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We establish the a priori estimates for three kinds of degenerate Hessian-type equations arising in conformal geometry. Based on these a priori estimates, we obtain an existence result using the continuity method.

1. Introduction

Let (M, g_0) be an n -dimensional closed smooth Riemannian manifold, $n \geq 3$. Consider a symmetric polynomial of degree k

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_k,$$

where

$$\Gamma_k = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, 1 \leq j \leq k\}, \quad 1 \leq k \leq n.$$

For any $n \times n$ matrix U , $U \in \Gamma_k$ means the eigenvalues of U lie in Γ_k and $\sigma_k(U)$ means that σ_k is applied to the eigenvalues of U . In this paper, we derive the a priori estimates for three kinds of degenerate equations arising in conformal geometry. The existence of $C^{1,1}$ solutions, which is the central issue for these degenerate equations, has been obtained by using these estimates.

The degenerate Hessian equations

$$(1-1) \quad \sigma_k(D^2u) = f(x, u) \geq 0, \quad \Omega \subset \mathbb{R}^n,$$

have attracted much attention recently. For $k = n$, the degenerate Monge–Ampère-type equations, Guan and Li studied the degenerate Weyl problem and the degenerate Gauss curvature problem in [16; 17]. They found that the conditions

$$(1-2) \quad \Delta f^{\frac{1}{n-1}} \geq -A \quad \text{and} \quad |D(f^{1/(n-1)})| \leq A$$

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for some constant A are sufficient to get $C^{1,1}$ estimates. Using (1-2), Guan [14] established $C^{1,1}$ estimates for the Dirichlet problem of the degenerate Monge–Ampère equations with homogeneous boundary condition. Guan, Trudinger and Wang [20] established $C^{1,1}$ estimates for the Dirichlet problem with nonhomogeneous boundary condition given by

$$(1-3) \quad f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega}).$$

Here, (1-3) implies (1-2).

For general $1 < k < n$, Dong [9] obtained C^2 estimates for the homogeneous Dirichlet problem with homogeneous boundary by

$$(1-4) \quad |D(f^{\frac{1}{k-1}})| \leq C f^{\frac{1}{2k-2}} \quad \text{in } \bar{\Omega}$$

and

$$(1-5) \quad f^{\frac{1}{k-1}} \in C^{1,1}(\bar{\Omega}).$$

Jiao and Wang [28] generalized the results in [9], and obtained $C^{1,1}$ estimates for k -curvature problems with homogeneous boundary by $f^{\frac{1}{k-1}} \in C^{1,1}(\mathbb{R}^n)$. They also found convex solutions to the Dirichlet problem of degenerate k -Hessian equations with nonhomogeneous boundary condition by (1-5) in [29].

Motivated by [28], it is natural to study the degenerate equations in conformal geometry. One of the central issues in conformal geometry is the k -Yamabe problem. The k -Yamabe problem was raised in [41] in order to find a conformal metric g so that $\sigma_k(A_g) = \text{const.}$, where $A_g = \frac{1}{n-2}(\text{Ric}_g - \frac{R_g}{2(n-1)}g)$, Ric_g , R_g are the Schouten tensor, the Ricci curvature and scalar curvature of (M, g) , respectively. The general prescribing curvature problem is to find a conformal metric such that $\sigma_k(A_g)$ equals a given function f ,

$$(1-6) \quad \sigma_k(A_g) = f(x).$$

We say (1-6) is *nondegenerate* if $f > 0$. Fruitful results have been achieved on the k -Yamabe problem and the prescribing curvature problem. Here, we just mention some existence results of the k -Yamabe problem and the prescribing curvature problem. The research of the nondegenerate prescribing curvature problem on manifolds without boundary can be tracked back to Chang, Gursky and Yang [1; 2]. They proved that if the Yamabe constant and $\int_M \sigma_2(A_{g_0})$ are both positive, then there exists a conformal metric g such that $\sigma_2(A_g)$ is a positive constant; see also [22]. Later on, Guan and Wang [18] and Li and Li [31] concluded that on locally conformally flat manifolds, the k -Yamabe problem is solvable for $2 \leq k < n/2$; see also [38]. Gursky and Viaclovsky confirmed the existence of a solution and compactness of solution set to $\sigma_k(A_g) = f > 0$ in the case of $n \geq k > n/2$ when $A_{g_0} \in \Gamma_k$ and (M, g_0) is not conformally equivalent to a sphere. Li and Nguyen [34]

obtained that if $A_{g_0} \in \Gamma_k, k = n/2$ and (M, g_0) is not conformally equivalent to a sphere, then there is a conformal metric \tilde{g} such that $\sigma_k(A_{\tilde{g}}) = 1$, and the solution set is compact. Other relevant conclusions are included in [3; 4; 10; 11; 12; 32; 40].

In 2003, Gursky and Viaclovsky [23] introduced the modified Schouten tensor

$$A_g^t = \frac{1}{n-2} \left(\text{Ric}_g - \frac{tR_g}{2(n-1)}g \right).$$

This tensor is, in fact, a constant multiple of the tensor $sA_g - \frac{(1-s)R_g}{2(n-1)}g$ which is introduced in [31]. It is also a meaningful problem to find a conformal metric g with

$$(1-7) \quad \sigma_k(-A_g^t) = f(x).$$

When $t = 0, A_g^t = \frac{1}{n-2} \text{Ric}_g$; when $t = 1, A_g^t$ is just the Schouten tensor A_g ; when $t = n - 1, A_g^t$ is the Einstein tensor. Gursky and Viaclovsky [23] and Li and Sheng [35] proved that when $t < 1$ every compact manifold with $-A_{g_0}^t \in \Gamma_k$ is conformal to a metric g with $\sigma_k(-A_g^t) = f(x) > 0$.

Another interesting problem is to study the boundary value problem for (1-6) and (1-7) on manifolds (M^n, g) with boundary ∂M . Guan [15] studied the existence problem for (1-6) under the Dirichlet boundary condition. Li and Sheng [36] considered the Dirichlet problem for $\sigma_k(\text{Ric}_g - Rg) = f$. Under various conditions, the Neumann problem arising in conformal geometry is derived in [6; 7; 24; 30; 33; 37].

A key issue for the study of fully nonlinear equations in conformal geometry is the a priori estimates. Thus, there is much profound research on the a priori estimates for (1-6) and (1-7) and their further generalizations. We refer to [5; 18; 19; 25; 26; 27; 31; 38; 42; 43].

It is noticed that most of the results focus on nondegenerate cases in conformal geometry. For the degenerate case, Ge, Lin and Wang [13] obtained that if $f = 0, k = 2$, there exists a $C^{1,1}$ metric such that $\sigma_2(A_g) = 0$. A natural question is whether the existence is still valid if f is nonnegative.

We consider an existence for the solutions to degenerate equations in the negative cone $-\Gamma_k = \{\lambda \mid -\lambda \in \Gamma_k\}$. We raise the following problem,

Problem. Let f be a nonnegative function in M . Is it true that there exists a conformal metric g on M with $\sigma_k(-A_g^t) = f \geq 0$?

The question is partly answered in this paper. The results are as follows. Let $g = e^{2w}g_0$. Then

$$-A_g^t = \frac{1-t}{n-2} \Delta w + \nabla^2 w - \nabla w \otimes \nabla w + \frac{2-t}{2} |\nabla w|^2 g_0 - A_{g_0}^t.$$

Here, w is said to be *admissible* if $-A_g^t \in \Gamma_k$. Note that [38] provides a counterexample to show that the regularity for $\det(\nabla^2 w + |\nabla w|^2 + S) = f$ in \mathbb{R}^2 fails where $S(x)$ is some 2×2 matrix. We only consider $\sigma_k(-A_g^t) = f \geq 0$ with $t < 1$.

Theorem 1.1. *Let (M, g_0) be an n -dimensional closed smooth Riemannian manifold, $n \geq 3$, $-A_{g_0}^t \in \Gamma_k$, $2 \leq k \leq n$, $t < 1$. Suppose that $f^{\frac{1}{k-1}} \in C^{1,1}$ is a nonnegative function in M . Equation (1-7) has an admissible supersolution \bar{w} :*

$$\sigma_k(-A_{e^{2\bar{w}g_0}}^t) \leq f(x), \quad -A_{e^{2\bar{w}g_0}}^t \in \Gamma_k.$$

Then, there is a $C^{1,1}$ solution w of (1-7) satisfying $A_{e^{-2w}g_0} \in \bar{\Gamma}_k$.

Perturbation techniques are often used to deal with degenerate problems. First, we transform the original degenerate problem into a nondegenerate equation by adding a positive number ϵ to the right-hand function f . Next, we establish the a priori estimates, which are independent of ϵ . Finally, we find the existence of the solution u_ϵ for each ϵ and then let $\epsilon \rightarrow 0$ to derive the existence result for the degenerate problem (1-7).

Since the a priori estimates are the core of the existence theorems, we need the following C^1 and C^2 estimates.

Theorem 1.2. *Let (M, g_0) be an n -dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$, $t < 1$. Suppose that f is a nonnegative function in M and $f^{\frac{1}{k-1}} \in C^{1,1}$. Let w be a C^4 solution to*

$$(1-8) \quad \sigma_k \left(-A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2} \Delta w - \nabla w \otimes \nabla w + \frac{2-t}{2} |\nabla w|^2 g_0 \right) \\ = (f^{\frac{1}{k-1}}(x) + \epsilon)^{k-1} e^{2kw}, \quad x \in M,$$

where $-A_{e^{2w}g_0}^t \in \Gamma_k$. Then, there is a positive constant C depending on g_0, n, k, t , $\|f^{\frac{1}{k-1}}\|_{C^1}$, $\|f\|_{C^0}$, $\|w\|_{C^0}$ but independent of ϵ , such that

$$(1-9) \quad \sup |\nabla w| \leq C.$$

Furthermore, there is a positive constant C depending on g_0, n, k, t , $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|\nabla w\|_{C^0}$ but independent of ϵ , such that

$$(1-10) \quad \sup |\nabla^2 w| \leq C(1 + e^{\frac{2k}{k-1} \sup w}).$$

The next theorems concern the estimates for $\sigma_k(A_g) = f \geq 0$. Before introducing the estimates, we give the following notation. Set $0 < \epsilon < 1$, and let $0 \leq \eta(t) \leq 1$ be a smooth function, which satisfies

$$(1-11) \quad \eta(t) = 1 \text{ for } t \leq \frac{1}{4}\theta; \quad \eta(t) = 0 \text{ for } t \geq \frac{1}{2}\theta; \quad \eta' \leq C_1\theta, \quad \eta'' \leq C_2\theta.$$

Here, $C_1, C_2, \theta > 0$ are constants.

Theorem 1.3. *Let (M, g_0) be an n -dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$. Suppose that f is a nonnegative function in M . Let u be a*

C^4 solution to

$$(1-12) \quad \begin{aligned} \sigma_k(A_{g_0} + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 g_0) \\ = (f^{\frac{1}{k-1}}(x) + \epsilon \eta(f^{\frac{1}{k-1}}(x)))^{k-1} e^{-2ku}, \quad x \in M, \end{aligned}$$

where $A_{e^{-2u}g_0} \in \Gamma_k$. If $f^{\frac{1}{k-1}} \in C^{1,1}$, then

$$(1-13) \quad \sup |\nabla u| \leq C,$$

where a positive constant C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|u\|_{C^0}$, g_0 , n , k , θ , C_1 , C_2 but is independent of ϵ .

Moreover,

$$(1-14) \quad \sup |\nabla^2 u| \leq C(1 + e^{-\frac{2k}{k-1} \inf u}),$$

where a positive constant C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|\nabla u\|_{C^0}$, g_0 , n , k , θ , C_1 , C_2 but is independent of ϵ .

The next theorem concerns the case of $f^{\frac{1}{k}} \in C^2$.

Theorem 1.4. *Let (M, g_0) be an n -dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$. Suppose that f is a nonnegative function in M . Let u be a C^4 solution to*

$$(1-15) \quad \begin{aligned} \sigma_k^{\frac{1}{k}}(A_{g_0} + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 g_0) \\ = (f^{\frac{1}{k}}(x) + \epsilon \eta(f^{\frac{1}{k}}(x))) e^{-2u}, \quad x \in M, \end{aligned}$$

where $A_{e^{-2u}g_0} \in \Gamma_k$. If $f^{\frac{1}{k}} \in C^2$, then

$$(1-16) \quad \sup(|\nabla u|^2 + |\nabla^2 u|) \leq C(1 + e^{-2 \inf u}),$$

where a positive constant C depends on g_0 , n , k , $\|f^{\frac{1}{k}}\|_{C^2}$, $\|f\|_{C^0}$, θ , C_1 , C_2 but is independent of ϵ .

The paper is built up as follows. In Section 2, we introduce the notation and the necessary formulas. In Sections 3, 4, 5, we establish the crucial estimates and prove Theorems 1.3, 1.4 and 1.2, respectively. Finally, the existence result of Theorem 1.1 is proved in Section 6.

2. Preliminaries

Throughout this paper,

$$\sigma_{k-1}(\lambda | i) = \frac{\partial \sigma_k}{\partial \lambda_i} \quad \text{and} \quad \sigma_{k-2}(\lambda | ij) = \frac{\partial^2 \sigma_k}{\partial \lambda_i \partial \lambda_j}.$$

We list some properties of σ_k , which will be used later.

Proposition 2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $1 \leq k \leq n$. Then we have*

- (1) $\Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_n$;
- (2) $\sigma_{k-1}(\lambda | i) > 0$ for $\lambda \in \Gamma_k$ and $1 \leq i \leq n$;
- (3) $\sigma_k(\lambda) = \sigma_k(\lambda | i) + \lambda_i \sigma_{k-1}(\lambda | i)$ for $1 \leq i \leq n$;
- (4) if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then $\sigma_{k-1}(\lambda | 1) \leq \sigma_{k-1}(\lambda | 2) \leq \dots \leq \sigma_{k-1}(\lambda | n)$ for $\lambda \in \Gamma_k$;
- (5) $\sum_{i=1}^n \sigma_{k-1}(\lambda | i) = (n - k + 1)\sigma_{k-1}(\lambda)$.

The generalized Newton–MacLaurin inequality is as follows, which will be used later.

Proposition 2.2. *For $\lambda \in \Gamma_m$ and $m > l \geq 0, r > s \geq 0, m \geq r, l \geq s$, we have*

$$\left[\frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l} \right]^{\frac{1}{m-l}} \leq \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s} \right]^{\frac{1}{r-s}}.$$

Proof. See [39]. □

The following lemma is the key in the proof of Theorems 1.3 and 1.2.

Lemma 2.3. *Let $\alpha = \frac{1}{k-1}$. If $U \in \Gamma_k$, then*

$$(2-1) \quad -\sigma_k^{ii,jj} U_{iip} U_{jjp} \geq \sigma_k \left[\frac{(\sigma_k)_p}{\sigma_k} - \frac{(\sigma_1)_p}{\sigma_1} \right] \left[(\alpha - 1) \frac{(\sigma_k)_p}{\sigma_k} - (\alpha + 1) \frac{(\sigma_1)_p}{\sigma_1} \right].$$

Proof. See [21]. □

In this paper ∇ denotes the Levi-Civita connection on (M, g_0) , and the curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Let e_1, e_2, \dots, e_n be local frames on M and define $g_{ij} = g_0(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, while the Christoffel symbols Γ_{ij}^k and curvature coefficients are given, respectively, by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ and

$$R_{ijkl} = g_0(R(e_k, e_l)e_j, e_i), \quad R_{jkl}^i = g^{im} R_{mjkl}.$$

We write $u_i = \nabla_i u = \nabla_{e_i} u$, $u_{ji} = [\nabla \nabla u](e_j, e_i) = [\nabla_i(\nabla u)](e_j) = \nabla_i(\nabla_j u) - \Gamma_{ij}^k u_k$, $u_{ijk} = [\nabla_k(\nabla \nabla u)](e_i, e_j)$, etc. Note that $u_{ij} = u_{ji}$,

$$(2-2) \quad u_{ijk} - u_{kij} = R_{ijk}^m u_m,$$

$$(2-3) \quad u_{kilj} - u_{klj} = R_{kil,j}^m u_m + R_{kil}^m u_{mj},$$

$$(2-4) \quad u_{kijl} - u_{kilj} = R_{ijl}^m u_{km} + R_{kjl}^m u_{mi}.$$

From (2-2), (2-3) and (2-4), we obtain

$$(2-5) \quad u_{ijkl} - u_{klij} = R_{ijk,l}^m u_m + R_{kil,j}^m u_m + R_{ijk}^m u_{ml} + R_{kil}^m u_{mj} + R_{ijl}^m u_{km} + R_{kjl}^m u_{mi}.$$

For convenience, we introduce the notation

$$F(U) = \sigma_k(U), \quad F^{ij} = \frac{\partial F}{\partial U_{ij}}, \quad F^{ij,rs} = \frac{\partial^2 F}{\partial U_{ij} \partial U_{rs}}.$$

Lemma 2.4. (1) *If $U = \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 g_0 + A_{g_0}$, then*

$$U_{ijp} = u_{ijp} + u_i u_{jp} + u_j u_{ip} - u_q u_{qp} g_{ij} + A_{ij,p},$$

and

$$U_{ijpp} = u_{ijpp} + 2u_{ip} u_{jp} + u_i u_{jpp} + u_j u_{ipp} - u_{qp} u_{qp} g_{ij} - u_q u_{qpp} g_{ij} + A_{ij,pp},$$

where A_{ij} are the components of A_{g_0} and g_{ij} are the components of g_0 .

(2) *If $W = -A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2} \Delta w g_0 - \nabla w \otimes \nabla w + \frac{2-t}{2} |\nabla w|^2 g_0$, then*

$$W_{ijp} = -A_{ij,p}^t + w_{ijp} + \frac{1-t}{n-2} w_{qpp} g_{ij} - w_i w_{jp} - w_j w_{ip} + (2-t) w_q w_{qp} g_{ij},$$

and

$$W_{ijpp} = -A_{ij,pp}^t + w_{ijpp} + \frac{1-t}{n-2} w_{qppp} g_{ij} - w_i w_{jpp} - w_j w_{ipp} - 2w_{ip} w_{jp} + (2-t) w_{qp} w_{qp} g_{ij} + (2-t) w_q w_{qpp} g_{ij},$$

where A_{ij}^t are the components of $A_{g_0}^t$ and g_{ij} are the components of g_0 .

The following two lemmas will be used in the proof of Theorems 1.3 and 1.2.

Lemma 2.5. *Assume that $-\mu \leq s \leq \mu$. Then, we may choose constants a , b , and p depending only on $\mu > 0$ so that $\gamma(s) = a(b+s)^p$ satisfies*

$$(2-6) \quad -\frac{1}{4} \gamma'(s) \geq \gamma''(s) + \gamma'(s) - \gamma'(s)^2 > 0.$$

Proof. We have

$$\gamma'(s) = pa(b+s)^{p-1}, \quad \gamma''(s) = p(p-1)a(b+s)^{p-2}.$$

Hence,

$$\begin{aligned} \frac{\gamma''(s) - \gamma'(s)^2}{-\gamma'(s)} &= \frac{p(p-1)a(b+s)^{p-2} - p^2 a^2 (b+s)^{2p-2}}{-pa(b+s)^{p-1}} \\ &= \frac{1-p}{b+s} + (b+s)^{p-1} ap. \end{aligned}$$

Now choose $b = 17\mu + 16$, $p = -\frac{35}{34}(b + \mu)$, $a = \frac{(b-\mu)^{1-p}}{-34p}$. Then

$$0 < \frac{1}{b + \mu} \leq \frac{1}{b + s} \leq \frac{1}{16}.$$

Moreover,

$$\frac{35}{34} = \frac{-p}{b + \mu} \leq \frac{-p}{b + s} \leq \frac{35}{34} \frac{b + \mu}{b - \mu} < \frac{9}{8} \cdot \frac{35}{34},$$

and

$$|(b + s)^{p-1}ap| \leq (b - \mu)^{p-1}|ap| = \frac{1}{34}.$$

Therefore,

$$1 < \frac{1-p}{b+s} + (b+s)^{p-1}ap \leq \frac{5}{4}.$$

Then, (2-6) is proved. \square

Lemma 2.6. *Assume that $-\mu \leq s \leq \mu$, $t < 1$. Then, we may choose constants a , b , and p depending on $\mu > 0$ and t so that $\gamma(s) = a(b + s)^p$ satisfies*

$$(2-7) \quad \frac{1-t}{2(n-2)}(\gamma''(s) - \gamma'(s)^2) \geq \gamma'(s) > 0.$$

Proof. It is easily seen that

$$\begin{aligned} \frac{\gamma''(s) - \gamma'(s)^2}{\gamma'(s)} &= \frac{p(p-1)a(b+s)^{p-2} - p^2a^2(b+s)^{2p-2}}{pa(b+s)^{p-1}} \\ &= \frac{p-1}{b+s} - (b+s)^{p-1}ap. \end{aligned}$$

Now choose $b = 2\mu + \frac{8(1-t)}{n-2}$, $p = \frac{9(n-2)}{4(1-t)}(b + \mu)$, $a = (b + \mu)^{1-p} \cdot \frac{n-2}{8p(1-t)}$. Similar to the calculation in Lemma 2.5, we have

$$\frac{2(n-2)}{1-t} \leq \frac{p-1}{b+s} - (b+s)^{p-1}ap.$$

Then, (2-7) is proved. \square

The following lemma will be used in the proof of Theorem 1.1.

Lemma 2.7. *Let*

$$\begin{aligned} W &= \frac{1-t}{n-2}\Delta w + \nabla^2 w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0 - A_{g_0}^t, \\ \bar{W} &= \frac{1-t}{n-2}\Delta \bar{w} + \nabla^2 \bar{w} - \nabla \bar{w} \otimes \nabla \bar{w} + \frac{2-t}{2}|\nabla \bar{w}|^2 g_0 - A_{g_0}^t. \end{aligned}$$

Suppose $\sigma_k(\bar{W}) \leq f e^{2k\bar{w}}$, $\sigma_k(\bar{W}) \neq 0$, $\sigma_k(W) = f e^{2kw}$. Then we have $w \leq \bar{w}$.

Proof. Let $t < 1$, $v^{\frac{4}{n-2}} = e^{2w}$, $\bar{v}^{\frac{4}{n-2}} = e^{2\bar{w}}$,

$$(2-8) \quad V = \frac{1-t}{n-2} \Delta v + \nabla^2 v - \frac{n}{n-2} \frac{\nabla v \otimes \nabla v}{v} + \frac{1}{n-2} \frac{|\nabla v|^2}{v} g_0 - \frac{n-2}{2} v A_{g_0}^t,$$

$$(2-9) \quad \bar{V} = \frac{1-t}{n-2} \Delta \bar{v} + \nabla^2 \bar{v} - \frac{n}{n-2} \frac{\nabla \bar{v} \otimes \nabla \bar{v}}{\bar{v}} + \frac{1}{n-2} \frac{|\nabla \bar{v}|^2}{\bar{v}} g_0 - \frac{n-2}{2} \bar{v} A_{g_0}^t.$$

It is enough to prove that $v \leq \bar{v}$. Suppose it is not true. Then by the positivity of v and \bar{v} , we find a number $\beta > 1$ such that $\beta \bar{v} \geq v$ and $\beta \bar{v}(\bar{x}) = v(\bar{x})$. Then

$$(2-10) \quad \nabla(\beta \bar{v})(\bar{x}) = \nabla v(\bar{x}), \quad [\nabla^2(\beta \bar{v}) - \nabla^2 v](\bar{x}) \geq 0.$$

Moreover,

$$\begin{aligned} \sigma_k(V)(\bar{x}) &\leq \sigma_k \left(\frac{1-t}{n-2} \Delta(\beta \bar{v}) + \nabla^2(\beta \bar{v}) - \frac{n}{n-2} \frac{\nabla(\beta \bar{v}) \otimes \nabla(\beta \bar{v})}{\beta \bar{v}} \right. \\ &\quad \left. + \frac{1}{n-2} \frac{|\nabla(\beta \bar{v})|^2}{\beta \bar{v}} g_0 - \frac{n-2}{2} \beta \bar{v} A_{g_0}^t \right) \Big|_{\bar{x}} \\ &= \beta^k \sigma_k(\bar{V})(\bar{x}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_k(V)(\bar{x}) &= \left(\frac{n-2}{2} v^{\frac{n+2}{n-2}} \right)^k f \Big|_{\bar{x}} = \left(\frac{n-2}{2} (\beta \bar{v})^{\frac{n+2}{n-2}} \right)^k f \Big|_{\bar{x}} \\ &= \left(\frac{n-2}{2} \beta^{\frac{n+2}{n-2}} \bar{v}^{\frac{n+2}{n-2}} \right)^k f \Big|_{\bar{x}} \\ &\geq \beta^{k \frac{n+2}{n-2}} \sigma_k(\bar{V})(\bar{x}), \end{aligned}$$

which is a contradiction. \square

3. The proof of Theorem 1.3

In this section, we obtain C^1 and C^2 estimates with the assumption that the maximum modulus estimation exists for (1-12). We define $\bar{f} = (f^\alpha + \epsilon \eta(f^\alpha))^{k-1}$, $\alpha = \frac{1}{k-1}$, $F = \sigma_k$, $U = A_{g_0} + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 g_0$, where u is a solution to (1-12).

First of all, we consider C^1 estimates of u . Let $K = (1 + \frac{1}{2} |\nabla u|^2) e^{\gamma(u)}$. Here γ is a function of the form $\gamma(s) = a(b+s)^p$. The constants a, b, p are chosen as in Lemma 2.5. Assume that K achieves the maximal value at some point \tilde{x} , which implies that $K_i(\tilde{x}) = 0$. In the rest of the proof, all calculations will be performed at the maximum point \tilde{x} . We may assume U_{ij} and F^{ij} are diagonal. Then, in an orthonormal frame,

$$(3-1) \quad 0 = K_i(\tilde{x}) = e^{\gamma(u)} \left(\left(1 + \frac{1}{2} u_i^2\right) \gamma' u_i + u_i u_{ii} \right)$$

and

$$K_{ij}(\tilde{x}) = e^{\gamma(u)} \left(\left(1 + \frac{1}{2}u_l^2\right) \left((\gamma')^2 u_i u_j + \gamma' u_{ij} + \gamma'' u_i u_j \right) \right. \\ \left. + u_l u_{lj} \gamma' u_i + u_l u_{li} \gamma' u_j + u_{lj} u_{li} + u_l u_{lij} \right).$$

Differentiating (1-12) gives

$$(3-2) \quad F^{ij} U_{ijl} = (\bar{f} e^{-2ku})_l.$$

Using (3-1), (3-2) and (2-2), we have

$$\begin{aligned} 0 &\geq e^{-\gamma(u)} F^{ij} K_{ij}(\tilde{x}) \\ &= F^{ij} (u_l u_{lij} + (1 + \frac{1}{2}u_l^2) ((\gamma')^2 + \gamma'') u_i u_j + \gamma' u_{ij}) + 2\gamma' u_l u_{li} u_j + u_l u_{lij} \\ &\geq F^{ij} (u_l u_{ijl} + (1 + \frac{1}{2}u_l^2) ((\gamma')^2 + \gamma'') u_i u_j + \gamma' u_{ij}) + 2\gamma' u_l u_{li} u_j \\ &\quad - C \sum F^{ii} (|\nabla u|^2 + 1) \\ &\geq u_l (\bar{f} e^{-2ku})_l + \gamma' k \bar{f} e^{-2ku} (1 + \frac{1}{2}u_l^2) \\ &\quad + F^{ij} \left(-2 \left(-\frac{1}{2} \right) u_l u_s u_{sl} \delta_{ij} - 2u_l u_i u_{lj} + \gamma' \left(-\left(-\frac{1}{2} \right) u_s^2 \delta_{ij} - u_i u_j \right) \left(1 + \frac{1}{2}u_l^2 \right) \right) \\ &\quad + F^{ij} \left(\left(1 + \frac{1}{2}u_l^2 \right) (\gamma'' - \gamma'^2) (u_i u_j) \right) - C \sum F^{ii} (|\nabla u|^2 + 1) \\ &\geq u_l (\bar{f} e^{-2ku})_l + \gamma' k \bar{f} e^{-2ku} (1 + \frac{1}{2}u_l^2) \\ &\quad + F^{ij} \left(\gamma' \left(1 + \frac{1}{2}u_l^2 \right) u_i u_j - \frac{1}{2} \gamma' \left(1 + \frac{1}{2}u_l^2 \right) u_s^2 \delta_{ij} + \left(1 + \frac{1}{2}u_l^2 \right) (\gamma'' - \gamma'^2) u_i u_j \right) \\ &\quad - C \sum F^{ii} (|\nabla u|^2 + 1). \end{aligned}$$

On the other hand, applying the Newton–MacLaurin inequality, we have

$$\sigma_{k-1} \geq C \sigma_k^{1-\frac{1}{k}}.$$

Thus

$$(3-3) \quad C \bar{f}^{1-\frac{1}{k}} e^{-2(k-1)u} \leq \sigma_{k-1} = C \sum F^{ii}.$$

Moreover, using $f^{\frac{1}{k-1}} \in C^{1,1}$, we have $\bar{f}^{\frac{1}{k-1}} \in C^{1,1}$, which implies

$$\bar{f}^{\frac{1}{k}} \in C^1.$$

Thus

$$(3-4) \quad |\bar{f}_i| \leq C \bar{f}^{1-\frac{1}{k}}.$$

Then, using (3-3) and (3-4), we obtain

$$(3-5) \quad \left| u_l (\bar{f} e^{-2ku})_l + \gamma' k \bar{f} e^{-2ku} \left(1 + \frac{1}{2}u_l^2 \right) \right| \leq C \bar{f}^{1-\frac{1}{k}} e^{-2ku} (|\nabla u|^2 + 1).$$

Let $\delta = \min\{-\frac{1}{8}\gamma'\}$. Combining (3-5) with (3-3), it follows that

$$(3-6) \quad 0 \geq F^{ii} \left(-\frac{1}{2} \gamma' u_s^2 + (\gamma'' + \gamma' - \gamma'^2) u_i^2 \right) \left(1 + \frac{1}{2}u_l^2 \right) - C \bar{f}^{1-\frac{1}{k}} e^{-2ku} (|\nabla u|^2 + 1) \\ - C \sum F^{ii} (|\nabla u|^2 + 1) \\ \geq C_3 \bar{f}^{1-\frac{1}{k}} e^{-2(k-1)u} (\delta |\nabla u|^4 - C(e^{-2u} + 1)(|\nabla u|^2 + 1)).$$

To derive (1-13), we divide the proof into two cases.

(A) $\bar{f} > 0$. Then, (1-13) follows from (3-6).

(B) $\bar{f} = 0$. Then, (3-6) implies

$$(3-7) \quad 0 \geq F^{ii} \left(-\frac{1}{2} \gamma' u_s^2 + (\gamma'' + \gamma' - \gamma'^2) u_i^2 \right) (1 + \frac{1}{2} u_l^2) - C \sum F^{ii} (|\nabla u|^2 + 1) \\ \geq \sum F^{ii} (\delta |\nabla u|^4 - C (|\nabla u|^2 + 1))$$

and (1-13) is proved.

Next, based on both maximum modulus estimation and gradient estimation, we discuss C^2 estimates of u . We may assume U is diagonal. It follows from $U \in \Gamma_1$,

$$(3-8) \quad 0 \leq \text{tr } U \leq \Delta u + \left(1 - \frac{1}{2}n\right) |\nabla u|^2 + C, \quad |\nabla u|^2 \leq C(\Delta u + 1).$$

Also, it follows from $U \in \Gamma_2$ that

$$(3-9) \quad |U_{ij}| \leq C\sigma_1, \quad |u_{ij}| \leq C(\Delta u + 1).$$

Thus, we may assume that Δu is sufficiently large and $\Delta u \geq \{1, 2\mu_1\}$, where $\mu_1 = (-1 + \frac{1}{2}n) \sup |\nabla u|^2 + \sup |\text{tr } A_{g_0}|$. Then, we have

$$(3-10) \quad 2\Delta u \geq \sigma_1 \geq \frac{1}{2}\Delta u.$$

Let $H = \Delta u + |\nabla u|^2$, and let \check{x} be the maximum point of H . By rotating the coordinates, we may assume $U_{ij}(\check{x})$ is diagonal. Differentiating H at \check{x} , we have

$$(3-11) \quad 0 = H_i(\check{x}) = u_{qqi} + 2u_{qi}u_q$$

and

$$(3-12) \quad 0 \geq H_{ii}(\check{x}) = u_{qqii} + 2u_q u_{qii} + 2u_{qi}^2.$$

Differentiating (1-12) gives

$$(3-13) \quad F^{ij} U_{ijl} = (\bar{f} e^{-2ku})_l.$$

By (3-11), (3-12), (3-13), (2-2), and (2-5), we obtain

$$(3-14) \quad 0 \geq F^{ii} H_{ii}(\check{x}) = F^{ii} (u_{qqii} + 2u_{qii}u_q + 2u_{qi}^2) \\ \geq F^{ii} (u_{iiqq} + 2u_{iiq}u_q + 2u_{qi}^2) - C \sum F^{ii} (\Delta u + 1) \\ \geq F^{ii} u_{iiqq} + F^{ii} (2u_q (U_{iiq} - (-\frac{1}{2}u_l^2 + u_i^2)_q) + 2u_{qi}^2) - C \sum F^{ii} (\Delta u + 1) \\ \geq F^{ii} (2u_q u_l u_{lq} - 4u_q u_{iq} u_i + U_{iiqq} - (-\frac{1}{2}u_l^2 + u_i^2)_{qq} + 2u_{qi}^2) \\ \quad + 2u_q (\bar{f} e^{-2ku})_q - C \sum F^{ii} (\Delta u + 1) \\ \geq F^{ii} (2u_q u_l u_{lq} - 4u_q u_{iq} u_i + U_{iiqq} - (-u_{lq}^2 - u_l u_{lqq} + 2u_i u_{iqq})) \\ \quad + 2u_q (\bar{f} e^{-2ku})_q - C \sum F^{ii} (\Delta u + 1) \\ \geq F^{ii} (U_{iiqq} + u_{qi}^2) + 2u_q (\bar{f} e^{-2ku})_q - C \sum F^{ii} (\Delta u + 1).$$

Then, using the Newton–MacLaurin inequality, we have

$$\sigma_{k-1} \geq C\sigma_k^{1-\alpha}\sigma_1^\alpha.$$

Thus

$$(3-15) \quad \bar{f}^{1-\alpha} \leq Ce^{2k(1-\alpha)u}\sigma_1^{-\alpha}\sigma_{k-1} = Ce^{2k(1-\alpha)u}\sigma_1^{-\alpha} \sum F^{ii}.$$

Hence, combining (3-15), $|\bar{f}_i| \leq C\bar{f}^{1-\alpha}$ and $\sigma_1 > \frac{1}{2}$, we have

$$|u_q(\bar{f}e^{-2ku})_q| \leq Ce^{-2k\alpha u} \sum F^{ii}(\Delta u + 1).$$

Thus, (3-14) becomes

$$(3-16) \quad 0 \geq F^{ii}(U_{iiqq} + u_{q1}^2) - C(1 + e^{-2k\alpha u}) \sum F^{ii}(\Delta u + 1).$$

On the other hand, recall that

$$-\sigma_k^{ii,jj}U_{iip}U_{jjp} \geq \sigma_k \left[\frac{(\sigma_k)_p}{\sigma_k} - \frac{(\sigma_1)_p}{\sigma_1} \right] \left[(\alpha - 1) \frac{(\sigma_k)_p}{\sigma_k} - (\alpha + 1) \frac{(\sigma_1)_p}{\sigma_1} \right].$$

Using (3-9), (3-10), and (3-11), we have

$$(3-17) \quad |(\sigma_1)_p| = \left| u_{qqp} + 2 \left(1 - \frac{n}{2} \right) u_{qp}u_q + A_{qq,p} \right| \\ = | -nu_{qp}u_q + A_{qq,p} | \leq C_0\sigma_1,$$

where C_0 depends on $\|\nabla u\|_{C^0}$, n , k , g_0 , μ_1 , A_{ij} are the components of A_{g_0} . Here, we have used the fact that Δu is sufficiently large. From $|\bar{f}_i| \leq C\bar{f}^{1-\alpha}$, (3-15) and (3-17), noticing that $\sigma_k = \bar{f}e^{-2ku}$, we obtain

$$(3-18) \quad -\sigma_k^{ii,jj}U_{iip}U_{jjp} \\ \geq (\alpha - 1) \left(e^{-2ku} \frac{\bar{f}_p^2 - 4ku_p\bar{f}\bar{f}_p + 4k^2\bar{f}^2u_p^2}{\bar{f}} \right) - Ce^{-2k\alpha u} \sum F^{ii}(\Delta u + 1) \\ \geq (\alpha - 1) \frac{|\nabla \bar{f}|^2}{\bar{f}} e^{-2ku} - Ce^{-2k\alpha u} \sum F^{ii}(\Delta u + 1).$$

Moreover, it follows from $f^{\frac{1}{k-1}} \in C^{1,1}$ that

$$\bar{f}^{\frac{1}{k-1}} \in C^{1,1}, \quad (\bar{f}^{\frac{1}{k-1}})_{qq} \geq -C$$

and

$$(3-19) \quad \bar{f}_{qq} \geq (1 - \alpha) \frac{|\nabla \bar{f}|^2}{\bar{f}} - C\bar{f}^{1-\alpha}.$$

Note that

$$(3-20) \quad F^{ii}U_{iiqq} \geq (\bar{f}e^{-2ku})_{qq} - \sigma_k^{ii,jj}U_{iip}U_{jjp}.$$

Thus, putting (3-19) and (3-18) into (3-20), we have

$$F^{ii}U_{iiqq} \geq -Ce^{-2ku}\bar{f}^{1-\alpha} - Ce^{-2k\alpha u} \sum F^{ii}(\Delta u + 1).$$

Therefore, (3-16) becomes

$$0 \geq \sum F^{ii} \frac{1}{n^2} (\Delta u)^2 - Ce^{-2ku}\bar{f}^{1-\alpha} - C(1 + e^{-2k\alpha u}) \sum F^{ii}(\Delta u + 1).$$

Then, using (3-15), we obtain

$$0 \geq \sum F^{ii} \left[\frac{1}{n^2} (\Delta u)^2 - Ce^{-2k\alpha u} - C(1 + e^{-2k\alpha u})(\Delta u + 1) \right].$$

According to $\sum F^{ii} > 0$, we have

$$0 \geq \frac{1}{n^2} (\Delta u)^2 - Ce^{-2k\alpha u} - C(1 + e^{-2k\alpha u})(\Delta u + 1),$$

hence (1-14) is proved.

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. To simplify notation, we define $\bar{f} = f^{1/k}(x) + \epsilon\eta(f^{1/k}(x))$, $F = \sigma_k^{1/k}$, $U = A_{g_0} + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 g_0$, where u is a solution to (1-15). It follows from $U \in \Gamma_1$,

$$(4-1) \quad 0 \leq \text{tr } U \leq \Delta u + \left(1 - \frac{n}{2}\right) |\nabla u|^2 + C, \quad |\nabla u|^2 \leq C(\Delta u + 1).$$

Also, $U \in \Gamma_2$ implies

$$(4-2) \quad |u_{ij}| \leq C(\Delta u + 1).$$

We may assume $\Delta u > 1$. Then let $H = \Delta u + |\nabla u|^2$, and let \tilde{x} be the maximum point of H . So

$$(4-3) \quad 0 = H_i(\tilde{x}) = u_{ppi} + 2u_{pi}u_p.$$

Let $U_{ij}(\tilde{x})$, $F^{ij}(\tilde{x})$ be diagonal. Differentiating (1-15) twice, we get

$$\begin{aligned} F^{ij}U_{ijp} &= (\bar{f}e^{-2u})_p, \\ F^{ij}U_{ijpp} &= (\bar{f}e^{-2u})_{pp} - F^{pq,rs}U_{pqp}U_{rsp} \geq (\bar{f}e^{-2u})_{pp}. \end{aligned}$$

We have, by (4-1), (4-2), and Ricci identities,

$$\begin{aligned}
0 &\geq F^{ii} H_{ii}(\tilde{x}) \geq F^{ii}(u_{ppii} + 2u_{iip}u_p + 2u_{pi}^2) - C \sum F^{ii}(\Delta u + 1) \\
&= F^{ii}(u_{ppii} + 2u_p(U_{iip} - (u_i u_i - \frac{1}{2}u_l^2 + A_{ii})_p) + 2u_{pi}^2) - C \sum F^{ii}(\Delta u + 1) \\
&\geq 2u_p(\bar{f}e^{-2u})_p + F^{ii}(U_{iipp} - (u_i^2 - \frac{1}{2}u_l^2 + A_{ii})_{pp} - 4u_{ip}u_i u_p + 2u_{lp}u_l u_p + 2u_{pi}^2) \\
&\quad - C \sum F^{ii}(\Delta u + 1) \\
&\geq 2u_p(\bar{f}e^{-2u})_p + (\bar{f}e^{-2u})_{pp} \\
&\quad + F^{ii}(-2u_{iipp}u_i - 2u_{ip}^2 + (u_{lp}u_l + u_{lp}^2) - 4u_{ip}u_i u_p + 2u_{lp}u_l u_p + 2u_{pi}^2) \\
&\quad - C \sum F^{ii}(\Delta u + 1).
\end{aligned}$$

Then, according to (4-3), we obtain

$$\begin{aligned}
0 &\geq 2u_p(\bar{f}e^{-2u})_p + (\bar{f}e^{-2u})_{pp} \\
&\quad + F^{ii}(4u_{ip}u_p u_i - 2u_{lp}u_p u_l + u_{lp}^2 - 4u_{ip}u_i u_p + 2u_{lp}u_l u_p) - C \sum F^{ii}(\Delta u + 1) \\
&= 2u_p(\bar{f}e^{-2u})_p + (\bar{f}e^{-2u})_{pp} + \sum F^{ii}u_{lp}^2 - C \sum F^{ii}(\Delta u + 1).
\end{aligned}$$

We notice that

$$|u_p(\bar{f}e^{-2u})_p + (\bar{f}e^{-2u})_{pp}| \leq Ce^{-2u}(\Delta u + 1),$$

where C depends on $\|f^{\frac{1}{k}}\|_{C^2}$, $\|f\|_{C^0}$, n , k , g_0 , θ , C_1 , C_2 . So

$$\begin{aligned}
0 &\geq \sum F^{ii}u_{lp}^2 - C(1 + e^{-2u}) \sum F^{ii}(\Delta u + 1) \\
&\geq \sum F^{ii} \cdot \left(\frac{1}{n^2}(\Delta u)^2 - C(1 + e^{-2u})(\Delta u + 1) \right).
\end{aligned}$$

Hence

$$\Delta u \leq C(1 + e^{-2u}).$$

Then, using (4-1) and (4-2), (1-16) is proved.

Remark. In this case, we do not need to discuss the gradient estimate alone by (4-1). Besides, in the proof of C^2 estimates, the third derivative term is treated by using the concavity of the operator F .

5. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2 under the assumption that the maximum modulus estimation holds. Since the proof is similar to the proof of Theorem 1.3, we present the outline for the proof and main differences here. Let

$$\alpha = \frac{1}{k-1}, \quad F = \sigma_k,$$

$$W = -A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2} \Delta w g_0 - \nabla w \otimes \nabla w + \frac{2-t}{2} |\nabla w|^2 g_0,$$

where w is a solution to (1-8).

First, we consider C^1 estimates of w . Let $K = (1 + \frac{|\nabla w|^2}{2})e^{\gamma(w)}$. Here γ is a function of the form $\gamma(s) = a(b+s)^p$. The constants a, b, p are chosen as in Lemma 2.6. Suppose $\max K = K(\tilde{x})$, W is diagonal at \tilde{x} . Differentiate K at \tilde{x} , by a direct calculation, we have

$$(5-1) \quad 0 = K_i(\tilde{x}) = e^{\gamma(w)} \left(\left(1 + \frac{w_l^2}{2} \right) \gamma' w_i + w_l w_{li} \right)$$

and

$$K_{ij}(\tilde{x}) = e^{\gamma(w)} \left(\left(1 + \frac{w_l^2}{2} \right) ((\gamma')^2 w_i w_j + \gamma' w_{ij} + \gamma'' w_i w_j) \right. \\ \left. + w_l w_{lj} \gamma' w_i + w_l w_{li} \gamma' w_j + w_{lj} w_{li} + w_l w_{lij} \right).$$

Define

$$P^{ij} = F^{ij} + \frac{1-t}{n-2} \sum_k F^{kk} \delta^{ij}.$$

Combining (5-1) with (1-8), we obtain

$$0 \geq e^{-\gamma(w)} P^{ii} K_{ii}(\tilde{x}) \\ = P^{ii} \left(w_l w_{lii} + \left(1 + \frac{w_l^2}{2} \right) (((\gamma')^2 + \gamma'') w_i^2 + \gamma' w_{ii}) + 2\gamma' w_l w_{li} w_i + w_{li}^2 \right) \\ \geq w_l (\bar{f} e^{2kw})_l + \gamma' k \bar{f} e^{2kw} \left(1 + \frac{w_l^2}{2} \right) \\ + F^{ii} \left(-\gamma' \left(1 + \frac{w_l^2}{2} \right) w_i^2 + \left(1 + \frac{w_l^2}{2} \right) (\gamma'' - \gamma'^2) \frac{(1-t)w_s^2}{n-2} \right) \\ - C \sum F^{ii} (|\nabla w|^2 + 1).$$

Similarly, we have

$$(5-2) \quad \sum F^{ii} = C \sigma_{k-1} \geq C \sigma_k^{1-\frac{1}{k}} = C (\bar{f} e^{2kw})^{1-\frac{1}{k}}$$

and

$$\left| w_l (\bar{f} e^{2kw})_l + \gamma' k \bar{f} e^{2kw} \left(1 + \frac{w_l^2}{2} \right) \right| \leq C \bar{f}^{1-\frac{1}{k}} e^{2kw} (|\nabla w|^2 + 1).$$

Then, letting $\delta = \min\left\{\frac{1-t}{4(n-2)}(\gamma'' - \gamma'^2)\right\}$, we obtain

$$(5-3) \quad 0 \geq F^{ii} \left((\gamma'' - \gamma'^2) \frac{(1-t)w_s^2}{n-2} - \gamma' w_i^2 \right) \left(1 + \frac{w_i^2}{2} \right) \\ - C \bar{f}^{1-\frac{1}{k}} e^{2kw} (|\nabla w|^2 + 1) - C \sum F^{ii} (|\nabla w|^2 + 1) \\ \geq C_4 \bar{f}^{1-\frac{1}{k}} e^{2(k-1)w} \left(\delta \frac{|\nabla w|^4}{2} - C(e^{2w} + 1)(|\nabla w|^2 + 1) \right).$$

To obtain (1-9), we divide the proof into two different cases.

(A) $\bar{f}(\check{x}) > 0$. Using (5-2) and (5-3), we have

$$\delta \frac{|\nabla w|^4}{2} - C(e^{2w} + 1)(|\nabla w|^2 + 1) \leq 0.$$

Thus, (1-9) can be obtained.

(B) $\bar{f}(\check{x}) = 0$. Applying (5-3), we see that

$$(5-4) \quad 0 \geq F^{ii} \left((\gamma'' - \gamma'^2) \frac{(1-t)w_s^2}{n-2} - \gamma' w_i^2 \right) \left(1 + \frac{w_i^2}{2} \right) - C \sum F^{ii} (|\nabla w|^2 + 1) \\ \geq \sum F^{ii} \left(\delta \frac{|\nabla w|^4}{2} - C(|\nabla w|^2 + 1) \right).$$

In view of $\sum F^{ii} > 0$, (1-9) follows from (5-4).

Next, we discuss C^2 estimates of w . Let $H = \Delta w + a|\nabla w|^2$, and let \check{x} be the maximum point of H , $a \geq \frac{2-t}{1-t}(2n-4)$. Since

$$|W_{ii}| \leq \sigma_1(W),$$

there exists C such that

$$(5-5) \quad |w_{ij}| \leq C \Delta w,$$

where C depends on $n, k, g_0, \|\nabla w\|_{C^0}$. Hence, we may assume $\Delta w \gg 1$ and

$$\frac{1}{2} \Delta w \leq \sigma_1 \leq 2 \Delta w.$$

We may further assume $W_{ij}(\check{x})$ is diagonal. Differentiating H at \check{x} , we obtain

$$(5-6) \quad 0 = H_i(\check{x}) = w_{qqi} + 2aw_{qi}w_q$$

and

$$(5-7) \quad 0 \geq H_{ii}(\check{x}) = w_{qqii} + 2aw_{qii}w_q + 2aw_{qi}^2.$$

By (5-7), (5-5), (5-6), (2-2), (2-5) and (1-8), we have

$$\begin{aligned}
 0 &\geq P^{ii} H_{ii}(\check{x}) = P^{ii} (u_{qqii} + 2aw_{qii}w_q + 2aw_{qi}^2) \\
 &\geq F^{ii} \left(-4a \frac{2-t}{2} w_q w_l w_{lq} + 4aw_q w_{iq} w_i \right. \\
 &\quad \left. + \frac{2a(1-t)}{n-2} w_{ql}^2 + W_{iiqq} - \left(\frac{2-t}{2} w_l^2 - w_i w_i \right)_{qq} \right) \\
 &\quad + 2aw_q (\bar{f} e^{2kw})_q - C \sum F^{ii} (\Delta w + 1) \\
 &\geq F^{ii} \left(4a \left(-\frac{2-t}{2} w_q w_l w_{lq} + w_q w_{iq} w_i \right) + \frac{2a(1-t)}{n-2} w_{ql}^2 \right. \\
 &\quad \left. - 2 \frac{2-t}{2} w_{lq}^2 + 2w_{iq}^2 + 4a \left(\frac{2-t}{2} w_l w_{lq} w_q - w_i w_{iq} w_q \right) \right) \\
 &\quad + F^{ii} W_{iiqq} + 2aw_q (\bar{f} e^{2kw})_q - C \sum F^{ii} (\Delta w + 1) \\
 &\geq F^{ii} \left(\frac{2a(1-t)}{n-2} w_{ql}^2 - (2-t) w_{lq}^2 \right) + F^{ii} W_{iiqq} + 2aw_q (\bar{f} e^{2kw})_q \\
 &\quad - C \sum F^{ii} (\Delta w + 1).
 \end{aligned}$$

We also have

$$(5-8) \quad C \bar{f}^{1-\alpha} e^{2k(1-\alpha)w} (\Delta w)^\alpha \leq \sigma_{k-1} = C \sum F^{ii},$$

$$(5-9) \quad |\bar{f}_i| \leq C \bar{f}^{1-\alpha} \leq C e^{-2k(1-\alpha)w} \sum F^{ii},$$

$$(5-10) \quad F^{ii} W_{iiqq} \geq -C e^{2kw} \bar{f}^{1-\alpha} - C e^{2k\alpha w} \sum F^{ii} (\Delta w + 1).$$

Using (5-8)–(5-10), and the definition of a ,

$$\begin{aligned}
 0 &\geq F^{ii} \left(\frac{2a(1-t)}{n-2} - (2-t) \right) w_{lq}^2 + F^{ii} W_{iiqq} - C(1 + e^{2k\alpha w}) \sum F^{ii} (\Delta w + 1) \\
 &\geq F^{ii} \left(\frac{2a(1-t)}{n-2} - (2-t) \right) w_{lq}^2 - C e^{2kw} \bar{f}^{1-\alpha} - C(1 + e^{2k\alpha w}) \sum F^{ii} (\Delta w + 1) \\
 &\geq F^{ii} \left(\frac{a(1-t)}{(n-2)n^2} \right) (\Delta w)^2 - C e^{2kw} \bar{f}^{1-\alpha} - C(1 + e^{2k\alpha w}) \sum F^{ii} (\Delta w + 1) \\
 &\geq F^{ii} \left[\left(\frac{a(1-t)}{(n-2)n^2} \right) (\Delta w)^2 - C e^{2k\alpha w} - C(1 + e^{2k\alpha w}) (\Delta w + 1) \right].
 \end{aligned}$$

Since $\sum F^{ii} > 0$, we have $\left(\frac{a(1-t)}{(n-2)n^2} \right) (\Delta w)^2 - C e^{2k\alpha w} - C(1 + e^{2k\alpha w}) (\Delta w + 1) \leq 0$.

Thus

$$\Delta w \leq C(1 + e^{2k\alpha w}),$$

and (1-10) is proved.

We remark that in this section $t < 1$ is the necessary condition since the Δw yields the dominating term in (5-3) and the penultimate display.

6. Proof of the existence result

In this section, we focus on dealing with the existence result for (1-7). We need to discuss the maximum modulus estimation under the assumption that the supersolution exists.

6.1. Proof of Theorem 1.1. Consider

$$(6-1) \quad \sigma_k(sW + (1-s)g_0) = (s\bar{f} + (1-s)C_n^k)e^{2kw},$$

where

$$W = -A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2} \Delta w - dw \otimes dw + \frac{2-t}{2} |\nabla w|^2 g_0,$$

$$s \in [0, 1], \bar{f} = (f^{\frac{1}{k-1}} + \epsilon)^{k-1}.$$

Claim: Equation (1-8) is solvable for each ϵ . In fact, define

$$(6-2) \quad \Phi_t[w] = \sigma_k(sW + (1-s)g_0) - (s\bar{f} + (1-s)C_n^k)e^{2kw}.$$

Note that the linearization of Φ_t is invertible, and $w = 0$ is a solution to (6-1) at $s = 0$. Thus, the problem is reduced to C^0 , C^1 , and C^2 estimates.

First, we consider C^0 estimates. Let \bar{x} be the maximum point of w . From (6-1) we have

$$e^{2kw(\bar{x})} \leq \frac{\max \sigma_k(-A_{g_0}^t + g_0)}{\min\{\bar{f}, C_n^k\}}.$$

Thus, it provides an upper bound for w . Let \hat{x} be the minimum point of w . Similarly, from (6-1) we have

$$e^{2kw(\hat{x})} \geq \frac{\min \sigma_k(-sA_{g_0}^t + (1-s)g_0)}{\max \bar{f} + C_n^k}.$$

Thus, it provides a lower bound for w .

Then, using Theorem 1.1 in [8], we get C^1 and C^2 estimates for solutions to (6-1). This proves the claim.

Next, we shall prove that (1-7) is solvable. Let us denote the solution derived in the claim by w_ϵ . If \bar{w} is the supersolution of (1-7), then it is the supersolution of (1-8) as well. Thus, by Lemma 2.7,

$$w_\epsilon \leq \bar{w}.$$

Moreover, from the equation, we have

$$e^{2k \min w_\epsilon} \geq \frac{\min \sigma_k(-A_{g_0}^t)}{(\max f^{\frac{1}{k-1}} + 1)^{k-1}}.$$

Then, by Theorem 1.2, we have $|\nabla w_\epsilon| \leq C$ and $|\nabla^2 w_\epsilon| \leq C$. Here, C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, g_0 , n , k , \bar{w} . So, $\lim_{\epsilon \rightarrow 0} w_\epsilon$ is a solution to (1-6).

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
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