagram of a homologically essential link, there will be a homologically essential unknot, and no 1-1 bifurcation will occur in the edge maps. Thus, instead of the extra map $f : V \to V$, what we require in the homologically essential case is a bimodule over the Frobenius algebra V, which will be assigned to the homologically essential unknot in each resolution.

- Naturally, one would like to ask how this Bar-Natan chain complex $\mathrm{CBN}(L)$ is related to an equivariant version of the Bar-Natan chain complex $\mathrm{CBN}(\widetilde{L})$ of the double cover $\widetilde{L}$ of L in S^3, and also the similar question for the Khovanov chain complex. When $\widetilde{L}$ is periodic or strongly invertible, there is a lot of work relating the Khovanov chain complex/stable homotopy type of the quotient link to the corresponding chain complex/stable homotopy type of the link itself with the action.

See, for example, [3; 16] for periodic links, and [10; 11] for strongly invertible knots. Since the Bar-Natan homology is much easier to describe than the Khovanov homology, we do obtain an inequality at the level of Bar-Natan homology:

$$\dim(\mathrm{HBN}(L)) \leq \dim(\mathrm{HBN}(\widetilde{L})),$$

where $\mathrm{HBN}(L)$ is the Bar-Natan homology of a null homologous link L in $\mathbb{RP}^3$ defined in this paper, and $\mathrm{HBN}(\widetilde{L})$ is the usual Bar-Natan homology for links in S^3. The reason is simple: as proved in Proposition 2.16, $\mathrm{HBN}(L)$ has a basis identified with the set of twisted orientations on L, which is by definition a subset of orientations on $\widetilde{L}$, and $\mathrm{HBN}(\widetilde{L})$ has a basis identified with the set of all orientations on $\widetilde{L}$. This suggests a possibility of a spectral sequence relating $\mathrm{HBN}(L)$ and $\mathrm{HBN}(\widetilde{L})$, and perhaps for the Khovanov homology as well. One possible starting point is to look at a link diagram of $\widetilde{L}$ as the closure of $T \circ F(T)$, as in Figure 8, on which one can formally define an action of the involution τ on the chain complex level for $\mathrm{CBN}(\widetilde{L})$ and $\mathrm{CKh}(\widetilde{L})$.

- For knots in S^3, the Lee and Bar-Natan deformation of the usual Khovanov homology are closely related to each other. For example, if $\mathbb{F}$ is a field of characteristic other than 2, then the s-invariant defined using the Bar-Natan deformation agrees with that defined using the Lee deformation. See Proposition 3.1 in [12]. However, in $\mathbb{RP}^3$, the behavior is quite different. For example, the class of slice surfaces whose genus is bounded by the s-invariant is not the same. The reason for this difference lies in the assignment of the map to the 1-1 bifurcation: it is the identity map in the Bar-Natan deformation, while it is 0 in the Lee deformation. It might be interesting to look at other extensions of the s-invariants to 3-manifolds using the Lee deformation, e.g., [7] for $S^1 \times D^2$, [14] for connected sums of $S^1 \times S^2$, and see what happens if one uses the Bar-Natan deformation instead of the Lee deformation in these cases.

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