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WEIGHTED LOW-LYING ZEROS OF L-FUNCTIONS ATTACHED TO SIEGEL MODULAR FORMS

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We study weighted low-lying zeros of spinor and standard L-functions attached to degree 2 Siegel modular forms. We show that the symmetry type of weighted low-lying zeros of spinor L-functions is symplectic, for test functions whose Fourier transform have support in $(-1, 1)$, extending the previous range $\left(-\frac{4}{15}, \frac{4}{15}\right)$. We then show that the symmetry type of weighted low-lying zeros of standard L-functions is also symplectic. We further extend the range of support by performing an average over weight. As an application, we discuss nonvanishing of central values of those L-functions.

1. Introduction

D. Hilbert and G. Pólya suggested that nontrivial zeros of the Riemann zeta function $\zeta(s)$ correspond to eigenvalues of a self-adjoint operator on some Hilbert space. The first evidence of such a connection was found by H. L. Montgomery [1973], who investigated the pair correlation of nontrivial zeros of $\zeta(s)$ and conjectured that it is, as pointed out by F. J. Dyson, the same as the pair correlation of eigenvalues of random Hermitian or unitary matrices of large order, also known as the gaussian unitary ensemble (GUE) model. This conjecture of Montgomery was later supported by numerical results by A. M. Odlyzko [1987], based on values for the first 10^5 zeros and for zeros number $10^{12}+1$ to $10^{12}+10^5$. The local spacing between these sample zeros matches the prediction by the GUE model quite well.

Z. Rudnick and P. Sarnak [1996] extended Montgomery's work by computing the general n-level correlation function of zeros of any principal L-function $L(s,\pi)$ attached to a cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ (in a restricted range). Their answer is universal and is precisely the one predicted by the GUE model. Numerical evidence was found by R. Rumely [1993] for primitive Dirichlet L-functions, and by M. O. Rubinstein [1998] for Hasse–Weil L-functions of three distinct elliptic curves and for the Hecke L-function associated to Ramanujan's τ-function.

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Although the n-level correlation statistic of zeros of any fixed principal automorphic L-function obeys the universal GUE law, there is another statistic, called the n-level density of low-lying zeros, that is sensitive to families. N. Katz and Sarnak [1999a] studied low-lying zeros of zeta functions of varieties over finite fields (the "function field" analogue). For these they indicated that a spectral interpretation exists in terms of eigenvalues of Frobenius on cohomology groups. On the number field side, although many results concerning low-lying zeros have been proved, it is still not clear where their spectral nature comes from. See also [Katz and Sarnak 1999b] for a nice survey on these topics.

Before stating our results, we first describe the problem in general terms. Let $\mathcal{F}_Q$ be a family of automorphic forms, ordered by conductor $Q \geq 1$. To each $f \in \mathcal{F}_Q$ one associates an L-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \tag{1-1}$$

which converges absolutely for $s \in \mathbb{C}$ in some right half-plane. We assume that $L(s, f)$ admits meromorphic continuation to the whole complex plane $\mathbb{C}$. We also assume that $L(s, f)$ satisfies a functional equation

$$\Lambda(s, f) = L_\infty(s, f) L(s, f) = \varepsilon_f \Lambda(1-s, f), \tag{1-2}$$

where $\varepsilon_f = \pm 1$ is the root number.

We assume the generalized Riemann hypothesis (GRH) for $L(s, f)$. That is, nontrivial zeros of $L(s, f)$ all lie on the critical line. We may denote those zeros by

$$\rho_f = \tfrac{1}{2} + i\gamma_f, \quad \gamma_f \in \mathbb{R}. \tag{1-3}$$

Let $\Phi \in \mathcal{S}(\mathbb{R})$ be an even Schwartz function (called "test function" throughout) whose Fourier transform $\hat{\Phi}$ has compact support. To this end we define the 1-level density of low-lying zeros of $L(s, f)$, with respect to the test function Φ, to be

$$D(f; \Phi) = \sum_{\rho_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right), \tag{1-4}$$

where ρ_f runs through nontrivial zeros of $L(s, f)$, counted with multiplicity, and c_f is a parameter associated with $f \in \mathcal{F}_Q$, comparable to the analytic conductor of f (specified later). The density conjecture for low-lying zeros of $L(s, f)$ asserts that:

Conjecture 1.1 (density conjecture). *For any even Schwartz function Φ whose Fourier transform $\hat{\Phi}$ has compact support, we have*

$$\lim_{Q\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{f \in \mathcal{F}_Q} D(f; \Phi) = \int_{-\infty}^{\infty} \Phi(x) W(\mathcal{F})(x)\, dx \tag{1-5}$$

for some distribution $W(\mathcal{F})$ depending only on $\mathcal{F}$.

Many observations and results in [Katz and Sarnak 1999a] suggest that the distribution $W(\mathcal{F})$ depends on the family $\mathcal{F}$ through a symmetry group $G(\mathcal{F})$. Possible symmetry types are orthogonal O, special orthogonal even SO(even), special orthogonal odd SO(odd), symplectic Sp and unitary U. The corresponding distributions and their Fourier transforms are

$$\begin{aligned} W(\mathrm{O})(x) &= 1+\tfrac{1}{2}\delta_0(x), & \hat{W}(\mathrm{O})(y) &= \delta_0(y)+\tfrac{1}{2}, \\ W(\mathrm{SO(even)})(x) &= 1+\frac{\sin 2\pi x}{2\pi x}, & \hat{W}(\mathrm{SO(even)})(y) &= \delta_0(y)+\tfrac{1}{2}\eta(y), \\ W(\mathrm{SO(odd)})(x) &= 1-\frac{\sin 2\pi x}{2\pi x}+\delta_0(x), & \hat{W}(\mathrm{SO(odd)})(y) &= \delta_0(y)-\tfrac{1}{2}\eta(y)+1, \\ W(\mathrm{Sp})(x) &= 1-\frac{\sin 2\pi x}{2\pi x}, & \hat{W}(\mathrm{Sp})(y) &= \delta_0(y)-\tfrac{1}{2}\eta(y), \\ W(\mathrm{U})(x) &= 1, & \hat{W}(\mathrm{U})(y) &= \delta_0(y), \end{aligned} \tag{1-6}$$

where δ_0 is the Dirac distribution at 0, and $\eta(y)=1, \frac{1}{2}, 0$ for $|y|<1$, $|y|=1$ and $|y|>1$ respectively. The first three distributions of different orthogonal symmetry type have indistinguishable Fourier transforms within $(-1, 1)$, while the symplectic and unitary symmetry types are distinguishable from the orthogonal ones.

The density conjecture (Conjecture 1.1) has been verified for many families (in restricted ranges). See [Iwaniec et al. 2000; Rubinstein 2001; Fouvry and Iwaniec 2003; Guloglu 2005; Young 2006; Dueñez and Miller 2006; Gao and Zhao 2011; Cho and Kim 2015; Shin and Templier 2016; Liu and Miller 2017; Kim et al. 2020], to name a few. In all results towards this direction, the support of Fourier transform of the test function Φ is restricted within certain range. One important question in this topic is how to extend the range as large as possible, for the full density Conjecture 1.1 does not require any condition on the compact support of $\hat{\Phi}$.

One can also consider "weighted" distribution of low-lying zeros by allowing certain weights ω_f. The weighted average density under consideration is

$$\Bigl(\sum_{f\in\mathcal{F}_Q}\omega_f\Bigr)^{-1}\sum_{f\in\mathcal{F}_Q}\omega_f D(f;\Phi). \tag{1-7}$$

Often these weights ω_f contain important arithmetic information such as central values of L-functions, and including them may possibly change the symmetry type. Recent results in this direction include [Kowalski et al. 2012; Knightly and Reno 2019; Sugiyama and Suriajaya 2022; Fazzari 2024].

In this article we study weighted low-lying zeros of spinor and standard L-functions attached to degree 2 Siegel modular forms. For a general introduction on Siegel modular forms, we refer readers to [Klingen 1990; Pitale 2019].

We proceed to describe our results. Let $k \geq 6$ be an even integer. Let $S_k(\Gamma_2)$ be the space of degree 2 holomorphic Siegel cusp forms of weight k for the symplectic group $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$. Each form $F \in S_k(\Gamma_2)$ is a holomorphic function on the Siegel upper half-plane

$$\mathbb{H}_2 = \{Z = X + iY \in M_2(\mathbb{C}) : Z = Z^T, Y > 0\}, \tag{1-8}$$

which satisfies the automorphy condition

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2,\ Z \in \mathbb{H}_2. \tag{1-9}$$

Here and after we use $M_n(R)$ to denote the ring of $n \times n$ matrices over a ring R.

The Fourier expansion of F is

$$F(Z) = \sum_{T \in \mathcal{T}} a_F(T)(\det T)^{\frac{k}{2}-\frac{3}{4}} e(\mathrm{Tr}(TZ)), \quad Z \in \mathbb{H}_2, \tag{1-10}$$

where the summation is taken over the set

$$\mathcal{T} = \{T = (t_{ij}) \in M_2(\mathbb{R}) : T > 0,\ t_{11} \in \mathbb{Z},\ t_{22} \in \mathbb{Z},\ 2t_{12} = 2t_{21} \in \mathbb{Z}\}. \tag{1-11}$$

We call $a_F(T)$ the (normalized) Fourier coefficient of F at T. It is known that $a_F(T) \in \mathbb{R}$.

We use I to denote the 2×2 identity matrix. For $F \in S_k(\Gamma_2)$ we set

$$\omega_F = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\,\Gamma\big(k - \tfrac{3}{2}\big)\,\Gamma(k-2)\frac{a_F(I)^2}{\|F\|^2} \tag{1-12}$$

to be the "harmonic" weight attached to F, where $\|F\|$ is the Petersson norm of F defined by

$$\|F\| = \left(\int_{\Gamma_2\backslash\mathbb{H}_2} |F(Z)|^2 (\det Y)^k \frac{dX\,dY}{(\det Y)^3}\right)^{1/2}. \tag{1-13}$$

We now choose a basis $H_k(\Gamma_2)$ of $S_k(\Gamma_2)$ consisting of eigenforms for all Hecke operators (we call such a form a Hecke eigenform). It is known (see, e.g., (1.8) in [Blomer 2019]) that

$$\sum_{F \in H_k(\Gamma_2)} \omega_F = 1 + O(e^{-k}). \tag{1-14}$$

Note that the above sum is independent of the choice of basis $H_k(\Gamma_2)$.

To each form $F \in H_k(\Gamma_2)$ we can attach a degree 4 spinor L-function $L(s, F; \mathrm{spin})$ and a degree 5 standard L-function $L(s, F; \mathrm{std})$, both normalized so that the central point is $s = \frac{1}{2}$. The analytic conductors of those L-functions are of size k^2 and k^4, respectively. Further properties of these L-functions are discussed in Section 2.

We assume GRH for both spinor and standard L-functions, and denote their nontrivial zeros on the critical line by

$$\rho_{F,\mathrm{spin}} = \tfrac{1}{2} + i\gamma_{F,\mathrm{spin}}, \quad \rho_{F,\mathrm{std}} = \tfrac{1}{2} + i\gamma_{F,\mathrm{std}}. \tag{1-15}$$

The corresponding density functions with respect to a test function Φ are

$$D(F;\Phi;\mathrm{spin}) = \sum_{\rho_{F,\mathrm{spin}}} \Phi\biggl(\frac{\gamma_{F,\mathrm{spin}}}{2\pi}\log c_{F;\mathrm{spin}}\biggr), \tag{1-16}$$

$$D(F;\Phi;\mathrm{std}) = \sum_{\rho_{F,\mathrm{std}}} \Phi\biggl(\frac{\gamma_{F,\mathrm{std}}}{2\pi}\log c_{F;\mathrm{std}}\biggr). \tag{1-17}$$

Our first result concerning low-lying zeros of spinor L-functions is as follows:

Theorem 1.2. *Let Φ be an even Schwartz function whose Fourier transform has support in $(-1,1)$. For $F \in H_k(\Gamma_2)$, define $D(F;\Phi;\mathrm{spin})$ as in (1-16) with $c_{F;\mathrm{spin}} = k^2$ and ω_F as in (1-12). Assume GRH for $L(s,F;\mathrm{spin})$. Then we have*

$$\lim_{k\to\infty} \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F;\Phi;\mathrm{spin}) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\,dx. \tag{1-18}$$

Remark 1.3. The result above has been obtained from [Kowalski et al. 2012], but only for test functions Φ with $\mathrm{supp}(\hat{\Phi}) \subset \bigl(-\frac{4}{15}, \frac{4}{15}\bigr)$, as an application of their quantitative local equidistribution result. Here we extend the range of support to $(-1,1)$. This improvement is crucial in application to nonvanishing problems, as we will explain in Section 5.

Let $H_k^*(\Gamma_2) \subset H_k(\Gamma_2)$ denote a Hecke basis of the space of Saito–Kurokawa lifts (these concepts will be discussed in Section 2). As a direct corollary of Theorem 1.2 we can establish the following nonvanishing result:

Corollary 1.4. *Assume GRH for $L(s,F;\mathrm{spin})$. Then we have*

$$\liminf_{k\to\infty} \sum_{\substack{F\in H_k(\Gamma_2)\setminus H_k^*(\Gamma_2)\\ L(1/2,F;\mathrm{spin})\neq 0}} \omega_F \geq \tfrac{3}{4}. \tag{1-19}$$

Remark 1.5. For comparison, it is shown in [Blomer 2019] that

$$\sum_{\substack{F\in H_k(\Gamma_2)\setminus H_k^*(\Gamma_2)\\ L(1/2,F;\mathrm{spin})\neq 0}} \omega_F \gg \frac{1}{\log k}, \tag{1-20}$$

unconditionally for large k. This follows from asymptotic formulas for the first and second moments of central values. Although it is not surprising that GRH would yield much stronger result, one still needs the range of support in Theorem 1.2 not to be too small to carry out the argument. It was also pointed out in [Blomer 2019, page 1756, Remark (b)] that it is possible to use the mollifier technique to obtain a positive proportional result unconditionally. The exact (unconditional) proportion would not be as large as our (conditional) proportion $\frac{3}{4}$ though.

For low-lying zeros of standard L-functions, we have the following result:

Theorem 1.6. *Let Φ be an even Schwartz function whose Fourier transform has support in $\left(-\frac{1}{4}, \frac{1}{4}\right)$. For $F \in H_k(\Gamma_2)$, define $D(F; \Phi; \mathrm{std})$ as in (1-17) with $c_{F;\mathrm{std}} = k^4$ and ω_F as in (1-12). Assume GRH for $L(s, F; \mathrm{std})$. Then we have*

$$(1\text{-}21)\quad \lim_{k\to\infty} \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F; \Phi; \mathrm{std}) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\, dx.$$

Remark 1.7. An unweighted version of Theorem 1.6 was established in [Kim et al. 2020], for test functions Φ whose Fourier transforms have sufficiently small support (for a precise range of support, see Proposition 9.3 in [Kim et al. 2020]). The (unweighted) symmetry type is also symplectic. For comparison, the symmetry type of low-lying zeros of spinor L-functions changes from orthogonal to symplectic when weighted by ω_F.

We may further extend the range of support in Theorem 1.6 from $\left(-\frac{1}{4}, \frac{1}{4}\right)$ to $\left(-\frac{5}{18}, \frac{5}{18}\right)$ by performing an extra (smooth) average over weight k. Our result is:

Theorem 1.8. *Let $\Omega \in C_c^\infty(0, \infty)$ be such that $\Omega \geq 0$, not identically 0. Let Φ be an even Schwartz function whose Fourier transform has support in $\left(-\frac{5}{18}, \frac{5}{18}\right)$. For $F \in H_k(\Gamma_2)$ and large parameter $K > 0$, define $D(F; \Phi; \mathrm{std})$ as in (1-17) with $c_{F;\mathrm{std}} = K^4$ and ω_F as in (1-12). Assume GRH for $L(s, F; \mathrm{std})$. Then we have*

$$(1\text{-}22)\quad \lim_{K\to\infty} \left(\sum_k \Omega\left(\frac{k}{K}\right)\right)^{-1} \sum_k \Omega\left(\frac{k}{K}\right) \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F; \Phi; \mathrm{std}) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\, dx.$$

where the summation in k is over even integers.

This article is organized as follows: In Section 2, we first review some facts about spinor and standard L-functions. We then work out the combinatorial relations between certain functions in Satake parameters of a form $F \in H_k(\Gamma_2)$ and its Fourier coefficients at scalar matrices. These relations allow us to apply Kitaoka's formula, which we state in Section 3. In Section 3 we also take average over weight k in Kitaoka's formula and give an upper bound for the off-diagonal term. In Section 4 we apply the results established in previous sections, as well as the explicit formula to prove Theorems 1.2–1.8. In Section 5 we prove Corollary 1.4 and discuss some other issues concerning nonvanishing of central L-values.

2. Spinor and standard L-functions

Let $F \in H_k(\Gamma_2)$ be a Hecke eigenform. It is known that for each prime p there are three complex numbers $\alpha_{F,0}(p)$, $\alpha_{F,1}(p)$, $\alpha_{F,2}(p)$, called the Satake parameters of F at p, with certain prescribed properties. See Chapter 3 in [Pitale 2019] for a detailed discussion. In particular, these Satake parameters satisfy the relation

$$\alpha_{F,0}(p)^2 \alpha_{F,1}(p)\, \alpha_{F,2}(p) = 1. \tag{2-1}$$

Let $S_{2k-2}(\Gamma_1)$ denote the space of holomorphic cusp forms of weight $2k-2$ for the full modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$. There is an injective Hecke-equivariant linear map

$$SK : S_{2k-2}(\Gamma_1) \to S_k(\Gamma_2), \quad f \mapsto F_f, \tag{2-2}$$

called the Saito–Kurokawa lifting. We denote the image of SK by $S_k^*(\Gamma_2)$ and call forms in $S_k^*(\Gamma_2)$ Saito–Kurokawa lifts. We also use $H_k^*(\Gamma_2)$ for a basis of $S_k^*(\Gamma_2)$ consisting of Hecke eigenforms. There are various ways to construct such a lifting map. For a construction using half-integral weight modular forms, see Section 2.1.3 in [Pitale 2019].

For $F \in H_k(\Gamma_2)$ that is not a Saito–Kurokawa lift, it is known that $|\alpha_{F,i}(p)| = 1$ for all prime p, by a result in [Weissauer 2009]. However, this is not true for Saito–Kurokawa lifts $F_f \in H_k^*(\Gamma_2)$. We will see this in Andrianov's explicit formula (2-8) stated below.

2.1. ***The spinor L-function.*** The spinor L-function attached to a Hecke eigenform $F \in H_k(\Gamma_2)$ is defined by a degree 4 Euler product

$$\begin{aligned} L(s, F; \text{spin}) &= \prod_p \left(1 - \frac{\alpha_{F,0}(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{F,0}(p)\, \alpha_{F,1}(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{F,0}(p)\, \alpha_{F,2}(p)}{p^s}\right)^{-1} \\ &\qquad \times \left(1 - \frac{\alpha_{F,0}(p)\, \alpha_{F,1}(p)\, \alpha_{F,2}(p)}{p^s}\right)^{-1}, \end{aligned} \tag{2-3}$$

which converges absolutely in some right half-plane. By setting

$$\alpha_F(p) = \alpha_{F,0}(p), \quad \beta_F(p) = \alpha_{F,0}(p)\, \alpha_{F,1}(p), \tag{2-4}$$

we may rewrite the above Euler product as

$$\begin{aligned} L(s, F; \text{spin}) &= \prod_p \left(1 - \frac{\alpha_F(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_F(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_F(p)^{-1}}{p^s}\right)^{-1} \left(1 - \frac{\beta_F(p)^{-1}}{p^s}\right)^{-1}, \end{aligned} \tag{2-5}$$

in view of the relation (2-1).

It is proved by A. N. Andrianov [1974] that $L(s, F; \text{spin})$ extends to a meromorphic function on $\mathbb{C}$, which has a simple pole at $s = \frac{3}{2}$ if F is a Saito–Kurokawa lift, and is entire otherwise. Its functional equation takes the form

$$\Lambda(s, F; \text{spin}) = \Gamma_{\mathbb{C}}\big(s+\tfrac{1}{2}\big)\Gamma_{\mathbb{C}}\big(s+k-\tfrac{3}{2}\big)L(s, F; \text{spin}) = \Lambda(1-s, F; \text{spin}), \tag{2-6}$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. For $F_f \in H_k^*(\Gamma_2)$ a Saito–Kurokawa lift, its spinor L-function decomposes as

$$L(s, F_f; \text{spin}) = \zeta\big(s+\tfrac{1}{2}\big)\zeta\big(s-\tfrac{1}{2}\big)L(s, f), \tag{2-7}$$

where $L(s, f)$ is the Hecke L-function of the elliptic cusp form f.

For $F \in H_k(\Gamma_2)$ we have Andrianov's explicit formula [Andrianov 1974]:

$$a_F(I)\,L(s, F; \text{spin}) = \zeta\big(s+\tfrac{1}{2}\big)L\big(s+\tfrac{1}{2}, \chi_{-4}\big)\sum_{n=1}^{\infty}\frac{a_F(nI)}{n^s}, \tag{2-8}$$

where χ_{-4} is the nontrivial Dirichlet character modulo 4. From this formula it follows

$$a_F(I) = 0 \implies a_F(nI) = 0, \quad n \geq 1. \tag{2-9}$$

2.2. ***The standard L-function.*** The standard L-function attached to a Hecke eigenform $F \in H_k(\Gamma_2)$ is defined by a degree 5 Euler product

$$\begin{aligned} L(s, F; \text{std}) &= \prod_p \Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_{F,1}(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_{F,1}(p)^{-1}}{p^s}\Big)^{-1} \\ &\qquad\times\Big(1-\frac{\alpha_{F,2}(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_{F,2}(p)^{-1}}{p^s}\Big)^{-1}, \end{aligned} \tag{2-10}$$

which converges absolutely in some right half-plane. Using (2-1), we rewrite this Euler product as

$$\begin{aligned} L(s, F; \text{std}) &= \prod_p \Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_F(p)\beta_F(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_F(p)^{-1}\beta_F(p)}{p^s}\Big)^{-1} \\ &\qquad\times\Big(1-\frac{\alpha_F(p)\beta_F(p)^{-1}}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_F(p)^{-1}\beta_F(p)^{-1}}{p^s}\Big)^{-1}. \end{aligned} \tag{2-11}$$

The analytic continuation and functional equation of standard L-functions were worked out by S. Böcherer [1985]. He proved that $L(s, F; \text{std})$ extends to an entire function and satisfies a functional equation

$$\begin{aligned}\Lambda(s, F; \mathrm{std}) &= \Gamma_{\mathbb{R}}(s)\,\Gamma_{\mathbb{C}}(s+k-1)\,\Gamma_{\mathbb{C}}(s+k-2)\,L(s, F; \mathrm{std}) \\ &= \Lambda(1-s, F; \mathrm{std}),\end{aligned} \tag{2-12}$$

where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

2.3. ***Combinatorial relations.*** For $F \in H_k(\Gamma_2)$, $m \geq 1$ and prime p, we set

$$c_m(p; F) = \alpha_F(p)^m + \alpha_F(p)^{-m} + \beta_F(p)^m + \beta_F(p)^{-m}, \tag{2-13}$$

$$\begin{aligned}\tau_{2m}(p; F) = 1 &+ \alpha_F(p)^m\beta_F(p)^m + \alpha_F(p)^m\beta_F(p)^{-m} \\ &+ \alpha_F(p)^{-m}\beta_F(p)^m + \alpha_F(p)^{-m}\beta_F(p)^{-m}\end{aligned} \tag{2-14}$$

to be the m-th power sum of local parameters of $L(s, F; \mathrm{spin})$ and $L(s, F; \mathrm{std})$ at p respectively.

The main goal of this section is to find expressions of these power sums in terms of Fourier coefficients of F at scalar matrices, for $m = 1, 2$, under the assumption that $a_F(I) \neq 0$. Note that the condition $a_F(I) \neq 0$ is not a direct consequence of $F \neq 0$, unlike in the elliptic case, where a primitive form f vanishes if and only if its first Fourier coefficient vanishes. In fact, determining whether $a_F(I) = 0$ or not is a difficult problem because ω_F is intimately connected to central values of spinor L-functions (Böcherer's conjecture, now a theorem proved by M. Furusawa and K. Morimoto [2021]). However, as we shall see later in Section 4, making this assumption here does no harm to our argument. Our result is as follows:

Lemma 2.1. *Let $F \in H_k(\Gamma_2)$ be a Hecke eigenform. For any prime p and $m \geq 1$, define $c_m(p; F)$ and $\tau_{2m}(p; F)$ as in (2-13) and (2-14). Assume that $a_F(I) \neq 0$, and set $U_m(p; F) = a_F(p^m I)/a_F(I)$. Also set $\lambda_p = 1 + \chi_{-4}(p)$ and $\mu_p = \chi_{-4}(p)$, where χ_{-4} is the nontrivial Dirichlet character modulo 4. Then we have*

$$c_1(p; F) = U_1(p; F) + \frac{\lambda_p}{\sqrt{p}},$$

$$c_2(p; F) = -U_1(p; F)^2 + 2U_2(p; F) + \frac{\lambda_p^2 - 2\mu_p}{p},$$

$$\tau_2(p; F) = U_1(p; F)^2 - U_2(p; F) + \frac{\lambda_p}{\sqrt{p}}U_1(p; F) + \frac{\mu_p}{p} - 1,$$

$$\begin{aligned}\tau_4(p; F) = &-U_3(p; F)\,U_1(p; F) + U_2(p; F)^2 + \frac{\lambda_p}{\sqrt{p}}U_2(p; F)\,U_1(p; F) \\ &+ \left(\frac{\mu_p}{p} - 1\right)U_1(p; F)^2 - \frac{\lambda_p}{\sqrt{p}}U_3(p; F) + \left(\frac{\lambda_p^2 - 2\mu_p}{p}\right)U_2(p; F) \\ &+ \left(\frac{\lambda_p\mu_p}{p^{3/2}} - 2\frac{\lambda_p}{\sqrt{p}}\right)U_1(p; F) + \frac{\mu_p^2}{p^2} - \frac{\lambda_p^2}{p} + 1.\end{aligned}$$

Remark 2.2. The key feature of Lemma 2.1 is that we are able to express $\tau_4(p;F)$ using polynomials of $U_m(p;F)$ of degree 2 (and not of higher degree). This is essential when we deal with weighted low-lying zeros of standard L-functions using Kitaoka's formula.

Proof. Throughout the proof, F and p are fixed. To save notation we use c_m, τ_{2m}, U_m, α_p, β_p to denote $c_m(p;F)$, $\tau_{2m}(p;F)$, $U_m(p;F)$, $\alpha_F(p)$, $\beta_F(p)$ respectively, with the understanding that they depend on F and p.

We start with Andrianov's explicit formula (2-8):

$$a_F(I)\,L(s,F;\text{spin})=\zeta\bigl(s+\tfrac{1}{2}\bigr)\,L\bigl(s+\tfrac{1}{2},\chi_{-4}\bigr)\sum_{n=1}^{\infty}\frac{a_F(nI)}{n^s}.$$

Using Euler product expansions for the L-functions involved, we see that the two Dirichlet series

$$\biggl(\prod_p\Bigl(1-\frac{\alpha_p}{p^s}\Bigr)\Bigl(1-\frac{\beta_p}{p^s}\Bigr)\Bigl(1-\frac{\alpha_p^{-1}}{p^s}\Bigr)\Bigl(1-\frac{\beta_p^{-1}}{p^s}\Bigr)\biggr)\cdot\biggl(\sum_{n=1}^{\infty}\frac{a_F(nI)\,a_F(I)^{-1}}{n^s}\biggr)\tag{2-15}$$

and

$$\prod_p\Bigl(1-\frac{1}{p^{s+1/2}}\Bigr)\Bigl(1-\frac{\chi_{-4}(p)}{p^{s+1/2}}\Bigr)\tag{2-16}$$

both converge absolutely in some right half-plane and are equal. Comparing coefficients of p^{-as}, $a=1,2,3,4$, we obtain

$$-\frac{\lambda_p}{\sqrt{p}}=U_1-c_1,\tag{2-17}$$

$$\frac{\mu_p}{p}=U_2-U_1c_1+\tau_2+1,\tag{2-18}$$

$$0=U_3-U_2c_1+U_1(\tau_2+1)-c_1,\tag{2-19}$$

$$0=U_4-U_3c_1+U_2(\tau_2+1)-U_1c_1+1.\tag{2-20}$$

We also have elementary relations

$$c_1^2=c_2+2(\tau_2+1),\tag{2-21}$$

$$(\tau_2+1)^2=3+\tau_4+4\tau_2+2c_2.\tag{2-22}$$

From (2-17) and (2-18) we obtain directly

$$c_1=U_1+\frac{\lambda_p}{\sqrt{p}},\tag{2-23}$$

$$\tau_2=U_1^2-U_2+\frac{\lambda_p}{\sqrt{p}}U_1+\frac{\mu_p}{p}-1.\tag{2-24}$$

Combining (2-21), (2-23) and (2-24) we have

$$c_2 = -U_1^2 + 2U_2 + \frac{\lambda_p^2 - 2\mu_p}{p}. \tag{2-25}$$

Using (2-22), (2-24) and (2-25) we express τ_4 as

$$\begin{aligned}\tau_4 = U_1^4 - 2U_2U_1^2 + 2\frac{\lambda_p}{\sqrt{p}}U_1^3 + U_2^2 - 2\frac{\lambda_p}{\sqrt{p}}U_2U_1 + \left(\frac{\lambda_p^2 + 2\mu_p}{p} - 2\right)U_1^2 \\ - 2\frac{\mu_p}{p}U_2 + \left(2\frac{\lambda_p\mu_p}{p^{3/2}} - 4\frac{\lambda_p}{\sqrt{p}}\right)U_1 + \frac{\mu_p^2}{p^2} - 2\frac{\lambda_p^2}{p} + 1.\end{aligned} \tag{2-26}$$

However, to get the final form of τ_4, we must express U_1^4, $U_2U_1^2$ and U_1^3 using degree 2 polynomials in U_a $(a = 1, 2, 3, 4)$. Combining (2-19), (2-23) and (2-24), we have

$$U_1^3 = 2U_2U_1 - \frac{\lambda_p}{\sqrt{p}}U_1^2 - U_3 + \frac{\lambda_p}{\sqrt{p}}U_2 + \left(1 - \frac{\mu_p}{p}\right)U_1 + \frac{\lambda_p}{\sqrt{p}}. \tag{2-27}$$

Likely, equations (2-20), (2-23) and (2-24) give us

$$U_2U_1^2 = U_3U_1 + U_2^2 - \frac{\lambda_p}{\sqrt{p}}U_2U_1 + U_1^2 - U_4 + \frac{\lambda_p}{\sqrt{p}}U_3 - \frac{\mu_p}{p}U_2 + \frac{\lambda_p}{\sqrt{p}}U_1 - 1. \tag{2-28}$$

Further, we multiply (2-27) by U_1 and apply (2-27), (2-28) to get

$$\begin{aligned}U_1^4 = U_3U_1 + 2U_2^2 - 3\frac{\lambda_p}{\sqrt{p}}U_2U_1 + \left(3 + \frac{\lambda_p^2 - \mu_p}{p}\right)U_1^2 - 2U_4 + 3\frac{\lambda_p}{\sqrt{p}}U_3 \\ - \left(\frac{\lambda_p^2 + 2\mu_p}{p}\right)U_2 + \left(2\frac{\lambda_p}{\sqrt{p}} + \frac{\lambda_p\mu_p}{p^{3/2}}\right)U_1 - \left(2 + \frac{\lambda_p^2}{p}\right).\end{aligned} \tag{2-29}$$

Finally, we insert (2-27), (2-28) and (2-29) into (2-26) to get

$$\begin{aligned}\tau_4 = -U_3U_1 + U_2^2 + \frac{\lambda_p}{\sqrt{p}}U_2U_1 + \left(\frac{\mu_p}{p} - 1\right)U_1^2 - \frac{\lambda_p}{\sqrt{p}}U_3 + \left(\frac{\lambda_p^2 - 2\mu_p}{p}\right)U_2 \\ + \left(\frac{\lambda_p\mu_p}{p^{3/2}} - 2\frac{\lambda_p}{\sqrt{p}}\right)U_1 + \frac{\mu_p^2}{p^2} - \frac{\lambda_p^2}{p} + 1. \quad \square\end{aligned} \tag{2-30}$$

3. Kitaoka's formula

The main tool used in this paper is a spectral summation formula of Petersson type. This formula was first proved by Y. Kitaoka [1984] by computing Fourier coefficients of Siegel Poincaré series. In this section we introduce this formula and consider an averaged (over weight) version of it.

We begin by introducing some notations. For $k \geq 6$ even, we set

$$c_k = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\,\Gamma\Big(k-\frac{3}{2}\Big)\,\Gamma(k-2). \tag{3-1}$$

For $T, Q \in \mathcal{T}$ we define

$$\Delta_k(T, Q) = \sum_{F \in H_k(\Gamma_2)} c_k \frac{a_F(T)\, a_F(Q)}{\|F\|^2}. \tag{3-2}$$

For a matrix $C \in M_2(\mathbb{Z})$ with $\det C \neq 0$ (we denote the set of such matrices by $\mathcal{C}$) and $Q, T \in \mathcal{T}$, define the symplectic Kloosterman sum to be

$$K(Q, T; C) = \sum_{D} e\big(\mathrm{Tr}(AC^{-1}Q + C^{-1}DT)\big), \tag{3-3}$$

where D runs through the set

$$\left\{ D \in M_2(\mathbb{Z}) \bmod C\Lambda : \begin{pmatrix} A & * \\ C & D \end{pmatrix} \in \Gamma_2 \right\}, \tag{3-4}$$

and Λ is the set of 2×2 symmetric integral matrices. By elementary divisor theory and Weil's bound for classical Kloosterman sums one has [Kitaoka 1984]:

$$|K(Q, T; C)| \leq |\det C|^{3/2}. \tag{3-5}$$

Remark 3.1. Optimal bounds for these symplectic Kloosterman sums were obtained in [Tóth 2013]. However, since applying Tóth's optimal bound does not improve our result, Kitaoka's bound suffices for our purpose.

For $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix}$, $S = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4 \end{pmatrix} \in \mathcal{T}$ and $c \geq 1$, we define another exponential sum:

$$\begin{aligned} H^{\pm}(P, S; c) \\ = \delta_{s_4 = p_4} \sideset{}{^*}\sum_{d_1 \bmod c} \sum_{d_2 \bmod c} e\left(\frac{\bar{d}_1 s_4 d_2^2 \mp \bar{d}_1 p_2 d_2 + s_2 d_2 + \bar{d}_1 p_1 + d_1 s_1}{c} \mp \frac{p_2 s_2}{2cs_4} \right). \end{aligned} \tag{3-6}$$

For these we have the trivial bound

$$|H^{\pm}(P, S; c)| \leq c^2. \tag{3-7}$$

For $P \in M_2(\mathbb{R})$ with positive eigenvalues $\lambda_1, \lambda_2 > 0$ we set

$$\mathcal{J}_{k-3/2}(P) = \int_0^{\pi/2} J_{k-3/2}(4\pi\sqrt{\lambda_1}\sin\theta)\, J_{k-3/2}(4\pi\sqrt{\lambda_2}\sin\theta)\sin\theta\, d\theta, \tag{3-8}$$

where $J_{k-3/2}$ is the usual J-Bessel function of half-integral order $k - \frac{3}{2}$. With these notation, we can now state Kitaoka's formula.

Lemma 3.2. *For $T, Q \in \mathcal{T}$ and $k \geq 6$ even, define $\Delta_k(T, Q)$ as in (3-2). Then*

$$\Delta_k(T, Q) = \frac{1}{8}|\mathrm{Aut}(T)|\left(\frac{\det Q}{\det T}\right)^{\frac{k}{2}-\frac{3}{4}} \delta_{Q\sim T} + \frac{\sqrt{2}\pi}{8} G_{1,k}(T, Q) + \pi^2 G_{2,k}(T, Q), \tag{3-9}$$

where $\mathrm{Aut}(T)$, $G_{1,k}(T, Q)$ *and* $G_{2,k}(T, Q)$ *are defined by*

$$\mathrm{Aut}(T) = \{U \in \mathrm{GL}_2(\mathbb{Z}) : U^T T U = T\}, \tag{3-10}$$

$$G_{1.k}(T, Q) = \sum_{\pm} \sum_{s=1}^{\infty} \sum_{c=1}^{\infty} \sum_{U,V} \frac{(-1)^{k/2}}{c^{3/2} s^{1/2}} \times H^{\pm}(UQU^T, V^{-1}TV^{-T}; c)\, J_{k-3/2}\left(\frac{4\pi\sqrt{\det(TQ)}}{cs}\right), \tag{3-11}$$

$$G_{2,k}(T, Q) = \sum_{C\in\mathcal{C}} \frac{K(Q, T; C)}{|\det C|^{3/2}} \mathcal{J}_{k-3/2}(TC^{-1}QC^{-T}). \tag{3-12}$$

Here $\sum_{U,V}$ in (3-11) is over $U = (u_{ij})/\{\pm I\}$, $V = (v_{ij}) \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$(u_{21}, u_{22})\, Q (u_{21}, u_{22})^T = (-v_{21}, v_{11})\, T (-v_{21}, v_{11})^T = s. \tag{3-13}$$

The delta symbol $\delta_{Q\sim T}$ is equal to 1 if Q and T are equivalent in the sense of quadratic forms, and is equal to 0 otherwise.

Remark 3.3. Following Kitaoka [1984], we call the three terms in (3-9) containing $\delta_{Q\sim T}$, $G_{1,k}(T, Q)$ and $G_{2,k}(T, Q)$ the diagonal term, the rank 1 term and the rank 2 term respectively. Note that the classical Petersson formula for elliptic modular forms contains only a diagonal term and an off-diagonal term.

Remark 3.4. As pointed out by V. Blomer (see Remark 1 in [Blomer 2019]), there are some numerical errors in Kitaoka's original derivation of Kitaoka's formula. The version that we present here is based on Lemma 1 in [Blomer 2019]. However, our results do not depend on exact values of those constants.

The main purpose of this section is to establish the following averaged Kitaoka's formula, which is asymptotic in nature.

Lemma 3.5. *Let $m, n \geq 1$ be positive integers such that $m\,|\,n$. For $k \geq 6$ even, define $\Delta_k(mI, nI)$ as in (3-2). Let $\Omega \in C_c^{\infty}(0, \infty)$ be such that $\Omega \geq 0$, not identically 0. Then for large $K > 0$ we have*

$$\left(\sum_k \Omega\left(\frac{k}{K}\right)\right)^{-1} \sum_k \Omega\left(\frac{k}{K}\right) \Delta_k(mI, nI) = \delta_{m=n} + O_{j,\epsilon,\Omega}\left(\left(\frac{m^{(3/2)-\epsilon} n^{-(1/2)+\epsilon}}{K^4}\right) + \left(\frac{(mn)^{2+\epsilon}}{K^{5+2\epsilon}}\right) + \left(\frac{(mn)^{(j/2)+1}}{K^{2j+3}}\right)\right) \tag{3-14}$$

for any $j \geq 3$ and $\epsilon > 0$ small. Here $\sum_k$ is over positive even integers $k \geq 6$.

Remark 3.6. Our Lemma 3.5 can be viewed as a GSp_4 analogue of the classical averaged Petersson formula on GL_2; see (5.81) in [Iwaniec 1997]. The main difficulty is the presence of a product of two Bessel functions (instead of a single Bessel function), each of half-integral order (instead of integral order). As we shall see in the proof below, this can be overcome by applying an integral representation (3-33) of a product of two Bessel functions. A related averaged Kitaoka's formula was also discussed in recent work of G. Felber [2023].

Proof. After applying Kitaoka's formula (3-9), we divide the left side of (3-14) into three terms. We also set $g(x) = \Omega(x/K)$ and $\ell = k - \frac{3}{2}$ to save notation.

We denote the contribution of the diagonal term by R_0. Thus

$$R_0 = \tfrac{1}{8}\Bigl(\sum_k g(k)\Bigr)^{-1} \sum_k g(k)|\mathrm{Aut}(mI)|\Bigl(\frac{n}{m}\Bigr)^{\ell} \delta_{mI\sim nI}. \tag{3-15}$$

Note that mI and nI define the same quadratic form if and only if $m = n$, and that $|\mathrm{Aut}(mI)| = 8$. Thus the above expression reduces to $R_0 = \delta_{m=n}$.

Denote by R_1 the sum of the rank 1 term over k. We have

$$R_1 = \sum_k g(k) \sum_{\pm} \sum_{s=1}^{\infty} \sum_{c=1}^{\infty} \sum_{U,V} \frac{(-1)^{k/2}}{c^{3/2} s^{1/2}} \times H^{\pm}(nUU^T, mV^{-1}V^{-T}; c)\, J_\ell\Bigl(\frac{4\pi mn}{cs}\Bigr), \tag{3-16}$$

where the sum $\sum_{U,V}$ is over

$$n(u_{21}^2 + u_{22}^2) = m(v_{11}^2 + v_{21}^2) = s. \tag{3-17}$$

So in particular $n \mid s$. Making change of variable $s \mapsto ns$, we may rewrite R_1 as

$$R_1 = \sum_k g(k) \sum_{\pm} \sum_{s=1}^{\infty} \sum_{c=1}^{\infty} \sum_{U,V} \frac{(-1)^{k/2}}{c^{3/2} (ns)^{1/2}} \times H^{\pm}(nUU^T, mV^{-1}V^{-T}; c)\, J_\ell\Bigl(\frac{4\pi m}{cs}\Bigr), \tag{3-18}$$

where $\sum_{U,V}$ is over

$$u_{21}^2 + u_{22}^2 = s, \qquad v_{11}^2 + v_{22}^2 = \frac{n}{m} s. \tag{3-19}$$

These equations have $O(s^\epsilon)$ and $O\bigl((\frac{n}{m}s)^\epsilon\bigr)$ integral solutions, respectively, for any $\epsilon > 0$, by the fact

$$|\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = s\}| = O(s^\epsilon). \tag{3-20}$$

In view of the estimate

$$J_\ell(x) \ll \left(\frac{x}{\ell}\right)^{\ell}, \quad x > 0,\ \ell > \tfrac{1}{2}, \tag{3-21}$$

which follows immediately from the integral representation [Gradshteyn and Ryzhik 2015, (8.411.4)], we may cut-off the sum in (3-18) by $sc \ll m/K$ up to a negligible error. In this range we change summation order and deal with the inner sum

$$\sum_k g(k)(-1)^{k/2} J_\ell\left(\frac{4\pi m}{cs}\right) \tag{3-22}$$

by applying Lemma 20 of [Blomer and Corbett 2022] to obtain

$$\begin{aligned}\sum_k g(k)(-1)^{k/2} J_\ell\left(\frac{4\pi m}{cs}\right) \\ = \omega_0\left(\frac{4\pi m}{cs}\right) + e^{\frac{4\pi i m}{cs}}\omega_+\left(\frac{4\pi m}{cs}\right) + e^{-\frac{4\pi i m}{cs}}\omega_-\left(\frac{4\pi m}{cs}\right),\end{aligned} \tag{3-23}$$

where $\omega_0(x)$, $\omega_\pm(x)$ are some smooth functions on $(0, \infty)$ satisfying

$$\omega_0(x) \ll_A K^{-A}, \tag{3-24}$$

$$\omega_\pm(x) \ll_A \left(1 + \frac{K^2}{x}\right)^{-A} \tag{3-25}$$

for any $A > 0$. The contribution of the ω_0 term is negligible, while the contribution of $\omega_\pm$ term depends on the size of $x = 4\pi m/cs$. For example, for $x \le K^2$ (i.e., $cs \gg m/K^2$), we have

$$\omega_+(x) \ll_A \left(1 + \frac{K^2}{x}\right)^{-A} \le \left(\frac{K^2}{x}\right)^{-A} \ll_A K^{-2A} m^A (cs)^{-A} \tag{3-26}$$

for any $A > 0$. This estimate, together with (3-7) and (3-20), give rise to

$$\begin{aligned} &R_1^{+, cs \gg \frac{m}{K^2}} \\ &= \sum_{\pm} \sum_{\frac{m}{K^2} \ll sc \ll \frac{m}{K}} \sum_{U,V} \frac{1}{c^{3/2}(ns)^{1/2}} H^\pm(nUU^T, mV^{-1}V^{-T}; c)\, \omega_+\left(\frac{4\pi m}{cs}\right) \\ &\ll_{\epsilon, A} \sum_{\frac{m}{K^2} \ll sc \ll \frac{m}{K}} c^{-3/2}(ns)^{-1/2} s^\epsilon \left(\frac{n}{m} s\right)^\epsilon c^2 K^{-2A} m^A c^{-A} s^{-A} \\ &\ll_\epsilon m^{(3/2)-\epsilon} n^{-(1/2)+\epsilon} K^{-3}\end{aligned} \tag{3-27}$$

for any small $\epsilon > 0$ if one fixes some $A > \frac{3}{2}$. The case where $cs \ll m/K^2$ is analyzed similarly, and its contribution to R_1 is again at most $m^{\frac{3}{2}-\epsilon} n^{-\frac{1}{2}+\epsilon} K^{-3}$. Therefore, we have obtained that

$$R_1 \ll_\epsilon m^{\frac{3}{2}-\epsilon} n^{-\frac{1}{2}+\epsilon} K^{-3} \tag{3-28}$$

for any small $\epsilon > 0$.

Denote by R_2 the sum of the rank 2 term over k. Explicitly,

$$R_2 = \sum_k g(k) \sum_{C \in \mathcal{C}} \frac{K(nI, mI; C)}{|\det C|^{3/2}} \int_0^{\pi/2} J_\ell(4\pi\sqrt{\lambda_1}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_2}\sin\theta)\sin\theta\, d\theta, \tag{3-29}$$

where λ_1, λ_2 are eigenvalues of the matrix $mnC^{-1}C^{-T}$. We set $\lambda_{\min}$ and $\lambda_{\max}$ to be the smaller and the larger eigenvalue of $mnC^{-1}C^{-T}$ respectively. Denote by $\|\cdot\|_F$ the Frobenius matrix norm. Then by Lemma 2 in [Blomer 2019] we have

$$\lambda_{\min} \ll \frac{mn}{\|C\|_F^2}. \tag{3-30}$$

Applying this estimate and (3-21) to $J_\ell(4\pi\sqrt{\lambda_{\min}}\sin\theta)$, and applying the estimate

$$J_\ell(x) \ll 1, \quad x > 0,\ \ell > \tfrac{1}{2} \tag{3-31}$$

that follows from [Gradshteyn and Ryzhik 2015, (8.411.13)] to $J_\ell(4\pi\sqrt{\lambda_{\max}}\sin\theta)$, we may cut-off the sum in R_2 by $\|C\|_F \ll \sqrt{mn}/K$ up to an negligible error. In this range we change the summation order and deal with the inner sum

$$\sum_k g(k)\, J_\ell(4\pi\sqrt{\lambda_1}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_2}\sin\theta) \tag{3-32}$$

by making use of the following integral representation of product of two Bessel functions [Erdélyi et al. 1981, page 47, (8)]:

$$J_v(z)\, J_v(\zeta) = \frac{2}{\pi}\int_0^{\pi/2} \cos((z-\zeta)\cos\alpha)\, J_{2v}(2\sqrt{z\zeta}\sin\alpha)\, d\alpha, \tag{3-33}$$

when $\Re(v) > -\frac{1}{2}$, $z > 0$, $\zeta > 0$. Choosing $v = \ell$, $z = 4\pi\sqrt{\lambda_1}\sin\theta$, $\zeta = 4\pi\sqrt{\lambda_2}\sin\theta$, and setting

$$\xi = 2\sqrt{z\zeta}\sin\alpha = \frac{8\pi\sqrt{mn}}{\sqrt{|\det C|}}\sin\theta\sin\alpha, \tag{3-34}$$

we obtain

$$\sum_k g(k)\, J_\ell(z)\, J_\ell(\zeta) = \frac{2}{\pi}\int_0^{\pi/2} \cos((z-\zeta)\cos\alpha)\left(\sum_k g(k)\, J_{2k-3}(\xi)\right) d\alpha. \tag{3-35}$$

Let $r = 2k-3$. We have $r \equiv 1 \mod 4$, since k is even. Setting $g_1(x) = g\left(\frac{x+3}{2}\right)$, we have

$$\sum_k g(k)\, J_{2k-3}(\xi) = \sum_{r \equiv 1 \mod 4} g_1(r)\, J_r(\xi). \tag{3-36}$$

From here the method of Neumann series can be applied, in view of the following integral representation of Bessel functions of integral order [Gradshteyn and Ryzhik 2015, (8.411.1)]:

$$J_r(x) = \int_{-1/2}^{1/2} e(rt)\, e^{-ix \sin 2\pi t}\, dt. \tag{3-37}$$

We quote the following result (Lemma 5.8 in [Iwaniec 1997]):

$$4 \sum_{r \equiv 1 \mod 4} g_1(r)\, J_r(\xi) = g_1(\xi) + h(\xi) + O(\xi c_3(g_1)), \tag{3-38}$$

where $h(\xi)$ and $c_3(g_1)$ are defined by

$$h(\xi) = \int_0^\infty g_1(\sqrt{2\xi y}) \sin\left(\xi + y - \frac{\pi}{4}\right)(\pi y)^{-1/2}\, dy, \tag{3-39}$$

$$c_3(g_1) = \int_{-\infty}^{\infty} |\hat{g_1}(t)\, t^3|\, dt. \tag{3-40}$$

We refer readers to Section 5.5 in [Iwaniec 1997] for a proof of (3-38). Recall that for g_1 we have

$$g_1^{(j)}(x) \ll_j K^{-j} \tag{3-41}$$

for any $j \geq 0$. Thus by repeated partial integration we have

$$h(\xi) \ll_j (\xi K^{-2})^j, \tag{3-42}$$

$$c_3(g_1) \ll K^{-3}. \tag{3-43}$$

See also (5.73) and (5.74) in [Iwaniec 1997].

The contribution of $g_1(\xi)$ to R_2 is

$$R_2^{g_1(\xi)} = \sum_{\|C\|_F \ll \frac{\sqrt{mn}}{K}} \frac{K(nI, mI; C)}{|\det C|^{3/2}} \int_0^{\pi/2} \int_0^{\pi/2} \cos((z-\zeta)\cos\alpha) g_1(\xi)\, d\alpha \sin\theta\, d\theta. \tag{3-44}$$

In view of the support of g_1, the sum in (3-44) is confined in the range

$$\xi = \frac{8\pi\sqrt{mn}}{\sqrt{|\det C|}} \sin\theta \sin\alpha \gg K. \tag{3-45}$$

Thus we have $\det(C) \ll mn/K^2$. By the estimate (3-5), we obtain

$$R_2^{g_1(\xi)} \ll \sum_{\substack{0\neq|\det C|\ll \frac{mn}{K^2} \\ \|C\|_F \ll \frac{\sqrt{mn}}{K}}} 1 = \sum_{0\neq|d|\ll \frac{mn}{K^2}} \sum_{\substack{\det C=d \\ \|C\|\ll \frac{\sqrt{mn}}{K}}} 1 = \sum_{0\neq|d|\ll \frac{mn}{K^2}} P_d\Big(C\cdot \frac{mn}{K^2}\Big), \tag{3-46}$$

where $P_d(X)$ is the hyperbolic lattice counting function

$$P_d(X) = |\{(\alpha,\beta,\gamma,\delta)\in\mathbb{Z}^4 : \alpha\delta-\beta\gamma=d,\ \alpha^2+\beta^2+\gamma^2+\delta^2\le X\}| \tag{3-47}$$

and $C>0$ is some constant. For $1\le d\le X$ we have the following asymptotic formula (see Theorem 12.4 in [Iwaniec 2002]):

$$P_d(X) = 6\Big(\sum_{\tau|d}\tau^{-1}\Big)(X+O(d^{\frac{1}{3}}X^{\frac{2}{3}})) \ll X\log|d|. \tag{3-48}$$

This estimate also applies to $-X\le d\le -1$ by symmetry. Thus we have

$$R_2^{g_1(\xi)} \le \frac{mn}{K^2}\sum_{0\neq|d|\ll\frac{mn}{K^2}} \log|d| \ll \frac{m^2n^2}{K^4}\log\frac{mn}{K^2} \ll_\epsilon \Big(\frac{(mn)^{2+\epsilon}}{K^{4+2\epsilon}}\Big). \tag{3-49}$$

The contributions of $h(\xi)$ and $O(\xi c_3(g_1))$ are analyzed similarly, making use of the bounds (3-42) and (3-43). We have

$$R_2^{h(\xi)} \ll_j \frac{(mn)^{(j/2)+1}}{K^{2j+2}}, \tag{3-50}$$

$$R_2^{O(\xi c_3(g_1))} \ll_\epsilon \frac{(mn)^{2+\epsilon}}{K^{6+2\epsilon}} \tag{3-51}$$

for any $j\ge 3$ and small $\epsilon>0$. Thus we obtain

$$R_2 \ll_{j,\epsilon} \frac{(mn)^{2+\epsilon}}{K^{4+2\epsilon}} + \frac{(mn)^{(j/2)+1}}{K^{2j+2}}. \tag{3-52}$$

Combining the estimates of R_1 and R_2 above, and that

$$\sum_k g(k) = \sum_k \Omega\Big(\frac{k}{K}\Big) \gg K, \tag{3-53}$$

by our choice of Ω, the proof is now complete. □

4. Proof of main theorems

In this section we prove Theorems 1.2–1.8. We assume $\hat{\Phi}$ is supported in $(-\alpha,\alpha)$. We also set $\ell = k-\frac{3}{2}$ to save notation.

4.1. ***Proof of Theorem 1.2.*** By standard argument using the explicit formula [Iwaniec et al. 2000, Section 4], we write the density function $D(F;\Phi;\mathrm{spin})$ as

$$D(F;\Phi;\mathrm{spin}) = \frac{2}{2\pi i}\int_{(2)} \Phi\left(\frac{s-1/2}{2\pi i}\log k^2\right)\frac{\Lambda'}{\Lambda}(s,F;\mathrm{spin})\,ds + 2\Phi\left(\frac{\log k^2}{2\pi i}\right)\delta_{F\in H_k^*(\Gamma_2)}, \tag{4-1}$$

where $\delta_{F\in H_k^*(\Gamma_2)} = 1$ if $F \in H_k^*(\Gamma_2)$ is a Saito–Kurokawa lift (in which case $L(s,F;\mathrm{spin})$ has a pole at $s=\frac{3}{2}$) and is 0 otherwise. By (2-6) and (2-5) we may further write

$$\begin{aligned} D(F;\Phi;\mathrm{spin}) &= \frac{2}{\log k^2}\int_{\mathbb{R}} \Phi(x)\left(-\log(2\pi)^2 + \frac{\Gamma'}{\Gamma}\left(1+\frac{2\pi i x}{\log k^2}\right) + \frac{\Gamma'}{\Gamma}\left(k-1+\frac{2\pi i x}{\log k^2}\right)\right)dx \\ &\quad - \frac{2}{\log k^2}\sum_{m=1}^{\infty}\sum_p c_m(p;F)\frac{\log p}{p^{m/2}}\hat{\Phi}\left(\frac{m\log p}{\log k^2}\right) + 2\Phi\left(\frac{\log k^2}{2\pi i}\right)\delta_{F\in H_k^*(\Gamma_2)} \end{aligned} \tag{4-2}$$

by shifting contour from $\sigma = 2$ to $\sigma = \frac{1}{2}$.

For the integral involving gamma factors, we use the following estimate [Gradshteyn and Ryzhik 2015, (8.363.4)]:

$$\frac{\Gamma'}{\Gamma}(a+bi) + \frac{\Gamma'}{\Gamma}(a-bi) = 2\frac{\Gamma'}{\Gamma}(a) + O\left(\frac{b^2}{a^2}\right), \quad a>0,\ b\in\mathbb{R} \tag{4-3}$$

and the fact that $\Phi(x)$ is even to get

$$\frac{2}{\log k^2}\int_{\mathbb{R}} \Phi(x)\left(-\log(2\pi)^2 + \frac{\Gamma'}{\Gamma}\left(1+\frac{2\pi i x}{\log k^2}\right) + \frac{\Gamma'}{\Gamma}\left(k-1+\frac{2\pi i x}{\log k^2}\right)\right)dx = \hat{\Phi}(0) + o(1). \tag{4-4}$$

This is done by splitting the integral over $\mathbb{R}$ to two integrals on $(-\infty,0)$ and $(0,\infty)$. Then we use the fact that $\Phi(x)$ is even, estimate (4-3), and the estimate $\frac{\Gamma'}{\Gamma}(k) = \log k + O(1)$.

For $c_1(p;F)$ and $c_2(p;F)$ we sum over F against the weight ω_F. Using (1-12), Lemma 2.1 and equation (2-9) we obtain

$$\sum_{F\in H_k(\Gamma_2)} \omega_F\, c_1(p;F) = \Delta_k(pI,I) + \frac{\lambda_p}{\sqrt{p}}\Delta_k(I,I), \tag{4-5}$$

$$\sum_{F\in H_k(\Gamma_2)} \omega_F\, c_2(p;F) = -\Delta_k(pI,pI) + 2\Delta_k(p^2I,I) + O\left(\frac{1}{p}\right)\Delta_k(I,I). \tag{4-6}$$

Collecting these we have the “explicit formula”:

$$\begin{aligned}
(4\text{-}7)\qquad & \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F;\Phi;\text{spin}) \\
&= \hat{\Phi}(0)+o(1)-\frac{2}{\log k^2}\sum_p\Big(\Delta_k(pI,I)+\frac{\lambda_p}{\sqrt{p}}\Delta_k(I,I)\Big)\frac{\log p}{\sqrt{p}}\hat{\Phi}\Big(\frac{\log p}{\log k^2}\Big) \\
&\quad -\frac{2}{\log k^2}\sum_p\Big(-\Delta_k(pI,pI)+2\Delta_k(p^2I,I)+O\Big(\frac{1}{p}\Big)\Delta_k(I,I)\Big) \\
&\qquad \times\frac{\log p}{p}\hat{\Phi}\Big(\frac{2\log p}{\log k^2}\Big) \\
&\quad -\frac{2}{\log k^2}\sum_{m=3}^{\infty}\sum_p\Big(\sum_{F\in H_k(\Gamma_2)}\omega_F\, c_m(p;F)\Big)\frac{\log p}{p^{m/2}}\hat{\Phi}\Big(\frac{m\log p}{\log k^2}\Big) \\
&\quad +2\Phi\Big(\frac{\log k^2}{2\pi i}\Big)\sum_{F_f\in H_k^*(\Gamma_2)}\omega_{F_f}.
\end{aligned}$$

We treat the terms $\Delta_k(pI,I)$, $\Delta_k(pI,pI)$ and $\Delta_k(p^2I,I)$ using Kitaoka's formula (3-9). Take the term $\Delta_k(pI,I)$ for example:

$$\Delta_k(pI,I)=\frac{\sqrt{2}\pi}{8}G_{1,k}(pI,I)+\pi^2G_{2,k}(pI,I). \tag{4-8}$$

The rank 1 term $G_{1,k}(pI,I)$ is

$$G_{1,k}(pI,I)=\sum_{\pm}\sum_{s=1}^{\infty}\sum_{c=1}^{\infty}\sum_{U,V}\frac{(-1)^{k/2}}{c^{3/2}(ps)^{1/2}}H^{\pm}(UU^T,pV^{-1}V^{-T};c)\,J_\ell\Big(\frac{4\pi}{cs}\Big),$$

after a change of variable $s\mapsto ps$, where the summation $\sum_{U,V}$ is over

$$u_{21}^2+u_{22}^2=ps,\qquad v_{11}^2+v_{21}^2=s. \tag{4-9}$$

By the estimates (3-7), (3-21) and (3-20), we bound $G_{1,k}(pI,I)$ as

$$G_{1,k}(pI,I)\ll\sum_{s=1}^{\infty}\sum_{c=1}^{\infty}(ps)^{\epsilon}s^{\epsilon}c^{-\frac{3}{2}}p^{-\frac{1}{2}}s^{-\frac{1}{2}}c^2\Big(\frac{4\pi}{cs\ell}\Big)^{\ell}\ll p^{-\frac{1}{2}+\epsilon}\Big(\frac{4\pi}{\ell}\Big)^{\ell} \tag{4-10}$$

for k sufficiently large. Thus its contribution to (4-7) is at most

$$\begin{aligned}
(4\text{-}11)\qquad & \frac{1}{\log k}\sum_p p^{-\frac{1}{2}+\epsilon}\Big(\frac{4\pi}{\ell}\Big)^{\ell}\frac{\log p}{\sqrt{p}}\hat{\Phi}\Big(\frac{\log p}{\log k^2}\Big) \\
&\ll\frac{1}{\log k}\Big(\frac{4\pi}{\ell}\Big)^{\ell}\sum_{p\le k^{2\alpha}}p^{-1+\epsilon}\log p\ll\frac{1}{\log k}\Big(\frac{4\pi}{\ell}\Big)^{\ell}k^{2\alpha\epsilon}=o(1)
\end{aligned}$$

for any $\alpha>0$ as $k\to\infty$.

The rank 2 term $G_{2,k}(pI, I)$ is

$$\text{(4-12)}\quad G_{2,k}(pI, I) = \sum_{C\in\mathcal{C}} \frac{K(I, pI; C)}{|\det C|^{3/2}} \int_0^{\pi/2} J_\ell(4\pi\sqrt{\lambda_{\min}}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_{\max}}\sin\theta)\sin\theta\, d\theta,$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smaller and larger eigenvalues of $pC^{-1}C^{-T}$ respectively. By estimates (3-5), (3-21), (3-31) and (3-30), we have

$$\text{(4-13)}\quad G_{2,k}(pI, I) \ll \sum_{C\in\mathcal{C}} \left(\frac{4\pi\sqrt{p}\sin\theta}{\ell\|C\|_F}\right)^\ell \ll p^{\ell/2}\left(\frac{4\pi}{\ell}\right)^\ell$$

for k sufficiently large. Thus its contribution to (4-7) is at most

$$\text{(4-14)}\quad \frac{1}{\log k}\sum_p p^{\ell/2}\left(\frac{4\pi}{\ell}\right)^\ell \frac{\log p}{\sqrt{p}}\hat{\Phi}\left(\frac{\log p}{\log k^2}\right) \ll \frac{1}{\log k}\left(\frac{4\pi}{\ell}\right)^\ell \sum_{p\le k^{2\alpha}} p^{\frac{\ell-1}{2}}\log p \ll k^\alpha\left(\frac{4\pi k^\alpha}{\ell}\right)^\ell,$$

which goes to 0 as $k\to\infty$ when $\alpha<1$. Thus we have proved the contribution of $\Delta_k(I, pI)$ to (4-7) is small when $\alpha<1$. Other off-diagonal contributions are estimated similarly, and are all small when $\alpha<1$. We skip the details here.

The diagonal contribution of $\frac{\lambda_p}{\sqrt{p}}\Delta_k(I, I)$ to (4-7) from the $m=1$ term is

$$\begin{aligned}\text{(4-15)}\quad -\frac{2}{\log k^2}\sum_p \frac{\lambda_p \log p}{p}\hat{\Phi}\left(\frac{\log p}{\log k^2}\right) &= -\frac{2}{\log k^2}\int_1^\infty \frac{\log x}{x}\hat{\Phi}\left(\frac{\log x}{\log k^2}\right) d\pi(x) + o(1)\\ &= -\frac{2}{\log k^2}\int_1^\infty \frac{\log x}{x}\hat{\Phi}\left(\frac{\log x}{\log k^2}\right)\frac{1}{\log x}dx + o(1)\\ &= -2\int_0^\infty \hat{\Phi}(y)\,dy + o(1)\\ &= -\Phi(0) + o(1).\end{aligned}$$

Here we have used the prime number theorem (PNT) for the prime counting function $\pi(x)$ and the fact the $\lambda_p = 1+\chi_{-4}(p)$ takes values 0 and 2 for primes p with density $\frac{1}{2}$ each.

The diagonal contribution of $-\Delta_k(pI, pI)$ from the $m=2$ term is

$$\text{(4-16)}\quad \frac{2}{\log k^2}\sum_p \frac{\log p}{p}\hat{\Phi}\left(\frac{2\log p}{\log k^2}\right) = \frac{\Phi(0)}{2} + o(1).$$

It is shown in [Kowalski et al. 2012] that the $m \geq 3$ terms in (4-7) contribute at most $O\left(\frac{1}{\log k}\right)$. For non-Saito–Kurokawa lifts F this follows from the Ramanujan bound $|c_m(p; F)| \leq 4$ and the fact

$$\sum_{m=3}^{\infty} \sum_{p} \frac{\log p}{p^{m/2}} < \infty. \tag{4-17}$$

For a treatment of Saito–Kurokawa lifts, we refer readers to Section 5 in [Kowalski et al. 2012].

For the last term in (4-7) we use the fact that (see, for example, page 1754 in [Blomer 2019])

$$\omega_{F_f} \ll \frac{1}{k^3} \frac{L\left(\frac{1}{2}, f \times \chi_{-4}\right)}{L(1, \operatorname{sym}^2 f)}. \tag{4-18}$$

This, combined with the convexity bound for $L\left(\frac{1}{2}, f \times \chi_{-4}\right)$ and the lower bound [Hoffstein and Lockhart 1994]

$$L(1, \operatorname{sym}^2 f) \gg k^{-\epsilon} \tag{4-19}$$

give us

$$\sum_{F_f \in H_k^*(\Gamma_2)} \omega_{F_f} = o(1). \tag{4-20}$$

Combining all results above, we finally have

$$\begin{aligned} \sum_{F \in H_k(\Gamma_2)} \omega_F \, D(F; \Phi; \text{spin}) &= \hat{\Phi}(0) - \Phi(0) + \frac{\Phi(0)}{2} + o(1) \\ &= \hat{\Phi}(0) - \frac{\Phi(0)}{2} + o(1) \end{aligned} \tag{4-21}$$

for $\alpha < 1$, as $k \to \infty$. This completes the proof of Theorem 1.2.

4.2. *Proof of Theorem 1.6.* The proof is similar to that of Theorem 1.2. The explicit formula for $D(F; \Phi; \text{std})$ is

$$\begin{aligned} &D(F; \Phi; \text{std}) \\ &= \frac{2}{\log k^4} \int_{\mathbb{R}} \Phi(x) \left(-2\log(2\pi) - \tfrac{1}{2}\log \pi + \tfrac{1}{2} \frac{\Gamma'}{\Gamma}\left(\tfrac{1}{4} + \frac{\pi i x}{\log k^4}\right)\right) dx \\ &\quad + \frac{2}{\log k^4} \int_{\mathbb{R}} \Phi(x) \left(\frac{\Gamma'}{\Gamma}\left(k - \tfrac{1}{2} + \frac{2\pi i x}{\log k^4}\right) + \frac{\Gamma'}{\Gamma}\left(k - \tfrac{3}{2} + \frac{2\pi i x}{\log k^4}\right)\right) dx \\ &\quad - \frac{2}{\log k^4} \sum_{m=1}^{\infty} \sum_{p} \tau_{2m}(p; F) \frac{\log p}{p^{m/2}} \hat{\Phi}\left(\frac{m \log p}{\log k^4}\right). \end{aligned} \tag{4-22}$$

The integrals of gamma factors and the sum over $m \geq 3$ can be computed similarly as before. We thus have the explicit formula for the weighted sum

$$\sum_{F \in H_k(\Gamma_2)} \omega_F D(F; \Phi; \text{std}) = \hat{\Phi}(0) - \frac{2}{\log k^4} \sum_{m=1}^{\infty} \sum_p \sum_{F \in H_k(\Gamma_2)} \omega_F \tau_{2m}(p; F) \frac{\log p}{p^{m/2}} \hat{\Phi}\left(\frac{m \log p}{\log k^4}\right) + o(1). \tag{4-23}$$

We have seen in the proof of Theorem 1.2 that the symmetry type is determined by diagonal contributions. So we shall concentrate on those terms and be brief about the rest.

By Lemma 2.1, the $m = 1$ term is

$$\sum_{F \in H_k(\Gamma_2)} \omega_F \tau_2(p; F) = \Delta_k(pI, pI) - \Delta_k(p^2 I, I) + \frac{\lambda_p}{\sqrt{p}} \Delta_k(pI, I) + \left(\frac{\mu_p}{p} - 1\right) \Delta_k(I, I), \tag{4-24}$$

in which the diagonal contribution of $\Delta_k(pI, pI)$ and $-\Delta_k(I, I)$ cancel each other.

The $m = 2$ term is

$$\begin{aligned} \sum_{F \in H_k(\Gamma_2)} \omega_F \tau_4(p; F) = {} & -\Delta_k(p^3 I, pI) + \Delta_k(p^2 I, p^2 I) \\ & + \frac{\lambda_p}{\sqrt{p}} \Delta_k(p^2 I, pI) + \left(\frac{\mu_p}{p} - 1\right) \Delta_k(pI, pI) \\ & - \frac{\lambda_p}{\sqrt{p}} \Delta_k(p^3 I, I) + O\left(\frac{1}{p}\right) \Delta_k(p^2 I, I) \\ & + O\left(\frac{1}{\sqrt{p}}\right) \Delta_k(pI, I) + \left(O\left(\frac{1}{p}\right) + 1\right) \Delta_k(I, I). \end{aligned} \tag{4-25}$$

The diagonal contribution from $\Delta_k(p^2 I, p^2 I)$, $-\Delta_k(pI, pI)$ and $\Delta_k(I, I)$ combined is

$$-\frac{2}{\log k^4} \sum_p \frac{\log p}{p} \hat{\Phi}\left(\frac{2 \log p}{\log k^4}\right) = -\frac{\Phi(0)}{2} + o(1). \tag{4-26}$$

To illustrate why the range of support is restricted to $\left(-\frac{1}{4}, \frac{1}{4}\right)$, we analyze the contribution of the rank 2 term $G_{2,k}(pI, pI)$:

$$G_{2,k}(pI, pI) = \sum_{C \in \mathcal{C}} \frac{K(pI, pI; C)}{|\det C|^{3/2}} \int_0^{\pi/2} J_\ell(4\pi \sqrt{\lambda_{\min}} \sin\theta) J_\ell(4\pi \sqrt{\lambda_{\max}} \sin\theta) \sin\theta \, d\theta, \tag{4-27}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are eigenvalues of $p^2C^{-1}C^{-T}$. We estimate as before to get

$$G_{2,k}(pI, pI) \ll \sum_{C\in\mathcal{C}} \left(\frac{4\pi p \sin\theta}{\ell \|C\|_F}\right)^{\ell} \ll p^{\ell}\left(\frac{4\pi}{\ell}\right)^{\ell}. \tag{4-28}$$

Thus it contributes at most

$$\begin{aligned} \frac{1}{\log k}\sum_p p^{\ell}\left(\frac{4\pi}{\ell}\right)^{\ell} \frac{\log p}{\sqrt{p}} \hat{\Phi}\left(\frac{\log p}{\log k^4}\right) \\ \ll \frac{1}{\log k}\left(\frac{4\pi}{\ell}\right)^{\ell} \sum_{p\le k^{4\alpha}} p^{\ell-\frac{1}{2}} \log p \ll k^{2\alpha}\left(\frac{4\pi k^{4\alpha}}{\ell}\right)^{\ell}, \end{aligned} \tag{4-29}$$

which is $o(1)$ as $k\to\infty$ if $\alpha<\frac{1}{4}$. Other off-diagonal terms are estimated similarly.

4.3. ***Proof of Theorem 1.8.*** The contribution of gamma factors and the diagonal contribution do not change upon averaging over k with respect to Ω. To illustrate how we may extend the range of support form $\left(-\frac{1}{4}, \frac{1}{4}\right)$ to $\left(-\frac{5}{18}, \frac{5}{18}\right)$, we take the term $\Delta_k(pI, pI)$ for example.

By Lemma 3.5, the off-diagonal part of

$$\left(\sum_k \Omega\left(\frac{k}{K}\right)\right)^{-1} \sum_k \Omega\left(\frac{k}{K}\right)\Delta_k(pI, pI) \tag{4-30}$$

is at most

$$\frac{p}{K^4} + \frac{p^{4+2\epsilon}}{K^{5+2\epsilon}} + \frac{p^{j+2}}{K^{2j+3}} \tag{4-31}$$

for any $j\ge 3$ and small $\epsilon>0$. It contributes at most

$$\begin{aligned} \frac{1}{\log K}\sum_{p\le K^{4\alpha}} \left(\frac{p}{K^4} + \frac{p^{4+2\epsilon}}{K^{5+2\epsilon}} + \frac{p^{j+2}}{K^{2j+3}}\right)\frac{\log p}{\sqrt{p}} \\ \ll K^{6\alpha-4} + K^{18\alpha-5+6\epsilon} + K^{(4j+10)\alpha-(2j+3)}, \end{aligned} \tag{4-32}$$

which is $o(1)$ if $\alpha<\frac{5}{18}$, by taking j sufficiently large.

Finally, to see

$$\hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\, dx, \tag{4-33}$$

we use the Plancherel formula

$$\int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\, dx = \int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{W}(\mathrm{Sp})(y)\, dy \tag{4-34}$$

and the Fourier pair (1-6). The proof is now complete.

5. Application to nonvanishing of central values

5.1. ***Proof of Corollary 1.4.*** By Theorem 1.2 we have

$$\sum_{F\in H_k(\Gamma_2)} \omega_F \sum_{\rho_{F,\mathrm{spin}}} \Phi\left(\frac{\gamma_{F,\mathrm{spin}}}{2\pi}\log k^2\right) < \hat{\Phi}(0) - \frac{\Phi(0)}{2} + \epsilon \tag{5-1}$$

for any $\epsilon > 0$ and k large enough. We further assume

$$\Phi(x) \geq 0, \quad \Phi(0) = 1. \tag{5-2}$$

By these conditions we may pick up only the zeros $\rho_{F,\mathrm{spin}} = \frac{1}{2}$ to get

$$\begin{aligned}\sum_{F\in H_k(\Gamma_2)} \omega_F \sum_{\rho_{F,\mathrm{spin}}} \Phi\left(\frac{\gamma_{F,\mathrm{spin}}}{2\pi}\log k^2\right) &\geq \sum_{F\in H_k(\Gamma_2)} \omega_F \cdot \mathrm{ord}_{s=1/2} L(s, F; \mathrm{spin}) \\ &= \sum_{m=2}^{\infty} m \sum_{\mathrm{ord}_{s=1/2} L(s,F;\mathrm{spin})=m} \omega_F \\ &\geq 2 \sum_{\mathrm{ord}_{s=1/2} L(s,F;\mathrm{spin})\geq 2} \omega_F .\end{aligned} \tag{5-3}$$

Here we have used the fact that the root number of $L(s, F; \mathrm{spin})$ is always $+1$. Thus the vanishing order of $L(s, F; \mathrm{spin})$ at $s = \frac{1}{2}$ is even. These inequalities, together with (1-14), give us

$$\sum_{L(1/2,F;\mathrm{spin})\neq 0} \omega_F > 1 - \tfrac{1}{2}\left(\hat{\Phi}(0) - \frac{\Phi(0)}{2}\right) - \epsilon. \tag{5-4}$$

It is discussed in [Iwaniec et al. 2000, Appendix A] that the Fourier pair

$$\Phi(x) = \left(\frac{\sin \pi v x}{\pi v x}\right)^2, \qquad \hat{\Phi}(y) = \frac{1}{v}\left(1 - \frac{|y|}{v}\right), \quad |y| < v \ \ (v > 0). \tag{5-5}$$

gives essentially the optimal bound. With this choice we have

$$\sum_{L(1/2,F;\mathrm{spin})\neq 0} \omega_F > \frac{5}{4} - \frac{1}{2v} - \epsilon \tag{5-6}$$

for any $0 < v < 1$. Taking liminf in k and $v \to 1$, we have

$$\liminf_{k\to\infty} \sum_{L(1/2,F;\mathrm{spin})\neq 0} \omega_F \geq \frac{3}{4}. \tag{5-7}$$

We can further ignore the contribution of Saito–Kurokawa lifts, in view of (4-20). This completes the proof of Corollary 1.4.

5.2. ***Further discussion.*** From the proof of Corollary 1.4 we see that in order to obtain any result on nonvanishing of central values of $L(s, F; \text{spin})$ or $L(s, F; \text{std})$, the range of support in the corresponding Density Theorem must go beyond $\left(-\frac{2}{5}, \frac{2}{5}\right)$. This range is by setting

$$\frac{5}{4} - \frac{1}{2v} = 0. \tag{5-8}$$

The previous range of support $\left(-\frac{4}{15}, \frac{4}{15}\right)$ obtained in [Kowalski et al. 2012] for spinor L-functions is not large enough, for $\frac{4}{15} < \frac{2}{5}$. Thus our extension to $(-1, 1)$ is significant for the purpose of nonvanishing.

Unfortunately, for standard L-functions, our range of support is still not large enough to obtain a nonvanishing result, even after performing an average over weight $\left(\frac{5}{18} < \frac{2}{5}\right)$. The author would like to address this problem by establishing a more refined version of Lemma 3.5 in the future.

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