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HYPERBOLIC L-SPACE KNOTS NOT CONCORDANT TO ALGEBRAIC KNOTS

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We construct infinitely many hyperbolic L-space knots for which $-3\int\Upsilon$ is not an integer, where Υ is the Ozsváth–Stipsicz–Szabó upsilon function. None of these knots can be concordant to a linear combination of algebraic knots.

1. Overview

Describing knot concordance groups and understanding $\mathbb{Z}$-homology cobordism of 3-manifolds are listed among the most important problems in low-dimensional topology. Among many questions, a particular interest is about the position of various classes of knots, respectively 3-manifolds, in the concordance group, respectively in the group of $\mathbb{Z}$-homology cobordisms.

It is well known [30] that each 3-manifold is $\mathbb{Z}$-homology cobordant to a hyperbolic 3-manifold. On the other hand, there are 3-manifolds that are not $\mathbb{Z}$-homology cobordant to Seifert fibered manifolds [10]; recently it is proved [20] that the Seifert fibered manifold span a subgroup of the group of $\mathbb{Z}$-homology cobordism with $\mathbb{Z}^\infty$-summand as a quotient.

For link cobordisms, there is an abundance of similar questions. There are knots that are not topologically concordant to alternating knots [16]. A refinement of the argument in [16] shows that there exist knots that are not topologically concordant to L-space knots [41]. On the other hand, all knots are topologically concordant to strongly quasipositive knots [5], a statement that is definitely false in the smooth category, because for all strongly quasipositive knots, all slice torus invariants are equal, compare [14].

An algebraic link is defined as a link of a plane curve singularity. All such links are graph links [12]. Also, all such links are L-space links by [17; 18]. Studying the position of algebraic knots in the whole knot concordance group seems to bring an immediate answer: while it is not stated explicitly in [41], the methods in that paper suggest that the quotient of the topological concordance group by the group of L-space knots has an infinite $\mathbb{Z}^\infty$ summand. However, the position of algebraic

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knots in the concordance subgroup spanned by all L-space knots seems rather mysterious. This question appears to be a natural generalization of a question on the position of graph manifolds in the $\mathbb{Z}$-homology cobordism group of 3-manifolds.

The main result of the paper is the following.

Theorem 1.1. *There exist infinitely many hyperbolic L-space knots that are not smoothly concordant to any linear combination of algebraic knots.*

Plan of the proof. In Section 4 we construct an infinite family of knots K_n. By Montesinos trick we show that each K_n admits a positive L-space surgery. The braid index of K_n is bounded by 4 by construction. Combining this fact with the lack of semigroup property of K_n, we show that K_n are hyperbolic.

The proof that K_n is not concordant to any linear combination of algebraic knots involves computing the function Υ of Ozsváth, Stipsicz and Szabó [33]. For L-space knots, this function is determined from the Alexander polynomial of K_n, and to pass from that polynomial to Υ, we use an explicit algorithm described in [6]. The computation of the Alexander polynomial of K_n, conducted in Section 4 involves a surgery description of K_n, and Torres formula.

The main obstruction is the following. It was shown in [36] that if K is an algebraic knot, then $-3\int_0^2 \Upsilon_K(t)\,dt$ is an integer. In fact, echoing the calculation of [4], one can express this integral via invariants of the underlying plane curves singularity, compare Section 3.

Conversely, Theorem 4.1 shows that $-3\int_0^2 \Upsilon_{K_n}$ is a nonintegral fraction of 5. So, none of the knots K_n can be expressed as linear combinations of algebraic knots.

The methods do not show whether the knots K_n generate a $\mathbb{Z}^\infty$ summand in the quotient subgroup generated by the concordance classes of all L-space knots modulo algebraic knots. However, we know that a subsequence of K_n knots is linearly independent in the topological concordance group, see Theorem 6.1.

The structure of the paper is the following. Section 2 gives a necessary background on the Υ function. In Section 3, we recall Tange's calculations of the integral of the Υ function and show some properties of the integral. In particular, we introduce a purely Floer-theoretic invariant ω and relate it to invariants of plane curve singularities. These results are of independent interest, especially that they give an algebrogeometric motivation for studying $-3\int \Upsilon$ as a knot invariant.

The family of knots K_n is constructed in Section 4. Their main properties are stated in Theorem 4.1, whose part is proved in that section. The most difficult part, showing that K_n are indeed L-space knots, is proved in Section 5.

Section 6 addresses the question of linear independence of knots K_n. We study roots of the Alexander polynomial of K_n and show that a subsequence of K_n is linearly independent. We also show that Υ function alone cannot prove that K_n are independent modulo the group of algebraic knots.

Finally, in Section 7 we provide a table of those L-space knots of [1] for which $-3\int\Upsilon$ is not integral.

2. Review of the Υ function

Recall that to a knot K in the 3-sphere, knot Floer homology associates a complex $\mathrm{CFK}^\infty(K)$ over the ring $\mathbb{Z}_2[U, U^{-1}]$, where U is a formal variable. The complex is $\mathbb{Z}\oplus\mathbb{Z}$–filtered, $\mathbb{Z}$-graded, and the multiplication by U lowers the grading by 2 and the filtration by $(1, 1)$. The complex is defined up to bifiltered chain homotopy equivalence.

In [33], a concordance invariant Υ was extracted from this chain complex; see also [26]. In short, for each $t \in [0, 2]$, one associates a collapsed filtration. If $x \in \mathrm{CFK}^\infty(K)$ is at bifiltration level (a, b), we define its $\mathcal{F}_t$-filtration level by $\frac{t}{2}a + \left(1 - \frac{t}{2}\right)b$. Denote by $\mathcal{C}_{s,t}$ the subcomplex of $\mathrm{CFK}^\infty(K)$ of elements at $\mathcal{F}_t$-filtration level $\le s$. As $\mathcal{C}_{s,t}$ is a subcomplex of $\mathrm{CFK}^\infty(K)$, there is a map $H_i(\mathcal{C}_{s,t}) \to H_i(\mathrm{CFK}^\infty(K))$, where the subscript i denotes the homological grading (the $\mathbb{Z}$-grading) of the complex $\mathrm{CFK}^\infty(K)$. Set

$$\nu(t) = \min\{s : H_0(\mathcal{C}_{s,t}) \to H_0(\mathrm{CFK}^\infty(K)) \text{ is surjective}\}.$$

The function Υ is defined by

$$\Upsilon(t) = -2\nu(t).$$

Among many properties of the function Υ, the most important for the sake of this paper is that it is a concordance invariant. That is to say, if K_1 is smoothly concordant to K_2, then $\Upsilon_{K_1}(t) = \Upsilon_{K_2}(t)$ for all $t \in [0, 2]$.

If K is an L-space knot, the complex $\mathrm{CFK}^\infty(K)$ is determined by the Alexander polynomial, via so-called *staircase complex*. The Υ function for such knots was computed in [33]. To be more precise, write the Alexander polynomial

$$\Delta_K = 1 + (t-1)(t^{c_1} + \cdots + t^{c_\ell}),$$

with $1 \le c_1 < \cdots < c_\ell$ (such presentation is possible for all L-space knots, see [31]). Let $S_K = \mathbb{Z}_{\ge 0} \setminus \{c_1, \ldots, c_\ell\}$.

Definition 2.1 (see [40]). The set S_K is called the *formal semigroup* of the L-space knot K.

An equivalent definition of S_K is via the power series expansion:

$$\frac{\Delta_K(t)}{1-t} = \sum_{s\in S_K} t^s.$$

If K is algebraic, the set S_K is the semigroup of the underlying singularity. It is an interesting question to study nonalgebraic L-space knots for which S_K has the structure of a semigroup [37; 40].

The Υ function of an L-space knot is related to the formal semigroup via the Fenchel–Legendre transform of an extension of the function $I : \mathbb{Z} \to \mathbb{Z}$ such that $I : n \mapsto \#\{S_K \cap [0, n)\}$; see [6]. In particular, for an L-space knot, Υ is convex. It is interesting to see for which knots the Υ function is convex [21].

3. The integral of Υ in algebraic geometry

It was proved by Tange [36] that $-3\int\Upsilon$ for an algebraic knot is an integer. We recall his computations. We also show that the quantity $-2\tau - 3\int\Upsilon$ has a special interpretation in singularity theory. This explains our initial interest in the invariant $-3\int\Upsilon$. We begin by recalling a standard definition; see [9; 12] for more details.

Let $z \in \mathbb{C}^2$ be a singular point of a complex algebraic curve C. Recall that the *multiplicity* of a singular point, denoted m, is the minimal positive number that can be obtained as a local intersection of C at z with another algebraic curve. The fact that the point z is singular means that $m > 1$.

Suppose z has a single branch. Set $m_1 = m$. Blow up a singular point to obtain a new curve $\tilde{C}$ and denote by E the exceptional divisor of the blow-up. Usually $\tilde{C}$ will have a singular point at $\tilde{C} \cap E$. Denote by m_2 its multiplicity. If $\tilde{C}$ is smooth, then $m_2 = 1$ and we stop the procedure. The procedure can be iterated until some strict transform $\tilde{C}$ is smooth. Eventually we obtain a finite sequence $(m_1, m_2, m_3, \dots, m_n)$ of integers such that $m_i > 1$ (usually we discard the last 1 from the sequence).

Definition 3.1. The sequence $(m_1, \dots, m_n)$ is called the *multiplicity sequence* of a unibranched singular point.

The multiplicity sequence is a complete topological invariant of a singular point, in the sense that any two unibranched singular points with the same multiplicity sequences are topologically equivalent. All topological invariants of the singularity can be computed from the multiplicity sequence. For example, the following formula was proved by Milnor [27], compare [9; 42]:

$$\mu_z = \sum_{i=1}^{n} m_i(m_i - 1), \tag{3.2}$$

where μ_z is the Milnor number of the critical point. The Milnor number is equal to twice the genus of the link of the singular point.

We introduce the quantity

$$\omega(K) = -3\int_0^2 \Upsilon_K(t)\,dt - 2\tau(K). \tag{3.3}$$

As we see, ω is defined purely from Heegaard Floer-type invariants.

Theorem 3.4. *Let z be a singular point with one branch. Let K be the link of singularity. Then $\omega(K) = \sum(m_i - 1)$.*

Proof. The proof relies on the following result of Tange [36], based on [13] and [3].

Lemma 3.5. *For $m > 1$ define the function*

$$\Upsilon_m(t) = -i(i+1) - \tfrac{1}{2}m(m-1-2i)t \quad \text{if } t \in \left[\frac{2i}{m}, \frac{2i+2}{m}\right].$$

If a singular point z has link K and multiplicity sequence $(m_1, \dots, m_n)$, then

$$\Upsilon_K = \sum_{i=1}^{n} \Upsilon_{m_i}.$$

Continuing the proof of Theorem 3.4, we use the following formula of [36]:

$$\int_0^2 \Upsilon_m(t)\,dt = \frac{-m^2+1}{3}. \tag{3.6}$$

From (3.6) and Lemma 3.5 we conclude that

$$-3\int_0^2 \Upsilon_K(t) = \sum(m_i^2 - 1).$$

For algebraic knots, $2\tau(K) = \mu_z$ is the Milnor number of the underlying singular point. We conclude the proof by (3.2). □

Corollary 3.7 (see [36]). *For an algebraic knot, $-3\int_0^2 \Upsilon(t)\,dt$ is an integer.*

We have the following interpretation of $\omega(K)$, which is due to Zaĭdenberg and Orevkov [42]. Blow up the critical point z until the reduced inverse image $D = \pi^{-1}(C)_{\text{red}}$ is a normal crossing divisor. Let $E_1, \dots, E_s$ be the exceptional divisors. Define the canonical divisor $K_z = \sum \alpha_i E_i$ by the condition that

$$K_z \cdot E_i + E_i \cdot E_i = -2 \quad \text{for all } i.$$

Let C_z' be the strict transform of C.

Proposition 3.8 (see [42, Lemma 4]). *For K the link of singularity at z, if z has one branch, then $\omega(K) = K_z \cdot (K_z + C_z')$.*

We conclude this section with a simple estimate for quasihomogeneous singular points:

Proposition 3.9. *If z is a quasihomogeneous singular point, topologically equivalent to $x^p - y^q = 0$ with p, q coprime, then*

$$\omega(K) < p + q,$$

where K is the link of the singularity (in this situation, K is the T_{pq}-torus knot). Moreover, if $p = 2$, then $\omega(K) = \frac{1}{2}(q-1)$.

Proof. The second part follows from the fact that the multiplicity sequence for the singular point $x^2 - y^{2k+1} = 0$ is a length k sequence $(2, \dots, 2)$. For the first part we observe that the multiplicity sequence is constructed as follows. Suppose $p < q$. Then $m_1 = p$. The blow-up replaces (p, q) by $(p, q - p)$ or $(q - p, p)$ depending on whether $p < q - p$ or $q - p < p$. It follows that $\sum_{i=2}^{n} m_i = q$, and hence

$$\sum_{i=1}^{n} m_i = p + q.$$

Therefore $\omega(K) \leq p + q - 1$. □

The implication of Proposition 3.9 is that ω is a *linear* invariant, that is, its value for the T_{pq}-torus knot grows like the sum of p and q, not like the product. The latter behavior is more typical, the genus and the signature are examples, in particular, for slice-torus invariants.

Another inequality involving $\omega(K)$ is a generalization of the Zaĭdenberg–Orevkov inequality [42, Section 11].

Proposition 3.10. *For a cuspidal singular point z with Milnor number μ and multiplicity m we have $\mu \leq m\omega$.*

Proof. Let $m_1, \dots, m_n$ be the multiplicity sequence of z. Then $m_1 \geq m_2 \geq \dots \geq m_n$. This means that $m_1 \sum (m_i - 1) \geq \sum m_i(m_i - 1)$, but in light of (3.2), this is precisely the statement of the proposition. □

4. The family K_n

For $n \geq 1$, our knot K_n is given by the surgery description shown in Figure 1. Let $L = K \cup C_1 \cup C_2$ be the oriented link as shown in Figure 1. If we perform (-1)-surgery on C_1 and $\left(-\frac{1}{n+1}\right)$-surgery on C_2, then K is changed into K_n. Thus K_n is the closure of the 4-braid:

$$[2, 1, 3, 2, (3, 2, 1)^4, 3^{2(n+1)}, 2],$$

where an integer k denotes the standard braid generator σ_k of the 4-string braid group. In particular, K_1 is $m211$, and K_2 is $t09284$ in the SnapPy census [11]. Since K_n is the closure of a positive braid, K_n is fibered and its genus is equal to $n + 8$. The diagram of K_1 is given in Figure 1.

Theorem 4.1. *For $n \geq 1$, the knot K_n enjoys the following.*

(1) *K_n is hyperbolic.*

(2) *$(4n + 24)$-surgery on K_n gives an L-space, so K_n is an L-space knot.*

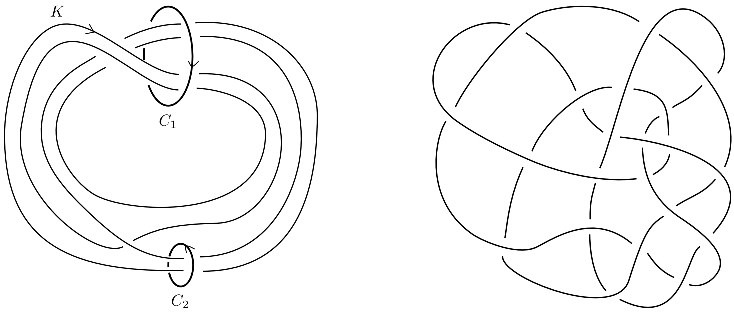

Figure 1. The surgery description of K_n (left) and the knot $K_1 = m211$ (right).

(3) *For the upsilon invariant* $\Upsilon_{K_n}(t)$ *of* K_n,

$$I = \int_0^2 \Upsilon_{K_n}(t)\,dt = -\big(n + \tfrac{34}{5}\big).$$

Thus $-3I = 3n + \frac{102}{5} \notin \mathbb{Z}$.

The proof of the second property is deferred to Section 5. Our first step towards proving Theorem 4.1 is to compute the Alexander polynomial. Once we know it, using the fact that K_n are L-space knots, we compute the integral proving (3). Moreover, the Alexander polynomial will allow us to prove that K_n is not hyperbolic.

Lemma 4.2. *The Alexander polynomial of* K_n *is given by*

$$\begin{aligned}\Delta_{K_n}(t) = (t^{2n+16} - t^{2n+15}) + (t^{2n+12} - t^{2n+11}) + (t^{2n+10} - t^{2n+9})\\ + (t^{2n+7} - t^{2n+6}) + (t^{2n+5} - t^{2n+4}) + \cdots + (t^{11} - t^{10})\\ + t^9 - t^7 + t^6 - t^5 + t^4 - t + 1.\end{aligned}$$

Proof. Let L be the link as in Figure 1. The multivariable Alexander polynomial of L can be readily computed using [11; 24]:

$$\begin{aligned}\Delta_L(x, y, z) = x^7y^3z - x^5y^3z + x^5y^2z + x^5y^2 - x^4y^2 - x^5y\\ - 2x^3y^2z + 2x^4y + x^2y^2z + x^3yz - x^2yz - x^2y + x^2 - 1,\end{aligned}$$

where the variables x, y, z correspond to the (oriented) meridians of K, C_1, C_2.

If we perform (-1)-surgery on C_1 and $\big(-\frac{1}{n+1}\big)$-surgery on C_2, then the link $K \cup C_1 \cup C_2$ is changed into $K_n \cup C_1^n \cup C_2^n$. Since these links have homeomorphic exteriors, the induced isomorphism on the homology groups relates their Alexander polynomials; compare [15; 29].

Let μ_K, μ_{C_1} and μ_{C_2} be the homology classes of meridians of K, C_1, C_2, respectively. Each meridian has linking number one with the corresponding component. Furthermore, let λ_{C_1} and λ_{C_2} be the homology classes of their oriented longitudes. We see that $\lambda_{C_1} = 4\mu_K$ and $\lambda_{C_2} = 2\mu_K$.

Next, let μ_{K_n}, $\mu_{C_1^n}$ and $\mu_{C_2^n}$ be the homology classes of meridians of K_n, C_1^n and C_2^n. Then we have that $\mu_{K_n}=\mu_K$, $\mu_{C_1^n}=-\mu_{C_1}+\lambda_{C_1}$, $\mu_{C_2^n}=-\mu_{C_2}+(n+1)\lambda_{C_2}$. Hence

$$\mu_K=\mu_{K_n},\quad \mu_{C_1}=-\mu_{C_1^n}+4\mu_{K_n},\quad \mu_{C_2}=-\mu_{C_2^n}+2(n+1)\mu_{K_n}.$$

Thus, we have the relation between the Alexander polynomials as

$$\Delta_{K_n\cup C_1^n\cup C_2^n}(x,y,z)=\Delta_L(x,x^4y^{-1},x^{2(n+1)}z^{-1}). \tag{4.3}$$

Since $\operatorname{lk}(K_n,C_2^n)=\operatorname{lk}(K,C_2)=2$ and $\operatorname{lk}(C_1^n,C_2^n)=\operatorname{lk}(C_1,C_2)=0$, the Torres condition [39] gives

$$\Delta_{K_n\cup C_1^n\cup C_2^n}(x,y,1)=(x^2y^0-1)\Delta_{K_n\cup C_1^n}(x,y)=(x^2-1)\Delta_{K_n\cup C_1^n}(x,y).$$

Furthermore, since $\operatorname{lk}(K_n,C_1^n)=\operatorname{lk}(K,C_1)=4$,

$$\Delta_{K_n\cup C_1^n}(x,1)=\frac{x^4-1}{x-1}\Delta_{K_n}(x).$$

Thus

$$\Delta_{K_n}(x)=\frac{x-1}{x^4-1}\Delta_{K_n\cup C_1^n}(x,1)=\frac{x-1}{(x^4-1)(x^2-1)}\Delta_{K_n\cup C_1^n\cup C_2^n}(x,1,1).$$

Then the relation (4.3) gives

$$\begin{aligned}
\Delta_{K_n}(t)&=\frac{t-1}{(t^4-1)(t^2-1)}\Delta_L(t,t^4,t^{2(n+1)})\\
&=\frac{1}{(t^4-1)(t+1)}(t^{2n+21}-t^{2n+19}+t^{2n+15}-2t^{2n+13}+t^{2n+12}+t^{2n+9}-t^{2n+8}\\
&\qquad\qquad +t^{13}-t^{12}-t^9+2t^8-t^6+t^2-1)\\
&=\frac{1}{(t^4-1)(t+1)}\big(t^{2n+13}(t^8-1)-t^{2n+15}(t^4-1)-t^{2n+9}(t^4-1)+t^{2n+8}(t^4-1)\\
&\qquad\qquad +t^9(t^4-1)-t^8(t^4-1)-t^2(t^4-1)+(t^8-1)\big)\\
&=\frac{1}{t+1}\big(t^{2n+13}(t^4+1)-t^{2n+15}-t^{2n+9}+t^{2n+8}+t^9-t^8-t^2+(t^4+1)\big)\\
&=\frac{1}{t+1}\big(t^{2n+15}(t^2-1)+t^{2n+9}(t^4-1)+t^9(t^{2n-1}+1)-t^4(t^4-1)-(t^2-1)\big)\\
&=t^{2n+15}(t-1)+t^{2n+9}(t^2+1)(t-1)+t^9\frac{t^{2n-1}+1}{t+1}-t^4(t^2+1)(t-1)-(t-1)\\
&=t^{2n+16}-t^{2n+15}+t^{2n+12}-t^{2n+11}+t^{2n+10}-t^{2n+9}\\
&\qquad\qquad +t^9\left(\sum_{i=1}^{n-1}(t^{2i}-t^{2i-1})+1\right)-t^7+t^6-t^5+t^4-t+1. \qquad\square
\end{aligned}$$

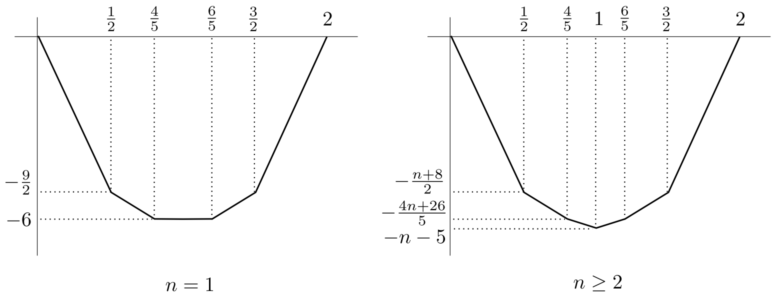

Figure 2. The upsilon function $\Upsilon_{K_n}(t)$: the case $n = 1$ (left) and $n \geq 2$ (right).

In Definition 2.1 we have recalled the notion of a formal semigroup of an L-space knot. While the proof that K_n are L-space knots is given only in Section 5, it is convenient to determine the Υ function of K_n from the Alexander polynomial we just computed.

First, we show that K_n is not algebraic.

Lemma 4.4. *The formal semigroup S_{K_n} of K_n is not closed under addition.*

Proof. By Lemma 4.2, the formal semigroup S_{K_n} of K_n starts:

$$0, 4, 6, 9, 11, 12, \dots, 2n+5, \dots.$$

Thus $4 \in S_{K_n}$, but $8 \notin S_{K_n}$. □

Lemma 4.5. *For $n \geq 1$, the upsilon function of K_n is given as*

$$\Upsilon_{K_n}(t) = \begin{cases} -(n+8)t & \left(0 \leq t \leq \frac{1}{2}\right), \\ -(n+4)t - 2 & \left(\frac{1}{2} \leq t \leq \frac{4}{5}\right), \\ -(n-1)t - 6 & \left(\frac{4}{5} \leq t \leq 1\right). \end{cases}$$

For $t \in [1, 2]$, we have $\Upsilon_{K_n}(t) = \Upsilon_{K_n}(2-t)$.

Figure 2 shows $\Upsilon_{K_n}(t)$ when $n = 1$ and $n \geq 2$.

Proof. By [6], the upsilon function is the Fenchel–Legendre transform of the gap function $G(x) = 2J(-x)$, in their notation, determined by the Alexander polynomial. In fact, there is a handy description of the graph of $G(x)$. Let us write $\Delta_{K_n}(t)$ as $t^{a_0} - t^{a_1} + t^{a_2} - \cdots + t^{a_{2n}}$. Then the sequence of the jumps in the exponents is

$$a_1 - a_0, a_2 - a_1, \dots, a_{2n} - a_{2n-1}. \tag{4.6}$$

Consider the vectors $\boldsymbol{u} = (1, 2)$ and $\boldsymbol{h} = (1, 0)$ on $\mathbb{R}^2$. Then [38, Lemma 2.2] shows that the graph of $G(x)$ restricted on $[-g, g]$ has a form of staircase specified by (4.6). More precisely, we start at the point $(-g, 0)$, and move $a_1 - a_0$ times along $\boldsymbol{u}$, then $a_2 - a_1$ times along $\boldsymbol{h}$, and so on. Finally, we reach the point $(g, 2g)$. The function $G(x)$ is 0 for $x \leq -g$, and $2x$ for $x \geq g$.

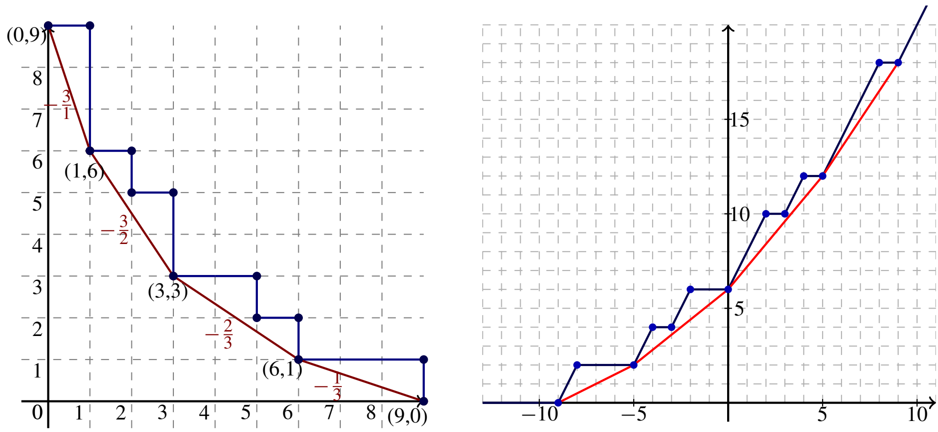

Figure 3. The knot K_1: the staircase (left) and the graph of the gap function (right). The red lines are piecewise-linear convex supporting functions, used to compute the Fenchel–Legendre transform.

By Lemma 4.2, the sequence of the jumps is

$$1, 3, 1, 1, 1, 2, \underbrace{1, 1, \dots, 1, 1}_{2n-2}, 2, 1, 1, 1, 3, 1.$$

Thus the graph of $G(x)$ goes through the points $(-g, 0) = (-n-8, 0)$, $(-n-4, 2)$, $(-n+1, 6)$, $(n-1, 2n+4)$, $(n+4, 2n+10)$ and $(g, 2g) = (n+8, 2n+16)$, which determine the convex hull. More precisely, the convex hull of $G(x)$ is determined by the following:

$$\begin{cases} y = 0 & (x \leq -n-8), \\ y = \frac{x+n+8}{2} & (-n-8 \leq x \leq -n-4), \\ y = \frac{4}{5}(x+n+4)+2 & (-n-4 \leq x \leq -n+1), \\ y = x+n+5 & (-n+1 \leq x \leq n-1), \\ y = \frac{6(x-n+1)}{5} + 2n+4 & (n-1 \leq x \leq n+4), \\ y = \frac{3(x-n-4)}{2} + 2n+10 & (n+4 \leq x \leq n+8), \\ y = 2x & (x \geq n+8). \end{cases}$$

See Figure 3 for the case $n = 1$. Then the Fenchel–Legendre transformation immediately gives the conclusion. □

In Section 5, we prove that $(4n+24)$-surgery on K_n yields an L-space by using the Montesinos trick. Once we admit it, it is easy to prove that K_n is hyperbolic.

Lemma 4.7. *K_n is hyperbolic.*

Proof. Recall that a torus knot of type (p, q) for $0 < p < q$ has the formal semigroup $\langle p, q\rangle = \{ap + bq \mid a, b \geq 0\}$, which is closed under addition. Hence K_n is not a torus knot by Lemma 4.4.

By [25], K_n is prime. Assume that K_n is a satellite knot for a contradiction. Since the bridge number of K_n is at most four, the companion is a 2-bridge knot and the pattern has wrapping number two [35]. Since K_n is an L-space knot, the companion and the pattern knot are also L-space knots [2; 22]. Furthermore, the pattern is braided. Thus the companion is a 2-bridge torus knot by [31], and K_n is its 2-cable. In other words, K_n is an iterated torus L-space knot. Finally, Wang [40] shows that the formal semigroup of such a knot is closed under addition. This contradicts Lemma 4.4. □

Proof of Theorem 4.1. This immediately follows from Lemmas 4.5 and 4.7, and Proposition 5.1 below. □

5. Montesinos trick

We use the Montesinos trick [28] to prove that the $(4n+24)$-surgery on K_n yields an L-space. For a surgery diagram of a strongly invertible knot or link, the Montesinos trick describes the resulting closed 3-manifold as the double branched cover of another knot or link obtained from the tangle replacements corresponding to the surgery coefficients.

Figure 4 shows a surgery diagram in a strongly invertible position with the axis A. After performing (-1)-surgery, we have K_n with surgery coefficient $4n+24$.

Take the quotient of $K_n \cup A$ under the involution along A. Then we obtain a two-component link shown in Figure 5. The Montesinos trick claims that the resulting manifold of $(4n+24)$-surgery on K_n is the double branched cover of S^3 branched over this link. Figures 6 and 7 show the deformations of the link.

Let us denote this link by ℓ_n. We perform two resolutions as shown in Figure 8 at a crossing located in the box with n half twists. Let ℓ_∞ and ℓ_{n-1} be the resulting links. Then it is straightforward to calculate $\det \ell_n = 4n+24$ and $\det \ell_\infty = 4$. (For example, use the checkerboard pattern of the diagrams in Figures 7 and 9.) This shows the equation $\det \ell_n = \det \ell_{n-1} + \det \ell_\infty$. Thus if the double branched covers of ℓ_{n-1} and ℓ_∞ are L-spaces, then so is the double branched cover of ℓ_n [8; 31; 32].

As shown in Figure 9, the link ℓ_∞ is the Montesinos link $M\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$. Thus the double branched cover of ℓ_∞ is a Seifert fibered space over the 2-sphere with three exceptional fibers of indices 2, 2, 2, which is elliptic. Then it is an L-space [31].

Inductively, it is enough to show that the double branched cover of ℓ_1 is an L-space. However, as shown in Figure 10, ℓ_1 is the Montesinos link $M\left(\frac{1}{2}, \frac{1}{2}, -\frac{4}{11}\right)$. The double branched cover is also an elliptic Seifert fibered space, which is an L-space.

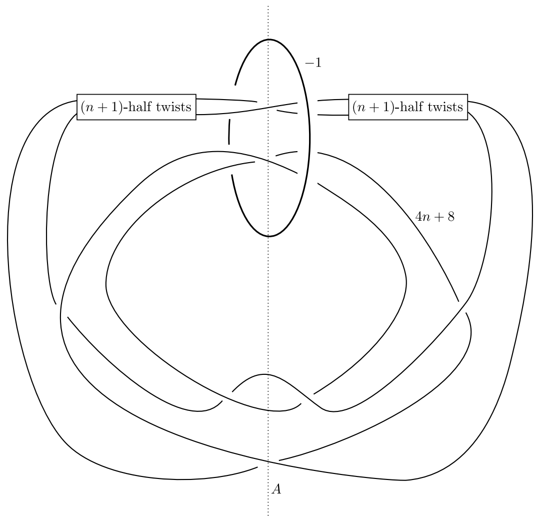

Figure 4. After (-1)-surgery, the surgery diagram gives $(4n+24)$-surgery on K_n. Each rectangle box contains right-handed $(n+1)$-half twists.

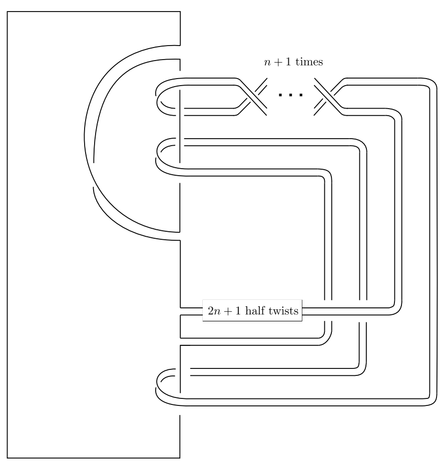

Figure 5. The double branched cover of S^3 branched over this link gives the resulting manifold of $(4n+24)$-surgery on K_n.

Thus, we have shown that:

Proposition 5.1. *For K_n, $(4n+24)$-surgery yields an L-space.*

6. Linear independence of K_n

Theorem 6.1. *There is an increasing sequence a_n such that the knots $K_{a_1}, K_{a_2}, \ldots$ are linearly independent in the topological concordance group.*

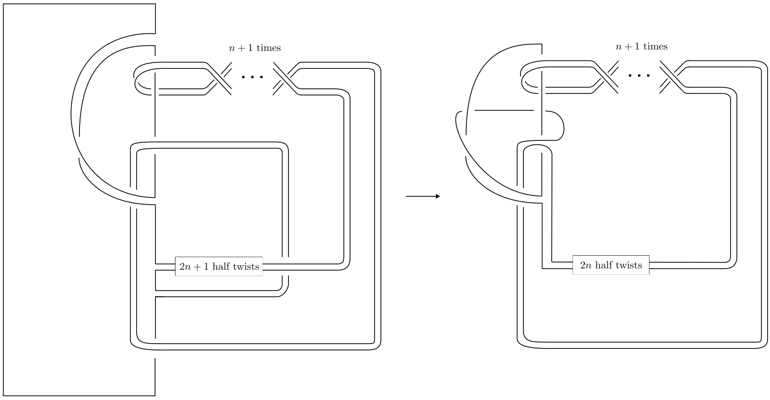

Figure 6. Deformation of the link.

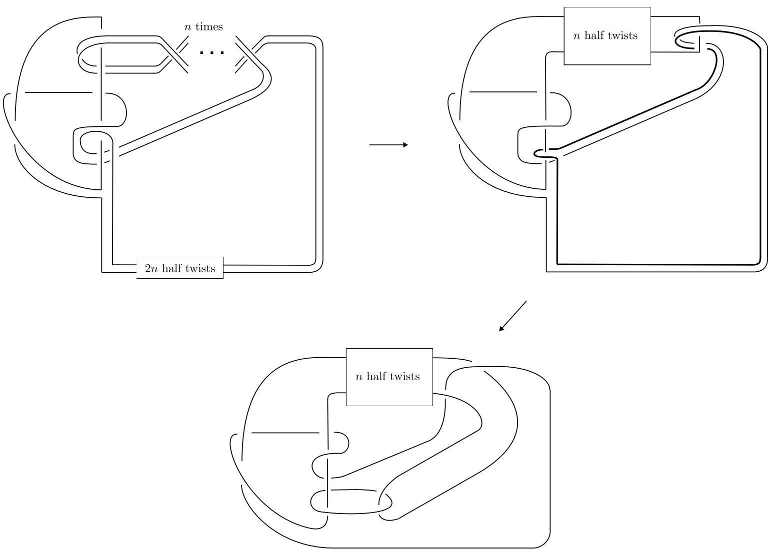

Figure 7. Deformation of the link (continued).

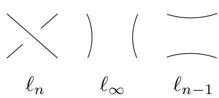

Figure 8. Two resolutions.

Proof. Recall that for any knot K, the Tristram–Levine signature function σ_K is a piecewise constant function from S^1 to $\mathbb{Z}$ with discontinuities only at the roots of the Alexander polynomial. We have the following classical result; see [23, Chapter 12] for a classical approach and [34] for the proof under topological concordance.

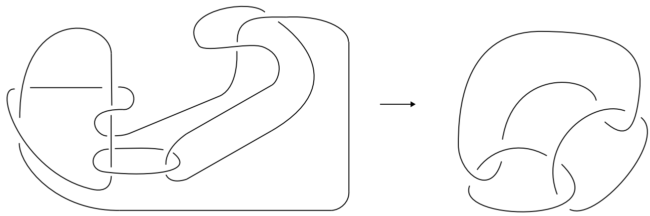

Figure 9. The link ℓ_∞ with $\det \ell_\infty = 4$ is the Montesinos link $M\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$. The double branched cover over this link is a Seifert fibered space over the 2-sphere with three exceptional fibers of indices 2, 2, 2, which is elliptic.

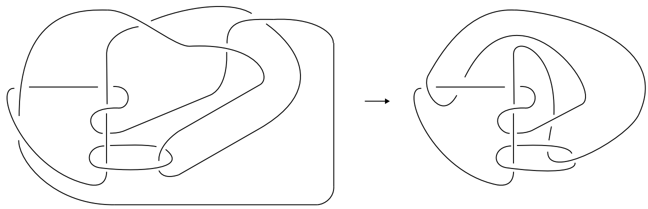

Figure 10. The link ℓ_1 is the Montesinos link $M\left(\frac{1}{2}, \frac{1}{2}, -\frac{4}{11}\right)$.

Proposition 6.2. *If K_1 and K_2 are topologically concordant, then $\sigma_{K_1}(t) = \sigma_{K_2}(t)$ for all but finitely many t in S^1.*

The next result is a consequence of the definition of the signature, it can be deduced from the dévissage of the Blanchfield form. We refer an interested reader to [7, Section 5].

Lemma 6.3. *Suppose $\Delta_K(t)$ has a zero at $t_0 \in S^1$ of odd multiplicity. Then, the signature function has jump at t_0, that is,*

$$\left| \lim_{u \to 0^+} \left(\sigma(t_0 e^{iu}) - \sigma(t_0 e^{-iu}) \right) \right| \geq 2.$$

Our goal is to find, for all knots K_n with $n \geq 11$, a zero of Δ of odd multiplicity of Δ on S^1, and very close to 1. First, we symmetrize the Alexander polynomial for K_n computed in Lemma 4.2 as

$$\begin{aligned}\psi_n = t^{-n-8}\big((t^{2n+16} - t^{2n+15}) + (t^{2n+12} - t^{2n+11}) + (t^{2n+10} - t^{2n+9}) \\ + (t^{2n+7} - t^{2n+6}) + (t^{2n+5} - t^{2n+4}) + \cdots + (t^{11} - t^{10}) \\ + t^9 - t^7 + t^6 - t^5 + t^4 - t + 1\big).\end{aligned}$$

Decompose $\psi_n = \psi_n^1 + \psi_n^2$, where

$$\begin{aligned}\psi_n^1 = t^{n+8} + t^{-n-8} - (t^{n+7} + t^{-n-7}) + t^{n+4} + t^{-n-4} - (t^{n+3} + t^{-n-3}) \\ + (t^{n+2} + t^{-n-2}) - (t^{n+1} + t^{-n-1}) + (t^{n-1} + t^{-n+1}) - (t^{n-2} + t^{-n+2})\end{aligned}$$

and

$$\psi_n^2 = t^{-n-8}(t^{11} - t^{12} + \cdots - t^{2n+4} + t^{2n+5}).$$

It is now convenient to consider both functions on $[0, 2\pi]$. To this end, write $t = e^{iu}$, and set

$$\alpha_n(u) = \tfrac{1}{2}\psi_n^1(e^{iu}), \quad \beta_n(u) = \tfrac{1}{2}\psi_n^2(e^{iu}).$$

We first deal with α_n. We have

$$\alpha_n(u) = \cos(n+8)\,u - \cos(n+7)\,u + \cos(n+4)\,u - \cos(n+3)\,u \\ + \cos(n+2)\,u - \cos(n+1)\,u + \cos(n-1)\,u - \cos(n-2)\,u.$$

From the cosine sum formula we get

$$\alpha_n(u) = -2\sin\frac{u}{2}\Big(\sin\frac{2n+15}{2}u + \sin\frac{2n+7}{2}u + \sin\frac{2n+3}{2}u + \sin\frac{2n-3}{2}u\Big).$$

Applying the sine sum formula we obtain

$$\alpha_n(u) = -4\sin\frac{u}{2}\Big(\sin\big(n+\tfrac{11}{2}\big)\,u\cos 2u + \sin nu\cos\tfrac{3}{2}u\Big). \tag{6.4}$$

The sign of the function α_n near 0 can be examined using (6.4). Indeed, the sine function is positive on $(0, \pi)$, and hence $\sin\big(n+\frac{11}{2}\big)u$ is positive on $\big(0, \frac{2\pi}{2n+11}\big)$. On that interval, also $\cos 2u$, $\sin nu$ and $\cos\frac{3}{2}u$ are positive. That is,

$$\alpha_n(u) < 0 \quad \text{for } u \in \Big(0, \frac{2\pi}{2n+11}\Big).$$

As for the expression ψ_n^2, we note that

$$\psi_n^2 = t^{-n-8}\frac{t^{2n+6}+t^{11}}{t+1} = \frac{t^{n-5/2}+t^{-n+5/2}}{t^{1/2}+t^{-1/2}}.$$

Hence

$$\beta_n(u) = \frac{\cos(n-5/2)\,u}{2\cos(u/2)}.$$

Then, $\beta_n(0) = \frac{1}{2}$ and $\beta_n\big(\frac{\pi}{2n-5}\big) = 0$. We have assumed that $n \geq 11$. This implies that $\frac{\pi}{2n-5} \in \big(0, \frac{2\pi}{2n+11}\big)$. Define $\gamma_n = \alpha_n + \beta_n$, so that $\gamma_n(u) = \frac{1}{2}\psi_n(e^{iu})$. We have

$$\gamma_n(0) = \tfrac{1}{2}, \quad \gamma_n\Big(\frac{\pi}{2n-5}\Big) < 0.$$

Conversely, γ_n changes sign on the interval $\big(0, \frac{\pi}{2n-5}\big)$. Let u_n be the smallest positive zero of γ_n of odd multiplicity. Note that e^{iu_n} is a root of the Alexander polynomial.

Remark 6.5. Computer experiments suggest that there is only one zero of γ_n in that interval and that this zero is simple. We will not need this in the proof.

We can now define an infinite increasing sequence a_m such that K_{a_m} are linearly independent. To this end set $a_1 = 11$. Suppose $a_1, \dots, a_m$ are already defined. As the function $u \mapsto \gamma_{a_m}(u)$ is continuous, and $\gamma_{a_m}(0) = 1$, there is $\lambda_m > 0$ such that $\gamma_{a_m}(u) > 0$ for all $u \in (0, \lambda_m)$. We choose λ_m in such a way that λ_m form a decreasing sequence of real positive numbers.

Positivity of γ_{a_m} implies in particular that there are no jumps of the signature function of K_{a_m}, i.e., the signature jumps at all values of e^{iu} for $u \in [0, \lambda_m)$ are zero.

Choose a_{m+1} by the condition that $\frac{\pi}{2a_{m+1}-5} < \lambda_m$. This means that $u_{a_{m+1}} \in (0, \lambda_m)$. Then, $\gamma_{a_{m+1}}$ vanishes on $u_{a_{m+1}}$, but for all $j \leq m$, γ_{a_m} is positive near $u_{a_{m+1}}$. In particular, the signature jump at $e^{iu_{m+1}}$ for $K_{a_{m+1}}$ is not zero. That is, the map Ψ_{m+1} assigning to a knot half its signature jump at $e^{ia_{m+1}}$ is a homomorphism from the topological concordance group to $\mathbb{Z}$ that vanishes on the subgroup spanned by $K_{a_1}, \dots, K_{a_m}$, and is not zero on $K_{a_{m+1}}$.

The maps $\Psi_1, \dots,$ define an isomorphism between the subgroup spanned by $K_{a_1}, \dots,$ and $\mathbb{Z}^\infty$. □

We conclude the section by the following statement.

Lemma 6.6. *Suppose that Υ' is the Υ function for the positive trefoil. Then, $\Upsilon_{K_{n+1}} = \Upsilon_{K_n} + \Upsilon'$.*

Proof. Follows immediately from Lemma 4.5. □

Corollary 6.7. *The Υ function alone is not sufficient to show that K_n are independent in the concordance group modulo the subgroup generated by the algebraic knots.*

We remark that the signature jumps at $\zeta = e^{2\pi i/6}$ show that the trefoil does not belong to the subgroup spanned by all the K_{a_i}, not even to the subgroup spanned by all the K_n. Note that Lemma 4.2 computes the Alexander polynomial of K_n from the Alexander polynomial Δ_L, which is common for all n. It follows that the value of the symmetrized Alexander polynomial of K_n, $\psi_n(\zeta)$, depends only on n mod 6. For $n = 1, \dots, 6$, we can show that $\psi_n(\zeta) \neq 0$ by direct computations. Hence, none of the ψ_n vanishes on ζ. The signature jump at ζ is equal to zero for all the K_n, but it is not zero for the trefoil.

We do not continue this argument, because these methods alone are insufficient to prove independence of K_n modulo the subgroup generated by the algebraic knots. In fact, there exist linear combinations of algebraic knots with vanishing signature jumps; see [19].

7. Specific knots

In [1], Baker and Kegel shown a list of 632 hyperbolic L-space knots from the SnapPy census. For all of them, we have computed the Alexander polynomial using SnapPy [11], and by a simple algorithm we have determined the Υ function. The expression $-3\int\Upsilon$ turned out to be nonintegral for 96 knots, with the denominators in the set $\{3, 5, 7, 10, 11, 14, 15, 21, 30, 35, 42, 70, 105, 385\}$ (see below).

m211	$\frac{117}{5}$	s560	$\frac{192}{5}$	v0319	$\frac{292}{5}$	v0545	$\frac{173}{5}$
v0830	$\frac{237}{5}$	v1359	$\frac{1874}{35}$	v1423	$\frac{326}{7}$	v1565	$\frac{267}{5}$
v2900	$\frac{389}{7}$	v3070	$\frac{445}{7}$	v3335	$\frac{188}{5}$	t00621	$\frac{467}{5}$
t01966	$\frac{357}{5}$	t03106	$\frac{999}{14}$	t03710	$\frac{342}{5}$	t03843	$\frac{293}{5}$
t04927	$\frac{571}{7}$	t06246	$\frac{3554}{35}$	t06637	$\frac{5987}{70}$	t06957	$\frac{2068}{21}$
t08114	$\frac{148}{5}$	t08184	$\frac{3099}{35}$	t08936	$\frac{1979}{35}$	t09284	$\frac{132}{5}$
t09633	$\frac{1566}{35}$	t09882	$\frac{108}{5}$	t10177	$\frac{690}{7}$	t11887	$\frac{308}{5}$
t12288	$\frac{634}{7}$	t12533	$\frac{157}{5}$	o9_01175	$\frac{642}{5}$	o9_02383	$\frac{413}{5}$
o9_02909	$\frac{334}{7}$	o9_04054	$\frac{348}{5}$	o9_04060	$\frac{477}{5}$	o9_07044	$\frac{4674}{35}$
o9_07152	$\frac{687}{5}$	o9_07401	$\frac{767}{7}$	o9_08402	$\frac{417}{5}$	o9_09271	$\frac{233}{3}$
o9_09731	$\frac{867}{14}$	o9_10192	$\frac{12346}{105}$	o9_10213	$\frac{3727}{42}$	o9_11556	$\frac{592}{5}$
o9_11658	$\frac{816}{7}$	o9_12079	$\frac{787}{5}$	o9_12253	$\frac{413}{5}$	o9_12477	$\frac{1881}{14}$
o9_13054	$\frac{662}{7}$	o9_16431	$\frac{5234}{35}$	o9_17382	$\frac{637}{5}$	o9_19247	$\frac{1067}{10}$
o9_19645	$\frac{517}{5}$	o9_20029	$\frac{1082}{7}$	o9_21620	$\frac{1671}{14}$	o9_22252	$\frac{1138}{7}$
o9_23032	$\frac{1769}{35}$	o9_23461	$\frac{6837}{70}$	o9_23723	$\frac{4324}{35}$	o9_24069	$\frac{9347}{70}$
o9_24126	$\frac{59548}{385}$	o9_24407	$\frac{3293}{30}$	o9_24946	$\frac{268}{5}$	o9_25110	$\frac{2005}{21}$
o9_27371	$\frac{935}{7}$	o9_27767	$\frac{15704}{105}$	o9_28751	$\frac{3557}{30}$	o9_29551	$\frac{1691}{15}$
o9_29648	$\frac{725}{7}$	o9_30142	$\frac{1570}{11}$	o9_31440	$\frac{503}{5}$	o9_32065	$\frac{7047}{70}$
o9_32314	$\frac{1041}{14}$	o9_33380	$\frac{252}{5}$	o9_33430	$\frac{312}{5}$	o9_33486	$\frac{3391}{21}$
o9_33801	$\frac{3204}{35}$	o9_33959	$\frac{6477}{70}$	o9_34689	$\frac{428}{5}$	o9_35720	$\frac{363}{5}$
o9_36380	$\frac{1482}{11}$	o9_36544	$\frac{2579}{35}$	o9_37482	$\frac{849}{10}$	o9_37551	$\frac{228}{5}$
o9_38287	$\frac{963}{14}$	o9_38679	$\frac{879}{7}$	o9_39162	$\frac{821}{14}$	o9_39859	$\frac{2159}{35}$
o9_40026	$\frac{1656}{35}$	o9_40363	$\frac{5427}{70}$	o9_40487	$\frac{173}{5}$	o9_42493	$\frac{2684}{35}$
o9_42675	$\frac{203}{5}$	o9_42961	$\frac{355}{7}$	o9_43750	$\frac{357}{5}$	o9_43857	$\frac{2269}{35}$

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