Pacific Journal of Mathematics

EXISTENCE FOR SOME DEGENERATE HESSIAN-TYPE EQUATIONS ARISING IN CONFORMAL GEOMETRY

YAN HE, QIANG TU AND NI XIANG

Volume 335 No. 1 March 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 335, No. 1, 2025
https://doi.org/10.2140/pjm.2025.335.97

EXISTENCE FOR SOME DEGENERATE HESSIAN-TYPE EQUATIONS ARISING IN CONFORMAL GEOMETRY

YAN HE, QIANG TU AND NI XIANG

We establish the a priori estimates for three kinds of degenerate Hessian-type equations arising in conformal geometry. Based on these a priori estimates, we obtain an existence result using the continuity method.

1. Introduction

Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$. Consider a symmetric polynomial of degree k

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k,$$

where

$$\Gamma_k = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ 1 \leq j \leq k\}, \quad 1 \leq k \leq n.$$

For any $n \times n$ matrix U, $U \in \Gamma_k$ means the eigenvalues of U lie in Γ_k and $\sigma_k(U)$ means that σ_k is applied to the eigenvalues of U. In this paper, we derive the a priori estimates for three kinds of degenerate equations arising in conformal geometry. The existence of $C^{1,1}$ solutions, which is the central issue for these degenerate equations, has been obtained by using these estimates.

The degenerate Hessian equations

$$\sigma_k(D^2 u) = f(x, u) \geq 0, \quad \Omega \subset \mathbb{R}^n, \tag{1-1}$$

have attracted much attention recently. For $k = n$, the degenerate Monge–Ampère-type equations, Guan and Li studied the degenerate Weyl problem and the degenerate Gauss curvature problem in [16; 17]. They found that the conditions

$$\Delta f^{\frac{1}{n-1}} \geq -A \quad \text{and} \quad |D(f^{1/(n-1)})| \leq A \tag{1-2}$$

This research was supported by funds from the National Natural Science Foundation of China Nos. 11971157, 12101206.

MSC2020: primary 53C21; secondary 35B45.

Keywords: conformal geometry, degenerate Hessian-type equations, a priori estimates.

for some constant A are sufficient to get $C^{1,1}$ estimates. Using (1-2), Guan [14] established $C^{1,1}$ estimates for the Dirichlet problem of the degenerate Monge–Ampère equations with homogeneous boundary condition. Guan, Trudinger and Wang [20] established $C^{1,1}$ estimates for the Dirichlet problem with nonhomogeneous boundary condition given by

$$f^{\frac{1}{n-1}} \in C^{1,1}(\overline{\Omega}). \tag{1-3}$$

Here, (1-3) implies (1-2).

For general $1 < k < n$, Dong [9] obtained C^2 estimates for the homogeneous Dirichlet problem with homogeneous boundary by

$$|D(f^{\frac{1}{k-1}})| \le C f^{\frac{1}{2k-2}} \quad \text{in } \overline{\Omega} \tag{1-4}$$

and

$$f^{\frac{1}{k-1}} \in C^{1,1}(\overline{\Omega}). \tag{1-5}$$

Jiao and Wang [28] generalized the results in [9], and obtained $C^{1,1}$ estimates for k-curvature problems with homogeneous boundary by $f^{\frac{1}{k-1}} \in C^{1,1}(\mathbb{R}^n)$. They also found convex solutions to the Dirichlet problem of degenerate k-Hessian equations with nonhomogeneous boundary condition by (1-5) in [29].

Motivated by [28], it is natural to study the degenerate equations in conformal geometry. One of the central issues in conformal geometry is the k-Yamabe problem. The k-Yamabe problem was raised in [41] in order to find a conformal metric g so that $\sigma_k(A_g) = \text{const.}$, where $A_g = \frac{1}{n-2}\big(\text{Ric}_g - \frac{R_g}{2(n-1)} g\big)$, Ric_g, R_g are the Schouten tensor, the Ricci curvature and scalar curvature of (M, g), respectively. The general prescribing curvature problem is to find a conformal metric such that $\sigma_k(A_g)$ equals a given function f,

$$\sigma_k(A_g) = f(x). \tag{1-6}$$

We say (1-6) is *nondegenerate* if $f > 0$. Fruitful results have been achieved on the k-Yamabe problem and the prescribing curvature problem. Here, we just mention some existence results of the k-Yamabe problem and the prescribing curvature problem. The research of the nondegenerate prescribing curvature problem on manifolds without boundary can be tracked back to Chang, Gursky and Yang [1; 2]. They proved that if the Yamabe constant and $\int_M \sigma_2(A_{g_0})$ are both positive, then there exists a conformal metric g such that $\sigma_2(A_g)$ is a positive constant; see also [22]. Later on, Guan and Wang [18] and Li and Li [31] concluded that on locally conformally flat manifolds, the k-Yamabe problem is solvable for $2 \le k < n/2$; see also [38]. Gursky and Viaclovsky confirmed the existence of a solution and compactness of solution set to $\sigma_k(A_g) = f > 0$ in the case of $n \ge k > n/2$ when $A_{g_0} \in \Gamma_k$ and (M, g_0) is not conformally equivalent to a sphere. Li and Nguyen [34] obtained that if $A_{g_0} \in \Gamma_k$, $k = n/2$ and (M, g_0) is not conformally equivalent to a sphere, then there is a conformal metric $\tilde{g}$ such that $\sigma_k(A_{\tilde{g}}) = 1$, and the solution set is compact. Other relevant conclusions are included in [3; 4; 10; 11; 12; 32; 40].

In 2003, Gursky and Viaclovsky [23] introduced the modified Schouten tensor

$$A_g^t = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{tR_g}{2(n-1)}g\right).$$

This tensor is, in fact, a constant multiple of the tensor $sA_g - \frac{(1-s)R_g}{2(n-1)}g$ which is introduced in [31]. It is also a meaningful problem to find a conformal metric g with

$$\sigma_k(-A_g^t) = f(x). \tag{1-7}$$

When $t = 0$, $A_g^t = \frac{1}{n-2}\mathrm{Ric}_g$; when $t = 1$, A_g^t is just the Schouten tensor A_g; when $t = n-1$, A_g^t is the Einstein tensor. Gursky and Viaclovsky [23] and Li and Sheng [35] proved that when $t < 1$ every compact manifold with $-A_{g_0}^t \in \Gamma_k$ is conformal to a metric g with $\sigma_k(-A_g^t) = f(x) > 0$.

Another interesting problem is to study the boundary value problem for (1-6) and (1-7) on manifolds (M^n, g) with boundary ∂M. Guan [15] studied the existence problem for (1-6) under the Dirichlet boundary condition. Li and Sheng [36] considered the Dirichlet problem for $\sigma_k(\mathrm{Ric}_g - Rg) = f$. Under various conditions, the Neumann problem arising in conformal geometry is derived in [6; 7; 24; 30; 33; 37].

A key issue for the study of fully nonlinear equations in conformal geometry is the a priori estimates. Thus, there is much profound research on the a priori estimates for (1-6) and (1-7) and their further generalizations. We refer to [5; 18; 19; 25; 26; 27; 31; 38; 42; 43].

It is noticed that most of the results focus on nondegenerate cases in conformal geometry. For the degenerate case, Ge, Lin and Wang [13] obtained that if $f = 0$, $k = 2$, there exists a $C^{1,1}$ metric such that $\sigma_2(A_g) = 0$. A natural question is whether the existence is still valid if f is nonnegative.

We consider an existence for the solutions to degenerate equations in the negative cone $-\Gamma_k = \{\lambda \mid -\lambda \in \Gamma_k\}$. We raise the following problem,

Problem. Let f be a nonnegative function in M. Is it true that there exists a conformal metric g on M with $\sigma_k(-A_g^t) = f \geq 0$?

The question is partly answered in this paper. The results are as follows. Let $g = e^{2w}g_0$. Then

$$-A_g^t = \frac{1-t}{n-2}\Delta w + \nabla^2 w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0 - A_{g_0}^t.$$

Here, w is said to be *admissible* if $-A_g^t \in \Gamma_k$. Note that [38] provides a counterexample to show that the regularity for $\det(\nabla^2 w + |\nabla w|^2 + S) = f$ in $\mathbb{R}^2$ fails where $S(x)$ is some 2×2 matrix. We only consider $\sigma_k(-A_g^t) = f \geq 0$ with $t < 1$.

Theorem 1.1. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$, $-A^t_{g_0} \in \Gamma_k$, $2 \leq k \leq n$, $t < 1$. Suppose that $f^{\frac{1}{k-1}} \in C^{1,1}$ is a nonnegative function in M. Equation (1-7) has an admissible supersolution $\overline{w}$:*

$$\sigma_k(-A^t_{e^{2\overline{w}}g_0}) \leq f(x), \quad -A^t_{e^{2\overline{w}}g_0} \in \Gamma_k.$$

Then, there is a $C^{1,1}$ solution w of (1-7) satisfying $A_{e^{-2w}g_0} \in \bar{\Gamma}_k$.

Perturbation techniques are often used to deal with degenerate problems. First, we transform the original degenerate problem into a nondegenerate equation by adding a positive number ϵ to the right-hand function f. Next, we establish the a priori estimates, which are independent of ϵ. Finally, we find the existence of the solution u_ϵ for each ϵ and then let $\epsilon \to 0$ to derive the existence result for the degenerate problem (1-7).

Since the a priori estimates are the core of the existence theorems, we need the following C^1 and C^2 estimates.

Theorem 1.2. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$, $t < 1$. Suppose that f is a nonnegative function in M and $f^{\frac{1}{k-1}} \in C^{1,1}$. Let w be a C^4 solution to*

$$\sigma_k\left(-A^t_{g_0} + \nabla^2 w + \frac{1-t}{n-2}\Delta w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0\right) = (f^{\frac{1}{k-1}}(x) + \epsilon)^{k-1} e^{2kw}, \quad x \in M, \tag{1-8}$$

where $-A^t_{e^{2w}g_0} \in \Gamma_k$. Then, there is a positive constant C depending on g_0, n, k, t, $\|f^{\frac{1}{k-1}}\|_{C^1}$, $\|f\|_{C^0}$, $\|w\|_{C^0}$ but independent of ϵ, such that

$$\sup |\nabla w| \leq C. \tag{1-9}$$

Furthermore, there is a positive constant C depending on g_0, n, k, t, $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|\nabla w\|_{C^0}$ but independent of ϵ, such that

$$\sup |\nabla^2 w| \leq C(1 + e^{\frac{2k}{k-1}\sup w}). \tag{1-10}$$

The next theorems concern the estimates for $\sigma_k(A_g) = f \geq 0$. Before introducing the estimates, we give the following notation. Set $0 < \epsilon < 1$, and let $0 \leq \eta(t) \leq 1$ be a smooth function, which satisfies

$$\eta(t) = 1 \ \text{ for } t \leq \tfrac{1}{4}\theta; \quad \eta(t) = 0 \ \text{ for } t \geq \tfrac{1}{2}\theta; \quad \eta' \leq C_1\theta, \quad \eta'' \leq C_2\theta. \tag{1-11}$$

Here, $C_1, C_2, \theta > 0$ are constants.

Theorem 1.3. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$. Suppose that f is a nonnegative function in M. Let u be a*

C^4 solution to

$$\sigma_k\big(A_{g_0}+\nabla^2 u+\nabla u\otimes\nabla u-\tfrac{1}{2}|\nabla u|^2 g_0\big) = \big(f^{\frac{1}{k-1}}(x)+\epsilon\eta(f^{\frac{1}{k-1}}(x))\big)^{k-1}e^{-2ku}, \quad x\in M, \tag{1-12}$$

where $A_{e^{-2u}g_0}\in\Gamma_k$. If $f^{\frac{1}{k-1}}\in C^{1,1}$, then

$$\sup|\nabla u|\le C, \tag{1-13}$$

where a positive constant C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|u\|_{C^0}$, g_0, n, k, θ, C_1, C_2 but is independent of ϵ.

Moreover,

$$\sup|\nabla^2 u|\le C(1+e^{-\frac{2k}{k-1}\inf u}), \tag{1-14}$$

where a positive constant C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|\nabla u\|_{C^0}$, g_0, n, k, θ, C_1, C_2 but is independent of ϵ.

The next theorem concerns the case of $f^{\frac{1}{k}}\in C^2$.

Theorem 1.4. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n\ge 3$, $2\le k\le n$. Suppose that f is a nonnegative function in M. Let u be a C^4 solution to*

$$\sigma_k^{\frac{1}{k}}\big(A_{g_0}+\nabla^2 u+\nabla u\otimes\nabla u-\tfrac{1}{2}|\nabla u|^2 g_0\big) = \big(f^{\frac{1}{k}}(x)+\epsilon\eta(f^{\frac{1}{k}}(x))\big)e^{-2u}, \quad x\in M, \tag{1-15}$$

where $A_{e^{-2u}g_0}\in\Gamma_k$. If $f^{\frac{1}{k}}\in C^2$, then

$$\sup(|\nabla u|^2+|\nabla^2 u|)\le C(1+e^{-2\inf u}), \tag{1-16}$$

where a positive constant C depends on g_0, n, k, $\|f^{\frac{1}{k}}\|_{C^2}$, $\|f\|_{C^0}$, θ, C_1, C_2 but is independent of ϵ.

The paper is built up as follows. In Section 2, we introduce the notation and the necessary formulas. In Sections 3, 4, 5, we establish the crucial estimates and prove Theorems 1.3, 1.4 and 1.2, respectively. Finally, the existence result of Theorem 1.1 is proved in Section 6.

2. Preliminaries

Throughout this paper,

$$\sigma_{k-1}(\lambda\mid i)=\frac{\partial\sigma_k}{\partial\lambda_i} \quad\text{and}\quad \sigma_{k-2}(\lambda\mid ij)=\frac{\partial^2\sigma_k}{\partial\lambda_i\partial\lambda_j}.$$

We list some properties of σ_k, which will be used later.

Proposition 2.1. *Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ *and* $1 \le k \le n$. *Then we have*

(1) $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n$;

(2) $\sigma_{k-1}(\lambda \mid i) > 0$ *for* $\lambda \in \Gamma_k$ *and* $1 \le i \le n$;

(3) $\sigma_k(\lambda) = \sigma_k(\lambda \mid i) + \lambda_i \sigma_{k-1}(\lambda \mid i)$ *for* $1 \le i \le n$;

(4) *if* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *then* $\sigma_{k-1}(\lambda \mid 1) \le \sigma_{k-1}(\lambda \mid 2) \le \cdots \le \sigma_{k-1}(\lambda \mid n)$ *for* $\lambda \in \Gamma_k$;

(5) $\sum_{i=1}^n \sigma_{k-1}(\lambda \mid i) = (n-k+1)\sigma_{k-1}(\lambda)$.

The generalized Newton–MacLaurin inequality is as follows, which will be used later.

Proposition 2.2. *For* $\lambda \in \Gamma_m$ *and* $m > l \ge 0$, $r > s \ge 0$, $m \ge r$, $l \ge s$, *we have*

$$\left[\frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{m-l}} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}.$$

Proof. See [39]. □

The following lemma is the key in the proof of Theorems 1.3 and 1.2.

Lemma 2.3. *Let* $\alpha = \frac{1}{k-1}$. *If* $U \in \Gamma_k$, *then*

$$-\sigma_k^{ii,jj} U_{iip} U_{jjp} \ge \sigma_k \left[\frac{(\sigma_k)_p}{\sigma_k} - \frac{(\sigma_1)_p}{\sigma_1}\right]\left[(\alpha-1)\frac{(\sigma_k)_p}{\sigma_k} - (\alpha+1)\frac{(\sigma_1)_p}{\sigma_1}\right]. \tag{2-1}$$

Proof. See [21]. □

In this paper ∇ denotes the Levi-Civita connection on (M, g_0), and the curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let $e_1, e_2, \dots, e_n$ be local frames on M and define $g_{ij} = g_0(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, while the Christoffel symbols Γ_{ij}^k and curvature coefficients are given, respectively, by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ and

$$R_{ijkl} = g_0(R(e_k, e_l)e_j, e_i), \quad R^i_{jkl} = g^{im} R_{mjkl}.$$

We write $u_i = \nabla_i u = \nabla_{e_i} u$, $u_{ji} = [\nabla\nabla u](e_j, e_i) = [\nabla_i(\nabla u)](e_j) = \nabla_i(\nabla_j u) - \Gamma_{ij}^k u_k$, $u_{ijk} = [\nabla_k(\nabla\nabla u)](e_i, e_j)$, etc. Note that $u_{ij} = u_{ji}$,

$$u_{ijk} - u_{kij} = R^m_{ijk} u_m, \tag{2-2}$$

$$u_{kilj} - u_{klij} = R^m_{kil,j} u_m + R^m_{kil} u_{mj}, \tag{2-3}$$

$$u_{kijl} - u_{kilj} = R^m_{ijl} u_{km} + R^m_{kjl} u_{mi}. \tag{2-4}$$

From (2-2), (2-3) and (2-4), we obtain

(2-5) $$u_{ijkl}-u_{klij}=R^m_{ijk,l}u_m+R^m_{kil,j}u_m+R^m_{ijk}u_{ml}+R^m_{kil}u_{mj}+R^m_{ijl}u_{km}+R^m_{kjl}u_{mi}.$$

For convenience, we introduce the notation

$$F(U)=\sigma_k(U),\qquad F^{ij}=\frac{\partial F}{\partial U_{ij}},\qquad F^{ij,rs}=\frac{\partial^2 F}{\partial U_{ij}\partial U_{rs}}.$$

Lemma 2.4. (1) *If* $U=\nabla^2 u+\nabla u\otimes\nabla u-\frac{1}{2}|\nabla u|^2 g_0+A_{g_0}$, *then*

$$U_{ijp}=u_{ijp}+u_iu_{jp}+u_ju_{ip}-u_qu_{qp}g_{ij}+A_{ij,p},$$

and

$$U_{ijpp}=u_{ijpp}+2u_{ip}u_{jp}+u_iu_{jpp}+u_ju_{ipp}-u_{qp}u_{qp}g_{ij}-u_qu_{qpp}g_{ij}+A_{ij,pp},$$

where A_{ij} *are the components of* A_{g_0} *and* g_{ij} *are the components of* g_0.

(2) *If* $W=-A^t_{g_0}+\nabla^2 w+\frac{1-t}{n-2}\Delta w g_0-\nabla w\otimes\nabla w+\frac{2-t}{2}|\nabla w|^2 g_0$, *then*

$$W_{ijp}=-A^t_{ij,p}+w_{ijp}+\frac{1-t}{n-2}w_{qqp}g_{ij}-w_iw_{jp}-w_jw_{ip}+(2-t)w_qw_{qp}g_{ij},$$

and

$$W_{ijpp}=-A^t_{ij,pp}+w_{ijpp}+\frac{1-t}{n-2}w_{qqpp}g_{ij}-w_iw_{jpp}-w_jw_{ipp}\\ -2w_{ip}w_{jp}+(2-t)w_{qp}w_{qp}g_{ij}+(2-t)w_qw_{qpp}g_{ij},$$

where A^t_{ij} *are the components of* $A^t_{g_0}$ *and* g_{ij} *are the components of* g_0.

The following two lemmas will be used in the proof of Theorems 1.3 and 1.2.

Lemma 2.5. *Assume that* $-\mu\le s\le\mu$. *Then, we may choose constants* a, b, *and* p *depending only on* $\mu>0$ *so that* $\gamma(s)=a(b+s)^p$ *satisfies*

(2-6) $$-\tfrac{1}{4}\gamma'(s)\ge\gamma''(s)+\gamma'(s)-\gamma'(s)^2>0.$$

Proof. We have

$$\gamma'(s)=pa(b+s)^{p-1},\quad \gamma''(s)=p(p-1)a(b+s)^{p-2}.$$

Hence,

$$\frac{\gamma''(s)-\gamma'(s)^2}{-\gamma'(s)}=\frac{p(p-1)a(b+s)^{p-2}-p^2a^2(b+s)^{2p-2}}{-pa(b+s)^{p-1}}\\ =\frac{1-p}{b+s}+(b+s)^{p-1}ap.$$

Now choose $b = 17\mu + 16$, $p = -\frac{35}{34}(b+\mu)$, $a = \frac{(b-\mu)^{1-p}}{-34p}$. Then

$$0 < \frac{1}{b+\mu} \le \frac{1}{b+s} \le \frac{1}{16}.$$

Moreover,

$$\frac{35}{34} = \frac{-p}{b+\mu} \le \frac{-p}{b+s} \le \frac{35}{34}\frac{b+\mu}{b-\mu} < \frac{9}{8}\cdot\frac{35}{34},$$

and

$$|(b+s)^{p-1}ap| \le (b-\mu)^{p-1}|ap| = \frac{1}{34}.$$

Therefore,

$$1 < \frac{1-p}{b+s} + (b+s)^{p-1}ap \le \frac{5}{4}.$$

Then, (2-6) is proved. □

Lemma 2.6. *Assume that $-\mu \le s \le \mu$, $t < 1$. Then, we may choose constants a, b, and p depending on $\mu > 0$ and t so that $\gamma(s) = a(b+s)^p$ satisfies*

$$\frac{1-t}{2(n-2)}(\gamma''(s) - \gamma'(s)^2) \ge \gamma'(s) > 0. \tag{2-7}$$

Proof. It is easily seen that

$$\begin{aligned}\frac{\gamma''(s)-\gamma'(s)^2}{\gamma'(s)} &= \frac{p(p-1)a(b+s)^{p-2} - p^2a^2(b+s)^{2p-2}}{pa(b+s)^{p-1}}\\ &= \frac{p-1}{b+s} - (b+s)^{p-1}ap.\end{aligned}$$

Now choose $b = 2\mu + \frac{8(1-t)}{n-2}$, $p = \frac{9(n-2)}{4(1-t)}(b+\mu)$, $a = (b+\mu)^{1-p}\cdot\frac{n-2}{8p(1-t)}$. Similar to the calculation in Lemma 2.5, we have

$$\frac{2(n-2)}{1-t} \le \frac{p-1}{b+s} - (b+s)^{p-1}ap.$$

Then, (2-7) is proved. □

The following lemma will be used in the proof of Theorem 1.1.

Lemma 2.7. *Let*

$$\begin{aligned}W &= \frac{1-t}{n-2}\Delta w + \nabla^2 w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0 - A^t_{g_0},\\ \overline{W} &= \frac{1-t}{n-2}\Delta \overline{w} + \nabla^2 \overline{w} - \nabla \overline{w} \otimes \nabla \overline{w} + \frac{2-t}{2}|\nabla \overline{w}|^2 g_0 - A^t_{g_0}.\end{aligned}$$

Suppose $\sigma_k(\overline{W}) \le fe^{2k\overline{w}}$, $\sigma_k(\overline{W}) \ne 0$, $\sigma_k(W) = fe^{2kw}$. Then we have $w \le \overline{w}$.

Proof. Let $t < 1$, $v^{\frac{4}{n-2}} = e^{2w}$, $\bar{v}^{\frac{4}{n-2}} = e^{2\overline{w}}$,

$$V = \frac{1-t}{n-2}\Delta v + \nabla^2 v - \frac{n}{n-2}\frac{\nabla v \otimes \nabla v}{v} + \frac{1}{n-2}\frac{|\nabla v|^2}{v} g_0 - \frac{n-2}{2} v A^t_{g_0}, \tag{2-8}$$

$$\overline{V} = \frac{1-t}{n-2}\Delta \bar{v} + \nabla^2 \bar{v} - \frac{n}{n-2}\frac{\nabla \bar{v} \otimes \nabla \bar{v}}{\bar{v}} + \frac{1}{n-2}\frac{|\nabla \bar{v}|^2}{\bar{v}} g_0 - \frac{n-2}{2} \bar{v} A^t_{g_0}. \tag{2-9}$$

It is enough to prove that $v \le \bar{v}$. Suppose it is not true. Then by the positivity of v and $\bar{v}$, we find a number $\beta > 1$ such that $\beta\bar{v} \ge v$ and $\beta\bar{v}(\bar{x}) = v(\bar{x})$. Then

$$\nabla(\beta\bar{v})(\bar{x}) = \nabla v(\bar{x}), \quad [\nabla^2(\beta\bar{v}) - \nabla^2 v](\bar{x}) \ge 0. \tag{2-10}$$

Moreover,

$$\begin{aligned}\sigma_k(V)(\bar{x}) \le \sigma_k\biggl(&\frac{1-t}{n-2}\Delta(\beta\bar{v}) + \nabla^2(\beta\bar{v}) - \frac{n}{n-2}\frac{\nabla(\beta\bar{v}) \otimes \nabla(\beta\bar{v})}{\beta\bar{v}} \\ &+ \frac{1}{n-2}\frac{|\nabla(\beta\bar{v})|^2}{\beta\bar{v}} g_0 - \frac{n-2}{2}\beta\bar{v}A^t_{g_0}\biggr)\Big|_{\bar{x}} \\ = \beta^k \sigma_k(\overline{V})(\bar{x}).&\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_k(V)(\bar{x}) &= \left(\frac{n-2}{2} v^{\frac{n+2}{n-2}}\right)^k f\Big|_{\bar{x}} = \left(\frac{n-2}{2}(\beta\bar{v})^{\frac{n+2}{n-2}}\right)^k f\Big|_{\bar{x}} \\ &= \left(\frac{n-2}{2}\beta^{\frac{n+2}{n-2}}\bar{v}^{\frac{n+2}{n-2}}\right)^k f\Big|_{\bar{x}} \\ &\ge \beta^{k\frac{n+2}{n-2}}\sigma_k(\overline{V})(\bar{x}),\end{aligned}$$

which is a contradiction. □

3. The proof of Theorem 1.3

In this section, we obtain C^1 and C^2 estimates with the assumption that the maximum modulus estimation exists for (1-12). We define $\bar{f} = (f^\alpha + \epsilon\eta(f^\alpha))^{k-1}$, $\alpha = \frac{1}{k-1}$, $F = \sigma_k$, $U = A_{g_0} + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2}|\nabla u|^2 g_0$, where u is a solution to (1-12).

First of all, we consider C^1 estimates of u. Let $K = \bigl(1 + \frac{1}{2}|\nabla u|^2\bigr)e^{\gamma(u)}$. Here γ is a function of the form $\gamma(s) = a(b+s)^p$. The constants a, b, p are chosen as in Lemma 2.5. Assume that K achieves the maximal value at some point $\tilde{x}$, which implies that $K_i(\tilde{x}) = 0$. In the rest of the proof, all calculations will be performed at the maximum point $\tilde{x}$. We may assume U_{ij} and F^{ij} are diagonal. Then, in an orthonormal frame,

$$0 = K_i(\tilde{x}) = e^{\gamma(u)}\bigl(\bigl(1 + \tfrac{1}{2}u_l^2\bigr)\gamma' u_i + u_l u_{li}\bigr) \tag{3-1}$$

and

$$K_{ij}(\tilde{x}) = e^{\gamma(u)}\Big(\big(1+\tfrac{1}{2}u_l^2\big)((\gamma')^2u_iu_j+\gamma'u_{ij}+\gamma''u_iu_j) + u_lu_{lj}\gamma'u_i+u_lu_{li}\gamma'u_j+u_{lj}u_{li}+u_lu_{lij}\Big).$$

Differentiating (1-12) gives

$$F^{ij}U_{ijl} = (\bar{f}e^{-2ku})_l. \tag{3-2}$$

Using (3-1), (3-2) and (2-2), we have

$$\begin{aligned}
0 &\geq e^{-\gamma(u)}F^{ij}K_{ij}(\tilde{x})\\
&= F^{ij}\big(u_lu_{lij}+\big(1+\tfrac{1}{2}u_l^2\big)\big(((\gamma')^2+\gamma'')u_iu_j+\gamma'u_{ij}\big)+2\gamma'u_lu_{li}u_j+u_{li}u_{lj}\big)\\
&\geq F^{ij}\big(u_lu_{ijl}+\big(1+\tfrac{1}{2}u_l^2\big)\big(((\gamma')^2+\gamma'')u_iu_j+\gamma'u_{ij}\big)+2\gamma'u_lu_{li}u_j\big)\\
&\qquad -C\sum F^{ii}(|\nabla u|^2+1)\\
&\geq u_l(\bar{f}e^{-2ku})_l+\gamma'k\bar{f}e^{-2ku}\big(1+\tfrac{1}{2}u_l^2\big)\\
&\qquad +F^{ij}\big(-2\big(-\tfrac{1}{2}\big)u_lu_su_{sl}\delta_{ij}-2u_lu_iu_{lj}+\gamma'\big(-\big(-\tfrac{1}{2}\big)u_s^2\delta_{ij}-u_iu_j\big)\big(1+\tfrac{1}{2}u_l^2\big)\big)\\
&\qquad +F^{ij}\big(\big(1+\tfrac{1}{2}u_l^2\big)(\gamma''-\gamma'^2)(u_iu_j)\big)-C\sum F^{ii}(|\nabla u|^2+1)\\
&\geq u_l(\bar{f}e^{-2ku})_l+\gamma'k\bar{f}e^{-2ku}\big(1+\tfrac{1}{2}u_l^2\big)\\
&\qquad +F^{ij}\big(\gamma'\big(1+\tfrac{1}{2}u_l^2\big)u_iu_j-\tfrac{1}{2}\gamma'\big(1+\tfrac{1}{2}u_l^2\big)u_s^2\delta_{ij}+\big(1+\tfrac{1}{2}u_l^2\big)(\gamma''-\gamma'^2)u_iu_j\big)\\
&\qquad -C\sum F^{ii}(|\nabla u|^2+1).
\end{aligned}$$

On the other hand, applying the Newton–MacLaurin inequality, we have

$$\sigma_{k-1}\geq C\sigma_k^{1-\frac{1}{k}}.$$

Thus

$$C\bar{f}^{1-\frac{1}{k}}e^{-2(k-1)u}\leq\sigma_{k-1}=C\sum F^{ii}. \tag{3-3}$$

Moreover, using $f^{\frac{1}{k-1}}\in C^{1,1}$, we have $\bar{f}^{\frac{1}{k-1}}\in C^{1,1}$, which implies

$$\bar{f}^{\frac{1}{k}}\in C^1.$$

Thus

$$|\bar{f}_i|\leq C\bar{f}^{1-\frac{1}{k}}. \tag{3-4}$$

Then, using (3-3) and (3-4), we obtain

$$\big|u_l(\bar{f}e^{-2ku})_l+\gamma'k\bar{f}e^{-2ku}\big(1+\tfrac{1}{2}u_l^2\big)\big|\leq C\bar{f}^{1-\frac{1}{k}}e^{-2ku}(|\nabla u|^2+1). \tag{3-5}$$

Let $\delta=\min\big\{-\frac{1}{8}\gamma'\big\}$. Combining (3-5) with (3-3), it follows that

$$\begin{aligned}
0 &\geq F^{ii}\big(-\tfrac{1}{2}\gamma'u_s^2+(\gamma''+\gamma'-\gamma'^2)u_i^2\big)\big(1+\tfrac{1}{2}u_l^2\big)-C\bar{f}^{1-\frac{1}{k}}e^{-2ku}(|\nabla u|^2+1)\\
&\qquad -C\sum F^{ii}(|\nabla u|^2+1)\\
&\geq C_3\bar{f}^{1-\frac{1}{k}}e^{-2(k-1)u}\big(\delta|\nabla u|^4-C(e^{-2u}+1)(|\nabla u|^2+1)\big).
\end{aligned} \tag{3-6}$$

To derive (1-13), we divide the proof into two cases.

(A) $\bar{f} > 0$. Then, (1-13) follows from (3-6).

(B) $\bar{f} = 0$. Then, (3-6) implies

$$\begin{aligned} 0 &\geq F^{ii}\big(-\tfrac{1}{2}\gamma' u_s^2 + (\gamma'' + \gamma' - \gamma'^2)u_i^2\big)\big(1 + \tfrac{1}{2}u_l^2\big) - C\sum F^{ii}(|\nabla u|^2 + 1) \\ &\geq \sum F^{ii}\big(\delta|\nabla u|^4 - C(|\nabla u|^2 + 1)\big) \end{aligned} \tag{3-7}$$

and (1-13) is proved.

Next, based on both maximum modulus estimation and gradient estimation, we discuss C^2 estimates of u. We may assume U is diagonal. It follows from $U \in \Gamma_1$,

$$0 \leq \operatorname{tr} U \leq \Delta u + \big(1 - \tfrac{1}{2}n\big)|\nabla u|^2 + C, \quad |\nabla u|^2 \leq C(\Delta u + 1). \tag{3-8}$$

Also, it follows from $U \in \Gamma_2$ that

$$|U_{ij}| \leq C\sigma_1, \quad |u_{ij}| \leq C(\Delta u + 1). \tag{3-9}$$

Thus, we may assume that Δu is sufficiently large and $\Delta u \geq \{1, 2\mu_1\}$, where $\mu_1 = \big(-1 + \tfrac{1}{2}n\big)\sup|\nabla u|^2 + \sup|\operatorname{tr} A_{g_0}|$. Then, we have

$$2\Delta u \geq \sigma_1 \geq \tfrac{1}{2}\Delta u. \tag{3-10}$$

Let $H = \Delta u + |\nabla u|^2$, and let $\breve{x}$ be the maximum point of H. By rotating the coordinates, we may assume $U_{ij}(\breve{x})$ is diagonal. Differentiating H at $\breve{x}$, we have

$$0 = H_i(\breve{x}) = u_{qqi} + 2u_{qi}u_q \tag{3-11}$$

and

$$0 \geq H_{ii}(\breve{x}) = u_{qqii} + 2u_q u_{qii} + 2u_{qi}^2. \tag{3-12}$$

Differentiating (1-12) gives

$$F^{ij}U_{ijl} = (\bar{f}e^{-2ku})_l. \tag{3-13}$$

By (3-11), (3-12), (3-13), (2-2), and (2-5), we obtain

$$\begin{aligned} 0 &\geq F^{ii}H_{ii}(\breve{x}) = F^{ii}(u_{qqii} + 2u_{qii}u_q + 2u_{qi}^2) \\ &\geq F^{ii}(u_{iiqq} + 2u_{iiq}u_q + 2u_{qi}^2) - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}u_{iiqq} + F^{ii}\big(2u_q\big(U_{iiq} - \big(-\tfrac{1}{2}u_l^2 + u_i^2\big)_q\big) + 2u_{qi}^2\big) - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}\big(2u_q u_l u_{lq} - 4u_q u_{iq} u_i + U_{iiqq} - \big(-\tfrac{1}{2}u_l^2 + u_i^2\big)_{qq} + 2u_{qi}^2\big) \\ &\qquad\qquad + 2u_q(\bar{f}e^{-2ku})_q - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}(2u_q u_l u_{lq} - 4u_q u_{iq} u_i + U_{iiqq} - (-u_{lq}^2 - u_l u_{lqq} + 2u_i u_{iqq})) \\ &\qquad\qquad + 2u_q(\bar{f}e^{-2ku})_q - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}(U_{iiqq} + u_{ql}^2) + 2u_q(\bar{f}e^{-2ku})_q - C\sum F^{ii}(\Delta u + 1). \end{aligned} \tag{3-14}$$

Then, using the Newton–MacLaurin inequality, we have

$$\sigma_{k-1} \geq C\sigma_k^{1-\alpha}\sigma_1^{\alpha}.$$

Thus

$$\bar{f}^{1-\alpha} \leq Ce^{2k(1-\alpha)u}\sigma_1^{-\alpha}\sigma_{k-1} = Ce^{2k(1-\alpha)u}\sigma_1^{-\alpha}\sum F^{ii}. \tag{3-15}$$

Hence, combining (3-15), $|\bar{f}_i| \leq C\bar{f}^{1-\alpha}$ and $\sigma_1 > \frac{1}{2}$, we have

$$|u_q(\bar{f}e^{-2ku})_q| \leq Ce^{-2k\alpha u}\sum F^{ii}(\Delta u + 1).$$

Thus, (3-14) becomes

$$0 \geq F^{ii}(U_{iiqq} + u_{ql}^2) - C(1 + e^{-2k\alpha u})\sum F^{ii}(\Delta u + 1). \tag{3-16}$$

On the other hand, recall that

$$-\sigma_k^{ii,jj}U_{iip}U_{jjp} \geq \sigma_k\left[\frac{(\sigma_k)_p}{\sigma_k} - \frac{(\sigma_1)_p}{\sigma_1}\right]\left[(\alpha-1)\frac{(\sigma_k)_p}{\sigma_k} - (\alpha+1)\frac{(\sigma_1)_p}{\sigma_1}\right].$$

Using (3-9), (3-10), and (3-11), we have

$$\begin{aligned} |(\sigma_1)_p| &= \left|u_{qqp} + 2\left(1 - \frac{n}{2}\right)u_{qp}u_q + A_{qq,p}\right| \\ &= |-nu_{qp}u_q + A_{qq,p}| \leq C_0\sigma_1, \end{aligned} \tag{3-17}$$

where C_0 depends on $\|\nabla u\|_{C^0}$, n, k, g_0, μ_1, A_{ij} are the components of A_{g_0}. Here, we have used the fact that Δu is sufficiently large. From $|\bar{f}_i| \leq C\bar{f}^{1-\alpha}$, (3-15) and (3-17), noticing that $\sigma_k = \bar{f}e^{-2ku}$, we obtain

$$\begin{aligned} -\sigma_k^{ii,jj}&U_{iip}U_{jjp} \\ &\geq (\alpha-1)\left(e^{-2ku}\frac{\bar{f}_p^2 - 4ku_p\bar{f}\bar{f}_p + 4k^2\bar{f}^2u_p^2}{\bar{f}}\right) - Ce^{-2k\alpha u}\sum F^{ii}(\Delta u + 1) \\ &\geq (\alpha-1)\frac{|\nabla\bar{f}|^2}{\bar{f}}e^{-2ku} - Ce^{-2k\alpha u}\sum F^{ii}(\Delta u + 1). \end{aligned} \tag{3-18}$$

Moreover, it follows from $f^{\frac{1}{k-1}} \in C^{1,1}$ that

$$\bar{f}^{\frac{1}{k-1}} \in C^{1,1}, \quad (\bar{f}^{\frac{1}{k-1}})_{qq} \geq -C$$

and

$$\bar{f}_{qq} \geq (1-\alpha)\frac{|\nabla\bar{f}|^2}{\bar{f}} - C\bar{f}^{1-\alpha}. \tag{3-19}$$

Note that

$$F^{ii}U_{iiqq} \geq (\bar{f}e^{-2ku})_{qq} - \sigma_k^{ii,jj}U_{iip}U_{jjp}. \tag{3-20}$$

Thus, putting (3-19) and (3-18) into (3-20), we have

$$F^{ii}U_{iiqq} \geq -Ce^{-2ku}\bar{f}^{1-\alpha} - Ce^{-2k\alpha u}\sum F^{ii}(\Delta u+1).$$

Therefore, (3-16) becomes

$$0 \geq \sum F^{ii}\frac{1}{n^2}(\Delta u)^2 - Ce^{-2ku}\bar{f}^{1-\alpha} - C(1+e^{-2k\alpha u})\sum F^{ii}(\Delta u+1).$$

Then, using (3-15), we obtain

$$0 \geq \sum F^{ii}\left[\frac{1}{n^2}(\Delta u)^2 - Ce^{-2k\alpha u} - C(1+e^{-2k\alpha u})(\Delta u+1)\right].$$

According to $\sum F^{ii} > 0$, we have

$$0 \geq \frac{1}{n^2}(\Delta u)^2 - Ce^{-2k\alpha u} - C(1+e^{-2k\alpha u})(\Delta u+1),$$

hence (1-14) is proved.

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. To simplify notation, we define $\bar{f} = f^{1/k}(x)+\epsilon\eta(f^{1/k}(x))$, $F=\sigma_k^{1/k}$, $U=A_{g_0}+\nabla^2 u+\nabla u\otimes\nabla u-\frac{1}{2}|\nabla u|^2 g_0$, where u is a solution to (1-15). It follows from $U\in\Gamma_1$,

$$0 \leq \operatorname{tr} U \leq \Delta u + \left(1-\frac{n}{2}\right)|\nabla u|^2 + C, \quad |\nabla u|^2 \leq C(\Delta u+1). \tag{4-1}$$

Also, $U\in\Gamma_2$ implies

$$|u_{ij}| \leq C(\Delta u+1). \tag{4-2}$$

We may assume $\Delta u > 1$. Then let $H=\Delta u+|\nabla u|^2$, and let $\tilde{x}$ be the maximum point of H. So

$$0 = H_i(\tilde{x}) = u_{ppi} + 2u_{pi}u_p. \tag{4-3}$$

Let $U_{ij}(\tilde{x})$, $F^{ij}(\tilde{x})$ be diagonal. Differentiating (1-15) twice, we get

$$F^{ij}U_{ijp} = (\bar{f}e^{-2u})_p,$$
$$F^{ij}U_{ijpp} = (\bar{f}e^{-2u})_{pp} - F^{pq,rs}U_{pqp}U_{rsp} \geq (\bar{f}e^{-2u})_{pp}.$$

We have, by (4-1), (4-2), and Ricci identities,

$$
\begin{aligned}
0 &\geq F^{ii} H_{ii}(\tilde{x}) \geq F^{ii}(u_{ppii}+2u_{iip}u_p+2u_{pi}^2)-C\sum F^{ii}(\Delta u+1)\\
&= F^{ii}\big(u_{ppii}+2u_p\big(U_{iip}-\big(u_iu_i-\tfrac{1}{2}u_l^2+A_{ii}\big)_p\big)+2u_{pi}^2\big)-C\sum F^{ii}(\Delta u+1)\\
&\geq 2u_p(\bar{f}e^{-2u})_p+F^{ii}\big(U_{iipp}-\big(u_i^2-\tfrac{1}{2}u_l^2+A_{ii}\big)_{pp}-4u_{ip}u_iu_p+2u_{lp}u_lu_p+2u_{pi}^2\big)\\
&\qquad\qquad -C\sum F^{ii}(\Delta u+1)\\
&\geq 2u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}\\
&\qquad +F^{ii}(-2u_{ipp}u_i-2u_{ip}^2+(u_{lpp}u_l+u_{lp}^2)-4u_{ip}u_iu_p+2u_{lp}u_lu_p+2u_{pi}^2)\\
&\qquad\qquad -C\sum F^{ii}(\Delta u+1).
\end{aligned}
$$

Then, according to (4-3), we obtain

$$
\begin{aligned}
0 &\geq 2u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}\\
&\qquad +F^{ii}(4u_{ip}u_pu_i-2u_{lp}u_pu_l+u_{lp}^2-4u_{ip}u_iu_p+2u_{lp}u_lu_p)-C\sum F^{ii}(\Delta u+1)\\
&= 2u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}+\sum F^{ii}u_{lp}^2-C\sum F^{ii}(\Delta u+1).
\end{aligned}
$$

We notice that

$$
|u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}| \leq Ce^{-2u}(\Delta u+1),
$$

where C depends on $\|f^{\frac{1}{k}}\|_{C^2}$, $\|f\|_{C^0}$, n, k, g_0, θ, C_1, C_2. So

$$
\begin{aligned}
0 &\geq \sum F^{ii}u_{lp}^2 - C(1+e^{-2u})\sum F^{ii}(\Delta u+1)\\
&\geq \sum F^{ii}\cdot\Big(\frac{1}{n^2}(\Delta u)^2 - C(1+e^{-2u})(\Delta u+1)\Big).
\end{aligned}
$$

Hence

$$
\Delta u \leq C(1+e^{-2u}).
$$

Then, using (4-1) and (4-2), (1-16) is proved.

Remark. In this case, we do not need to discuss the gradient estimate alone by (4-1). Besides, in the proof of C^2 estimates, the third derivative term is treated by using the concavity of the operator F.

5. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2 under the assumption that the maximum modulus estimation holds. Since the proof is similar to the proof of Theorem 1.3, we present the outline for the proof and main differences here. Let

$\alpha = \frac{1}{k-1}$, $F = \sigma_k$,

$$W = -A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2}\Delta w g_0 - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0,$$

where w is a solution to (1-8).

First, we consider C^1 estimates of w. Let $K = \left(1 + \frac{|\nabla w|^2}{2}\right)e^{\gamma(w)}$. Here γ is a function of the form $\gamma(s) = a(b+s)^p$. The constants a, b, p are chosen as in Lemma 2.6. Suppose $\max K = K(\tilde{x})$, W is diagonal at $\tilde{x}$. Differentiate K at $\tilde{x}$, by a direct calculation, we have

$$0 = K_i(\tilde{x}) = e^{\gamma(w)}\left(\left(1 + \frac{w_l^2}{2}\right)\gamma' w_i + w_l w_{li}\right) \tag{5-1}$$

and

$$\begin{aligned} K_{ij}(\tilde{x}) = e^{\gamma(w)}\Bigg(\left(1 + \frac{w_l^2}{2}\right)&((\gamma')^2 w_i w_j + \gamma' w_{ij} + \gamma'' w_i w_j) \\ &+ w_l w_{lj}\gamma' w_i + w_l w_{li}\gamma' w_j + w_{lj} w_{li} + w_l w_{lij}\Bigg). \end{aligned}$$

Define

$$P^{ij} = F^{ij} + \frac{1-t}{n-2}\sum_k F^{kk}\delta^{ij}.$$

Combining (5-1) with (1-8), we obtain

$$\begin{aligned} 0 &\geq e^{-\gamma(w)} P^{ii} K_{ii}(\tilde{x}) \\ &= P^{ii}\left(w_l w_{lii} + \left(1 + \frac{w_l^2}{2}\right)\left(((\gamma')^2 + \gamma'')w_i^2 + \gamma' w_{ii}\right) + 2\gamma' w_l w_{li} w_i + w_{li}^2\right) \\ &\geq w_l(\bar{f}e^{2kw})_l + \gamma' k \bar{f} e^{2kw}\left(1 + \frac{w_l^2}{2}\right) \\ &\qquad + F^{ii}\left(-\gamma'\left(1 + \frac{w_l^2}{2}\right)w_i^2 + \left(1 + \frac{w_l^2}{2}\right)(\gamma'' - \gamma'^2)\frac{(1-t)w_s^2}{n-2}\right) \\ &\qquad\qquad - C\sum F^{ii}(|\nabla w|^2 + 1). \end{aligned}$$

Similarly, we have

$$\sum F^{ii} = C\sigma_{k-1} \geq C\sigma_k^{1-\frac{1}{k}} = C(\bar{f}e^{2kw})^{1-\frac{1}{k}} \tag{5-2}$$

and

$$\left| w_l(\bar{f}e^{2kw})_l + \gamma' k \bar{f} e^{2kw}\left(1 + \frac{w_l^2}{2}\right)\right| \leq C\bar{f}^{1-\frac{1}{k}} e^{2kw}(|\nabla w|^2 + 1).$$

Then, letting $\delta = \min\{\frac{1-t}{4(n-2)}(\gamma'' - \gamma'^2)\}$, we obtain

$$0 \geq F^{ii}\Big((\gamma'' - \gamma'^2)\frac{(1-t)w_s^2}{n-2} - \gamma' w_i^2\Big)\Big(1 + \frac{w_l^2}{2}\Big) - C\bar{f}^{1-\frac{1}{k}}e^{2kw}(|\nabla w|^2+1) - C\sum F^{ii}(|\nabla w|^2+1)$$
$$\geq C_4\bar{f}^{1-\frac{1}{k}}e^{2(k-1)w}\Big(\delta\frac{|\nabla w|^4}{2} - C(e^{2w}+1)(|\nabla w|^2+1)\Big). \tag{5-3}$$

To obtain (1-9), we divide the proof into two different cases.

(A) $\bar{f}(\tilde{x}) > 0$. Using (5-2) and (5-3), we have

$$\delta\frac{|\nabla w|^4}{2} - C(e^{2w}+1)(|\nabla w|^2+1) \leq 0.$$

Thus, (1-9) can be obtained.

(B) $\bar{f}(\tilde{x}) = 0$. Applying (5-3), we see that

$$0 \geq F^{ii}\Big((\gamma'' - \gamma'^2)\frac{(1-t)w_s^2}{n-2} - \gamma' w_i^2\Big)\Big(1 + \frac{w_l^2}{2}\Big) - C\sum F^{ii}(|\nabla w|^2+1)$$
$$\geq \sum F^{ii}\Big(\delta\frac{|\nabla w|^4}{2} - C(|\nabla w|^2+1)\Big). \tag{5-4}$$

In view of $\sum F^{ii} > 0$, (1-9) follows from (5-4).

Next, we discuss C^2 estimates of w. Let $H = \Delta w + a|\nabla w|^2$, and let $\check{x}$ be the maximum point of H, $a \geq \frac{2-t}{1-t}(2n-4)$. Since

$$|W_{ii}| \leq \sigma_1(W),$$

there exists C such that

$$|w_{ij}| \leq C\Delta w, \tag{5-5}$$

where C depends on n, k, g_0, $\|\nabla w\|_{C^0}$. Hence, we may assume $\Delta w \gg 1$ and

$$\tfrac{1}{2}\Delta w \leq \sigma_1 \leq 2\Delta w.$$

We may further assume $W_{ij}(\check{x})$ is diagonal. Differentiating H at $\check{x}$, we obtain

$$0 = H_i(\check{x}) = w_{qqi} + 2aw_{qi}w_q \tag{5-6}$$

and

$$0 \geq H_{ii}(\check{x}) = w_{qqii} + 2aw_{qii}w_q + 2aw_{qi}^2. \tag{5-7}$$

By (5-7), (5-5), (5-6), (2-2), (2-5) and (1-8), we have

$$
\begin{aligned}
0 \ge P^{ii} H_{ii}(\check{x}) &= P^{ii}(u_{qqii} + 2a w_{qii} w_q + 2a w_{qi}^2) \\
&\ge F^{ii}\biggl(-4a\frac{2-t}{2} w_q w_l w_{lq} + 4a w_q w_{iq} w_i \\
&\qquad\qquad + \frac{2a(1-t)}{n-2} w_{ql}^2 + W_{iiqq} - \Bigl(\frac{2-t}{2} w_l^2 - w_i w_i\Bigr)_{qq}\biggr) \\
&\qquad\qquad\qquad + 2a w_q (\bar{f} e^{2kw})_q - C\sum F^{ii}(\Delta w + 1) \\
&\ge F^{ii}\biggl(4a\Bigl(-\frac{2-t}{2} w_q w_l w_{lq} + w_q w_{iq} w_i\Bigr) + \frac{2a(1-t)}{n-2} w_{ql}^2 \\
&\qquad\qquad - 2\frac{2-t}{2} w_{lq}^2 + 2 w_{iq}^2 + 4a\Bigl(\frac{2-t}{2} w_l w_{lq} w_q - w_i w_{iq} w_q\Bigr)\biggr) \\
&\qquad\qquad\qquad + F^{ii} W_{iiqq} + 2a w_q (\bar{f} e^{2kw})_q - C\sum F^{ii}(\Delta w + 1) \\
&\ge F^{ii}\Bigl(\frac{2a(1-t)}{n-2} w_{ql}^2 - (2-t) w_{lq}^2\Bigr) + F^{ii} W_{iiqq} + 2a w_q (\bar{f} e^{2kw})_q \\
&\qquad\qquad\qquad - C\sum F^{ii}(\Delta w + 1).
\end{aligned}
$$

We also have

$$C\bar{f}^{1-\alpha} e^{2k(1-\alpha)w} (\Delta w)^\alpha \le \sigma_{k-1} = C\sum F^{ii}, \tag{5-8}$$

$$|\bar{f}_i| \le C\bar{f}^{1-\alpha} \le C e^{-2k(1-\alpha)w} \sum F^{ii}, \tag{5-9}$$

$$F^{ii} W_{iiqq} \ge -C e^{2kw} \bar{f}^{1-\alpha} - C e^{2k\alpha w} \sum F^{ii}(\Delta w + 1). \tag{5-10}$$

Using (5-8)–(5-10), and the definition of a,

$$
\begin{aligned}
0 &\ge F^{ii}\Bigl(\frac{2a(1-t)}{n-2} - (2-t)\Bigr) w_{lq}^2 + F^{ii} W_{iiqq} - C(1+e^{2k\alpha w})\sum F^{ii}(\Delta w + 1) \\
&\ge F^{ii}\Bigl(\frac{2a(1-t)}{n-2} - (2-t)\Bigr) w_{lq}^2 - C e^{2kw}\bar{f}^{1-\alpha} - C(1+e^{2k\alpha w})\sum F^{ii}(\Delta w + 1) \\
&\ge F^{ii}\Bigl(\frac{a(1-t)}{(n-2)n^2}\Bigr)(\Delta w)^2 - C e^{2kw}\bar{f}^{1-\alpha} - C(1+e^{2k\alpha w})\sum F^{ii}(\Delta w + 1) \\
&\ge F^{ii}\Bigl[\Bigl(\frac{a(1-t)}{(n-2)n^2}\Bigr)(\Delta w)^2 - C e^{2k\alpha w} - C(1+e^{2k\alpha w})(\Delta w + 1)\Bigr].
\end{aligned}
$$

Since $\sum F^{ii} > 0$, we have $\bigl(\frac{a(1-t)}{(n-2)n^2}\bigr)(\Delta w)^2 - C e^{2k\alpha w} - C(1+e^{2k\alpha w})(\Delta w + 1) \le 0$. Thus

$$\Delta w \le C(1 + e^{2k\alpha w}),$$

and (1-10) is proved.

We remark that in this section $t < 1$ is the necessary condition since the Δw yields the dominating term in (5-3) and the penultimate display.

6. Proof of the existence result

In this section, we focus on dealing with the existence result for (1-7). We need to discuss the maximum modulus estimation under the assumption that the supersolution exists.

6.1. ***Proof of Theorem 1.1.*** Consider

(6-1)
$$\sigma_k(sW + (1-s)g_0) = (s\bar{f} + (1-s)C_n^k)e^{2kw},$$

where

$$W = -A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2}\Delta w - dw \otimes dw + \frac{2-t}{2}|\nabla w|^2 g_0,$$
$$s \in [0,1], \ \bar{f} = (f^{\frac{1}{k-1}} + \epsilon)^{k-1}.$$

<u>Claim</u>: Equation (1-8) is solvable for each ϵ. In fact, define

(6-2)
$$\Phi_t[w] = \sigma_k(sW + (1-s)g_0) - (s\bar{f} + (1-s)C_n^k)e^{2kw}.$$

Note that the linearization of Φ_t is invertible, and $w = 0$ is a solution to (6-1) at $s = 0$. Thus, the problem is reduced to C^0, C^1, and C^2 estimates.

First, we consider C^0 estimates. Let $\bar{x}$ be the maximum point of w. From (6-1) we have

$$e^{2kw(\bar{x})} \le \frac{\max \sigma_k(-A_{g_0}^t + g_0)}{\min\{\bar{f}, C_n^k\}}.$$

Thus, it provides an upper bound for w. Let $\hat{x}$ be the minimum point of w. Similarly, from (6-1) we have

$$e^{2kw(\hat{x})} \ge \frac{\min \sigma_k(-sA_{g_0}^t + (1-s)g_0)}{\max \bar{f} + C_n^k}.$$

Thus, it provides a lower bound for w.

Then, using Theorem 1.1 in [8], we get C^1 and C^2 estimates for solutions to (6-1). This proves the claim.

Next, we shall prove that (1-7) is solvable. Let us denote the solution derived in the claim by w_ϵ. If $\overline{w}$ is the supersolution of (1-7), then it is the supersolution of (1-8) as well. Thus, by Lemma 2.7,

$$w_\epsilon \le \overline{w}.$$

Moreover, from the equation, we have

$$e^{2k \min w_\epsilon} \geq \frac{\min \sigma_k(-A^t_{g_0})}{(\max f^{\frac{1}{k-1}} + 1)^{k-1}}.$$

Then, by Theorem 1.2, we have $|\nabla w_\epsilon| \leq C$ and $|\nabla^2 w_\epsilon| \leq C$. Here, C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, g_0, n, k, $\overline{w}$. So, $\lim_{\epsilon\to 0} w_\epsilon$ is a solution to (1-6).

References

[1] S.-Y. A. Chang, M. J. Gursky, and P. Yang, "An a priori estimate for a fully nonlinear equation on four-manifolds", *J. Anal. Math.* **87** (2002), 151–186. MR Zbl

[2] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, "An equation of Monge–Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature", *Ann. of Math.* (2) **155**:3 (2002), 709–787. MR Zbl

[3] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, "A conformally invariant sphere theorem in four dimensions", *Publ. Math. Inst. Hautes Études Sci.* 98 (2003), 105–143. MR Zbl

[4] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, "Entire solutions of a fully nonlinear equation", pp. 43–60 in *Lectures on partial differential equations*, edited by S.-Y. A. Chang et al., New Stud. Adv. Math. **2**, International Press, Somerville, MA, 2003. MR Zbl

[5] S.-y. S. Chen, "Local estimates for some fully nonlinear elliptic equations", *Int. Math. Res. Not.* **2005**:55 (2005), 3403–3425. MR Zbl

[6] S.-y. S. Chen, "Boundary value problems for some fully nonlinear elliptic equations", *Calc. Var. Partial Differential Equations* **30**:1 (2007), 1–15. MR Zbl

[7] S.-y. S. Chen, "Conformal deformation on manifolds with boundary", *Geom. Funct. Anal.* **19**:4 (2009), 1029–1064. MR Zbl

[8] L. Chen, X. Guo, and Y. He, "A class of fully nonlinear equations arising in conformal geometry", *Int. Math. Res. Not.* **2022**:5 (2022), 3651–3676. MR Zbl

[9] H. Dong, "Hessian equations with elementary symmetric functions", *Comm. Partial Differential Equations* **31**:7-9 (2006), 1005–1025. MR Zbl

[10] Y. Ge and G. Wang, "On a fully nonlinear Yamabe problem", *Ann. Sci. École Norm. Sup.* (4) **39**:4 (2006), 569–598. MR Zbl

[11] Y. Ge and G. Wang, "On a conformal quotient equation", *Int. Math. Res. Not.* **2007**:6 (2007), art. id. 019. MR Zbl

[12] Y. Ge and G. Wang, "On a conformal quotient equation, II", *Comm. Anal. Geom.* **21**:1 (2013), 1–38. MR Zbl

[13] Y. Ge, C. S. Lin, and G. Wang, "On the σ_2-scalar curvature", *J. Differential Geom.* **84**:1 (2010), 45–86.

[14] P. Guan, "C^2 a priori estimates for degenerate Monge–Ampère equations", *Duke Math. J.* **86**:2 (1997), 323–346. MR Zbl

[15] B. Guan, "Conformal metrics with prescribed curvature functions on manifolds with boundary", *Amer. J. Math.* **129**:4 (2007), 915–942. MR Zbl

[16] P. Guan and Y. Y. Li, "The Weyl problem with nonnegative Gauss curvature", *J. Differential Geom.* **39**:2 (1994), 331–342. MR Zbl

[17] P. Guan and Y. Li, "$C^{1,1}$ estimates for solutions of a problem of Alexandrov", *Comm. Pure Appl. Math.* **50**:8 (1997), 789–811. MR Zbl

[18] P. Guan and G. Wang, "A fully nonlinear conformal flow on locally conformally flat manifolds", *J. Reine Angew. Math.* **557** (2003), 219–238. MR Zbl

[19] P. Guan and G. Wang, "Local estimates for a class of fully nonlinear equations arising from conformal geometry", *Int. Math. Res. Not.* **2003**:26 (2003), 1413–1432. MR Zbl

[20] P. Guan, N. S. Trudinger, and X.-J. Wang, "On the Dirichlet problem for degenerate Monge–Ampère equations", *Acta Math.* **182**:1 (1999), 87–104. MR Zbl

[21] P. Guan, J. Li, and Y. Li, "Hypersurfaces of prescribed curvature measure", *Duke Math. J.* **161**:10 (2012), 1927–1942. MR Zbl

[22] M. J. Gursky and J. A. Viaclovsky, "A fully nonlinear equation on four-manifolds with positive scalar curvature", *J. Differential Geom.* **63**:1 (2003), 131–154. MR Zbl

[23] M. J. Gursky and J. A. Viaclovsky, "Fully nonlinear equations on Riemannian manifolds with negative curvature", *Indiana Univ. Math. J.* **52**:2 (2003), 399–419. MR Zbl

[24] Y. He and W. Sheng, "Local estimates for elliptic equations arising in conformal geometry", *Int. Math. Res. Not.* **2013**:2 (2013), 258–290. MR Zbl

[25] F. Jiang and N. S. Trudinger, "Oblique boundary value problems for augmented Hessian equations, II", *Nonlinear Anal.* **154** (2017), 148–173. MR Zbl

[26] F. Jiang and N. S. Trudinger, "Oblique boundary value problems for augmented Hessian equations, I", *Bull. Math. Sci.* **8**:2 (2018), 353–411. MR Zbl

[27] F. Jiang and N. S. Trudinger, "Oblique boundary value problems for augmented Hessian equations, III", *Comm. Partial Differential Equations* **44**:8 (2019), 708–748. MR Zbl

[28] H. Jiao and Z. Wang, "The Dirichlet problem for degenerate curvature equations", *J. Funct. Anal.* **283**:1 (2022), art. id. 109485. MR Zbl

[29] H. Jiao and Z. Wang, "Second order estimates for convex solutions of degenerate k-Hessian equations", *J. Funct. Anal.* **286**:3 (2024), art. id. 110248. MR Zbl

[30] Q. Jin, A. Li, and Y. Y. Li, "Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary", *Calc. Var. Partial Differential Equations* **28**:4 (2007), 509–543. MR Zbl

[31] A. Li and Y. Li, "On some conformally invariant fully nonlinear equations", *Comm. Pure Appl. Math.* **56**:10 (2003), 1416–1464. MR Zbl

[32] A. Li and Y. Y. Li, "On some conformally invariant fully nonlinear equations, II: Liouville, Harnack and Yamabe", *Acta Math.* **195** (2005), 117–154. MR

[33] A. Li and Y. Y. Li, "A fully nonlinear version of the Yamabe problem on manifolds with boundary", *J. Eur. Math. Soc.* (*JEMS*) **8**:2 (2006), 295–316. MR Zbl

[34] Y. Li and L. Nguyen, "A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound", *J. Funct. Anal.* **266**:6 (2014), 3741–3771. MR Zbl

[35] J. Li and W. Sheng, "Deforming metrics with negative curvature by a fully nonlinear flow", *Calc. Var. Partial Differential Equations* **23**:1 (2005), 33–50. MR Zbl

[36] Q.-R. Li and W. Sheng, "Some Dirichlet problems arising from conformal geometry", *Pacific J. Math.* **251**:2 (2011), 337–359. MR Zbl

[37] W. Sheng and L.-X. Yuan, "A class of Neumann problems arising in conformal geometry", *Pacific J. Math.* **270**:1 (2014), 211–235. MR Zbl

[38] W.-M. Sheng, N. S. Trudinger, and X.-J. Wang, "The Yamabe problem for higher order curvatures", *J. Differential Geom.* **77**:3 (2007), 515–553. MR Zbl

[39] J. Spruck, “Geometric aspects of the theory of fully nonlinear elliptic equations”, pp. 283–309 in *Global theory of minimal surfaces*, edited by D. Hoffman, Clay Math. Proc. **2**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl

[40] N. S. Trudinger and X.-J. Wang, “On Harnack inequalities and singularities of admissible metrics in the Yamabe problem”, *Calc. Var. Partial Differential Equations* **35**:3 (2009), 317–338. MR Zbl

[41] J. A. Viaclovsky, “Conformal geometry, contact geometry, and the calculus of variations”, *Duke Math. J.* **101**:2 (2000), 283–316. MR Zbl

[42] J. A. Viaclovsky, “Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds”, *Comm. Anal. Geom.* **10**:4 (2002), 815–846. MR Zbl

[43] X.-J. Wang, “A priori estimates and existence for a class of fully nonlinear elliptic equations in conformal geometry”, *Chinese Ann. Math. Ser. B* **27**:2 (2006), 169–178. MR Zbl

Received March 18, 2023. Revised February 5, 2025.

YAN HE
FACULTY OF MATHEMATICS AND STATISTICS
HUBEI UNIVERSITY
WUHAN
CHINA
helenaig@hotmail.com

QIANG TU
FACULTY OF MATHEMATICS AND STATISTICS
HUBEI UNIVERSITY
WUHAN
CHINA
qiangtu@hubu.edu.cn

NI XIANG
FACULTY OF MATHEMATICS AND STATISTICS
HUBEI UNIVERSITY
WUHAN
CHINA
nixiang@hubu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 335 No. 1 March 2025
