Pacific Journal of Mathematics

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Volume 335 No. 1 March 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 335, No. 1, 2025
https://doi.org/10.2140/pjm.2025.335.1

CENTRAL NILPOTENCY OF LEFT SKEW BRACES AND SOLUTIONS OF THE YANG–BAXTER EQUATION

ADOLFO BALLESTER-BOLINCHES, RAMÓN ESTEBAN-ROMERO, MARIA FERRARA, VICENT PÉREZ-CALABUIG AND MARCO TROMBETTI

This paper delves into the study of centrally nilpotent skew braces. In particular, we study their torsion theory, we introduce an "index" for subbraces, but we also show that the product of centrally nilpotent ideals need not be centrally nilpotent. To cope with these examples, we introduce a special type of nilpotent ideal, using which, we define a *good* Fitting ideal. Also, a Frattini ideal is defined and its relationship with the Fitting ideal is investigated. A key ingredient is the characterisation of the commutator of ideals in terms of star products, which solves a problem of Bonatto and Jedlička (*J. Algebra Appl.* 22:12 (2023), art. id. 2350255). Moreover, we provide an example showing that the idealiser of a subbrace does not exist in general.

1. Introduction

The study of set-theoretic solutions of the Yang–Baxter equation (YBE) provides a common framework for a multidisciplinary approach from different areas including knot theory, braid theory, and Garside theory among others (see [10; 12] for example). The main challenge in this area is to classify all set-theoretic solutions with prescribed properties. The algebraic structure of left skew braces plays a fundamental role in this classification. Nondegenerate set-theoretic solutions of the YBE naturally lead to left skew braces (see [18]), and conversely, every left skew brace B defines a solution (B, r_B) of the YBE (see [14]). Nowadays, we are far from being able to understand arbitrary solutions of the YBE. But it is becoming more and more clear that almost every nondegenerate solution is a multipermutation, and nilpotency of left skew braces is precisely introduced to deal with multipermutation solutions (see [7; 13; 16] for example). In this paper, we provide a deep and complete study of central nilpotency with new results that could be a reference point for all future work on the argument.

We now highlight some of the main aspects of nilpotency of left skew braces we deal with (see the next sections for the definitions): In Section 4.1, we study

MSC2020: primary 16T25, 20F18; secondary 81R50.

Keywords: brace, Yang–Baxter equation, nilpotency, Fitting ideal, commutator of ideals, idealiser.

its torsion theory, providing great extensions of some of the results in [16] (see, in particular, Theorem 4.14). In Section 4.2, we deal with the problem of defining an index for subbraces, proving that this is possible in the context of locally centrally nilpotent left skew braces. Also we provide many examples showing how peculiar is the behaviour of centrally nilpotent left skew braces if compared to that of groups and rings. Among these examples, two are the most striking (see Section 6): Example A shows that, contrary to what is claimed in [16], it is not always possible to define the idealiser of a subbrace of a left skew brace of abelian type. Example B shows that the product of two centrally nilpotent ideals of a left skew brace (of abelian type) need not be centrally nilpotent. In order to cope with the above examples, we introduce a new type of nilpotency for ideals (see Section 5). Using this new concept we are able to define a well-behaving *Fitting ideal* for left skew braces. In turns, the Fitting ideal makes it possible to give a good definition of *Frattini ideal* for left skew braces. Using these definitions we are able to prove an analogue of a celebrated result of Gaschütz that relates the Frattini and the Fitting subgroups of a group (see Theorem 5.10). Finally, we note that one of the key steps in our work is showing that the two known definitions of commutators of ideals (inspired by distinct universal algebra approaches) coincide, thus solving Problem 3.4 of [4] (see Section 3).

2. Preliminaries

From now on, the word "brace" will mean "left skew brace". We refer to [1; 2; 3; 20] for the standard preliminaries about braces. However, we recall that if E is any subset of a brace B, then $\langle E\rangle$ is the subbrace generated by E in B, E^B is the smallest ideal of B containing E, and E_B is the maximal ideal of B contained in E. Moreover, $C \leq B$ (resp., $C \trianglelefteq B$) denotes that C is a subbrace (resp., an ideal) of B, while $C \lessdot B$ denotes that C is a maximal subbrace of B. The following commutator of ideals is introduced in [5] and plays a key role in the study of nilpotency and solubility in braces.

Definition 2.1. Let I, J be ideals of a brace B. The *commutator* $[I, J]^B$ is defined as $[I, J]^B := \big\langle [I, J]_+ \cup [I, J]_\cdot \cup \{ij - (i + j) : i \in I,\ j \in J\}\big\rangle^B$. Clearly, $[I, J]^B = [J, I]^B$ and $[I, J]^B \leq I \cap J$.

A brace B is said to be *abelian* if $[B, B]^B = 0$, i.e., if it is a trivial brace of abelian type. Using this commutator, solubility of braces has been introduced in [2]. If I is an ideal of B, the *commutator* or *derived ideal of I with respect to B* is defined as $\partial_B(I) = [I, I]^B$. If $B = I$, then we say that $\partial_B(B) = \partial(B) = \partial_1(B)$ is the *commutator* or *derived ideal* of B. By repeatedly forming derived ideals, we recursively obtain a descending sequence of ideals with $\partial_n(I) = \partial_B(\partial_{n-1}(I))$ for every $n \in \mathbb{N}$. We call this series the *derived series of I with respect to B*. Clearly,

$\partial_{n-1}(B)/\partial_n(B)$ is an abelian brace for every $n \in \mathbb{N}$, and $B/\partial(B)$ is the greatest abelian quotient in B. We say that an ideal I of B is *soluble with respect to* B, if there exists a nonnegative integer n such that $\partial_n(I) = 0$. If $I = B$, we simply say that B is a *soluble brace*, and the smallest nonnegative integer m for which $\partial_m(B) = 0$ is the *derived length* of B.

3. Unifying universal algebra definitions of commutators of ideals in braces

The definition of commutator of ideals given in Section 2 was inspired by the *Huq = Smith condition* for the category of braces (see [5] for further details). On the other hand, following ideas of universal algebra, a definition of a commutator of ideals of a brace in terms of absorbing polynomials is provided in [4] as a brace-theoretic version of a commutator of congruences given by Freese and McKenzie in [11]. In case of modular lattices of congruences, it turns out that the Freese and McKenzie commutator coincides with the Hagemann and Herrmann commutator (see [11, Theorem 4.9]). According to [15], the Hagemann and Herrmann commutator was introduced as an extension of the Smith commutator so that both commutators coincide in case of permutable lattices of congruences.

Observe that the lattice of ideals of a brace is modular and permutable, since the product of ideals commute and for all ideals I, J, K with $I \leq J$, the Dedekind identity holds. Hence, the definitions of commutator of ideals of a brace given in [4; 5] coincide.

In this context, the following questions were raised in [4, Problem 3.4].

Problem 3.1. Let B be a brace.

(1) Does the equality $[I, J]^B = \langle I * J + J * I + [I, J]_+\rangle^B$ hold?

(2) Does the equality $[I, J]^B = I * J + J * I + [I, J]_+$ hold?

Our next theorem gives a positive answer to the first question.

Remark 3.2. Let B be a brace. Then, for every $i, j, x, y, z \in (B, +)$, we have that

$$i + x + j + y - i + z - j = [-i, -x]_+ + x + [-i, -j]_+ + [-i, -y]_+^{-j,+} + [-j, -y]_+ + y + [-j, -z]_+ + z.$$

Theorem 3.3. *Let I, J be ideals of a brace B.*

(1) *$I * J + J * I + [I, J]_+$ is the least left ideal containing*

$$X_{I,J} = [I, J]_+ \cup [I, J]_\cdot \cup \{ij - (i + j) : i \in I,\ j \in J\}.$$

(2) *$[I, J]^B = \langle I * J + J * I + [I, J]_+\rangle^B$. Thus, $[I, J]^B = I * J + J * I + [I, J]_+$ if and only if $I * J + J * I + [I, J]_+$ is an ideal of B.*

Proof. (1) By [4, Proposition 1.4], $I * J + J * I$ and $[I, J]_+$ are left ideals. Since $[I, J]_+$ is a normal subgroup of $(B, +)$, it follows that $I * J + J * I + [I, J]_+$ is also a left ideal.

For the inclusion $X_{I,J} \subseteq I * J + J * I + [I, J]_+$, we need to prove that

$$\{ij - (i + j) : i \in I,\ \ j \in J\},\ [I, J]_{\cdot} \subseteq I * J + J * I + [I, J]_+.$$

Let $i \in I$ and $j \in J$. For the former, observe that $ij - j - i = i + i * j - i$. Thus, $ij - j - i = i + i * j - i - i * j + i * j \in [I, J]_+ + I * J$. For the latter, it follows that

$$\begin{aligned} iji^{-1}j^{-1} &= iji^{-1} + (iji^{-1}) * j^{-1} + j^{-1} \\ &= ij + (ij) * i^{-1} + i^{-1} + (iji^{-1}) * j^{-1} + j^{-1} \\ &= i + i * j + j + (ij) * i^{-1} + i^{-1} + (iji^{-1}) * j^{-1} + j^{-1}. \end{aligned}$$

Now, the above equation becomes

$$\begin{aligned} &i + i * j + j + i * (j * i^{-1}) + j * i^{-1} + i * i^{-1} + i^{-1} \\ &\quad + (ij) * (i^{-1} * j^{-1}) + i^{-1} * j^{-1} + (ij) * j^{-1} + j^{-1} \\ &\qquad = i + i * j + j + i * (j * i^{-1}) + j * i^{-1} + i * i^{-1} + i^{-1} \\ &\qquad\quad + i * (j * (i^{-1} * j^{-1})) + j * (i^{-1} * j^{-1}) + i * (i^{-1} * j^{-1}) + i^{-1} * j^{-1} \\ &\qquad\quad + i * (j * j^{-1}) + j * j^{-1} + i * j^{-1} + j^{-1}. \end{aligned}$$

Observe that $i * i^{-1} + i^{-1} = -i + 0 - i^{-1} + i^{-1} = -i$ and $j * j^{-1} = -j - j^{-1}$. Thus, we have

$$(\star)\quad iji^{-1}j^{-1} = i + i * j + j + i * (j * i^{-1}) + j * i^{-1} - i + i * (j * (i^{-1} * j^{-1}))$$
$$(\diamond)\quad + j * (i^{-1} * j^{-1}) + i * (i^{-1} * j^{-1}) + i^{-1} * j^{-1} + i * (j * j^{-1}) - j - j^{-1} + i * j^{-1} + j^{-1}.$$

Since $I * J,\ J * I \subseteq I \cap J$, Remark 3.2 applied on $(\star) + (\diamond)$ yields

$$iji^{-1}j^{-1} \in I * J + J * I + [I, J]_+ + (-j^{-1} + i * j^{-1} + j^{-1}).$$

But, $-j^{-1} + i * j^{-1} + j^{-1} = [j^{-1}, -i * j^{-1}]_+ + i * j^{-1} \in [I, J]_+ + I * J$. Hence, $[I, J]_{\cdot} \subseteq I * J + J * I + [I, J]_+$ follows.

Now, let L be the least left ideal of B containing $X_{I,J}$ (note that the arbitrary intersection of left ideals is a left ideal). In order to prove that $I * J + J * I + [I, J]_+ = L$, we only need to show that $J * I \subseteq L$.

Let $j \in J$ and $i \in I$. Then, $j * i = [-j * i, -j]_+ + (ji - i - j) \in X_{I,J} + (ji - i - j)$, as $j * i \in I \cap J$, so it suffices to prove that $(ji - i - j) \in L$. Since $[j, i]_{\cdot} \in X_{I,J} \subseteq L$ and L is λ-invariant, it follows that

$$ji = ij[j, i]_{\cdot} = ij + \lambda_{ij}([j, i]_{\cdot}) = ij + x$$

g	$\delta(g)$	g	$\delta(g)$	g	$\delta(g)$	g	$\delta(g)$
1	0	b	y	c	$x+y$	bc	$3x$
a	x	b^3	$2x+3y$	c^3	$3x+3y$	bc^{-1}	$x+2y$
a^2	$2x+2y$	ab	$x+3y$	ac	$2x+y$	abc	$2y$
a^3	$3x+2y$	ab^{-1}	$3x+y$	ac^{-1}	$3y$	abc^{-1}	$2x$

Table 1. Bijective 1-cocycle associated with the brace of Example 3.4.

for some $x \in L \cap I \cap J$. Therefore,

$$ji - i - j = ij + x - ([-j, -i]_+ + i + j) = ij + x - (i + j) + [-i, -j]_+$$
$$= ij - (i + j) + [-(i + j), -x]_+ + x + [-i, -j]_+ \in L.$$

Consequently, $I * J + J * I + [I, J]_+ = L$.

(2) By definition, $[I, J]^B = \langle X_{I,J}\rangle^B$. Then, the previous statement yields $[I, J]^B = \langle I * J + J * I + [I, J]_+\rangle^B$. □

The next example gives a negative answer to Problem 3.1(2).

Example 3.4. Let $(B, +) = \langle x, y \mid 4x = 4y = 0,\ x + y = y + x\rangle$ and $(C, \cdot) = \langle a, b, c \mid c^4 = 1,\ a^2 = b^2 = c^2,\ (ab^{-1})^2 = 1,\ b^{-1}cb = c,\ a^{-1}ca = c^{-1}\rangle$. We see that C acts on B via $a(x) = x + 2y$, $b(x) = x + 2y$, $c(x) = 3x + 2y$, $a(y) = -y$, $b(y) = 2x + y$, $c(y) = 2x + 3y$. Consider the semidirect product G of B by C with respect to this action $\lambda : C \to \operatorname{Aut}(B)$. For the sake of clarity, we use multiplicative notation for B in G. Let $D = \langle xa, yb, xyc\rangle \le G$. It follows that G is a trifactorised group as $BD = DC$, $D \cap C = D \cap B = 1$. By [1, Lemma 3.2], there exists a bijective 1-cocycle $\delta : C \to (B, +)$ given by $D = \{\delta(c)c : c \in C\}$ (see Table 1). This yields a product in B and we get a brace of abelian type, $(B, +, \cdot)$, which corresponds to `SmallBrace(16, 73)` in the YangBaxter library [21] for GAP.

Let $I = \langle 2x, y\rangle \le (B, +)$. Then $\lambda(I) = I$ and $|I| = 8$, so I is an ideal of B. Since B is of abelian type, we compute

$$I * I + [I, I]_+ = I * I = \langle u * v \mid u, v \in I\rangle_+ = \langle 2x\rangle_+ = \{1, 2x\}.$$

Therefore, $I * I$ is not an ideal of B, as $\delta(abc^{-1}) = 2x$ and $\{1, abc^{-1}\}$ is not a normal subgroup of C. Hence, $I * I \subsetneq [I, I]^B = I$.

4. Central nilpotency of braces

In this section we develop a standard theory of central nilpotency of braces. We start by introducing the basic definitions. Central nilpotency of braces was first introduced by using a brace-theoretic analogue of the centre of a group (see [4; 16]). The *centre* of a brace B (also known as the *annihilator ideal* of B) is the ideal of B defined as $\zeta(B) := \{a \in B \mid a+b = b+a = ab = ba \text{ for all } b \in B\}$ (see [6]). Thus, abelian braces B are precisely those ones satisfying $\zeta(B) = B$. In [8; 20], the definition of central nilpotency has been extended to include more types of braces (see also [9], where similar concepts for braces of abelian type are considered).

Definition 4.1. Let B be a brace. If $J \leq I$ are ideals of B satisfying $I/J \leq \zeta(B/J)$, we say that I/J is a *central factor* of B.

A *c-series* of B is a chain $\mathcal{I}$ of ideals of B such that 0, $B \in \mathcal{I}$ and whose factors are central, that is, $I/J \leq \zeta(B/J)$ for all *consecutive* elements $J \leq I$ of $\mathcal{I}$ (meaning that there is no $C \in \mathcal{I}$ satisfying $J < C < I$). A *complete* c-series is just a c-series containing the unions and the intersections of all its members. Since every c-series can be completed, we usually consider every c-series to be complete. We say that B is *ζ-nilpotent* if it admits a c-series.

If B has an ascending c-series

$$0 = I_0 \leq I_1 \leq \cdots I_\alpha \leq I_{\alpha+1} \leq \cdots I_\mu = B$$

(here $\alpha < \mu$ are ordinal numbers), then B is *hypercentral*, and the smallest ordinal number μ for which such an ascending c-series exists is the *hypercentral length* $n_c(B)$ of B. (Note that $I_{\alpha+1}/I_\alpha \leq \zeta(B/I_\alpha)$ for all ordinals $\alpha < \mu$.)

If B has a finite c-series $0 = I_0 \leq I_1 \leq \cdots \leq I_n = B$, then B is *centrally nilpotent*; in this case, the smallest nonnegative integer for which such a c-series exists is referred to as the *class* $n_c(B)$ of B. (Note that $I_{i+1}/I_i \leq \zeta(B/I_i)$ for all $0 \leq i < n$.)

Of course, centrally nilpotent braces are hypercentral, and hypercentral braces are ζ-nilpotent, but the converses do not hold. Moreover, subbraces of centrally nilpotent (resp. hypercentral, ζ-nilpotent) braces are centrally nilpotent (resp. hypercentral, ζ-nilpotent). Also, quotients of centrally nilpotent (resp. hypercentral) braces are still centrally nilpotent (resp. hypercentral), but the consideration of the infinite dihedral group shows that quotients of a ζ-nilpotent brace can be non-ζ-nilpotent. Finally, direct products of hypercentral braces are hypercentral; direct products of finitely many centrally nilpotent braces are centrally nilpotent; and restricted direct products of ζ-nilpotent braces are ζ-nilpotent.

Following [4; 8], canonical c-series are introduced for a brace B.

(▲) The *upper central series* of B is recursively defined by putting $\zeta_0(B) = 0$, $\zeta_{\alpha+1}(B)/\zeta_\alpha(B) = \zeta(B/\zeta_\alpha(B))$ for every ordinal α, and $\zeta_\lambda(B) = \bigcup_{\beta<\lambda} \zeta_\beta(B)$ for every limit ordinal λ. The last term of the upper central series is the *hypercentre* $\bar{\zeta}(B)$ of B.

(▼) The *lower central series* of B is recursively defined by putting $\Gamma_1(B) = B$, $\Gamma_{\alpha+1}(B) = \langle \Gamma_\alpha(B)*B,\, B*\Gamma_\alpha(B),\, [\Gamma_\alpha(B),\, B]_+\rangle_+$ for every ordinal α, and $\Gamma_\lambda(B) = \bigcap_{\beta<\lambda} \Gamma_\beta(B)$ for every limit ordinal λ. Note that since $\Gamma_\alpha(B)$ is an ideal for every

$\alpha \leq \mu$, we have that

$$\Gamma_{\alpha+1}(B) = \Gamma_\alpha(B) * B + B * \Gamma_\alpha(B) + [\Gamma_\alpha(B), B]_+ = [\Gamma_\alpha(B), B]^B,$$

for every ordinal $\alpha < \mu$ (see Theorem 3.3). The last term of the lower central series is the *hypocentre* $\overline{\Gamma}(B)$ of B.

The following easily provable all-in-one result shows the relations between the concepts we just introduced (see [17] for the definition of the upper central series $\{Z_\alpha(G)\}_{\alpha\in\mathrm{Ord}(G)}$ of a group G).

Proposition 4.2. *Let B be a brace.*

(1) (See [2, Proposition 17].) *If $J \leq I$ are ideals of B, then $I/J \leq \zeta(B/J)$ if and only if $[I, B]^B \leq J$. In particular, if $0 = I_0 \leq I_1 \leq \cdots \leq I_n = B$ is a finite c-series, then*:

- (a) $\Gamma_j(B) \leq I_{n-j+1}$ *for every* $1 \leq j \leq n+1$; *so* $\Gamma_{n+1}(B) = 0$.
- (b) $I_j \leq \zeta_j(B)$ *for every* $0 \leq j \leq n$; *so* $\zeta_n(B) = B$.
- (c) $n_c(B)$ *is the smallest number n such that $\zeta_n(B) = B$, and the smallest number n such that $\Gamma_{n+1}(B) = 0$.*

(2) *B is hypercentral if and only if $B = \bar{\zeta}(B)$. Moreover, in this case $n_c(B)$ is the smallest ordinal α for which $B = \zeta_\alpha(B)$.*

(3) $\zeta_\alpha(B) \subseteq Z_\alpha(B,+) \cap Z_\alpha(B,\cdot)$ *for every ordinal α. In particular, centrally nilpotent* (*resp. hypercentral*) *braces are braces of nilpotent* (*resp. hypercentral*) *type whose additive group is nilpotent* (*resp. hypercentral*).

(4) $\partial_n(B) \leq \Gamma_{n+1}(B)$ *for every $n \in \mathbb{N}$. In particular, centrally nilpotent braces are soluble with derived length less than or equal to their class.*

The following result generalises Grün's lemma for groups (see, for instance, [17, Part 1, p. 48]).

Theorem 4.3. *Let B be a brace such that $\zeta_2(B) > \zeta(B)$. Then either $[B, B]_+$ or $[B, B]_\cdot$ is a proper subset of B.*

Proof. We may assume by Grün's lemma that

$$Z(B,+) = Z_2(B,+) \quad \text{and} \quad Z(B,\cdot) = Z_2(B,\cdot).$$

By Proposition 4.2, $\zeta_2(B) \subseteq Z_2(B,+) \cap Z_2(B,\cdot) = Z(B,+) \cap Z(B,\cdot)$. Now, choose $c \in \zeta_2(B) \setminus \zeta(B)$ and consider the map $\varphi_c : b \in B \mapsto c * b \in \zeta(B)$. Let $b_1, b_2 \in B$. Then $c * (b_1 + b_2) = c * b_1 + c * b_2$ because $c * b_2 \in \zeta(B)$, so φ_c is a homomorphism with respect to $+$. Since $c \notin \zeta(B)$, we have that $c \notin \mathrm{Ker}(\lambda)$ and consequently $\varphi_c(B) \neq 0$. Thus $[B, B]_+$ is properly contained in B. $\square$

In order to see if a brace is centrally nilpotent (resp. hypercentral) or not, we only need to look at its countable subbraces: this is the content of our next result.

Theorem 4.4. *Let B be a brace.*

(1) *B is centrally nilpotent of class at most c if and only if its finitely generated subbraces are centrally nilpotent of class at most c.*

(2) *B is centrally nilpotent* (*hypercentral*) *if and only if its countable subbraces are centrally nilpotent* (*hypercentral*).

Proof. For each $u, v \in B$, the symbol $u \circ v$ means that we are performing one (but we do not know which) of the following operations $[u, v]_{\cdot}$, $[u, v]_{+}$, $u * v$.

(1) The first statement is a direct consequence of the fact that $\zeta_c(B)$ is easily characterised as the set of all elements $b \in B$ with $\big(\cdots((b \circ b_1) \circ \cdots) \circ b_c\big) = 0$ for all $b_1, \ldots, b_c \in B$.

(2) We start by considering central nilpotency. Only one implication is in doubt. Thus, assume all countable subbraces of B are centrally nilpotent but B is not centrally nilpotent. By (1), for each $c \in \mathbb{N}$, there is a finitely generated centrally nilpotent brace B_c whose class is at least c. Then $C = \langle B_i : i \in \mathbb{N}\rangle$ is a countable subbrace of B that is not centrally nilpotent, a contradiction.

Now, we turn to hypercentrality. Again, only one implication is in doubt. Thus, assume all countable subbraces of B are hypercentral, and that B is not hypercentral. We may further assume that $\zeta(B) = 0$. Let $0 \neq x \in B$. Then there are sequences of nonzero elements $a_1, a_2, \ldots, a_n, \ldots$ and $x = b_1, b_2 \ldots, b_n, \ldots$ of B with $b_{n+1} = b_n \circ a_n$ for all $n \in \mathbb{N}$. Let $C = \langle a_i, b_j : i, j \in \mathbb{N}\rangle$. Thus, C is countable and so it is hypercentral. Now, for each $i \in \mathbb{N}$, let α_i be the smallest ordinal β such that $b_i \in \zeta_\beta(C)$. Then $\alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots$ is a strictly descending chain of ordinal numbers, a contradiction. □

In studying the structure of an arbitrary group, local analogues of nilpotence are very useful. Similarly, the following definition is crucial for us in studying centrally nilpotent braces (see [8]). A brace B is *locally centrally nilpotent* if every finitely generated subbrace is centrally nilpotent. Of course, every subbrace/quotient of a locally centrally nilpotent brace is still locally centrally nilpotent, and also restricted direct products of locally centrally nilpotent braces are locally centrally nilpotent. Moreover, by Corollary 3.6 of [20], every hypercentral brace is locally centrally nilpotent, but the converse does not hold. As a consequence of the following result, we see that every locally centrally nilpotent brace is ζ-nilpotent.

Theorem 4.5. *Let B be a locally centrally nilpotent brace.*

(1) *If I is any minimal ideal of B, then $I \leq \zeta(B)$.*

(2) *If M is any maximal subbrace of B, then M is an ideal of B.*

In particular, every minimal ideal of B has prime order, and $\partial(B)$ is contained in every maximal subbrace of B.

Proof. (1) Suppose that $I \not\leq \zeta(B)$. Then there exist elements $b \in B$ and $x \in I$ such that $\mathcal{S} = \{[b, x]_+, [b, x]_\cdot, x * b\} \neq \{0\}$. Let $c \in \mathcal{S} \setminus \{0\}$. Since I is a minimal ideal of B, the ideal generated by c in B is I, so there are elements $y_1, \dots, y_n \in B$ such that x belongs to the ideal generated by c in $S = \langle b, c, y_1, \dots, y_n\rangle$.

Let $J = x^S$. Since S is centrally nilpotent, there is a finite chain $0 = J_0 < J_1 < \cdots < J_m = J$ of ideals of S such that $J_{i+1}/J_i \leq \zeta(S/J_i)$. Choose $\ell \in \mathbb{N}$ with $c \in J_\ell \setminus J_{\ell-1}$; clearly, $\ell \neq 0, m$ because c is a nonzero element of one of the following types: $[b, x]_+$, $[b, x]_\cdot$, $x * b$. Now,

$$x^S \leq c^S \leq c^S + J_{\ell-1} = \langle c\rangle + J_{\ell-1} \leq J_\ell < J_m = J = x^S,$$

a contradiction.

(2) Assume M is not an ideal. If $B * B \leq M$, then $(M, +)/(B * B, +)$ is a maximal subgroup of the locally nilpotent group $(B, +)/(B * B, +)$, so it is even normal, and it follows that M is an ideal of B, a contradiction. Thus, there exists an element $x \in B * B \setminus M$. Since M is a maximal subbrace of B, we have that $B = \langle M, x\rangle$. Then there is a finite subset L of B such that $x \in \langle L\rangle * \langle L\rangle$. For each $y \in L$, let B_y be a finite subset of M such that $y \in \langle B_y \cup \{x\}\rangle$. Put $D = \langle B_y : y \in L\rangle$ and $E = \langle D, x\rangle$, so E is finitely generated and $L \subseteq E$. Now, E is centrally nilpotent, and $x \notin D \subseteq M$. Let N be a subbrace of E which is maximal with respect to containing D but not x. Since $E = \langle D, x\rangle$, we see that N is actually a maximal subbrace of E. Since E is centrally nilpotent, there is $n \in \mathbb{N}$ such that $\zeta_n(E) \leq N$ but $\zeta_{n+1}(E) \not\leq N$. Then N is a proper ideal of $\zeta_{n+1}(E) + N$ and so $N \trianglelefteq E$, since N is a maximal subbrace of E. Therefore E/N is centrally nilpotent, and so $x \in E * E \leq N$, a contradiction. □

In [3], chief factors of braces are introduced and shown to play a key role in its ideal structure. Let I and J be ideals of a brace B such that $J \leq I$. The quotient I/J is said to be a *chief factor* of B if I/J is a minimal ideal of B/J. A chain $\mathcal{C}$ of ideals of B is a *chief series* of B if $0, B \in \mathcal{C}$ and I/J is a chief factor of B whenever $J < I$ are consecutive terms of $\mathcal{C}$. By Zorn's lemma, every brace has a (possibly infinite) chief series. In [3], a brace B is proved to have a finite chief series if and only if it is *noetherian* (that is, every ascending chain of ideals is eventually stationary) and *artinian* (that is, every descending chain of ideals is eventually stationary). By Theorem 4.5, every chief series of a locally centrally nilpotent brace is a c-series, so we have the following result.

Corollary 4.6. *Let B be a locally centrally nilpotent brace. Then B is ζ-nilpotent.*

Remark 4.7. The proof of Theorem 4.5(1) proves much more than we stated. In fact, let $\mathfrak{Z}$ be the class of all braces in which every chief factor is central. Moreover, let $\boldsymbol{L}\mathfrak{Z}$ be the class of all braces in which every finite subset F is contained in a subbrace $C_F \in \mathfrak{Z}$. The proof of Theorem 4.5(1) can be modified to show that $\boldsymbol{L}\mathfrak{Z} = \mathfrak{Z}$.

More in detail, using the notation of the first half of the proof of Theorem 4.5(1), we get that S is contained in a subbrace $T \in \mathfrak{Z}$. Let $J = x^T$.

Since $[J, T]^T$ is an ideal of T containing c, we have that $x \in c^T \leq [J, T]^T$ and so $J = x^T = [J, T]^T$. Finally, let M be a maximal ideal of T contained in J and such that $x \notin M$. Then J/M is a chief factor of T and so $[J, T]^T \leq M$, a contradiction.

It follows from Corollary 4.6 that any nonzero ideal of a locally centrally nilpotent brace contains a (nonzero) central factor of the whole brace. In case of a hypercentral brace we can say more.

Lemma 4.8. *Let B be a hypercentral brace. If I is any nonzero ideal of B, then $I \cap \zeta(B) \neq 0$.*

Proof. Let α be the smallest ordinal number such that $J = I \cap \zeta_\alpha(B) \neq 0$. Then α is successor and $I \cap \zeta_{\alpha-1}(B) = 0$. Now, $[J, B]^B \leq J \cap \zeta_{\alpha-1}(B) = 0$ and so $J \leq \zeta(B)$. □

Corollary 4.9. *Let B be a brace having a finite chief series* (*resp. an ascending chief series*) *$\mathcal{I}$. Then B is centrally nilpotent* (*resp. hypercentral*) *if and only if every chief factor of $\mathcal{I}$ is central.*

Our next two subsections deal with the torsion theory of locally centrally nilpotent braces and with the problem of defining a suitable index for subbraces. Before delving into them, we note that some important results for nilpotent groups do not hold for braces.

(1) Bearing in mind the normaliser condition for nilpotent groups, the idealiser of a subbrace is introduced in [16]: given a subbrace S of a brace B, the idealiser of S is defined as the largest subbrace N of B such that S is an ideal of N. It is then stated that every subideal is properly contained in its idealiser (see Section 4.2 for the definition of subideal). Example A in Section 6 shows that the idealiser of a subbrace does not exist in general. We note however that if C is a subbrace of a brace B, then one can define the largest strong left ideal $N_B(C)$ of B additively and multiplicatively normalising B and such that $\lambda_x(C) = C$ for every $x \in N_B(C)$ — but such a strong left ideal need not contain C.

(2) Example B shows that there is no analogue of Fitting's theorem for centrally nilpotent ideals. Moreover, Example C shows that an abelian subideal need not be contained in a centrally nilpotent ideal.

(3) The ideal structure of the brace listed as `SmallBrace(32, 24003)` in the YangBaxter library [21] for GAP is described in [2]. This brace B has only a unique maximal subbrace I, which is also its only nonzero proper ideal. Moreover, $\partial(B) = I$. Nevertheless, B is not centrally nilpotent as it is not even soluble. This

shows that a finite brace whose maximal subbraces are ideals need not be soluble. The same example shows that a finite brace whose subbraces are subideals need not be centrally nilpotent (see Example D for more details), although an easy induction shows that they are at least weakly soluble in the sense explained in [2].

4.1. ***Torsion theory.*** The aim of this subsection is to establish a torsion theory for locally centrally nilpotent braces. We start with some definitions.

Definition 4.10. Let B be a brace. The subset of all periodic elements of $(B,+)$ is denoted by $T_+(B)$, while that of all periodic elements of $(B,\cdot)$ is denoted by $T_\cdot(B)$. Moreover, an element b of B is *periodic* if $\langle b\rangle$ is finite. The *order* of b is $|\langle b\rangle|$. If π is any set of prime numbers, then b is a π*-element* if its order is a π-number. A π*-subbrace* is just a subbrace containing only π-elements. Finally, B is *periodic* if every element of B is periodic. B is *torsion-free* if every element $b\in B$ is either zero or is nonperiodic. B is *locally finite* if every finitely generated subbrace of B is finite. B has *finite exponent* n if B is periodic and n is the smallest positive integer such that $b^n = nb = 0$ for all $b\in B$.

Clearly, every locally finite brace is periodic but the converse does not hold. The following result shows that in the context of locally centrally nilpotent braces we can precisely identify the set of all periodic elements of B.

Theorem 4.11. *Let B be a locally centrally nilpotent brace.*

(1) $T_+(B) = T_\cdot(B)$.
(2) $T_+(B)$ *is an ideal of* B.
(3) $T_+(B/T_+(B)) = 0$.
(4) *If B is periodic, then B is locally finite.*
(5) *B is locally finite if and only if $(B,+)$ is locally finite if and only if $(B,\cdot)$ is locally finite.*

Proof. The proof of (1)–(3) is an easy consequence of Proposition 4.2 of [16]. Let us prove (4). Assume B is periodic and finitely generated. Then $(B,+)$ is a periodic nilpotent group. Moreover, by Theorem 3.7 of [20], $(B,+)$ is also finitely generated. Thus $(B,+)$ is finite and (4) is proved. Finally, (5) is an obvious consequence of Theorem 3.7 of [20]. □

Let B be a brace, and let p be a prime. The *Sylow p-subbrace* of B is just a maximal element of the set of all its p-subbraces with respect to the inclusion. Our next result shows that the Sylow subbraces of a locally centrally nilpotent brace are ideals and that they coincide with the additive/multiplicative Sylow subgroups.

Theorem 4.12. *Let B be a locally finite brace. Then, B is locally centrally nilpotent if and only if, for every prime p, $\mathrm{Syl}_p(B,+) = \mathrm{Syl}_p(B,\cdot) = \{B_p\}$, B_p is locally centrally nilpotent and B is the direct product of the B_p's.*

Proof. Only one direction is in doubt. Since B is locally finite, we may assume B is finite and centrally nilpotent. Let p be a prime. Since both $(B,+)$ and $(B,\cdot)$ are nilpotent groups, there exist Sylow p-subgroups $B_p \trianglelefteq (B,+)$ and $\bar{B}_p \trianglelefteq (B,\cdot)$. Observe that B_p is also λ-invariant, as it is a characteristic subgroup of $(B,+)$. Therefore, $B_p = \bar{B}_p$ is an ideal of B. □

Proposition 4.13. *Let B be a brace whose additive and multiplicative groups are cyclic. Then there is $x \in B$ which is a generator of both $(B,+)$ and $(B,\cdot)$.*

Proof. By Theorem 4.6 of [19], we may assume B is finite. If $\operatorname{Ker}(\lambda)=0$, then $(B,\cdot)$ embeds into $\operatorname{Aut}(B,+)$, a contradiction. Thus, $\zeta(B)\neq 0$. Iterating this argument, we see that B is centrally nilpotent, so B factorises into the direct product of its Sylow p-subgroups. It is therefore possible to assume that B has prime power order p^n.

Let I be a subbrace of $\zeta(B)$ of order p. By induction there is an element $x \in B$ such that $x+I$ is both a generator of $(B/I,+)$ and $(B/I,\cdot)$. If $\langle x^{p^{n-1}}\rangle_{\cdot} \cap I = 0$, then $(B,\cdot)$ is not cyclic, a contradiction. Thus, $\langle x^{p^{n-1}}\rangle_{\cdot} = I$ and x is a generator of $(B,\cdot)$. Similarly, x is a generator of $(B,+)$. □

Our next result is a huge generalisation of Lemma 4.1 of [16]. In order to state it, we need the following definition. Let B be a brace, and let π be a set of prime numbers. We say that B is *π-free* if it does not contain π-elements. Obviously, a trivial brace B is π-free if and only if $(B,+)$ and/or $(B,\cdot)$ are π-free as groups.

Theorem 4.14. *Let B be a brace and let π be a set of primes. If $\zeta(B)$ is π-free, then each factor of the upper central series of B, and therefore the hypercentre of B, is π-free.*

Proof. Suppose the theorem is false and let α be the first ordinal such that $\zeta_{\alpha+1}(B)/\zeta_\alpha(B)$ is not π-free; in particular, there is $x \in \zeta_{\alpha+1}(B)\setminus\zeta_\alpha(B)$ such that $x^m \in \zeta_\alpha(B)$ for some positive π-number m. We divide the proof in two parts according to α being limit or not.

Suppose first α is limit. Then $x^m \in \zeta_{\beta+1}(B)\setminus\zeta_\beta(B)$ for some $\beta<\alpha$. Since $x \notin \zeta_\alpha(B)$, there is $b \in B$ and $\gamma \geq \beta$ such that one of the elements $x*b$, $[x,b]_+$, $[x,b]_\cdot$ belongs to $\zeta_{\gamma+1}(B)\setminus\zeta_\gamma(B)$; call c such an element. Assume $c = x*b$. Then

$$(x*b)^m \equiv x^m * b \pmod{\zeta_\gamma(B)}$$

and so $x^m * b \in \zeta_\beta(B) \leq \zeta_\gamma(B)$. Therefore $(x*b)^m \in \zeta_\gamma(B)$. But $x*b \in \zeta_{\gamma+1}(B)$ and $\zeta_{\gamma+1}(B)/\zeta_\gamma(B)$ is π-free, so $c = x*b \in \zeta_\gamma(B)$, a contradiction. Similarly, we deal with the cases in which $c=[b,x]_\cdot$ and $c=[b,x]_+$.

Suppose now that α is successor; we may assume $\alpha=1$, so $x \in \zeta_2(B)\setminus\zeta(B)$, $\zeta(B)$ is π-free, and $x^m \in \zeta(B)$. Put $C=\langle x\rangle+\zeta(B)$. By Theorem 3.5 of [8], $|C*C|$ is a π-number. On the other hand, $C*C \leq \zeta(B)$, and so $C*C=0$. Thus, $x^m = mx$.

Let b be any element of B. Then $m[x,b]_+ = [mx,b]_+ = 0$, so $[x,b]_+ = 0$. Similarly, $[x,b]_\cdot = x*b = 0$. Therefore x belongs to $\zeta(B)$, the final contradiction. □

Corollary 4.15. *Let B be a brace. If $\zeta(B)$ is torsion-free, then $\zeta_{\alpha+1}(B)/\zeta_\alpha(B)$ is torsion-free for every ordinal α.*

Conversely, if we have information on the exponent of $\zeta(B)$, then we can obtain information on the exponent of the factors of the upper central series.

Theorem 4.16. *Let B be a brace. If $\zeta(B)$ has exponent n, then $\zeta_{i+1}(B)/\zeta_i(B)$ has exponent dividing n^{2^i} for each positive integer i.*

Proof. It is enough to show that $\zeta_2(B)/\zeta(B)$ has exponent dividing n^2. Let $b \in \zeta_2(B)$ and $a \in B$. Then $b * a \in \zeta(B)$, so $b^n * a = n(b*a) = 0$ and $[a, b^n]_\cdot = [a,b]_\cdot^n = 0$. Thus, if we put $c = nb^n = b^{n^2}$, then $c \in \mathrm{Ker}(\lambda) \cap Z(B,\cdot)$. But also $[a, nb^n]_+ = n[a,b^n]_+ = 0$ and so $c \in \zeta(B)$. □

Corollary 4.17. *Let B be a brace such that $\zeta(B)$ has exponent n. If B is centrally nilpotent of class c, then B has exponent at most n^{2^c-1}.*

4.2. ***The index of a subbrace.*** The following definition provides us with an invaluable tool in studying the "index" of a subbrace.

Definition 4.18. Let C be a subbrace of the brace B. We say that C is *serial* in B if there is a chain of subbraces $\mathcal{C}$ connecting C to B such that if $D < E$ are consecutive elements of $\mathcal{C}$, then $D \trianglelefteq E$ — as in the case of c-series, we usually assume that these chains of subbraces are *complete*, meaning that they contain arbitrary unions and intersections of their members.

Now, C is *ascendant* (resp. *descendant*) if $\mathcal{C}$ can be well ordered (resp. inversely well ordered) with respect to the inclusion and its order type is λ (resp. the inverse of λ) for some ordinal number λ. If C is ascendant, then $\mathcal{C}$ can be written as

$$C = C_0 \trianglelefteq C_1 \trianglelefteq \cdots C_\alpha \trianglelefteq C_{\alpha+1} \trianglelefteq \cdots C_\lambda = B, \qquad (\triangle)$$

where $\alpha < \lambda$ are ordinal numbers; while, if C is descendant, then $\mathcal{C}$ takes the form

$$C = C_\lambda \cdots \trianglelefteq C_{\alpha+1} \trianglelefteq C_\alpha \cdots \trianglelefteq C_1 \trianglelefteq C_0 = B, \qquad (\square)$$

where $\alpha < \lambda$ are ordinal numbers. If $\mathcal{I}$ is finite, we say that C is *subideal*.

If C is ascendant (resp. descendant) in B, then the smallest ordinal number λ for which there is a chain of subbraces of type $(\triangle)$ (resp. of type $(\square)$) is the *ascendant length* (resp. *descendant length*) of C in B. In case C is subideal, the ascendant length of C in B is finite and is also called the *subideal defect* of C in B.

Let C be a subbrace of a brace B. Put $C^{B,0} := B$ and recursively define $C^{B,\alpha+1} = C^{C^{B,\alpha}}$ for every ordinal α, and $C^{B,\lambda} = \bigcap_{\alpha<\lambda} C^{B,\alpha}$ for every limit ordinal λ. The family $\{C^{B,\alpha}\}_{\alpha\in \mathrm{Ord}}$ is the *ideal closure series* of C in B. It is easy to show that C is descendant (resp. subideal) in B if and only if $C = C^{B,\mu}$ for some ordinal μ (resp. for some finite ordinal μ). If C is descendant (resp. subideal), then the descendant length (resp. the subideal defect) of C is the smallest ordinal number λ for which $C = C^{B,\lambda}$.

Lemma 4.19. *Let B be a brace.*

- *Every subideal of B is ascendant, descendant and serial. Moreover, ascendant (resp. descendant) subbraces are serial.*
- *If C is subideal (resp. ascendant, descendant, serial) in B, and $D \leq B$, then $C \cap D$ is subideal (resp. ascendant, descendant, serial) in D.*
- *If C is subideal (resp. ascendant), then CI/I is subideal (resp. ascendant) in B/I for every ideal I of B.*
- *If C is subideal in B of defect n, then C is subideal in C^B of defect $n-1$.*

Lemma 4.20. *Let B be a brace.*

(1) *If B is hypercentral, then every subbrace C of B is ascendant.*

(2) *If B is centrally nilpotent, then every subbrace C of B is subideal.*

(3) *If B is locally centrally nilpotent, then every subbrace C of B is serial.*

Proof. (1)–(2) We only prove (1). Let λ be the hypercentral length of B. Since C is an ideal of $C + \zeta(B)$, we see that

$$C \trianglelefteq C + \zeta(B) \trianglelefteq \cdots \trianglelefteq C + \zeta_\alpha(B) \trianglelefteq C + \zeta_{\alpha+1}(B) \trianglelefteq \cdots \trianglelefteq C + \zeta_\lambda(B) = B$$

is an ascending chain of subbraces of B connecting C to B.

(3) Zorn's lemma implies that there is a maximal chain of subbraces between C and B. By Theorem 4.5(2), if $D < E$ are consecutive terms of this chain, then D is an ideal of E. Therefore C is serial in B. □

Remark 4.21. Let B be a brace and let C be a (strong) left ideal of B. The proof of Lemma 4.20 can actually be employed to prove that if we have an ascending c-series of B, then there is an ascending chain of (strong) left ideals connecting C to B.

Definition 4.22. Let B be a brace. A subbrace C of B is said to have *finite index* in B if both $n_+ = |(B,+):(C,+)|$ and $n_\cdot = |(B,\cdot):(C,\cdot)|$ are finite; if $n_+ = n_\cdot = n$, we define the *index* $|B:C|$ of C in B as n. If C does not have finite index, we say that C has *infinite index*.

Lemma 4.23. *Let B be a brace, $C, D \leq B$ and $I \trianglelefteq B$.*

(1) *If C and D have finite index in B, then $C \cap D$ has finite index in B.*

(2) *Suppose $C \leq D$. If C has finite index in D, and D has finite index in B, then C has finite index in B. Moreover, if $|D:C|$ and $|B:D|$ exist, then also $|B:C| = |B:D| \cdot |D:C|$ exists.*

(3) *If I has finite index, then $|B:I|$ exists and is equal to $|B/I|$.*

Lemma 4.24. *Let C be a serial subbrace of the brace B. The following conditions are equivalent*:

(1) $|(B,+):(C,+)|<\infty$.

(2) $|(B,\cdot):(C,\cdot)|<\infty$.

(3) *C has finite index in B.*

(4) *$|B:C|$ exists.*

In particular, if any of these equivalent statements hold, then all the indices are equal.

Proof. Clearly, (4) $\Longrightarrow$ (1), (2) and (3). Assume (1). Since C is serial in B, there is a chain $\mathcal{C}$ of subbraces connecting C to B, and in which $E \trianglelefteq F$ whenever $E \lessdot F$. Looking at the corresponding additive parts of the members of $\mathcal{C}$, we see that $\mathcal{C}$ is actually finite, so C is subideal in B. We prove the result by induction on the subideal defect n of C in B. If $n \leq 1$, then C is an ideal of B such that $(B,+)/(C,+)$ is finite, so B/C is finite and we are done. Assume $n > 1$ and let $D = C^B$. Then the subideal defect of C in D is strictly less than n and so induction yields that $|D:C|$ exists. Since $|B:D|$ trivially exists, we have that $|B:C|$ exists by Lemma 4.23. Thus, (4) is proved. Similarly, we can prove that (2) implies (4), and that (3) implies (4). □

A combination of Lemmas 4.24 and 4.20 shows that every finite-index subbrace of a locally centrally nilpotent brace has a well-defined index. Our next result is a considerable extension of this fact.

Theorem 4.25. *Let B be a brace having an ascendant chain of ideals*

$$0 = I_0 \leq I_1 \leq \cdots I_\alpha \leq I_{\alpha+1} \leq \cdots I_\lambda = B$$

such that $I_{\beta+1}/I_\beta$ is either finite or locally centrally nilpotent for all ordinal numbers $\beta < \lambda$. If C is a subbrace of B, then the following are equivalent:

(1) $|(B,+):(C,+)|<\infty$.

(2) $|(B,\cdot):(C,\cdot)|<\infty$.

(3) *C has finite index in B.*

(4) *$|B:C|$ exists.*

Proof. We prove the result by induction on λ. To this aim it is sufficient to show that (1) implies (4).

If $\lambda \leq 1$, then B is either finite or locally centrally nilpotent. The former case is obvious, while the latter is a consequence of Lemmas 4.20(3) and 4.24. Assume $\lambda > 1$.

Suppose λ is successor. Since $(C, +)$ has finite index in $(B, +)$, it follows that $(C \cap I_{\lambda-1}, +)$ has finite index n in $(I_{\lambda-1}, +)$. By induction,

$$n = |(I_{\lambda-1}, \cdot) : (C \cap I_{\lambda-1}, \cdot)| = |(C I_{\lambda-1}, \cdot) : (C, \cdot)| = |(C + I_{\lambda-1}, +) : (C, +)|.$$

Thus the index $|C + I_{\lambda-1} : C|$ exists. Since also the index $|B : C + I_{\lambda-1}|$ exists, it follows that the index $|B : C|$ exists.

Now, assume λ is limit. Let F_+ be a transversal for $(C, +)$ in $(B, +)$; in particular, F is finite. Also, let $F.$ be a transversal for $(C, \cdot)$ in $(B, \cdot)$. Suppose $|F.| > |F_+|$, and let $E.$ be a finite subset of $F.$ such that $|E.| > |F_+|$. Then there is an ordinal number $\mu < \lambda$ such that $F_+ \cup E. \subseteq I_\mu$. By induction,

$$|(B, +) : (C, +)| = |(I_\mu, +) : (C \cap I_\mu, +)| = |(I_\mu, \cdot) : (C \cap I_\mu, \cdot)|,$$

a contradiction. Thus $|F.| \leq |F_+|$. By a symmetric argument, $|F_+| \leq |F.|$ and hence the index $|B : C|$ exists. □

The range of applicability of Theorem 4.25 is not restricted to local centrally nilpotent brace. It follows in fact from Theorems 3.14 of [8] that Theorem 4.25 applies even to any *good* brace with *property* (S) (see [8] for the definitions).

We end this discussion by showing that subbraces of finite index can sometimes be employed to prove the existence of large proper ideals.

Theorem 4.26. *Let B be a brace such that $B/\zeta_2(B)$ is finite. If C is any finite-index subbrace of B, then B/C_B is finite.*

Proof. Without loss of generality we may assume $C_B = 0$; thus, in particular, $C \cap \zeta(B) = 0$ and $\zeta(B)$ is finite. Moreover, we may replace C by $C \cap \zeta_2(B)$, assuming $C \leq \zeta_2(B)$. Then $C + \zeta(B)$ is an ideal of B.

Since $C \simeq C + \zeta(B)/\zeta(B)$, we have that C is an abelian brace. Let $n = |\zeta(B)|$. Then Theorem 4.16 shows that

$$C^{n^2,\cdot} \leq \zeta(B) \cap C = 0,$$

so C is periodic. Thus, as a group, C can be described as a direct product of infinitely many cyclic subgroups $\langle b_i \rangle$, $i \in I$, of order dividing n.

Let F be a finitely generated subbrace of B such that $C + \zeta(B) + F = B$. Since B is periodic, it is locally finite by Lemma 3.1 of [20], which implies that F is finite.

For every $b \in F$, $b_i^{b,+} = b_i + u_{b,i,+}$ for some $u_{b,i,+} \in \zeta(B)$. On the other hand, $\zeta(B)$ is finite, so there is an infinite subset J_1 of I such that $b_i^{b,+} = b_i + u_{+,b}$ for all $i \in J_1$, and for a fixed $u_{+,b} \in \zeta(B)$. Repeating this argument for all $b \in F$, we may assume $b_i^{b,+} = b_i + u_{+,b}$ for some $u_{+,b} \in \zeta(B)$ and for all $i \in J_1$, $b \in F$. Similarly, there is an infinite subset J_2 of J_1 such that $b_i^{b,\cdot} = b_i + u_{\cdot,b}$ for some $u_{\cdot,b} \in \zeta(B)$ and for all $i \in J_2$, $b \in F$. Finally, there is an infinite subset J_3 of J_2 such that $b_i * b = b_j * b$ for all i, $j \in J_3$ and $b \in F$.

Now, for each $i,\, j \in J_3$ with $i \neq j$, we have that $d_i = b_i - b_j = b_i \cdot b_j^{-1}$ is additively and multiplicatively centralised by F, and that $d_i * F = 0$. Since $B = C + \zeta(B) + F$, it follows that $d_i \in \zeta(B)$ for all $i \in J_3$, a contradiction. □

5. Central nilpotency for ideals

A celebrated result of Fitting states that a product of nilpotent normal subgroups of a group is nilpotent. Example B in Section 6 shows that the product of two centrally nilpotent ideals is not centrally nilpotent in general. In this section, we define a nilpotency concept for ideals that allows us to define a suitable Fitting ideal. It turns out that, for such a Fitting ideal, it is possible to generalise remarkable results of group theory concerned with the Fitting subgroup (see Theorems 5.6, 5.7, and 5.10).

Let B be a brace. We start by defining B-centrally nilpotent braces. Let I be an ideal of B. We can define the *lower central series of I with respect to B*, or simply the *lower B-central series of I*, as follows: take $\Gamma_1(I)^B = I$ and $\Gamma_{n+1}(I)^B = [\Gamma_n(I),\, I]^B$, for every $n \geq 1$. Therefore,

$$I = \Gamma_1(I)^B \geq \Gamma_2(I)^B \geq \cdots \geq \Gamma_n(I)^B \geq \cdots$$

is a descending chain of ideals of B with $\Gamma_n(I)^B / \Gamma_{n+1}(I)^B \leq \zeta(I/\Gamma_{n+1}(I)^B)$ for every $n \in \mathbb{N}$. Similarly, we may define the *upper central series of I with respect to B* (or simply the *upper B-central series of I*), as follows: take $\zeta_0^B(I) = 0$ and let $\zeta_{n+1}(I)^B$ satisfy $\zeta_{n+1}(I)^B/\zeta_n(I)^B = \zeta(I/\zeta_n(I)^B)_{B/\zeta_n(I)^B}$. Then $\zeta_0(I)^B \leq \zeta_1(I)^B \leq \cdots \leq \zeta_n(I)^B \leq \cdots$ is an ascending chain of ideals of B.

Definition 5.1. An ideal I of a brace B is defined to be *centrally nilpotent with respect of B*, or simply a *B-centrally nilpotent ideal*, if there exists $n \in \mathbb{N}$ such that $\Gamma_{n+1}(I)^B = 0$, or, equivalently, $\zeta_n(I)^B = 0$. For practical purposes, we often use the following equivalent definition: I is B-centrally nilpotent if there exists a chain $0 = J_0 \leq J_1 \leq \cdots \leq J_n = I$ of ideals of I such that $J_i/J_{i-1} \leq \zeta(I/J_{i-1})$, for every $1 \leq i \leq n$.

To simplify notation, if J is an ideal of B contained in I and such that I/J is B/I-centrally nilpotent, we just say that I/J is *centrally nilpotent with respect to B*, or *B-centrally nilpotent*. If I/J is B-centrally nilpotent, then the smallest n such that $\Gamma_{n+1}(I/J)^{B/J} = 0$ is referred to as its *class*.

Clearly, a brace B is centrally nilpotent if and only if it is B-centrally nilpotent; in this trivial case, the *upper central series* (resp. *lower central series*) and the *upper B-central series* (resp. *lower B-central series*) coincide. Moreover, if I is a B-centrally nilpotent ideal of a brace B, and J is any ideal of B, then $(I+J)/J$ is B-centrally nilpotent (of class less than or equal to that of I), and $I \cap C$ is C-centrally nilpotent for any subbrace C of B (also in this case the class of $I \cap C$ is less than or equal to that of I). Our next result shows that an analogue of Fitting's theorem holds for B-centrally nilpotent ideals, but first, we need the following property of commutators of ideals in braces.

Lemma 5.2. *Let B be a brace and let I, J, K be ideals of B. Then*

$$[I, JK]_B = [I, J+K]_B = [I, J]_B + [I, K]_B = [I, J]_B[I, K]_B.$$

Proof. We prove the equality for the sum. Observe that only one inclusion is in doubt so, by Theorem 3.3, it suffices to show that

$$[I, J+K]_+,\ I*(J+K),\ (J+K)*I \subseteq [I, J]_B + [I, K]_B.$$

Since $(I, +)$, $(J, +)$ and $(K, +)$ are normal subgroups of $(B, +)$, we have that $[I, J+K]_+ = [I, J]_+ + [I, K]_+$ is contained in $[I, J]_B + [I, K]_B$. Then we have that for every $i \in I$, $j \in J$ and $k \in K$,

$$i*(j+k) = i*j + j + i*k - j \in [I, J]_B + [I, K]_B.$$

Finally, note that $(J+K)*I = (JK)*I$, so

$$(jk)*i = j*(k*i) + k*i + j*i \in J*I + K*I + J*I \subseteq [I, J]_B + [I, K]_B$$

for every $j \in J$, $k \in K$, $i \in I$. □

We also need this notation in the proof: if $I_1, \dots, I_n$ are ideals of a brace B, we put $[I_1]^B = I_1$, and then, recursively, $[I_1, \dots, I_k]^B := [[I_1, \dots, I_{k-1}]^B, I_k]^B$ for every $2 \le k \le n$.

Theorem 5.3 (see also Theorem 5.13). *Let I, J be B-centrally nilpotent ideals of a brace B. If I and J have classes n_0 and m_0, respectively, then $I+J$ is B-centrally nilpotent of class at most n_0+m_0.*

Proof. Set $K = I+J$. First, we show by induction that for every $n \in \mathbb{N}$, $\Gamma_n(K)^B$ is the sum of all commutators of the form $[L_1, \dots, L_n]^B$ with either $L_i = I$ or $L_i = J$, for every $1 \le i \le n$. The base case is clear. Assume the assertion is true for some $1 \le n \in \mathbb{N}$. Then,

$$\Gamma_{n+1}(K)^B = [\Gamma_n(K), K]^B = [\Gamma_n(K), I]^B + [\Gamma_n(K), J]^B$$

by Lemma 5.2. Using iteratively Lemma 5.2, the assertion also holds for $n+1$.

In particular, for $r = n_0+m_0+1$, $\Gamma_r(K)^B$ is the sum of all commutators of the form $[L_1, \dots, L_r]^B$, where either I occurs n_0+1 times or J occurs m_0+1 times. Thus, it follows that each $[L_1, \dots, L_r]^B$ is contained in either $\Gamma_{n_0+1}(I)^B = 0$ or $\Gamma_{m_0+1}(J)^B = 0$. Hence, $\Gamma_r(K)^B = 0$ and so K is B-centrally nilpotent. □

Let B be a brace. The *Fitting ideal* $\mathrm{Fit}(B)$ of B is the ideal generated by all B-centrally nilpotent ideals of B. It follows from Theorem 5.3 that in a finite brace B, $\mathrm{Fit}(B)$ is B-centrally nilpotent. More generally, the same result shows that this is true for a broader class of braces.

Corollary 5.4. *Let B be a noetherian brace. Then* $\mathrm{Fit}(B)$ *is a B-centrally nilpotent ideal.*

Now, in order to obtain a characterisation of the Fitting ideal in terms of chief factors, we need the following definition (recall Proposition 4.2). In [5, Propostion 4.19], the *centraliser of an ideal I* of a brace B, $\mathrm{C}_B(I)$, is defined as the largest ideal that *centralises* I, i.e., $[\mathrm{C}_B(I), I]^B = 0$.

Moreover, if I/J is a chief factor of B, we define the *centraliser in B of I/J* as the ideal $\mathrm{C}_B(I/J)$ of B satisfying $\mathrm{C}_{B/J}(I/J) = \mathrm{C}_B(I/J)/J$. Equivalently, $\mathrm{C}_B(I/J)$ is the largest ideal of B such that $[\mathrm{C}_B(I/J), I]^B \leq J$.

Lemma 5.5. *Let I be a B-centrally nilpotent ideal of a brace B. If J is a minimal ideal of B, then $[J, I]^B = 0$.*

Proof. Since J is a minimal ideal of B, either $[J, I]^B = 0$ or $[J, I]^B = J$. However, in the latter case we contradict Definition 5.1. □

Theorem 5.6. *Let B be a brace with a finite chief series $\mathcal{S}$. Then* $\mathrm{Fit}(B)$ *is the intersection of the centralisers in B of the factors of $\mathcal{S}$.*

Proof. Let $0 = I_0 \leq I_1 \leq \cdots \leq I_n = B$ be a finite chief series of B and set $C := \bigcap \{\mathrm{C}_B(I_k/I_{k-1}) : 1 \leq k \leq n\}$. Then C is an ideal of B and $0 = C \cap I_0 \leq C \cap I_1 \leq \cdots \leq C \cap I_n = C$ is a finite chain of ideals of B such that $(I_i \cap C)/(I_{i-1} \cap C) \leq \zeta(C/I_{i-1} \cap C)$ for all $1 \leq i \leq n$. Thus, C is B-centrally nilpotent, and hence $C \leq \mathrm{Fit}(B)$.

Conversely, B is noetherian (see [3]) and so $F := \mathrm{Fit}(B)$ is B-centrally nilpotent by Corollary 5.4. If I/J is any chief factor of B, then I/J is centralised by $(F + J)/J$ by Lemma 5.5. □

Theorem 5.7. *Let B be a brace and put $F = \mathrm{Fit}(B)$. Then, $(\mathrm{C}_B(F) + F)/F$ does not contain any nonzero soluble ideal with respect of B/F. In particular, if B is a soluble brace, then $\mathrm{C}_B(F) = \zeta(F)$.*

Proof. Assume that $(\mathrm{C}_B(F) + F)/F$ contains a nonzero soluble ideal I/F with respect to B/F. Then it also contains a nonzero ideal J/F which is an abelian brace. Let $C = \mathrm{C}_B(F)$. Then $J \cap C \cap F \leq \zeta(J \cap C)$ and $(J \cap C)/(J \cap C \cap F)$ is an abelian brace. Thus, $J \cap C$ is B-centrally nilpotent and so $J \cap C \leq F$. Finally, $J = J \cap (C + F) = (J \cap C) + F = F$, a contradiction. If B is a soluble brace, then B/F is soluble, and therefore $(\mathrm{C}_B(F) + F)/F$ must be zero. Hence, $\mathrm{C}_B(F) \leq F$ which yields $\mathrm{C}_B(F) = \zeta(F)$. □

It is well known that the Fitting subgroup of a finite group modulo its Frattini subgroup is the product of all its abelian minimal normal subgroups. The following Frattini-like ideal leads to a brace-theoretic analogue of this result. Let B be a finite brace. We define the *Frattini ideal* of F as

$$\mathrm{Frat}(B) := \bigcap \{\, L \mid L \text{ is a maximal left ideal of } B \,\} \cap \mathrm{Fit}(B).$$

Clearly, the Frattini ideal of a finite brace is a left ideal, but the following result shows that it is actually an ideal (hence providing a justification for its name).

Lemma 5.8. *Let B be a finite brace. If L is any maximal left ideal of B, then $L \cap \mathrm{Fit}(B)$ is an ideal of B.*

Proof. Let $F = \mathrm{Fit}(B)$ and assume $F \not\leq L$, so in particular $B = FL$.

Since $F \cap L \trianglelefteq (L,+)$, it follows that $L \leq \mathrm{N}_{(B,+)}(F \cap L)$. Moreover, $F \cap L$ is properly contained in $\mathrm{N}_{(F,+)}(F \cap L)$, as $(F,+)$ is nilpotent. Therefore, L is properly contained in $\mathrm{N}_{(B,+)}(F \cap L)$.

Because $F \cap L$ is λ-invariant, we have $\mathrm{N}_{(B,+)}(F \cap L)$ is also λ-invariant. Thus, $\mathrm{N}_{(B,+)}(F \cap L) = B$, and so $F \cap L$ is a strong left ideal. Then, by Lemma 4.20 and Remark 4.21, we can find a strong left ideal T of B contained in F and such that $F \cap L$ is a proper ideal of T. Therefore, $F \cap L$ is an ideal of $T + L = TL$, with $T + L$ being a left ideal of B. Hence, $TL = B$ by the maximality of L and the result follows. □

Corollary 5.9. *Let B be a finite brace. Then* $\mathrm{Frat}(B)$ *is an ideal of B.*

Theorem 5.10. *Let B be a finite brace with* $\mathrm{Frat}(B) = 0$*. Then* $\mathrm{Fit}(B)$ *is the product of all the abelian minimal ideals of B.*

Proof. Let $F = \mathrm{Fit}(B)$. We claim that $\partial_B(F) = 0$. Indeed, suppose that L is a maximal left ideal of B such that $\partial_B(F)$ is not included in L. Then, $B = \partial_B(F)L$, so $F = F \cap \partial_B(F)L = (F \cap L)\partial_B(F)$. By Lemma 5.8, $F \cap L$ is an ideal of B. Since $F/(F \cap L)$ is nonzero and B-centrally nilpotent, we have that $\partial_B(F)(F \cap L)/(F \cap L) \leq \partial_{B/(F\cap L)}(F/(F \cap L)) < F/(F \cap L)$, a contradiction. Thus, F is an abelian brace.

Let N be the product of all abelian minimal ideals of B. Then $N \leq F$. For the other inclusion, take S a minimal subbrace subject to $B = SN$. Consider $X = \bigcap \{L \mid L \text{ is a maximal left ideal of } S\}$, a left ideal of S. If $S \cap N \not\leq X$, then there exists a maximal left ideal L of S such that $S \cap N$ is not included in L. Thus, $(S \cap N)L = S$ and then $B = SN = (S \cap N)LN = LN$, which contradicts the minimality of S. Therefore, $S \cap N \leq X$.

Now, $S \cap N$ is an ideal of B, as N is abelian and $B = SN$ (see [2, Lemma 27]). Assume that there exists a maximal left ideal L of B such that $S \cap N \not\leq L$. Thus, $B = (S \cap N)L$ and then, $S = S \cap (S \cap N)L = (S \cap L)(S \cap N)$. Take L' a left ideal of S maximal subject to $S \cap L \leq L'$ and $S \cap N$ not included in L'. Then, L' is indeed a maximal left ideal of S, because the existence of a left ideal L'' of S such that $L' < L'' \leq S$ yields $S = (S \cap L)(S \cap N) \leq L''$. Therefore, $S \cap N \leq X \leq L'$, a contradiction. Thus, $S \cap N = 0$.

Finally, $S \cap F$ is an ideal of B, as F is abelian and $SF = B$ (see again [2, Lemma 27]), and consequently $S \cap F$ contains an abelian minimal ideal of B, contradicting $S \cap N = 0$. □

In a centrally nilpotent brace, the Frattini ideal behaves pretty well. For example, it is possible to prove that it coincides with the set of nongenerators. Let B be a brace. We say that an element $b \in B$ is a *nongenerator* of B if for all $S \leq B$ such that $B = \langle b, S\rangle$, we have $B = S$.

Theorem 5.11. *Let B be a centrally nilpotent finite brace. Then* $\mathrm{Frat}(B)$ *coincides with the set of all nongenerators of B.*

Proof. Since in a centrally nilpotent brace the maximal left ideals coincide with the maximal ideals and with the maximal subbraces, the usual group-theoretic proof adapts to prove the result. □

However, we note that there exists a brace B of order 6 in which $\mathrm{Fit}(B) = \mathrm{Frat}(B)$ is the only nonzero proper left ideal of B and has order 3. This shows that there is no possible analogue of these two well-known group-theoretic theorems concerning the Frattini subgroup of a group G:

(1) $G/\mathrm{Frat}(G)$ is nilpotent implies G is nilpotent.

(2) If p is a prime dividing $|G|$, then p divides $|G/\mathrm{Frat}(G)|$ too.

In the final part of this section we discuss further aspects of B-centrally nilpotence and hypercentral/locally nilpotent concepts for (sub)ideals.

The definition of upper B-central series (and lower B-central series) for an ideal I of a brace B can be extended by using transfinite numbers (just how we did in Section 4), and this allows us to define B-hypercentral ideals of braces. However, for our purposes, the following equivalent definition is more convenient.

Definition 5.12. Let B be a brace. An ideal I of B is said to be *B-hypercentral* if there is an ascending chain of ideals of B

$$0 =: I_0 \leq I_1 \leq \cdots I_\alpha \leq I_{\alpha+1} \leq \cdots I_\lambda = I$$

such that $I_{\alpha+1}/I_\alpha \leq \zeta(B/I_\alpha)$ for all ordinals $\alpha < \lambda$. The smallest λ for which such a chain exists is the *length* of I.

Clearly, if B is a brace, then B is hypercentral if and only if B is B-hypercentral, and every B-centrally nilpotent ideal is B-hypercentral. The following result generalises Theorem 5.3.

Theorem 5.13. *Let B be a brace.*

(1) *If C and D are ideals of B which are B-hypercentral of lengths α and β, respectively, then $C+D$ is a B-hypercentral ideal of length at most $\beta\alpha+\max\{\alpha, \beta\}$.*

(2) *Suppose C is subideal of defect n and centrally nilpotent of class c, and D is a B-centrally nilpotent ideal of class d. Then $C+D$ is centrally nilpotent of class at most $(c+n)d+c$.*

(3) *Suppose C is ascendant of length μ and hypercentral of length γ, and D is a B-hypercentral ideal of length δ. Then $C+D$ is hypercentral of length at most $(\gamma+\mu)\delta+\gamma$.*

Proof. (1) Let $E=C+D$. Then E is an ideal of B and to show that E is B-hypercentral, it suffices to prove that $\zeta(E)$ contains a nonzero ideal I of B. To this aim we may certainly assume that C and D are nonzero.

Suppose first $C\cap D=0$. By hypothesis $\zeta(C)$ contains a nonzero ideal I of B. On the other hand, $\zeta(C)\leq\zeta(E)$ and so we are done.

Assume $C\cap D\neq 0$. Then $\zeta(C)\cap D$ contains a nonzero ideal I of B, and $I\cap\zeta(D)$ contains a nonzero ideal J of B. Thus $J\leq\zeta(C)\cap\zeta(D)\leq\zeta(E)$ and we are done. The bound on the hypercentral length can be easily deduced from the proof.

(2)–(3) We only prove (3), the proof of (2) being similar. Since D is B-hypercentral of length δ, there is an ascending chain of ideals of B

$$0=D_0<D_1<\cdots D_\alpha<D_{\alpha+1}<\cdots D_\delta=D$$

such that $D_{\beta+1}/D_\beta\leq\zeta(D/D_\beta)$ for all ordinals $\beta<\delta$.

Let $E=C+D$. Since C is hypercentral of length γ, it follows that $C\cap D_1\leq\zeta_\gamma(E)$. Thus, we may factor out $C\cap D_1$ and assume $C\cap D_1=0$. Let $F=\langle C, D_1\rangle=CD_1$. Now, since C is ascendant of length μ, there is an ascending chain $C=C_0\trianglelefteq C_1\trianglelefteq\cdots\trianglelefteq C_\alpha\trianglelefteq C_{\alpha+1}\trianglelefteq\cdots\trianglelefteq C_\mu=F$ connecting C to F. It is easy to see that $(C_{\beta+1}\cap D_1)/(C_\beta\cap D_1)\leq\zeta(E/(C_\beta\cap D_1))$ for all $\beta<\mu$. Therefore $D_1\leq\zeta_\mu(E)$.

We factor D_1 out and we repeat the above argument. This shows that $D\leq\zeta_\rho(E)$, where $\rho=(\gamma+\mu)\delta$, so E is hypercentral of length at most $\rho+\gamma$. □

If B satisfies the maximal condition on ideals, then the product of all B-hypercentral ideals of a brace is clearly B-hypercentral. Moreover, the idea of the proof of Theorem 5.6 yields that if B is a brace having an ascending chief series $\mathcal{S}$ (in particular, if B satisfies the minimal condition on ideals), then the maximal ideal centralising all factors of $\mathcal{S}$ is precisely the unique maximal B-hypercentral ideal of B.

The following result shows that in the universe of locally centrally nilpotent braces, the class of B-hypercentral braces is closed with respect to forming extensions by finitely generated hypercentral braces.

Theorem 5.14. *Let N be an ideal of the locally centrally nilpotent brace B. If N is B-hypercentral and B/N is finitely generated, then B is hypercentral.*

Proof. Let S be a finite subset of B that generates B modulo N, and let $Z = \bar{\zeta}(B)$ be the hypercentre of B. Assume by contradiction $Z \neq B$. Now, B/N is centrally nilpotent, so $N \not\leq Z$, and hence the ideal $K := Z \cap N$ of B is strictly contained in N. Since N/K is B/K-hypercentral, there is a nonzero ideal A/K of B/K such that $A/K \leq \zeta(N/K)$. Let $a \in A \setminus K$ and $U = \langle a, S, K \rangle$; in particular, U/K is centrally nilpotent. Since $(A \cap U)/K$ is a nonzero ideal of U/K, we have that $V/K := \zeta(U/K) \cap ((A \cap U)/K) \neq 0$. Now, the fact that $V \leq A$ implies that $[V, N]_+$, $[V, N]_.$ and $V * N$ are all contained in K. Similarly, the fact that $V/K \leq \zeta(U/K)$ shows that $[V, T]_+$, $[V, T]_.$ and $V * T$ are all contained in K, where $T = \langle S \rangle$. Since $B = N + T = NT$, we easily see that $[V, B]_+$ and $[V, B]_.$ are contained in K. Moreover, if $u \in N$, $v \in T$, and $a \in V$, then $a * (u + v) = a * u + u + a * v - u \in K$. This shows that $V * B \leq K$ and proves that $V/K \leq \zeta(B/K)$. Since $K \leq Z$, it follows that $V \leq Z$, so $V \leq Z \cap N$, a contradiction. □

Also B-central nilpotency (resp. hypercentrality) can be locally detected, and our next result is in fact a generalisation of Theorem 4.4.

Theorem 5.15. *Let B be a brace and let $I \trianglelefteq B$.*

(1) *I is B-centrally nilpotent of class at most c if and only if $I \cap F$ is F-centrally nilpotent of class at most c for every finitely generated subbrace F of B.*

(2) *I is B-centrally nilpotent if and only if $I \cap C$ is C-centrally nilpotent for every countable subbrace C of B.*

(3) *I is B-hypercentral if and only if $I \cap C$ is C-hypercentral for every countable subbrace C of B.*

Proof. We only deal with the proof of (1), since (2) and (3) then follow in a similar fashion using ideas from Theorem 4.4.

For each $u, v \in B$, we write $u \circ v$ to denote one (but we do not know which one) of the operations $[u, v]_.$, $[u, v]_+$, $u * v$. Then (1) is a direct consequence of the fact that $\zeta_c(I)^B$ can be easily characterised as the set of all elements $b \in I$ such that $\big\langle\big(\cdots((b \circ b_1) \circ \cdots) \circ b_{c-i}\big)\big\rangle^B \leq \zeta_i(I)$ for all $i = 0, 1, \ldots, c-1$ and for all $b_1, \ldots, b_{c-i} \in B$. □

To provide a good definition of "locally B-nilpotent ideal" is not an obvious task. We now concern ourselves with a couple of possible definitions, mostly sketching proofs and results. The first idea that comes in mind is that of using central chains of ideals, just as we did for B-hypercentral and B-centrally nilpotent ideals. In fact, it follows from Theorem 4.5 (and Zorn's lemma) that in every chief series of a locally centrally nilpotent brace B, two consecutive ideals $K \leq H$ satisfy $H/K \leq \zeta(B/K)$.

Definition 5.16. Let B be a brace. An ideal I of B is said to be *ζ_B-nilpotent* if every quotient I/J of I by an ideal J of B admits a maximal chain $\mathcal{S}$ of ideals of B in which $H/K \leq \zeta(I/K)$ for all consecutive terms $K/J \leq H/J$ of the chain $\mathcal{S}$.

Using ideas from the proof of Theorem 5.13, it is not difficult to see that the product of arbitrarily many ζ_B-nilpotent ideals is ζ_B-nilpotent. Thus, any brace B has a unique maximal ζ_B-nilpotent ideal, we call it the *ζ_B-radical* of B: it turns out that the largest ideal of B centralising all quotients of a chief series of B is precisely the ζ_B-radical of B. The following result shows that an analogue of Theorem 5.7 is possible for the ζ_B-radical.

Theorem 5.17. *Let B be a brace admitting an ascending chain of ideals*

$$0 = B_0 \leq B_1 \leq \cdots B_\alpha \leq B_{\alpha+1} \leq \cdots B_\lambda = B$$

such that $B_{\beta+1}/B_\beta$ is a ζ_B-nilpotent ideal of B/B_β for all $\beta < \lambda$. Then $C_B(H) \leq H$, where H is the ζ_B-radical of B.

Proof. Suppose $C = C_B(H) \not\leq H$. Then $(C+H)/H$ contains a nonzero ζ_B-nilpotent ideal I/H of B/H. Since

$$I \cap C \cap H \leq \zeta(I \cap C),$$

it follows that $I \cap C$ is a ζ_B-nilpotent ideal of B. Thus,

$$I \cap C \leq H \quad \text{and} \quad I = I \cap (C+H) = (I+C) \cap H = H,$$

a contradiction. □

However, there is one reason for which this is not a convincingly good definition of "locally B-nilpotent ideal": a ζ_B-ideal could not be locally centrally nilpotent (there are examples even among groups) — although if B is locally finite, then a ζ_B-ideal is locally centrally nilpotent.

A more fruitful approach could deal with finitely generated subbraces and the way the ideal embeds into them. There are several ways in which this case may be achieved, but the most reasonable one seems to be the following. Let B be a brace. An ideal I of B is *locally B-nilpotent* if the following property holds: for every finitely generated subbrace F of B, the finitely generated subbraces of $I \cap F$ are contained in F-centrally nilpotent ideals of F.

Trivially, every locally B-nilpotent ideal is locally centrally nilpotent, so this solves the previous issue for ζ_B-nilpotency.

Theorem 5.18. *Let B be a brace. The sum of arbitrarily many locally B-nilpotent ideals of B is locally B-nilpotent.*

Proof. It is clearly enough to prove the statement for two locally B-nilpotent ideals I and J. Let F be a finitely generated subbrace of B. Choose a finitely generated subbrace E of $F \cap (I+J)$. In order to prove that E is contained in an F-centrally nilpotent ideal of F, we may assume $E = E_1 \cup E_2$, where $E_1 \subseteq I$ and $E_2 \subseteq J$, by suitably replacing F. Now, E_1 and E_2 are respectively contained in F-centrally nilpotent ideals I_1 and I_2 of F. Since $I_1 + I_2$ is F-centrally nilpotent, we are done. □

By Theorem 5.18, every brace admits a unique maximal locally B-nilpotent ideal; we call it the *Hirsch–Plotkin radical* of B. Using ideas from the proof of Theorem 4.5(1), we see that every locally B-nilpotent ideal is actually ζ_B-nilpotent. Finally, we note that for a locally finite brace B, the concepts of locally B-nilpotent ideal and ζ_B-ideal coincide.

6. Worked examples

In this section we describe the main examples of the paper. These examples are all constructed in a similar fashion (see [1]), which we now explain, and all the computations can be done with the computer algebra system GAP and the functions of its package YangBaxter [21]. The first of them shows that the idealiser of a subbrace (as introduced in [16]) does not exist in general (even for braces of abelian type).

Example A. Let $B = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \simeq C_4 \times C_4 \times C_2$, whose operation is additively written, and

$$\begin{aligned} C = \big\langle m_1, m_2, m_3, m_4, m_5 \mid m_1^2 = m_4,\ m_2^2 = 1,\ m_3^2 = m_4^2 = m_5,\ m_5^2 = 1,\\ m_1m_2m_1^{-1} = m_3m_2,\ m_1m_3m_1^{-1} = m_2m_3m_2^{-1} = m_5m_3,\\ m_1m_4m_1^{-1} = m_2m_4m_2^{-1} = m_3m_4m_3^{-1} = m_4,\\ m_1m_5m_1^{-1} = m_2m_5m_2^{-1} = m_3m_5m_3^{-1} = m_4m_5m_4^{-1} = m_5\big\rangle. \end{aligned}$$

We note that B and C are groups of order 32. Since $m_3 = m_1m_2m_1^{-1}m_2^{-1}$, $m_4 = m_1^2$, $m_5 = m_1m_3m_1^{-1}m_3^{-1} = m_3^2$, we have that $\langle m_3, m_4, m_5\rangle$ is contained in $\operatorname{Frat}(C)$ (in fact, they coincide) and so $C = \langle m_1, m_2\rangle$. Then C acts on B by means of an action λ defined by

$$\begin{aligned} \lambda_{m_1}(a_1) &= 3a_1 + a_3, & \lambda_{m_2}(a_1) &= 3a_1,\\ \lambda_{m_1}(a_2) &= a_1 + a_2 + a_3, & \lambda_{m_2}(a_2) &= a_1 + a_2 + a_3,\\ \lambda_{m_1}(a_3) &= a_3, & \lambda_{m_2}(a_3) &= a_3. \end{aligned}$$

We note that

$$\begin{aligned} \lambda_{m_3}(a_1) &= \lambda_{m_1m_2m_1^{-1}m_2^{-1}}(a_1) = a_1, & \lambda_{m_4}(a_1) &= \lambda_{m_1^2}(a_1) = a_1,\\ \lambda_{m_3}(a_2) &= \lambda_{m_1m_2m_1^{-1}m_2^{-1}}(a_2) = a_2 + a_3, & \lambda_{m_4}(a_2) &= \lambda_{m_1^2}(a_2) = a_2 + a_3,\\ \lambda_{m_3}(a_3) &= \lambda_{m_1m_2m_1^{-1}m_2^{-1}}(a_3) = a_3, & \lambda_{m_4}(a_3) &= \lambda_{m_1^2}(a_3) = a_3, \end{aligned}$$

and λ_{m_5} is the identity map on B.

We can consider the semidirect product $G = [B]C$ with respect to this action. Then G turns out to be a trifactorised group as it possesses a subgroup $D = \langle a_1a_2^3m_1, a_1m_2\rangle$ such that $D \cap C = D \cap B = 1$, $DC = BD = G$. Thus, the bijective

c	$\delta(c)$	c	$\delta(c)$
1	0	m_1	a_1+3a_2
m_5	$2a_1$	m_1m_5	$3a_1+3a_2$
m_4	$3a_1+2a_2+a_3$	m_1m_4	a_2
m_4m_5	$a_1+2a_2+a_3$	$m_1m_4m_5$	$2a_1+a_2$
m_3	$3a_1+a_3$	m_1m_3	$2a_1+3a_2$
m_3m_5	a_1+a_3	$m_1m_3m_5$	$3a_2$
m_3m_4	$2a_1+2a_2$	$m_1m_3m_4$	a_1+a_2
$m_3m_4m_5$	$2a_2$	$m_1m_3m_4m_5$	$3a_1+a_2$
m_2	a_1	m_1m_2	$3a_2+a_3$
m_2m_5	$3a_1$	$m_1m_2m_5$	$2a_1+3a_2+a_3$
m_2m_4	$2a_2+a_3$	$m_1m_2m_4$	$3a_1+a_2+a_3$
$m_2m_4m_5$	$2a_1+2a_2+a_3$	$m_1m_2m_4m_5$	$a_1+a_2+a_3$
m_2m_3	$2a_1+a_3$	$m_1m_2m_3$	$3a_1+3a_2+a_3$
$m_2m_3m_5$	a_3	$m_1m_2m_3m_5$	$a_1+3a_2+a_3$
$m_2m_3m_4$	a_1+2a_2	$m_1m_2m_3m_4$	$2a_1+a_2+a_3$
$m_2m_3m_4m_5$	$3a_1+2a_2$	$m_1m_2m_3m_4m_5$	a_2+a_3

Table 2. Associated bijective 1-cocycle.

1-cocycle $\delta: C \to B$ with respect to λ given by Table 2 yields a brace of abelian type $(B,+,\cdot)$ of order 32. This brace corresponds to `SmallBrace(32, 14649)` in the YangBaxter library for GAP (this method of construction of braces will be used in the subsequent examples without a further note).

We have that $\langle 2a_1+2a_2\rangle_+ \leq (B,+)$, corresponding to $\langle m_3m_4\rangle \leq C$ (through δ), defines a subbrace S of B of order 2. We also have that $\langle 2a_1, 2a_2, a_1+a_2+a_3\rangle_+ \leq (B,+)$, corresponding to $\langle m_5, m_3m_4m_5, m_1m_2m_4m_5\rangle \leq C$, defines a subbrace T of B of order 8. Furthermore $\langle 2a_1, 2a_2, a_1+a_3\rangle_+ \leq (B,+)$, corresponding to $\langle m_5, m_3m_4m_5, m_3m_5\rangle \leq C$, defines another subbrace U of B of order 8.

We note that S is not a left ideal of B, because

$$\lambda_{m_1}(2a_1+2a_2) = 2(3a_1+a_3)+2(a_1+a_2+a_3) = 2a_2 \notin S.$$

On the other hand, S is a left ideal of T, since $\lambda_{m_5}(2a_1+2a_2) = \lambda_{m_3m_4m_5}(2a_1+2a_2) = \lambda_{m_1m_2m_4m_5}(2a_1+2a_2) = 2a_1+2a_2$. Furthermore, $\langle m_3m_4\rangle$ is a normal subgroup of $\langle m_5, m_3m_4m_5, m_1m_2m_4m_5\rangle$. Therefore, S is an ideal of T. We also have that S is a left ideal of U, since $\lambda_{m_5}(2a_1+2a_2) = \lambda_{m_3m_4m_5}(2a_1+2a_2) = \lambda_{m_3m_5}(2a_1+2a_2) = 2a_1+2a_2$. Moreover, $\langle m_3m_4\rangle$ is a normal subgroup of $\langle m_5, m_3m_4m_5, m_3m_5\rangle$. Therefore, S is an ideal of U.

We prove now that the subbrace $D = \langle T, U\rangle$ of B generated by T and U is B. Let H be the additive group of D. Then $H \geq \langle 2a_1, a_2, a_1+a_3\rangle_+$. Thus, if $R = \delta^{-1}(H)$

x	$\delta(x)$	x	$\delta(x)$	x	$\delta(x)$		
1	0	$g_5g_3g_2$	$a+e$	g_1	$2a+c$	$g_3g_2g_1$	$3a+e$
g_2	$c+d+e$	$g_4g_2g_1$	$2a+d+e$	g_3	$3a+c$	$g_5g_2g_1$	$c+e$
g_4	c	$g_4g_3g_1$	$a+c+d$	g_5	d	$g_5g_3g_1$	$3a$
g_2g_1	$2a+c+d+e$	$g_5g_4g_1$	$2a+d$	g_3g_1	$3a+d$	$g_4g_3g_2$	$a+c+d+e$
g_4g_1	$2a$	$g_5g_4g_2$	$2a+e$	g_5g_1	$2a+c+d$	$g_5g_4g_3$	$a+d$
g_3g_2	$3a+d+e$	$g_4g_3g_2g_1$	$a+c+e$	g_4g_2	$d+e$	$g_5g_3g_2g_1$	$a+d+e$
g_5g_2	$2a+c+d$	$g_5g_4g_2g_1$	e	g_4g_3	a	$g_5g_4g_3g_1$	$a+c$
g_5g_3	$3a+c+d$	$g_5g_4g_3g_2$	$3a+c+e$	g_5g_4	$c+d$	$g_5g_4g_3g_2g_1$	$3a+c+d+e$

Table 3. Associated bijective 1-cocycle.

is the corresponding multiplicative group, then

$$\delta^{-1}(2a_1) = m_5 \in R, \quad \delta^{-1}(a_2) = m_1m_4 \in R, \quad \delta^{-1}(2a_2) = m_3m_4m_5 \in R,$$
$$\delta^{-1}(a_1+a_3) = m_3m_5 \in R, \quad \delta^{-1}(a_1+a_2+a_3) = m_1m_2m_4m_5 \in R,$$

which implies that $C = \langle m_1, m_2, m_3, m_4, m_5\rangle = R$. Thus, $H = (B,+)$ and $\langle T, U\rangle = B$.

Finally, suppose that S possesses an idealiser in B. Since it must contain every subbrace of B in which S is an ideal, it must contain T and U. It follows that the idealiser of S in B must be B, but S is not an ideal of B.

Our second example shows that there is no analogue of Fitting's theorem for central nilpotency, even for braces of abelian type.

Example B. Let $B = \langle a\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle \simeq C_4 \times C_2 \times C_2 \times C_2$, written additively. Let us consider the automorphisms g_1, g_2, g_3 of B given by $g_1(a) = 3a+d$, $g_1(c) = c$, $g_1(d) = d$, $g_1(e) = c+e$, $g_2(a) = a$, $g_2(c) = c$, $g_2(d) = 2a+d$, $g_2(e) = 2a+e$, $g_3(a) = a$, $g_3(c) = 2a+c$, $g_3(d) = d$, $g_3(e) = 2a+e$. If $g_4 = g_1g_2g_1^{-1}g_2^{-1}$ and $g_5 = g_1g_3g_1^{-1}g_3^{-1}$, then their action on $(B,+)$ is the following one: g_4 maps a to $3a$ and fixes c, d, e, while g_5 maps e to $2a+e$ and fixes a, c, d. We have that $C = \langle g_1, g_2, g_3\rangle = \langle g_1, g_2, g_3, g_4, g_5\rangle$ satisfies the following relations: $g_1^2 = 1$, $g_2^2 = 1$, $g_1g_2g_1^{-1} = g_4g_2$, $g_3^2 = 1$, $g_1g_3g_1^{-1} = g_5g_3$, $g_2g_3g_2^{-1} = g_3$, $g_4^2 = 1$, $g_1g_4g_1^{-1} = g_4$, $g_2g_4g_2^{-1} = g_4$, $g_3g_4g_3^{-1} = g_4$, $g_5^2 = 1$, $g_1g_5g_1^{-1} = g_5$, $g_2g_5g_2^{-1} = g_5$, $g_3g_5g_3^{-1} = g_5$, $g_4g_5g_4^{-1} = g_5$. It follows that C is a group of order 32. The bijective 1-cocycle $\delta\colon C \to B$ given in Table 3 yields a brace of abelian type on $(B,+,\cdot)$ of order 32. This brace corresponds to `SmallBrace(32, 23060)` in the YangBaxter library for GAP.

Since C has order 32, we have that $\operatorname{Ker}\lambda = 1$. In particular, we have that $\zeta(B) = 0$, and B is not centrally nilpotent.

Now, let us compute the ideals of B. Suppose that I is a nonzero ideal of B with additive group L and multiplicative group E. Since E is a normal subgroup of C, it must contain a minimal normal subgroup of E. All minimal normal subgroups of C are contained in $\mathrm{Z}(C) = \langle g_4, g_5\rangle$. Hence E must contain $\langle g_4\rangle$, $\langle g_5\rangle$ or $\langle g_4 g_5\rangle$. In the first case, L must contain $\delta(g_4) = c$. Since L must be invariant under the action of C, it should contain $g_3(c) = 2a + c$. Consequently, $\langle 2a, c\rangle_+ \leq L$. In particular, $\delta^{-1}(2a) = g_4 g_1 \in E$ and $\langle g_1, g_4\rangle \leq E$. Since $E \trianglelefteq C$, we have that $g_3 g_1 g_3^{-1} = g_5 g_1 \in E$, so $\langle g_1, g_4, g_5\rangle \leq E$.

Similarly, if $g_5 \in E$, then $\delta(g_5) = d \in L$. Thus, $g_2(d) = 2a + d \in L$, and hence $\langle 2a, d\rangle \leq L$. Now, $g_1 g_4 \in E$, so $g_3^{-1} g_1 g_4 g_3 = g_1 g_4 g_5 \in E$ and also $g_5 \in E$. Therefore

$$\langle g_1, g_4, g_5\rangle \leq E.$$

Finally, if $g_4 g_5 \in E$, then $\delta(g_4 g_5) = c + d \in L$, so $g_2(c + d) = 2a + c + d \in L$, and hence $\langle 2a, c + d\rangle_+ \leq L$. Again, $g_1 g_4 \in E$, so $g_3^{-1} g_1 g_4 g_3 = g_1 g_4 g_5 \in E$ and also $g_5 \in E$. Thus,

$$\langle g_1, g_4, g_5\rangle \leq E.$$

In all cases, we found that $\langle g_1, g_4, g_5\rangle \leq E$. Since $\delta(\langle g_1, g_4, g_5\rangle) = \langle 2a, c, d\rangle_+ \leq (B, +)$ is a δ-invariant subgroup and $\langle g_1, g_4, g_5\rangle \trianglelefteq C$, we have that $J = \langle 2a, c, d\rangle_+$ is the unique ideal of B of order 8. We observe that B/J is abelian. Therefore, the only three ideals of order 16 of B are $I_1 = \langle 2a, c, d, e\rangle_+$, $I_2 = \langle a + e, 2a, c, d\rangle_+$, $I_3 = \langle a, c, d\rangle_+$.

It can be easily seen that $0 \leq \langle c\rangle \leq \langle 2a, c\rangle_+ \leq I_1$, $0 \leq \langle c+d\rangle_+ \leq \langle 2a, c+d\rangle_+ \leq I_2$ and $0 \leq \langle d\rangle_+ \leq \langle 2a, d\rangle_+ \leq I_3$ are c-series of I_1, I_2 and I_3, respectively. In particular, I_1, I_2 and I_3 are centrally nilpotent braces. However, $B = I_1 + I_2 = I_1 + I_3 = I_2 + I_3$, but, as we have mentioned, B is not centrally nilpotent.

Our third example shows that there may be abelian subideals that are not contained in any centrally nilpotent ideals.

Example C. Let $(B, +) = \langle a\rangle \times \langle b\rangle \simeq \mathrm{C}_2 \times \mathrm{C}_{12}$ and $(C, \cdot) = [\langle\sigma\rangle]\langle\tau\rangle \simeq \mathrm{Dih}_{24}$. We have that C acts on B by means of the action λ defined by $\lambda_\sigma(a) = a + 6b$, $\lambda_\tau(a) = a$, $\lambda_\sigma(b) = a + b$, $\lambda_\tau(b) = a - b$. The bijective 1-cocycle $\delta: C \to B$ with respect to λ given by Table 4 yields a brace of abelian type $(B, +, \cdot)$ of order 24. This brace corresponds to `SmallBrace(24, 57)` in the YangBaxter library for GAP.

Let I be any ideal of B of order 12, and put $E = \delta^{-1}(I, +)$. Since $(I, +)$ is a maximal subgroup of $(B, +)$, it must contain its Frattini subgroup, which is $\langle 6b\rangle$. As $\delta^{-1}(6b) = \tau$ and $E \trianglelefteq C$, it follows that $\sigma\tau\sigma^{-1} = \sigma^2\tau \in E$. Therefore, $\delta(\sigma^2\tau) = a + 2b \in I$ and then $(I, +) = \langle a, 2b\rangle_+$. Since I is λ-invariant, we get that $I = \langle a, 2b\rangle_+$ is the only ideal of order 12 of B.

c	$\delta(c)$	c	$\delta(c)$	c	$\delta(c)$	c	$\delta(c)$
1	0	σ^6	a	τ	$6b$	$\sigma^6\tau$	$a+6b$
σ	$a+7b$	σ^7	b	$\sigma\tau$	$a+b$	$\sigma^7\tau$	$7b$
σ^2	$a+8b$	σ^8	$8b$	$\sigma^2\tau$	$a+2b$	$\sigma^8\tau$	$2b$
σ^3	$9b$	σ^9	$a+3b$	$\sigma^3\tau$	$3b$	$\sigma^9\tau$	$a+9b$
σ^4	$4b$	σ^{10}	$a+4b$	$\sigma^4\tau$	$10b$	$\sigma^{10}\tau$	$a+10b$
σ^5	$11b$	σ^{11}	$5b$	$\sigma^5\tau$	$a+5b$	$\sigma^{11}\tau$	$11b$

Table 4. Associated bijective 1-cocycle.

Note that I is not abelian as $\mathrm{Soc}(I) = \langle a+4b\rangle_+$. Thus,

$$(I, \cdot) \simeq \mathrm{Dih}_{12}$$

and so I is not centrally nilpotent. Hence, $\mathrm{Soc}(I)$ is an abelian subideal of B of order 6 such that it is not contained in any centrally nilpotent ideal of B.

Our last example shows that there are noncentrally nilpotent braces whose subbraces are subideals.

Example D. Let $(B,+,\cdot)$ be the brace of abelian type of order 32 studied in [2, Example 37]. It corresponds to `SmallBrace(32, 24003)` in the YangBaxter library for GAP, so $(B,+) = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \simeq C_4 \times C_2 \times C_2 \times C_2$, and $(B,\cdot) \simeq \langle e, f, h\rangle \simeq [C_2 \times \mathrm{Q}_8]C_2$ with bijective 1-cocycle given by Table 5 and associated action given by

$$\begin{aligned} &e(a) = a+c+d, && e(b) = 2a+c, && e(c) = b, && e(d) = 2a+c+d,\\ &f(a) = a+b+c, && f(b) = 2a+b, && f(c) = c, && f(d) = c+d,\\ &h(a) = a, && h(b) = b, && h(c) = c, && h(d) = 2a+d. \end{aligned}$$

x	$\delta(x)$	x	$\delta(x)$	x	$\delta(x)$	x	$\delta(x)$
1	0	h	c	f	a	fh	$a+c$
e	$3a+b$	eh	$3a$	ef	$b+c+d$	efh	$c+d$
e^2	$b+c$	e^2h	$2a+b+c+d$	e^2f	$3a+d$	e^2fh	$a+c+d$
e^3	$3a+b+d$	e^3h	$a+d$	e^3f	b	e^3fh	$2a$
e^4	$2a+b+c$	e^4h	$2a+b$	e^4f	$a+b+c$	e^4fh	$a+b$
e^5	$3a+c$	e^5h	$3a+b+c$	e^5f	$2a+d$	e^5fh	$2a+b+d$
e^6	$2a+c+d$	e^6h	d	e^6h	$3a+b+c+d$	e^6fh	$a+b+d$
e^7	$3a+c+d$	e^7h	$a+b+c+d$	e^7f	$2a+c$	e^7fh	$b+d$

Table 5. Associated bijective 1-cocycle.

We start by providing all subbraces of order 2. These are generated by those elements $x \in (B, +)$ of order 2 such that $\lambda_x(x) = x$. We have $S_1 = \{1, 2a+b+d\}$, $S_2 = \{1, c\}$, $S_3 = \{1, b+c\}$, $S_4 = \{1, c+d\}$, $S_5 = \{1, 2a\}$, $S_6 = \{1, 2a+b\}$, $S_7 = \{1, 2a+b+c\}$ (here, S_5 is the only left ideal). For subbraces of order 4, we need those subgroups $H \le (B, +)$ of order 4 such that $\delta^{-1}(H)$ is also a subgroup of $\langle e, f, h\rangle$. We find

H	$\delta^{-1}(H)$	H	$\delta^{-1}(H)$
$S_8 = \langle 2a, b+c\rangle$	$\langle e^3 fh, e^7 fh\rangle$	$S_9 = \langle 2a+b, c\rangle$	$\langle e^4, h\rangle$
$S_{10} = \langle b, 2a+c\rangle$	$\langle e^3 f\rangle$	$S_{11} = \langle c+d, 2a+b+d\rangle$	$\langle efh, e^5 fh\rangle$
$S_{12} = \langle 2a+d, b+c+d\rangle$	$\langle e^5 f\rangle$	$S_{13} = \langle d, 2a+b+c\rangle$	$\langle e^6 h\rangle$
$S_{14} = \langle 2a+b+c, b+d\rangle$	$\langle e^2\rangle$		

(here, S_8 is the only left ideal). For subbraces of order 8, we need those subgroups $H \le (B, +)$ of order 8 such that $\delta^{-1}(H) \le \langle e, f, h\rangle$. Thus,

$$\begin{aligned} S_{15} &= \langle 2a, b, c\rangle, & \delta^{-1}(S_{15}) &= \langle e^3 fh, e^3 f\rangle, \\ S_{16} &= \langle 2a, b+c, b+d\rangle, & \delta^{-1}(S_{16}) &= \langle e^3 fh, e^2\rangle, \\ S_{17} &= \langle 2a, b+c, d\rangle, & \delta^{-1}(S_{17}) &= \langle e^3 fh, e^6 h\rangle \end{aligned}$$

(here, S_{15} is the only left ideal). The only subbrace of order 16 is the only nonzero proper ideal of B, that is, $S_{18} = \langle 2a, b, c, d\rangle$. The following relations can be easily checked to hold:

$$\begin{aligned} &S_1, S_4, S_7 \trianglelefteq S_{11} \trianglelefteq S_{15} \trianglelefteq S_{18} \trianglelefteq B, && S_3, S_5, S_7 \trianglelefteq S_8 \trianglelefteq S_{16} \trianglelefteq S_{18} \trianglelefteq B, \\ &S_2, S_6, S_7 \trianglelefteq S_9 \trianglelefteq S_{17} \trianglelefteq S_{18} \trianglelefteq B, && S_{14} \trianglelefteq S_{15} \trianglelefteq S_{18} \trianglelefteq B, \\ &S_{12}, S_{13} \trianglelefteq S_{17} \trianglelefteq S_{18} \trianglelefteq B, && S_{10} \trianglelefteq S_{17} \trianglelefteq S_{18} \trianglelefteq B. \end{aligned}$$

Therefore, all subbraces are subideals but B is not soluble.

Acknowledgements

All authors are members of the nonprofit association "Advances in Group Theory and Applications". Ferrara and Trombetti are supported by GNSAGA (INdAM) and wish to express their gratitude to the Department of Mathematics of the University of Valencia for its support during their stay. Ferrara has been supported by a research visiting grant issued by the Istituto Nazionale di Alta Matematica (INdAM).

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Received March 18, 2024. Revised January 13, 2025.

Adolfo Ballester-Bolinches
Departament de Matemàtiques
Universitat de València
València
Spain

adolfo.ballester@uv.es

Ramón Esteban-Romero
Departament de Matemàtiques
Universitat de València
València
Spain

ramon.esteban@uv.es

Maria Ferrara
Dipartimento di Matematica e Fisica
Università degli Studi della Campania "Luigi Vanvitelli"
Caserta
Italy

maria.ferrara1@unicampania.it

Vicent Pérez-Calabuig
Departament de Matemàtiques
Universitat de València
València
Spain

vicent.perez-calabuig@uv.es

Marco Trombetti
Dipartimento di Matematica e Applicazioni "Renato Caccioppoli"
Università degli Studi di Napoli Federico II
Napoli
Italy

marco.trombetti@unina.it

PACIFIC JOURNAL OF MATHEMATICS

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 335 No. 1 March 2025
