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Ribbon concordances between knots generalize the notion of ribbon knots. Agol (2022) proved ribbon concordance which gives a partial order on knots in S^3, and Boninger and Greene (2024) conjectured that positive knots are minimal in this ordering. In this article we prove this conjecture for a large class of positive knots, and show that a positive knot cannot be expressed as a nontrivial band sum. Both results extend earlier theorems of Boninger and Greene for special alternating knots. In a related direction, we prove that if positive knots K and K' are concordant and $|\sigma(K)| \geq 2g(K) - 2$, then K and K' have isomorphic rational Alexander modules. This strengthens a result of Stoimenow, and gives evidence toward a conjecture that any concordance class contains at most one positive knot.

1. Introduction

A *smooth concordance* between knots $K_0, K_1 \subset S^3$ is a smooth, properly embedded cylinder $C \subset S^3 \times I$ such that $C \cap (S^3 \times \{i\}) = K_i$ for $i = 0, 1$. Here $I = [0, 1]$ is the unit interval. Perturbing C if necessary, we assume the height function $h : C \hookrightarrow S^3 \times I \to I$ is Morse, and we say C is a *ribbon concordance from K_1 to K_0* if h has no critical points of index two. If such a concordance exists, we say K_1 is *ribbon concordant* to K_0 and we write $K_0 \leq K_1$. This terminology generalizes the notion of a ribbon knot, since a knot is ribbon if and only if it is ribbon concordant to the unknot.

Gordon conjectured, in a now-classic paper [15], that ribbon concordance induces a partial ordering on the set of knots. This conjecture was settled in the affirmative by Agol [1], and many authors have shown that ribbon concordance places strong constraints on knot invariants. To give just a few examples, if $K_0 \leq K_1$, then:

- The Alexander polynomial Δ_{K_0} of K_0 divides Δ_{K_1} [12; 14].
- The genus $g(K_0)$ of K_0 is less than or equal to $g(K_1)$ [42].
- If K_1 is fibered, then K_0 is as well [25; 42].

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Here, we consider the class of knots which are minimal under ribbon concordance. Gordon proved this class includes torus knots [15], and in a more recent paper Boninger and Greene [6] proved many special alternating knots are also ribbon concordance minimal. Torus knots and special alternating knots are both examples of *positive knots*, which are knots admitting a diagram in which all crossings are positive. This motivated Boninger and Greene to conjecture:

Conjecture 1.1 [6, Conjecture 1.6; 41, Question 1.3]. *If $K_1 \subset S^3$ is a positive knot and $K_0 \leq K_1$, then $K_0 \cong K_1$.*

Conjecture 1.1 was posed independently by Tagami, who proved it for positive two-bridge knots [41]. Boninger and Greene also observed that the conjecture holds for fibered positive knots [6, Proposition 1.7], adapting an argument used by Baker and Motegi [5, Theorem 1.1].

In this article we prove two theorems in support of Conjecture 1.1. First, we verify Conjecture 1.1 for a large class of positive knots.

Theorem 1.2. *Let $K \subset S^3$ be a positive knot. If the leading coefficient of Δ_K is a prime power, then K is ribbon concordance minimal.*

Second, given a two-component split link $K_0 \sqcup K_1 \subset S^3$ and an embedded band $b = I \times I \subset S^3$ satisfying $b \cap K_i = I \times \{i\}$ for $i = 0, 1$, we define the *band sum* $K_0 \#_b K_1 \subset S^3$ by

$$K_0 \#_b K_1 = (K_0 \cup K_1 - I \times \partial I) \cup \partial I \times I.$$

This band sum is *trivial* if there exists a sphere $\Sigma \subset S^3 - (K_0 \cup K_1)$ which intersects b in a single arc — in this case $K_0 \#_b K_1 \cong K_0 \# K_1$ — the ordinary connect sum. We say a knot is *band prime* if it cannot be written as a nontrivial band sum.

Miyazaki [24] proved any band sum $K_0 \#_b K_1$ is ribbon concordant to the connect sum $K_0 \# K_1$. Thus, band sums are a natural way in which ribbon concordances arise. Additionally, if $K' \leq K$ via a ribbon concordance with only two critical points, then Morse theory shows K is equivalent to a band sum of K' and an unknot. We prove:

Theorem 1.3. *Positive knots are band prime.*

The proof of Theorem 1.3 is somewhat more involved than that of Theorem 1.2.

Boninger and Greene [6] proved Theorems 1.2 and 1.3 for all special alternating knots, and our results here broadly extend that work. Additionally, while the proof that special alternating knots are band prime in [6] has a combinatorial flavor, our proof of Theorem 1.3 is more geometric. We use a theorem of Ozawa [28] stating that any incompressible Seifert surface of a positive knot is *free*, and we show that such a Seifert surface cannot witness a nontrivial, genus-preserving band sum.

In a related direction, we consider the following question of Gordon:

Question 1.4 [15, Question 6.1]. *Does every smooth concordance class contain a unique representative which is minimal with respect to ribbon concordance*?

Affirmative answers to this question and to [15, Question 6.2] would imply a generalization of the slice-ribbon conjecture. Question 1.4 also seems closely related to conjectures made independently by other authors: Rudolph [32] conjectured that each concordance class contains at most one algebraic knot and Baker [4] conjectured each concordance class contains at most one fibered knot supporting the tight contact structure. Most relevant to us, Stoimenow [38] conjectured each concordance class contains finitely many positive knots — this was verified by Baader, Dehornoy and Liechti [2]. Considering Conjecture 1.1 and Question 1.4, it is natural to posit:

Conjecture 1.5 [38]. *Every smooth concordance class contains at most one positive knot.*

We credit Stoimenow since Conjecture 1.5 seems implicit in his work. For any knot K, let $d(K)$ denote the degree of Δ_K when normalized to have no negative exponents. As evidence of Conjecture 1.5, we prove:

Theorem 1.6. *Let K and K' be* (*topologically or algebraically*) *concordant positive knots. If K satisfies $|\sigma(K)| \geq d(K) - 2$, where σ denotes the signature, then the rational Alexander modules of K and K' are isomorphic.*

By the *rational Alexander module* of K we mean the cohomology ring $H^*(\overline{X};\mathbb{Q})$, where $\overline{X}$ denotes the infinite cyclic cover of the exterior of K, viewed as a module over the group ring of deck transformations. Theorem 1.6 strengthens a result of Stoimenow, which concluded under the above hypotheses that K and K' have the same Alexander polynomial [38, Theorem 4.5]. Additionally, although the hypothesis that

$$|\sigma(K)| \geq d(K) - 2 \tag{1}$$

is somewhat restrictive, it is known that the signatures of positive knots are linearly bounded from below by their genus [2]. In fact, (1) holds for all positive knots with genus less than or equal to four with the single exception of the knot 14_{45657} [8; 37; 38, Theorem 2.4; 39].

Corollary 1.7. *Let K and K' be* (*topologically or algebraically*) *concordant positive knots. If $g(K) \leq 4$, then the rational Alexander modules of K and K' are isomorphic.*

Condition (1) also includes all special alternating knots, since these satisfy $|\sigma(K)| = d(K)$. We prove Theorem 1.6 by showing that positive knots which satisfy (1) are $\mathbb{Q}$-*anisotropic* — for a definition of $\mathbb{Q}$-anisotropy, see Section 5 below. By a classical result of Kervaire and Gilmer, algebraically concordant knots which are $\mathbb{Q}$-anisotropic and admit nonsingular Seifert matrices have isomorphic rational Alexander modules [14, Proposition 4.2; 18]. Thus, we could remove the requirement (1) from Theorem 1.6 if we knew that:

Conjecture 1.8. *Positive knots are $\mathbb{Q}$-anisotropic.*

By work of Gilmer, Conjecture 1.8 may be thought of as the statement that positive knots are *algebraically* ribbon concordance minimal [14, Theorem 0.1]. Scharlemann proved Conjecture 1.8 for the case of torus knots [36, Proposition 2.3].

Further discussion. Section 5 below contains some results on roots of Alexander polynomials which may be of independent interest — for example, we show that Alexander polynomials of positive knots have no rational roots. We also remark that our proof of Theorem 1.6 extends to *almost positive* knots, knots which admit a diagram with one negative crossing, using results of Tagami [40] and Stoimenow [38, Theorem 2.3]. It may be interesting to consider whether Theorems 1.2 and 1.3 could also be extended to almost positive knots.

Outline. In Section 2 we recall relevant properties of positive knots, in Section 3 we prove Theorem 1.3 and in Section 4 we prove Theorem 1.2. In Section 5 we discuss $\mathbb{Q}$-anisotropy and prove Theorem 1.6 and Corollary 1.7.

2. Properties of positive knots

We gather some useful facts about positive knots. First, by results of Rudolph, the genus $g(K)$ and slice genus $g_4(K)$ of a positive knot K are equal [33; 34]. This motivates the following lemma, well known to experts:

Lemma 2.1. *Let $K_0, K_1 \subset S^3$ be such that $K_0 \leq K_1$ and K_1 satisfies $g(K_1) = g_4(K_1)$. Then $g(K_0) = g(K_1)$.*

In particular, the conclusion of Lemma 2.1 holds if $K_0 \leq K_1$ and K_1 is positive.

Proof. Since genus is nonincreasing under ribbon concordance [42] and slice genus is a concordance invariant, we have

$$g_4(K_1) = g_4(K_0) \leq g(K_0) \leq g(K_1) = g_4(K_1). \qquad \square$$

Second, we will need a theorem of Ozawa [28]. A Seifert surface $S \subset S^3$ is called *free* if $S^3 - \nu(S)$ is a handlebody, where ν denotes a regular open neighborhood; equivalently, S is free if $\pi_1(S^3 - S)$ is a free group [27, Lemma 2.2].

Theorem 2.2 [28, Corollary 1.2]. *If K is a positive knot, then every incompressible Seifert surface of K is free.*

Finally, we will use the fact that positive knots are *pseudoalternating* (not be confused with *quasialternating*!). The precise definition of pseudoalternating will not be important to us (see [21, Section 4]), but for experts we note that pseudoalternating links are those links which can be built from Murasugi sums of special alternating links. Positive knots are pseudoalternating because they are homogeneous [9].

3. Positive knots are band prime

In this section we prove Theorem 1.3, first recalling some standard definitions from three-manifold topology. Let Y be a three-manifold and $\Sigma \subset Y$ a properly embedded surface. A *compressing disk for* Σ is an embedded disk $D \subset Y$ with $D \cap \Sigma = \partial D$, such that ∂D does not bound a disk in Σ. Similarly, a *boundary-compressing disk for* Σ is a disk $D \subset Y$ such that:

- $D \cap \Sigma \subset \partial D$.
- ∂D consists of an arc in ∂Y and an arc in Σ which is not boundary-parallel in Σ.

The arc $\partial D \cap \Sigma$ is called a *boundary-compressing arc*. The surface Σ is called *compressible* (respectively *boundary-compressible*) if it admits a compressing disk (respectively boundary-compressing disk). Conversely, Σ is *incompressible* if it is not compressible, and is also not a two-sphere bounding a three-ball in Y. A nonboundary-compressible surface is called *boundary-incompressible*.

We will need a couple of lemmas on surfaces in handlebodies — the first is a classical fact.

Lemma 3.1 [16, Example III.13]. *Let H be a handlebody. If $\Sigma \subset H$ is a connected surface which is incompressible and boundary-incompressible, then Σ is a disk.*

In the next lemma, by a *planar surface* we mean a compact surface which can be embedded in $\mathbb{R}^2$.

Lemma 3.2. *Let H be a handlebody with boundary $F = \partial H$. Let $\Sigma \subset H$ be a properly embedded, connected planar surface such that:*

- *Σ is incompressible in H.*
- *The components of $\partial\Sigma$ are parallel to one another in F.*
- *Each component of $\partial\Sigma$ separates F.*

Then Σ is either a disk or a boundary-parallel annulus.

Proof. We will suppose Σ has more than one boundary component (i.e., Σ is not a disk) and show Σ is a boundary-parallel annulus. By Lemma 3.1, since Σ is incompressible it is boundary-compressible. Let D be a boundary-compressing disk for Σ, so that $\partial D = \alpha \cup \alpha'$ with $\alpha' \subset F$ and α a properly embedded arc in Σ which is not boundary-parallel.

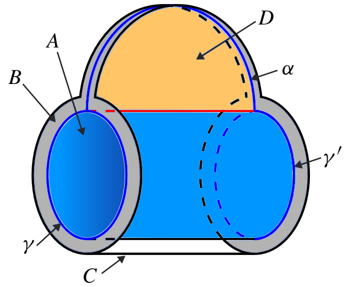

Figure 1. A boundary-compressing disk for Σ, intersecting distinct boundary components.

We first suppose the two boundary points $\partial\alpha = \partial\alpha'$ lie in distinct boundary components γ and γ' of $\partial\Sigma$. Since γ and γ' each separate F, $F - (\gamma \cup \gamma')$ has three components, and α' lies in the unique component with boundary $\gamma \cup \gamma'$. Since γ and γ' are parallel in F, this component is an annulus which we denote by A. Let N be a regular neighborhood of $A \cup D$ in $H - \Sigma$, and let

$$B = N \cap \Sigma = \partial N \cap \Sigma.$$

Equivalently, B is a regular neighborhood of $\gamma \cup \alpha \cup \gamma'$ in Σ. Let $C = \partial N - (A \cup B)$, as in Figure 1.

The neighborhood N is a thickened annulus, so ∂N is a torus. Since $A \cup B$ is a torus with a disk removed, as Figure 1 shows, it follows that C is a disk. Now

$$\partial C = \partial B \subset \Sigma,$$

and the incompressibility of Σ implies ∂C bounds a disk C' in Σ. Thus $\Sigma = B \cup C'$, and in this case Σ is an annulus with boundary $\gamma \cup \gamma'$. Additionally the union of the disks C and C' is a two-sphere, which bounds a three-ball in H by Lemma 3.1. It follows that C' and C are isotopic, so Σ is boundary-parallel.

We have shown that if Σ admits a boundary-compressing disk intersecting distinct components of $\partial\Sigma$, then Σ is a boundary-parallel annulus. We thus assume that each boundary-compressing disk intersects only one component of $\partial\Sigma$. The planarity of Σ then implies that, if D is such a disk with $\partial D = \alpha \cup \alpha'$ as above, then α separates Σ. We therefore choose a boundary-compressing disk D whose compressing arc $\alpha \subset \Sigma$ is "outermost", i.e., one of the two components of $\Sigma - \alpha$ does not contain any boundary-compressing arcs which are not isotopic to α. Let Σ' denote this component of $\Sigma - \alpha$.

Finally, we consider the surface $\Sigma'' = \Sigma' \cup_\alpha D$. Since D is a boundary-compressing disk α is not boundary-parallel in Σ, so neither Σ' nor Σ'' is a disk. Additionally, Σ'' is incompressible: suppose, toward a contradiction, that Σ'' admits a compressing disk D'. Then $\partial D'$ may be isotoped into $\Sigma' \subset \Sigma$, and the incompressibility of Σ implies $\partial D'$ bounds a disk in Σ. Since the component of $\Sigma - \partial D'$ not contained in Σ' has nonempty boundary, $\partial D'$ in fact bounds a disk in $\Sigma' \subset \Sigma''$, contradicting the compression assumption. A similar argument, using the fact that α is outermost in Σ, shows Σ'' is boundary-incompressible. Then Σ'' is incompressible, boundary-incompressible, and not a disk, contradicting Lemma 3.1. We conclude Σ must admit at least one boundary-compressing disk intersecting two boundary components, so Σ is a boundary-parallel annulus. □

Theorem 1.3 now follows from the more general Theorem 3.3 below. Positive knots satisfy the hypotheses of Theorem 3.3 by the discussion in Section 2.

Theorem 3.3. *Let $K \subset S^3$ be a knot such that*:

- $g(K) = g_4(K)$.
- *Every minimal genus Seifert surface of K is free.*

Then K is band prime.

Proof. Let K be a knot satisfying the given hypotheses, and suppose K can be written as a band sum of knots K_0, $K_1 \subset S^3$. We will show K is equivalent to the standard connect sum $K_0 \# K_1$; by a result of Miyazaki, this occurs if and only if the band sum is trivial [26] (compare [11, Theorem 2]).

As mentioned in the introduction, Miyazaki also showed K is ribbon concordant to the connect sum $K_0 \# K_1$ [24]. Thus, by Lemma 2.1,

$$g(K) = g(K_0 \# K_1) = g(K_0) + g(K_1).$$

Gabai proved that a band sum preserves genus in the above sense if and only if there exists a minimal genus Seifert surface S for K, such that S is a band sum

$$S = S_0 \#_b S_1$$

of Seifert surfaces S_i for K_i, $i = 0, 1$ [13]. In other words, S is the result of joining the split union $S_0 \sqcup S_1$ along a band b. The band b may be different from the band in our initial band sum representation of K, but this is not an issue since we will ultimately show $K = K_0 \# K_1$. By the discussion in the first paragraph, this will imply *any* representation of K as a band sum of K_0 and K_1 is trivial.

Fix such a surface $S = S_0 \#_b S_1$, and let $\Sigma \subset S^3$ be a sphere separating the component surfaces S_0 and S_1. We choose Σ transverse to S so that the number of intersection components $|\Sigma \cap S|$ is minimal among all such spheres; then $\Sigma \cap S$ consists of a set of parallel cocores of b, and $|\Sigma \cap S|$ is odd since Σ separates the feet of b. Let $\nu(S)$ be a regular neighborhood of S, and let $H = S^3 - \nu(S)$. We claim the planar surface

$$\Sigma \cap H = \Sigma - \nu(S)$$

is incompressible in H. Suppose not: then $\Sigma \cap H$ admits a compressing disk D. The curve ∂D separates Σ into components Σ_0 and Σ_1, and each component Σ_i contains some intersection with S since ∂D does not bound a disk in $\Sigma - \nu(S)$. Because Σ separates S_0 and S_1, one of the spheres $\Sigma_0 \cup D$ or $\Sigma_1 \cup D$ does as well. Assuming the former without loss of generality, we conclude that $\Sigma_0 \cup D$ is a sphere separating S_0 and S_1 with

$$|(\Sigma_0 \cup D) \cap S| < |\Sigma \cap S|.$$

This contradicts the minimality of $|\Sigma \cap S|$, proving the claim.

The surface S has minimal genus, so H is a handlebody by hypothesis. Now $\Sigma \cap H$ is an incompressible planar surface in H, and since $\Sigma \cap S$ consists of a set of parallel cocores of b, $\partial(\Sigma \cap H)$ consists of a set of parallel curves which are separating on ∂H. From Lemma 3.2, since $|\Sigma \cap S|$ is odd, we conclude that $|\Sigma \cap S| = 1$ and $\Sigma - S$ is a disk. Thus the band sum $K_0 \#_b K_1$ is trivial and K is band prime. □

Remark 3.4. It is not true that *strongly quasipositive* knots are band prime, even though a strongly quasipositive knot K satisfies $g(K) = g_4(K)$, see [5, Section 4.1]. It is true, however, that fibered strongly quasipositive knots are band prime by a result of Baker and Motegi [5, Theorem 1.1]. In fact, Baker and Motegi's argument can be modified to show that fibered strongly quasipositive knots — like fibered positive knots — are ribbon concordance minimal.

4. A condition for minimality

In this section we prove Theorem 1.2, which involves piecing together several existing results. First, the *lower central series* $\{\gamma_i\}_{i \geq 0}$ of a group G is defined recursively by

$$\gamma_0 = G, \qquad \gamma_i = [\gamma_{i-1}, G] \quad \text{for all } i > 0,$$

where $[*, *]$ indicates the commutator. The group G is *residually nilpotent* if $\bigcap_{i=0}^{\infty} \gamma_i = \{1\}$, and following Gordon [15] we say a knot $K \subset S^3$ is *residually nilpotent* if the commutator subgroup of the knot group is residually nilpotent. As in the introduction, let $d(K)$ denote the degree of Δ_K. Then Gordon proves:

Lemma 4.1 [15, Lemma 3.4]. *Let $K_0, K_1 \subset S^3$ be knots with $K_0 \leq K_1$. If K_1 is residually nilpotent and $d(K_0) = d(K_1)$, then $K_0 \cong K_1$.*

Fibered knots are examples of residually nilpotent knots, since their commutator subgroups are free (see [20, Chapter 5]), but little is known in general about which knots are residually nilpotent. Mayland and Murasugi [21] proved:

Theorem 4.2 [21]. *Let K be a pseudoalternating knot such that the leading coefficient of Δ_K is a prime power. Then K is residually nilpotent.*

Next, we recall some background on knot Floer homology [30; 31]. The hat version of knot Floer homology, $\widehat{HFK}$, associates a finitely generated, bigraded $\mathbb{F}_2$-vector space to any knot K:

$$\widehat{HFK}(K) = \bigoplus_{i,j\in\mathbb{Z}} \widehat{HFK}_i(K, j).$$

The i and j gradings are called the *Maslov* and *Alexander* gradings respectively, and the graded Euler characteristic of $\widehat{HFK}$ is the symmetrized Alexander polynomial:

$$\Delta_K(t) = \sum_{i,j} (-1)^i \dim(\widehat{HFK}_i(K, j))\, t^j. \tag{2}$$

Generalizing the classical fact that $d(K) \leq 2g(K)$, knot Floer homology detects knot genus in the following sense [29]:

$$g(K) = \max\{j \mid \widehat{HFK}(K, j) \neq 0\}.$$

Any concordance $C \subset S^3 \times I$ between knots $K_0 \subset S^3 \times \{0\}$ and $K_1 \subset S^3 \times \{1\}$ induces a bigrading-preserving homomorphism

$$C_* : \widehat{HFK}(K_0) \to \widehat{HFK}(K_1),$$

and Zemke [42] proved that if C is a ribbon concordance from K_1 to K_0, so $K_0 \leq K_1$, then the map C_* is injective. Finally, we require the following theorem of Cheng, Hedden and Sarkar [7]:

Theorem 4.3 [7, Corollary 1.6]. *If K is a pseudoalternating link, then the top Alexander grading $\widehat{HFK}(K, g(K))$ of $\widehat{HFK}(K)$ is supported in a single Maslov grading.*

Theorem 1.2 now follows easily from these results and the next proposition (compare [6, Propositon 1.4]).

Proposition 4.4. *Let K_1 be a pseudoalternating knot such that $g(K_1) = g_4(K_1)$, and suppose $K_0 \leq K_1$. Then $\Delta_{K_0} = \Delta_{K_1}$.*

Proof. By Lemma 2.1, $g(K_0) = g(K_1) = g$ for some $g \in \mathbb{N}$. Fix a ribbon concordance $C \subset S^3 \times I$ from K_1 to K_0, and let C_* denote the induced map

$$C_* : \widehat{HFK}(K_0, g) \to \widehat{HFK}(K_1, g).$$

This map is injective by Zemke's result, and both groups are nonzero since the knots have genus g. By Theorem 4.3 $\widehat{HFK}(K_1; g)$ is supported in a single Maslov grading, and thus $\widehat{HFK}(K_0; g)$ is as well. Consequently, (2) implies that

$$d(K_0) = 2g(K_0) = 2g(K_1) = d(K_1).$$

Since Δ_{K_0} divides Δ_{K_1}, we have $\Delta_{K_0} = m\Delta_{K_1}$ for some $m \in \mathbb{Z}$ [14]. But

$$\Delta_{K_0}(1) = \Delta_{K_1}(1) = 1,$$

so $\Delta_{K_0} = \Delta_{K_1}$. □

Proof of Theorem 1.2.. Suppose K_0 and K_1 are knots such that K_1 satisfies the hypotheses of the theorem and $K_0 \leq K_1$. By Proposition 4.4, $\Delta_{K_0} = \Delta_{K_1}$. Additionally K_1 is residually nilpotent by Theorem 4.2, so Lemma 4.1 implies $K_1 \cong K_0$. □

Remark 4.5. Two-bridge knots are residually nilpotent by a theorem of Johnson [17]. Thus our proof of Theorem 1.2 also gives an alternate proof of Tagami's theorem that positive two-bridge knots are ribbon concordance minimal, using [17, Corollary 1.3] in place of Theorem 4.2.

5. Alexander modules and $\mathbb{Q}$-anisotropy

We now work toward the proofs of Theorem 1.6 and Corollary 1.7. We expect that the following proposition is known to experts, but we have not been able to find it in the literature.

Proposition 5.1. *Let $K \subset S^3$ be a knot such that Δ_K has a rational root q. Then $q = (m-1)/m$ for some integer $m \notin \{0, 1\}$. In particular, q is positive.*

Proof. Let $q = a/b$ for $a, b \in \mathbb{Z}$. Fix an oriented Seifert surface S for K, and let

$$\iota_\pm : H_1(S) \to H_1(S^3 - S)$$

be the maps induced by pushing curves off S to the $\pm$-component of the unit normal bundle of S, with sign determined by the orientation. We represent these maps by Seifert matrices, which we also denote by ι_+ and ι_-.

Now

$$\Delta_K(t) = \det(\iota_+ - t\iota_-),$$

and $\Delta_K(q) = 0$ implies that

$$0 = \det(b\iota_+ - a\iota_-).$$

Therefore there exists a nonzero vector $v \in H_1(S)$ such that

$$b\iota_+(v) = a\iota_-(v).$$

Dividing by a scalar if necessary, we assume v is primitive, i.e., that v extends to a basis of $H_1(S)$.

The intersection pairing

$$\cdot : H_1(S) \times H_1(S) \to \mathbb{Z}$$

satisfies the identity

$$v_1 \cdot v_2 = \mathrm{lk}(v_1, (\iota_+ - \iota_-)(v_2)) \quad \text{for all } v_1, v_2 \in H_1(S),$$

where lk indicates the linking number. Since v is a primitive homology class on a once-punctured surface, v is representable by a simple, closed nonseparating curve on S [22; 35], and it follows that there exists $w \in H_1(S)$ such that $w \cdot v = 1$.

We have

$$1 = \operatorname{lk}(w, (\iota_+ - \iota_-)(v)) = \operatorname{lk}(w, \iota_+(v)) - \operatorname{lk}(w, \iota_-(v))$$

and therefore

$$a \operatorname{lk}(w, \iota_+(v)) = b \operatorname{lk}(w, \iota_-(v)) = b(\operatorname{lk}(w, \iota_+(v)) - 1).$$

Since a and b are both nonzero, $\operatorname{lk}(w, \iota_+(v)) \notin \{0, 1\}$. We conclude that

$$q = \frac{a}{b} = \frac{\operatorname{lk}(w, \iota_+(v)) - 1}{\operatorname{lk}(w, \iota_+(v))}$$

as desired. □

Corollary 5.2. *If K is a positive knot, then Δ_K has no rational roots.*

Proof. Let K be positive. The *Conway polynomial* of K, $\nabla_K(z) \in \mathbb{Z}[z^2]$, is the unique polynomial satisfying

$$\nabla_K(x - x^{-1}) = \Delta_K(x^2),$$

where Δ_K is the symmetrized Alexander polynomial. Cromwell proved that for a positive knot K, the coefficients of ∇_K are nonnegative [9, Corollary 2.1]. Also

$$\nabla_K(0) = \Delta_K(1) = 1,$$

so ∇_K has no real roots. It follows that Δ_K has no positive real roots, since a positive real root q of Δ_K would yield a real root $\sqrt{q} - 1/\sqrt{q}$ of ∇_K. Therefore, by Proposition 5.1, Δ_K has no rational roots. □

Let $\overline{X}$ denote the infinite cyclic cover of the exterior of a knot K. For any field $\mathbb{F}$, an *invariant $\mathbb{F}$-isotropic subspace* of $H^1(\overline{X}; \mathbb{F}) \cong H^1(\overline{X}, \partial\overline{X}; \mathbb{F})$ is one which is preserved by the action of deck transformations and self-annihilating with respect to the cup product

$$\smile : H^1(\overline{X}; \mathbb{F}) \times H^1(\overline{X}, \partial\overline{X}; \mathbb{F}) \to H^2(\overline{X}, \partial\overline{X}; \mathbb{F}) \cong \mathbb{F}.$$

The knot K is called $\mathbb{F}$-*anisotropic* if $H^1(\overline{X}; \mathbb{F})$ does not contain a nontrivial invariant $\mathbb{F}$-isotropic subspace, see [15] for more background. As the introduction discusses, $\mathbb{Q}$-anisotropy can be used to restrict changes to the Alexander module under concordance: Kervaire [18] and Gilmer [14, Proposition 4.2] proved that if (algebraically) concordant knots K and K' are $\mathbb{Q}$-anisotropic and admit Seifert matrices which are invertible over $\mathbb{Q}$, then the rational Alexander modules of K and K' are isomorphic.

Proposition 5.3. *If a positive knot K satisfies $|\sigma(K)| \geq d(K) - 2$, then K is $\mathbb{Q}$-anisotropic.*

Proof. Let $\overline{X}$ denote the infinite cyclic cover of the exterior of K, and let

$$t : H^1(\overline{X}; \mathbb{Q}) \to H^1(\overline{X}; \mathbb{Q})$$

be the map induced by a primitive deck transformation. Let $\Lambda \subset H^1(\overline{X}; \mathbb{Q})$ be a nontrivial invariant $\mathbb{Q}$-isotropic subspace of $H^1(\overline{X}; \mathbb{Q})$.

Up to a scalar in $\mathbb{Q}$, the characteristic polynomial of t coincides with Δ_K [19, Theorem 6.17]. Since the cup product is skew-symmetric, it is straightforward to check that $H^1(\overline{X}; \mathbb{Q})$ contains a one-dimensional invariant $\mathbb{Q}$-isotropic subspace if and only if Δ_K has a rational root, i.e., if and only if t has a rational eigenvalue (see [15, Proposition 4.3]). Thus, Corollary 5.2 implies $\dim(\Lambda) \geq 2$.

We now recall the Milnor form μ on $H^1(\overline{X}; \mathbb{Q})$, defined by

$$\mu(v, w) = t(v) \smile w + t(w) \smile v$$

for $v, w \in H^1(\overline{X}; \mathbb{Q})$. This is a nondegenerate symmetric bilinear form satisfying $\sigma(\mu) = \sigma(K)$, where σ denotes the signature [10; 23]. As Gordon observes and is easy to check, Λ is also a self-annihilating subspace for μ [15, Proposition 4.5]. Let $V_\pm$ denote a maximal subspace of $H^1(\overline{X}; \mathbb{Q})$ on which μ is $\pm$-definite. Then $V_\pm \cap \Lambda = \{0\}$, so

$$\dim V_\pm \leq \dim H^1(\overline{X}; \mathbb{Q}) - \dim \Lambda \leq \dim H^1(\overline{X}; \mathbb{Q}) - 2 = d(K) - 2.$$

For the last equality, we again use the fact that Δ_K is the characteristic polynomial of t. It follows that

$$|\sigma(K)| = |\sigma(\mu)| \leq d(K) - 4,$$

and since $d(K)$ and $\sigma(K)$ are even this implies the desired inequality. □

Remark 5.4. It is not true that positive knots are $\mathbb{R}$-anisotropic, even if they satisfy the hypothesis of Proposition 5.3: for example, the Alexander polynomial of the positive knot 10_{139} has a negative real root. It is also not true that strongly quasipositive knots are $\mathbb{Q}$-anisotropic, since there exist strongly quasipositive knots which are topologically slice, see, for example, [3].

Proof of Theorem 1.6. Suppose K and K' are concordant positive knots such that $|\sigma(K)| \geq d(K) - 2$. Then $\sigma(K) = \sigma(K')$, and since K and K' are positive we have

$$d(K) = g_4(K) = g_4(K') = d(K').$$

It follows that $|\sigma(K')| \geq d(K') - 2$, so K and K' are both $\mathbb{Q}$-anisotropic by Proposition 5.3. Since $g(K) = d(K)$ and $g(K') = d(K')$, any Seifert matrix of a minimal genus Seifert surface for K or K' is invertible over $\mathbb{Q}$. Thus, by the result of Kervaire and Gilmer discussed before Proposition 5.3, the rational Alexander modules of K and K' are isomorphic. □

Proof of Corollary 1.7. Since the knot 14_{45657} is only the positive knot satisfying $g(K)=4$ and $\sigma(K)=-4$, 14_{45657} is not concordant to any other positive knot. The corollary then follows from Theorem 1.6 and the discussion in the introduction. □

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