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THE WEIGHTS OF ISOLATED CURVE SINGULARITIES ARE DETERMINED BY HODGE IDEALS

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We calculate Hodge ideals and Hodge moduli algebras for three types of isolated quasihomogeneous curve singularities. We show that Hodge ideals and Hodge moduli algebras of the singularities can determine the weights of the polynomials defining the singularities. We give some examples to explain why Hodge moduli algebras and the Hodge moduli sequence are better invariants than the characteristic polynomial (a topological invariant of the singularity) for nondegenerate quasihomogeneous singularities, in the sense that the characteristic polynomial cannot determine the weight type of the singularity.

1. Introduction

In [15; 16], the authors ask whether the topology of the singularity determines the weights of the polynomial defining the singularity. They showed that this is valid in the category of isolated singularities of Brieskorn–Pham type and isolated quasihomogeneous curve singularities.

Theorem 1.1 [15]. *The topology of a singularity of Brieskorn–Pham type determines the exponents* (*weight*) *of the polynomial defining the singularity.*

Theorem 1.2 [16]. *Let* $f_i(z_1, z_2)$, $i = 1, 2$, *be nondegenerate quasihomogeneous polynomials of weight* $(r_{i1}, r_{i2}; 1)$, $0 \le r_{i1} \le r_{i2} \le \frac{1}{2}$, *and let* V_i *be the germ of* $f_i(z_1, z_2) = 0$ *at the origin of* $\mathbb{C}^2$. *Then if* $(\mathbb{C}^2, V_1, 0) \simeq (\mathbb{C}^2, V_2, 0)$, *homeomorphically*, *we have* $(r_{11}, r_{12}) = (r_{21}, r_{22})$.

For quasihomogeneous surface singularities, there are some relevant results. Arnold [1, pages 91–131] and Orlik and Wagreich [9] showed that if $h(z_0, z_1, z_2)$ is a quasihomogeneous polynomial in $\mathbb{C}^3$ and $V = \{h(z) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the following seven classes below while keeping the differentiable structure of the link $K_V = S_\epsilon^{2n+1} \cap V$ constant:

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(I) $V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2}\}$, $a_0, a_1, a_2 > 1$,

(II) $V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}\}$, $a_0, a_1 > 1, a_2 > 0$,

(III) $V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2}\}$, $a_0 > 1, a_1, a_2 > 0$,

(IV) $V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2}\}$, $a_0 > 1, a_1, a_2 > 0$,

(V) $V(a_0, a_1, a_2; 5) = \{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}\}$, $a_0, a_1, a_2 > 0$,

(VI) $V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}\}$, $a_0 > 1$, $a_1, a_2, b_1, b_2 > 0$ satisfy $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$,

(VII) $V(a_0, a_1, a_2; 7) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}\}$, $a_0, a_1, a_2, b_1, b_2 > 0$ satisfy $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2(a_0 a_1 - 1)$.

Xu and Yau [14] proved that the above deformation is actually a topological trivial deformation as a pair (S^{2n+1}, K_V). Therefore any isolated quasihomogeneous surface singularity has the same topological type of one of the seven classes above. Let $\Delta_V(z)$ denote the characteristic polynomial of the *Milnor* fibration of $(V, 0)$.

Theorem 1.3 [14]. *If* $(V, 0)$ *and* $(W, 0)$ *are among the seven classes above*, *then* $(\mathbb{C}^3, V, 0)$ *is biholomorphic to* $(\mathbb{C}^3, W, 0)$ *if and only if* $(\mathbb{C}^3, V, 0)$ *is homeomorphic to* $(\mathbb{C}^3, W, 0)$ *with some exceptional cases. And* $(\mathbb{C}^3, V, 0)$ *is homeomorphic to* $(\mathbb{C}^3, W, 0)$ *if and only if* $\pi_1(K_V) \simeq \pi_1(K_W)$ *and* $\Delta_V(z) = \Delta_W(z)$.

The following are direct corollaries of the above theorem:

Corollary 1.4 [14]. *Let* $(V, 0)$ *and* $(W, 0)$ *be two isolated quasihomogeneous surface singularities in* $\mathbb{C}^3$. *Then* $(\mathbb{C}^3, V, 0)$ *is homeomorphic to* $(\mathbb{C}^3, W, 0)$ *if and only if* $\pi_1(K_V) \simeq \pi_1(K_W)$ *and* $\Delta_V(z) = \Delta_W(z)$.

Corollary 1.5 [14]. *Let* $(V, 0)$ *be an isolated quasihomogeneous surface singularity with weights* (w_0, w_1, w_2). *Then the topological type of* $(V, 0)$ *determines and is determined by its weights* (w_0, w_1, w_2).

Corollary 1.6 [14]. *Let* $(V, 0)$ *be an isolated singularity defined by a quasihomogeneous polynomial in* $\mathbb{C}^3$ *with weights* (w_0, w_1, w_2). *Then the fundamental group of the link* $\pi_1(K_V)$ *and the characteristic polynomial* $\Delta_V(z)$ *determine and are determined by the weights* (w_0, w_1, w_2).

The original motivation of [14] was to prove the *Zariski* conjecture (see [17]) for isolated quasihomogeneous surface singularities in $\mathbb{C}^3$: multiplicity is an invariant of topological type. As a corollary, they proved:

Corollary 1.7 [14]. *Let* $(V, 0)$ *and* $(W, 0)$ *be two isolated quasihomogeneous surface singularities in* $\mathbb{C}^3$. *If* $(\mathbb{C}^3, V, 0)$ *is homeomorphic to* $(\mathbb{C}^3, W, 0)$, *then* V *and* W *have the same multiplicity at the origin.*

Recall that in [3], we proved that a series of new invariants, Hodge moduli algebras and the Hodge moduli sequence, of the singularity are complete contact invariants for simple surface singularities. And our final aim is to extend this result to isolated quasihomogeneous surface singularities or even more general types of singularity. Note that in the proof of the above theorems, the characteristic polynomial of the singularity plays a fundamental role, since the characteristic polynomial is a topological invariant of the singularity. Motivated by these results and our former results, it is natural to ask whether we can replace the characteristic polynomial by Hodge ideals and Hodge moduli algebras of the singularity to determine the weights of the polynomials defining the singularities. That is, we want to prove if the i-th Hodge moduli algebras of two isolated quasihomogeneous curve singularities are isomorphic for all $i \geq 0$, then the weights of these two singularities are the same.

If $h(x, y)$ is a quasihomogeneous polynomial in $\mathbb{C}^2$ and $V = \{h(x, y) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the following three classes below while keeping the differentiable structure of the link $K_V = S_\epsilon^{2n+1} \cap V$ constant:

$$\begin{aligned} \mathbb{F}_1(x, y) &= x^a + y^b, && a, b \geq 2, \\ \mathbb{F}_2(x, y) &= x^a + xy^b, && a \geq 2,\ b \geq 1, \\ \mathbb{F}_3(x, y) &= x^a y + xy^b, && a, b \geq 1. \end{aligned}$$

After a tedious calculation for Hodge ideals and Hodge moduli algebras of isolated quasihomogeneous curve singularities of the above three types, we obtain the following.

Main Theorem A (0-th and 1-st Hodge moduli algebras determine weight type).
(1) *For isolated quasihomogeneous curve singularities*

$$D_1^{(a_1,b_1)} = \{x^{a_1} + y^{b_1} = 0\}, \quad 2 \leq a_1 \leq b_1,$$

and

$$D_2^{(a_2,b_2)} = \{x^{a_2} + xy^{b_2} = 0\}, \quad 1 \leq a_2 - 1 \leq b_2,$$

if their 0*-th and* 1*-st Hodge moduli algebras* (*taking* $\alpha = 1$ *in their Hodge ideals*) *are isomorphic*, *i.e.*,

$$M_0(D_1^{(a_1,b_1)}) \simeq M_0(D_2^{(a_2,b_2)}), \quad M_1(D_1^{(a_1,b_1)}) \simeq M_1(D_2^{(a_2,b_2)}),$$

then the weight types of $D_1^{(a_1,b_1)}$ *and* $D_2^{(a_2,b_2)}$ *are the same*, *i.e.*,

$$\mathrm{wt}(F_1^{(a_1,b_1)}) = \mathrm{wt}(F_2^{(a_2,b_2)}).$$

(2) *For isolated quasihomogeneous curve singularities*

$$D_2^{(a_2,b_2)} = \{x^{a_2} + xy^{b_2} = 0\}, \quad a_2 - 1 \geq b_2 \geq 1,$$

and

$$D_3^{(a_3,b_3)} = \{x^{a_3}y + xy^{b_3} = 0\}, \quad 1 \leq a_3 \leq b_3,$$

if their 0*-th and* 1*-st Hodge moduli algebras* (*taking* $\alpha = 1$ *in their Hodge ideals*) *are isomorphic*, *i.e.*,

$$M_0(D_2^{(a_2,b_2)}) \simeq M_0(D_3^{(a_3,b_3)}), \quad M_1(D_2^{(a_2,b_2)}) \simeq M_1(D_3^{(a_3,b_3)}),$$

then the weight types of $D_2^{(a_2,b_2)}$ *and* $D_3^{(a_3,b_3)}$ *are the same*, *i.e.*,

$$\mathrm{wt}(F_1^{(a_1,b_1)}) = \mathrm{wt}(F_3^{(a_3,b_3)}).$$

(3) *For isolated quasihomogeneous curve singularities*

$$D_1^{(a_1,b_1)} = \{x^{a_1} + y^{b_1} = 0\}, \quad a_1, b_1 \geq 2,$$

and

$$D_3^{(a_3,b_3)} = \{x^{a_3}y + xy^{b_3} = 0\}, \quad a_3, b_3 \geq 1,$$

their i*-th Hodge moduli algebras* (*taking* $\alpha = 1$ *in their Hodge ideals*) *are not isomorphic*, *for* $i = 0, 1$, *respectively.*

As a by-product, we obtain an inequality of the δ-invariant, 0-th Hodge moduli number and multiplicity for isolated quasihomogeneous curve singularities of the above three types:

Main Theorem B. (1) *For isolated quasihomogeneous curve singularities* $D_1^{(a,b)} = \{x^a + y^b = 0\}$, $a, b \geq 2$, *we have*

$$0 \leq \delta_1(a, b) - m_0(D_1^{(a,b)}) \leq \mathrm{mt}(D_1^{(a,b)}),$$

where $\delta_1(a, b)$ *is the* δ*-invariant of* $D_1^{(a,b)}$, $m_0(D_1^{(a,b)})$ *is the* 0*-th Hodge moduli number of the divisor* $D_1^{(a,b)}$ *for* $\alpha = 1$ *and* $\mathrm{mt}(D_1^{(a,b)})$ *is the multiplicity of* $D_1^{(a,b)}$.

(2) *For isolated quasihomogeneous curve singularities* $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, $a \geq 2, b \geq 1$, *we have*

$$1 \leq \delta_2(a, b) - m_0(D_2^{(a,b)}) \leq \mathrm{mt}(D_2^{(a,b)}),$$

where $\delta_2(a, b)$ *is the* δ*-invariant of* $D_2^{(a,b)}$, $m_0(D_2^{(a,b)})$ *is the* 0*-th Hodge moduli number of the divisor* $D_2^{(a,b)}$ *for* $\alpha = 1$ *and* $\mathrm{mt}(D_2^{(a,b)})$ *is the multiplicity of* $D_2^{(a,b)}$.

(3) *For isolated quasihomogeneous curve singularities* $D_3^{(a,b)} = \{x^a y + xy^b = 0\}$, $a, b \geq 1$, *we have*

$$2 \leq \delta_3(a, b) - m_0(D_3^{(a,b)}) \leq \mathrm{mt}(D_3^{(a,b)}),$$

where $\delta_3(a,b)$ *is the* δ*-invariant of* $D_3^{(a,b)}$, $m_0(D_3^{(a,b)})$ *is the* 0*-th Hodge moduli number of the divisor* $D_3^{(a,b)}$ *for* $\alpha = 1$ *and* $\mathrm{mt}(D_3^{(a,b)})$ *is the multiplicity of* $D_3^{(a,b)}$.

In Section 2, we recall a number of classical results on the Hodge ideals of effective $\mathbb{Q}$-divisors and the δ-invariants of curve singularities. We also collect some important lemmas and theorems that will be used in the following parts. In Section 3, we explicitly calculate Hodge ideals and Hodge moduli algebras of isolated quasihomogeneous curve singularities of three types. In Sections 4 and 5 we prove Main Theorems A and B by the results in Section 3. Finally, in Section 6 we give some examples to illustrate that Hodge moduli algebras and the Hodge moduli sequence are better invariants than the characteristic polynomial (a topological invariant of singularity) for nondegenerate quasihomogeneous singularities. Furthermore, from the observation of some examples, we raise a conjecture that the Hodge moduli numbers of isolated quasihomogeneous curve singularities remain constant under quasihomogeneous deformation. That is, Hodge moduli numbers of isolated quasihomogeneous curve singularities only depend on the weights of the singularities.

2. Preliminaries

2.1. ***Hodge ideals.*** In [7; 8], the authors extend the notion of Hodge ideals to the case when D is an arbitrary effective $\mathbb{Q}$-divisor on X, where X is a smooth complex variety. Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ are defined in terms of the Hodge filtration $F_\bullet$ on some $\mathscr{D}_X$-module associated with D (see [7, §2–§4] for more details). When D is an integral and reduced divisor, this recovers the definition of Hodge ideals $I_k(D)$ in [6].

Let X be a smooth complex variety, and $\mathscr{D}_X$ be the sheaf of differential operators on X. If H is an integral and reduced effective divisor on X, $D = \alpha H$, $\alpha \in \mathbb{Q} \cap (0, 1]$, let $\mathcal{O}_X(*D)$ be the sheaf of rational functions with poles along D. It is also a left $\mathscr{D}_X$-module underlying the mixed Hodge module $j_*\mathbb{Q}_U^H[n]$, where $U = X \setminus D$ and $j : U \hookrightarrow X$ is the inclusion map. Any $\mathscr{D}_X$-module associated with a mixed Hodge module has a good filtration $F_\bullet$, the Hodge filtration of the mixed Hodge module [12].

To study the Hodge filtration of $\mathcal{O}_X(*D)$, it seems easier to consider a series of ideal sheaves, defined by Mustață and Popa [6], which can be considered to be a generalization of multiplier ideals of divisors. The Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ of the divisor D are defined by

$$F_k\mathscr{O}_X(*D) = I_k(D) \otimes \mathscr{O}_X((k+1)D) \quad \text{for all } k \in \mathbb{N}.$$

These are coherent sheaves of ideals. See [6] for details and an extensive study of the ideals $I_k(D)$. Hodge ideals are indexed by the nonnegative integers; at the 0-th step, they essentially coincide with multiplier ideals. It turns out that $I_0(D) = \mathscr{J}((1-\epsilon)D)$, the multiplier ideal of the divisor $(1-\epsilon)D$, $0 < \epsilon \ll 1$. The multiplier ideal sheaves are ubiquitous objects in birational geometry, encoding local numerical invariants of singularities, and satisfying Kodaira-type vanishing theorems in the global setting. The Hodge ideals are interesting invariants of the singularities, they have similar properties as multiplier ideals.

We summarize the properties and results (see [7; 5]) of Hodge ideals as follows.

Given a reduced effective divisor H on a smooth complex variety X, $D=\alpha H$, $\alpha \in \mathbb{Q} \cap (0, 1]$, we also denote by Z the support of D. The sequence of Hodge ideals $I_k(D)$, with $k \geq 0$, satisfies these properties:

- $I_0(D)$ is the multiplier ideal $\mathcal{I}((1-\epsilon)D)$, so in particular $I_0(D)=\mathscr{O}_X$ if and only if the pair(X, D) is log canonically.

- When Z has simple normal crossings,

$$I_k(D)=I_k(Z)\otimes\mathscr{O}_X(Z-\lceil D\rceil),$$

where $I_k(Z)$ can be computed explicitly as in [6]. If Z is smooth, then $I_k(D)=\mathscr{O}_X(Z-\lceil D\rceil)$.

- The Hodge filtration is generated at level $n-1$, where $n=\dim X$, i.e.,

$$F_\ell\mathscr{D}_X\cdot(I_k(D)\otimes\mathscr{O}_X(kZ)h^{-\alpha})=I_{k+\ell}(D)\otimes\mathscr{O}_X((k+\ell)Z)h^{-\alpha}$$

for all $k\geqslant n-1$ and $\ell\geqslant 0$.

- There are nontriviality criteria for $I_k(D)$ at a point $x\in D$ in terms of the multiplicity of D at x.

- If X is projective, $I_k(D)$ satisfy a vanishing theorem analogous to Nadel vanishing for multiplier ideals.

- If Y is a smooth divisor in X such that $Z|_Y$ is reduced, then $I_k(D)$ satisfy

$$I_k(D|_Y)\subseteq I_k(D)\cdot\mathscr{O}_Y,$$

with equality when Y is general.

- If $X\to T$ is a smooth family with a section $s:T\to X$, and D is a relative divisor on X that satisfies a suitable condition then

$$\{t\in T\mid I_k(D_t)\nsubseteq\mathfrak{m}^q_{s(t)}\}$$

is an open subset of T, for each $q\geqslant 1$.

- If D_1 and D_2 are $\mathbb{Q}$-divisors with supports Z_1 and Z_2, such that Z_1+Z_2 is also reduced, then we have the subadditivity property

$$I_k(D_1+D_2)\subseteq I_k(D_1)\cdot I_k(D_2)$$

For comparison, the list of properties of Hodge ideals in the case when D is reduced is summarized in [10]. The setting of $\mathbb{Q}$-divisors is more intricate. For instance, the bounds for the generation level of the Hodge filtration can become worse. Moreover, it is not known whether the inclusions $I_k(D) \subseteq I_{k-1}(D)$ continue to hold for arbitrary $\mathbb{Q}$-divisors. New phenomena appear as well: given two rational numbers $\alpha_1 < \alpha_2$, usually the ideals $I_k(\alpha_1 Z)$ and $I_k(\alpha_2 Z)$ cannot be compared for $k \geqslant 1$, unlike in the case of multiplier ideals.

We recall the following definition.

Definition 2.1. Let $f, g \in R = \mathbb{C}\{x_1, \dots, x_n\}$ which is the convergent power series ring. We say f and g are contact equivalent if the local $\mathbb{C}$-algebras $R/(f)$ and $R/(g)$ are isomorphic.

Definition 2.2. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $n \geq 2$, be an isolated hypersurface singularity. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f, $D^\alpha = \alpha H$, $\alpha \in \mathbb{Q} \cap (0, 1]$. We define the i-th Hodge moduli algebra of D^α to be the moduli algebra of the ideal $J_i(D^\alpha) := (f) + I_i(D^\alpha)$ (or J_i for short)

$$M_i(D^\alpha) := \mathbb{C}\{x_1, \dots, x_n\}/J_i(D^\alpha)$$

for $i \geq 0$ (or M_i for short), where $I_i(D^\alpha)$ is the i-th Hodge ideal (or I_i for short). The i-th Hodge moduli number of D^α is defined to be

$$m_i(D^\alpha) := \dim_{\mathbb{C}}(M_i(D^\alpha))$$

for $i \geq 0$ (or m_i for short). We define the Hodge moduli sequence of D to be the sequence

$$\{m_i\} := \{m_0, m_1, m_2, \dots\}.$$

Definition 2.3. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called weighted homogeneous if there exists positive rational numbers $w_1, \dots, w_n$ (that is, weights of $x_1, \dots, x_n$) and d such that $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with a nonzero coefficient. The number d is called the weighted homogeneous degree (w-deg) of f for weights w_j, $1 \leq j \leq n$. These w_j, $1 \leq j \leq n$, are called the weight type of f.

The Hodge filtration $F_\bullet$ of $\mathcal{O}_X(*D)$ is usually hard to describe. However, it does have an explicit formula in the case when D is defined by a reduced weighted homogeneous polynomial f which has an isolated singularity at the origin, which is proved by M. Saito [13]. To state Saito's result, we first clarify the notation as follows.

- Denote by $\mathcal{O} = \mathbb{C}\{x_1, \dots, x_n\}$ the ring of germs of holomorphic function for local coordinates $x_1, \dots, x_n$.
- Denote by $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of a holomorphic function that is quasihomogeneous, i.e., $f \in \mathcal{J}(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$, and with an isolated singularity at the origin. Kyoji Saito [11] showed that after a biholomorphic coordinate change, we can assume f is a weighted homogeneous polynomial with an isolated singularity at the origin. We will keep this assumption for f unless otherwise stated.

- Denote by $w = w(f) = (w_1, \dots, w_n)$ the weights of the weighted homogeneous polynomial f.
- Denote by $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of a holomorphic function, and we write

$$g = \sum_{A \in \mathbb{N}^n} g_A x^A,$$

where $A = (a_1, \dots, a_n)$, $g_A \in \mathbb{C}$ and $x^A = x_1^{a_1} \cdots x_n^{a_n}$.
- Denote by $\rho(g)$ the weight of an element $g \in \mathcal{O}$ defined by

$$\rho(g) = \left(\sum_{i=1}^{m} w_i \right) + \inf\{\langle w, A\rangle : g_A \neq 0\}.$$

The weight function ρ defines a filtration on $\mathcal{O}$ as

$$\mathcal{O}^{>k} = \{u \in \mathcal{O} : \rho(u) > k\},$$
$$\mathcal{O}^{\geq k} = \{u \in \mathcal{O} : \rho(u) \geq k\}.$$

Since we consider $\mathscr{D}_X$-modules locally around the isolated singularity, we can assume $X = \mathbb{C}^n$ and identify the stalk at the singularity to be that of $\mathscr{D}_X$-modules on $\mathbb{C}^n$. For example, we replace $F_k \mathcal{O}_{X,0}(*D)$ with $F_k \mathcal{O}_X(*D)$. Now we can state the formula proved by M. Saito (see [13, Theorem 0.7]):

$$F_k \mathcal{O}_X(*D) = \sum_{i=0}^{k} F_{k-i} \mathscr{D}_X \left(\frac{\mathcal{O}^{\geq i+1}}{f^{i+1}} \right) \quad \text{for all } k \in \mathbb{N}. \tag{1}$$

Since the Hodge filtration can be constructed on analogous $\mathscr{D}_X$-modules associated with any effective $\mathbb{Q}$-divisor D, so it satisfies a similar formula in the case when D is supported on a hypersurface defined by such a polynomial f.

Assume that the divisor is $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = (f = 0)$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin. In this case, the associated $\mathscr{D}_X$-module is the well-known twisted localization $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha}) := \mathcal{O}_X(*Z) f^{1-\alpha}$ (see more details in [7] about how to construct the Hodge filtration $F_\bullet \mathcal{M}(f^{1-\alpha})$). With new ingredients from Mustaţă and Popa [8], where this Hodge filtration is compared to the V-filtration on $\mathcal{M}(f^{1-\alpha})$, M. Zhang generalized Saito's formula and proved the following theorem:

Theorem 2.4 (Zhang, [18]). *If $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have*

$$F_k \mathcal{M}(f^{1-\alpha}) = \sum_{i=0}^{k} F_{k-i} \mathscr{D}_X \left(\frac{\mathcal{O}^{\geq \alpha+i}}{f^{i+1}} f^{1-\alpha} \right),$$

where the action $\cdot$ of $\mathscr{D}_X$ on the right-hand side is the action on the left $\mathscr{D}_X$-module $\mathcal{M}(f^{1-\alpha})$ defined by

$$D\cdot(wf^{1-\alpha}) := \left(D(w) + w\frac{(1-\alpha)D(f)}{f}\right)f^{1-\alpha} \quad \textit{for any } D \in \mathrm{Der}_{\mathbb{C}}\,\mathcal{O}_X.$$

Notice that if we set $\alpha = 1$, Theorem 2.4 recovers Saito's formula (1) mentioned above. For any polynomial f with an isolated singularity at the origin, it is well known that the *Milnor* algebra

$$\mathcal{A}_f := \mathbb{C}\{x_1, \dots, x_n\}/(\partial_1 f, \dots, \partial_n f)$$

is a finite-dimensional $\mathbb{C}$-vector space. Fix a monomial basis $\{v_1, \dots, v_\mu\}$ for this vector space, where μ is the dimension of $\mathcal{A}_f$ (i.e., *Milnor* number). The next theorem follows from Theorem 2.4.

Theorem 2.5 (Zhang, [18]). *If $D = \alpha Z$, where $0 < \alpha \le 1$ and $Z = \{f = 0\}$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have*

$$F_0\mathcal{M}(f^{1-\alpha}) = f^{-1}\cdot\mathcal{O}^{\ge\alpha} f^{1-\alpha}$$

and

$$F_k\mathcal{M}(f^{1-\alpha}) = \left(f^{-1}\cdot \sum_{v_j\in\mathcal{O}^{\ge k+1+\alpha}} \mathcal{O}_X\cdot v_j\right)f^{1-\alpha} + F_1\mathfrak{D}_X\cdot F_{k-1}\mathcal{M}(f^{1-\alpha}).$$

Alternatively, in terms of Hodge ideals, these formulas say that

$$I_0(D) = \mathcal{O}^{\ge\alpha}$$

and

$$I_{k+1}(D) = \sum_{v_j\in\mathcal{O}^{\ge k+1+\alpha}} \mathcal{O}_X\cdot v_j + \sum_{1\le i\le n,\, a\in I_k(D)} \mathcal{O}_X(f\partial_i a - (\alpha+k)a\partial_i f).$$

2.2. *Delta invariant of curve singularities.*

Definition 2.6 (δ-invariant). Let $f \in \mathbb{C}\{x, y\}$ be a reduced convergent power series, and let

$$\mathcal{O} = \mathbb{C}\{x, y\}/\langle f\rangle \hookrightarrow \overline{\mathcal{O}}$$

denote the normalization. Then we call

$$\delta(f) := \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O}$$

the δ-invariant of f.

Although we can explicitly calculate the δ-invariants of isolated quasihomogeneous singularities of three types F_1, F_2, F_3, by blowing up singularities and using the above theorem. We use Lemma 2.7, which is a very useful equality of the *Milnor* number, the δ-invariant and the number of irreducible factors of a curve singularity $\{f=0\}$, to show Lemmas 2.8, 2.9, and 2.10. And we only give proof for Lemma 2.8 for simplicity, since the proofs for Lemmas 2.9 and 2.10 are similar.

Lemma 2.7 [2, Proposition 3.35]. *Let $f \in \mathfrak{m} \subseteq \mathbb{C}\{x, y\}$ be reduced. Then*

$$\mu(f)=2\delta(f)-r(f)+1,$$

*where $\mu(f)$ is the **Milnor** number of f, $\delta(f)$ is the δ-invariant of f and $r(f)$ is the number of irreducible factors of f.*

Lemma 2.8. *For an isolated quasihomogeneous curve singularity of the form $D_1^{(a,b)}=\{x^a+y^b=0\}$, defined by $F_1^{(a,b)}=x^a+y^b$, $a, b \geq 2$, its δ-invariant is*

$$\delta_1(a, b)=\frac{(a-1)(b-1)+\operatorname{gcd}(a, b)-1}{2}.$$

In particular, $\delta_1(a, b)=\frac{(a-1)(b-1)}{2}$, if $\operatorname{gcd}(a, b)=1$.

Proof. Since $\mu(f)=(a-1)(b-1)$ and $r(f)=\operatorname{gcd}(a, b)$,

$$\delta_1(a, b)=\frac{\mu(f)+r(f)-1}{2}=\frac{(a-1)(b-1)+\operatorname{gcd}(a, b)-1}{2}. \qquad \square$$

Lemma 2.9. *For an isolated quasihomogeneous curve singularity of the form $D_2^{(a,b)}=\{x^a+xy^b=0\}$, defined by $F_2^{(a,b)}=x^a+xy^b$, $a \geq 2$, $b \geq 1$, its δ-invariant is*

$$\delta_2(a, b)=\frac{a(b-1)+\operatorname{gcd}(a-1, b)+1}{2}.$$

In particular, $\delta_2(a, b)=\frac{a(b-1)+2}{2}$, if $\operatorname{gcd}(a-1, b)=1$.

Lemma 2.10. *For an isolated quasihomogeneous curve singularity of the form $D_3^{(a,b)}=\{x^a y+xy^b=0\}$, defined by $F_3^{(a,b)}=x^a y+xy^b$, $a, b \geq 1$, its δ-invariant is*

$$\delta_3(a, b)=\frac{ab+\operatorname{gcd}(a-1, b-1)+1}{2}.$$

In particular, $\delta_3(a, b)=\frac{ab+2}{2}$, if $\operatorname{gcd}(a-1, b-1)=1$.

3. The first two Hodge ideals of three types of isolated quasihomogeneous curve singularities

In this section, we compute Hodge ideals of three types of isolated quasihomogeneous curve singularities for $\alpha=1$ in Theorem 2.5. And $\mathcal{O}_X=\mathbb{C}\{x, y\}$ in the following computation. The following lemma is used in the computation of dimensions of Hodge moduli algebras.

Lemma 3.1. *For $n, m \in \mathbb{N}$, $n, m \geq 1$,*

$$\sum_{i=1}^{n-1}\left[\frac{mi}{n}\right]=\frac{(m-1)(n-1)+\operatorname{gcd}(m, n)-1}{2},$$

Consider isolated quasihomogeneous curve singularities $D_1^{(a,b)} = \{x^a + y^b = 0\}$, defined by $F_1^{(a,b)} = x^a + y^b$. If $a \le b$, let $r = \frac{a}{\gcd(a,b)}$; then $1 \le r \le a$. Since $\frac{1+a-1}{a} + \frac{1}{b} \ge 1$, we have $x^{a-1} \in I_0(D_1)$. And we have

$$x^{k-1} y^{[\frac{b(a-k)}{a}]} \in I_0(D_1) \quad \text{for all } 1 \le k \le a-1,\ r \nmid k,$$

and

$$x^{ir-1} y^{b-\frac{ibr}{a}-1} \in I_0(D_1) \quad \text{for all } 1 \le i \le \gcd(a,b)-1.$$

So the 0-th Hodge ideal for D_1 is

$$\begin{aligned} J_0(D_1) = I_0(D_1) = \big(&x^{a-1}, x^{a-2} y^{[\frac{b}{a}]}, \dots, x^{a-r} y^{[\frac{b(r-1)}{a}]},\\ &\dots, x^{ir-1} y^{b-\frac{ibr}{a}-1}, x^{ir-2} y^{[\frac{b(a-ir+1)}{a}]}, \dots, x^{(i-1)r} y^{[\frac{b(a-(i-1)r-1)}{a}]},\\ &\dots, x^{r-1} y^{b-\frac{br}{a}-1}, x^{r-2} y^{[\frac{b(a-r+1)}{a}]}, \dots, y^{[\frac{b(a-1)}{a}]}\big), \end{aligned}$$

where $1 \le i \le \gcd(a,b)-1$. Its multiplicity $\mathrm{mt}(J_0(D_1))$ equals $a-1$. Using Lemma 3.1, we obtain the dimension of the 0-th Hodge moduli algebra $M_0(D_1) = \mathcal{O}_X / J_0(D_1)$:

$$\begin{aligned} m_0(D_1) &= \sum_{i=1}^{a-1} \left[\frac{bi}{a}\right] - (\gcd(a,b)-1)\\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} - (\gcd(a,b)-1)\\ &= \frac{(a-1)(b-1)-\gcd(a,b)+1}{2}. \end{aligned}$$

The 1-st Hodge ideal of D_1 is

$$\begin{aligned} J_1(D_1) &= (f) + I_0(D_1) \cdot (Jf)\\ &= \big(x^a + y^b, x^{a-2} y^b, x^{a-3} y^{[\frac{b}{a}]+b}, \dots, x^{a-r-1} y^{[\frac{b(r-1)}{a}]+b},\\ &\qquad \dots, x^{ir-2} y^{2b-\frac{ibr}{a}-1}, \dots, x^{(i-1)r-1} y^{[\frac{b(a-(i-1)r-1)}{a}]+b},\\ &\qquad \dots, x^{r-2} y^{2b-\frac{br}{a}-1}, \dots, y^{[\frac{b(a-2)}{a}]+b}, x^{a-1} y^{[\frac{b(a-1)}{a}]}\big), \end{aligned}$$

where $2 \le i \le \gcd(a,b)$. Its multiplicity $\mathrm{mt}(J_1(D_1))$ equals a. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_1) = \mathcal{O}_X / J_1(D_1)$ is

$$\begin{aligned} m_1(D_1) &= \sum_{i=1}^{a-2} \left(\left[\frac{bi}{a}\right] + b\right) - (\gcd(a,b)-1) + b + \left[\frac{b(a-1)}{a}\right]\\ &= \sum_{i=1}^{a-1} \left[\frac{bi}{a}\right] + (a-1)b - (\gcd(a,b)-1)\\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} + (a-1)b - (\gcd(a,b)-1)\\ &= \frac{(a-1)(3b-1)-\gcd(a,b)+1}{2}. \end{aligned}$$

If $a \geq b$, let $r = \frac{b}{\gcd(a,b)}$; then $1 \leq r \leq b$. By symmetry of a, b, we obtain the 0-th Hodge ideal for D_1:

$$\begin{aligned} J_0(D_1) &= I_0(D_1) \\ &= \big(y^{b-1}, y^{b-2}x^{[\frac{a}{b}]}, \dots, y^{b-r}x^{[\frac{a(r-1)}{b}]}, \\ &\qquad \dots, y^{ir-1}x^{a-\frac{iar}{b}-1}, y^{ir-2}x^{[\frac{a(b-ir+1)}{b}]}, \dots, y^{(i-1)r}x^{[\frac{a(b-(i-1)r-1)}{b}]}, \\ &\qquad\qquad \dots, y^{r-1}x^{a-\frac{ar}{b}-1}, y^{r-2}x^{[\frac{a(b-r+1)}{b}]}, \dots, x^{[\frac{a(b-1)}{b}]}\big), \end{aligned}$$

where $1 \leq i \leq \gcd(a,b)-1$. Its multiplicity $\mathrm{mt}(J_0(D_1))$ equals $b-1$. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_0(D_1) = \mathcal{O}_X/J_0(D_1)$ is

$$\begin{aligned} m_0(D_1) &= \sum_{i=1}^{b-1}\left[\frac{ai}{b}\right] - (\gcd(a,b)-1) \\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} - (\gcd(a,b)-1) \\ &= \frac{(a-1)(b-1)-\gcd(a,b)+1}{2}. \end{aligned}$$

And the 1-st Hodge ideal for D_1 is

$$\begin{aligned} J_1(D_1) &= (f) + I_0(D_1)\cdot(Jf) \\ &= \big(x^a+y^b, y^{b-2}x^a, y^{b-3}x^{[\frac{a}{b}]+a}, \dots, y^{b-r-1}x^{[\frac{a(r-1)}{b}]+a}, \\ &\qquad \dots, y^{ir-2}x^{2a-\frac{iar}{b}-1}, \dots, y^{(i-1)r-1}x^{[\frac{a(b-(i-1)r-1)}{b}]+a}, \\ &\qquad\qquad \dots, y^{r-2}x^{2a-\frac{ar}{b}-1}, \dots, x^{[\frac{a(b-2)}{b}]+a}, y^{b-1}x^{[\frac{a(b-1)}{b}]}\big), \end{aligned}$$

where $2 \leq i \leq \gcd(a,b)$. Its multiplicity $\mathrm{mt}(J_1(D_1))$ equals b. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_1) = \mathcal{O}_X/J_1(D_1)$ is

$$\begin{aligned} m_1(D_1) &= \sum_{i=1}^{b-2}\left(\left[\frac{ai}{b}\right]+a\right) - (\gcd(a,b)-1) + a + \left[\frac{a(b-1)}{b}\right] \\ &= \sum_{i=1}^{b-1}\left[\frac{ai}{b}\right] + (b-1)a - (\gcd(a,b)-1) \\ &= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} + (b-1)a - (\gcd(a,b)-1) \\ &= \frac{(3a-1)(b-1)-\gcd(a,b)+1}{2}. \end{aligned}$$

Consider isolated quasihomogeneous curve singularities $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, defined by $F_2^{(a,b)} = x^a + xy^b$. If $a-1 \leq b$, let $r = \frac{a-1}{\gcd(a-1,b)}$; then $1 \leq r \leq a-1$.

Since $\frac{1+a-1}{a}+\frac{a-1}{ab}\geq 1$, we have $x^{a-1}\in I_0(D_2)$. And we have

$$x^k y^{[\frac{b(a-k-1)}{a-1}]}\in I_0(D_2)\quad \text{for all } 1\leq k\leq a-1,\ r\nmid k,$$

and

$$x^{ir}y^{b-\frac{ibr}{a-1}-1}\in I_0(D_2)\quad \text{for all } 1\leq i\leq \gcd(a-1,b)-1.$$

Since $\frac{1}{a}+\frac{(a-1)(1+b-1)}{ab}\geq 1$, we have $y^{b-1}\in I_0(D_2)$. So the 0-th Hodge ideal for D_2 is

$$\begin{aligned}J_0(D_2)&=I_0(D_2)\\&=\big(x^{a-1},x^{a-2}y^{[\frac{b}{a-1}]},\dots,x^{a-r}y^{[\frac{b(r-1)}{a-1}]},\\&\qquad\dots,x^{ir}y^{b-\frac{ibr}{a-1}-1},x^{ir-1}y^{[\frac{b(a-ir)}{a-1}]},\dots,x^{(i-1)r+1}y^{[\frac{b(a-((i-1)r+1)-1)}{a-1}]},\\&\qquad\qquad\dots,x^ry^{b-\frac{br}{a-1}-1},x^{r-1}y^{[\frac{b(a-r)}{a-1}]},\dots,xy^{[\frac{b(a-2)}{a-1}]},y^{b-1}\big),\end{aligned}$$

where $1\leq i\leq \gcd(a-1,b)-1$. Its multiplicity $\mathrm{mt}(J_0(D_2))$ equals $a-1$. The dimension of the 0-th Hodge moduli algebra $M_0(D_2)=\mathcal{O}_X/J_0(D_2)$, by Lemma 3.1, is

$$\begin{aligned}m_0(D_2)&=\sum_{i=1}^{a-2}\left[\frac{bi}{a-1}\right]-(\gcd(a-1,b)-1)+b-1\\&=\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}-(\gcd(a,b)-1)+b-1\\&=\frac{a(b-1)-\gcd(a-1,b)+1}{2}.\end{aligned}$$

And the 1-st Hodge ideal of D_2 is

$$\begin{aligned}J_1(D_2)&=(f)+I_0(D_2)\cdot(Jf)\\&=\big(x^a+xy^b,ax^{a-1}y^{b-1}+y^{2b-1}x^{a-1}y^b,x^{a-2}y^{[\frac{b}{a-1}]+b},\dots,x^{a-r}y^{[\frac{b(r-1)}{a-1}]+b},\\&\qquad\dots,x^{ir}y^{2b-\frac{ibr}{a-1}-1},\dots,x^{(i-1)r+1}y^{[\frac{b(a-((i-1)r+1)-1)}{a-1}]+b},\\&\qquad\qquad\dots,x^ry^{2b-\frac{br}{a-1}-1},\dots,xy^{[\frac{b(a-2)}{a-1}]+b}\big),\end{aligned}$$

where $1\leq i\leq \gcd(a-1,b)-1$. Its multiplicity $\mathrm{mt}(J_1(D_2))$ equals a. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_2)=\mathcal{O}_X/J_1(D_2)$ is

$$\begin{aligned}m_1(D_2)&=2b-1+\sum_{i=1}^{a-2}\left(\left[\frac{bi}{a-1}\right]+b\right)-(\gcd(a-1,b)-1)+b\\&=\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}+(a-2)b+3b-1-(\gcd(a-1,b)-1)\\&=\frac{a(3b-1)-\gcd(a-1,b)+1}{2}.\end{aligned}$$

If $a-1\geq b$, let $r=\frac{b}{\gcd(a-1,b)}$; then $1\leq r\leq b$. Since $\frac{1}{a}+\frac{(a-1)(1+b-1)}{ab}\geq 1$, we have $y^{b-1}\in I_0(D_2)$. And we have

$$x^{[\frac{(a-1)(b-k)}{b}]+1}y^{k-1}\in I_0(D_2)\quad\text{for all }\ 1\leq k\leq b,\ r\nmid k,$$

and

$$x^{a-1-\frac{i(a-1)r}{b}}\in I_0(D_2)\quad\text{for all }\ 1\leq i\leq \gcd(a-1,b).$$

So the 0-th Hodge ideal for D_2 is

$$\begin{aligned}J_0(D_2)&=I_0(D_2)\\&=\big(x^{[\frac{(a-1)(b-1)}{b}]+1},x^{[\frac{(a-1)(b-2)}{b}]+1}y,\dots,x^{a-1-\frac{(a-1)r}{b}}y^{r-1},\dots,\\&\qquad x^{[\frac{(a-1)(b-(i-1)r-1)}{b}]+1}y^{(i-1)r},x^{[\frac{(a-1)(b-(i-1)r-2)}{b}]+1}y^{(i-1)r+1},\dots,x^{a-1-\frac{i(a-1)r}{b}}y^{ir-1},\\&\qquad\dots,x^{[\frac{(a-1)(r-1)}{b}]+1}y^{b-r},x^{[\frac{(a-1)(r-2)}{b}]+1}y^{b-r+1},\dots,y^{b-1}\big),\end{aligned}$$

where $1\leq i\leq\gcd(a-1,b)$. Its multiplicity $\mathrm{mt}(J_0(D_2))$ equals $b-1$. By Lemma 3.1, the dimension of the 0-th Hodge moduli algebra $M_0(D_2)=\mathcal{O}_X/J_0(D_2)$ is

$$\begin{aligned}m_0(D_2)&=\sum_{i=1}^{b-1}\left(\left[\frac{(a-1)i}{b}\right]+1\right)-(\gcd(a-1,b)-1)\\&=\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}+b-1-(\gcd(a-1,b)-1)\\&=\frac{a(b-1)-\gcd(a-1,b)+1}{2}.\end{aligned}$$

And the 1-st Hodge ideal of D_2 is

$$\begin{aligned}J_1(D_2)&=(f)+I_0(D_2)\cdot(Jf)\\&=\big(x^a+xy^b,ax^{a-1}y^{b-1}+y^{2b-1},x^{[\frac{(a-1)(b-1)}{b}]+2}y^{b-1},x^{[\frac{(a-1)(b-2)}{b}]+2}y^{b},\dots,\\&\qquad x^{a-\frac{(a-1)r}{b}}y^{b+r-2},\dots,x^{[\frac{(a-1)(b-(i-1)r-1)}{b}]+2}y^{b+(i-1)r-1},\\&\qquad x^{[\frac{(a-1)(b-(i-1)r-2)}{b}]+2}y^{b+(i-1)r},\dots,x^{a-\frac{i(a-1)r}{b}}y^{b+ir-2},\dots,\\&\qquad x^{[\frac{(a-1)(r-1)}{b}]+2}y^{2b-r-1},x^{[\frac{(a-1)(r-2)}{b}]+2}y^{2b-r},\dots,xy^{2b-2}\big),\end{aligned}$$

where $1\leq i\leq\gcd(a-1,b)$. Its multiplicity $\mathrm{mt}(J_1(D_2))$ equals $b+1$. By Lemma 3.1, the dimension of the 1-st Hodge moduli algebra $M_1(D_2)=\mathcal{O}_X/J_1(D_2)$ is

$$\begin{aligned}m_1(D_2)&=a(b-1)+\sum_{i=1}^{b-1}\left(\left[\frac{(a-1)i}{b}\right]+2\right)-(\gcd(a-1,b)-1)+1\\&=(a+2)(b-1)+\frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}-(\gcd(a-1,b)-1)+1\\&=\frac{(3a+2)(b-1)-\gcd(a-1,b)+3}{2}.\end{aligned}$$

Consider isolated quasihomogeneous curve singularities $D_3^{(a,b)} = \{x^a y + x y^b = 0\}$, defined by $F_3^{(a,b)} = x^a y + x y^b$. If $a \le b$, let $r = \frac{a-1}{\gcd(a-1,b-1)}$; then $1 \le r \le a-1$. Since

$$\frac{(b-1)(1+a-1)}{ab-1} + \frac{a-1}{ab-1} \ge 1,$$

we have $x^{a-1} \in I_0(D_3)$. And we have

$$x^k y^{[\frac{(b-1)(a-1-k)}{a-1}]+1} \in I_0(D_3) \quad \text{for all } 1 \le k \le a-2,\ r \nmid k,$$

and

$$x^{ir} y^{b-1-\frac{i(b-1)r}{a-1}} \in I_0(D_3) \quad \text{for all } 1 \le i \le \gcd(a-1, b-1) - 1.$$

So the 0-th Hodge ideal for D_3 is

$$\begin{aligned} J_0(D_3) &= I_0(D_3) \\ &= \big(x^{a-1}, x^{a-2} y^{[\frac{b-1}{a-1}]+1}, \dots, x^{a-r} y^{[\frac{(b-1)(r-1)}{a-1}]+1}, \dots, \\ &\qquad x^{ir} y^{b-1-\frac{i(b-1)r}{a-1}}, x^{ir-1} y^{[\frac{(b-1)(a-ir)}{a-1}]+1}, \dots, x^{(i-1)r+1} y^{[\frac{(b-1)(a-1-(i-1)r-1)}{a-1}]+1}, \\ &\qquad\qquad \dots, x^r y^{b-1-\frac{(b-1)r}{a-1}}, x^{r-1} y^{[\frac{(b-1)(a-r)}{a-1}]+1}, \dots, x y^{[\frac{(b-1)(a-2)}{a-1}]+1}, y^{b-1}\big), \end{aligned}$$

where $1 \le i \le \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_0(D_3))$ equals $a-1$. The dimension of the 0-th Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X / J_0(D_3)$, by Lemma 3.1, is

$$\begin{aligned} m_0(D_3) &= \sum_{i=1}^{a-2} \left(\left[\frac{(b-1)i}{a-1} \right] + 1 \right) - (\gcd(a-1, b-1) - 1) + b - 1 \\ &= \frac{ab - \gcd(a-1, b-1) - 1}{2}. \end{aligned}$$

And the 1-st Hodge ideal of D_3 is

$$\begin{aligned} J_1(D_3) &= (f) + I_0(D_3) \cdot (Jf) \\ &= \big(x^a y + x y^b, x^{2a-1}, y^{2b-1} x^{2a-2} y, \dots, x^{2a-r-1} y^{[\frac{(b-1)(r-1)}{a-1}]+2}, \\ &\qquad \dots, x^{a-1+ir} y^{b-\frac{i(b-1)r}{a-1}}, \dots, x^{a+(i-1)r} y^{[\frac{(b-1)(a-1-(i-1)r-1)}{a-1}]+2}, \\ &\qquad\qquad \dots, x^{a-1+r} y^{b-\frac{(b-1)r}{a-1}}, \dots, x^a y^{[\frac{(b-1)(a-2)}{a-1}]+2}\big), \end{aligned}$$

where $1 \le i \le \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_1(D_3))$ equals $a+1$. The dimension of the 1-st Hodge moduli algebra $M_1(D_3) = \mathcal{O}_X / J_1(D_3)$, by Lemma 3.1, is

$$\begin{aligned} m_1(D_3) &= (2b-1) + (a-2)b + b + \sum_{i=1}^{a-2} \left(\left[\frac{(b-1)i}{a-1} \right] + 2 \right) - (\gcd(a-1, b-1) - 1) + 1 \\ &= \frac{a(3b+2) - \gcd(a-1, b-1) - 3}{2}. \end{aligned}$$

If $a \geq b$, let $r = \frac{b-1}{\gcd(a-1,b-1)}$; then $1 \leq r \leq b-1$. By symmetry of a, b, we obtain the 0-th Hodge ideal

$$\begin{aligned}
J_0(D_3) &= I_0(D_3)\\
&= \big(y^{b-1}, y^{b-2}x^{[\frac{a-1}{b-1}]+1}, \dots, y^{b-r}x^{[\frac{(a-1)(r-1)}{b-1}]+1}, \dots,\\
&\qquad y^{ir}x^{a-1-\frac{i(a-1)r}{b-1}}, y^{ir-1}x^{[\frac{(a-1)(b-ir)}{b-1}]+1}, \dots, y^{(i-1)r+1}x^{[\frac{(a-1)(b-1-(i-1)r-1)}{b-1}]+1},\\
&\qquad\qquad \dots, y^r x^{a-1-\frac{(a-1)r}{b-1}}, y^{r-1}x^{[\frac{(a-1)(b-r)}{b-1}]+1}, \dots, yx^{[\frac{(a-1)(b-2)}{b-1}]+1}, x^{a-1}\big),
\end{aligned}$$

where $1 \leq i \leq \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_0(D_3))$ equals $b-1$. The dimension of the 0-th Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X/J_0(D_3)$, by Lemma 3.1, is

$$\begin{aligned}
m_0(D_3) &= \sum_{i=1}^{b-2}\left(\left[\frac{(a-1)i}{b-1}\right]+1\right) - (\gcd(a-1, b-1)-1) + a - 1\\
&= \frac{ab - \gcd(a-1, b-1) - 1}{2}.
\end{aligned}$$

And the 1-st Hodge ideal of D_3 is

$$\begin{aligned}
J_1(D_3) &= (f) + I_0(D_3)\cdot(Jf)\\
&= \big(x^a y + xy^b, x^{2a-1}, y^{2b-1}y^{2b-2}x, \dots, y^{2b-r-1}x^{[\frac{(a-1)(r-1)}{b-1}]+2},\\
&\qquad \dots, y^{b-1+ir}x^{a-\frac{i(a-1)r}{b-1}}, \dots, y^{b+(i-1)r}x^{[\frac{(a-1)(b-1-(i-1)r-1)}{b-1}]+2},\\
&\qquad\qquad \dots, y^{b-1+r}x^{a-\frac{(a-1)r}{b-1}}, \dots, y^a x^{[\frac{(a-1)(b-2)}{b-1}]+2}\big),
\end{aligned}$$

where $1 \leq i \leq \gcd(a-1, b-1)$. Its multiplicity $\mathrm{mt}(J_1(D_3))$ equals $b+1$. The dimension of the 1-st Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X/J_1(D_3)$, by Lemma 3.1, is

$$\begin{aligned}
m_1(D_3) &= (2a-1)+(b-2)a+a+\sum_{i=1}^{b-2}\left(\left[\frac{(a-1)i}{b-1}\right]+2\right)-(\gcd(a-1, b-1)-1)+1\\
&= \frac{(3a+2)b-\gcd(a-1, b-1)-3}{2}.
\end{aligned}$$

4. Proof of Main Theorem A

I. We compare singularities of types $\mathbb{F}_1$ and $\mathbb{F}_2$:

(1) Suppose for singularities

$$\begin{aligned}
D_1 &= \{x^{a_1} + y^{b_1} = 0\}, \quad && 2 \leq a_1 \leq b_1,\\
D_2 &= \{x^{a_2} + xy^{b_2} = 0\}, \quad && 1 \leq a_2 - 1 \leq b_2,
\end{aligned}$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_2), \quad M_1(D_1) \simeq M_1(D_2).$$

By our computation in Section 3 we have

$$\begin{aligned}\mathrm{mt}(J_0(D_1)) &= a_1 - 1,\\ \mathrm{mt}(J_1(D_1)) &= a_1,\\ m_0(D_1) &= \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\ m_1(D_1) &= \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.\end{aligned}$$

And

$$\begin{aligned}\mathrm{mt}(J_0(D_2)) &= a_2 - 1,\\ \mathrm{mt}(J_1(D_2)) &= a_2,\\ m_0(D_2) &= \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\ m_1(D_2) &= \frac{a_2(3b_2-1)-\gcd(a_2-1,b_2)+1}{2}.\end{aligned}$$

Hence we obtain the equations

$$\begin{gathered}a_1 - 1 = a_2 - 1,\\ a_1 = a_2,\\ \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2},\\ \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{a_2(3b_2-1)-\gcd(a_2-1,b_2)+1}{2},\end{gathered}$$

that is,

$$\begin{gathered}a_1 = a_2,\\ (a_1-1)b_1 = a_2b_2,\\ \gcd(a_1,b_1) - \gcd(a_2-1,b_2) = a_2 - (a_1-1).\end{gathered}$$

Its solutions are $(a_1, b_1) = (a_2, a_2m)$, $(a_2, b_2) = (a_2, (a_2-1)m)$, where $a_2, m \in \mathbb{N}$, $a_2 \geq 2$, $m \geq 1$. And we have

$$\begin{aligned}\mathrm{wt}(F_1) &= \left\{\frac{1}{a_1}, \frac{1}{b_1}\right\} = \left\{\frac{1}{a_2}, \frac{1}{a_2m}\right\},\\ \mathrm{wt}(F_2) &= \left\{\frac{1}{a_2}, \frac{a_2-1}{a_2b_2}\right\} = \left\{\frac{1}{a_2}, \frac{1}{a_2m}\right\}.\end{aligned}$$

It follows that $\mathrm{wt}(F_1) = \mathrm{wt}(F_2)$. Under these conditions, we obtain

$$\begin{aligned}J_0(D_1) &= (x^{a_2-1}, x^{a_2-2}y^{m-1}, \dots, y^{(a_2-1)m-1}),\\ J_0(D_2) &= (x^{a_2-1}, x^{a_2-2}y^{m-1}, \dots, y^{(a_2-1)m-1}),\end{aligned}$$

i.e., $J_0(D_1) = J_0(D_2)$, which shows $M_0(D_1) \simeq M_0(D_2)$ directly.

(2) Suppose for singularities

$$D_1 = \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \le a_1 \le b_1,$$
$$D_2 = \{x^{a_2} + xy^{b_2} = 0\}, \quad a_2 - 1 \ge b_2 \ge 1,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_2), \quad M_1(D_1) \simeq M_1(D_2).$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_1)) &= a_1 - 1,\\
\mathrm{mt}(J_1(D_1)) &= a_1,\\
m_0(D_1) &= \frac{(a_1-1)(b_1-1) - \gcd(a_1, b_1) + 1}{2},\\
m_1(D_1) &= \frac{(a_1-1)(3b_1-1) - \gcd(a_1, b_1) + 1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_2)) &= b_2 - 1,\\
\mathrm{mt}(J_1(D_2)) &= b_2 + 1,\\
m_0(D_2) &= \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},\\
m_1(D_2) &= \frac{(3a_2+2)(b_2-1) - \gcd(a_2-1, b_2) + 3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{gathered}
a_1 - 1 = b_2 - 1,\\
a_1 = b_2 + 1,\\
\frac{(a_1-1)(b_1-1) - \gcd(a_1, b_1) + 1}{2} = \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},\\
\frac{(a_1-1)(3b_1-1) - \gcd(a_1, b_1) + 1}{2} = \frac{(3a_2+2)(b_2-1) - \gcd(a_2-1, b_2) + 3}{2}.
\end{gathered}$$

It has no solution.

II. We compare singularities of types $\mathbb{F}_2$ and $\mathbb{F}_3$:

(1) Suppose for singularities

$$D_2 = \{x^{a_2} + xy^{b_2} = 0\}, \quad a_2 - 1 \ge b_2 \ge 1,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \qquad 1 \le a_3 \le b_3,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_2) \simeq M_0(D_3), \quad M_1(D_2) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_2)) &= b_2 - 1,\\
\mathrm{mt}(J_1(D_2)) &= b_2 + 1,\\
m_0(D_2) &= \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},\\
m_1(D_2) &= \frac{(3a_2+2)(b_2-1) - \gcd(a_2-1, b_2) + 3}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_3)) &= a_3 - 1,\\
\mathrm{mt}(J_1(D_3)) &= a_3 + 1,\\
m_0(D_3) &= \frac{a_3 b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
m_1(D_3) &= \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{gathered}
b_2 - 1 = a_3 - 1,\\
b_2 + 1 = a_3 + 1,\\
\frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2} = \frac{a_3 b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
\frac{(3a_2+2)(b_2-1) - \gcd(a_2-1, b_2) + 3}{2} = \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2},
\end{gathered}$$

that is,

$$\begin{aligned}
b_2 &= a_3,\\
a_2 b_2 + b_2 - a_2 &= a_3 b_3 + a_3 - 1,\\
\gcd(a_2-1, b_2) - \gcd(a_3-1, b_3-1) &= a_3 - (b_2 - 1).
\end{aligned}$$

Its solutions are $(a_2, b_2) = (mb_2+1, b_2)$, $(a_3, b_3) = (b_2, m(b_2-1)+1)$, where $b_2, m \in \mathbb{N}$, $b_2 \geq 2$, $m \geq 1$. And we have

$$\begin{aligned}
\mathrm{wt}(F_2) &= \left\{\frac{1}{a_2}, \frac{a_2-1}{a_2 b_2}\right\} = \left\{\frac{1}{mb_2+1}, \frac{m}{mb_2+1}\right\},\\
\mathrm{wt}(F_3) &= \left\{\frac{b_3-1}{a_3 b_3 - 1}, \frac{a_3-1}{a_3 b_3 - 1}\right\} = \left\{\frac{m}{mb_2+1}, \frac{1}{mb_2+1}\right\}.
\end{aligned}$$

It follows that $\mathrm{wt}(F_2) = \mathrm{wt}(F_3)$. Under these conditions, we obtain

$$\begin{aligned}
J_0(D_2) &= (x^{m(b_2-1)}, x^{m(b_2-2)}y, \dots, y^{b_2-1}),\\
J_0(D_3) &= (y^{m(b_2-1)}, y^{m(b_2-2)}x, \dots, x^{b_2-1}),
\end{aligned}$$

i.e., $J_0(D_2) \cong J_0(D_3)$, $x \mapsto y$, which shows $M_0(D_2) \cong M_0(D_3)$ directly.

(2) Suppose for singularities

$$D_2 = \{x^{a_2} + xy^{b_2} = 0\}, \qquad 1 \le a_2 - 1 \le b_2,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \quad 1 \le a_3 \le b_3,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_2) \simeq M_0(D_3), \quad M_1(D_2) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\mathrm{mt}(J_0(D_2)) &= a_2 - 1,\\
\mathrm{mt}(J_1(D_2)) &= a_2,\\
m_0(D_2) &= \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},\\
m_1(D_2) &= \frac{a_2(3b_2-1) - \gcd(a_2-1, b_2) + 1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_3)) &= a_3 - 1,\\
\mathrm{mt}(J_1(D_3)) &= a_3 + 1,\\
m_0(D_3) &= \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
m_1(D_3) &= \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_2 - 1 &= a_3 - 1,\\
a_2 &= a_3 + 1,\\
\frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2} &= \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
\frac{a_2(3b_2-1) - \gcd(a_2-1, b_2) + 1}{2} &= \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

It has no solution.

III. We compare singularities of types $\mathbb{F}_1$ and $\mathbb{F}_3$:

(1) Suppose for singularities

$$D_1 = \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \le a_1 \le b_1,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \quad 1 \le a_3 \le b_3,$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_3), \quad M_1(D_1) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\operatorname{mt}(J_0(D_1)) &= a_1 - 1,\\
\operatorname{mt}(J_1(D_1)) &= a_1,\\
m_0(D_1) &= \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\
m_1(D_1) &= \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\operatorname{mt}(J_0(D_3)) &= a_3 - 1,\\
\operatorname{mt}(J_1(D_3)) &= a_3 + 1,\\
m_0(D_3) &= \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2},\\
m_1(D_3) &= \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_1 - 1 &= a_3 - 1,\\
a_1 &= a_3 + 1,\\
\frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2} &= \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2},\\
\frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} &= \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2}.
\end{aligned}$$

It has no solutions.

(2) Suppose for singularities

$$\begin{aligned}
D_1 &= \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \le a_1 \le b_1,\\
D_3 &= \{x^{a_3}y + xy^{b_3} = 0\}, \quad a_3 \ge b_3 \ge 1,
\end{aligned}$$

their 0-th and 1-st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_3), \quad M_1(D_1) \simeq M_1(D_3)$$

By our computation in Section 3, we have

$$\begin{aligned}
\operatorname{mt}(J_0(D_1)) &= a_1 - 1,\\
\operatorname{mt}(J_1(D_1)) &= a_1,\\
m_0(D_1) &= \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2},\\
m_1(D_1) &= \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{mt}(J_0(D_3)) &= b_3 - 1,\\
\mathrm{mt}(J_1(D_3)) &= b_3 + 1,\\
m_0(D_3) &= \frac{a_3b_3-\gcd(a_3-1, b_3-1)-1}{2},\\
m_1(D_3) &= \frac{(3a_3+2)b_3-\gcd(a_3-1, b_3-1)-3}{2}.
\end{aligned}$$

Hence we obtain the equations

$$\begin{aligned}
a_1 - 1 &= b_3 - 1,\\
a_1 &= b_3 + 1,\\
\frac{(a_1-1)(b_1-1)-\gcd(a_1, b_1)+1}{2} &= \frac{a_3b_3-\gcd(a_3-1, b_3-1)-1}{2},\\
\frac{(a_1-1)(3b_1-1)-\gcd(a_1, b_1)+1}{2} &= \frac{(3a_3+2)b_3-\gcd(a_3-1, b_3-1)-3}{2}.
\end{aligned}$$

It has no solution.

5. Proof of Main Theorem B

(1) For isolated quasihomogeneous curve singularity

$$D_1^{(a,b)} = \{x^a + y^b = 0\}, \quad a, b \geq 2,$$

since the 0-th Hodge moduli number is

$$m_0(D_1^{(a,b)}) = \frac{(a-1)(b-1)-\gcd(a, b)+1}{2},$$

we have

$$\delta_1(a, b) - m_0(D_1^{(a,b)}) = \gcd(a, b) - 1 \geq 0.$$

And we also have

$$\delta_1(a, b) - m_0(D_1^{(a,b)}) = \gcd(a, b) - 1 \leq \min\{a, b\} - 1 = \mathrm{mt}(D_1^{(a,b)}) - 1.$$

The equality holds if and only if $\min\{a, b\} = \gcd(a, b)$, i.e., $(a, b) = (a, am)$ or $(a, b) = (bm', b)$ for some $m, m' \in \mathbb{N}$.

(2) For isolated quasihomogeneous curve singularity

$$D_2^{(a,b)} = \{x^a + xy^b = 0\}, \quad a \geq 2, b \geq 1,$$

since the 0-th Hodge moduli number is

$$m_0(D_2^{(a,b)}) = \frac{a(b-1)-\gcd(a-1, b)+1}{2},$$

we have

$$\delta_2(a, b) - m_0(D_2^{(a,b)}) = \gcd(a-1, b) \geq 1.$$

And we also have

$$\delta_2(a,b) - m_0(D_2^{(a,b)}) = \gcd(a-1,b) \le \min\{a-1,b\} = \mathrm{mt}(D_2^{(a,b)}) - 1.$$

The equality holds if and only if $\min\{a-1,b\} = \gcd(a-1,b)$, i.e., $(a,b) = (a,(a-1)m)$ or $(a,b) = (bm'+1,b)$ for some $m,m' \in \mathbb{N}$.

(3) For isolated quasihomogeneous curve singularity

$$D_3^{(a,b)} = \{x^a y + xy^b = 0\}, \quad a,b \ge 1,$$

since the 0-th Hodge moduli number is

$$m_0(D_3^{(a,b)}) = \frac{ab - \gcd(a-1,b-1)+1}{2},$$

we have

$$\delta_3(a,b) - m_0(D_3^{(a,b)}) = \gcd(a-1,b-1)+1 \ge 2.$$

And we also have

$$\begin{aligned}\delta_3(a,b) - m_0(D_3^{(a,b)}) &= \gcd(a-1,b-1)+1 \le \min\{a-1,b-1\}+1 \\ &= \min\{a,b\} = \mathrm{mt}(D_3^{(a,b)}) - 1.\end{aligned}$$

The equality holds if and only if $\min\{a-1,b-1\} = \gcd(a-1,b-1)$, i.e., $(a,b) = (a,(a-1)m+1)$ or $(a,b) = ((b-1)m'+1,b)$ for some $m,m' \in \mathbb{N}$.

6. Some examples and conjectures

Example 6.1. Let curve singularities $H_1 = \{x^2y + xy^6 = 0\}$ be defined by a polynomial $f(x,y) = x^2y + xy^6$ and $H_2 = \{x^3y + xy^4 = 0\}$ be defined by a polynomial $g(x,y) = x^3y + xy^4$. Then f is quasihomogeneous of weight type $\left(\frac{5}{11}, \frac{1}{11}; 1\right)$ and g is quasihomogeneous of weight type $\left(\frac{3}{11}, \frac{2}{11}; 1\right)$. In [16], the characteristic polynomials of f and g coincide:

$$\Delta_f(t) = (t-1)(t^{11}-1) = \Delta_g(t).$$

So this tells us that the characteristic polynomial does not determine the weights of the nondegenerate quasihomogeneous polynomial defining the singularity.

However, their i-th Hodge moduli algebras $M_i(D^\alpha)$ are not isomorphic for $i \ge i_0(\alpha)$, for a big enough $i_0(\alpha)$. Precisely speaking,

$$M_i(D_1^\alpha) \not\cong M_i(D_2^\alpha) \quad \text{for all } i \ge 1,$$

where $D_j^\alpha = \alpha H_j$, $j = 1, 2$, for $\alpha = 1$. In fact, we just simply observe this result by their Hodge moduli numbers are different for $i \ge 1$ as follows:

singularity	weight type	$m_0(D)$	$m_1(D)$	$m_2(D)$	$m_3(D)$	$m_4(D)$	$m_5(D)$
$x^2y + xy^6$	$\left(\frac{5}{11}, \frac{1}{11}; 1\right)$	5	18	32	46	60	74
$x^3y + xy^4$	$\left(\frac{3}{11}, \frac{2}{11}; 1\right)$	5	19	39	49	64	79

Example 6.2. Let $f(z_1, \ldots, z_n, w_1, w_2) = z_1^2 + \cdots + z_n^2 + w_1^3 + w_2^{2p}$ be a quasihomogeneous polynomial of weight type $\left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{3}, \frac{1}{2p}; 1\right)$ with an isolated singularity at the origin for any $p \in \mathbb{N}$. Let $n \geq 0$, even and $\gcd(3, p) = 1$. Then we know their characteristic polynomials are

$$\Delta_f(t) = \frac{t^{4p} + t^{2p} + 1}{t^2 + t + 1} \quad \text{for all } n \geq 0, \text{even.}$$

Hence $\Delta_f(1) = 1$. By Theorem 8.5 in [4], each of their links $K_f = S_\epsilon \cap \{f(z, w) = 0\}$ is a topological sphere. Thus all K_f for all p, $(3, p) = 1$, are homeomorphic to each other though $f(z_1, \ldots, z_n, w_1, w_2)$ are of the different quasihomogeneous types for all p.

However, their i-th Hodge moduli algebras $M_i(D^\alpha)$ are not isomorphic for all $i \geq 1$ and $n \geq 2$, where $D^\alpha = \{f(z_1, \ldots, z_n, w_1, w_2) = 0\}$ for $\alpha = 1$. In fact, we have their 0-th Hodge ideal

$$I_0(D) = \begin{cases} (1) & \text{if } n \geq 2, \text{ or } n = 0,\ p = 1, 2, \\ (w_1, w_2^{i_0}) & \text{if } n = 0,\ p \geq 4, \end{cases}$$

where $i_0 = \left\lceil \frac{4p}{3} \right\rceil - 1$, is the smallest integer bigger than or equal to $\frac{4p}{3} - 1$. Then we compute their 1-st Hodge ideal as follows. For example, if $n = 2$, we have

$$\begin{aligned} I_1^{(2)}(D) &= \sum_{v_j \in \mathcal{O}^{\geq 2}} \mathcal{O}_X \cdot v_j + \sum_{\substack{1 \leq i \leq 4 \\ a \in I_0(D)}} \mathcal{O}_X(f\partial_i a - \alpha a \partial_i f) \\ &= (w_2^{i_1}, w_1 w_2^{j_1}) + (z_1, z_2, w_1^2, w_2^{2p-1}) \\ &= (z_1, z_2, w_1^2, w_1 w_2^{j_1}, w_2^{i_1}), \\ J_1^{(2)}(D) &= (f) + I_1^{(2)}(D) \\ &= (z_1, z_2, w_1^2, w_1 w_2^{j_1}, w_2^{i_1}), \end{aligned}$$

where $i_1 = \left\lceil \frac{4p}{3} \right\rceil - 1$ and $j_1 = \left\lceil \frac{2p}{3} \right\rceil - 1$. And if $n = 4$, we have

$$\begin{aligned} I_1^{(4)}(D) &= \sum_{v_j \in \mathcal{O}^{\geq 2}} \mathcal{O}_X \cdot v_j + \sum_{\substack{1 \leq i \leq 6 \\ a \in I_0(D)}} \mathcal{O}_X(f\partial_i a - \alpha a \partial_i f) \\ &= (z_1, z_2, z_3, z_4, w_1^2, w_2^{2p-1}), \\ J_1^{(4)}(D) &= (f) + I_1^{(4)}(D) \\ &= (z_1, z_2, z_3, z_4, w_1^2, w_2^{2p-1}). \end{aligned}$$

So we obtain their corresponding Hodge moduli algebras

$$\begin{aligned} M_1^{(2)}(D) &= \mathbb{C}\{z_1, z_2, w_1, w_2\}/I_1^{(2)}(D) = \mathbb{C}\{w_1, w_2\}/(w_1^2, w_1 w_2^{j_1}, w_2^{i_1}), \\ M_2^{(4)}(D) &= \mathbb{C}\{z_1, z_2, z_3, z_4, w_1, w_2\}/I_1^{(4)}(D) = \mathbb{C}\{w_1, w_2\}/(w_1^2, w_2^{2p-1}), \end{aligned}$$

which are not isomorphic obviously, since one can verify

$$\dim_{\mathbb{C}} M_1^{(2)}(D) = i_1 + j_1 < 2(2p-1) = \dim_{\mathbb{C}} M_1^{(4)}(D).$$

Thus these examples imply that Hodge moduli algebras and Hodge moduli numbers (or the Hodge moduli sequence) are better invariants than the characteristic polynomial (a topological invariant of the singularity) for nondegenerate quasihomogeneous singularities.

It is an interesting question interesting whether Hodge numbers, Hodge ideals and Hodge moduli algebras of singularities remain constant or isomorphic under some deformations, like quasihomogeneous or semiquasihomogeneous deformations (or, more generally, μ-constant deformations). We give an example to explain that the Hodge moduli numbers of isolated singularities may remain constant under quasihomogeneous deformation.

Example 6.3. For quasihomogeneous polynomial

$$f = x^2 + y^4$$

of weight $\operatorname{wt}(f) = \left(\frac{1}{2}, \frac{1}{4}; 1\right)$, let divisor $D_1^{\alpha} = \{f = 0\}$, where $\alpha = 1$. Then its 1-st Hodge ideal and Hodge moduli algebra are

$$J_1(D_1) = (x^2, xy, y^4),$$
$$M_1(D_1) = \mathbb{C}\{x, y\}/(x^2, xy, y^4).$$

And for quasihomogeneous polynomial $g = x^2 + y^4 + xy^2$ of weight $\operatorname{wt}(g) = \left(\frac{1}{2}, \frac{1}{4}; 1\right)$, which is a quasihomogeneous deformation of f, let divisor $D_2^{\alpha} = \{g = 0\}$, where $\alpha = 1$. Then its 1-st Hodge ideal and Hodge moduli algebra are

$$J_1(D_2) = (x^2, xy^2, 2xy + y^3, y^4),$$
$$M_1(D_2) = \mathbb{C}\{x, y\}/(x^2, xy^2, 2xy + y^3, y^4).$$

As $\mathbb{C}$-vector spaces, $M_1(D_1)$ and $M_1(D_2)$ have the same the $\mathbb{C}$-basis:

$$1, x, y, y^2, y^3.$$

Thus, the Hodge moduli numbers of D_1 and D_2 are the same.

So we raise a conjecture from the above example.

Conjecture 6.4. Suppose F_i is one of the three types[1] of quasihomogeneous polynomial in $\mathbb{C}^2$, $1 \le i \le 3$. Let $H_{i,t} = F_i + tG_i$ be a semiquasihomogeneous deformation of F_i, $t \in \mathbb{C}$, $1 \le i \le 3$. Then the k-th Hodge moduli algebras of the divisors $D_H^{\alpha} = \{H_{i,t} = 0\}$ for $\alpha = 1$, and $D_F^{\alpha} = \{F_i = 0\}$ for $\alpha = 1$, have the same basis

[1]See pages 351–352 in the introduction.

over $\mathbb{C}$ for all $k \geq 0$. Hence, their dimensions, i.e., their k-th Hodge moduli numbers, are the same,

$$m_k(D_H^\alpha) = m_k(D_F^\alpha) \quad \forall k \geq 0, \forall 1 \leq i \leq 3, \forall t \in \mathbb{C}.$$

And we can also ask whether the inequalities in Main Theorem B can be extended to more general singularities. Suppose F_i is one of the three types of quasihomogeneous curve singularities, $1 \leq i \leq 3$. Consider a μ-constant deformation $H_{i,t} = F_i + tG_i$ of F_i, $t \in \mathbb{C}$. If we furthermore assume $H_{i,t}$ is reduced, i.e., all distinct irreducible factors of $H_{i,t}$ have multiplicity 1, we have

$$\mu(H_{i,t}) = \mu(F_i),$$
$$r(H_{i,t}) \leq r(F_i).$$

By Lemma 2.7, we have

$$\delta(H_{i,t}) \leq \delta(F_i),$$

where μ, δ and r are the same notation as in Lemma 2.7. So we have a corollary:

Corollary 6.5. *Suppose the above Conjecture 6.4 is true. For a semiquasihomogeneous deformation $H_{i,t}$, $t \in \mathbb{C}$, of F_i, we have*

$$\delta(D_{i,t}) - m_0(D_{i,t}) \leq \mathrm{mt}(D_{i,t}),$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$, where $D_{i,t} = \{H_{i,t} = 0\}$ is the corresponding singularity, $\delta(D_{i,t})$ is the δ-invariant of $D_{i,t}$, $m_0(D_{i,t})$ is the 0-th Hodge moduli number of $D_{i,t}$, and $\mathrm{mt}(D_{i,t})$ is the multiplicity of $D_{i,t}$.

Proof. In fact, one can verify the multiplicity of F_i is not decreasing under the above semiquasihomogeneous deformation for all $1 \leq i \leq 3$, i.e.,

$$\mathrm{mt}(D_{i,t}) \geq \mathrm{mt}(D_{i,0})$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. And by Conjecture 6.4, we have

$$m_0(D_{i,t}) = m_0(D_{i,0})$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. Finally, by the discussion after Conjecture 6.4, we have

$$\delta(D_{i,t}) \leq \delta(D_{i,0})$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. So we have

$$\delta(D_{i,t}) - m_0(D_{i,t}) \leq \delta(D_{i,0}) - m_0(D_{i,0}) \leq \mathrm{mt}(D_{i,0}) \leq \mathrm{mt}(D_{i,t}),$$

for any $t \in \mathbb{C}$ such that $H_{i,t}$ is a reduced polynomial, $1 \leq i \leq 3$. $\square$

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YANG WANG
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA
wangyang21@mails.tsinghua.edu.cn

STEPHEN S.-T. YAU
BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS (BIMSA)
BEIJING
CHINA
yau@uic.edu
and
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA

HUAIQING ZUO
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA
hqzuo@mail.tsinghua.edu.cn

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