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TRANSPOSED POISSON SUPERALGEBRA STRUCTURES ON TWISTED $N = 1$ BLOCK–LIE SUPERALGEBRA

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We investigate the transposed Poisson superalgebra structures on the twisted $N = 1$ Block–Lie superalgebra $\mathcal{S}(p, q)$, where p and q are arbitrary complex numbers. We obtain that $\mathcal{S}(p, q)$ admits only trivial transposed Poisson superalgebra structure for $q \neq 0$ or $p \notin \mathbb{Z}$, while $\mathcal{S}(p, q)$ has nontrivial transposed Poisson superalgebra structures for $q = 0$ and $p \in \mathbb{Z}$, which are nonisomorphic with respect to $p \in \mathbb{Z}$.

1. Introduction

Poisson algebras have their roots in Hamiltonian mechanics and have become a significant research topic in mathematics and physics. In 1809, D. Poisson introduced the operation of Poisson brackets on smooth functions while studying Lagrangian mechanics, thus giving smooth functions a Poisson structure. Thereafter, many researchers realized the importance of Poisson algebras and conducted extensive research from different perspectives. From string theory, quantum groups, and differential geometry, to integrable systems, algebraic geometry and representation theory, especially with the rise of noncommutative geometry, Poisson algebras have become an important branch of algebraic research (see [2; 4; 12]). More precisely, every Poisson algebra $(\mathcal{L}, \cdot, [\cdot, \cdot])$ satisfies the *Leibniz rule*:

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z] \quad \text{for } x, y, z \in \mathcal{L},$$

Transposed Poisson algebras (see [1]) are a generalization of Poisson algebras. In the definition of Poisson algebra, the Leibniz rule regards the Lie bracket operation as a derivation in an associative algebra. In the definition of transposed Poisson algebra, the transposed Leibniz rule treats the associative operation as a $\frac{1}{2}$-derivation in a Lie algebra. The transposed Leibniz rule, the compatibility condition of the transposed Poisson algebra $(\mathcal{L}, \cdot, [\cdot, \cdot])$, is articulated by

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y] \quad \text{for } x, y, z \in \mathcal{L}.$$

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Therefore, the transposed Poisson algebra can be viewed as a dual structure of a Poisson algebra. In addition, the connection between the $\frac{1}{2}$-derivation of a Lie algebra and the transposed Poisson algebra is established in [6]. By applying this connection, all the transposed Poisson algebra structures can be obtained on the Witt algebra, Virasoro algebra, twisted Heisenberg–Virasoro algebra, Schrödinger–Virasoro algebra and extended Schrödinger–Virasoro algebra (see [6; 22]). The definition of the transposed Poisson superalgebra is also provided in [6]. A method for determining transposed Poisson algebra structures via the Kantor product is presented in [5]. The transposed Hom-Poisson algebras and the transposed BiHom-Poisson algebras are considered in [11; 13].

The study of antiderivation is generalized by Filippov (see [7]), in which the primary focus is on the δ-derivations of prime Lie algebras with a nondegenerated symmetric invariant bilinear form. Specifically for a fixed element δ in the ground field, a linear map φ on a Lie algebra $(\mathcal{L}, [\cdot, \cdot])$ is called a *δ-derivation* of $\mathcal{L}$ if

$$\varphi([x, y]) = \delta\big([\varphi(x), y] + [x, \varphi(y)]\big) \quad \text{for } x, y \in \mathcal{L}.$$

The usual derivations can be viewed as 1-derivations. The centroid of a Lie algebra $\mathcal{L}$, $\mathrm{Cent}(\mathcal{L})$, is the space of linear maps χ on $\mathcal{L}$ satisfying $\chi([x, y]) = [\chi(x), y] = [x, \chi(y)]$ for all $x, y \in \mathcal{L}$. It is easy to see that elements of the centroid are $\frac{1}{2}$-derivations. If $\delta = 0$ and $[\mathcal{L}, \mathcal{L}] \neq 0$, then every nonzero linear map is called a *0-derivation*. If $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$, in particular, $\mathcal{L}$ is a simple Lie algebra, then $\mathcal{L}$ has no nonzero 0-derivations. If $\delta = -1$, then the linear map is called an *antiderivation*. The concept of the δ-superderivation on nonassociative superalgebras is introduced by Kaygorodov, and it is proved that simple finite-dimensional Lie superalgebras over an algebraically closed field of characteristic 0 do not have nontrivial δ-superderivations (see [8], [9]).

Block–Lie algebras are a class of infinite-dimensional simple Lie algebras introduced by Block in 1958 (see [3]). Since then, several generalizations of these algebras have been proposed (see [14; 15; 21]). The Block algebra $B(q)$ for a fixed complex number q is defined in [18]. The notion of the Block superalgebra $\mathfrak{K}(p)$ is introduced, and its finite-dimensional irreducible conformal modules are classified for any nonzero parameter p (see [17]). Based on the twisted rules from Ramond superalgebras to Neveu–Schwarz superalgebras, a twisted version of the $\mathbb{Z}$-graded conformal superalgebra $\mathfrak{T}(p)$ is introduced in [16], where p is a nonzero parameter. Precisely speaking, the subscripts of the odd generators of the original $\mathbb{Z}$-graded algebra are shifted by $\frac{1}{2}$. Motivated by [16; 19], the parameters of twisted $N = 1$ Block superalgebra are extended to include two nonzero parameters, p and q. Note that the special case of $p \in \mathbb{C}$ and $q = 1$ is considered in [16]. The nonweight modules over $N = 1$ Lie superalgebras of Block type is studied in [19]. In particular, transposed Poisson algebra structures on Block–Lie algebras $\mathcal{B}(q)$ and

Block–Lie superalgebras $\mathcal{S}(q)$ are described in [10]. This work propels us to delve into transposed Poisson superalgebra structures of the twisted $N = 1$ Block–Lie superalgebra $\mathcal{S}(p, q)$ with two parameters.

This paper is organized as follows. In Section 2, we recall some basic definitions and the relation between transposed Poisson superalgebra structures and $\frac{1}{2}$-superderivations. In Section 3, we characterize all $\frac{1}{2}$-superderivations of the twisted $N = 1$ Block–Lie superalgebra $\mathcal{S}(p, q)$. In Section 4, we present our main theorem about the transposed Poisson superalgebra structures on $\mathcal{S}(p, q)$; see Theorem 4.1. More precisely, we prove in Theorem 4.1 that there is only nontrivial transposed Poisson superalgebra structure for $q = 0$ and $p \in \mathbb{Z}$, otherwise, such structures are trivial. Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}^*$ the complex numbers, nonzero complex numbers, rational numbers, integers and nonzero integers, respectively.

2. Preliminaries

In this section, we recall some definitions and notation for future convenience.

Motivated by the definitions of the Block-type algebra and the $N = 1$ Lie superalgebra of Block type from [19; 20], we arrive at the subsequent definition.

Definition 2.1. Let p and q be fixed complex numbers. The *twisted* $N = 1$ *Block–Lie superalgebra* $\mathcal{S}(p, q)$ is defined as an infinite-dimensional Lie superalgebra over $\mathbb{C}$ with the even part $\mathcal{S}(p, q)_{\bar{0}} = \{L_{m,i} \mid m, i \in \mathbb{Z}\}$ and the odd part $\mathcal{S}(p, q)_{\bar{1}} = \{G_{m+\frac{1}{2},i+\frac{1}{2}} \mid m, i \in \mathbb{Z}\}$ together with the relations

$$\begin{aligned}
(2\text{-}1)\qquad [L_{m,i}, L_{n,j}] &= ((n+p)(i+q)-(j+q)(m+p))L_{m+n,i+j} \\
&= \begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} L_{m+n,i+j}, \\
[L_{m,i}, G_{n+\frac{1}{2},j+\frac{1}{2}}] &= \big((n+\tfrac{p}{2}+\tfrac{1}{2})(i+q)-\big(j+\tfrac{q}{2}+\tfrac{1}{2}\big)(m+p)\big)G_{m+n+\frac{1}{2},i+j+\frac{1}{2}} \\
&= \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}, \\
[G_{m+\frac{1}{2},i+\frac{1}{2}}, G_{n+\frac{1}{2},j+\frac{1}{2}}] &= 2qL_{m+n+1,i+j+1}
\end{aligned}$$

for $m, n, i, j \in \mathbb{Z}$.

Let $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. For $i \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, if $x \in \mathcal{L}_i$, then $|x|$ denotes the parity of x, that is, $|x| = 0$ if $x \in \mathcal{L}_{\bar{0}}$ or $|x| = 1$ if $x \in \mathcal{L}_{\bar{1}}$.

We will briefly recall some definitions from [10].

Definition 2.2. Let $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space equipped with two nonzero bilinear superoperations $\cdot$ and $[\cdot\,,\cdot]$. The triple $(\mathcal{L}, \cdot, [\cdot\,,\cdot])$ is called a *transposed Poisson superalgebra* if $(\mathcal{L}, \cdot\,)$ is a supercommutative associative superalgebra and $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ is a Lie superalgebra that satisfies the compatibility condition

$$2z\cdot[x, y]=[z\cdot x, y]+(-1)^{|x|\,|z|}[x, z\cdot y] \quad \text{for } x, z\in\mathcal{L}_i,\ y\in\mathcal{L},\ i\in\mathbb{Z}_2. \tag{2-2}$$

Definition 2.3. Let $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ be a Lie superalgebra. A transposed Poisson superalgebra structure on $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ is a supercommutative associative multiplication $\cdot$ on $\mathcal{L}$ which makes $(\mathcal{L}, \cdot\,, [\,\cdot\,,\cdot\,])$ a transposed Poisson superalgebra.

If φ is a homogeneous linear map $\mathcal{L}\to\mathcal{L}$, then $|\varphi|=0$ for $\varphi(\mathcal{L}_i)\subseteq\mathcal{L}_i$ or $|\varphi|=1$ for $\varphi(\mathcal{L}_i)\subseteq\mathcal{L}_{\bar{1}-i}$, $i\in\mathbb{Z}_2$.

Definition 2.4. Let $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ be a superalgebra and φ a homogeneous linear map $\mathcal{L}\to\mathcal{L}$. Then φ is called $\frac{1}{2}$*-superderivation* if it satisfies

$$\varphi([x, y])=\tfrac{1}{2}\big([\varphi(x), y]+(-1)^{|\varphi|\,|x|}[x, \varphi(y)]\big) \quad \text{for } x, y\in\mathcal{L}_i,\ i\in\mathbb{Z}_2. \tag{2-3}$$

We will use the notation $\Delta(\mathcal{L})$ for the space of $\frac{1}{2}$-superderivations of Lie superalgebra $\mathcal{L}$.

Let L_z denote the operator of the left multiplication by an element $z\in\mathcal{L}$, that is,

$$L_z(x)=z\cdot x \quad \text{for } x\in\mathcal{L}.$$

Definitions 2.2 and 2.4 immediately imply the following key lemma.

Lemma 2.5. *Let $(\mathcal{L}, \cdot\,, [\,\cdot\,,\cdot\,])$ be a transposed Poisson superalgebra and $z\in\mathcal{L}_i$, $i\in\mathbb{Z}_2$. Then the left multiplication L_z of $(\mathcal{L}, \cdot\,)$ is a $\frac{1}{2}$-superderivation of $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ and $|L_z|=|z|$.*

The basic example of a $\frac{1}{2}$-superderivation is the multiplication by a field element. Such $\frac{1}{2}$-superderivations will be called *trivial*.

Theorem 2.6. *Let $\mathcal{L}$ be a Lie superalgebra without nontrivial $\frac{1}{2}$-superderivations. Then all transposed Poisson superalgebra structures on $\mathcal{L}$ are trivial.*

Let $\cdot$ be a transposed Poisson superalgebra structure on a Lie superalgebra $(\mathcal{L}, [\,\cdot\,,\cdot\,])$. Then any automorphism ϕ of $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ induces another transposed Poisson superalgebra structure $*$ on $(\mathcal{L}, [\,\cdot\,,\cdot\,])$ given by

$$x*y=\phi(\phi^{-1}(x)\cdot\phi^{-1}(y)) \quad \text{for } x, y\in\mathcal{L}.$$

Clearly, ϕ is an isomorphism of transposed Poisson superalgebras $(\mathcal{L}, \cdot\,, [\,\cdot\,,\cdot\,])$ and $(\mathcal{L}, *, [\,\cdot\,,\cdot\,])$.

3. $\frac{1}{2}$-superderivations of the twisted $N=1$ Block–Lie superalgebra

In this section, we will investigate and describe all the $\frac{1}{2}$-superderivations of the twisted $N=1$ Block–Lie superalgebra $\mathcal{S}(p, q)$.

3.1. ***Even $\frac{1}{2}$-superderivations of $\mathcal{S}(p,q)$.*** In this subsection we consider only *even* linear maps $\varphi : \mathcal{S}(p,q) \to \mathcal{S}(p,q)$, i.e., those which satisfy $\varphi(\mathcal{S}(p,q)_i) \subseteq \mathcal{S}(p,q)_i$ for $i \in \mathbb{Z}_2$. We thus have $|\varphi| = 0$, so φ is a $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$ if and only if φ is a usual $\frac{1}{2}$-derivation of $\mathcal{S}(p,q)$. We now write

$$\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s},$$

where

$$\varphi_{r,s}(L_{m,i}) = d^{\bar{0}}_{r,s}(m,i) L_{m+r,i+s}, \tag{3-1}$$

$$\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}) = d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) G_{m+r+\frac{1}{2},i+s+\frac{1}{2}} \tag{3-2}$$

for some $d^{\bar{0}}_{r,s}(m,i)$, $d^{\bar{1}}_{r,s}\big(m+\frac{1}{2}, i+\frac{1}{2}\big) \in \mathbb{C}$, $m, i, r, s \in \mathbb{Z}$. Then we have $\varphi \in \Delta^{\bar{0}}(\mathcal{S}(p,q))$ if and only if $\varphi_{r,s} \in \Delta^{\bar{0}}(\mathcal{S}(p,q))$ for all $r, s \in \mathbb{Z}$.

Lemma 3.1. *Let $\varphi_{r,s} : \mathcal{S}(p,q) \to \mathcal{S}(p,q)$, $r, s \in \mathbb{Z}$, be a linear map satisfying* (3-1) *and* (3-2). *Then $\varphi_{r,s} \in \Delta^{\bar{0}}(\mathcal{S}(p,q))$ if and only if the these three conditions hold*:

$$2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r,s}(m+n, i+j) = \begin{vmatrix} n+p & j+q \\ m+r+p & i+s+q \end{vmatrix} d^{\bar{0}}_{r,s}(m,i) + \begin{vmatrix} n+r+p & j+s+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r,s}(n,j), \tag{3-3}$$

$$\begin{aligned} 2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) &= \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+r+p & i+s+q \end{vmatrix} d^{\bar{0}}_{r,s}(m,i) \\ &\quad + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big), \end{aligned} \tag{3-4}$$

$$2q d^{\bar{0}}_{r,s}(m+n+1, i+j+1) = q\Big(d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\Big). \tag{3-5}$$

Proof. By Definition 2.1, $\varphi_{r,s}$ is an even $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$ if and only if (2-3) hold on the basis $\{L_{m,i}, G_{m+\frac{1}{2},i+\frac{1}{2}} \mid m, i \in \mathbb{Z}\}$. Namely, the subsequent three equations are satisfied:

$$2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r,s}(L_{m+n,i+j}) = [\varphi_{r,s}(L_{m,i}), L_{n,j}] + [L_{m,i}, \varphi_{r,s}(L_{n,j})], \tag{3-6}$$

$$\begin{aligned} 2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r,s}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}) &= [\varphi_{r,s}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] \\ &\quad + [L_{m,i}, \varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})], \end{aligned} \tag{3-7}$$

$$\begin{aligned} 4q\varphi_{r,s}(L_{m+n+1,i+j+1}) &= [\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}] \\ &\quad + [G_{m+\frac{1}{2},i+\frac{1}{2}}, \varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})]. \end{aligned} \tag{3-8}$$

In view of (3-6), we can see that

$$\begin{aligned}2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r,s}(m+n,i+j)L_{m+n+r,i+j+s} &= 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r,s}(L_{m+n,i+j}) \\ &= [\varphi_{r,s}(L_{m,i}), L_{n,j}] + [L_{m,i}, \varphi_{r,s}(L_{n,j})] \\ &= [d^{\bar{0}}_{r,s}(m,i)L_{m+r,i+s}, L_{n,j}] + [L_{m,i}, d^{\bar{0}}_{r,s}(n,j)L_{n+r,j+s}] \\ &= \begin{vmatrix} n+p & j+q \\ m+r+p & i+s+q \end{vmatrix} d^{\bar{0}}_{r,s}(m,i)L_{m+n+r,i+j+s} \\ &\qquad + \begin{vmatrix} n+r+p & j+s+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r,s}(n,j)L_{m+n+r,i+j+s}.\end{aligned}$$

Thus, we come to (3-3). By (3-7), we observe

$$\begin{aligned}&2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big)G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} \\ &\quad= 2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r,s}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}) \\ &\quad= [\varphi_{r,s}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [L_{m,i}, \varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})] \\ &\quad= [d^{\bar{0}}_{r,s}(m,i)L_{m+r,i+s}, G_{n+\frac{1}{2},j+\frac{1}{2}}] + \big[L_{m,i}, d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)G_{n+r+\frac{1}{2},j+s+\frac{1}{2}}\big] \\ &\quad= \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+r+p & i+s+q \end{vmatrix} d^{\bar{0}}_{r,s}(m,i)G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} \\ &\qquad\qquad + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}}.\end{aligned}$$

Accordingly, we arrive at (3-4). With the use of (3-8), we have

$$\begin{aligned}&4q\,d^{\bar{0}}_{r,s}(m+n+1, i+j+1)L_{m+n+r+1,i+j+s+1} \\ &\qquad= 4q\,\varphi_{r,s}(L_{m+n+1,i+j+1}) \\ &\qquad= [\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [G_{m+\frac{1}{2},i+\frac{1}{2}}, \varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})] \\ &\qquad= \big[d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big)(G_{m+r+\frac{1}{2},i+s+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}\big] \\ &\qquad\qquad + \big[G_{m+\frac{1}{2},i+\frac{1}{2}}, d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)(G_{n+r+\frac{1}{2},j+s+\frac{1}{2}})\big] \\ &\qquad= 2q\big(d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\big)L_{m+n+r+1,i+j+s+1}.\end{aligned}$$

So, we get (3-5). □

Inspired by [10], we proceed to analyze the $\frac{1}{2}$-superderivatives of $\mathcal{S}(p,q)$, considering whether p and q are zero or not.

3.1.1. *The case $p, q \neq 0$.*

Lemma 3.2. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be an even $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$ and $(r,s) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. Then*

$$\varphi_{r,s} = 0.$$

Proof. Taking $n = j = 0$ in (3-3), we obtain

$$(p(i-s)-q(m-r))d^{\bar{0}}_{r,s}(m,i) = ((r+p)(i+q)-(s+q)(m+p))d^{\bar{0}}_{r,s}(0,0). \tag{3-9}$$

(1) For $\frac{p}{q} \in \mathbb{Q}$, there exists some $k \in \mathbb{C}^*$ satisfying $kp, kq \in \mathbb{Z}$. Putting $m = kp+r$ and $i = kq+s$ in (3-9), it follows that

$$(qr-ps)d^{\bar{0}}_{r,s}(0,0) = 0.$$

Clearly, we will discuss $qr - ps$ being zero or nonzero.

Case 1: $qr - ps \neq 0$. We have $d^{\bar{0}}_{r,s}(0,0) = 0$. And hence by (3-9), we can obtain

$$d^{\bar{0}}_{r,s}(m,i) = 0 \quad \text{if } i \neq \frac{q(m-r)}{p} + s. \tag{3-10}$$

If $\frac{q(m-r)}{p} + s \in \mathbb{Z}$, we set $i = \frac{q(m-r)}{p} + s$, $n = -m$, $j = -i$. Since

$$j = -i = -\frac{q(m-r)}{p} - s = \frac{q(n+r)}{p} - s \neq \frac{q(n-r)}{p} + s,$$

we can apply (3-10). It follows that

$$d^{\bar{0}}_{r,s}(n,j) = d^{\bar{0}}_{r,s}(-m,-i) = 0.$$

Substituting it into (3-3), we get

$$(3p+r-m)d^{\bar{0}}_{r,s}\left(m, \frac{q(m-r)}{p} + s\right) = 0.$$

Therefore, it leads to

$$d^{\bar{0}}_{r,s}\left(m, \frac{q(m-r)}{p} + s\right) = 0 \quad \text{if } m \neq 3p+r.$$

Taking $m = 3p+r$, $i = q(m-r)/p + s = 3q+s$ and $n \notin \{0, 3p+r\}$ in (3-3), for $d^{\bar{0}}_{r,s}(m+n, i+j) = 0$ and $d^{\bar{0}}_{r,s}(n,j) = 0$, we obtain

$$((n+p)(2q+s)-(j+q)(2p+r))d^{\bar{0}}_{r,s}(3p+r, 3q+s) = 0. \tag{3-11}$$

If $r+2p\neq 0$, then we take $j\neq \frac{(n+p)(2q+s)}{r+2p}-q$. So, we obtain

$$d^{\bar{0}}_{r,s}(3p+r,3q+s)=0.$$

If $r+2p=0$, it follows from (3-11) that

$$(n+p)(2q+s)d^{\bar{0}}_{-2p,s}(3p+r,3q+s)=0.$$

Due to $qr-ps\neq 0$ and $r=-2p$, then we have $2q+s\neq 0$. It only needs to take $n\neq -p$, then we get

$$d^{\bar{0}}_{-2p,s}(p,3q+s)=0.$$

Case 2: $qr-ps=0$. Since $\frac{p}{q}=\frac{r}{s}\in\mathbb{Q}$, there exists $t\in\mathbb{C}^*$ satisfying $tp,tq\in\mathbb{Z}$. Taking $r=tp$, $s=tq$ in (3-3), we have

$$2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{tp,tq}(m+n,i+j)=\begin{vmatrix} n+p & j+q \\ m+(t+1)p & i+(t+1)q \end{vmatrix} d^{\bar{0}}_{tp,tq}(m,i) + \begin{vmatrix} n+(t+1)p & j+(t+1)q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{tp,tq}(n,j). \tag{3-12}$$

Considering $n=j=0$, $m\neq 0$ in (3-12), we get

$$(pi-qm)d^{\bar{0}}_{tp,tq}(m,i)=(t+1)(pi-qm)d^{\bar{0}}_{tp,tq}(0,0).$$

Clearly, our discussion will proceed based on whether $pi-qm$ is zero or not.

Subcase 1: $pi-qm\neq 0$. We have

$$d^{\bar{0}}_{tp,tq}(m,i)=(t+1)d^{\bar{0}}_{tp,tq}(0,0). \tag{3-13}$$

Taking $m=-n\neq 0$, $j=-i$ and $pi+qn\neq 0$ in (3-12), we obtain

$$4d^{\bar{0}}_{tp,tq}(0,0)=(t+2)d^{\bar{0}}_{tp,tq}(-n,i)+(t+2)d^{\bar{0}}_{tp,tq}(n,-i). \tag{3-14}$$

Then applying (3-13), we have $d^{\bar{0}}_{tp,tq}(n,-i)=d^{\bar{0}}_{tp,tq}(-n,i)=(t+1)d^{\bar{0}}_{tp,tq}(0,0)$. As a result, (3-14) can be simplified as

$$(t+3)d^{\bar{0}}_{tp,tq}(0,0)=0.$$

If $t\neq -3$, then we arrive at $d^{\bar{0}}_{tp,tq}(0,0)=0$. And hence by (3-13), it follows that

$$d^{\bar{0}}_{tp,tq}(m,i)=0.$$

If $t=-3$, by (3-13), we derive

$$d^{\bar{0}}_{-3p,-3q}(m,i)=-2d^{\bar{0}}_{-3p,-3q}(0,0). \tag{3-15}$$

Letting $i = j = 0$ and $m,\ n,\ m \pm n \neq 0$ in (3-12), using (3-15), we are able to conclude that

$$2(n-m)d^{\bar{0}}_{-3p,-3q}(m+n,0) = -(m+2n)d^{\bar{0}}_{-3p,-3q}(m,0)+(2m+n)d^{\bar{0}}_{-3p,-3q}(-n,0),$$
$$d^{\bar{0}}_{-3p,-3q}(0,0) = 0.$$

Substituting it into (3-15), we see that

$$d^{\bar{0}}_{-3p,-3q}(m,i) = 0.$$

Hence, we can deduce that

$$d^{\bar{0}}_{tp,tq}(m,i) = 0 \quad \text{for } pi - qm \neq 0. \tag{3-16}$$

Subcase 2: $pi - qm = 0$. Since $\frac{p}{q} = \frac{m}{i} \in \mathbb{Q}$, there exists $l \in \mathbb{C}^*$ satisfying $lp, lq \in \mathbb{Z}$. By substituting $m = j = 0$, $n = lp$ and $i = lq$ into (3-12), since $pi - qm \neq 0$, $pj - qn \neq 0$ and (3-16), we arrive at $d^{\bar{0}}_{tp,tq}(lp,0) = d^{\bar{0}}_{tp,tq}(0,lq) = 0$. So (3-12) can be written as

$$(l+2)d^{\bar{0}}_{tp,tq}(lp,lq) = 0.$$

Then, it yields

$$d^{\bar{0}}_{tp,tq}(lp,lq) = 0, \text{ if } l \neq -2.$$

If $l = -2$, then taking $m = -2p$, $n = 2p$, $i = -2q$ and $j = 0$ in (3-12) and considering (3-16), it follows that

$$(t-1)pq d^{\bar{0}}_{tp,tq}(-2p,-2q) = 0.$$

Since $p, q \neq 0$, then $d^{\bar{0}}_{tp,tq}(-2p,-2q) = 0$ for $t \neq 1$ is derived.

Therefore, we can conclude that $d^{\bar{0}}_{r,s}(m,i) = 0$ for all $(m,i) \in \mathbb{Z}\times\mathbb{Z}$, $(r,s) \notin \{(0,0),(p,q)\}$ and $d^{\bar{0}}_{p,q}(m,i) = 0$ unless $(m,i) = (-2p,-2q)$.

(2) For $\frac{p}{q} \notin \mathbb{Q}$, we can obtain, from (3-9),

$$d^{\bar{0}}_{r,s}(m,i) = \frac{(r+p)(i+q)-(s+q)(m+p)}{p(i-s)-q(m-r)} d^{\bar{0}}_{r,s}(0,0) \quad \text{for } (m,i) \neq (r,s). \tag{3-17}$$

In the following analysis, we will proceed by the conditions whether r and s are zero or not.

Case 1: $r, s \neq 0$. Taking $m = -r$ and $j = -s$ into (3-9), we can see that

$$d^{\bar{0}}_{r,s}(-r,-s) = d^{\bar{0}}_{r,s}(0,0).$$

Given the conditions $m = 0$, $n = -r$, $i = -s$ and $j = 0$ in (3-3), it follows that

$$2((p-r)(q-s)-pq)d^{\bar{0}}_{r,s}(-r,-s) = -2qr d^{\bar{0}}_{r,s}(0,-s) - 2ps d^{\bar{0}}_{r,s}(-r,0).$$

By combining

$$(qr-2ps)d^{\bar{0}}_{r,s}(0,-s)=(qr-2ps-rs)d^{\bar{0}}_{r,s}(0,0),$$
$$(ps-2qr)d^{\bar{0}}_{r,s}(-r,0)=(ps-2qr-rs)d^{\bar{0}}_{r,s}(0,0),$$

we can deduce that

$$d^{\bar{0}}_{r,s}(0,0)=0.$$

So it leads to

$$d^{\bar{0}}_{r,s}(m,i)=0 \quad \text{for } (m,i)\neq(r,s).$$

Taking $m=r$, $i=s$, $j\notin\{0,s\}$ and $n\notin\{0,r,2r\}$ into (3-3), we can get

$$((n+p)(2s+q)-(j+q)(2r+p))d^{\bar{0}}_{r,s}(r,s)=0.$$

Since $\frac{p}{q}\neq\mathbb{Q}$, we have $d^{\bar{0}}_{r,s}(r,s)=0$.

Case 2: $r=0$, $s\neq 0$. In this case, (3-17) becomes

$$d^{\bar{0}}_{0,s}(m,i)=\frac{p(i+q)-(s+q)(m+p)}{p(i-s)-qr}d^{\bar{0}}_{0,s}(0,0) \quad \text{for } (m,i)\neq(0,s). \tag{3-18}$$

Substituting $i=s$ and $m\neq 0$ in (3-18), we have

$$d^{\bar{0}}_{0,s}(m,s)=\frac{s+q}{q}d^{\bar{0}}_{0,s}(0,0) \quad \text{for } m\neq 0.$$

Putting $m=0$ and $i=2s$ into (3-18), we obtain

$$d^{\bar{0}}_{0,s}(0,2s)=d^{\bar{0}}_{0,s}(0,0).$$

Furthermore, considering $m+n=0$, $m,n\neq 0$ and $i=j=0$ in (3-3), we can deduce that

$$4n(s+q)d^{\bar{0}}_{0,s}(0,0)=((n+p)(2s+q)-(-n+p)(s+q))d^{\bar{0}}_{0,s}(m,s)$$
$$+((n+p)(s+q)-(-n+p)(2s+q))d^{\bar{0}}_{0,s}(n,s).$$

Therefore, it can be checked that

$$d^{\bar{0}}_{0,s}(m,i)=0 \quad \text{for } (m,i)\neq(0,s).$$

Taking $m=0$, $i=s$, $j\notin\{0,s\}$ and $n\neq 0$ into (3-3), we can get

$$((n+p)(2s+q)-p(j+q))d^{\bar{0}}_{0,s}(0,s)=0.$$

Since $\frac{p}{q}\neq\mathbb{Q}$, we have $d^{\bar{0}}_{0,s}(0,s)=0$.

Case 3: $r \neq 0$, $s = 0$. Similarly, it can be shown that $d^{\bar{0}}_{r,0}(m, i) = 0$ for $(m, i) \in \mathbb{Z}\times\mathbb{Z}$.

Subsequently, we consider the value of $d^{\bar{1}}_{r,s}\big(m+\frac{1}{2}, i+\frac{1}{2}\big)$. If $(r, s) \notin \{(0, 0), (p, q)\}$, (3-5) becomes

$$d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0.$$

Taking $m = n$ and $i = j$, we get

$$d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (m, i) \in \mathbb{Z}\times\mathbb{Z} \text{ and } (r, s) \notin \{(0, 0), (p, q)\}.$$

If $(r, s) = (p, q)$, it implies $p, q \in \mathbb{Z}$. Then taking $m = n$ and $i = j$ in (3-5), we get

$$d^{\bar{1}}_{p,q}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = d^{\bar{0}}_{r,s}(2m+1, 2i+1) = 0 \quad \text{for all } (m, i) \in \mathbb{Z}\times\mathbb{Z}.$$

Considering $m = 0$, $i = 0$, $n = -2p-1$ and $j = -2q-1$ in (3-5), we get

$$2d^{\bar{0}}_{p,q}(-2p, -2q) = d^{\bar{1}}_{p,q}\big(\tfrac{1}{2}, \tfrac{1}{2}\big) + d^{\bar{1}}_{p,q}\big(-2p-\tfrac{1}{2}, -2q-\tfrac{1}{2}\big) = 0.$$

So we obtain $d^{\bar{0}}_{r,s}(m, i) = 0$ for all $(m, i) \in \mathbb{Z}\times\mathbb{Z}$ and $(r, s) \neq (0, 0)$. □

Lemma 3.3. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, q)$ and $(r, s) = (0, 0)$. Then*

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{1}}_{0,0}\big(m'+\tfrac{1}{2}, i'+\tfrac{1}{2}\big)$$

for all $(m, i), (m', i') \in \mathbb{Z}\times\mathbb{Z}$.

Proof. Writing (3-3) with $(r, s) = (0, 0)$, we have

$$((n+p)(i+q) - (m+p)(j+q)) \times (2d^{\bar{0}}_{0,0}(m+n, i+j) - d^{\bar{0}}_{0,0}(m, i) - d^{\bar{0}}_{0,0}(n, j)) = 0. \tag{3-19}$$

Taking $n = j = 0$ in (3-19), we obtain

$$(pi - qm)(d^{\bar{0}}_{0,0}(m, i) - d^{\bar{0}}_{0,0}(0, 0)) = 0.$$

Therefore, we derive

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{0}}_{0,0}(0, 0) \quad \text{if } pi - qm \neq 0. \tag{3-20}$$

Now, if $pi - qm = 0$, then we can assume $m = kp$ and $i = kq$, $k \in \mathbb{C}^*$. We choose $pj - qn \neq 0$ in (3-19), it becomes

$$(k+1)((n+p)q - (j+q)p) \times (2d^{\bar{0}}_{0,0}(kp+n, kq+j) - d^{\bar{0}}_{0,0}(kp, kq) - d^{\bar{0}}_{0,0}(n, j)) = 0. \tag{3-21}$$

Since $p(kq+j) - q(kp+n) \neq 0$, by (3-20) we obtain

$$d^{\bar{0}}_{0,0}(kp+n, kq+j) = d^{\bar{0}}_{0,0}(n, j) = d^{\bar{0}}_{0,0}(0, 0).$$

Substituting it into (3-21), we have

$$(k+1)(d^{\bar{0}}_{0,0}(kp,kq)-d^{\bar{0}}_{0,0}(0,0))=0.$$

Hence, it follows that

$$d^{\bar{0}}_{0,0}(kp,kq)=d^{\bar{0}}_{0,0}(0,0)\quad\text{if } k\neq -1.$$

We substitute $m=0$, $n=-p$, $i=-q$ and $j=0$ into (3-19). By (3-20), it leads to

$$d^{\bar{0}}_{0,0}(-p,-q)=d^{\bar{0}}_{0,0}(0,0). \tag{3-22}$$

Consequently, we can conclude that $d^{\bar{0}}_{0,0}(m,i)=d^{\bar{0}}_{0,0}(0,0)$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$.

Therefore, (3-5) becomes

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)+d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)=2d^{\bar{0}}_{0,0}(0,0).$$

Taking $m=n$ and $i=j$, we get

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(0,0)\quad\text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}. \tag{3-23}$$

Combining with (3-22) and (3-23), we conclude that $\varphi_{0,0}=d^{\bar{0}}_{0,0}(0,0)id$. □

From Lemmas 3.2 and 3.3, the subsequent proposition is directly derived.

Proposition 3.4. *Let $p,q\in\mathbb{C}^*$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(p,q))=\langle id\rangle.$$

3.1.2. *The case $p=0$, $q\neq 0$.*

Lemma 3.5. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,q)$, $r\in\mathbb{Z}^*$ and $s\in\mathbb{Z}$. Then*

$$\varphi_{r,s}=0.$$

Proof. According to Lemma 2.3 from [10], we can obtain $d^{\bar{0}}_{r,s}(m,i)=0$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$. So, (3-5) becomes

$$d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)+d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)=0.$$

Taking $m=n$ and $i=j$, we get $d^{\bar{1}}_{r,s}\big(m+\frac{1}{2},i+\frac{1}{2}\big)=0$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$. □

Lemma 3.6. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,q)$, $r=0$ and $s\in\mathbb{Z}^*$. Then*

$$\varphi_{0,s}=0.$$

Proof. If $s \neq q$, then we can obtain that $d^{\bar{0}}_{0,s}(m, i) = 0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$ is established based on Lemma 2.4 in [10]. Hence, (3-5) becomes

$$d^{\bar{1}}_{0,s}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) + d^{\bar{1}}_{0,s}\big(n + \tfrac{1}{2}, j + \tfrac{1}{2}\big) = 0.$$

Taking $m = n$ and $i = j$, we get

$$d^{\bar{1}}_{0,s}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) = 0 \quad \text{for all } (m, i) \in \mathbb{Z} \times \mathbb{Z}.$$

If $s = q$, then we arrive at $d^{\bar{0}}_{0,q}(m, i) = 0$ unless $(m, i) = (0, -2q)$, which has been proved in Lemma 2.4 of [10]. Therefore (3-5) becomes

$$d^{\bar{1}}_{0,q}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big)+d^{\bar{1}}_{0,q}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)=0 \quad \text{for } (m+n+1, i+j+1) \neq (0, -2q), \tag{3-24}$$

$$d^{\bar{1}}_{0,q}\big(-n-\tfrac{1}{2}, -j-2q-\tfrac{1}{2}\big)+d^{\bar{1}}_{0,q}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)=2d^{\bar{0}}_{0,q}(0, -2q). \tag{3-25}$$

By setting $m = n = i = j$ into (3-24) and observing $(m + n + 1, i + j + 1) = (2m + 1, 2i + 1) \neq (0, -2q)$, we deduce, from (3-24),

$$d^{\bar{1}}_{0,q}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) = 0 \quad \text{for all } (m, i) \in \mathbb{Z} \times \mathbb{Z}.$$

In the end, with the use of (3-25), we can obtain $d^{\bar{0}}_{0,q}(0, -2q) = 0$. □

Lemma 3.7. *Let $\varphi = \sum_{r,s \in \mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0, q)$ and $(r, s) = (0, 0)$. Then*

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{1}}_{0,0}\big(m' + \tfrac{1}{2}, i' + \tfrac{1}{2}\big)$$

for all $(m, i), (m', i') \in \mathbb{Z} \times \mathbb{Z}$.

Proof. As stated in Lemma 2.7 of [10], we can deduce $d^{\bar{0}}_{0,0}(m, i) = d^{\bar{0}}_{0,0}(0, 0)$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. Then, (3-5) becomes

$$d^{\bar{1}}_{0,0}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) + d^{\bar{1}}_{0,0}\big(n + \tfrac{1}{2}, j + \tfrac{1}{2}\big) = 2d^{\bar{0}}_{0,0}(0, 0).$$

Taking $m = n = i = j$, we have

$$d^{\bar{1}}_{0,0}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) = d^{\bar{0}}_{0,0}(0, 0)$$

for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. □

Lemmas 3.5, 3.6 and 3.7 jointly yield the following proposition.

Proposition 3.8. *Let $p = 0$ and $q \in \mathbb{C}^*$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(0, q)) = \langle id \rangle.$$

3.1.3. *The case* $p \neq 0$, $q = 0$. For this case, (3-3) and (3-4) are expressed as, respectively,

$$2\begin{vmatrix} n+p & j \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r,s}(m+n, i+j) = \begin{vmatrix} n+p & j \\ m+r+p & i+s \end{vmatrix} d^{\bar{0}}_{r,s}(m, i) + \begin{vmatrix} n+r+p & j+s \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r,s}(n, j), \tag{3-26}$$

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+r+p & i+s \end{vmatrix} d^{\bar{0}}_{r,s}(m, i) + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big). \tag{3-27}$$

Lemma 3.9. *Let* $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ *be a* $\frac{1}{2}$*-derivation of* $\mathcal{S}(p, 0)$, $r \in \mathbb{Z}$ *and* $s \in \mathbb{Z}^*$. *Then*

$$\varphi_{r,s} = 0.$$

Proof. Taking $n = j = 0$ in (3-26), we can see that

$$p(i-s)d^{\bar{0}}_{r,s}(m, i) = (i(r+p) - s(m+p))d^{\bar{0}}_{r,s}(0, 0). \tag{3-28}$$

Letting $i = s$ and $m \neq r$ in (3-28), we obtain

$$d^{\bar{0}}_{r,s}(0, 0) = 0.$$

Furthermore, substituting it into (3-28), it leads to

$$(i-s)d^{\bar{0}}_{r,s}(m, i) = 0 \quad \text{and} \quad d^{\bar{0}}_{r,s}(m, i) = 0 \qquad \text{for } i \neq s.$$

Putting $i = s$, $j \notin \{0, s\}$ and $n \neq \frac{j(m+r+p)}{2s} - p$ in (3-26), then we have

$$d^{\bar{0}}_{r,s}(m+n, i+j) = 0 \quad \text{and} \quad d^{\bar{0}}_{r,s}(n, j) = 0.$$

Hence we obtain

$$(2s(n+p) - j(m+r+p))d^{\bar{0}}_{r,s}(m, s) = 0 \quad \text{and} \quad d^{\bar{0}}_{r,s}(m, s) = 0.$$

Therefore, we derive

$$d^{\bar{0}}_{r,s}(m, i) = 0 \quad \text{for all } (m, i) \in \mathbb{Z} \times \mathbb{Z}. \tag{3-29}$$

Taking into account both (3-27) and (3-29), then we infer

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big).$$

Substituting $m=i=0$, we get

$$\big(j-s+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\, j+\tfrac{1}{2}\big)=0.$$

Since $j, s \in \mathbb{Z}$, then we have $j-s+\frac{1}{2}\neq 0$. Hence it follows that

$$d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\, j+\tfrac{1}{2}\big)=0$$

for all $(n, j)\in\mathbb{Z}\times\mathbb{Z}$. □

Lemma 3.10. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p,0)$, $r\in\mathbb{Z}^*$ and $s=0$.*

(1) *If $r\neq p$, then $\varphi_{r,0}=0$.*

(2) *If $r=p$, then $d^{\bar{0}}_{p,0}(m,i)=0$ for all $(m,i)\neq(-2p,0)$ and $d^{\bar{1}}_{p,0}\big(m+\frac{1}{2}, i+\frac{1}{2}\big)=0$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$.*

Proof. With $s=0$, (3-26) can be expressed as

$$2\begin{vmatrix} n+p & j\\ m+p & i\end{vmatrix} d^{\bar{0}}_{r,0}(m+n, i+j) = \begin{vmatrix} n+p & j\\ m+r+p & i\end{vmatrix} d^{\bar{0}}_{r,0}(m,i)+\begin{vmatrix} n+r+p & j\\ m+p & i\end{vmatrix} d^{\bar{0}}_{r,0}(n,j). \tag{3-30}$$

Taking $n=j=0$ in (3-30), we get

$$pid^{\bar{0}}_{r,0}(m,i)=(p+r)id^{\bar{0}}_{r,0}(0,0).$$

Thus, we have

$$d^{\bar{0}}_{r,0}(m,i)=(1+rp^{-1})d^{\bar{0}}_{r,0}(0,0) \text{ for } i\neq 0. \tag{3-31}$$

We proceed with a classification discussion based on whether or not $p\in\mathbb{Z}$.

Case 1: $p\in\mathbb{Z}$. Choosing $m=-n$ and $j=-i\neq 0$ in (3-30) and applying (3-31), then we obtain

$$\begin{aligned} 4pd^{\bar{0}}_{r,0}(0,0) &= (r+2p)(d^{\bar{0}}_{r,0}(-n,i)+d^{\bar{0}}_{r,0}(n,-i))\\ &= 2(r+2p)(1+rp^{-1})d^{\bar{0}}_{r,0}(0,0),\\ r(r+3p)d^{\bar{0}}_{r,0}(0,0) &= 0. \end{aligned}$$

Thus, we get

$$d^{\bar{0}}_{r,0}(0,0)=0 \quad \text{for } r\neq -3p.$$

We take $r=-3p$ and use (3-31). Hence it follows that

$$d^{\bar{0}}_{r,0}(m,i)=-2d^{\bar{0}}_{r,0}(0,0) \quad \text{for } i\neq 0. \tag{3-32}$$

Putting $m=n=0$, i, j, $i\pm j\neq 0$ in (3-30) and applying (3-32), then we can derive

$$\begin{aligned}2p(i-j)d^{\bar{0}}_{r,0}(0,i+j)&=p(i+2j)d^{\bar{0}}_{r,0}(0,i)-p(2i+j)d^{\bar{0}}_{r,0}(0,j),\\ 2p(i-j)d^{\bar{0}}_{r,0}(0,0)&=p(i+2j)d^{\bar{0}}_{r,0}(0,0)-p(2i+j)d^{\bar{0}}_{r,0}(0,0),\\ d^{\bar{0}}_{r,0}(0,0)&=0.\end{aligned}$$

Case 2: $p\notin\mathbb{Z}$. Observing that $1+rp^{-1}\neq 0$ in this case, we take $m=n=0$ in (3-30), then we can obtain

$$2p(i-j)d^{\bar{0}}_{r,0}(0,i+j)=(p(i-j)-rj)d^{\bar{0}}_{r,0}(0,i)+(p(i-j)+ri)d^{\bar{0}}_{r,0}(0,j).$$

We choose i, j, $i\pm j\neq 0$. Then owing to (3-31) and $1+rp^{-1}\neq 0$, we have

$$d^{\bar{0}}_{r,0}(0,0)=0.$$

Hence by (3-31), we can deduce that

$$d^{\bar{0}}_{r,0}(m,i)=0\quad\text{for } i\neq 0. \tag{3-33}$$

It remains to analyze $d^{\bar{0}}_{r,0}(m,0)$. To this end, putting $m=0$ and $j=-i\neq 0$ in (3-30) and applying (3-33), hence we have

$$\begin{aligned}2(2p+n)id^{\bar{0}}_{r,0}(n,0)&=(2p+n+r)i(d^{\bar{0}}_{r,0}(0,i)+d^{\bar{0}}_{r,0}(n,-i)),\\ (2p+n)d^{\bar{0}}_{r,0}(n,0)&=0.\end{aligned}$$

Therefore, we can conclude

$$d^{\bar{0}}_{r,0}(n,0)=0\quad\text{for } n\neq -2p.$$

Taking $m=-2p$, $i=0$ and $j\neq 0$ in (3-30) and using (3-33), it follows that

$$\begin{aligned}2pjd^{\bar{0}}_{r,0}(-2p+n,j)&=(p-r)jd^{\bar{0}}_{r,0}(-2p,0)+pjd^{\bar{0}}_{r,0}(n,j),\\ (p-r)d^{\bar{0}}_{r,0}(-2p,0)&=0.\end{aligned}$$

So, $d^{\bar{0}}_{r,0}(-2p,0)=0$ for $r\neq p$ is derived.

If $r\neq p$, then (3-27) becomes

$$2\begin{vmatrix} n+\frac{p}{2}+\frac12 & j+\frac12\\ m+p & i\end{vmatrix} d^{\bar{1}}_{r,0}\big(m+n+\tfrac12, i+j+\tfrac12\big) = \begin{vmatrix} n+r+\frac{p}{2}+\frac12 & j+\frac12\\ m+p & i\end{vmatrix} d^{\bar{1}}_{r,0}\big(n+\tfrac12, j+\tfrac12\big). \tag{3-34}$$

By taking $m=i=0$ into (3-34), we get

$$\big(j+\tfrac12\big)d^{\bar{1}}_{r,0}\big(n+\tfrac12, j+\tfrac12\big)=0.$$

Since $j \in \mathbb{Z}$, then $j+\frac{1}{2} \neq 0$ is inferred. Furthermore, we have

$$d^{\bar{1}}_{r,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)=0 \quad \text{for all } (n, j) \in \mathbb{Z}\times\mathbb{Z} \text{ and } r \neq p.$$

If $r=p$, the term in (3-27) that includes $d^{\bar{0}}_{p,0}(m,i)$ is

$$\big((n+\tfrac{p}{2}+\tfrac{1}{2})i-(j+\tfrac{1}{2})(m+2p)\big)d^{\bar{0}}_{p,0}(m,i)=0 \quad \text{for } (m,i)\neq(-2p,0),$$
$$\big(j+\tfrac{1}{2}\big)(-2p+p+p)d^{\bar{0}}_{p,0}(-2p,0)=0 \quad \text{for } (m,i)=(-2p,0).$$

Therefore, the term with $d^{\bar{0}}_{p,0}(m,i)$ in (3-27) vanishes. Equation (3-27) becomes

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{p,0}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+\frac{3p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{p,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big). \tag{3-35}$$

Taking $m=i=0$ into (3-35), so we obtain

$$d^{\bar{1}}_{p,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)=0$$

for all $(n, j) \in \mathbb{Z}\times\mathbb{Z}$. □

According to this conclusion, we can get the following lemma.

Lemma 3.11. *Let $p \in \mathbb{Z}^*$. Then the linear map $\alpha : \mathcal{S}(p,0) \to \mathcal{S}(p,0)$ is a $\frac{1}{2}$-derivation of $\mathcal{S}(p,0)$ such that*

$$\alpha(L_{m,i})=\begin{cases} 0, & (m,i)\neq(-2p,0), \\ L_{-p,0}, & (m,i)=(-2p,0), \end{cases} \qquad \alpha(G_{m+\frac{1}{2},i+\frac{1}{2}})=0$$

for $(m,i) \in \mathbb{Z}\times\mathbb{Z}$.

Proof. We observe that $\alpha=\sum_{r,s\in\mathbb{Z}}\alpha_{r,s}=\alpha_{p,0}$. In view of Lemma 3.1 we need to check (3-3) and (3-4) for $(r,s)=(p,0)$ and

$$d^{\bar{0}}_{p,0}(m,i)=\begin{cases} 0, & (m,i)\neq(-2p,0), \\ 1, & (m,i)=(-2p,0), \end{cases} \tag{3-36}$$

$$d^{\bar{1}}_{p,0}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big)=0, \quad (m,i)\in\mathbb{Z}\times\mathbb{Z}. \tag{3-37}$$

Firstly, we prove that (3-3) holds.

Case 1: $(m,i), (n,j), (m+n,i+j)\neq(-2p,0)$. Then both sides of (3-3) are zero.

Case 2: $(m,i)=(-2p,0)$. Then (3-3) becomes

$$2pjd^{\bar{0}}_{p,0}(-2p+n, j)=pjd^{\bar{0}}_{p,0}(n,j).$$

If $j=0$, then it is trivially satisfied, otherwise both sides are zero by (3-36).

Case 3: $(n, j) = (-2p, 0)$. Then (3-3) becomes

$$2pid^{\bar{0}}_{p,0}(-2p+m, i) = pid^{\bar{0}}_{p,0}(m, i),$$

so this case is similar to Case 2.

Case 4: $(m+n, i+j) = (-2p, 0)$. Then (3-3) becomes

$$0 = pid^{\bar{0}}_{p,0}(-2p-n, i) + pid^{\bar{0}}_{p,0}(n, -i),$$

and again this holds by (3-36).

Next, we proceed to prove that (3-4) holds. Due to (3-37), (3-4) becomes

$$\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+2p & i \end{vmatrix} d^{\bar{0}}_{p,0}(m, i) = 0.$$

Given (3-36), it holds. □

The corollary is an immediate consequence of Lemmas 3.9 and 3.11.

Corollary 3.12. *Let φ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, 0)$, $r, s \in \mathbb{Z}$.*

(1) *If $p \notin \mathbb{Z}$, then $\varphi_{r,s} = 0$ for $(r, s) \neq (0, 0)$.*

(2) *If $p \in \mathbb{Z}$, then $\varphi_{r,s} = 0$ for all $(r, s) \notin \{(0, 0), (p, 0)\}$ and $\varphi_{p,0} \in \langle\alpha\rangle$.*

Finally we proceed to prove the case of $r = s = 0$.

Lemma 3.13. *Let φ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, 0)$ and $r = s = 0$. Then*

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{1}}_{0,0}\big(m' + \tfrac{1}{2}, i' + \tfrac{1}{2}\big)$$

for all $(m, i), (m', i') \in \mathbb{Z} \times \mathbb{Z}$.

Proof. In this situation, (3-3) and (3-4) are presented as, respectively,

$$(i(n+p) - j(m+p))(2d^{\bar{0}}_{0,0}(m+n, i+j) - d^{\bar{0}}_{0,0}(m, i) - d^{\bar{0}}_{0,0}(n, j) = 0, \tag{3-38}$$

$$\begin{aligned}&\big(i\big(n+\tfrac{p}{2}+\tfrac{1}{2}\big) - \big(j+\tfrac{1}{2}\big)(m+p)\big) \\ &\quad\times\big(2d^{\bar{1}}_{0,0}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) - d^{\bar{0}}_{0,0}(m, i) - d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\big) = 0.\end{aligned} \tag{3-39}$$

Taking $n = j = 0$ and $i \neq 0$ in (3-38), we obtain

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{0}}_{0,0}(0, 0) \quad \text{for } i \neq 0. \tag{3-40}$$

We choose $j \neq 0$ and $n = -m$ in (3-38), then we arrive at

$$(m+p)(2d^{\bar{0}}_{0,0}(0, j) - d^{\bar{0}}_{0,0}(m, 0) - d^{\bar{0}}_{0,0}(-m, j)) = 0.$$

Due to (3-40), it follows that

$$d^{\bar{0}}_{0,0}(m, 0) = d^{\bar{0}}_{0,0}(0, 0) \quad \text{for } m \neq -p.$$

Substituting $m=-p$, $n=0$ and $j=-i\neq 0$ into (3-38) and using (3-40), we can derive that

$$2d^{\bar 0}_{0,0}(-p,0)-d^{\bar 0}_{0,0}(-p,i)-d^{\bar 0}_{0,0}(0,-i)=0,$$
$$d^{\bar 0}_{0,0}(-p,0)=d^{\bar 0}_{0,0}(0,0).$$

Taking $m=i=0$ in (3-39), we can see that

$$p\big(j+\tfrac12\big)\big(2d^{\bar 1}_{0,0}\big(n+\tfrac12,j+\tfrac12\big)-d^{\bar 0}_{0,0}(0,0)-d^{\bar 1}_{0,0}\big(n+\tfrac12,j+\tfrac12\big)\big)=0.$$

Since $p\neq 0$ and $j\in\mathbb{Z}$, then we have $p\big(j+\frac12\big)\neq 0$. Furthermore, we are able to conclude that

$$d^{\bar 1}_{0,0}\big(n+\tfrac12,j+\tfrac12\big)=d^{\bar 0}_{0,0}(0,0)$$

for all $(n,j)\in\mathbb{Z}\times\mathbb{Z}$. □

Directly from Lemmas 3.9, 3.13 and Corollary 3.12, we deduce the following proposition.

Proposition 3.14. *Let $p\in\mathbb{C}^*$ and $q=0$. Then*

$$\Delta^{\bar 0}(\mathcal{S}(p,0))=\begin{cases}\langle id\rangle, & p\notin\mathbb{Z},\\ \langle id,\alpha\rangle, & p\in\mathbb{Z}^*,\end{cases}$$

where α is as in Lemma 3.11.

3.1.4. *The case $p=q=0$.* Rewriting (3-4) with $p=q=0$, we obtain

$$2\begin{vmatrix}n+\frac12 & j+\frac12\\ m & i\end{vmatrix}d^{\bar 1}_{r,s}\big(m+n+\tfrac12,i+j+\tfrac12\big)=\begin{vmatrix}n+\frac12 & j+\frac12\\ m+r & i+s\end{vmatrix}d^{\bar 0}_{r,s}(m,i)+\begin{vmatrix}n+r+\frac12 & j+s+\frac12\\ m & i\end{vmatrix}d^{\bar 1}_{r,s}\big(n+\tfrac12,j+\tfrac12\big). \tag{3-41}$$

Lemma 3.15. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac12$-derivation of $\mathcal{S}(0,0)$ and let $(r,s)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. Then*

$$\varphi_{r,s}=0.$$

Proof. $d^{\bar 0}_{r,s}(m,i)=0$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$ is derived from Lemma 2.9 in [10]. Then, (3-41) becomes

$$2\begin{vmatrix}n+\frac12 & j+\frac12\\ m & i\end{vmatrix}d^{\bar 1}_{r,s}\big(m+n+\tfrac12,i+j+\tfrac12\big)=\begin{vmatrix}n+r+\frac12 & j+s+\frac12\\ m & i\end{vmatrix}d^{\bar 1}_{r,s}\big(n+\tfrac12,j+\tfrac12\big). \tag{3-42}$$

Taking $m = j = 0$ in (3-42), we obtain

$$2i\big(n+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = i\big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big),$$

$$2\big(n+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = \big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) \quad \text{for } i \neq 0. \tag{3-43}$$

It implies that $d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, i+\tfrac{1}{2}\big)$ is independent of i for $i \neq 0$. Substitution $m = 0$, $j = -i \neq 0$ into (3-42), it leads to

$$\begin{aligned}(2n+1)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) &= \big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, -i+\tfrac{1}{2}\big)\\ &= \big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, i+\tfrac{1}{2}\big).\end{aligned} \tag{3-44}$$

Multiplying (3-44) by $2n+1$ and using (3-43), we get

$$(2n+1)^2 d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) = \big(n+r+\tfrac{1}{2}\big)^2 d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big).$$

Assuming $d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) \neq 0$, we obtain $(2n+1)^2 = \big(n+r+\tfrac{1}{2}\big)^2$, whence $2n+1 = \pm\big(n+r+\tfrac{1}{2}\big)$. It follows that $n = r - \tfrac{1}{2}$ or $n = -\tfrac{r}{3} - \tfrac{1}{2}$, which contradicts $n \in \mathbb{Z}$.

Then utilizing (3-44) and $n+r+\tfrac{1}{2} \neq 0$, we have

$$d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) = \frac{2n+1}{n+r+\tfrac{1}{2}} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) = 0$$

for all $(n, i) \in \mathbb{Z} \times \mathbb{Z}$. □

Lemma 3.16. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be a $\tfrac{1}{2}$-derivation of $\mathcal{S}(0,0)$ and $(r,s) = (0,0)$. Then*

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = d^{\bar{0}}_{0,0}(m', i')$$

for $(m,i) \in \mathbb{Z}\times\mathbb{Z}$, $(m', i') \in \mathbb{Z}\times\mathbb{Z} \setminus \{(0,0)\}$.

Proof. According to Lemma 2.10 from [10], we can obtain

$$d^{\bar{0}}_{0,0}(m,i) = d^{\bar{0}}_{0,0}(m', i') \quad \text{for all } (m,i), (m', i') \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}. \tag{3-45}$$

By substituting $(r,s) = (0,0)$ into (3-41), we get

$$\begin{vmatrix} n+\frac{1}{2} & j+\frac{1}{2} \\ m & i \end{vmatrix} \big(2d^{\bar{1}}_{0,0}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) - d^{\bar{0}}_{0,0}(m,i) - d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\big) = 0. \tag{3-46}$$

We take $m = j = 0$ and $i \neq 0$ in (3-46). And since $n + \tfrac{1}{2} \neq 0$, we obtain

$$2d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = d^{\bar{0}}_{0,0}(0,i) + d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, \tfrac{1}{2}\big) \quad \text{for } i \neq 0. \tag{3-47}$$

Letting $n = i = 0$ and $m \neq 0$ in (3-46), and because of $j + \tfrac{1}{2} \neq 0$, we have

$$2d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = d^{\bar{0}}_{0,0}(m,0) + d^{\bar{1}}_{0,0}\big(12, j+\tfrac{1}{2}\big) \quad \text{for } m \neq 0.$$

Hence, due to (3-45), we arrive at

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)\quad\text{for } n,i\neq 0. \tag{3-48}$$

Substituting $m=-n\neq 0$, $i\neq 0$ and $j=0$ into (3-46), it leads to

$$\big(ni+\tfrac{1}{2}(n+i)\big)\big(2d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)-d^{\bar{0}}_{0,0}(-n,i)-d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)\big)=0.$$

If $(n,i)\notin\{(0,0),(-1,-1)\}$, then we have $ni+\frac{1}{2}(n+i)\neq 0$. Therefore, it follows that

$$2d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(-n,i)+d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } (n,i)\neq(0,0),(-1,-1). \tag{3-49}$$

Now, combining (3-47), (3-48) and (3-49), we get

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(-n,i)\quad\text{for } n,i\neq 0 \text{ and } (n,i)\neq(-1,-1),$$

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(-n,i)=d^{\bar{0}}_{0,0}(n',i')$$
$$\text{for } (n,i)\neq(0,0),(-1,-1) \text{ and } (n',i')\neq(0,0).$$

In particular, taking $n=i=-1$ in (3-47), we have

$$2d^{\bar{1}}_{0,0}\big(-\tfrac{1}{2},-\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(0,-1)+d^{\bar{1}}_{0,0}\big(-\tfrac{1}{2},\tfrac{1}{2}\big)=2d^{\bar{0}}_{0,0}(0,-1)=2d^{\bar{0}}_{0,0}(n',i')$$

for $(n',i')\neq(0,0)$. Therefore, we can conclude

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(m',i')\quad\text{for } (m,i),(m',i')\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\} \tag{3-50}$$

Taking $n=j=0$ in (3-46), we can deduce

$$(i-m)\big(2d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)-d^{\bar{0}}_{0,0}(m,i)-d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\big)=0. \tag{3-51}$$

If $m\neq i$, then we get $(m,i)\neq(0,0)$. Consequently, it follows that $d^{\bar{0}}_{0,0}(m,i)=d^{\bar{1}}_{0,0}\big(m+\frac{1}{2},i+\frac{1}{2}\big)$. Hence, from (3-51) we obtain that

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } m\neq i.$$

Using (3-50), we have

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } (m,i)\in\mathbb{Z}\times\mathbb{Z}.\qquad\square$$

According to this conclusion, we are able to deduce the subsequent lemma.

Lemma 3.17. *Let $p=q=0$. Then the linear map $\beta:\mathcal{S}(0,0)\to\mathcal{S}(0,0)$ is a $\frac{1}{2}$-derivation of $\mathcal{S}(0,0)$ such that*

$$\beta(L_{m,i})=\begin{cases}0, & (m,i)\neq(0,0),\\ L_{0,0}, & (m,i)=(0,0),\end{cases}\qquad \beta(G_{m+\frac{1}{2},i+\frac{1}{2}})=0$$

for $(m,i)\in\mathbb{Z}\times\mathbb{Z}$.

Proof. We observe that $\beta = \sum_{r,s\in\mathbb{Z}} \beta_{r,s} = \beta_{0,0}$. In view of Lemma 3.1, we need to check (3-3) and (3-4) for $(r, s) = (0, 0)$ and

$$d^{\bar{0}}_{0,0}(m, i) = \begin{cases} 0, & (m, i) \neq (0, 0), \\ 1, & (m, i) = (0, 0), \end{cases} \tag{3-52}$$

$$d^{\bar{1}}_{0,0}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) = 0 \tag{3-53}$$

for $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.

Firstly, we prove (3-3). In fact, we can split this proof into four cases:

(1) $(m, i), (n, j), (m + n, i + j) = (0, 0)$,

(2) $(m, i) = (0, 0)$,

(3) $(n, j) = (0, 0)$,

(4) $(m + n, i + j) \neq (0, 0)$.

Obviously, under all these different cases, (3-3) is trivially satisfied.

Subsequently, we prove (3-4). Due to (3-53), (3-4) becomes

$$\begin{vmatrix} n+\frac{1}{2} & j+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{0}}_{0,0}(m, i) = 0.$$

It clearly holds by (3-52). □

With Lemmas 3.15 and 3.17, we can directly infer the subsequent proposition.

Proposition 3.18. *Let $p = q = 0$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(0, 0)) = \langle id, \beta\rangle,$$

where β is as in Lemma 3.17.

By integrating Propositions 3.4, 3.8, 3.14 and 3.18, we can deduce the following corollary.

Corollary 3.19. *Let $p, q \in \mathbb{C}$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(p, q)) = \begin{cases} \langle id\rangle, & q \neq 0, \\ \langle id, \alpha\rangle, & p \neq 0, q = 0, \\ \langle id, \beta\rangle, & p = q = 0. \end{cases}$$

where α and β are as in Lemmas 3.17 and 3.11.

3.2. ***Odd $\frac{1}{2}$-derivations of $\mathcal{S}(p, q)$.*** In this subsection we consider that a linear map $\varphi : \mathcal{S}(p, q) \to \mathcal{S}(p, q)$ is *odd*, if $\varphi(\mathcal{S}(p, q)_i) \subseteq \mathcal{S}(p, q)_{\bar{1}-i}$ for $i \in \mathbb{Z}_2$. In this case $|\varphi| = 1$, so φ is a $\frac{1}{2}$-superderivation of $\mathcal{S}(p, q)$ if and only if

$$\varphi([x, y]) = \tfrac{1}{2}\big([\varphi(x), y] + [x, \varphi(y)]\big), \quad x \in \mathcal{S}(p, q)_{\bar{0}},$$
$$\varphi([x, y]) = \tfrac{1}{2}\big([\varphi(x), y] - [x, \varphi(y)]\big), \quad x \in \mathcal{S}(p, q)_{\bar{1}}.$$

Denote by $\Delta^{\bar{1}}(\mathcal{S}(p,q))$ the space of odd $\frac{1}{2}$-superderivations of $\mathcal{S}(p,q)$. As usual, for any $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$, we write

$$\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r+\frac{1}{2},s+\frac{1}{2}},$$

where

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}) = d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i)G_{m+r+\frac{1}{2},i+s+\frac{1}{2}}, \tag{3-54}$$

$$\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}) = d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big)L_{m+r+1,i+s+1} \tag{3-55}$$

for some $d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i)$, $d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\frac{1}{2}, i+\frac{1}{2}\big) \in \mathbb{C}$, $m, i, r, s \in \mathbb{Z}$. We have

$$\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p,q)) \quad \text{if and only if} \quad \varphi_{r+\frac{1}{2},s+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q)) \ \text{ for all } r, s \in \mathbb{Z}.$$

Lemma 3.20. *Let*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} : \mathcal{S}(p,q) \to \mathcal{S}(p,q)$$

be a linear map satisfying (3-54) *and* (3-55). *Then*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$$

if and only if the following three conditions hold:

$$\begin{aligned} 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n, i+j) &= \begin{vmatrix} n+p & j+q \\ m+r+\frac{p}{2}+\frac{1}{2} & i+s+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) \\ &\quad + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j), \end{aligned} \tag{3-56}$$

$$\begin{aligned} 2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) \\ = 2q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) + \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big), \end{aligned} \tag{3-57}$$

$$\begin{aligned} 4q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1, i+j+1) &= \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+r+p+1 & i+s+q+1 \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) \\ &\quad - \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big). \end{aligned} \tag{3-58}$$

Proof. By Definition 2.1, $\varphi_{r+\frac{1}{2},s+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$ if and only if the following three expressions are valid:

$$(3\text{-}59)\quad 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n,i+j}) = [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), L_{n,j}] + [L_{m,i} + \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{n,j})],$$

$$(3\text{-}60)\quad 2\begin{vmatrix} n+\frac{1}{2}+\frac{p}{2} & j+\frac{1}{2}+\frac{q}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}) = [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [L_{m,i}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})],$$

$$(3\text{-}61)\quad 4q\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n+1,i+j+1}) = [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}] - [G_{m+\frac{1}{2},i+\frac{1}{2}}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})].$$

Due to (3-59), we can see that

$$\begin{aligned}
2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} & d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n, i+j) G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} \\
&= 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n,i+j}) \\
&= [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), L_{n,j}] + [L_{m,i}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{n,j})] \\
&= [d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) G_{m+\frac{1}{2},i+\frac{1}{2}}, L_{n,j}] + [L_{m,i}, d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j) G_{n+r+\frac{1}{2},j+r+\frac{1}{2}}] \\
&= \begin{vmatrix} n+p & j+q \\ m+r+\frac{p}{2}+\frac{1}{2} & i+s+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} \\
&\qquad + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j) G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}}.
\end{aligned}$$

Thus, (3-56) is obtained. By (3-60), we have

$$\begin{aligned}
2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} & d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) L_{m+n+r+1,i+j+s+1} \\
&= 2\begin{vmatrix} n+\frac{1}{2}+\frac{p}{2} & j+\frac{1}{2}+\frac{q}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}) \\
&= [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [L_{m,i}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})] \\
&= [d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) G_{m+\frac{1}{2},i+\frac{1}{2}}, G_{n+\frac{1}{2},j+\frac{1}{2}}] \\
&\qquad\qquad + [L_{m,i}, d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) L_{n+r+1,j+s+1}] \\
&= 2q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) L_{m+n+r+1,i+j+s+1} \\
&\qquad + \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) L_{m+n+r+1,i+j+s+1}.
\end{aligned}$$

Consequently, we come to (3-57). Because of (3-61), we observe

$$
\begin{aligned}
4qd^{\bar 0}_{r+\frac12,s+\frac12}&(m+n+1,i+j+1)G_{m+n+r+\frac32,i+j+s+\frac32}\\
&=4q\varphi_{r+\frac12,s+\frac12}(L_{m+n+1,i+j+1})\\
&=[\varphi_{r+\frac12,s+\frac12}(G_{m+\frac12,i+\frac12}),G_{n+\frac12,j+\frac12}]-[G_{m+\frac12,i+\frac12},\varphi_{r+\frac12,s+\frac12}(G_{n+\frac12,j+\frac12})]\\
&=[d^{\bar 1}_{r+\frac12,s+\frac12}(G_{m+\frac12,i+\frac12}),G_{n+\frac12,j+\frac12}]+[G_{m+\frac12,i+\frac12},d^{\bar 1}_{r+\frac12,s+\frac12}(G_{n+\frac12,j+\frac12})]\\
&=\begin{vmatrix} n+\frac{p}{2}+\frac12 & j+\frac{q}{2}+\frac12\\ m+r+p+1 & i+s+q+1\end{vmatrix} d^{\bar 1}_{r+\frac12,s+\frac12}\big(m+\tfrac12,i+\tfrac12\big)G_{m+n+r+\frac32,i+j+s+\frac32}\\
&\qquad-\begin{vmatrix} n+r+p+1 & j+s+q+1\\ m+\frac{p}{2}+\frac12 & i+\frac{q}{2}+\frac12\end{vmatrix} d^{\bar 1}_{r+\frac12,s+\frac12}\big(n+\tfrac12,j+\tfrac12\big)G_{m+n+r+\frac32,i+j+s+\frac32}.
\end{aligned}
$$

Hence, it follows that (3-58) holds truly. □

Let $\varphi\in\Delta^{\bar 1}(\mathcal{S}(p,q))$ and $\psi_{m+\frac12,i+\frac12}$ be the left multiplication by $G_{m+\frac12,i+\frac12}$ in $\mathcal{S}(p,q)$, $m,i,r,s\in\mathbb{Z}$. Since $|\psi_{m+\frac12,i+\frac12}|=1$ and we have, for all $x,y\in\{L_{n,j},G_{n+\frac12,j+\frac12}\mid n,j\in\mathbb{Z}\}$,

$$
\begin{aligned}
\psi_{m+\frac12,i+\frac12}([x,y])&=[G_{m+\frac12,i+\frac12},[x,y]]\\
&=[[G_{m+\frac12,i+\frac12},x],y]+(-1)^{|x|}[x,[G_{m+\frac12,i+\frac12},y]]\\
&=[\psi_{m+\frac12,i+\frac12}(x),y]+(-1)^{|x|}[x,\psi_{m+\frac12,i+\frac12}(y)],
\end{aligned}
$$

$\psi_{m+\frac12,i+\frac12}$ is an odd superderivation of $\mathcal{S}(p,q)$.

Since the supercommutator

$$[\varphi_{r+\frac12,s+\frac12},\psi_{m+\frac12,i+\frac12}]=\varphi_{r+\frac12,s+\frac12}\psi_{m+\frac12,i+\frac12}+\psi_{m+\frac12,i+\frac12}\varphi_{r+\frac12,s+\frac12}$$

satisfies

$$
\begin{aligned}
&[\varphi_{r+\frac12,s+\frac12},\psi_{m+\frac12,i+\frac12}](L_{n,j})\\
&\quad=\varphi_{r+\frac12,s+\frac12}([G_{m+\frac12,i+\frac12},L_{n,j}])+[G_{m+\frac12,i+\frac12},\varphi_{r+\frac12,s+\frac12}(L_{n,j})]\\
&\quad=\begin{vmatrix} n+p & j+q\\ m+\frac{p}{2}+\frac12 & i+\frac{q}{2}+\frac12\end{vmatrix} d^{\bar 1}_{r+\frac12,s+\frac12}\big(m+n+\tfrac12,i+j+\tfrac12\big)L_{m+n+r+1,i+j+s+1}\\
&\qquad\qquad+2qd^{\bar 1}_{r+\frac12,s+\frac12}(n,j)L_{m+n+r+1,i+j+s+1},
\end{aligned}\tag{3-62}
$$

$$
\begin{aligned}
&[\varphi_{r+\frac12,s+\frac12},\psi_{m+\frac12,i+\frac12}](G_{n+\frac12,j+\frac12})\\
&\quad=\varphi_{r+\frac12,s+\frac12}([G_{m+\frac12,i+\frac12},G_{n+\frac12,j+\frac12}])+[G_{m+\frac12,i+\frac12},\varphi_{r+\frac12,s+\frac12}(G_{n+\frac12,j+\frac12})]\\
&\quad=2qd^{\bar 0}_{r+\frac12,s+\frac12}(m+n+1,i+j+1)G_{m+n+r+\frac32,i+j+s+\frac32}\\
&\qquad+\begin{vmatrix} n+r+p+1 & j+s+q+1\\ m+\frac{p}{2}+\frac12 & i+\frac{q}{2}+\frac12\end{vmatrix} d^{\bar 1}_{r+\frac12,s+\frac12}\big(n+\tfrac12,j+\tfrac12\big)G_{m+n+r+\frac32,i+j+s+\frac32},
\end{aligned}\tag{3-63}
$$

for all $x, y \in \{L_{n,j}, G_{n+\frac{1}{2},j+\frac{1}{2}} \mid n, j \in \mathbb{Z}\}$, it follows that

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}]([x, y]) = \varphi_{r+\frac{1}{2},s+\frac{1}{2}}\big(\psi_{m+\frac{1}{2},i+\frac{1}{2}}([x, y])\big) + \psi_{m+\frac{1}{2},i+\frac{1}{2}}\big(\varphi_{r+\frac{1}{2},s+\frac{1}{2}}([x, y])\big) = \tfrac{1}{2}\big([[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](x), y] + [x, [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](y)]\big).$$

Therefore, we can conclude that $[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}]$ is an even $\frac{1}{2}$-superderivation of $\mathcal{S}(p, q)$, which was given in Corollary 3.19.

So, if $q \neq 0$, then we arrive at

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ cL_{n,j}, & (m+r+1, i+s+1) = (0,0), \end{cases} \tag{3-64}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ cL_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-65}$$

for some constant $c \in \mathbb{C}$.

If $q = 0$ and $p \notin \mathbb{Z}$, then we get

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_1 L_{n,j}, & (m+r+1, i+s+1) = (0,0), \end{cases} \tag{3-66}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_1 L_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-67}$$

for some constant $c_1 \in \mathbb{C}$.

If $q = 0$ and $p \in \mathbb{Z}^*$, then we have

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), (0, p), \\ c_2 L_{n,j}, & (m+r+1, i+s+1) = (0,0), \\ 0, & (m+r+1, i+s+1) = (0, p) \text{ and } (n, j) \neq (-2p, 0), \\ c_3 L_{-p,0}, & (m+r+1, i+s+1) = (0, p) \text{ and } (n, j) = (-2p, 0), \end{cases} \tag{3-68}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_2 L_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-69}$$

for some constants $c_2, c_3 \in \mathbb{C}$.

If $p = q = 0$, then we obtain

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_4 L_{n,j}, & (m+r+1, i+s+1) = (0,0) \text{ and } (n, j) \neq (0,0), \\ c_5 L_{n,j}, & (m+r+1, i+s+1) = (0,0) \text{ and } (n, j) = (0,0), \end{cases} \tag{3-70}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_4 L_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-71}$$

for some constants $c_4, c_5 \in \mathbb{C}$.

3.2.1. *The case* $p, q \neq 0$. By (3-62)–(3-65), we have

$$\begin{vmatrix} n+p & j+q \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) + 2qd^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(n, j) = 0, \tag{3-72}$$

$$2qd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1, i+j+1) + \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \tag{3-73}$$

for all $(m, i) \neq (-r-1, -s-1)$.

Lemma 3.21. *Let* $r, s \in \mathbb{Z}$, *and* $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p, q))$.

(1) *If* $p, q \in 2\mathbb{Z}+1$, *then*

$$\begin{aligned} \varphi_{r+\frac{1}{2},s+\frac{1}{2}} &= 0 \quad \textit{for } (r, s) \neq \big(\tfrac{p}{2}-\tfrac{1}{2}, \tfrac{q}{2}-\tfrac{1}{2}\big), \\ \varphi_{\frac{p}{2},\frac{q}{2}}(L_{m,i}) &= 0 \quad \textit{for all } (m, i) \in \mathbb{Z}\times\mathbb{Z}, \\ \varphi_{\frac{p}{2},\frac{q}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}) &= 0 \quad \textit{for } (m, i) \neq \big(-\tfrac{3p}{2}-\tfrac{1}{2}, -\tfrac{3q}{2}-\tfrac{1}{2}\big). \end{aligned}$$

(2) *Otherwise*, $\varphi_{r+\frac{1}{2},s+\frac{1}{2}} = 0$.

Proof. Taking $m = i = 0$ in (3-56), we obtain

$$(q(n-r) - p(j-s))d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 2qd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0, 0). \tag{3-74}$$

Then taking $n = r$ and $j = s$ in (3-74), it gives $d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0, 0) = 0$. Hence, (3-74) can be given by

$$(p(i-s)-q(m-r))d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (r, s), (m, i) \in \mathbb{Z}\times\mathbb{Z}. \tag{3-75}$$

Taking $n = j = 0$ in (3-72), it yields

$$\big(p\big(i+\tfrac{1}{2}\big) - q\big(m+\tfrac{1}{2}\big)\big)d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \tag{3-76}$$

for all $(r, s) \in \mathbb{Z}\times\mathbb{Z}$ and $(m, i) \neq (-r-1, -s-1)$. When $(m, i) = (-r-1, -s-1)$, (3-75) can be written

$$\big(p\big(s+\tfrac{1}{2}\big) - q\big(r+\tfrac{1}{2}\big)\big)d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(-r-\tfrac{1}{2}, -s-\tfrac{1}{2}\big) = 0.$$

In addition, subtracting (3-75) from (3-76), we get

$$\big(p\big(s+\tfrac{1}{2}\big) - q\big(r+\tfrac{1}{2}\big)\big)d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0$$

for all $(r, s), (m, i) \in \mathbb{Z}\times\mathbb{Z}$. Clearly, we will analyze whether $p\big(s+\frac{1}{2}\big) - q\big(r+\frac{1}{2}\big)$ is zero or not.

Case 1: $p\big(s+\frac{1}{2}\big)-q\big(r+\frac{1}{2}\big)\neq 0$. We have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0 \quad \text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}.$$

So (3-58) becomes

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1,i+j+1)=0 \quad \text{for all } (m,i),(n,j)\in\mathbb{Z}\times\mathbb{Z}.$$

Letting $u=m+n+1$ and $v=i+j+1$, we obtain

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(u,v)=0 \quad \text{for all } (u,v)\in\mathbb{Z}\times\mathbb{Z}.$$

Case 2: $p\big(s+\frac{1}{2}\big)-q\big(r+\frac{1}{2}\big)=0$. There exists some $k\in\mathbb{C}$ satisfying $r=kp-\frac{1}{2}$, $s=kq-\frac{1}{2}\in\mathbb{Z}$. Now taking into (3-75), we have

$$\big(p\big(i+\tfrac{1}{2}\big)-q\big(m+\tfrac{1}{2}\big)\big)d^{\bar{1}}_{kp,kq}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0.$$

Accordingly, it follows that

$$d^{\bar{1}}_{kp,kq}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0 \quad \text{for } (m,i)\in\mathbb{Z}\times\mathbb{Z} \text{ and } \frac{m+\frac{1}{2}}{i+\frac{1}{2}}\neq\frac{p}{q}. \tag{3-77}$$

Letting $u=m+n+1$ and $v=i+j+1$ with $\frac{m+\frac{1}{2}}{i+\frac{1}{2}}\neq\frac{p}{q}$ and $\frac{n+\frac{1}{2}}{j+\frac{1}{2}}\neq\frac{p}{q}$, we can obtain, by (3-58) and (3-77),

$$d^{\bar{0}}_{kp,kq}(u,v)=0 \quad \text{for all } (u,v)\in\mathbb{Z}\times\mathbb{Z}.$$

In conclusion, we obtain

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(u,v)=0 \quad \text{for all } (u,v),(r,s)\in\mathbb{Z}\times\mathbb{Z}. \tag{3-78}$$

Next, we will calculate

$$d^{\bar{1}}_{kp,kq}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0 \quad \text{with } \frac{m+\frac{1}{2}}{i+\frac{1}{2}}=\frac{p}{q}.$$

We choose $(m,i)\in\mathbb{Z}\times\mathbb{Z}$ with $\frac{m+\frac{1}{2}}{i+\frac{1}{2}}=\frac{p}{q}$, then there exists some $l\in\mathbb{C}$ satisfying $m=lp-\frac{1}{2}, i=lq-\frac{1}{2}\in\mathbb{Z}$. When $p\neq q$, taking $n=lp-\frac{1}{2}$, $j=lq-\frac{1}{2}$, $(m,i)\in\mathbb{Z}\times\mathbb{Z}$ with $\frac{m}{i}=\frac{p}{q}$ and $m\neq -r-1$ in (3-73), we have

$$(k+l+1)(p-q)d^{\bar{1}}_{kp,kq}(lp,lq)=0.$$

So, we get

$$d^{\bar{1}}_{kp,kq}(lp,lq)=0 \quad \text{for } k+l+1\neq 0 \text{ and } p\neq q.$$

When $p=q$, taking $n=lp-\frac{1}{2}$, $j=lq-\frac{1}{2}$ and $m\neq i\in\mathbb{Z}$ in (3-73), we have

$$(k+l+1)d^{\bar{1}}_{kp,kp}(lp,lp)=0.$$

Therefore, we can conclude

$$d^{\bar{1}}_{kp,kp}(lp, lp) = 0 \quad \text{for } k + l + 1 \neq 0.$$

If k, l satisfy $k+l+1=0$, i.e., $(r, s)=\big(kp-\frac{1}{2}, kq-\frac{1}{2}\big)$ and $(m, i)=\big(lp-\frac{1}{2}, lq-\frac{1}{2}\big)=\big(-kp - p - \frac{1}{2}, -kq - q - \frac{1}{2}\big)$, then we can deduce $p, q \in \mathbb{Z}$. Furthermore, taking $n = -p$, $m \neq -\frac{p+1}{2}$, $j \neq -q$ and $i \neq -s - 1$ in (3-72), we have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m - p + \tfrac{1}{2}, i + j + \tfrac{1}{2}\big) = 0.$$

Letting $u = m - p$ and $v = i + j$, we get $d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(u+\frac{1}{2}, v+\frac{1}{2}\big) = 0$ for $u \neq -\frac{3p}{2} - \frac{1}{2}$. And hence, we can arrive at

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(u + \tfrac{1}{2}, v + \tfrac{1}{2}\big) = 0$$

for $(r, s) \neq \big(\frac{p}{2} - \frac{1}{2}, \frac{q}{2} - \frac{1}{2}\big)$ and $(u, v) \neq \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big)$. □

According to this conclusion, we can obtain the following lemma.

Lemma 3.22. *Let $p, q \in 2\mathbb{Z} + 1$. Then the linear map $\gamma : \mathcal{S}(p, q) \to \mathcal{S}(p, q)$ is a $\frac{1}{2}$-derivation of $\mathcal{S}(p, q)$ such that*

$$\gamma(L_{m,i}) = 0,$$

$$\gamma(G_{m+\frac{1}{2},i+\frac{1}{2}}) = \begin{cases} 0, & (m, i) \neq \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big), \\ L_{-p,-q}, & (m, i) = \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big) \end{cases}$$

for $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.

Proof. We observe that

$$\gamma = \sum_{r,s\in\mathbb{Z}} \gamma_{r+\frac{1}{2},s+\frac{1}{2}} = \gamma_{\frac{p}{2},\frac{q}{2}}.$$

Given Lemma 3.20, we need to check (3-56)–(3-58) for $(r, s) = \big(\frac{p}{2} - \frac{1}{2}, \frac{q}{2} - \frac{1}{2}\big)$ and

(3-79) $$d^{\bar{0}}_{\frac{p}{2},\frac{q}{2}}(m, i) = 0, \quad (m, i) \in \mathbb{Z} \times \mathbb{Z},$$

(3-80) $$d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) = \begin{cases} 0, & (m, i) \neq \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big), \\ 1, & (m, i) = \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big). \end{cases}$$

Evidently, (3-56) is satisfied. The next step is to check (3-57).

Case 1: $(n, j), (m + n, i + j) \neq \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big)$. Clearly, both sides of (3-57) are zero by (3-80).

Case 2: $(n, j) = \big(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\big)$. Then (3-57) becomes

$$(pi - qm)d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\big(m - \tfrac{3p}{2}, i - \tfrac{3q}{2}\big) = 0.$$

If $(m, i) = (0, 0)$, then it is trivially satisfied, otherwise both sides are zero by (3-80).

Case 3: $(m+n, i+j) = \left(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\right)$, and $(n, j) \neq \left(-\frac{3p}{2} - \frac{1}{2}, -\frac{3q}{2} - \frac{1}{2}\right)$. Then (3-57) becomes

$$\left(\left(n + \tfrac{p}{2} + \tfrac{1}{2}\right)\left(-j - \tfrac{q}{2} - \tfrac{1}{2}\right) - \left(-n - \tfrac{p}{2} - \tfrac{1}{2}\right)\left(j + \tfrac{q}{2} + \tfrac{1}{2}\right)\right) d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(-\tfrac{3p}{2}, -\tfrac{3q}{2}\right) = 0.$$

Both sides of (3-57) are zero by (3-80).

It remains to check (3-58). Due to (3-79) and (3-80), (3-58) becomes

$$\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+\frac{3p}{2}+\frac{1}{2} & i+\frac{3q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(m + \tfrac{1}{2}, i + \tfrac{1}{2}\right) - \begin{vmatrix} n+\frac{3p}{2}+\frac{1}{2} & j+\frac{3q}{2}+\frac{1}{2} \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(n + \tfrac{1}{2}, j + \tfrac{1}{2}\right) = 0.$$

It evidently holds by (3-80). □

With Lemma 3.22, we can directly infer the proposition below.

Proposition 3.23. *Let $p, q \in \mathbb{C}^*$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p, q)) = \begin{cases} \langle \gamma \rangle, & p, q \in 2\mathbb{Z}+1, \\ \{0\}, & \textit{otherwise}, \end{cases}$$

where γ is as in Lemma 3.22.

3.2.2. *The case $p \neq 0$, $q = 0$.* In this case, (3-56) becomes

$$2\begin{vmatrix} n+p & j \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n, i+j) = \begin{vmatrix} n+p & j \\ m+r+\frac{p}{2}+\frac{1}{2} & i+s+\frac{1}{2} \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m, i) + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n, j). \tag{3-81}$$

On the other hand, by using (3-62)–(3-63) and (3-66)–(3-69), respectively, we have

$$\begin{vmatrix} n+p & j \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(m + n + \tfrac{1}{2}, i + j + \tfrac{1}{2}\right) = 0 \tag{3-82}$$

for $(m, i) \neq (-r-1, -s-1), (-p-r-1, -s-1)$, and

$$\begin{vmatrix} n+r+p+1 & j+s+1 \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(n + \tfrac{1}{2}, j + \tfrac{1}{2}\right) = 0$$

for $(m, i) \neq (-r-1, -s-1)$.

Lemma 3.24. *Let $r, s \in \mathbb{Z}$ and $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p, 0))$. Then*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} = 0.$$

Proof. Taking $n \notin \{-p, 0\}$, $j = 0$ and $m \notin \{-p-r-1, -r-1\}$ in (3-82), we have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0.$$

Since any $k \in \mathbb{Z}$ can be written as $m+n$ with $m \notin \{-p-r-1, -r-1\}$ and $n \notin \{-p, 0\}$ (by choosing $n \notin \{-p, 0, k+p+r+1, k+r+1\}$), we obtain

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(k+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (k,i) \in \mathbb{Z}\times\mathbb{Z}.$$

Choosing $n = j = 0$ in (3-81), we obtain

$$p\big(i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) = \big(i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)-(m+p)\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0). \tag{3-83}$$

Substituting $j = -i \neq 0$ and $m = n = 0$ into (3-81), it leads to

$$\begin{aligned}4pid^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0) = \big(p\big(i+s+\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)\\ +\big(p\big(i-s-\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-i).\end{aligned}$$

Given (3-83), then we get

$$\begin{aligned}&4pip\big(i-s-\tfrac{1}{2}\big)\big(-i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0)\\ &=\Big(\big(p\big(i+s+\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)\big(i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)-p\big(s+\tfrac{1}{2}\big)\big)\big(-i-s-\tfrac{1}{2}\big)\\ &\quad+\big(p\big(i-s-\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)\big(-i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)-p\big(s+\tfrac{1}{2}\big)\big)\big(i-s-\tfrac{1}{2}\big)\Big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0).\end{aligned}$$

Thus, we have

$$\big(r+\tfrac{5p}{2}+\tfrac{1}{2}\big)\big(r-\tfrac{p}{2}+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0) = 0.$$

We will discuss whether r is an element of $\big\{-\frac{5p}{2}-\frac{1}{2}, \frac{p}{2}-\frac{1}{2}\big\}$.

Case 1: $r \neq \{-\frac{5p}{2}-\frac{1}{2}, \frac{p}{2}-\frac{1}{2}\}$. It follows that

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0) = 0 \quad \text{for } r \notin \big\{-\tfrac{5p}{2}-\tfrac{1}{2}, \tfrac{p}{2}-\tfrac{1}{2}\big\}.$$

From (3-83), we can get

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) = 0 \quad \text{for } r \notin \big\{-\tfrac{5p}{2}-\tfrac{1}{2}, \tfrac{p}{2}-\tfrac{1}{2}\big\}.$$

Case 2: $r = -\frac{5p}{2}-\frac{1}{2}$. Then (3-81) becomes

$$\begin{aligned}2(i(n+p)-j(m+p))d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m+n,i+j)\\ =\big((n+p)\big(i+s+\tfrac{1}{2}\big)-j(m-2p)\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m,i)\\ +\big(i(n-2p)-(m+p)\big(j+s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(n,j).\end{aligned} \tag{3-84}$$

And (3-83) becomes

$$p\big(i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m,i)=-\big(2pi+(m+p)\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,0). \tag{3-85}$$

Taking $m=n=0$ and $i, j, i\pm j\neq 0$ in (3-84), we have

$$2(i-j)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,i+j)=\big(i+2j+s+\tfrac{1}{2}\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,i)-\big(2i+j+s+\tfrac{1}{2}\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,j).$$

Because of (3-85), we can obtain

$$d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,0)=0.$$

Then we use (3-85), it follows that

$$d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m,i)=0 \quad \text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}.$$

Case 3: $r=\frac{p}{2}-\frac{1}{2}$. In this case, (3-81) becomes

$$\begin{aligned}2(i(n+p)-j(m+p))d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m+n,i+j)&=\big((n+p)\big(i+s+\tfrac{1}{2}\big)-j(m+p)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m,i)\\&\quad+\big(i(n+p)-(m+p)\big(j+s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,j).\end{aligned}\tag{3-86}$$

And (3-83) can be expressed as

$$p\big(i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m,i)=\big(pi-(m+p)\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0). \tag{3-87}$$

Choosing $m=0$ in (3-87) and utilizing $i-s-\frac{1}{2}\neq 0$, we can see

$$d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,i)=d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0) \quad \text{for all } i\in\mathbb{Z}. \tag{3-88}$$

Substituting $m=j=0$ in (3-86), we have

$$2i(n+p)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i)=(n+p)\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,i)+\big(i(n+p)-p\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,0). \tag{3-89}$$

Furthermore, taking $i=0$, it follows that

$$pd^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,0)=(n+p)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0). \tag{3-90}$$

Multiplying (3-89) by p and using (3-88) and (3-90), we get

$$2pi(n+p)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i)=p(n+p)\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0)+(n+p)\big(i(n+p)-p\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0).$$

Additionally, we arrive at

$$2pid^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i)=i(2p+n)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0)\quad\text{for } n\neq -p.$$

And then, we can deduce that

$$2pd^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i)=(2p+n)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0)\quad\text{for } n\neq -p \text{ and } i\neq 0. \tag{3-91}$$

Taking $n\notin\{-p,0\}$ and $i\neq 0$ and combining (3-87) and (3-91), we have

$$d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0)=0.$$

Due to (3-87), we obtain

$$d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m,i)=0$$

for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$. □

By Lemma 3.24, we directly obtain the following proposition.

Proposition 3.25. *Let $p\in\mathbb{C}^*$ and $q=0$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,0))=0.$$

3.2.3. *The case $p=0$, $q\neq 0$.* By (3-62)–(3-63) and (3-64)–(3-65), respectively, we have

$$\begin{vmatrix} n & j+q \\ m+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\big)+2qd^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j)=0, \tag{3-92}$$

$$2qd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1,i+j+1)+\begin{vmatrix} n+r+1 & j+s+q+1 \\ m+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)=0$$

for all $(m,i)\neq(-r-1,-s-1)$.

Lemma 3.26. *Let $r,s\in\mathbb{Z}$ and $\varphi\in\Delta^{\bar{1}}(\mathcal{S}(0,q))$. Then*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}}=0.$$

Proof. Applying the proof of Lemma 3.24 concerning $d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\frac{1}{2},i+\frac{1}{2}\big)$, we can conclude that

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i)=0\quad\text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}.$$

Then (3-92) becomes

$$\begin{vmatrix} n & j+q \\ m+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\big)=0.$$

Taking $n = 0$, $j \neq -q$ and $i \neq -s-1$, we have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = 0.$$

Since any $k \in \mathbb{Z}$ can be written as $i+j$ with $i \neq -s-1$ and $j \neq -q$, by choosing $j \notin \{-q, k+s+1\}$, we obtain

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, k+\tfrac{1}{2}\big) = 0$$

for all $(m,k) \in \mathbb{Z} \times \mathbb{Z}$. □

A direct result is the subsequent proposition.

Proposition 3.27. *Let $p = 0$ and $q \in \mathbb{C}^*$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,0)) = 0.$$

3.2.4. *The case $p = q = 0$.* In this case, (3-56) becomes

$$2\begin{vmatrix} n & j \\ m & i \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n, i+j) = \begin{vmatrix} n & j \\ m+r+\frac{1}{2} & i+s+\frac{1}{2} \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) + \begin{vmatrix} n+r+\frac{1}{2} & j+s+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j). \tag{3-93}$$

On the other hand, by using (3-62)–(3-63) and (3-70)–(3-71), respectively, we have

$$\begin{vmatrix} n & j \\ m+\frac{1}{2} & i+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = 0, \quad (m,i) \neq (-r-1,-s-1), \tag{3-94}$$

$$\begin{vmatrix} n+r+1 & j+s+1 \\ m+\frac{1}{2} & i+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0, \quad (m,i) \neq (-r-1,-s-1). \tag{3-95}$$

Lemma 3.28. *Let $r, s \in \mathbb{Z}$ and $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(0,0))$. Then*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} = 0.$$

Proof. If $j \neq -s-1$, then we take $m \notin \big\{-r-1, \frac{(n+r+1)(i+\frac{1}{2})}{j+s+1} - \frac{1}{2}\big\}$ in (3-95). It follows that

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \quad \text{for all } n \in \mathbb{Z} \text{ and } j \neq -s-1.$$

If $n \neq -r-1$, then we choose $i \notin \big\{-s-1, \frac{(j+s+1)(m+\frac{1}{2})}{n+r+1} - \frac{1}{2}\big\}$ in (3-95). We can arrive at

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \quad \text{for all } j \in \mathbb{Z} \text{ and } n \neq -r-1.$$

Hence, we can conclude

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \quad \text{for } (n,j) \neq (-r-1,-s-1).$$

Substituting $m=-n-r-1$, $j=-i-s-1$ and $i\neq -s-1$ into (3-94), it leads to

$$\begin{vmatrix} -r-\frac{1}{2} & -s-\frac{1}{2} \\ -n-r-\frac{1}{2} & i+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(-r-\tfrac{1}{2},\,-s-\tfrac{1}{2}\big)=0.$$

Since $r\in\mathbb{Z}$ and $r+\frac{1}{2}\neq 0$, then we take $i\notin\big\{-\frac{n(s+\frac{1}{2})}{r+\frac{1}{2}}-s-1,\,-s-1\big\}$. It follows that

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(-r-\tfrac{1}{2},\,-s-\tfrac{1}{2}\big)=0.$$

Therefore, we arrive at

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2},\,j+\tfrac{1}{2}\big)=0 \quad \text{for all } (n,j),(r,s)\in\mathbb{Z}\times\mathbb{Z}.$$

Choosing $n=j=0$ in (3-93), we have

$$\big(i\big(r+\tfrac{1}{2}\big)-m\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0)=0.$$

Since $s+\frac{1}{2}\neq 0$, then taking $m\neq\frac{i(r+\frac{1}{2})}{s+\frac{1}{2}}$, we have

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0)=0.$$

Substituting $m=j=0$ into (3-93), we obtain

$$2ni\,d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,i)=n\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)+i\big(n+r+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0). \tag{3-96}$$

Putting $m=-n$ and $j=0$ into (3-93), we get

$$\begin{aligned} 2ni\,d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)&=n\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-n,i)\\ &\quad+\big(i\big(n+r+\tfrac{1}{2}\big)+n\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0). \end{aligned} \tag{3-97}$$

Taking $i=0$ in (3-97), we can conclude

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-n,0)=-d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0) \quad \text{for all } n\in\mathbb{Z}. \tag{3-98}$$

Substituting $j=-i$ and $m=0$ into (3-93), it follows that

$$\begin{aligned} 2ni\,d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)&=\big(n\big(i+s+\tfrac{1}{2}\big)+i\big(r+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)\\ &\quad+i\big(n+r+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-i). \end{aligned} \tag{3-99}$$

Letting $n=0$ in (3-99), we have

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-i)=-d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i) \quad \text{for all } i\in\mathbb{Z}. \tag{3-100}$$

Choosing $n=-(2r+1)$ in (3-96), we have

$$4id^{0}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) = (2i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)+id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0). \tag{3-101}$$

Then we let $n=2r+1$ in (3-97) together with (3-98). It follows that

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)=(2i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) + (3i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(2r+1,0),$$

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)=(2i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) - (3i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0). \tag{3-102}$$

Replacing i by $-i$ and taking $n=-2r-1$ in (3-99) together with (3-100), it gives

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0) = -(2s-i+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-i)+id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i),$$

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0) = (2s-i+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)+id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i). \tag{3-103}$$

Combining (3-101) and (3-103), we can deduce

$$15id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0)=(10s-2i+5)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i), \tag{3-104}$$

$$15id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i)=(10s+7i+5)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i). \tag{3-105}$$

If $i\neq 0$, taking (3-104) and (3-105) in (3-102), we have

$$(10i-2s-1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)=0.$$

Since $s\in\mathbb{Z}$, then we obtain $\frac{1}{5}\big(s+\frac{1}{2}\big)\notin\mathbb{Z}$ and $i\neq\frac{1}{5}\big(s+\frac{1}{2}\big)$, i.e., $10i-2s-1\neq 0$. Hence, we arrive at

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)=0 \quad \text{for all } i\in\mathbb{Z}.$$

We take $i=-2s-1$ in (3-96). It follows that

$$4nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-2s-1) = nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-2s-1)+(2n+2r+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0). \tag{3-106}$$

Replacing n by $-n$ and taking $i=-2s-1$ in (3-97) together with (3-98), it gives

$$4nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-2s-1) = nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-2s-1)+(2r-n+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0). \tag{3-107}$$

Letting $i=2s+1$ in (3-99), we obtain

$$4nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)=(2n+2r+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-2s-1) - (3n+2r+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-2s-1). \tag{3-108}$$

Combining (3-106)–(3-108), we have

$$(10n-2r-1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)=0.$$

As the similar reason above, we can get

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)=0 \quad \text{for all } n\in\mathbb{Z}.$$

In this end, by (3-96), we obtain

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,i)=0 \quad \text{for } n,i\neq 0.$$

Therefore, we can conclude

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,i)=0$$

for $(n,j)\in\mathbb{Z}\times\mathbb{Z}$. □

It leads directly to the following proposition.

Proposition 3.29. *Let $p=q=0$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(0,0))=0.$$

By integrating Propositions 3.23, 3.25, 3.27 and 3.29, we can deduce the following corollary.

Corollary 3.30. *Let $p,q\in\mathbb{C}$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,q))=\begin{cases}\langle\gamma\rangle, & p,q\in 2\mathbb{Z}+1,\\ \{0\}, & \textit{otherwise},\end{cases}$$

where γ is as in Lemma 3.22.

4. Transposed Poisson superalgebra structures on $\mathcal{S}(p, q)$

Theorem 4.1. *Let p and q be fixed complex numbers.*

(1) *If $p \notin \mathbb{Z}$ or $q \neq 0$, then all the transposed Poisson superalgebra structures on $(\mathcal{S}(p,q), [\cdot,\cdot])$ are trivial.*

(2) *If $p \in \mathbb{Z}$ and $q = 0$, then the nontrivial transposed Poisson superalgebra structure $(\mathcal{S}(p,0), \cdot, [\cdot,\cdot])$ on $(\mathcal{S}(p,0), [\cdot,\cdot])$ is, up to an isomorphism,*

$$L_{-2p,0} \cdot L_{-2p,0} = L_{-p,0}. \tag{4-1}$$

Proof. Let $(\mathcal{S}(p,q), \cdot, [\cdot,\cdot])$ be a transposed Poisson superalgebra, i.e., $(\mathcal{S}(p,q), \cdot)$ is supercommutative and (2-2) holds. Given $(m,i) \in \mathbb{Z} \times \mathbb{Z}$, we denote by $\varphi^{m,i}$ and $\psi^{m+\frac{1}{2},i+\frac{1}{2}}$ the left multiplication by $L_{m,i}$ and $G_{m+\frac{1}{2},i+\frac{1}{2}}$, respectively, in $(\mathcal{S}(p,q), \cdot)$, that is,

$$\begin{aligned} L_{m,i} \cdot L_{n,j} &= \varphi^{m,i}(L_{n,j}), \qquad (4\text{-}2)\\ L_{m,i} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= \varphi^{m,i}(G_{n+\frac{1}{2},j+\frac{1}{2}}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot L_{n,j} &= \psi^{m+\frac{1}{2},i+\frac{1}{2}}(L_{n,j}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= \psi^{m+\frac{1}{2},i+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}}). \qquad (4\text{-}3) \end{aligned}$$

In view of supercommutativity of $(\mathcal{S}(p,q), \cdot)$, we have

$$\begin{aligned} L_{m,i} \cdot L_{n,j} &= \varphi^{n,j}(L_{m,i}), \qquad (4\text{-}4)\\ L_{m,i} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= \psi^{n+\frac{1}{2},j+\frac{1}{2}}(L_{m,i}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot L_{n,j} &= \varphi^{n,j}(G_{m+\frac{1}{2},i+\frac{1}{2}}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= -\psi^{n+\frac{1}{2},j+\frac{1}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}). \qquad (4\text{-}5) \end{aligned}$$

By using (2-2), we have $\varphi^{m,i} \in \Delta^{\bar{0}}(\mathcal{S}(p,q))$ and $\psi^{m+\frac{1}{2},i+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$. Furthermore, based on the Corollaries 3.19 and 3.30, we proceed to the following discussion.

Case 1: $p, q \neq 0$ and $p, q \notin 2\mathbb{Z}+1$. It is clear that $\varphi^{m,i} = a^{m,i} id$ for some $a^{m,i} \in \mathbb{C}$ by Proposition 3.4 and $\psi^{m+\frac{1}{2},i+\frac{1}{2}} = 0$ by Proposition 3.23 . It follows from (4-2) and (4-4) that $a^{m,i} = 0$ for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$. So, $\cdot$ is trivial whenever $p, q \neq 0$ and $p, q \notin 2\mathbb{Z}+1$.

Case 2: $p, q \neq 0$ and $p, q \in 2\mathbb{Z}+1$. We have $\varphi^{m,i} = a^{m,i} id$ for some $a^{m,i} \in \mathbb{C}$ by Proposition 3.4. Equations (4-2) and (4-4) imply $a^{m,i} = 0$, whence $\varphi^{m,i} = 0$. We arrive at $L_{m,i} \cdot L_{n,j} = L_{m,i} \cdot G_{n,j} = 0$. From Proposition 3.23, it is easy to observe that

$$\psi^{m+\frac{1}{2},i+\frac{1}{2}} = b^{m+\frac{1}{2},i+\frac{1}{2}} \gamma,$$

with $b^{m+\frac{1}{2},i+\frac{1}{2}} \in \mathbb{C}$. On the one hand, it follows from (3-80) and (4-3) that

$$\begin{aligned} G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= b^{m+\frac{1}{2},i+\frac{1}{2}} \gamma(G_{n+\frac{1}{2},j+\frac{1}{2}}) \\ &= \begin{cases} 0, & (n,j) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right), \\ b^{m+\frac{1}{2},i+\frac{1}{2}} L_{-p,-q}, & (n,j) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right). \end{cases} \end{aligned}$$

On the other hand, by using (3-80) and (4-5), we arrive at

$$\begin{aligned} G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= -b^{n+\frac{1}{2},j+\frac{1}{2}} \gamma(G_{m+\frac{1}{2},i+\frac{1}{2}}) \\ &= \begin{cases} 0, & (m,i) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right), \\ -b^{n+\frac{1}{2},j+\frac{1}{2}} L_{-p,-q}, & (m,i) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right). \end{cases} \end{aligned}$$

Thus, the product $G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}}$ is zero unless

$$(m,i) = (n,j) = \left(-\tfrac{3p}{2}-\tfrac{1}{2}, -\tfrac{3q}{2}-\tfrac{1}{2}\right).$$

But $G_{-\frac{3p}{2},-\frac{3q}{2}} \cdot G_{-\frac{3p}{2},-\frac{3q}{2}} = -G_{-\frac{3p}{2},-\frac{3q}{2}} \cdot G_{-\frac{3p}{2},-\frac{3q}{2}}$, then $G_{-\frac{3p}{2},-\frac{3q}{2}} \cdot G_{-\frac{3p}{2},-\frac{3q}{2}} = 0$. So, $\cdot$ is trivial.

Case 3: $p \notin \mathbb{Z}$ and $q = 0$. We obtain that $\varphi^{m,i} = a^{m,i} id$ for some $a^{m,i} \in \mathbb{C}$ by Proposition 3.14 and $\psi^{m+\frac{1}{2},i+\frac{1}{2}} = 0$ by Proposition 3.25 . It follows from (4-2) and (4-4) that $a^{m,i} = 0$ for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$. So, $\cdot$ is trivial.

Case 4: $p \in \mathbb{Z}^*$ and $q = 0$. We have $\psi^{m+\frac{1}{2},i+\frac{1}{2}} = 0$ by Proposition 3.25. It leads to $G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot L_{n,j} = 0$ and $G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} = 0$.

Because of Proposition 3.14, it is clear to show that $\varphi^{m,i} = f^{m,i} id + g^{m,i} \alpha$ for $f^{m,i}, g^{m,i} \in \mathbb{C}$. We can get from (3-36) and (4-2) that

$$\begin{aligned} L_{m,i} \cdot L_{n,j} &= f^{m,i} L_{n,j} + g^{m,i} \alpha(L_{n,j}) \\ &= \begin{cases} f^{m,i} L_{n,j}, & (n,j) \neq (-2p,0), \\ f^{m,i} L_{-2p,0} + g^{m,i} L_{-p,0}, & (n,j) = (-2p,0). \end{cases} \end{aligned}$$

On the other hand, by using (3-36) and (4-4), we have

$$\begin{aligned} L_{m,i} \cdot L_{n,j} &= f^{m,i} L_{n,j} + g^{m,i} \alpha(L_{n,j}) \\ &= \begin{cases} f^{n,i} L_{m,i}, & (m,i) \neq (-2p,0), \\ f^{n,j} L_{-2p,0} + g^{n,j} L_{-p,0}, & (m,i) = (-2p,0). \end{cases} \end{aligned}$$

Thus, we can discuss it into the following cases.

Subcase 1: $(m,i), (n,j) \neq (-2p,0)$. We get $f^{m,i} L_{n,j} = f^{n,j} L_{m,i}$. So taking $(m,i) \neq (n,j)$ we conclude that $f^{m,i} = f^{n,j} = 0$. Thus, $L_{m,i} \cdot L_{n,j} = 0$.

Subcase 2: $(m,i) = (-2p,0)$, $(n,j) \neq (-2p,0)$. We have

$$f^{-2p,0} L_{n,j} = f^{n,j} L_{-2p,0} + g^{n,j} L_{-p,0} = g^{n,j} L_{-p,0},$$

because of $f^{n,j}=0$ for $(n,\,j)\neq(-2p,0)$. So, taking $(n,\,j)\neq(-p,0)$, we conclude that $f^{-2p,0}=0$, whence $L_{-2p,0}\cdot L_{n,j}=0$.

Subcase 3: $(m,i)\neq(-2p,0)$, $(n,\,j)=(-2p,0)$. It implies $L_{m,i}\cdot L_{-2p,0}=0$.

Subcase 4: $(m,i)=(n,\,j)=(-2p,0)$. It leads to

$$L_{-2p,0}\cdot L_{-2p,0}=f^{-2p,0}L_{-2p,0}+g^{-2p,0}L_{-p,0}=g^{-2p,0}L_{-p,0},$$

because of $f^{-2p,0}=0$.

Thus, we can conclude that

$$L_{m,i}\cdot L_{n,j}=\begin{cases}0, & (m,i),(n,\,j)\neq(-2p,0),\\ g^{-2p,0}L_{-p,0}, & (m,i)=(n,\,j)=(-2p,0).\end{cases}$$

Therefore, the product $(\mathcal{S}(p,0),\cdot\,)$ is of the form

$$L_{-2p,0}\cdot L_{-2p,0}=cL_{-p,0}, \tag{4-6}$$

where $c\in\mathbb{C}$. Assume that $c\neq0$, otherwise the transposed Poisson superalgebra structure is trivial. We observe that $L_{-p,0}\in Z(\mathcal{S}(p,0))$, where $Z(\mathcal{S}(p,0))$ is the center of the Lie superalgebra $(\mathcal{S}(p,0),[\,\cdot\,,\cdot\,])$. Indeed,

$$[L_{m,i},L_{-p,0}]=((-p+p)\cdot0-(m+p)\cdot0)L_{m+n,i+j}=0$$

for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$. Hence the linear map ϕ such that $\phi(L_{m,i})=L_{m,i}$ for $(m,i)\neq(-p,0)$ and $\phi(L_{-p,0})=kL_{-p,0}$ is an automorphism of $(\mathcal{S}(p,0),[\,\cdot\,,\cdot\,])$ for any $k\in\mathbb{C}^*$. If $p\neq0$, then we take $k=c^{-1}$. Furthermore, we obtain an isomorphic transposed Poisson superalgebra structure $*$ on $(\mathcal{S}(p,0),[\,\cdot\,,\cdot\,])$ in which the only nonzero product is

$$\begin{aligned}L_{-2p,0}*L_{-2p,0}&=\phi(L_{-2p,0})*\phi(L_{-2p,0})=\phi(L_{-2p,0}\cdot L_{-2p,0})\\&=\phi(cL_{-2p,0})=c^{-1}\cdot cL_{-p,0}=L_{-p,0}.\end{aligned}$$

So, up to an isomorphism, we may consider $c=1$ in (4-6).

Case 5: $p=q=0$. We have $\psi^{m+\frac12,i+\frac12}=0$ by Proposition 3.29. It implies

$$G_{m+\frac12,i+\frac12}\cdot L_{n,j}=0\quad\text{and}\quad G_{m+\frac12,i+\frac12}\cdot G_{n+\frac12,j+\frac12}=0.$$

It is easy to show that $\varphi^{m,i}=x^{m,i}id+y^{m,i}\beta$ for $x^{m,i}$, $y^{m,i}\in\mathbb{C}$ by Proposition 3.18. We can get from (3-52) and (4-2) that

$$L_{m,i}\cdot L_{n,j}=x^{m,i}L_{n,j}+y^{m,i}\alpha(L_{n,j})=\begin{cases}x^{m,i}L_{n,j}, & (n,\,j)\neq(0,0),\\ (x^{m,i}+y^{m,i})L_{0,0}, & (n,\,j)=(0,0).\end{cases}$$

On the other hand, by using (3-52) and (4-4), we obtain

$$L_{m,i}\cdot L_{n,j}=x^{m,i}L_{n,j}+y^{m,i}\alpha(L_{n,j})=\begin{cases}x^{n,i}L_{m,i}, & (m,i)\neq(0,0),\\ (x^{n,j}+y^{n,j})L_{0,0}, & (m,i)=(0,0).\end{cases}$$

Thus, we can discuss it into the following cases.

Subcase 1: (m, i), $(n, j) \neq (0, 0)$. We get $x^{m,i} L_{n,j} = x^{n,j} L_{m,i}$. So taking $(m, i) \neq (n, j)$ we conclude that $x^{m,i} = x^{n,j} = 0$. Thus, we arrive at $L_{m,i} \cdot L_{n,j} = 0$.

Subcase 2: $(m, i) = (0, 0)$, $(n, j) \neq (0, 0)$. We have $x^{0,0} L_{n,j} = (x^{n,j} + y^{n,j}) L_{0,0} = y^{n,j} L_{0,0}$. So, we obtain $x^{0,0} = y^{n,j} = 0$, whence $L_{0,0} \cdot L_{n,j} = 0$.

Subcase 3: $(m, i) \neq (0, 0)$, $(n, j) = (0, 0)$. It implies $L_{m,i} \cdot L_{0,0} = 0$.

Subcase 4: $(m, i) = (n, j) = (0, 0)$. It leads to $L_{0,0} \cdot L_{0,0} = (x^{0,0} + y^{0,0}) L_{0,0} = y^{0,0} L_{0,0}$, because of $x^{0,0} = 0$.

Thus, we have

$$L_{m,i} \cdot L_{n,j} = \begin{cases} 0, & (m, i), (n, j) \neq (0, 0), \\ y^{0,0} L_{0,0}, & (m, i), (n, j) = (0, 0). \end{cases}$$

Therefore, the product $(\mathcal{S}(0, 0), \cdot\,)$ is of the form

$$L_{0,0} \cdot L_{0,0} = c L_{0,0}, \tag{4-7}$$

where $c \in \mathbb{C}$. Assume that $c \neq 0$, otherwise the transposed Poisson superalgebra structure is trivial. Observe that $L_{0,0} \in Z(\mathcal{S}(0, 0))$, where $Z(\mathcal{S}(0, 0))$ is the center of the Lie superalgebra $(\mathcal{S}(0, 0), [\,\cdot\,, \cdot\,])$. Indeed,

$$[L_{m,i}, L_{0,0}] = 0$$

for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. Hence the linear map ϕ such that $\phi(L_{m,i}) = L_{m,i}$ for $(m, i) \neq (0, 0)$ and $\phi(L_{0,0}) = k L_{-p,0}$ is an automorphism of $(\mathcal{S}(0, 0), [\,\cdot\,, \cdot\,])$ for any $k \in \mathbb{C}^*$. Then taking $k = c$, we obtain an isomorphic transposed Poisson superalgebra structure $*$ on $(\mathcal{S}(0, 0), [\,\cdot\,, \cdot\,])$ in which the only nonzero product is

$$\begin{aligned} L_{0,0} * L_{0,0} &= c^{-1}\phi(L_{0,0}) * c^{-1}\phi(L_{0,0}) = c^{-2}\phi(L_{0,0} \cdot L_{0,0}) \\ &= c^{-2}\phi(c L_{0,0}) = c^{-2} \cdot c^2 L_{0,0} = L_{0,0}. \end{aligned}$$

So, up to an isomorphism, we may consider $c = 1$ in (4-7).

Conversely, each of two associative and supercommutative multiplication (4-1) defines a transposed Poisson superalgebra structure on $\mathcal{S}(p, 0)$, $p \in \mathbb{Z}$. If $p \in \mathbb{Z}^*$, we can observe that $\mathcal{S}(p, 0) \cdot \mathcal{S}(p, 0) \subseteq \langle L_{-p,0} \rangle \subseteq \mathbb{Z}(\mathcal{S}(p, 0))$. Hence, the right-hand side of (2-2) is always zero. In fact, the left-hand side of (2-2) is zero as well, because of $[\mathcal{S}(p, 0), \mathcal{S}(p, 0)] \subseteq \mathrm{Ann}(\mathcal{S}(p, 0))$, where $\mathrm{Ann}(\mathcal{S}(p, 0))$ is the annihilator of $(\mathcal{S}(p, 0), \cdot\,)$. Assuming $[L_{m,i}, L_{n,j}] \in \langle L_{-2p,0} \rangle$, we obtain from (2-1) that $m + n = -2p$ and $i + j = 0$. But it leads to

$$-j(n + p) - j(m + p) = -j(n + p) - j(-2p - n + p) = 0,$$

so we have $[L_{m,i}, L_{n,j}] = 0$. Thus, we have $[L_{m,i}, L_{n,j}] \subseteq \mathrm{Ann}(\mathcal{S}(p, 0))$ for all (m, i), $(n, j) \in \mathbb{Z} \times \mathbb{Z}$, as needed. If $p = 0$, since $L_{0,0} \in \mathbb{Z}(\mathcal{S}(0, 0))$, then it leads to

$\mathcal{S}(0,0)\cdot\mathcal{S}(0,0)\subseteq\mathbb{Z}(\mathcal{S}(0,0))$. Hence the right-hand side of (2-2) is zero. Assuming $[L_{m,i}, L_{n,j}]\in\langle L_{0,0}\rangle$, we obtain from (2-1) that $m=-n$ and $j=-i$. But then we get

$$ni-(-n)(-i)=ni-ni=0.$$

So $[L_{m,i}, L_{n,j}]=0$. We get $[L_{m,i}, L_{n,j}]\subseteq \mathrm{Ann}(\mathcal{S}(0,0))$ for all $(m,i),(n,j)\in\mathbb{Z}\times\mathbb{Z}$. Then the left-hand side of (2-2) is zero as needed. □

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