

*Pacific  
Journal of  
Mathematics*

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Volume 335 No. 1

March 2025



# THE CR YAMABE CONSTANT AND INEQUIVALENT CR STRUCTURES

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**The CR Yamabe constant is an invariant of a compact strongly pseudoconvex CR manifold and plays an important role in CR geometry. We show some integral formulas of the CR Yamabe constant. We also construct an infinite-dimensional family of strongly pseudoconvex CR structures with varying CR Yamabe constants and a compact simply connected manifold admitting two strongly pseudoconvex CR structures with different signs of the CR Yamabe constant.**

## 1. Introduction

The Yamabe problem, which is one of the most important problems in conformal geometry, asks whether there exists a Riemannian metric in a given conformal class minimizing the Yamabe functional. The infimum of this functional defines a conformal invariant called the Yamabe constant. The Yamabe constant is a fundamental invariant in conformal geometry, and there is intensive research on this invariant. It is known that every compact manifold of dimension greater than 2 has a continuous family of conformal structures with all different Yamabe constants. Moreover, every compact manifold of dimension greater than 2 admits a conformal structure with negative Yamabe constant. Furthermore, there exist some integral formulas of this invariant, which have been useful for computing not only the Yamabe constants but also other curvature parts of 3- and 4-manifolds [LeBrun 1999; Sung 2012; 2021].

Jerison and Lee [1987] have considered a CR analog of the Yamabe problem, known as the *CR Yamabe problem*. The CR Yamabe problem asks whether on any compact strongly pseudoconvex CR manifold  $(X, H, J)$  of dimension  $2n + 1$ , there exists a contact form  $\theta$  minimizing the functional

$$\mathfrak{Y}(\theta) := \frac{\int_X R_\theta d\mu_\theta}{\left(\int_X d\mu_\theta\right)^{n/(n+1)}},$$

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Takeuchi was supported by JSPS KAKENHI grant number JP21K13792.

*MSC2020:* 32V05, 32V20, 57K43.

*Keywords:* CR Yamabe constant, CR manifold, pseudohermitian manifold.

where  $R_\theta$  is the pseudohermitian scalar curvature for  $\theta$  and  $d\mu_\theta = \theta \wedge (d\theta)^n$ . The infimum  $\lambda(X, H, J)$  of the above functional is an invariant of the CR manifold  $(X, H, J)$  and called the *CR Yamabe constant* of  $(X, H, J)$ ; we will simply write  $\lambda(X)$  when the CR structure on  $X$  is clear from the context. Just as in the Yamabe problem, one has

$$\lambda(X) \leq \lambda(S^{2n+1}) = 2n(n+1)\pi$$

for the standard CR structure on  $S^{2n+1}$ , and the CR Yamabe problem is solvable when  $\lambda(X) < \lambda(S^{2n+1})$  [Jerison and Lee 1987]. There is intensive research on conditions when  $\lambda(X) < \lambda(S^{2n+1})$  holds for  $X$  not CR equivalent to  $S^{2n+1}$ ; see [Jerison and Lee 1988; 1989; Cheng et al. 2014; 2017; 2023; Takeuchi 2020] for related results. Every minimizer  $\theta$  has constant  $R_\theta$ , and when  $\lambda(X) \leq 0$ , any  $\theta$  with constant  $R_\theta$  is a CR Yamabe minimizer, which is unique up to a constant. Moreover, an Einstein contact form is also a CR Yamabe minimizer; see Lemma 2.1. In a similar way to the Yamabe constant, the CR Yamabe constant can be written as various integral formulas; see Theorem 3.1. This may be useful for estimating norms of parts of pseudohermitian curvature tensor as in the Riemannian case [Sung 2021].

It is natural to ask whether a manifold admitting one CR structure has abundant other CR structures with all different CR Yamabe constants or CR structures with negative CR Yamabe constant. In comparison to the Riemannian case, the difficulty lies in imposing the integrability condition to an almost CR structure, which obstructs generic deformations of almost CR structures. In this paper, we will construct an infinite-dimensional family of strongly pseudoconvex CR structures with varying CR Yamabe constants. To this end, we consider an  $n$ -dimensional compact Hodge manifold  $(M, J, \omega)$  with constant scalar curvature. Denote the space of Kähler potentials in the class  $[\omega]$  by

$$\mathcal{K} := \{\varphi \in C^\infty(M) \mid \omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0\}$$

endowed with the  $C^4$ -topology, and write  $\mathcal{F}$  for the subset of  $\varphi \in \mathcal{K}$  such that  $\omega_\varphi$  has constant scalar curvature. Let  $p : P_M \rightarrow M$  be a principal  $S^1$ -bundle over  $M$  whose Euler class is  $-[\omega]$ . For any  $\varphi \in \mathcal{K}$ , there exists a principal connection  $\theta_\varphi$  on  $P_M$  such that  $d\theta_\varphi/2\pi = p^*\omega_\varphi$ . The complex structure  $J$  on  $M$  induces a strongly pseudoconvex CR structure  $p^*J$  on  $H_\varphi := \text{Ker } \theta_\varphi$ ; see Proposition 2.2. This gives an infinite-dimensional family of pseudohermitian manifolds  $(P_M, H_\varphi, p^*J, \theta_\varphi/2\pi)$  underlying the same manifold  $P_M$ .

**Theorem 1.1.** *Let  $(M, J, \omega)$  be an  $n$ -dimensional compact Hodge manifold. Then the map*

$$\mathcal{K} \rightarrow \mathbb{R}; \quad \varphi \mapsto \lambda(P_M, H_\varphi, p^*J)$$

is continuous. Moreover if  $\theta_0/2\pi$  is a CR Yamabe minimizer, then

$$\lambda(P_M, H_\varphi, p^*J) < \lambda(P_M, H_0, p^*J) = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{([\omega]^n)^{n/(n+1)}}$$

for any  $\varphi \in \mathcal{K} \setminus \mathcal{F}$ . The assumption holds if  $\omega$  has nonpositive constant scalar curvature or it defines a Kähler–Einstein metric.

We will also show the existence of a compact simply connected manifold admitting two strongly pseudoconvex CR structures with different signs of the CR Yamabe constant.

**Theorem 1.2.** *For each  $n \geq 3$ , there exists a compact simply connected  $(2n + 1)$ -manifold  $X$  admitting two strongly pseudoconvex CR structures  $(H, J)$  and  $(\tilde{H}, \tilde{J})$  such that they have different signs of the CR Yamabe constants, and  $(M, H)$  and  $(M, \tilde{H})$  are not isomorphic as cooriented contact manifolds.*

We remark that the existence of CR structures with different signs of the CR Yamabe constant on a fixed contact structure remains unsolved.

This paper is organized as follows. In Section 2, we recall basic facts on CR manifolds and show that a principal  $S^1$ -bundle over a Hodge manifold has a canonical strongly pseudoconvex CR structure. Some integral formulas of the CR Yamabe constant are given in Section 3. In Section 4, we prove the continuity of the CR Yamabe constant under suitable deformations of CR structures, which will be used for the proof of Theorem 1.1. Section 5 is devoted to constructions of deformations of strongly pseudoconvex CR structures with varying CR Yamabe constants. In Section 6, we give a proof of Theorem 1.2.

## 2. CR manifolds

An almost CR structure on a smooth  $(2n + 1)$ -manifold  $X$  is a pair  $(H, J)$  where  $H \subset TX$  is a codimension 1 smooth subbundle with an almost complex structure  $J$ . An almost CR structure is called integrable or a CR structure if

$$[\Gamma(T^{1,0}X), \Gamma(T^{1,0}X)] \subset \Gamma(T^{1,0}X)$$

for

$$T^{1,0}X := \{v - iJv \mid v \in H\} \subset H \otimes \mathbb{C}.$$

We shall consider only an orientable CR manifold. Then one can choose a smooth real-valued 1-form  $\theta$  annihilating exactly  $H$ , which is determined up to multiplication by a nowhere vanishing real-valued function on  $X$ . By the integrability condition,  $d\theta$  is  $J$ -invariant; i.e., a  $(1, 1)$ -form, and hence one can introduce the symmetric bilinear form

$$L_\theta := d\theta(\cdot, J\cdot)$$

defined on  $H$ , called the *Levi form*.

A CR manifold  $(X, H, J)$  is called *strongly pseudoconvex* if  $L_\theta$  is definite for some (and hence all)  $\theta$ , and a strongly pseudoconvex CR manifold  $(X, H, J)$  with a choice of  $\theta$  is called a *pseudohermitian* manifold. We shall always assume that  $X$  is strongly pseudoconvex and  $\theta$  is chosen so that  $L_\theta$  is positive definite, unless otherwise specified. In this case, the distribution  $H$  is a contact structure on  $X$  with a contact form  $\theta$ . Let  $T$  be its Reeb vector field; i.e., the unique vector field satisfying  $\theta(T) = 1$  and  $\iota(T) d\theta = 0$ . The Levi form induces a Hermitian metric  $L_\theta^*$  on  $H^*$ . The *sublaplacian*  $\Delta_b$  is defined by

$$\int_X (\Delta_b u) v \, d\mu_\theta = \int_X L_\theta^*(du|_H, dv|_H) \, d\mu_\theta.$$

A set of local 1-forms  $\{\theta^1, \dots, \theta^n\}$  of type  $(1, 0)$  is called *admissible*, if its restriction to  $T^{1,0}X$  forms a basis of  $(T^{1,0}X)^*$  at each point and  $\theta^\alpha(T) = 0$  for all  $\alpha$ . For an admissible coframe, we have

$$(2-1) \quad d\theta = ih_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}},$$

where  $(h_{\alpha\bar{\beta}})$  is a positive-definite hermitian matrix of functions and  $\theta^{\bar{\beta}} = \overline{\theta^\beta}$ . We shall always adopt the Einstein convention and use the matrix  $(h_{\alpha\bar{\beta}})$  and its inverse  $(h^{\alpha\bar{\beta}})$  to raise and lower indices. The integrability condition of  $J$  can be rephrased as

$$d\theta^\alpha \equiv 0 \pmod{\theta, \theta^\gamma}$$

along with (2-1).

A pseudohermitian manifold carries a canonical linear connection, the *Tanaka–Webster connection* [Tanaka 1975; Webster 1978], whose connection 1-forms  $\omega_\alpha^\beta$  and torsion forms  $\tau^\alpha$  of type  $(0, 1)$  are uniquely determined by the relations

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta \quad \text{and} \quad \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} = dh_{\alpha\bar{\beta}}$$

together with (2-1). We call  $\tau^\alpha$  the *pseudohermitian torsion*. The whole torsion tensor is composed of  $\theta \wedge \tau^\gamma$  and  $ih_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}$ , and so it is nowhere vanishing.

The covariant differentiation with respect to this connection is given by

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where a local frame  $\{Z_\alpha\}$  of  $T^{1,0}X$  is dual to  $\{\theta^\alpha\}$ . Its curvature 2-forms

$$\Omega_\alpha^\beta := d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$$

may have several types, but one considers only its  $(1, 1)$ -part:

$$R_\alpha^\beta{}_{\rho\bar{\sigma}} \theta^\rho \wedge \theta^{\bar{\sigma}}$$

to get its *pseudohermitian Ricci tensor*

$$R_{\rho\bar{\sigma}} := R_\alpha^\alpha{}_{\rho\bar{\sigma}}$$

by taking its contraction. Finally its *pseudohermitian scalar curvature* is the metric contraction  $R_\rho{}^\rho = h^{\rho\bar{\sigma}} R_{\rho\bar{\sigma}}$ . We say  $\theta$  to be *Einstein* if its pseudohermitian torsion is identically zero and its pseudohermitian Ricci curvature is a constant multiple of the Levi form.

**Lemma 2.1.** *Any Einstein contact form is a CR Yamabe minimizer.*

*Proof.* Let  $(X, H, J)$  be a compact strongly pseudoconvex CR manifold of dimension  $2n + 1$  and  $\theta$  be an Einstein contact form on  $X$ . If the pseudohermitian scalar curvature is nonpositive, then it must be a CR Yamabe minimizer. If the pseudohermitian scalar curvature is positive, we may assume that it is equal to  $n(n + 1)$  by homothety. Consider the Riemannian metric  $g_\theta$  on  $X$  given by

$$g_\theta(U, V) := \frac{1}{2}d\theta(U, JV) + \theta(U)\theta(V), \quad U, V \in TX.$$

Here we extend  $J$  to an endomorphism on  $TM$  by  $JT = 0$ . Note that the volume form of  $g_\theta$  coincides with  $(2^n n!)^{-1} d\mu_\theta$ . This  $g_\theta$  satisfies  $\text{Ric}_{g_\theta} = 2ng_\theta$ ; see [Takeuchi 2018, Proposition 2.9] for example. The Bishop inequality implies that

$$\begin{aligned} \mathfrak{F}(\theta) &= n(n + 1)(2^n n! \text{Vol}_{g_\theta}(X))^{\frac{1}{n+1}} \\ &\leq n(n + 1)(2^n n! \text{Vol}_{g_0}(S^{2n+1}))^{\frac{1}{n+1}} = 2n(n + 1)\pi = \lambda(S^{2n+1}), \end{aligned}$$

where  $g_0$  is the standard Riemannian metric on  $S^{2n+1}$ . Moreover, the equality holds if and only if  $(X, g_\theta)$  is isometric to  $(S^{2n+1}, g_0)$ . In this case,  $(X, H, J)$  is CR isomorphic to the standard CR sphere and  $\theta$  is a CR Yamabe minimizer; see the paragraph after the proof of Proposition 4 in [Wang 2015]. If  $\mathfrak{F}(\theta) < \lambda(S^{2n+1})$ , then  $(X, H, J)$  has a CR Yamabe minimizer by [Jerison and Lee 1987, Theorem 3.4(c)], and  $\theta$  is also a CR Yamabe minimizer by [Wang 2015, Theorem 3].  $\square$

An important example of a strongly pseudoconvex CR manifold is a principal  $S^1$ -bundle over a Hodge manifold. Given a Hodge manifold  $(M, J, \omega)$ ; that is, its Kähler class  $[\omega]$  is an integral cohomology class, we consider a principal  $S^1$ -bundle  $p : P_M \rightarrow M$  whose Euler class is  $-[\omega]$ . Recall that for any  $\mathbb{R}$ -valued principal connection  $\theta$  on  $P_M$ ,  $d\theta/2\pi$  descends to  $M$  and its cohomology class coincides with  $[\omega]$ . We take a principal connection  $\theta$  satisfying  $d\theta/2\pi = p^*\omega$ , and consider the lifted almost complex structure

$$p^*J : H := \text{Ker } \theta \rightarrow H.$$

**Proposition 2.2.** *The triple  $(P_M, H, p^*J)$  is a strongly pseudoconvex CR manifold. Moreover, the pseudohermitian scalar curvature of  $(P_M, H, p^*J, \theta/2\pi)$  is equal to  $p^*S(\omega)$ , where  $S(\omega)$  is the scalar curvature of  $(M, J, \omega)$ .*

*Proof.* This result is essentially well known; see [Webster 1978, Section 3] or [Wang 2019, Section 5] for example. However, we give a proof for the reader's

convenience. Let  $(z^1, \dots, z^n)$  be a holomorphic local coordinate of  $M$ . Then  $\theta^\alpha := p^* dz^\alpha$  defines an admissible coframe. Since  $d\theta^\alpha = p^* ddz^\alpha = 0$ , the almost CR structure  $(H, p^*J)$  is integrable. Moreover,

$$L_\theta(X, X) = d\theta(X, (p^*J)X) = 2\pi\omega(p_*X, J(p_*X)) > 0$$

for any nonzero  $X \in H$ , which implies that  $(M, H, p^*J)$  is strongly pseudoconvex.

Consider the Tanaka–Webster connection with respect to  $\theta/2\pi$ . The Kähler form  $\omega$  is written as  $\omega = i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ , where  $(g_{\alpha\bar{\beta}})$  is a positive definite Hermitian matrix. Since  $d\theta/2\pi = p^*\omega$ , we have

$$d\theta/2\pi = i(p^*g_{\alpha\bar{\beta}})\theta^\alpha \wedge \theta^{\bar{\beta}},$$

which implies  $h_{\alpha\bar{\beta}} = p^*g_{\alpha\bar{\beta}}$ . The structure equation on the Kähler manifold  $(M, J, \omega)$  says that

$$0 = d(dz^\beta) = dz^\alpha \wedge \phi_\alpha^\beta, \quad dg_{\alpha\bar{\beta}} = \phi_{\alpha\bar{\beta}} + \phi_{\bar{\beta}\alpha},$$

where  $\phi_\alpha^\beta$  is the Levi-Civita connection 1-form, and so

$$d\theta^\beta = \theta^\alpha \wedge (p^*\phi_\alpha^\beta), \quad d(p^*g_{\alpha\bar{\beta}}) = p^*\phi_{\alpha\bar{\beta}} + p^*\phi_{\bar{\beta}\alpha}.$$

Hence the pull-back connection given by  $p^*\phi_\alpha^\beta$  coincides with the Tanaka–Webster connection. The pseudohermitian torsion is equal to zero and  $P_M$  is Sasakian.

The curvature 2-forms of the Levi-Civita connection on  $(M, J, \omega)$  are given by

$$\Phi_\alpha^\beta := d\phi_\alpha^\beta - \phi_\alpha^\gamma \wedge \phi_\gamma^\beta = R_{\alpha\bar{\rho}\sigma}^\beta dz^\rho \wedge d\bar{z}^\sigma.$$

Its pull-back to  $P_M$  yields

$$p^*\Phi_\alpha^\beta = d(p^*\phi_\alpha^\beta) - (p^*\phi_\alpha^\gamma) \wedge (p^*\phi_\gamma^\beta) = (p^*R_{\alpha\bar{\rho}\sigma}^\beta)\theta^\rho \wedge \theta^{\bar{\sigma}},$$

which are the curvature 2-forms of the Tanaka–Webster connection. Hence the pseudohermitian Ricci tensor  $R_{\rho\bar{\sigma}}$  is given by

$$p^*R_{\alpha\bar{\rho}\sigma}^\alpha = p^*\text{Ric}_{\rho\bar{\sigma}},$$

where Ric is the Ricci tensor of  $(M, J, \omega)$ , and the pseudohermitian scalar curvature is given by

$$R_\rho^\rho = p^*\text{Ric}_\rho^\rho = p^*S(\omega). \quad \square$$

### 3. Formulas for the CR Yamabe constant

As with the Riemannian Yamabe problem, the functional  $\mathfrak{F}(\theta)$  can be rewritten as a functional on  $C^\infty(X, \mathbb{R}_+)$ :

$$(3-1) \quad \mathfrak{F}(u^{2/n}\theta) = \frac{\int_X ((2 + 2/n)|du|_\theta^2 + R_\theta u^2) d\mu_\theta}{\left(\int_X u^{2+2/n} d\mu_\theta\right)^{n/(n+1)}},$$

where  $\theta$  is a fixed contact form and  $|du|_{\tilde{\theta}}^2 = L_{\tilde{\theta}}^*(du|_H, du|_H)$  [Jerison and Lee 1987]. It follows from this that  $\lambda(X) \geq 0$  if  $R_{\theta} \geq 0$ . Moreover, consider the case  $R_{\theta} > 0$ . Suppose to the contrary that  $\lambda(X) = 0$ . Then there exists a CR Yamabe contact form  $\tilde{\theta} = u^{2/n}\theta$  satisfying  $R_{\tilde{\theta}} = 0$ , which contradicts (3-1). Therefore  $\lambda(X) > 0$  if  $R_{\theta} > 0$ ; see [Wang 2003, Proposition 3.1] for another proof.

**Theorem 3.1.** *Let  $(X, H, J)$  be a compact strongly pseudoconvex CR manifold of dimension  $2n + 1$ . If  $\lambda(X) \geq 0$ , then for any  $r \in [1, \infty]$*

$$\lambda(X) \leq \inf_{\tilde{\theta}} \|R_{\tilde{\theta}}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1} - \frac{1}{r}},$$

where  $\text{Vol}_{\theta}(X)$  is the volume of  $X$  with respect to  $d\mu_{\theta}$ . If the CR Yamabe problem is solvable on  $X$ , then the equality holds. If  $\lambda(X) \leq 0$ , then for any  $r \in [n + 1, \infty]$

$$(3-2) \quad \lambda(X) = - \inf_{\tilde{\theta}} \|R_{\tilde{\theta}}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1} - \frac{1}{r}},$$

$$(3-3) \quad = - \inf_{\tilde{\theta}} \|R_{\tilde{\theta}}^-\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1} - \frac{1}{r}},$$

where  $R_{\tilde{\theta}}^- := \min(R_{\tilde{\theta}}, 0)$ , and the two infima are realized only by a CR Yamabe minimizer.

*Proof.* When  $\lambda(X) \geq 0$ , the Hölder inequality implies

$$\lambda(X) \leq \frac{\int_X R_{\tilde{\theta}} d\mu_{\tilde{\theta}}}{\text{Vol}_{\tilde{\theta}}(X)^{n/(n+1)}} \leq \|R_{\tilde{\theta}}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1} - \frac{1}{r}},$$

and the equality holds if  $\tilde{\theta}$  is a CR Yamabe minimizer.

Now in the case of  $\lambda(X) \leq 0$ , we use the technique of Besson, Courtois, and Gallot [Besson et al. 1991]. Let  $\theta$  be a CR Yamabe minimizer, which is unique up to a constant in this case. Consider another contact form  $\tilde{\theta} = u^{2/n}\theta$ , where  $u$  is a positive smooth function. The Hölder inequality yields

$$\begin{aligned} \|R_{\tilde{\theta}}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1} - \frac{1}{r}} &= \left( \int_X |R_{\tilde{\theta}}|^r d\mu_{\tilde{\theta}} \right)^{\frac{1}{r}} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1} - \frac{1}{r}} \\ &\geq \left( \int_X |R_{\tilde{\theta}}^-|^r u^{2+2/n} d\mu_{\theta} \right)^{\frac{1}{r}} \left( \int_X u^{2+2/n} d\mu_{\theta} \right)^{\frac{1}{n+1} - \frac{1}{r}} \\ &\geq \left( \int_X (-R_{\tilde{\theta}}^-) u^{2/n} d\mu_{\theta} \right) \text{Vol}_{\theta}(X)^{-\frac{n}{n+1}} \\ &\geq \left( \int_X (-R_{\tilde{\theta}}) u^{2/n} d\mu_{\theta} \right) \text{Vol}_{\theta}(X)^{-\frac{n}{n+1}}. \end{aligned}$$

Here recall that

$$R_{\tilde{\theta}} = u^{-1-2/n}(R_{\theta} + (2 + 2/n)\Delta_b)u,$$

where  $\Delta_b$  is the sublaplacian [Jerison and Lee 1987]. Therefore

$$\begin{aligned} \|R_{\tilde{\theta}}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}} &\geq \left( \int_X (-R_{\theta} - (2+2/n)u^{-1}\Delta_b u) d\mu_{\theta} \right) \text{Vol}_{\theta}(X)^{-\frac{n}{n+1}} \\ &= \left( \int_X (-R_{\theta} + (2+2/n)u^{-2}|du|_{\tilde{\theta}}^2) d\mu_{\theta} \right) \text{Vol}_{\theta}(X)^{-\frac{n}{n+1}} \\ &\geq - \left( \int_X R_{\theta} d\mu_{\theta} \right) \text{Vol}_{\theta}(X)^{-\frac{n}{n+1}} \\ &= -\lambda(X). \end{aligned}$$

This proves

$$\lambda(X) \geq -\inf_{\tilde{\theta}} \|R_{\tilde{\theta}}^{-}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}} \geq -\inf_{\tilde{\theta}} \|R_{\tilde{\theta}}\|_{L^r} \text{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}}.$$

On the other hand,

$$\lambda(X) = -\|R_{\theta}^{-}\|_{L^r} \text{Vol}_{\theta}(X)^{\frac{1}{n+1}-\frac{1}{r}} = -\|R_{\theta}\|_{L^r} \text{Vol}_{\theta}(X)^{\frac{1}{n+1}-\frac{1}{r}}$$

since  $R_{\theta}$  is a nonpositive constant, which proves the two desired formulas together. It remains to decide when the infima are realized. The infimum of (3-2) or (3-3) is realized by  $\tilde{\theta}$  if and only if the above all inequalities are attained by equalities, which holds if and only if  $R_{\tilde{\theta}}^{-} \equiv R_{\tilde{\theta}}$  is a nonpositive constant (and hence  $u$  is a positive constant); i.e.,  $\tilde{\theta}$  is a CR Yamabe minimizer.  $\square$

#### 4. Continuity of the CR Yamabe constant

In this section, we prove the continuity of the CR Yamabe constant under suitable deformations of CR structures. Remark that Lemma 4.2 below is a generalization of [Dietrich 2021, Lemma 5.5].

**Proposition 4.1.** *Let  $(X, H, J, \theta)$  be a compact pseudohermitian manifold of dimension  $2n+1$ . Assume that  $(X, H_i, J_i, \theta_i)_{i \in \mathbb{N}}$  is a sequence of pseudohermitian structures on  $X$  such that  $\theta_i \rightarrow \theta$  in the  $C^2$ -topology, and  $J_i \rightarrow J$  and  $R_{\theta_i} \rightarrow R_{\theta}$  in the  $C^0$ -topology, where  $J_i$  and  $J$  extend to endomorphisms on  $TX$  in an obvious way. Then one has  $\lambda(X, H_i, J_i) \rightarrow \lambda(X, H, J)$ .*

*Proof.* Since  $\theta_i \rightarrow \theta$  in the  $C^2$ -topology, we may assume that  $\theta_i^t := t\theta_i + (1-t)\theta$  for  $t \in [0, 1]$  is a smooth family of contact forms on  $X$ . Let  $T_i^t$  be the Reeb vector field of  $\theta_i^t$ , which is determined by

$$\theta_i^t(T_i^t) = 1, \quad \iota(T_i^t)d\theta_i^t = 0.$$

Since  $\theta_i \rightarrow \theta$  in the  $C^2$ -topology,  $\sup_{t \in [0,1]} \|T_i^t - T\|_{C^1} \rightarrow 0$ . Take the time-dependent vector field  $V_i^t \in \Gamma(\text{Ker } \theta_i^t)$  satisfying

$$\iota(V_i^t) d\theta_i^t = (\theta_i - \theta)(T_i^t) \theta_i^t - (\theta_i - \theta).$$

It follows from  $\sup_{t \in [0,1]} \|\theta_i^t - \theta\|_{C^2} \rightarrow 0$  and  $\sup_{t \in [0,1]} \|T_i^t - T\|_{C^1} \rightarrow 0$  that

$$(4-1) \quad \sup_{t \in [0,1]} \|V_i^t\|_{C^1} \rightarrow 0.$$

The isotopy  $\psi_i^t : X \rightarrow X$  generated by  $V_i^t$  satisfies  $(\psi_i^t)^* H_i^t = H$  for any  $t \in [0, 1]$ ; see the proof of the Gray stability theorem [Geiges 2008, Theorem 2.2.2]. Equation (4-1) yields that  $\psi_i^1 \rightarrow \text{id}_X$  in the  $C^1$ -topology; see, e.g., [Zhang 2022] for a modern treatment of time-dependent vector fields with parameters and a proof of this fact. In particular,

$$\begin{aligned} \tilde{\theta}_i &:= (\psi_i^1)^* \theta_i \rightarrow \theta, & d\tilde{\theta}_i &:= (\psi_i^1)^* d\theta_i \rightarrow d\theta, \\ \tilde{J}_i &:= (\psi_i^1)^* J_i \rightarrow J, & R_{\tilde{\theta}_i} &:= (\psi_i^1)^* R_{\theta_i} \rightarrow R_\theta \end{aligned}$$

in the  $C^0$ -topology. Since  $\lambda(X, H_i, J_i) = \lambda(X, H, \tilde{J}_i)$ , the statement follows from the lemma below.  $\square$

**Lemma 4.2.** *Let  $(X, H, J, \theta)$  be a compact pseudohermitian manifold of dimension  $2n + 1$ . Assume that  $(X, H, J_i, \theta_i)_{i \in \mathbb{N}}$  is a sequence of pseudohermitian structures on  $X$  such that  $\theta_i \rightarrow \theta$ ,  $d\theta_i \rightarrow d\theta$ ,  $J_i \rightarrow J$ , and  $R_{\theta_i} \rightarrow R_\theta$  in the  $C^0$ -topology. Then one has  $\lambda(X, H, J_i) \rightarrow \lambda(X, H, J)$ .*

*Proof.* Without loss of generality, we may assume that  $\text{Vol}_{\theta_i}(X) = \text{Vol}_\theta(X) = 1$ . Since  $R_{\theta_i} \rightarrow R_\theta$  in the  $C^0$ -topology, we can find  $K > 0$  such that  $|R_\theta| < K$  and  $|R_{\theta_i}| < K$  for any  $i$ . For each  $\varepsilon \in (0, 1)$ , take  $N(\varepsilon) \in \mathbb{Z}_+$  such that  $i \geq N(\varepsilon)$  implies

$$(1+\varepsilon)^{-1} L_\theta < L_{\theta_i} < (1+\varepsilon) L_\theta, \quad (1+\varepsilon)^{-1} d\mu_\theta < d\mu_{\theta_i} < (1+\varepsilon) d\mu_\theta, \quad |R_{\theta_i} - R_\theta| < \varepsilon.$$

For any  $f \in C^\infty(X, \mathbb{R})$ , we write  $f^+ := \max(f, 0)$  and  $f^- := \min(f, 0)$ . Let  $u \in C^\infty(X, \mathbb{R}_+)$ . The Hölder inequality yields

$$\int_X u^2 d\mu_\theta \leq \left( \int_X u^{2+2/n} d\mu_\theta \right)^{\frac{n}{n+1}} \text{Vol}_\theta(X)^{\frac{1}{n+1}} = \left( \int_X u^{2+2/n} d\mu_\theta \right)^{\frac{n}{n+1}}$$

and

$$\left| \int_X R_\theta^\pm u^2 d\mu_\theta \right| \leq K \int_X u^2 d\mu_\theta \leq K \left( \int_X u^{2+2/n} d\mu_\theta \right)^{\frac{n}{n+1}}.$$

It follows from the above inequalities that

$$\begin{aligned} \int_X R_{\theta_i} u^2 d\mu_{\theta_i} &\leq \int_X (R_{\theta} + \varepsilon) u^2 d\mu_{\theta_i} \\ &= \int_X (R_{\theta}^+ + \varepsilon) u^2 d\mu_{\theta_i} + \int_X R_{\theta}^- u^2 d\mu_{\theta_i} \\ &\leq (1 + \varepsilon) \int_X (R_{\theta}^+ + \varepsilon) u^2 d\mu_{\theta} + (1 + \varepsilon)^{-1} \int_X R_{\theta}^- u^2 d\mu_{\theta}, \end{aligned}$$

and

$$(1 + \varepsilon)^{-1} \int_X u^{2+2/n} d\mu_{\theta} \leq \int_X u^{2+2/n} d\mu_{\theta_i} \leq (1 + \varepsilon) \int_X u^{2+2/n} d\mu_{\theta}.$$

Thus we have

$$\begin{aligned} \mathfrak{F}(u^{2/n} \theta_i) &= \left[ \int_X \left( \left(2 + \frac{2}{n}\right) |du|_{\theta_i}^2 + R_{\theta_i} u^2 \right) d\mu_{\theta_i} \right] \left( \int_X u^{2+\frac{2}{n}} d\mu_{\theta_i} \right)^{-\frac{n}{n+1}} \\ &\leq \left[ (1 + \varepsilon)^{2+\frac{n}{n+1}} \int_X \left(2 + \frac{2}{n}\right) |du|_{\theta}^2 d\mu_{\theta} + (1 + \varepsilon)^{1+\frac{n}{n+1}} \int_X (R_{\theta}^+ + \varepsilon) u^2 d\mu_{\theta} \right. \\ &\quad \left. + (1 + \varepsilon)^{-1-\frac{n}{n+1}} \int_X R_{\theta}^- u^2 d\mu_{\theta} \right] \left( \int_X u^{2+\frac{2}{n}} d\mu_{\theta} \right)^{-\frac{n}{n+1}} \\ &\leq (1 + \varepsilon)^{2+\frac{n}{n+1}} \mathfrak{F}(u^{2/n} \theta) + C\varepsilon, \end{aligned}$$

where  $C$  is a positive constant independent of  $u$  and  $\varepsilon$ . Taking the infimum yields

$$\lambda(X, H, J_i) \leq (1 + \varepsilon)^{2+\frac{n}{n+1}} \lambda(X, H, J) + C\varepsilon.$$

Since  $(J, \theta)$  and  $(J_i, \theta_i)$  are symmetric, we also obtain

$$\lambda(X, H, J) \leq (1 + \varepsilon)^{2+\frac{n}{n+1}} \lambda(X, H, J_i) + C\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have

$$\limsup_{i \rightarrow \infty} \lambda(X, H, J_i) \leq \lambda(X, H, J) \leq \liminf_{i \rightarrow \infty} \lambda(X, H, J_i),$$

which implies  $\lambda(X, H, J_i) \rightarrow \lambda(X, H, J)$ . □

## 5. Deformations of CR structures with varying CR Yamabe constants

In this section, we construct a family of strongly pseudoconvex CR structures with varying CR Yamabe constants. Let  $(M, J, \omega)$  be an  $n$ -dimensional compact Hodge manifold with constant scalar curvature. Let us write

$$\mathcal{K} := \{\varphi \in C^\infty(M) \mid \omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0\}$$

for the space of Kähler potentials in the class  $[\omega]$  endowed with the  $C^4$ -topology. For any  $\varphi \in \mathcal{K}$ , we have

$$\int_M \omega_\varphi^n = [\omega]^n, \quad \int_M S(\omega_\varphi) \omega_\varphi^n = 2n\pi c_1(M) \cup [\omega]^{n-1},$$

which are independent of the choice of  $\varphi$ . In particular if  $\omega_\varphi$  is a constant scalar curvature Kähler metric, then

$$S(\omega_\varphi) = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{[\omega]^n} =: \hat{S}.$$

Set

$$\mathcal{F} := \{\varphi \in \mathcal{K} \mid S(\omega_\varphi) = \hat{S}\}$$

so that  $\omega_\varphi$  is not a constant scalar curvature Kähler metric for any  $\varphi \in \mathcal{K} \setminus \mathcal{F}$ . It is known that  $\omega_\varphi$  is a constant scalar curvature Kähler metric if and only if there exists  $F \in \text{Aut}^0(M)$  such that  $\omega_\varphi = F^*\omega$  [Berman and Berndtsson 2017, Theorem 1.3]. In particular if  $\text{Aut}(M)$  is discrete, or equivalently,  $M$  admits no nontrivial holomorphic vector fields, then  $\mathcal{F} = \mathbb{R}$ . More generally, if any holomorphic vector field is parallel, then  $\mathcal{F} = \mathbb{R}$ . This is because a parallel and holomorphic vector field preserves the Kähler form  $\omega$ .

For each  $\varphi \in \mathcal{K}$ , take a principal connection  $\theta_\varphi$  on  $P_M$  satisfying  $d\theta_\varphi/2\pi = p^*\omega_\varphi$ , which is given by

$$\theta_\varphi := \theta + \pi p^* d^c \varphi,$$

where  $d^c := i(\bar{\partial} - \partial)$ . This gives an infinite-dimensional family of pseudohermitian manifolds

$$X_\varphi := (P_M, H_\varphi := \text{Ker } \theta_\varphi, p^*J, \theta_\varphi/2\pi)$$

underlying the same manifold  $P_M$ . Integration along fibers yields that

$$\begin{aligned} \mathfrak{F}(\theta_\varphi/2\pi) &= \frac{\int_{P_M} p^* S(\omega_\varphi) (\theta_\varphi/2\pi) \wedge (p^* \omega_\varphi)^n}{\left(\int_{P_M} (\theta_\varphi/2\pi) \wedge (p^* \omega_\varphi)^n\right)^{n/(n+1)}} \\ &= \frac{\int_M S(\omega_\varphi) \omega_\varphi^n}{\left(\int_M \omega_\varphi^n\right)^{n/(n+1)}} = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{([\omega]^n)^{n/(n+1)}}, \end{aligned}$$

which is independent of  $\varphi$ .

*Proof of Theorem 1.1.* The continuity follows from Proposition 4.1. If  $\theta_0 = \theta$  is a CR Yamabe minimizer, then  $\lambda(P_M, H, p^*J) = \mathfrak{F}(\theta_0/2\pi)$ . On the other hand,  $p^*S(\omega_\varphi)$  is not constant for any  $\varphi \in \mathcal{K} \setminus \mathcal{F}$ , and so

$$\lambda(P_M, H_\varphi, p^*J) < \mathfrak{F}(\theta_\varphi/2\pi) = \mathfrak{F}(\theta_0/2\pi) = \lambda(P_M, H, p^*J),$$

which completes the first assertion. If  $\omega$  has nonpositive constant scalar curvature, then so does  $\theta/2\pi$ , which must be a CR Yamabe minimizer. If  $\omega$  is a Kähler–Einstein metric, then  $\theta/2\pi$  is an Einstein contact form. It follows from Lemma 2.1 that  $\theta/2\pi$  is a CR Yamabe minimizer.  $\square$

In general, it may be cumbersome to have nontrivial  $\mathcal{F}$ , since it can be singular and not easy to locate. In fact, we do not have many examples with nontrivial  $\mathcal{F}$ . We give some examples of Hodge manifolds with  $\mathcal{F} = \mathbb{R}$ .

**Example 5.1** (Kähler–Einstein manifolds with nonpositive scalar curvature). First, a compact complex manifold  $M$  with  $c_1(M) < 0$  admits a Kähler–Einstein metric in the Kähler class  $-c_1(M)$  by the Aubin–Yau theorem [Aubin 1976; Yau 1978]. Second, the celebrated Calabi–Yau theorem [Yau 1978] implies that any Kähler class on a compact complex manifold  $M$  with  $c_1(M) = 0$  in  $H^2(M; \mathbb{R})$  is represented by a unique Ricci-flat Kähler metric. Thus we can take any integral Kähler class for our purpose. In these cases, a constant scalar curvature Kähler metric in any Kähler class, if it exists, must be unique [Chen 2000, Theorem 7] and hence  $\mathcal{F} = \mathbb{R}$ .

**Example 5.2** (Fano manifolds). Let  $M$  be a Fano manifold; that is, a compact complex manifold with  $c_1(M) > 0$ . If  $M$  admits a Kähler–Einstein metric  $\omega$  in the Kähler class  $c_1(M)$ , then  $\theta/2\pi$  gives a CR Yamabe minimizer. For example, a complex surface given by a blow-up of  $\mathbb{C}P^2$  at  $m$  points in general position with  $3 \leq m \leq 8$  admits a Kähler–Einstein metric of positive scalar curvature [Tian and Yau 1987; Tian 1990]. Note that the automorphism groups of these surfaces are discrete. As a higher-dimensional example, consider the Fermat hypersurface  $F_{n,d} \subset \mathbb{C}P^{n+1}$  of degree  $3 \leq d \leq n+1$ , that is,

$$F_{n,d} := \left\{ [z_0 : \cdots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid \sum_{k=0}^{n+1} z_k^d = 0 \right\}.$$

This  $F_{n,d}$  admits a Kähler–Einstein metric [Tian 2000, Section 6.3]. Note that  $F_{n,d}$  has no nontrivial holomorphic vector field [Kodaira and Spencer 1958, Lemma 14.2]; see [Matsumura and Monsky 1964] for another proof.

**Example 5.3** (scalar-flat but not Ricci-flat surfaces). Take a complex surface  $S_m$  obtained by blowing up  $\mathbb{C}P^1 \times \Sigma$  at generic  $m$  points  $p_1, \dots, p_m$ , where  $\Sigma$  is a compact Riemann surface of genus  $g \geq 2$  and  $m \geq 3$ . It is known that  $S_m$  admits a scalar-flat Kähler form  $\tilde{\omega}$  [LeBrun and Singer 1993, Theorem 3.11]. Note that  $S_m$  does not admit a Ricci-flat Kähler metric since  $c_1(S_m)^2 = 8(1-g) - m < 0$ . Assume that the projection of  $\{p_1, \dots, p_m\}$  to  $\mathbb{C}P^1$  consists of at least 3 points. Since  $\Sigma$  has no nontrivial holomorphic vector fields and any holomorphic vector field on  $\mathbb{C}P^1$  vanishing at least 3 points must be trivial,  $S_m$  admits no nontrivial holomorphic vector fields also.

It still remains to show that  $S_m$  has a scalar-flat integral Kähler class. Since  $H^{2,0}(S_m) = 0$ , we have  $H^{1,1}(S_m; \mathbb{R}) \cong H^2(S_m; \mathbb{Z}) \otimes \mathbb{R}$ . The linear operator

$$c_1(S_m) \cup : H^{1,1}(S_m; \mathbb{R}) \rightarrow \mathbb{R}$$

has integer coefficients and satisfies  $c_1(S_m) \cup [\tilde{\omega}] = 0$ . Hence there exists a rational Kähler class  $\mu$  close enough to  $[\tilde{\omega}]$  satisfying  $c_1(S_m) \cup \mu = 0$ . [LeBrun and Simanca 1995, Theorem 1] implies that  $\mu$  contains a scalar-flat Kähler form  $\omega$ . Therefore we obtain a scalar-flat integral Kähler class by homothety.

## 6. Construction of a manifold with different signs of CR Yamabe constants

In this section, we construct a manifold admitting two strongly pseudoconvex CR structures with different signs of CR Yamabe constants. Our construction is based on an exotic smooth structure of a certain complex surface and an adaptation of the technique originated by Ruan [1994] and used by Kim and Sung [2016] to show the existence of inequivalent symplectic structures on certain 6-manifolds.

Let  $B$  be the Barlow surface [Barlow 1985; Kotschick 1989] and  $R_8$  be a complex surface given by a blow-up of  $\mathbb{C}P^2$  at 8 points in general position. The Barlow surface is a simply connected minimal surface of general type with  $q = p_g = 0$  and  $c_1(B)^2 = 1$ , and contains  $(-2)$ -curves so that its canonical line bundle is not ample. But as shown in [Catanese and LeBrun 1997, Theorem 7], it has a small deformation with ample canonical line bundle and hence admits a Kähler–Einstein metric of negative scalar curvature by the celebrated Aubin–Yau theorem. By the results in [Tian and Yau 1987; Tian 1990],  $R_8$  admits a Kähler–Einstein metric of positive scalar curvature. It is well-known that  $B$  and  $R_8$  are homeomorphic by Freedman’s classification [Freedman 1982, Theorem 1.5] while they are not diffeomorphic by Kotschick’s theorem [Kotschick 1989, Theorem 1].

**Remark 6.1.** An easier way of proving Kotschick’s theorem by using Seiberg–Witten invariant runs as follows. Since  $R_8$  and  $B$  have  $b_2^+ = 1$ , their Seiberg–Witten invariants for a  $\text{Spin}^c$  structure  $\xi$  with  $c_1(\xi)^2 > 0$  are well-defined for any small perturbation. The complex surface  $R_8$  admits a metric of positive scalar curvature, so its Seiberg–Witten invariants all vanish. However, the Seiberg–Witten invariant of  $B$  for the canonical  $\text{Spin}^c$  structure determined by the complex structure is  $\pm 1$  [Morgan 1996].

Since the intersection forms of both  $B$  and  $R_8$  are indefinite and odd, they are isomorphic to  $(1) \oplus 8(-1)$ . Wall [1962, p. 336] has proved that all characteristic vectors with square 1 in  $(1) \oplus 8(-1)$  are equivalent. Since the first Chern class of  $B$  and  $R_8$  are characteristic with square 1 by Wu’s formula, there is an isomorphism from  $H^2(R_8; \mathbb{Z})$  to  $H^2(B; \mathbb{Z})$  preserving the intersection form and the first

Chern class. This induces an isomorphism

$$\Psi : H^*(R_8 \times \mathbb{C}P^1; \mathbb{Z}) \rightarrow H^*(B \times \mathbb{C}P^1; \mathbb{Z})$$

preserving  $H^*(\mathbb{C}P^1; \mathbb{Z})$  in the obvious way. We claim that  $\Psi$  satisfies the conditions of the following theorem.

**Theorem 6.2** [Jupp 1973, Theorem 1]. *Let  $X$  and  $Y$  be smooth closed simply connected 6-manifolds with torsion-free homology. Suppose that there is an isomorphism from  $H^*(X; \mathbb{Z})$  to  $H^*(Y; \mathbb{Z})$  preserving the triple cup product structure  $\mu : H^2 \otimes H^2 \otimes H^2 \rightarrow \mathbb{Z}$ , the second Stiefel–Whitney class, and the first Pontryagin class. Then there exists an orientation-preserving diffeomorphism from  $X$  to  $Y$  realizing this algebraic isomorphism.*

It is enough to check that  $\Psi$  preserves the specified characteristic classes. By the product formula,

$$\begin{aligned} w_2(R_8 \times \mathbb{C}P^1) &= w_2(R_8) + w_1(R_8) w_1(\mathbb{C}P^1) + w_2(\mathbb{C}P^1) \\ &= c_1(R_8) + 0 + c_1(\mathbb{C}P^1) \pmod{2}, \end{aligned}$$

and likewise for  $B \times \mathbb{C}P^1$ . Since

$$\Psi(c_1(R_8) + c_1(\mathbb{C}P^1)) = c_1(B) + c_1(\mathbb{C}P^1),$$

$\Psi$  preserves the second Stiefel–Whitney class. Using the fact that  $p_1 = c_1^2 - 2c_2$  and the product formula, we have

$$\begin{aligned} p_1(R_8 \times \mathbb{C}P^1) &= c_1(R_8 \times \mathbb{C}P^1)^2 - 2c_2(R_8 \times \mathbb{C}P^1) \\ &= (c_1(R_8) + c_1(\mathbb{C}P^1))^2 - 2(c_2(R_8) + c_1(R_8) c_1(\mathbb{C}P^1)), \end{aligned}$$

and likewise for  $B \times \mathbb{C}P^1$ . Since  $\Psi$  preserves the Euler characteristic; i.e., the alternating sum of Betti numbers,  $\Psi$  maps  $e(R_8) = c_2(R_8)$  to  $e(B) = c_2(B)$ . Therefore  $\Psi$  preserves the first Pontryagin class too, and we have an orientation-preserving diffeomorphism

$$\psi : R_8 \times \mathbb{C}P^1 \rightarrow B \times \mathbb{C}P^1$$

satisfying  $\psi^*(c_1(B)) = c_1(R_8)$  and  $\psi^*(c_1(\mathbb{C}P^1)) = c_1(\mathbb{C}P^1)$ .

Let  $n \geq 3$ . Take Kähler forms  $\omega_1 \in c_1(R_8)$ ,  $\omega_2 \in c_1(\mathbb{C}P^1)$ , and  $\omega_3 \in nc_1(\mathbb{C}P^{n-3})$  such that

$$\text{Ric}(\omega_1) = 2\pi\omega_1, \quad \text{Ric}(\omega_2) = 2\pi\omega_2, \quad \text{Ric}(\omega_3) = \frac{2\pi}{n}\omega_3.$$

Then

$$(M := R_8 \times \mathbb{C}P^1 \times \mathbb{C}P^{n-3}, \omega := \omega_1 + \omega_2 + \omega_3)$$

is a Kähler manifold with positive Ricci curvature. Let  $P_M$  be the principal  $S^1$ -bundle over  $M$  whose Euler class is  $-\omega$ . This  $P_M$  admits a connection one-form  $\theta$

and the lifted CR structure  $J$  such that  $d\theta/2\pi$  projects down to  $\omega$  on  $M$ . The CR Yamabe constant of  $(P_M, H := \text{Ker } \theta, J)$  must be positive by the argument after formula (3-1).

**Lemma 6.3.** *The manifold  $P_M$  is simply connected.*

*Proof.* Let  $E \cong \mathbb{C}P^1$  be an exceptional divisor in  $R_8$  and consider it as a complex curve in  $M$ . Then the restriction of  $-\omega$  to  $E$  coincides with the first Chern class of the tautological line bundle over  $E$ . Hence  $(P_M)|_E \rightarrow E$  is a Hopf fibration. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \pi_2((P_M)|_E) & \longrightarrow & \pi_2(E) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1((P_M)|_E) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \pi_2(M) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(P_M) \longrightarrow \pi_1(M) = 0 \end{array}$$

Since  $(P_M)|_E \rightarrow E$  is a Hopf fibration,  $\pi_1((P_M)|_E) = 0$  and  $\pi_2((P_M)|_E) = 0$ , and so the map  $\pi_2(E) \rightarrow \pi_1(S^1)$  is an isomorphism. Thus we have  $\pi_2(M) \rightarrow \pi_1(S^1)$  is surjective and  $\pi_1(P_M) = 0$ .  $\square$

On the other hand, let  $\tilde{J}_1$  and  $-\tilde{\omega}_1$  be the complex structure and a Kähler form in the class  $-c_1(B)$  giving an Einstein metric of negative scalar curvature on  $B$ . Denote by  $\tilde{M}'$  the complex 3-manifold  $(B \times \mathbb{C}P^1, (-\tilde{J}_1) \times J_2)$ . The two-form  $\tilde{\omega}' = \tilde{\omega}_1 + \omega_2$  defines a Kähler form on  $\tilde{M}'$  with constant scalar curvature  $-2\pi$ . Consider the Kähler manifold

$$(\tilde{M} := \tilde{M}' \times \mathbb{C}P^{n-3}, \tilde{\omega} := \tilde{\omega}' + \omega_3).$$

The scalar curvature of this manifold is given by  $-2\pi + \frac{2(n-3)\pi}{n} < 0$ . Denote by  $\tilde{\psi}$  the diffeomorphism

$$\psi \times \text{id}_{\mathbb{C}P^{n-3}} : (R_8 \times \mathbb{C}P^1) \times \mathbb{C}P^{n-3} \rightarrow (B \times \mathbb{C}P^1) \times \mathbb{C}P^{n-3}.$$

Then  $\tilde{\psi}^*([\tilde{\omega}]) = [\omega]$  since  $\psi^*([\tilde{\omega}']) = [\omega_1] + [\omega_2]$ . Hence there exists a connection form  $\tilde{\theta}$  of  $P_M$  and the lifted CR structure  $\tilde{J}$  such that  $d\tilde{\theta}/2\pi = p^*\tilde{\psi}^*\tilde{\omega}$ . We derive from Proposition 2.2 that  $(P_M, \tilde{H} := \text{Ker } \tilde{\theta}, \tilde{J})$  has negative CR Yamabe constant.

Before the proof of Theorem 1.2, we recall some facts on contact geometry. Two cooriented contact manifolds  $(X, H)$  and  $(X', H')$  are isomorphic if there exists a diffeomorphism  $\psi : X \rightarrow X'$  preserving contact structures and coorientation. Moreover, the first Chern class of a strongly pseudoconvex CR manifold is an invariant of the underlying cooriented contact structure; see [Geiges 2008, Section 2.4] for example.

*Proof of Theorem 1.2.* It remains to show that  $(P_M, H)$  is not isomorphic to  $(P_M, \tilde{H})$  as cooriented contact manifolds. Denote by  $p : P_M \rightarrow M$  the projection from  $P_M$

to  $M$ . Then

$$c_1(P_M, H, J) = p^*c_1(M), \quad c_1(P_M, \tilde{H}, \tilde{J}) = p^*\tilde{\psi}^*c_1(\tilde{M}).$$

Now consider the Gysin exact sequence

$$H^0(M; \mathbb{Z}) = \mathbb{Z} \xrightarrow{-[\omega]} H^2(M; \mathbb{Z}) \xrightarrow{D^*} H^2(P_M; \mathbb{Z}) \rightarrow H^1(M; \mathbb{Z}) = 0.$$

We first consider the case  $n = 3$ . Since  $c_1(R_8)$  and  $[\omega] = c_1(R_8) + c_1(\mathbb{C}P^1)$  are linearly independent in  $H^2(M; \mathbb{Z})$ , we have

$$c_1(P_M, H, J) = p^*c_1(M) = 0, \quad c_1(P_M, \tilde{H}, \tilde{J}) = p^*\tilde{\psi}^*c_1(\tilde{M}) = -2p^*c_1(R_8) \neq 0.$$

Hence  $(P_M, H)$  is not isomorphic to  $(P_M, \tilde{H})$ .

In the remainder of the proof, we assume that  $n \geq 4$ . In this case,

$$c_1(P_M, H, J) = p^*c_1(M) = -(n-1)(n-2)p^*c_1(\mathcal{O}_{\mathbb{C}P^{n-3}}(1)).$$

In particular,  $[c_1(P_M, H, J)] = 0$  in  $H^2(P_M; \mathbb{Z})/(n-1)H^2(P_M; \mathbb{Z})$ . It suffices to show that  $[c_1(P_M, \tilde{H}, \tilde{J})] \neq 0$  in  $H^2(P_M; \mathbb{Z})/(n-1)H^2(P_M; \mathbb{Z})$ . Suppose to the contrary that  $[c_1(P_M, \tilde{H}, \tilde{J})] = [p^*\tilde{\psi}^*c_1(\tilde{M})] = 0$  in  $H^2(P_M; \mathbb{Z})/(n-1)H^2(P_M; \mathbb{Z})$ . Consider the following exact sequence:

$$H^0(M; \mathbb{Z}) = \mathbb{Z} \xrightarrow{-[\omega]} \frac{H^2(M; \mathbb{Z})}{(n-1)H^2(M; \mathbb{Z})} \xrightarrow{p^*} \frac{H^2(P_M; \mathbb{Z})}{(n-1)H^2(P_M; \mathbb{Z})}.$$

This yields that there exists  $k \in \mathbb{Z}$  such that

$$\tilde{\psi}^*c_1(\tilde{M}) + k[\omega] = -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega] \in (n-1)H^2(M; \mathbb{Z})$$

Hence

$$\langle -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega], a \rangle \equiv 0 \pmod{n-1}$$

for any  $a \in H_2(M; \mathbb{Z})$ . Let  $E \cong \mathbb{C}P^1$  be an exceptional divisor in  $R_8$  and consider it as a complex curve in  $M$ . Taking  $a = [E]$  gives that

$$0 \equiv \langle -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega], [E] \rangle = k - 1 \pmod{n-1}.$$

Consider also a projective line  $L \subset \mathbb{C}P^{n-3}$ , which is seen as a complex curve in  $M$ . Then

$$\begin{aligned} 0 &\equiv \langle -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega], [L] \rangle \\ &= (n-2) + kn(n-2) \equiv n-3 \pmod{n-1}, \end{aligned}$$

which is a contradiction. □

### Acknowledgements

Sung would like to thank Professor Pak Tung Ho for a kind discussion at KIAS, and Professor Chong-Kyu Han for introducing him to CR geometry and warm encouragement on this work.

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Received March 7, 2024. Revised February 1, 2025.

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
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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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Volume 335 No. 1 March 2025

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