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CENTRAL NILPOTENCY OF LEFT SKEW BRACES AND SOLUTIONS OF THE YANG–BAXTER EQUATION

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This paper delves into the study of centrally nilpotent skew braces. In particular, we study their torsion theory, we introduce an "index" for subbraces, but we also show that the product of centrally nilpotent ideals need not be centrally nilpotent. To cope with these examples, we introduce a special type of nilpotent ideal, using which, we define a *good* Fitting ideal. Also, a Frattini ideal is defined and its relationship with the Fitting ideal is investigated. A key ingredient is the characterisation of the commutator of ideals in terms of star products, which solves a problem of Bonatto and Jedlička (*J. Algebra Appl.* 22:12 (2023), art. id. 2350255). Moreover, we provide an example showing that the idealiser of a subbrace does not exist in general.

1. Introduction

The study of set-theoretic solutions of the Yang–Baxter equation (YBE) provides a common framework for a multidisciplinary approach from different areas including knot theory, braid theory, and Garside theory among others (see [10; 12] for example). The main challenge in this area is to classify all set-theoretic solutions with prescribed properties. The algebraic structure of left skew braces plays a fundamental role in this classification. Nondegenerate set-theoretic solutions of the YBE naturally lead to left skew braces (see [18]), and conversely, every left skew brace B defines a solution (B, r_B) of the YBE (see [14]). Nowadays, we are far from being able to understand arbitrary solutions of the YBE. But it is becoming more and more clear that almost every nondegenerate solution is a multipermutation, and nilpotency of left skew braces is precisely introduced to deal with multipermutation solutions (see [7; 13; 16] for example). In this paper, we provide a deep and complete study of central nilpotency with new results that could be a reference point for all future work on the argument.

We now highlight some of the main aspects of nilpotency of left skew braces we deal with (see the next sections for the definitions): In Section 4.1, we study

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its torsion theory, providing great extensions of some of the results in [16] (see, in particular, Theorem 4.14). In Section 4.2, we deal with the problem of defining an index for subbraces, proving that this is possible in the context of locally centrally nilpotent left skew braces. Also we provide many examples showing how peculiar is the behaviour of centrally nilpotent left skew braces if compared to that of groups and rings. Among these examples, two are the most striking (see Section 6): Example A shows that, contrary to what is claimed in [16], it is not always possible to define the idealiser of a subbrace of a left skew brace of abelian type. Example B shows that the product of two centrally nilpotent ideals of a left skew brace (of abelian type) need not be centrally nilpotent. In order to cope with the above examples, we introduce a new type of nilpotency for ideals (see Section 5). Using this new concept we are able to define a well-behaving *Fitting ideal* for left skew braces. In turns, the Fitting ideal makes it possible to give a good definition of *Frattini ideal* for left skew braces. Using these definitions we are able to prove an analogue of a celebrated result of Gaschütz that relates the Frattini and the Fitting subgroups of a group (see Theorem 5.10). Finally, we note that one of the key steps in our work is showing that the two known definitions of commutators of ideals (inspired by distinct universal algebra approaches) coincide, thus solving Problem 3.4 of [4] (see Section 3).

2. Preliminaries

From now on, the word "brace" will mean "left skew brace". We refer to [1; 2; 3; 20] for the standard preliminaries about braces. However, we recall that if E is any subset of a brace B, then $\langle E\rangle$ is the subbrace generated by E in B, E^B is the smallest ideal of B containing E, and E_B is the maximal ideal of B contained in E. Moreover, $C \leq B$ (resp., $C \trianglelefteq B$) denotes that C is a subbrace (resp., an ideal) of B, while $C \lessdot B$ denotes that C is a maximal subbrace of B. The following commutator of ideals is introduced in [5] and plays a key role in the study of nilpotency and solubility in braces.

Definition 2.1. Let I, J be ideals of a brace B. The *commutator* $[I, J]^B$ is defined as $[I, J]^B := \big\langle [I, J]_+ \cup [I, J]_\cdot \cup \{ij - (i + j) : i \in I,\ j \in J\}\big\rangle^B$. Clearly, $[I, J]^B = [J, I]^B$ and $[I, J]^B \leq I \cap J$.

A brace B is said to be *abelian* if $[B, B]^B = 0$, i.e., if it is a trivial brace of abelian type. Using this commutator, solubility of braces has been introduced in [2]. If I is an ideal of B, the *commutator* or *derived ideal of I with respect to B* is defined as $\partial_B(I) = [I, I]^B$. If $B = I$, then we say that $\partial_B(B) = \partial(B) = \partial_1(B)$ is the *commutator* or *derived ideal* of B. By repeatedly forming derived ideals, we recursively obtain a descending sequence of ideals with $\partial_n(I) = \partial_B(\partial_{n-1}(I))$ for every $n \in \mathbb{N}$. We call this series the *derived series of I with respect to B*. Clearly,

$\partial_{n-1}(B)/\partial_n(B)$ is an abelian brace for every $n \in \mathbb{N}$, and $B/\partial(B)$ is the greatest abelian quotient in B. We say that an ideal I of B is *soluble with respect to* B, if there exists a nonnegative integer n such that $\partial_n(I) = 0$. If $I = B$, we simply say that B is a *soluble brace*, and the smallest nonnegative integer m for which $\partial_m(B) = 0$ is the *derived length* of B.

3. Unifying universal algebra definitions of commutators of ideals in braces

The definition of commutator of ideals given in Section 2 was inspired by the *Huq* = *Smith condition* for the category of braces (see [5] for further details). On the other hand, following ideas of universal algebra, a definition of a commutator of ideals of a brace in terms of absorbing polynomials is provided in [4] as a brace-theoretic version of a commutator of congruences given by Freese and McKenzie in [11]. In case of modular lattices of congruences, it turns out that the Freese and McKenzie commutator coincides with the Hagemann and Herrmann commutator (see [11, Theorem 4.9]). According to [15], the Hagemann and Herrmann commutator was introduced as an extension of the Smith commutator so that both commutators coincide in case of permutable lattices of congruences.

Observe that the lattice of ideals of a brace is modular and permutable, since the product of ideals commute and for all ideals I, J, K with $I \leq J$, the Dedekind identity holds. Hence, the definitions of commutator of ideals of a brace given in [4; 5] coincide.

In this context, the following questions were raised in [4, Problem 3.4].

Problem 3.1. Let B be a brace.

(1) Does the equality $[I, J]^B = \langle I * J + J * I + [I, J]_+\rangle^B$ hold?

(2) Does the equality $[I, J]^B = I * J + J * I + [I, J]_+$ hold?

Our next theorem gives a positive answer to the first question.

Remark 3.2. Let B be a brace. Then, for every $i, j, x, y, z \in (B, +)$, we have that

$$i + x + j + y - i + z - j = [-i, -x]_+ + x + [-i, -j]_+ + [-i, -y]_+^{-j,+} + [-j, -y]_+ + y + [-j, -z]_+ + z.$$

Theorem 3.3. *Let I, J be ideals of a brace B.*

(1) *$I * J + J * I + [I, J]_+$ is the least left ideal containing*

$$X_{I,J} = [I, J]_+ \cup [I, J]_\cdot \cup \{ij - (i + j) : i \in I,\ j \in J\}.$$

(2) *$[I, J]^B = \langle I * J + J * I + [I, J]_+\rangle^B$. Thus, $[I, J]^B = I * J + J * I + [I, J]_+$ if and only if $I * J + J * I + [I, J]_+$ is an ideal of B.*

Proof. (1) By [4, Proposition 1.4], $I * J + J * I$ and $[I, J]_+$ are left ideals. Since $[I, J]_+$ is a normal subgroup of $(B, +)$, it follows that $I * J + J * I + [I, J]_+$ is also a left ideal.

For the inclusion $X_{I,J} \subseteq I * J + J * I + [I, J]_+$, we need to prove that

$$\{ij - (i + j) : i \in I,\ \ j \in J\},\ [I, J]_. \subseteq I * J + J * I + [I, J]_+.$$

Let $i \in I$ and $j \in J$. For the former, observe that $ij - j - i = i + i * j - i$. Thus, $ij - j - i = i + i * j - i - i * j + i * j \in [I, J]_+ + I * J$. For the latter, it follows that

$$\begin{aligned} iji^{-1}j^{-1} &= iji^{-1} + (iji^{-1}) * j^{-1} + j^{-1} \\ &= ij + (ij) * i^{-1} + i^{-1} + (iji^{-1}) * j^{-1} + j^{-1} \\ &= i + i * j + j + (ij) * i^{-1} + i^{-1} + (iji^{-1}) * j^{-1} + j^{-1}. \end{aligned}$$

Now, the above equation becomes

$$\begin{aligned} &i + i * j + j + i * (j * i^{-1}) + j * i^{-1} + i * i^{-1} + i^{-1} \\ &\quad + (ij) * (i^{-1} * j^{-1}) + i^{-1} * j^{-1} + (ij) * j^{-1} + j^{-1} \\ &\qquad = i + i * j + j + i * (j * i^{-1}) + j * i^{-1} + i * i^{-1} + i^{-1} \\ &\qquad\quad + i * (j * (i^{-1} * j^{-1})) + j * (i^{-1} * j^{-1}) + i * (i^{-1} * j^{-1}) + i^{-1} * j^{-1} \\ &\qquad\quad + i * (j * j^{-1}) + j * j^{-1} + i * j^{-1} + j^{-1}. \end{aligned}$$

Observe that $i * i^{-1} + i^{-1} = -i + 0 - i^{-1} + i^{-1} = -i$ and $j * j^{-1} = -j - j^{-1}$. Thus, we have

$$\begin{aligned} (\star)\quad & iji^{-1}j^{-1} = i + i * j + j + i * (j * i^{-1}) + j * i^{-1} - i + i * (j * (i^{-1} * j^{-1})) \\ (\diamond)\quad & \qquad + j * (i^{-1} * j^{-1}) + i * (i^{-1} * j^{-1}) + i^{-1} * j^{-1} + i * (j * j^{-1}) - j \\ & \qquad - j^{-1} + i * j^{-1} + j^{-1}. \end{aligned}$$

Since $I * J,\ J * I \subseteq I \cap J$, Remark 3.2 applied on $(\star) + (\diamond)$ yields

$$iji^{-1}j^{-1} \in I * J + J * I + [I, J]_+ + (-j^{-1} + i * j^{-1} + j^{-1}).$$

But, $-j^{-1} + i * j^{-1} + j^{-1} = [j^{-1}, -i * j^{-1}]_+ + i * j^{-1} \in [I, J]_+ + I * J$. Hence, $[I, J]_. \subseteq I * J + J * I + [I, J]_+$ follows.

Now, let L be the least left ideal of B containing $X_{I,J}$ (note that the arbitrary intersection of left ideals is a left ideal). In order to prove that $I * J + J * I + [I, J]_+ = L$, we only need to show that $J * I \subseteq L$.

Let $j \in J$ and $i \in I$. Then, $j * i = [-j * i, -j]_+ + (ji - i - j) \in X_{I,J} + (ji - i - j)$, as $j * i \in I \cap J$, so it suffices to prove that $(ji - i - j) \in L$. Since $[j, i]_. \in X_{I,J} \subseteq L$ and L is λ-invariant, it follows that

$$ji = ij[j, i]_. = ij + \lambda_{ij}([j, i]_.) = ij + x$$

g	$\delta(g)$	g	$\delta(g)$	g	$\delta(g)$	g	$\delta(g)$
1	0	b	y	c	$x+y$	bc	$3x$
a	x	b^3	$2x+3y$	c^3	$3x+3y$	bc^{-1}	$x+2y$
a^2	$2x+2y$	ab	$x+3y$	ac	$2x+y$	abc	$2y$
a^3	$3x+2y$	ab^{-1}	$3x+y$	ac^{-1}	$3y$	abc^{-1}	$2x$

Table 1. Bijective 1-cocycle associated with the brace of Example 3.4.

for some $x \in L \cap I \cap J$. Therefore,

$$\begin{aligned} ji - i - j &= ij + x - ([-j, -i]_+ + i + j) = ij + x - (i+j) + [-i, -j]_+ \\ &= ij - (i+j) + [-(i+j), -x]_+ + x + [-i, -j]_+ \in L. \end{aligned}$$

Consequently, $I * J + J * I + [I, J]_+ = L$.

(2) By definition, $[I, J]^B = \langle X_{I,J}\rangle^B$. Then, the previous statement yields $[I, J]^B = \langle I * J + J * I + [I, J]_+\rangle^B$. ☐

The next example gives a negative answer to Problem 3.1(2).

Example 3.4. Let $(B, +) = \langle x, y \mid 4x = 4y = 0,\ x + y = y + x\rangle$ and $(C, \cdot) = \langle a, b, c \mid c^4 = 1,\ a^2 = b^2 = c^2,\ (ab^{-1})^2 = 1,\ b^{-1}cb = c,\ a^{-1}ca = c^{-1}\rangle$. We see that C acts on B via $a(x) = x + 2y$, $b(x) = x + 2y$, $c(x) = 3x + 2y$, $a(y) = -y$, $b(y) = 2x + y$, $c(y) = 2x + 3y$. Consider the semidirect product G of B by C with respect to this action $\lambda : C \to \mathrm{Aut}(B)$. For the sake of clarity, we use multiplicative notation for B in G. Let $D = \langle xa, yb, xyc\rangle \leq G$. It follows that G is a trifactorised group as $BD = DC$, $D \cap C = D \cap B = 1$. By [1, Lemma 3.2], there exists a bijective 1-cocycle $\delta : C \to (B, +)$ given by $D = \{\delta(c)c : c \in C\}$ (see Table 1). This yields a product in B and we get a brace of abelian type, $(B, +, \cdot)$, which corresponds to `SmallBrace(16,73)` in the YangBaxter library [21] for GAP.

Let $I = \langle 2x, y\rangle \leq (B, +)$. Then $\lambda(I) = I$ and $|I| = 8$, so I is an ideal of B. Since B is of abelian type, we compute

$$I * I + [I, I]_+ = I * I = \langle u * v \mid u, v \in I\rangle_+ = \langle 2x\rangle_+ = \{1, 2x\}.$$

Therefore, $I * I$ is not an ideal of B, as $\delta(abc^{-1}) = 2x$ and $\{1, abc^{-1}\}$ is not a normal subgroup of C. Hence, $I * I \subsetneq [I, I]^B = I$.

4. Central nilpotency of braces

In this section we develop a standard theory of central nilpotency of braces. We start by introducing the basic definitions. Central nilpotency of braces was first introduced by using a brace-theoretic analogue of the centre of a group (see [4; 16]). The *centre* of a brace B (also known as the *annihilator ideal* of B) is the ideal of B defined as $\zeta(B) := \{a \in B \mid a+b = b+a = ab = ba \text{ for all } b \in B\}$ (see [6]). Thus, abelian braces B are precisely those ones satisfying $\zeta(B) = B$. In [8; 20], the definition of central nilpotency has been extended to include more types of braces (see also [9], where similar concepts for braces of abelian type are considered).

Definition 4.1. Let B be a brace. If $J \leq I$ are ideals of B satisfying $I/J \leq \zeta(B/J)$, we say that I/J is a *central factor* of B.

A *c-series* of B is a chain $\mathcal{I}$ of ideals of B such that 0, $B \in \mathcal{I}$ and whose factors are central, that is, $I/J \leq \zeta(B/J)$ for all *consecutive* elements $J \leq I$ of $\mathcal{I}$ (meaning that there is no $C \in \mathcal{I}$ satisfying $J < C < I$). A *complete* c-series is just a c-series containing the unions and the intersections of all its members. Since every c-series can be completed, we usually consider every c-series to be complete. We say that B is *ζ-nilpotent* if it admits a c-series.

If B has an ascending c-series

$$0 = I_0 \leq I_1 \leq \cdots I_\alpha \leq I_{\alpha+1} \leq \cdots I_\mu = B$$

(here $\alpha < \mu$ are ordinal numbers), then B is *hypercentral*, and the smallest ordinal number μ for which such an ascending c-series exists is the *hypercentral length* $n_c(B)$ of B. (Note that $I_{\alpha+1}/I_\alpha \leq \zeta(B/I_\alpha)$ for all ordinals $\alpha < \mu$.)

If B has a finite c-series $0 = I_0 \leq I_1 \leq \cdots \leq I_n = B$, then B is *centrally nilpotent*; in this case, the smallest nonnegative integer for which such a c-series exists is referred to as the *class* $n_c(B)$ of B. (Note that $I_{i+1}/I_i \leq \zeta(B/I_i)$ for all $0 \leq i < n$.)

Of course, centrally nilpotent braces are hypercentral, and hypercentral braces are ζ-nilpotent, but the converses do not hold. Moreover, subbraces of centrally nilpotent (resp. hypercentral, ζ-nilpotent) braces are centrally nilpotent (resp. hypercentral, ζ-nilpotent). Also, quotients of centrally nilpotent (resp. hypercentral) braces are still centrally nilpotent (resp. hypercentral), but the consideration of the infinite dihedral group shows that quotients of a ζ-nilpotent brace can be non-ζ-nilpotent. Finally, direct products of hypercentral braces are hypercentral; direct products of finitely many centrally nilpotent braces are centrally nilpotent; and restricted direct products of ζ-nilpotent braces are ζ-nilpotent.

Following [4; 8], canonical c-series are introduced for a brace B.

(▲) The *upper central series* of B is recursively defined by putting $\zeta_0(B) = 0$, $\zeta_{\alpha+1}(B)/\zeta_\alpha(B) = \zeta(B/\zeta_\alpha(B))$ for every ordinal α, and $\zeta_\lambda(B) = \bigcup_{\beta<\lambda} \zeta_\beta(B)$ for every limit ordinal λ. The last term of the upper central series is the *hypercentre* $\bar{\zeta}(B)$ of B.

(▼) The *lower central series* of B is recursively defined by putting $\Gamma_1(B) = B$, $\Gamma_{\alpha+1}(B) = \langle \Gamma_\alpha(B) * B,\, B * \Gamma_\alpha(B),\, [\Gamma_\alpha(B),\, B]_+ \rangle_+$ for every ordinal α, and $\Gamma_\lambda(B) = \bigcap_{\beta<\lambda} \Gamma_\beta(B)$ for every limit ordinal λ. Note that since $\Gamma_\alpha(B)$ is an ideal for every

$\alpha \le \mu$, we have that

$$\Gamma_{\alpha+1}(B) = \Gamma_\alpha(B) * B + B * \Gamma_\alpha(B) + [\Gamma_\alpha(B), B]_+ = [\Gamma_\alpha(B), B]^B,$$

for every ordinal $\alpha < \mu$ (see Theorem 3.3). The last term of the lower central series is the *hypocentre* $\overline{\Gamma}(B)$ of B.

The following easily provable all-in-one result shows the relations between the concepts we just introduced (see [17] for the definition of the upper central series $\{Z_\alpha(G)\}_{\alpha\in\mathrm{Ord}(G)}$ of a group G).

Proposition 4.2. *Let B be a brace.*

(1) (See [2, Proposition 17].) *If $J \le I$ are ideals of B, then $I/J \le \zeta(B/J)$ if and only if $[I, B]^B \le J$. In particular, if $0 = I_0 \le I_1 \le \cdots \le I_n = B$ is a finite c-series, then*:

- (a) $\Gamma_j(B) \le I_{n-j+1}$ *for every* $1 \le j \le n+1$; *so* $\Gamma_{n+1}(B) = 0$.
- (b) $I_j \le \zeta_j(B)$ *for every* $0 \le j \le n$; *so* $\zeta_n(B) = B$.
- (c) $n_c(B)$ *is the smallest number n such that $\zeta_n(B) = B$, and the smallest number n such that $\Gamma_{n+1}(B) = 0$.*

(2) *B is hypercentral if and only if $B = \bar{\zeta}(B)$. Moreover, in this case $n_c(B)$ is the smallest ordinal α for which $B = \zeta_\alpha(B)$.*

(3) $\zeta_\alpha(B) \subseteq \mathrm{Z}_\alpha(B, +) \cap \mathrm{Z}_\alpha(B, \cdot)$ *for every ordinal α. In particular, centrally nilpotent (resp. hypercentral) braces are braces of nilpotent (resp. hypercentral) type whose additive group is nilpotent (resp. hypercentral).*

(4) $\partial_n(B) \le \Gamma_{n+1}(B)$ *for every $n \in \mathbb{N}$. In particular, centrally nilpotent braces are soluble with derived length less than or equal to their class.*

The following result generalises Grün's lemma for groups (see, for instance, [17, Part 1, p. 48]).

Theorem 4.3. *Let B be a brace such that $\zeta_2(B) > \zeta(B)$. Then either $[B, B]_+$ or $[B, B]_\cdot$ is a proper subset of B.*

Proof. We may assume by Grün's lemma that

$$Z(B, +) = Z_2(B, +) \quad \text{and} \quad Z(B, \cdot) = Z_2(B, \cdot).$$

By Proposition 4.2, $\zeta_2(B) \subseteq Z_2(B, +) \cap Z_2(B, \cdot) = Z(B, +) \cap Z(B, \cdot)$. Now, choose $c \in \zeta_2(B) \setminus \zeta(B)$ and consider the map $\varphi_c : b \in B \mapsto c * b \in \zeta(B)$. Let $b_1, b_2 \in B$. Then $c * (b_1 + b_2) = c * b_1 + c * b_2$ because $c * b_2 \in \zeta(B)$, so φ_c is a homomorphism with respect to $+$. Since $c \notin \zeta(B)$, we have that $c \notin \mathrm{Ker}(\lambda)$ and consequently $\varphi_c(B) \neq 0$. Thus $[B, B]_+$ is properly contained in B. □

In order to see if a brace is centrally nilpotent (resp. hypercentral) or not, we only need to look at its countable subbraces: this is the content of our next result.

Theorem 4.4. *Let B be a brace.*

(1) *B is centrally nilpotent of class at most c if and only if its finitely generated subbraces are centrally nilpotent of class at most c.*

(2) *B is centrally nilpotent* (*hypercentral*) *if and only if its countable subbraces are centrally nilpotent* (*hypercentral*).

Proof. For each $u, v \in B$, the symbol $u \circ v$ means that we are performing one (but we do not know which) of the following operations $[u, v]_{\cdot}$, $[u, v]_{+}$, $u * v$.

(1) The first statement is a direct consequence of the fact that $\zeta_c(B)$ is easily characterised as the set of all elements $b \in B$ with $\big(\cdots((b \circ b_1) \circ \cdots) \circ b_c\big) = 0$ for all $b_1, \dots, b_c \in B$.

(2) We start by considering central nilpotency. Only one implication is in doubt. Thus, assume all countable subbraces of B are centrally nilpotent but B is not centrally nilpotent. By (1), for each $c \in \mathbb{N}$, there is a finitely generated centrally nilpotent brace B_c whose class is at least c. Then $C = \langle B_i : i \in \mathbb{N}\rangle$ is a countable subbrace of B that is not centrally nilpotent, a contradiction.

Now, we turn to hypercentrality. Again, only one implication is in doubt. Thus, assume all countable subbraces of B are hypercentral, and that B is not hypercentral. We may further assume that $\zeta(B) = 0$. Let $0 \neq x \in B$. Then there are sequences of nonzero elements $a_1, a_2, \dots, a_n, \dots$ and $x = b_1, b_2 \dots, b_n, \dots$ of B with $b_{n+1} = b_n \circ a_n$ for all $n \in \mathbb{N}$. Let $C = \langle a_i, b_j : i, j \in \mathbb{N}\rangle$. Thus, C is countable and so it is hypercentral. Now, for each $i \in \mathbb{N}$, let α_i be the smallest ordinal β such that $b_i \in \zeta_\beta(C)$. Then $\alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots$ is a strictly descending chain of ordinal numbers, a contradiction. □

In studying the structure of an arbitrary group, local analogues of nilpotence are very useful. Similarly, the following definition is crucial for us in studying centrally nilpotent braces (see [8]). A brace B is *locally centrally nilpotent* if every finitely generated subbrace is centrally nilpotent. Of course, every subbrace/quotient of a locally centrally nilpotent brace is still locally centrally nilpotent, and also restricted direct products of locally centrally nilpotent braces are locally centrally nilpotent. Moreover, by Corollary 3.6 of [20], every hypercentral brace is locally centrally nilpotent, but the converse does not hold. As a consequence of the following result, we see that every locally centrally nilpotent brace is ζ-nilpotent.

Theorem 4.5. *Let B be a locally centrally nilpotent brace.*

(1) *If I is any minimal ideal of B, then $I \leq \zeta(B)$.*

(2) *If M is any maximal subbrace of B, then M is an ideal of B.*

In particular, every minimal ideal of B has prime order, and $\partial(B)$ is contained in every maximal subbrace of B.

Proof. (1) Suppose that $I \not\leq \zeta(B)$. Then there exist elements $b \in B$ and $x \in I$ such that $\mathcal{S} = \{[b, x]_+, [b, x]_\cdot, x * b\} \neq \{0\}$. Let $c \in \mathcal{S} \setminus \{0\}$. Since I is a minimal ideal of B, the ideal generated by c in B is I, so there are elements $y_1, \dots, y_n \in B$ such that x belongs to the ideal generated by c in $S = \langle b, c, y_1, \dots, y_n\rangle$.

Let $J = x^S$. Since S is centrally nilpotent, there is a finite chain $0 = J_0 < J_1 < \cdots < J_m = J$ of ideals of S such that $J_{i+1}/J_i \leq \zeta(S/J_i)$. Choose $\ell \in \mathbb{N}$ with $c \in J_\ell \setminus J_{\ell-1}$; clearly, $\ell \neq 0, m$ because c is a nonzero element of one of the following types: $[b, x]_+$, $[b, x]_\cdot$, $x * b$. Now,

$$x^S \leq c^S \leq c^S + J_{\ell-1} = \langle c\rangle + J_{\ell-1} \leq J_\ell < J_m = J = x^S,$$

a contradiction.

(2) Assume M is not an ideal. If $B * B \leq M$, then $(M, +)/(B * B, +)$ is a maximal subgroup of the locally nilpotent group $(B, +)/(B * B, +)$, so it is even normal, and it follows that M is an ideal of B, a contradiction. Thus, there exists an element $x \in B * B \setminus M$. Since M is a maximal subbrace of B, we have that $B = \langle M, x\rangle$. Then there is a finite subset L of B such that $x \in \langle L\rangle * \langle L\rangle$. For each $y \in L$, let B_y be a finite subset of M such that $y \in \langle B_y \cup \{x\}\rangle$. Put $D = \langle B_y : y \in L\rangle$ and $E = \langle D, x\rangle$, so E is finitely generated and $L \subseteq E$. Now, E is centrally nilpotent, and $x \notin D \subseteq M$. Let N be a subbrace of E which is maximal with respect to containing D but not x. Since $E = \langle D, x\rangle$, we see that N is actually a maximal subbrace of E. Since E is centrally nilpotent, there is $n \in \mathbb{N}$ such that $\zeta_n(E) \leq N$ but $\zeta_{n+1}(E) \not\leq N$. Then N is a proper ideal of $\zeta_{n+1}(E) + N$ and so $N \trianglelefteq E$, since N is a maximal subbrace of E. Therefore E/N is centrally nilpotent, and so $x \in E * E \leq N$, a contradiction. □

In [3], chief factors of braces are introduced and shown to play a key role in its ideal structure. Let I and J be ideals of a brace B such that $J \leq I$. The quotient I/J is said to be a *chief factor* of B if I/J is a minimal ideal of B/J. A chain $\mathcal{C}$ of ideals of B is a *chief series* of B if $0, B \in \mathcal{C}$ and I/J is a chief factor of B whenever $J < I$ are consecutive terms of $\mathcal{C}$. By Zorn's lemma, every brace has a (possibly infinite) chief series. In [3], a brace B is proved to have a finite chief series if and only if it is *noetherian* (that is, every ascending chain of ideals is eventually stationary) and *artinian* (that is, every descending chain of ideals is eventually stationary). By Theorem 4.5, every chief series of a locally centrally nilpotent brace is a c-series, so we have the following result.

Corollary 4.6. *Let B be a locally centrally nilpotent brace. Then B is ζ-nilpotent.*

Remark 4.7. The proof of Theorem 4.5(1) proves much more than we stated. In fact, let $\mathfrak{Z}$ be the class of all braces in which every chief factor is central. Moreover, let $\boldsymbol{L}\mathfrak{Z}$ be the class of all braces in which every finite subset F is contained in a subbrace $C_F \in \mathfrak{Z}$. The proof of Theorem 4.5(1) can be modified to show that $\boldsymbol{L}\mathfrak{Z} = \mathfrak{Z}$.

More in detail, using the notation of the first half of the proof of Theorem 4.5(1), we get that S is contained in a subbrace $T \in \mathfrak{Z}$. Let $J = x^T$.

Since $[J, T]^T$ is an ideal of T containing c, we have that $x \in c^T \leq [J, T]^T$ and so $J = x^T = [J, T]^T$. Finally, let M be a maximal ideal of T contained in J and such that $x \notin M$. Then J/M is a chief factor of T and so $[J, T]^T \leq M$, a contradiction.

It follows from Corollary 4.6 that any nonzero ideal of a locally centrally nilpotent brace contains a (nonzero) central factor of the whole brace. In case of a hypercentral brace we can say more.

Lemma 4.8. *Let B be a hypercentral brace. If I is any nonzero ideal of B, then $I \cap \zeta(B) \neq 0$.*

Proof. Let α be the smallest ordinal number such that $J = I \cap \zeta_\alpha(B) \neq 0$. Then α is successor and $I \cap \zeta_{\alpha-1}(B) = 0$. Now, $[J, B]^B \leq J \cap \zeta_{\alpha-1}(B) = 0$ and so $J \leq \zeta(B)$. □

Corollary 4.9. *Let B be a brace having a finite chief series* (*resp. an ascending chief series*) *$\mathcal{I}$. Then B is centrally nilpotent* (*resp. hypercentral*) *if and only if every chief factor of $\mathcal{I}$ is central.*

Our next two subsections deal with the torsion theory of locally centrally nilpotent braces and with the problem of defining a suitable index for subbraces. Before delving into them, we note that some important results for nilpotent groups do not hold for braces.

(1) Bearing in mind the normaliser condition for nilpotent groups, the idealiser of a subbrace is introduced in [16]: given a subbrace S of a brace B, the idealiser of S is defined as the largest subbrace N of B such that S is an ideal of N. It is then stated that every subideal is properly contained in its idealiser (see Section 4.2 for the definition of subideal). Example A in Section 6 shows that the idealiser of a subbrace does not exist in general. We note however that if C is a subbrace of a brace B, then one can define the largest strong left ideal $N_B(C)$ of B additively and multiplicatively normalising B and such that $\lambda_x(C) = C$ for every $x \in N_B(C)$ — but such a strong left ideal need not contain C.

(2) Example B shows that there is no analogue of Fitting's theorem for centrally nilpotent ideals. Moreover, Example C shows that an abelian subideal need not be contained in a centrally nilpotent ideal.

(3) The ideal structure of the brace listed as `SmallBrace(32, 24003)` in the YangBaxter library [21] for GAP is described in [2]. This brace B has only a unique maximal subbrace I, which is also its only nonzero proper ideal. Moreover, $\partial(B) = I$. Nevertheless, B is not centrally nilpotent as it is not even soluble. This

shows that a finite brace whose maximal subbraces are ideals need not be soluble. The same example shows that a finite brace whose subbraces are subideals need not be centrally nilpotent (see Example D for more details), although an easy induction shows that they are at least weakly soluble in the sense explained in [2].

4.1. ***Torsion theory.*** The aim of this subsection is to establish a torsion theory for locally centrally nilpotent braces. We start with some definitions.

Definition 4.10. Let B be a brace. The subset of all periodic elements of $(B,+)$ is denoted by $T_+(B)$, while that of all periodic elements of $(B,\cdot)$ is denoted by $T_\cdot(B)$. Moreover, an element b of B is *periodic* if $\langle b\rangle$ is finite. The *order* of b is $|\langle b\rangle|$. If π is any set of prime numbers, then b is a π*-element* if its order is a π-number. A π*-subbrace* is just a subbrace containing only π-elements. Finally, B is *periodic* if every element of B is periodic. B is *torsion-free* if every element $b\in B$ is either zero or is nonperiodic. B is *locally finite* if every finitely generated subbrace of B is finite. B has *finite exponent* n if B is periodic and n is the smallest positive integer such that $b^n = nb = 0$ for all $b\in B$.

Clearly, every locally finite brace is periodic but the converse does not hold. The following result shows that in the context of locally centrally nilpotent braces we can precisely identify the set of all periodic elements of B.

Theorem 4.11. *Let B be a locally centrally nilpotent brace.*

(1) $T_+(B) = T_\cdot(B)$.
(2) $T_+(B)$ *is an ideal of* B.
(3) $T_+(B/T_+(B)) = 0$.
(4) *If B is periodic, then B is locally finite.*
(5) *B is locally finite if and only if $(B,+)$ is locally finite if and only if $(B,\cdot)$ is locally finite.*

Proof. The proof of (1)–(3) is an easy consequence of Proposition 4.2 of [16]. Let us prove (4). Assume B is periodic and finitely generated. Then $(B,+)$ is a periodic nilpotent group. Moreover, by Theorem 3.7 of [20], $(B,+)$ is also finitely generated. Thus $(B,+)$ is finite and (4) is proved. Finally, (5) is an obvious consequence of Theorem 3.7 of [20]. □

Let B be a brace, and let p be a prime. The *Sylow p-subbrace* of B is just a maximal element of the set of all its p-subbraces with respect to the inclusion. Our next result shows that the Sylow subbraces of a locally centrally nilpotent brace are ideals and that they coincide with the additive/multiplicative Sylow subgroups.

Theorem 4.12. *Let B be a locally finite brace. Then, B is locally centrally nilpotent if and only if, for every prime p, $\mathrm{Syl}_p(B,+) = \mathrm{Syl}_p(B,\cdot) = \{B_p\}$, B_p is locally centrally nilpotent and B is the direct product of the B_p's.*

Proof. Only one direction is in doubt. Since B is locally finite, we may assume B is finite and centrally nilpotent. Let p be a prime. Since both $(B,+)$ and $(B,\cdot)$ are nilpotent groups, there exist Sylow p-subgroups $B_p \trianglelefteq (B,+)$ and $\bar{B}_p \trianglelefteq (B,\cdot)$. Observe that B_p is also λ-invariant, as it is a characteristic subgroup of $(B,+)$. Therefore, $B_p = \bar{B}_p$ is an ideal of B. □

Proposition 4.13. *Let B be a brace whose additive and multiplicative groups are cyclic. Then there is $x \in B$ which is a generator of both $(B,+)$ and $(B,\cdot)$.*

Proof. By Theorem 4.6 of [19], we may assume B is finite. If $\operatorname{Ker}(\lambda)=0$, then $(B,\cdot)$ embeds into $\operatorname{Aut}(B,+)$, a contradiction. Thus, $\zeta(B)\neq 0$. Iterating this argument, we see that B is centrally nilpotent, so B factorises into the direct product of its Sylow p-subgroups. It is therefore possible to assume that B has prime power order p^n.

Let I be a subbrace of $\zeta(B)$ of order p. By induction there is an element $x \in B$ such that $x+I$ is both a generator of $(B/I,+)$ and $(B/I,\cdot)$. If $\langle x^{p^{n-1}}\rangle_\cdot \cap I = 0$, then $(B,\cdot)$ is not cyclic, a contradiction. Thus, $\langle x^{p^{n-1}}\rangle_\cdot = I$ and x is a generator of $(B,\cdot)$. Similarly, x is a generator of $(B,+)$. □

Our next result is a huge generalisation of Lemma 4.1 of [16]. In order to state it, we need the following definition. Let B be a brace, and let π be a set of prime numbers. We say that B is *π-free* if it does not contain π-elements. Obviously, a trivial brace B is π-free if and only if $(B,+)$ and/or $(B,\cdot)$ are π-free as groups.

Theorem 4.14. *Let B be a brace and let π be a set of primes. If $\zeta(B)$ is π-free, then each factor of the upper central series of B, and therefore the hypercentre of B, is π-free.*

Proof. Suppose the theorem is false and let α be the first ordinal such that $\zeta_{\alpha+1}(B)/\zeta_\alpha(B)$ is not π-free; in particular, there is $x \in \zeta_{\alpha+1}(B) \setminus \zeta_\alpha(B)$ such that $x^m \in \zeta_\alpha(B)$ for some positive π-number m. We divide the proof in two parts according to α being limit or not.

Suppose first α is limit. Then $x^m \in \zeta_{\beta+1}(B)\setminus\zeta_\beta(B)$ for some $\beta<\alpha$. Since $x \notin \zeta_\alpha(B)$, there is $b \in B$ and $\gamma \geq \beta$ such that one of the elements $x*b$, $[x,b]_+$, $[x,b]_\cdot$ belongs to $\zeta_{\gamma+1}(B)\setminus\zeta_\gamma(B)$; call c such an element. Assume $c = x*b$. Then

$$(x*b)^m \equiv x^m * b \pmod{\zeta_\gamma(B)}$$

and so $x^m * b \in \zeta_\beta(B) \leq \zeta_\gamma(B)$. Therefore $(x*b)^m \in \zeta_\gamma(B)$. But $x*b \in \zeta_{\gamma+1}(B)$ and $\zeta_{\gamma+1}(B)/\zeta_\gamma(B)$ is π-free, so $c = x*b \in \zeta_\gamma(B)$, a contradiction. Similarly, we deal with the cases in which $c=[b,x]_\cdot$ and $c=[b,x]_+$.

Suppose now that α is successor; we may assume $\alpha=1$, so $x \in \zeta_2(B)\setminus\zeta(B)$, $\zeta(B)$ is π-free, and $x^m \in \zeta(B)$. Put $C = \langle x\rangle + \zeta(B)$. By Theorem 3.5 of [8], $|C*C|$ is a π-number. On the other hand, $C*C \leq \zeta(B)$, and so $C*C=0$. Thus, $x^m = mx$.

Let b be any element of B. Then $m[x,b]_+ = [mx,b]_+ = 0$, so $[x,b]_+=0$. Similarly, $[x,b]_\cdot = x*b = 0$. Therefore x belongs to $\zeta(B)$, the final contradiction. □

Corollary 4.15. *Let B be a brace. If $\zeta(B)$ is torsion-free, then $\zeta_{\alpha+1}(B)/\zeta_\alpha(B)$ is torsion-free for every ordinal α.*

Conversely, if we have information on the exponent of $\zeta(B)$, then we can obtain information on the exponent of the factors of the upper central series.

Theorem 4.16. *Let B be a brace. If $\zeta(B)$ has exponent n, then $\zeta_{i+1}(B)/\zeta_i(B)$ has exponent dividing n^{2^i} for each positive integer i.*

Proof. It is enough to show that $\zeta_2(B)/\zeta(B)$ has exponent dividing n^2. Let $b \in \zeta_2(B)$ and $a \in B$. Then $b * a \in \zeta(B)$, so $b^n * a = n(b*a) = 0$ and $[a, b^n]_\cdot = [a,b]_\cdot^n = 0$. Thus, if we put $c = nb^n = b^{n^2}$, then $c \in \mathrm{Ker}(\lambda) \cap Z(B,\cdot)$. But also $[a, nb^n]_+ = n[a,b^n]_+ = 0$ and so $c \in \zeta(B)$. □

Corollary 4.17. *Let B be a brace such that $\zeta(B)$ has exponent n. If B is centrally nilpotent of class c, then B has exponent at most n^{2^c-1}.*

4.2. ***The index of a subbrace.*** The following definition provides us with an invaluable tool in studying the “index” of a subbrace.

Definition 4.18. Let C be a subbrace of the brace B. We say that C is *serial* in B if there is a chain of subbraces $\mathcal{C}$ connecting C to B such that if $D < E$ are consecutive elements of $\mathcal{C}$, then $D \trianglelefteq E$ — as in the case of c-series, we usually assume that these chains of subbraces are *complete*, meaning that they contain arbitrary unions and intersections of their members.

Now, C is *ascendant* (resp. *descendant*) if $\mathcal{C}$ can be well ordered (resp. inversely well ordered) with respect to the inclusion and its order type is λ (resp. the inverse of λ) for some ordinal number λ. If C is ascendant, then $\mathcal{C}$ can be written as

$$C = C_0 \trianglelefteq C_1 \trianglelefteq \cdots C_\alpha \trianglelefteq C_{\alpha+1} \trianglelefteq \cdots C_\lambda = B, \qquad (\triangle)$$

where $\alpha < \lambda$ are ordinal numbers; while, if C is descendant, then $\mathcal{C}$ takes the form

$$C = C_\lambda \cdots \trianglelefteq C_{\alpha+1} \trianglelefteq C_\alpha \cdots \trianglelefteq C_1 \trianglelefteq C_0 = B, \qquad (\square)$$

where $\alpha < \lambda$ are ordinal numbers. If $\mathcal{I}$ is finite, we say that C is *subideal*.

If C is ascendant (resp. descendant) in B, then the smallest ordinal number λ for which there is a chain of subbraces of type ($\triangle$) (resp. of type ($\square$)) is the *ascendant length* (resp. *descendant length*) of C in B. In case C is subideal, the ascendant length of C in B is finite and is also called the *subideal defect* of C in B.

Let C be a subbrace of a brace B. Put $C^{B,0} := B$ and recursively define $C^{B,\alpha+1} = C^{C^{B,\alpha}}$ for every ordinal α, and $C^{B,\lambda} = \bigcap_{\alpha<\lambda} C^{B,\alpha}$ for every limit ordinal λ. The family $\{C^{B,\alpha}\}_{\alpha\in\mathrm{Ord}}$ is the *ideal closure series* of C in B. It is easy to show that C is descendant (resp. subideal) in B if and only if $C = C^{B,\mu}$ for some ordinal μ (resp. for some finite ordinal μ). If C is descendant (resp. subideal), then the descendant length (resp. the subideal defect) of C is the smallest ordinal number λ for which $C = C^{B,\lambda}$.

Lemma 4.19. *Let B be a brace.*

- *Every subideal of B is ascendant, descendant and serial. Moreover, ascendant (resp. descendant) subbraces are serial.*
- *If C is subideal (resp. ascendant, descendant, serial) in B, and $D \leq B$, then $C \cap D$ is subideal (resp. ascendant, descendant, serial) in D.*
- *If C is subideal (resp. ascendant), then CI/I is subideal (resp. ascendant) in B/I for every ideal I of B.*
- *If C is subideal in B of defect n, then C is subideal in C^B of defect $n-1$.*

Lemma 4.20. *Let B be a brace.*

(1) *If B is hypercentral, then every subbrace C of B is ascendant.*

(2) *If B is centrally nilpotent, then every subbrace C of B is subideal.*

(3) *If B is locally centrally nilpotent, then every subbrace C of B is serial.*

Proof. (1)–(2) We only prove (1). Let λ be the hypercentral length of B. Since C is an ideal of $C + \zeta(B)$, we see that

$$C \trianglelefteq C + \zeta(B) \trianglelefteq \cdots \trianglelefteq C + \zeta_\alpha(B) \trianglelefteq C + \zeta_{\alpha+1}(B) \trianglelefteq \cdots \trianglelefteq C + \zeta_\lambda(B) = B$$

is an ascending chain of subbraces of B connecting C to B.

(3) Zorn's lemma implies that there is a maximal chain of subbraces between C and B. By Theorem 4.5(2), if $D < E$ are consecutive terms of this chain, then D is an ideal of E. Therefore C is serial in B. □

Remark 4.21. Let B be a brace and let C be a (strong) left ideal of B. The proof of Lemma 4.20 can actually be employed to prove that if we have an ascending c-series of B, then there is an ascending chain of (strong) left ideals connecting C to B.

Definition 4.22. Let B be a brace. A subbrace C of B is said to have *finite index* in B if both $n_+ = |(B,+):(C,+)|$ and $n_\cdot = |(B,\cdot):(C,\cdot)|$ are finite; if $n_+ = n_\cdot = n$, we define the *index* $|B:C|$ of C in B as n. If C does not have finite index, we say that C has *infinite index*.

Lemma 4.23. *Let B be a brace, $C, D \leq B$ and $I \trianglelefteq B$.*

(1) *If C and D have finite index in B, then $C \cap D$ has finite index in B.*

(2) *Suppose $C \leq D$. If C has finite index in D, and D has finite index in B, then C has finite index in B. Moreover, if $|D:C|$ and $|B:D|$ exist, then also $|B:C| = |B:D| \cdot |D:C|$ exists.*

(3) *If I has finite index, then $|B:I|$ exists and is equal to $|B/I|$.*

Lemma 4.24. *Let C be a serial subbrace of the brace B. The following conditions are equivalent*:

(1) $|(B,+):(C,+)| < \infty$.

(2) $|(B,\cdot):(C,\cdot)| < \infty$.

(3) *C has finite index in B.*

(4) *$|B:C|$ exists.*

In particular, if any of these equivalent statements hold, then all the indices are equal.

Proof. Clearly, (4) $\Longrightarrow$ (1), (2) and (3). Assume (1). Since C is serial in B, there is a chain $\mathcal{C}$ of subbraces connecting C to B, and in which $E \trianglelefteq F$ whenever $E \lessdot F$. Looking at the corresponding additive parts of the members of $\mathcal{C}$, we see that $\mathcal{C}$ is actually finite, so C is subideal in B. We prove the result by induction on the subideal defect n of C in B. If $n \leq 1$, then C is an ideal of B such that $(B,+)/(C,+)$ is finite, so B/C is finite and we are done. Assume $n > 1$ and let $D = C^B$. Then the subideal defect of C in D is strictly less than n and so induction yields that $|D:C|$ exists. Since $|B:D|$ trivially exists, we have that $|B:C|$ exists by Lemma 4.23. Thus, (4) is proved. Similarly, we can prove that (2) implies (4), and that (3) implies (4). □

A combination of Lemmas 4.24 and 4.20 shows that every finite-index subbrace of a locally centrally nilpotent brace has a well-defined index. Our next result is a considerable extension of this fact.

Theorem 4.25. *Let B be a brace having an ascendant chain of ideals*

$$0 = I_0 \leq I_1 \leq \cdots I_\alpha \leq I_{\alpha+1} \leq \cdots I_\lambda = B$$

such that $I_{\beta+1}/I_\beta$ is either finite or locally centrally nilpotent for all ordinal numbers $\beta < \lambda$. If C is a subbrace of B, then the following are equivalent:

(1) $|(B,+):(C,+)| < \infty$.

(2) $|(B,\cdot):(C,\cdot)| < \infty$.

(3) *C has finite index in B.*

(4) *$|B:C|$ exists.*

Proof. We prove the result by induction on λ. To this aim it is sufficient to show that (1) implies (4).

If $\lambda \leq 1$, then B is either finite or locally centrally nilpotent. The former case is obvious, while the latter is a consequence of Lemmas 4.20(3) and 4.24. Assume $\lambda > 1$.

Suppose λ is successor. Since $(C,+)$ has finite index in $(B,+)$, it follows that $(C\cap I_{\lambda-1},+)$ has finite index n in $(I_{\lambda-1},+)$. By induction,

$$n=|(I_{\lambda-1},\cdot):(C\cap I_{\lambda-1},\cdot)|=|(CI_{\lambda-1},\cdot):(C,\cdot)|=|(C+I_{\lambda-1},+):(C,+)|.$$

Thus the index $|C+I_{\lambda-1}:C|$ exists. Since also the index $|B:C+I_{\lambda-1}|$ exists, it follows that the index $|B:C|$ exists.

Now, assume λ is limit. Let F_+ be a transversal for $(C,+)$ in $(B,+)$; in particular, F is finite. Also, let $F_\cdot$ be a transversal for $(C,\cdot)$ in $(B,\cdot)$. Suppose $|F_\cdot|>|F_+|$, and let $E_\cdot$ be a finite subset of $F_\cdot$ such that $|E_\cdot|>|F_+|$. Then there is an ordinal number $\mu<\lambda$ such that $F_+\cup E_\cdot\subseteq I_\mu$. By induction,

$$|(B,+):(C,+)|=|(I_\mu,+):(C\cap I_\mu,+)|=|(I_\mu,\cdot):(C\cap I_\mu,\cdot)|,$$

a contradiction. Thus $|F_\cdot|\le|F_+|$. By a symmetric argument, $|F_+|\le|F_\cdot|$ and hence the index $|B:C|$ exists. □

The range of applicability of Theorem 4.25 is not restricted to local centrally nilpotent brace. It follows in fact from Theorems 3.14 of [8] that Theorem 4.25 applies even to any *good* brace with *property* (S) (see [8] for the definitions).

We end this discussion by showing that subbraces of finite index can sometimes be employed to prove the existence of large proper ideals.

Theorem 4.26. *Let B be a brace such that $B/\zeta_2(B)$ is finite. If C is any finite-index subbrace of B, then B/C_B is finite.*

Proof. Without loss of generality we may assume $C_B=0$; thus, in particular, $C\cap\zeta(B)=0$ and $\zeta(B)$ is finite. Moreover, we may replace C by $C\cap\zeta_2(B)$, assuming $C\le\zeta_2(B)$. Then $C+\zeta(B)$ is an ideal of B.

Since $C\simeq C+\zeta(B)/\zeta(B)$, we have that C is an abelian brace. Let $n=|\zeta(B)|$. Then Theorem 4.16 shows that

$$C^{n^2,\cdot}\le\zeta(B)\cap C=0,$$

so C is periodic. Thus, as a group, C can be described as a direct product of infinitely many cyclic subgroups $\langle b_i\rangle$, $i\in I$, of order dividing n.

Let F be a finitely generated subbrace of B such that $C+\zeta(B)+F=B$. Since B is periodic, it is locally finite by Lemma 3.1 of [20], which implies that F is finite.

For every $b\in F$, $b_i^{b,+}=b_i+u_{b,i,+}$ for some $u_{b,i,+}\in\zeta(B)$. On the other hand, $\zeta(B)$ is finite, so there is an infinite subset J_1 of I such that $b_i^{b,+}=b_i+u_{+,b}$ for all $i\in J_1$, and for a fixed $u_{+,b}\in\zeta(B)$. Repeating this argument for all $b\in F$, we may assume $b_i^{b,+}=b_i+u_{+,b}$ for some $u_{+,b}\in\zeta(B)$ and for all $i\in J_1$, $b\in F$. Similarly, there is an infinite subset J_2 of J_1 such that $b_i^{b,\cdot}=b_i+u_{\cdot,b}$ for some $u_{\cdot,b}\in\zeta(B)$ and for all $i\in J_2$, $b\in F$. Finally, there is an infinite subset J_3 of J_2 such that $b_i*b=b_j*b$ for all $i,j\in J_3$ and $b\in F$.

Now, for each $i, j \in J_3$ with $i \neq j$, we have that $d_i = b_i - b_j = b_i \cdot b_j^{-1}$ is additively and multiplicatively centralised by F, and that $d_i * F = 0$. Since $B = C + \zeta(B) + F$, it follows that $d_i \in \zeta(B)$ for all $i \in J_3$, a contradiction. □

5. Central nilpotency for ideals

A celebrated result of Fitting states that a product of nilpotent normal subgroups of a group is nilpotent. Example B in Section 6 shows that the product of two centrally nilpotent ideals is not centrally nilpotent in general. In this section, we define a nilpotency concept for ideals that allows us to define a suitable Fitting ideal. It turns out that, for such a Fitting ideal, it is possible to generalise remarkable results of group theory concerned with the Fitting subgroup (see Theorems 5.6, 5.7, and 5.10).

Let B be a brace. We start by defining B-centrally nilpotent braces. Let I be an ideal of B. We can define the *lower central series of I with respect to B*, or simply the *lower B-central series of I*, as follows: take $\Gamma_1(I)^B = I$ and $\Gamma_{n+1}(I)^B = [\Gamma_n(I), I]^B$, for every $n \geq 1$. Therefore,

$$I = \Gamma_1(I)^B \geq \Gamma_2(I)^B \geq \cdots \geq \Gamma_n(I)^B \geq \cdots$$

is a descending chain of ideals of B with $\Gamma_n(I)^B / \Gamma_{n+1}(I)^B \leq \zeta(I/\Gamma_{n+1}(I)^B)$ for every $n \in \mathbb{N}$. Similarly, we may define the *upper central series of I with respect to B* (or simply the *upper B-central series of I*), as follows: take $\zeta_0^B(I) = 0$ and let $\zeta_{n+1}(I)^B$ satisfy $\zeta_{n+1}(I)^B/\zeta_n(I)^B = \zeta(I/\zeta_n(I)^B)_{B/\zeta_n(I)^B}$. Then $\zeta_0(I)^B \leq \zeta_1(I)^B \leq \cdots \leq \zeta_n(I)^B \leq \cdots$ is an ascending chain of ideals of B.

Definition 5.1. An ideal I of a brace B is defined to be *centrally nilpotent with respect of B*, or simply a *B-centrally nilpotent ideal*, if there exists $n \in \mathbb{N}$ such that $\Gamma_{n+1}(I)^B = 0$, or, equivalently, $\zeta_n(I)^B = 0$. For practical purposes, we often use the following equivalent definition: I is B-centrally nilpotent if there exists a chain $0 = J_0 \leq J_1 \leq \cdots \leq J_n = I$ of ideals of I such that $J_i/J_{i-1} \leq \zeta(I/J_{i-1})$, for every $1 \leq i \leq n$.

To simplify notation, if J is an ideal of B contained in I and such that I/J is B/I-centrally nilpotent, we just say that I/J is *centrally nilpotent with respect to B*, or *B-centrally nilpotent*. If I/J is B-centrally nilpotent, then the smallest n such that $\Gamma_{n+1}(I/J)^{B/J} = 0$ is referred to as its *class*.

Clearly, a brace B is centrally nilpotent if and only if it is B-centrally nilpotent; in this trivial case, the *upper central series* (resp. *lower central series*) and the *upper B-central series* (resp. *lower B-central series*) coincide. Moreover, if I is a B-centrally nilpotent ideal of a brace B, and J is any ideal of B, then $(I+J)/J$ is B-centrally nilpotent (of class less than or equal to that of I), and $I \cap C$ is C-centrally nilpotent for any subbrace C of B (also in this case the class of $I \cap C$ is less than or equal to that of I). Our next result shows that an analogue of Fitting's theorem holds for B-centrally nilpotent ideals, but first, we need the following property of commutators of ideals in braces.

Lemma 5.2. *Let B be a brace and let I, J, K be ideals of B. Then*

$$[I, JK]_B = [I, J+K]_B = [I, J]_B + [I, K]_B = [I, J]_B [I, K]_B.$$

Proof. We prove the equality for the sum. Observe that only one inclusion is in doubt so, by Theorem 3.3, it suffices to show that

$$[I, J+K]_+,\ I * (J+K),\ (J+K) * I \subseteq [I, J]_B + [I, K]_B.$$

Since $(I, +)$, $(J, +)$ and $(K, +)$ are normal subgroups of $(B, +)$, we have that $[I, J+K]_+ = [I, J]_+ + [I, K]_+$ is contained in $[I, J]_B + [I, K]_B$. Then we have that for every $i \in I$, $j \in J$ and $k \in K$,

$$i * (j+k) = i * j + j + i * k - j \in [I, J]_B + [I, K]_B.$$

Finally, note that $(J+K) * I = (JK) * I$, so

$$(jk) * i = j * (k * i) + k * i + j * i \in J * I + K * I + J * I \subseteq [I, J]_B + [I, K]_B$$

for every $j \in J$, $k \in K$, $i \in I$. □

We also need this notation in the proof: if $I_1, \dots, I_n$ are ideals of a brace B, we put $[I_1]^B = I_1$, and then, recursively, $[I_1, \dots, I_k]^B := [[I_1, \dots, I_{k-1}]^B, I_k]^B$ for every $2 \le k \le n$.

Theorem 5.3 (see also Theorem 5.13). *Let I, J be B-centrally nilpotent ideals of a brace B. If I and J have classes n_0 and m_0, respectively, then $I + J$ is B-centrally nilpotent of class at most $n_0 + m_0$.*

Proof. Set $K = I + J$. First, we show by induction that for every $n \in \mathbb{N}$, $\Gamma_n(K)^B$ is the sum of all commutators of the form $[L_1, \dots, L_n]^B$ with either $L_i = I$ or $L_i = J$, for every $1 \le i \le n$. The base case is clear. Assume the assertion is true for some $1 \le n \in \mathbb{N}$. Then,

$$\Gamma_{n+1}(K)^B = [\Gamma_n(K), K]^B = [\Gamma_n(K), I]^B + [\Gamma_n(K), J]^B$$

by Lemma 5.2. Using iteratively Lemma 5.2, the assertion also holds for $n+1$.

In particular, for $r = n_0 + m_0 + 1$, $\Gamma_r(K)^B$ is the sum of all commutators of the form $[L_1, \dots, L_r]^B$, where either I occurs $n_0 + 1$ times or J occurs $m_0 + 1$ times. Thus, it follows that each $[L_1, \dots, L_r]^B$ is contained in either $\Gamma_{n_0+1}(I)^B = 0$ or $\Gamma_{m_0+1}(J)^B = 0$. Hence, $\Gamma_r(K)^B = 0$ and so K is B-centrally nilpotent. □

Let B be a brace. The *Fitting ideal* $\mathrm{Fit}(B)$ of B is the ideal generated by all B-centrally nilpotent ideals of B. It follows from Theorem 5.3 that in a finite brace B, $\mathrm{Fit}(B)$ is B-centrally nilpotent. More generally, the same result shows that this is true for a broader class of braces.

Corollary 5.4. *Let B be a noetherian brace. Then* $\mathrm{Fit}(B)$ *is a B-centrally nilpotent ideal.*

Now, in order to obtain a characterisation of the Fitting ideal in terms of chief factors, we need the following definition (recall Proposition 4.2). In [5, Proposition 4.19], the *centraliser of an ideal I* of a brace B, $\mathrm{C}_B(I)$, is defined as the largest ideal that *centralises* I, i.e., $[\mathrm{C}_B(I), I]^B = 0$.

Moreover, if I/J is a chief factor of B, we define the *centraliser in B of I/J* as the ideal $\mathrm{C}_B(I/J)$ of B satisfying $\mathrm{C}_{B/J}(I/J) = \mathrm{C}_B(I/J)/J$. Equivalently, $\mathrm{C}_B(I/J)$ is the largest ideal of B such that $[\mathrm{C}_B(I/J), I]^B \leq J$.

Lemma 5.5. *Let I be a B-centrally nilpotent ideal of a brace B. If J is a minimal ideal of B, then $[J, I]^B = 0$.*

Proof. Since J is a minimal ideal of B, either $[J, I]^B = 0$ or $[J, I]^B = J$. However, in the latter case we contradict Definition 5.1. □

Theorem 5.6. *Let B be a brace with a finite chief series $\mathcal{S}$. Then* $\mathrm{Fit}(B)$ *is the intersection of the centralisers in B of the factors of $\mathcal{S}$.*

Proof. Let $0 = I_0 \leq I_1 \leq \cdots \leq I_n = B$ be a finite chief series of B and set $C := \bigcap \{\mathrm{C}_B(I_k/I_{k-1}) : 1 \leq k \leq n\}$. Then C is an ideal of B and $0 = C \cap I_0 \leq C \cap I_1 \leq \cdots \leq C \cap I_n = C$ is a finite chain of ideals of B such that $(I_i \cap C)/(I_{i-1} \cap C) \leq \zeta(C/I_{i-1} \cap C)$ for all $1 \leq i \leq n$. Thus, C is B-centrally nilpotent, and hence $C \leq \mathrm{Fit}(B)$.

Conversely, B is noetherian (see [3]) and so $F := \mathrm{Fit}(B)$ is B-centrally nilpotent by Corollary 5.4. If I/J is any chief factor of B, then I/J is centralised by $(F + J)/J$ by Lemma 5.5. □

Theorem 5.7. *Let B be a brace and put $F = \mathrm{Fit}(B)$. Then, $(\mathrm{C}_B(F) + F)/F$ does not contain any nonzero soluble ideal with respect of B/F. In particular, if B is a soluble brace, then $\mathrm{C}_B(F) = \zeta(F)$.*

Proof. Assume that $(\mathrm{C}_B(F) + F)/F$ contains a nonzero soluble ideal I/F with respect to B/F. Then it also contains a nonzero ideal J/F which is an abelian brace. Let $C = \mathrm{C}_B(F)$. Then $J \cap C \cap F \leq \zeta(J \cap C)$ and $(J \cap C)/(J \cap C \cap F)$ is an abelian brace. Thus, $J \cap C$ is B-centrally nilpotent and so $J \cap C \leq F$. Finally, $J = J \cap (C + F) = (J \cap C) + F = F$, a contradiction. If B is a soluble brace, then B/F is soluble, and therefore $(\mathrm{C}_B(F) + F)/F$ must be zero. Hence, $\mathrm{C}_B(F) \leq F$ which yields $\mathrm{C}_B(F) = \zeta(F)$. □

It is well known that the Fitting subgroup of a finite group modulo its Frattini subgroup is the product of all its abelian minimal normal subgroups. The following Frattini-like ideal leads to a brace-theoretic analogue of this result. Let B be a finite brace. We define the *Frattini ideal* of F as

$$\mathrm{Frat}(B) := \bigcap\{\, L \mid L \text{ is a maximal left ideal of } B \,\} \cap \mathrm{Fit}(B).$$

Clearly, the Frattini ideal of a finite brace is a left ideal, but the following result shows that it is actually an ideal (hence providing a justification for its name).

Lemma 5.8. *Let B be a finite brace. If L is any maximal left ideal of B, then $L \cap \mathrm{Fit}(B)$ is an ideal of B.*

Proof. Let $F = \mathrm{Fit}(B)$ and assume $F \not\leq L$, so in particular $B = FL$.

Since $F \cap L \trianglelefteq (L,+)$, it follows that $L \leq \mathrm{N}_{(B,+)}(F \cap L)$. Moreover, $F \cap L$ is properly contained in $\mathrm{N}_{(F,+)}(F \cap L)$, as $(F,+)$ is nilpotent. Therefore, L is properly contained in $\mathrm{N}_{(B,+)}(F \cap L)$.

Because $F \cap L$ is λ-invariant, we have $\mathrm{N}_{(B,+)}(F \cap L)$ is also λ-invariant. Thus, $\mathrm{N}_{(B,+)}(F \cap L) = B$, and so $F \cap L$ is a strong left ideal. Then, by Lemma 4.20 and Remark 4.21, we can find a strong left ideal T of B contained in F and such that $F \cap L$ is a proper ideal of T. Therefore, $F \cap L$ is an ideal of $T + L = TL$, with $T + L$ being a left ideal of B. Hence, $TL = B$ by the maximality of L and the result follows. □

Corollary 5.9. *Let B be a finite brace. Then* $\mathrm{Frat}(B)$ *is an ideal of B.*

Theorem 5.10. *Let B be a finite brace with* $\mathrm{Frat}(B) = 0$. *Then* $\mathrm{Fit}(B)$ *is the product of all the abelian minimal ideals of B.*

Proof. Let $F = \mathrm{Fit}(B)$. We claim that $\partial_B(F) = 0$. Indeed, suppose that L is a maximal left ideal of B such that $\partial_B(F)$ is not included in L. Then, $B = \partial_B(F)L$, so $F = F \cap \partial_B(F)L = (F \cap L)\partial_B(F)$. By Lemma 5.8, $F \cap L$ is an ideal of B. Since $F/(F \cap L)$ is nonzero and B-centrally nilpotent, we have that $\partial_B(F)(F \cap L)/(F \cap L) \leq \partial_{B/(F\cap L)}(F/(F \cap L)) < F/(F \cap L)$, a contradiction. Thus, F is an abelian brace.

Let N be the product of all abelian minimal ideals of B. Then $N \leq F$. For the other inclusion, take S a minimal subbrace subject to $B = SN$. Consider $X = \bigcap\{L \mid L \text{ is a maximal left ideal of } S\}$, a left ideal of S. If $S \cap N \not\leq X$, then there exists a maximal left ideal L of S such that $S \cap N$ is not included in L. Thus, $(S \cap N)L = S$ and then $B = SN = (S \cap N)LN = LN$, which contradicts the minimality of S. Therefore, $S \cap N \leq X$.

Now, $S \cap N$ is an ideal of B, as N is abelian and $B = SN$ (see [2, Lemma 27]). Assume that there exists a maximal left ideal L of B such that $S \cap N \not\leq L$. Thus, $B = (S \cap N)L$ and then, $S = S \cap (S \cap N)L = (S \cap L)(S \cap N)$. Take L' a left ideal of S maximal subject to $S \cap L \leq L'$ and $S \cap N$ not included in L'. Then, L' is indeed a maximal left ideal of S, because the existence of a left ideal L'' of S such that $L' < L'' \leq S$ yields $S = (S \cap L)(S \cap N) \leq L''$. Therefore, $S \cap N \leq X \leq L'$, a contradiction. Thus, $S \cap N = 0$.

Finally, $S \cap F$ is an ideal of B, as F is abelian and $SF = B$ (see again [2, Lemma 27]), and consequently $S \cap F$ contains an abelian minimal ideal of B, contradicting $S \cap N = 0$. □

In a centrally nilpotent brace, the Frattini ideal behaves pretty well. For example, it is possible to prove that it coincides with the set of nongenerators. Let B be a brace. We say that an element $b \in B$ is a *nongenerator* of B if for all $S \leq B$ such that $B = \langle b, S\rangle$, we have $B = S$.

Theorem 5.11. *Let B be a centrally nilpotent finite brace. Then* $\mathrm{Frat}(B)$ *coincides with the set of all nongenerators of B.*

Proof. Since in a centrally nilpotent brace the maximal left ideals coincide with the maximal ideals and with the maximal subbraces, the usual group-theoretic proof adapts to prove the result. □

However, we note that there exists a brace B of order 6 in which $\mathrm{Fit}(B) = \mathrm{Frat}(B)$ is the only nonzero proper left ideal of B and has order 3. This shows that there is no possible analogue of these two well-known group-theoretic theorems concerning the Frattini subgroup of a group G:

(1) $G/\mathrm{Frat}(G)$ is nilpotent implies G is nilpotent.

(2) If p is a prime dividing $|G|$, then p divides $|G/\mathrm{Frat}(G)|$ too.

In the final part of this section we discuss further aspects of B-centrally nilpotence and hypercentral/locally nilpotent concepts for (sub)ideals.

The definition of upper B-central series (and lower B-central series) for an ideal I of a brace B can be extended by using transfinite numbers (just how we did in Section 4), and this allows us to define B-hypercentral ideals of braces. However, for our purposes, the following equivalent definition is more convenient.

Definition 5.12. Let B be a brace. An ideal I of B is said to be *B-hypercentral* if there is an ascending chain of ideals of B

$$0 =: I_0 \leq I_1 \leq \cdots I_\alpha \leq I_{\alpha+1} \leq \cdots I_\lambda = I$$

such that $I_{\alpha+1}/I_\alpha \leq \zeta(B/I_\alpha)$ for all ordinals $\alpha < \lambda$. The smallest λ for which such a chain exists is the *length* of I.

Clearly, if B is a brace, then B is hypercentral if and only if B is B-hypercentral, and every B-centrally nilpotent ideal is B-hypercentral. The following result generalises Theorem 5.3.

Theorem 5.13. *Let B be a brace.*

(1) *If C and D are ideals of B which are B-hypercentral of lengths α and β, respectively, then $C + D$ is a B-hypercentral ideal of length at most $\beta\alpha + \max\{\alpha, \beta\}$.*

(2) *Suppose C is subideal of defect n and centrally nilpotent of class c, and D is a B-centrally nilpotent ideal of class d. Then $C + D$ is centrally nilpotent of class at most $(c+n)d+c$.*

(3) *Suppose C is ascendant of length μ and hypercentral of length γ, and D is a B-hypercentral ideal of length δ. Then $C + D$ is hypercentral of length at most $(\gamma+\mu)\delta+\gamma$.*

Proof. (1) Let $E = C + D$. Then E is an ideal of B and to show that E is B-hypercentral, it suffices to prove that $\zeta(E)$ contains a nonzero ideal I of B. To this aim we may certainly assume that C and D are nonzero.

Suppose first $C \cap D = 0$. By hypothesis $\zeta(C)$ contains a nonzero ideal I of B. On the other hand, $\zeta(C) \leq \zeta(E)$ and so we are done.

Assume $C \cap D \neq 0$. Then $\zeta(C) \cap D$ contains a nonzero ideal I of B, and $I \cap \zeta(D)$ contains a nonzero ideal J of B. Thus $J \leq \zeta(C) \cap \zeta(D) \leq \zeta(E)$ and we are done. The bound on the hypercentral length can be easily deduced from the proof.

(2)–(3) We only prove (3), the proof of (2) being similar. Since D is B-hypercentral of length δ, there is an ascending chain of ideals of B

$$0 = D_0 < D_1 < \cdots D_\alpha < D_{\alpha+1} < \cdots D_\delta = D$$

such that $D_{\beta+1}/D_\beta \leq \zeta(D/D_\beta)$ for all ordinals $\beta < \delta$.

Let $E = C+D$. Since C is hypercentral of length γ, it follows that $C \cap D_1 \leq \zeta_\gamma(E)$. Thus, we may factor out $C \cap D_1$ and assume $C \cap D_1 = 0$. Let $F = \langle C, D_1\rangle = CD_1$. Now, since C is ascendant of length μ, there is an ascending chain $C = C_0 \trianglelefteq C_1 \trianglelefteq \cdots \trianglelefteq C_\alpha \trianglelefteq C_{\alpha+1} \trianglelefteq \cdots \trianglelefteq C_\mu = F$ connecting C to F. It is easy to see that $(C_{\beta+1} \cap D_1)/(C_\beta \cap D_1) \leq \zeta(E/(C_\beta \cap D_1))$ for all $\beta < \mu$. Therefore $D_1 \leq \zeta_\mu(E)$.

We factor D_1 out and we repeat the above argument. This shows that $D \leq \zeta_\rho(E)$, where $\rho = (\gamma+\mu)\delta$, so E is hypercentral of length at most $\rho+\gamma$. □

If B satisfies the maximal condition on ideals, then the product of all B-hypercentral ideals of a brace is clearly B-hypercentral. Moreover, the idea of the proof of Theorem 5.6 yields that if B is a brace having an ascending chief series $\mathcal{S}$ (in particular, if B satisfies the minimal condition on ideals), then the maximal ideal centralising all factors of $\mathcal{S}$ is precisely the unique maximal B-hypercentral ideal of B.

The following result shows that in the universe of locally centrally nilpotent braces, the class of B-hypercentral braces is closed with respect to forming extensions by finitely generated hypercentral braces.

Theorem 5.14. *Let N be an ideal of the locally centrally nilpotent brace B. If N is B-hypercentral and B/N is finitely generated, then B is hypercentral.*

Proof. Let S be a finite subset of B that generates B modulo N, and let $Z = \bar{\zeta}(B)$ be the hypercentre of B. Assume by contradiction $Z \neq B$. Now, B/N is centrally nilpotent, so $N \not\leq Z$, and hence the ideal $K := Z \cap N$ of B is strictly contained in N. Since N/K is B/K-hypercentral, there is a nonzero ideal A/K of B/K such that $A/K \leq \zeta(N/K)$. Let $a \in A \setminus K$ and $U = \langle a, S, K\rangle$; in particular, U/K is centrally nilpotent. Since $(A \cap U)/K$ is a nonzero ideal of U/K, we have that $V/K := \zeta(U/K) \cap ((A \cap U)/K) \neq 0$. Now, the fact that $V \leq A$ implies that $[V, N]_+$, $[V, N]_.$ and $V * N$ are all contained in K. Similarly, the fact that $V/K \leq \zeta(U/K)$ shows that $[V, T]_+$, $[V, T]_.$ and $V * T$ are all contained in K, where $T = \langle S\rangle$. Since $B = N + T = NT$, we easily see that $[V, B]_+$ and $[V, B]_.$ are contained in K. Moreover, if $u \in N$, $v \in T$, and $a \in V$, then $a * (u + v) = a * u + u + a * v - u \in K$. This shows that $V * B \leq K$ and proves that $V/K \leq \zeta(B/K)$. Since $K \leq Z$, it follows that $V \leq Z$, so $V \leq Z \cap N$, a contradiction. □

Also B-central nilpotency (resp. hypercentrality) can be locally detected, and our next result is in fact a generalisation of Theorem 4.4.

Theorem 5.15. *Let B be a brace and let $I \trianglelefteq B$.*

(1) *I is B-centrally nilpotent of class at most c if and only if $I \cap F$ is F-centrally nilpotent of class at most c for every finitely generated subbrace F of B.*

(2) *I is B-centrally nilpotent if and only if $I \cap C$ is C-centrally nilpotent for every countable subbrace C of B.*

(3) *I is B-hypercentral if and only if $I \cap C$ is C-hypercentral for every countable subbrace C of B.*

Proof. We only deal with the proof of (1), since (2) and (3) then follow in a similar fashion using ideas from Theorem 4.4.

For each $u, v \in B$, we write $u \circ v$ to denote one (but we do not know which one) of the operations $[u, v]_.$, $[u, v]_+$, $u * v$. Then (1) is a direct consequence of the fact that $\zeta_c(I)^B$ can be easily characterised as the set of all elements $b \in I$ such that $\big\langle\big(\cdots((b \circ b_1) \circ \cdots) \circ b_{c-i}\big)\big\rangle^B \leq \zeta_i(I)$ for all $i = 0, 1, \ldots, c-1$ and for all $b_1, \ldots, b_{c-i} \in B$. □

To provide a good definition of "locally B-nilpotent ideal" is not an obvious task. We now concern ourselves with a couple of possible definitions, mostly sketching proofs and results. The first idea that comes in mind is that of using central chains of ideals, just as we did for B-hypercentral and B-centrally nilpotent ideals. In fact, it follows from Theorem 4.5 (and Zorn's lemma) that in every chief series of a locally centrally nilpotent brace B, two consecutive ideals $K \leq H$ satisfy $H/K \leq \zeta(B/K)$.

Definition 5.16. Let B be a brace. An ideal I of B is said to be *ζ_B-nilpotent* if every quotient I/J of I by an ideal J of B admits a maximal chain $\mathcal{S}$ of ideals of B in which $H/K \leq \zeta(I/K)$ for all consecutive terms $K/J \leq H/J$ of the chain $\mathcal{S}$.

Using ideas from the proof of Theorem 5.13, it is not difficult to see that the product of arbitrarily many ζ_B-nilpotent ideals is ζ_B-nilpotent. Thus, any brace B has a unique maximal ζ_B-nilpotent ideal, we call it the *ζ_B-radical* of B: it turns out that the largest ideal of B centralising all quotients of a chief series of B is precisely the ζ_B-radical of B. The following result shows that an analogue of Theorem 5.7 is possible for the ζ_B-radical.

Theorem 5.17. *Let B be a brace admitting an ascending chain of ideals*

$$0 = B_0 \leq B_1 \leq \cdots B_\alpha \leq B_{\alpha+1} \leq \cdots B_\lambda = B$$

such that $B_{\beta+1}/B_\beta$ is a ζ_B-nilpotent ideal of B/B_β for all $\beta < \lambda$. Then $C_B(H) \leq H$, where H is the ζ_B-radical of B.

Proof. Suppose $C = C_B(H) \not\leq H$. Then $(C+H)/H$ contains a nonzero ζ_B-nilpotent ideal I/H of B/H. Since

$$I \cap C \cap H \leq \zeta(I \cap C),$$

it follows that $I \cap C$ is a ζ_B-nilpotent ideal of B. Thus,

$$I \cap C \leq H \quad \text{and} \quad I = I \cap (C+H) = (I+C) \cap H = H,$$

a contradiction. □

However, there is one reason for which this is not a convincingly good definition of "locally B-nilpotent ideal": a ζ_B-ideal could not be locally centrally nilpotent (there are examples even among groups) — although if B is locally finite, then a ζ_B-ideal is locally centrally nilpotent.

A more fruitful approach could deal with finitely generated subbraces and the way the ideal embeds into them. There are several ways in which this case may be achieved, but the most reasonable one seems to be the following. Let B be a brace. An ideal I of B is *locally B-nilpotent* if the following property holds: for every finitely generated subbrace F of B, the finitely generated subbraces of $I \cap F$ are contained in F-centrally nilpotent ideals of F.

Trivially, every locally B-nilpotent ideal is locally centrally nilpotent, so this solves the previous issue for ζ_B-nilpotency.

Theorem 5.18. *Let B be a brace. The sum of arbitrarily many locally B-nilpotent ideals of B is locally B-nilpotent.*

Proof. It is clearly enough to prove the statement for two locally B-nilpotent ideals I and J. Let F be a finitely generated subbrace of B. Choose a finitely generated subbrace E of $F \cap (I+J)$. In order to prove that E is contained in an F-centrally nilpotent ideal of F, we may assume $E = E_1 \cup E_2$, where $E_1 \subseteq I$ and $E_2 \subseteq J$, by suitably replacing F. Now, E_1 and E_2 are respectively contained in F-centrally nilpotent ideals I_1 and I_2 of F. Since $I_1 + I_2$ is F-centrally nilpotent, we are done. □

By Theorem 5.18, every brace admits a unique maximal locally B-nilpotent ideal; we call it the *Hirsch–Plotkin radical* of B. Using ideas from the proof of Theorem 4.5(1), we see that every locally B-nilpotent ideal is actually ζ_B-nilpotent. Finally, we note that for a locally finite brace B, the concepts of locally B-nilpotent ideal and ζ_B-ideal coincide.

6. Worked examples

In this section we describe the main examples of the paper. These examples are all constructed in a similar fashion (see [1]), which we now explain, and all the computations can be done with the computer algebra system GAP and the functions of its package YangBaxter [21]. The first of them shows that the idealiser of a subbrace (as introduced in [16]) does not exist in general (even for braces of abelian type).

Example A. Let $B = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \simeq C_4 \times C_4 \times C_2$, whose operation is additively written, and

$$\begin{aligned} C = \big\langle m_1, m_2, m_3, m_4, m_5 \mid m_1^2 = m_4,\ m_2^2 = 1,\ m_3^2 = m_4^2 = m_5,\ m_5^2 = 1, \\ m_1m_2m_1^{-1} = m_3m_2,\ m_1m_3m_1^{-1} = m_2m_3m_2^{-1} = m_5m_3, \\ m_1m_4m_1^{-1} = m_2m_4m_2^{-1} = m_3m_4m_3^{-1} = m_4, \\ m_1m_5m_1^{-1} = m_2m_5m_2^{-1} = m_3m_5m_3^{-1} = m_4m_5m_4^{-1} = m_5\big\rangle. \end{aligned}$$

We note that B and C are groups of order 32. Since $m_3 = m_1m_2m_1^{-1}m_2^{-1}$, $m_4 = m_1^2$, $m_5 = m_1m_3m_1^{-1}m_3^{-1} = m_3^2$, we have that $\langle m_3, m_4, m_5\rangle$ is contained in $\operatorname{Frat}(C)$ (in fact, they coincide) and so $C = \langle m_1, m_2\rangle$. Then C acts on B by means of an action λ defined by

$$\begin{aligned} &\lambda_{m_1}(a_1) = 3a_1 + a_3, && \lambda_{m_2}(a_1) = 3a_1, \\ &\lambda_{m_1}(a_2) = a_1 + a_2 + a_3, && \lambda_{m_2}(a_2) = a_1 + a_2 + a_3, \\ &\lambda_{m_1}(a_3) = a_3, && \lambda_{m_2}(a_3) = a_3. \end{aligned}$$

We note that

$$\begin{aligned} &\lambda_{m_3}(a_1) = \lambda_{m_1m_2m_1^{-1}m_2^{-1}}(a_1) = a_1, && \lambda_{m_4}(a_1) = \lambda_{m_1^2}(a_1) = a_1, \\ &\lambda_{m_3}(a_2) = \lambda_{m_1m_2m_1^{-1}m_2^{-1}}(a_2) = a_2 + a_3, && \lambda_{m_4}(a_2) = \lambda_{m_1^2}(a_2) = a_2 + a_3, \\ &\lambda_{m_3}(a_3) = \lambda_{m_1m_2m_1^{-1}m_2^{-1}}(a_3) = a_3, && \lambda_{m_4}(a_3) = \lambda_{m_1^2}(a_3) = a_3, \end{aligned}$$

and λ_{m_5} is the identity map on B.

We can consider the semidirect product $G = [B]C$ with respect to this action. Then G turns out to be a trifactorised group as it possesses a subgroup $D = \langle a_1a_2^3m_1, a_1m_2\rangle$ such that $D \cap C = D \cap B = 1$, $DC = BD = G$. Thus, the bijective

c	$\delta(c)$	c	$\delta(c)$
1	0	m_1	a_1+3a_2
m_5	$2a_1$	m_1m_5	$3a_1+3a_2$
m_4	$3a_1+2a_2+a_3$	m_1m_4	a_2
m_4m_5	$a_1+2a_2+a_3$	$m_1m_4m_5$	$2a_1+a_2$
m_3	$3a_1+a_3$	m_1m_3	$2a_1+3a_2$
m_3m_5	a_1+a_3	$m_1m_3m_5$	$3a_2$
m_3m_4	$2a_1+2a_2$	$m_1m_3m_4$	a_1+a_2
$m_3m_4m_5$	$2a_2$	$m_1m_3m_4m_5$	$3a_1+a_2$
m_2	a_1	m_1m_2	$3a_2+a_3$
m_2m_5	$3a_1$	$m_1m_2m_5$	$2a_1+3a_2+a_3$
m_2m_4	$2a_2+a_3$	$m_1m_2m_4$	$3a_1+a_2+a_3$
$m_2m_4m_5$	$2a_1+2a_2+a_3$	$m_1m_2m_4m_5$	$a_1+a_2+a_3$
m_2m_3	$2a_1+a_3$	$m_1m_2m_3$	$3a_1+3a_2+a_3$
$m_2m_3m_5$	a_3	$m_1m_2m_3m_5$	$a_1+3a_2+a_3$
$m_2m_3m_4$	a_1+2a_2	$m_1m_2m_3m_4$	$2a_1+a_2+a_3$
$m_2m_3m_4m_5$	$3a_1+2a_2$	$m_1m_2m_3m_4m_5$	a_2+a_3

Table 2. Associated bijective 1-cocycle.

1-cocycle $\delta : C \to B$ with respect to λ given by Table 2 yields a brace of abelian type $(B,+,\cdot)$ of order 32. This brace corresponds to `SmallBrace(32, 14649)` in the YangBaxter library for GAP (this method of construction of braces will be used in the subsequent examples without a further note).

We have that $\langle 2a_1+2a_2\rangle_+ \leq (B,+)$, corresponding to $\langle m_3m_4\rangle \leq C$ (through δ), defines a subbrace S of B of order 2. We also have that $\langle 2a_1, 2a_2, a_1+a_2+a_3\rangle_+ \leq (B,+)$, corresponding to $\langle m_5, m_3m_4m_5, m_1m_2m_4m_5\rangle \leq C$, defines a subbrace T of B of order 8. Furthermore $\langle 2a_1, 2a_2, a_1+a_3\rangle_+ \leq (B,+)$, corresponding to $\langle m_5, m_3m_4m_5, m_3m_5\rangle \leq C$, defines another subbrace U of B of order 8.

We note that S is not a left ideal of B, because

$$\lambda_{m_1}(2a_1+2a_2) = 2(3a_1+a_3)+2(a_1+a_2+a_3) = 2a_2 \notin S.$$

On the other hand, S is a left ideal of T, since $\lambda_{m_5}(2a_1+2a_2) = \lambda_{m_3m_4m_5}(2a_1+2a_2) = \lambda_{m_1m_2m_4m_5}(2a_1+2a_2) = 2a_1+2a_2$. Furthermore, $\langle m_3m_4\rangle$ is a normal subgroup of $\langle m_5, m_3m_4m_5, m_1m_2m_4m_5\rangle$. Therefore, S is an ideal of T. We also have that S is a left ideal of U, since $\lambda_{m_5}(2a_1+2a_2) = \lambda_{m_3m_4m_5}(2a_1+2a_2) = \lambda_{m_3m_5}(2a_1+2a_2) = 2a_1+2a_2$. Moreover, $\langle m_3m_4\rangle$ is a normal subgroup of $\langle m_5, m_3m_4m_5, m_3m_5\rangle$. Therefore, S is an ideal of U.

We prove now that the subbrace $D = \langle T, U\rangle$ of B generated by T and U is B. Let H be the additive group of D. Then $H \geq \langle 2a_1, a_2, a_1+a_3\rangle_+$. Thus, if $R = \delta^{-1}(H)$

x	$\delta(x)$	x	$\delta(x)$	x	$\delta(x)$		
1	0	$g_5g_3g_2$	$a+e$	g_1	$2a+c$	$g_3g_2g_1$	$3a+e$
g_2	$c+d+e$	$g_4g_2g_1$	$2a+d+e$	g_3	$3a+c$	$g_5g_2g_1$	$c+e$
g_4	c	$g_4g_3g_1$	$a+c+d$	g_5	d	$g_5g_3g_1$	$3a$
g_2g_1	$2a+c+d+e$	$g_5g_4g_1$	$2a+d$	g_3g_1	$3a+d$	$g_4g_3g_2$	$a+c+d+e$
g_4g_1	$2a$	$g_5g_4g_2$	$2a+e$	g_5g_1	$2a+c+d$	$g_5g_4g_3$	$a+d$
g_3g_2	$3a+d+e$	$g_4g_3g_2g_1$	$a+c+e$	g_4g_2	$d+e$	$g_5g_3g_2g_1$	$a+d+e$
g_5g_2	$2a+c+d$	$g_5g_4g_2g_1$	e	g_4g_3	a	$g_5g_4g_3g_1$	$a+c$
g_5g_3	$3a+c+d$	$g_5g_4g_3g_2$	$3a+c+e$	g_5g_4	$c+d$	$g_5g_4g_3g_2g_1$	$3a+c+d+e$

Table 3. Associated bijective 1-cocycle.

is the corresponding multiplicative group, then

$$\delta^{-1}(2a_1) = m_5 \in R, \quad \delta^{-1}(a_2) = m_1m_4 \in R, \quad \delta^{-1}(2a_2) = m_3m_4m_5 \in R,$$
$$\delta^{-1}(a_1+a_3) = m_3m_5 \in R, \quad \delta^{-1}(a_1+a_2+a_3) = m_1m_2m_4m_5 \in R,$$

which implies that $C = \langle m_1, m_2, m_3, m_4, m_5\rangle = R$. Thus, $H = (B,+)$ and $\langle T, U\rangle = B$.

Finally, suppose that S possesses an idealiser in B. Since it must contain every subbrace of B in which S is an ideal, it must contain T and U. It follows that the idealiser of S in B must be B, but S is not an ideal of B.

Our second example shows that there is no analogue of Fitting's theorem for central nilpotency, even for braces of abelian type.

Example B. Let $B = \langle a\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle \simeq C_4 \times C_2 \times C_2 \times C_2$, written additively. Let us consider the automorphisms g_1, g_2, g_3 of B given by $g_1(a) = 3a+d$, $g_1(c) = c$, $g_1(d) = d$, $g_1(e) = c+e$, $g_2(a) = a$, $g_2(c) = c$, $g_2(d) = 2a+d$, $g_2(e) = 2a+e$, $g_3(a) = a$, $g_3(c) = 2a+c$, $g_3(d) = d$, $g_3(e) = 2a+e$. If $g_4 = g_1g_2g_1^{-1}g_2^{-1}$ and $g_5 = g_1g_3g_1^{-1}g_3^{-1}$, then their action on $(B,+)$ is the following one: g_4 maps a to $3a$ and fixes c, d, e, while g_5 maps e to $2a+e$ and fixes a, c, d. We have that $C = \langle g_1, g_2, g_3\rangle = \langle g_1, g_2, g_3, g_4, g_5\rangle$ satisfies the following relations: $g_1^2 = 1$, $g_2^2 = 1$, $g_1g_2g_1^{-1} = g_4g_2$, $g_3^2 = 1$, $g_1g_3g_1^{-1} = g_5g_3$, $g_2g_3g_2^{-1} = g_3$, $g_4^2 = 1$, $g_1g_4g_1^{-1} = g_4$, $g_2g_4g_2^{-1} = g_4$, $g_3g_4g_3^{-1} = g_4$, $g_5^2 = 1$, $g_1g_5g_1^{-1} = g_5$, $g_2g_5g_2^{-1} = g_5$, $g_3g_5g_3^{-1} = g_5$, $g_4g_5g_4^{-1} = g_5$. It follows that C is a group of order 32. The bijective 1-cocycle $\delta : C \to B$ given in Table 3 yields a brace of abelian type on $(B,+,\cdot)$ of order 32. This brace corresponds to `SmallBrace(32, 23060)` in the YangBaxter library for GAP.

Since C has order 32, we have that $\operatorname{Ker}\lambda = 1$. In particular, we have that $\zeta(B) = 0$, and B is not centrally nilpotent.

Now, let us compute the ideals of B. Suppose that I is a nonzero ideal of B with additive group L and multiplicative group E. Since E is a normal subgroup of C, it must contain a minimal normal subgroup of E. All minimal normal subgroups of C are contained in $\mathrm{Z}(C) = \langle g_4, g_5\rangle$. Hence E must contain $\langle g_4\rangle$, $\langle g_5\rangle$ or $\langle g_4 g_5\rangle$. In the first case, L must contain $\delta(g_4) = c$. Since L must be invariant under the action of C, it should contain $g_3(c) = 2a + c$. Consequently, $\langle 2a, c\rangle_+ \leq L$. In particular, $\delta^{-1}(2a) = g_4 g_1 \in E$ and $\langle g_1, g_4\rangle \leq E$. Since $E \trianglelefteq C$, we have that $g_3 g_1 g_3^{-1} = g_5 g_1 \in E$, so $\langle g_1, g_4, g_5\rangle \leq E$.

Similarly, if $g_5 \in E$, then $\delta(g_5) = d \in L$. Thus, $g_2(d) = 2a + d \in L$, and hence $\langle 2a, d\rangle \leq L$. Now, $g_1 g_4 \in E$, so $g_3^{-1} g_1 g_4 g_3 = g_1 g_4 g_5 \in E$ and also $g_5 \in E$. Therefore

$$\langle g_1, g_4, g_5\rangle \leq E.$$

Finally, if $g_4 g_5 \in E$, then $\delta(g_4 g_5) = c + d \in L$, so $g_2(c + d) = 2a + c + d \in L$, and hence $\langle 2a, c + d\rangle_+ \leq L$. Again, $g_1 g_4 \in E$, so $g_3^{-1} g_1 g_4 g_3 = g_1 g_4 g_5 \in E$ and also $g_5 \in E$. Thus,

$$\langle g_1, g_4, g_5\rangle \leq E.$$

In all cases, we found that $\langle g_1, g_4, g_5\rangle \leq E$. Since $\delta(\langle g_1, g_4, g_5\rangle) = \langle 2a, c, d\rangle_+ \leq (B, +)$ is a δ-invariant subgroup and $\langle g_1, g_4, g_5\rangle \trianglelefteq C$, we have that $J = \langle 2a, c, d\rangle_+$ is the unique ideal of B of order 8. We observe that B/J is abelian. Therefore, the only three ideals of order 16 of B are $I_1 = \langle 2a, c, d, e\rangle_+$, $I_2 = \langle a + e, 2a, c, d\rangle_+$, $I_3 = \langle a, c, d\rangle_+$.

It can be easily seen that $0 \leq \langle c\rangle \leq \langle 2a, c\rangle_+ \leq I_1$, $0 \leq \langle c+d\rangle_+ \leq \langle 2a, c+d\rangle_+ \leq I_2$ and $0 \leq \langle d\rangle_+ \leq \langle 2a, d\rangle_+ \leq I_3$ are c-series of I_1, I_2 and I_3, respectively. In particular, I_1, I_2 and I_3 are centrally nilpotent braces. However, $B = I_1 + I_2 = I_1 + I_3 = I_2 + I_3$, but, as we have mentioned, B is not centrally nilpotent.

Our third example shows that there may be abelian subideals that are not contained in any centrally nilpotent ideals.

Example C. Let $(B, +) = \langle a\rangle \times \langle b\rangle \simeq \mathrm{C}_2 \times \mathrm{C}_{12}$ and $(C, \cdot) = [\langle\sigma\rangle]\langle\tau\rangle \simeq \mathrm{Dih}_{24}$. We have that C acts on B by means of the action λ defined by $\lambda_\sigma(a) = a + 6b$, $\lambda_\tau(a) = a$, $\lambda_\sigma(b) = a + b$, $\lambda_\tau(b) = a - b$. The bijective 1-cocycle $\delta : C \to B$ with respect to λ given by Table 4 yields a brace of abelian type $(B, +, \cdot)$ of order 24. This brace corresponds to `SmallBrace(24, 57)` in the YangBaxter library for GAP.

Let I be any ideal of B of order 12, and put $E = \delta^{-1}(I, +)$. Since $(I, +)$ is a maximal subgroup of $(B, +)$, it must contain its Frattini subgroup, which is $\langle 6b\rangle$. As $\delta^{-1}(6b) = \tau$ and $E \trianglelefteq C$, it follows that $\sigma\tau\sigma^{-1} = \sigma^2\tau \in E$. Therefore, $\delta(\sigma^2\tau) = a + 2b \in I$ and then $(I, +) = \langle a, 2b\rangle_+$. Since I is λ-invariant, we get that $I = \langle a, 2b\rangle_+$ is the only ideal of order 12 of B.

c	$\delta(c)$	c	$\delta(c)$	c	$\delta(c)$	c	$\delta(c)$
1	0	σ^6	a	τ	$6b$	$\sigma^6\tau$	$a+6b$
σ	$a+7b$	σ^7	b	$\sigma\tau$	$a+b$	$\sigma^7\tau$	$7b$
σ^2	$a+8b$	σ^8	$8b$	$\sigma^2\tau$	$a+2b$	$\sigma^8\tau$	$2b$
σ^3	$9b$	σ^9	$a+3b$	$\sigma^3\tau$	$3b$	$\sigma^9\tau$	$a+9b$
σ^4	$4b$	σ^{10}	$a+4b$	$\sigma^4\tau$	$10b$	$\sigma^{10}\tau$	$a+10b$
σ^5	$11b$	σ^{11}	$5b$	$\sigma^5\tau$	$a+5b$	$\sigma^{11}\tau$	$11b$

Table 4. Associated bijective 1-cocycle.

Note that I is not abelian as $\mathrm{Soc}(I) = \langle a+4b\rangle_+$. Thus,

$$(I,\cdot) \simeq \mathrm{Dih}_{12}$$

and so I is not centrally nilpotent. Hence, $\mathrm{Soc}(I)$ is an abelian subideal of B of order 6 such that it is not contained in any centrally nilpotent ideal of B.

Our last example shows that there are noncentrally nilpotent braces whose subbraces are subideals.

Example D. Let $(B,+,\cdot)$ be the brace of abelian type of order 32 studied in [2, Example 37]. It corresponds to `SmallBrace(32, 24003)` in the YangBaxter library for GAP, so $(B,+) = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \simeq C_4 \times C_2 \times C_2 \times C_2$, and $(B,\cdot) \simeq \langle e, f, h\rangle \simeq [C_2 \times \mathrm{Q}_8]C_2$ with bijective 1-cocycle given by Table 5 and associated action given by

$$\begin{aligned} &e(a) = a+c+d, && e(b) = 2a+c, && e(c) = b, && e(d) = 2a+c+d,\\ &f(a) = a+b+c, && f(b) = 2a+b, && f(c) = c, && f(d) = c+d,\\ &h(a) = a, && h(b) = b, && h(c) = c, && h(d) = 2a+d. \end{aligned}$$

x	$\delta(x)$	x	$\delta(x)$	x	$\delta(x)$	x	$\delta(x)$
1	0	h	c	f	a	fh	$a+c$
e	$3a+b$	eh	$3a$	ef	$b+c+d$	efh	$c+d$
e^2	$b+c$	e^2h	$2a+b+c+d$	e^2f	$3a+d$	e^2fh	$a+c+d$
e^3	$3a+b+d$	e^3h	$a+d$	e^3f	b	e^3fh	$2a$
e^4	$2a+b+c$	e^4h	$2a+b$	e^4f	$a+b+c$	e^4fh	$a+b$
e^5	$3a+c$	e^5h	$3a+b+c$	e^5f	$2a+d$	e^5fh	$2a+b+d$
e^6	$2a+c+d$	e^6h	d	e^6h	$3a+b+c+d$	e^6fh	$a+b+d$
e^7	$3a+c+d$	e^7h	$a+b+c+d$	e^7f	$2a+c$	e^7fh	$b+d$

Table 5. Associated bijective 1-cocycle.

We start by providing all subbraces of order 2. These are generated by those elements $x \in (B, +)$ of order 2 such that $\lambda_x(x) = x$. We have $S_1 = \{1, 2a+b+d\}$, $S_2 = \{1, c\}$, $S_3 = \{1, b+c\}$, $S_4 = \{1, c+d\}$, $S_5 = \{1, 2a\}$, $S_6 = \{1, 2a+b\}$, $S_7 = \{1, 2a+b+c\}$ (here, S_5 is the only left ideal). For subbraces of order 4, we need those subgroups $H \leq (B, +)$ of order 4 such that $\delta^{-1}(H)$ is also a subgroup of $\langle e, f, h\rangle$. We find

H	$\delta^{-1}(H)$	H	$\delta^{-1}(H)$
$S_8 = \langle 2a, b+c\rangle$	$\langle e^3 fh, e^7 fh\rangle$	$S_9 = \langle 2a+b, c\rangle$	$\langle e^4, h\rangle$
$S_{10} = \langle b, 2a+c\rangle$	$\langle e^3 f\rangle$	$S_{11} = \langle c+d, 2a+b+d\rangle$	$\langle efh, e^5 fh\rangle$
$S_{12} = \langle 2a+d, b+c+d\rangle$	$\langle e^5 f\rangle$	$S_{13} = \langle d, 2a+b+c\rangle$	$\langle e^6 h\rangle$
$S_{14} = \langle 2a+b+c, b+d\rangle$	$\langle e^2\rangle$		

(here, S_8 is the only left ideal). For subbraces of order 8, we need those subgroups $H \leq (B, +)$ of order 8 such that $\delta^{-1}(H) \leq \langle e, f, h\rangle$. Thus,

$$\begin{aligned} S_{15} &= \langle 2a, b, c\rangle, & \delta^{-1}(S_{15}) &= \langle e^3 fh, e^3 f\rangle, \\ S_{16} &= \langle 2a, b+c, b+d\rangle, & \delta^{-1}(S_{16}) &= \langle e^3 fh, e^2\rangle, \\ S_{17} &= \langle 2a, b+c, d\rangle, & \delta^{-1}(S_{17}) &= \langle e^3 fh, e^6 h\rangle \end{aligned}$$

(here, S_{15} is the only left ideal). The only subbrace of order 16 is the only nonzero proper ideal of B, that is, $S_{18} = \langle 2a, b, c, d\rangle$. The following relations can be easily checked to hold:

$$\begin{aligned} &S_1, S_4, S_7 \trianglelefteq S_{11} \trianglelefteq S_{15} \trianglelefteq S_{18} \trianglelefteq B, && S_3, S_5, S_7 \trianglelefteq S_8 \trianglelefteq S_{16} \trianglelefteq S_{18} \trianglelefteq B, \\ &S_2, S_6, S_7 \trianglelefteq S_9 \trianglelefteq S_{17} \trianglelefteq S_{18} \trianglelefteq B, && S_{14} \trianglelefteq S_{15} \trianglelefteq S_{18} \trianglelefteq B, \\ &S_{12}, S_{13} \trianglelefteq S_{17} \trianglelefteq S_{18} \trianglelefteq B, && S_{10} \trianglelefteq S_{17} \trianglelefteq S_{18} \trianglelefteq B. \end{aligned}$$

Therefore, all subbraces are subideals but B is not soluble.

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HYPERBOLIC L-SPACE KNOTS NOT CONCORDANT TO ALGEBRAIC KNOTS

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We construct infinitely many hyperbolic L-space knots for which $-3\int\Upsilon$ is not an integer, where Υ is the Ozsváth–Stipsicz–Szabó upsilon function. None of these knots can be concordant to a linear combination of algebraic knots.

1. Overview

Describing knot concordance groups and understanding $\mathbb{Z}$-homology cobordism of 3-manifolds are listed among the most important problems in low-dimensional topology. Among many questions, a particular interest is about the position of various classes of knots, respectively 3-manifolds, in the concordance group, respectively in the group of $\mathbb{Z}$-homology cobordisms.

It is well known [30] that each 3-manifold is $\mathbb{Z}$-homology cobordant to a hyperbolic 3-manifold. On the other hand, there are 3-manifolds that are not $\mathbb{Z}$-homology cobordant to Seifert fibered manifolds [10]; recently it is proved [20] that the Seifert fibered manifold span a subgroup of the group of $\mathbb{Z}$-homology cobordism with $\mathbb{Z}^\infty$-summand as a quotient.

For link cobordisms, there is an abundance of similar questions. There are knots that are not topologically concordant to alternating knots [16]. A refinement of the argument in [16] shows that there exist knots that are not topologically concordant to L-space knots [41]. On the other hand, all knots are topologically concordant to strongly quasipositive knots [5], a statement that is definitely false in the smooth category, because for all strongly quasipositive knots, all slice torus invariants are equal, compare [14].

An algebraic link is defined as a link of a plane curve singularity. All such links are graph links [12]. Also, all such links are L-space links by [17; 18]. Studying the position of algebraic knots in the whole knot concordance group seems to bring an immediate answer: while it is not stated explicitly in [41], the methods in that paper suggest that the quotient of the topological concordance group by the group of L-space knots has an infinite $\mathbb{Z}^\infty$ summand. However, the position of algebraic

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knots in the concordance subgroup spanned by all L-space knots seems rather mysterious. This question appears to be a natural generalization of a question on the position of graph manifolds in the $\mathbb{Z}$-homology cobordism group of 3-manifolds.

The main result of the paper is the following.

Theorem 1.1. *There exist infinitely many hyperbolic L-space knots that are not smoothly concordant to any linear combination of algebraic knots.*

Plan of the proof. In Section 4 we construct an infinite family of knots K_n. By Montesinos trick we show that each K_n admits a positive L-space surgery. The braid index of K_n is bounded by 4 by construction. Combining this fact with the lack of semigroup property of K_n, we show that K_n are hyperbolic.

The proof that K_n is not concordant to any linear combination of algebraic knots involves computing the function Υ of Ozsváth, Stipsicz and Szabó [33]. For L-space knots, this function is determined from the Alexander polynomial of K_n, and to pass from that polynomial to Υ, we use an explicit algorithm described in [6]. The computation of the Alexander polynomial of K_n, conducted in Section 4 involves a surgery description of K_n, and Torres formula.

The main obstruction is the following. It was shown in [36] that if K is an algebraic knot, then $-3\int_0^2 \Upsilon_K(t)\,dt$ is an integer. In fact, echoing the calculation of [4], one can express this integral via invariants of the underlying plane curves singularity, compare Section 3.

Conversely, Theorem 4.1 shows that $-3\int_0^2 \Upsilon_{K_n}$ is a nonintegral fraction of 5. So, none of the knots K_n can be expressed as linear combinations of algebraic knots.

The methods do not show whether the knots K_n generate a $\mathbb{Z}^\infty$ summand in the quotient subgroup generated by the concordance classes of all L-space knots modulo algebraic knots. However, we know that a subsequence of K_n knots is linearly independent in the topological concordance group, see Theorem 6.1.

The structure of the paper is the following. Section 2 gives a necessary background on the Υ function. In Section 3, we recall Tange's calculations of the integral of the Υ function and show some properties of the integral. In particular, we introduce a purely Floer-theoretic invariant ω and relate it to invariants of plane curve singularities. These results are of independent interest, especially that they give an algebrogeometric motivation for studying $-3\int\Upsilon$ as a knot invariant.

The family of knots K_n is constructed in Section 4. Their main properties are stated in Theorem 4.1, whose part is proved in that section. The most difficult part, showing that K_n are indeed L-space knots, is proved in Section 5.

Section 6 addresses the question of linear independence of knots K_n. We study roots of the Alexander polynomial of K_n and show that a subsequence of K_n is linearly independent. We also show that Υ function alone cannot prove that K_n are independent modulo the group of algebraic knots.

Finally, in Section 7 we provide a table of those L-space knots of [1] for which $-3\int\Upsilon$ is not integral.

2. Review of the Υ function

Recall that to a knot K in the 3-sphere, knot Floer homology associates a complex $\mathrm{CFK}^\infty(K)$ over the ring $\mathbb{Z}_2[U, U^{-1}]$, where U is a formal variable. The complex is $\mathbb{Z}\oplus\mathbb{Z}$–filtered, $\mathbb{Z}$-graded, and the multiplication by U lowers the grading by 2 and the filtration by $(1, 1)$. The complex is defined up to bifiltered chain homotopy equivalence.

In [33], a concordance invariant Υ was extracted from this chain complex; see also [26]. In short, for each $t \in [0, 2]$, one associates a collapsed filtration. If $x \in \mathrm{CFK}^\infty(K)$ is at bifiltration level (a, b), we define its $\mathcal{F}_t$-filtration level by $\frac{t}{2}a + \left(1 - \frac{t}{2}\right)b$. Denote by $\mathcal{C}_{s,t}$ the subcomplex of $\mathrm{CFK}^\infty(K)$ of elements at $\mathcal{F}_t$-filtration level $\leq s$. As $\mathcal{C}_{s,t}$ is a subcomplex of $\mathrm{CFK}^\infty(K)$, there is a map $H_i(\mathcal{C}_{s,t}) \to H_i(\mathrm{CFK}^\infty(K))$, where the subscript i denotes the homological grading (the $\mathbb{Z}$-grading) of the complex $\mathrm{CFK}^\infty(K)$. Set

$$\nu(t) = \min\{s : H_0(\mathcal{C}_{s,t}) \to H_0(\mathrm{CFK}^\infty(K)) \text{ is surjective}\}.$$

The function Υ is defined by

$$\Upsilon(t) = -2\nu(t).$$

Among many properties of the function Υ, the most important for the sake of this paper is that it is a concordance invariant. That is to say, if K_1 is smoothly concordant to K_2, then $\Upsilon_{K_1}(t) = \Upsilon_{K_2}(t)$ for all $t \in [0, 2]$.

If K is an L-space knot, the complex $\mathrm{CFK}^\infty(K)$ is determined by the Alexander polynomial, via so-called *staircase complex*. The Υ function for such knots was computed in [33]. To be more precise, write the Alexander polynomial

$$\Delta_K = 1 + (t-1)(t^{c_1} + \cdots + t^{c_\ell}),$$

with $1 \leq c_1 < \cdots < c_\ell$ (such presentation is possible for all L-space knots, see [31]). Let $S_K = \mathbb{Z}_{\geq 0} \setminus \{c_1, \ldots, c_\ell\}$.

Definition 2.1 (see [40]). The set S_K is called the *formal semigroup* of the L-space knot K.

An equivalent definition of S_K is via the power series expansion:

$$\frac{\Delta_K(t)}{1-t} = \sum_{s\in S_K} t^s.$$

If K is algebraic, the set S_K is the semigroup of the underlying singularity. It is an interesting question to study nonalgebraic L-space knots for which S_K has the structure of a semigroup [37; 40].

The Υ function of an L-space knot is related to the formal semigroup via the Fenchel–Legendre transform of an extension of the function $I : \mathbb{Z} \to \mathbb{Z}$ such that $I : n \mapsto \#\{S_K \cap [0, n)\}$; see [6]. In particular, for an L-space knot, Υ is convex. It is interesting to see for which knots the Υ function is convex [21].

3. The integral of Υ in algebraic geometry

It was proved by Tange [36] that $-3\int\Upsilon$ for an algebraic knot is an integer. We recall his computations. We also show that the quantity $-2\tau - 3\int\Upsilon$ has a special interpretation in singularity theory. This explains our initial interest in the invariant $-3\int\Upsilon$. We begin by recalling a standard definition; see [9; 12] for more details.

Let $z \in \mathbb{C}^2$ be a singular point of a complex algebraic curve C. Recall that the *multiplicity* of a singular point, denoted m, is the minimal positive number that can be obtained as a local intersection of C at z with another algebraic curve. The fact that the point z is singular means that $m > 1$.

Suppose z has a single branch. Set $m_1 = m$. Blow up a singular point to obtain a new curve $\tilde{C}$ and denote by E the exceptional divisor of the blow-up. Usually $\tilde{C}$ will have a singular point at $\tilde{C} \cap E$. Denote by m_2 its multiplicity. If $\tilde{C}$ is smooth, then $m_2 = 1$ and we stop the procedure. The procedure can be iterated until some strict transform $\tilde{C}$ is smooth. Eventually we obtain a finite sequence $(m_1, m_2, m_3, \ldots, m_n)$ of integers such that $m_i > 1$ (usually we discard the last 1 from the sequence).

Definition 3.1. The sequence $(m_1, \ldots, m_n)$ is called the *multiplicity sequence* of a unibranched singular point.

The multiplicity sequence is a complete topological invariant of a singular point, in the sense that any two unibranched singular points with the same multiplicity sequences are topologically equivalent. All topological invariants of the singularity can be computed from the multiplicity sequence. For example, the following formula was proved by Milnor [27], compare [9; 42]:

$$\mu_z = \sum_{i=1}^{n} m_i(m_i - 1), \tag{3.2}$$

where μ_z is the Milnor number of the critical point. The Milnor number is equal to twice the genus of the link of the singular point.

We introduce the quantity

$$\omega(K) = -3\int_0^2 \Upsilon_K(t)\,dt - 2\tau(K). \tag{3.3}$$

As we see, ω is defined purely from Heegaard Floer-type invariants.

Theorem 3.4. *Let z be a singular point with one branch. Let K be the link of singularity. Then $\omega(K) = \sum(m_i - 1)$.*

Proof. The proof relies on the following result of Tange [36], based on [13] and [3].

Lemma 3.5. *For $m > 1$ define the function*

$$\Upsilon_m(t) = -i(i+1) - \tfrac{1}{2}m(m-1-2i)t \quad \textit{if } t \in \left[\frac{2i}{m}, \frac{2i+2}{m}\right].$$

If a singular point z has link K and multiplicity sequence $(m_1, \dots, m_n)$, then

$$\Upsilon_K = \sum_{i=1}^{n} \Upsilon_{m_i}.$$

Continuing the proof of Theorem 3.4, we use the following formula of [36]:

$$\int_0^2 \Upsilon_m(t)\, dt = \frac{-m^2+1}{3}. \tag{3.6}$$

From (3.6) and Lemma 3.5 we conclude that

$$-3\int_0^2 \Upsilon_K(t) = \sum(m_i^2 - 1).$$

For algebraic knots, $2\tau(K) = \mu_z$ is the Milnor number of the underlying singular point. We conclude the proof by (3.2). □

Corollary 3.7 (see [36]). *For an algebraic knot, $-3\int_0^2 \Upsilon(t)\, dt$ is an integer.*

We have the following interpretation of $\omega(K)$, which is due to Zaĭdenberg and Orevkov [42]. Blow up the critical point z until the reduced inverse image $D = \pi^{-1}(C)_{\text{red}}$ is a normal crossing divisor. Let $E_1, \dots, E_s$ be the exceptional divisors. Define the canonical divisor $K_z = \sum \alpha_i E_i$ by the condition that

$$K_z \cdot E_i + E_i \cdot E_i = -2 \quad \text{for all } i.$$

Let C'_z be the strict transform of C.

Proposition 3.8 (see [42, Lemma 4]). *For K the link of singularity at z, if z has one branch, then $\omega(K) = K_z \cdot (K_z + C'_z)$.*

We conclude this section with a simple estimate for quasihomogeneous singular points:

Proposition 3.9. *If z is a quasihomogeneous singular point, topologically equivalent to $x^p - y^q = 0$ with p, q coprime, then*

$$\omega(K) < p + q,$$

where K is the link of the singularity (in this situation, K is the T_{pq}-torus knot). Moreover, if $p = 2$, then $\omega(K) = \frac{1}{2}(q-1)$.

Proof. The second part follows from the fact that the multiplicity sequence for the singular point $x^2 - y^{2k+1} = 0$ is a length k sequence $(2, \dots, 2)$. For the first part we observe that the multiplicity sequence is constructed as follows. Suppose $p < q$. Then $m_1 = p$. The blow-up replaces (p, q) by $(p, q-p)$ or $(q-p, p)$ depending on whether $p < q-p$ or $q-p < p$. It follows that $\sum_{i=2}^n m_i = q$, and hence

$$\sum_{i=1}^n m_i = p + q.$$

Therefore $\omega(K) \leq p+q-1$. □

The implication of Proposition 3.9 is that ω is a *linear* invariant, that is, its value for the T_{pq}-torus knot grows like the sum of p and q, not like the product. The latter behavior is more typical, the genus and the signature are examples, in particular, for slice-torus invariants.

Another inequality involving $\omega(K)$ is a generalization of the Zaĭdenberg–Orevkov inequality [42, Section 11].

Proposition 3.10. *For a cuspidal singular point z with Milnor number μ and multiplicity m we have $\mu \leq m\omega$.*

Proof. Let $m_1, \dots, m_n$ be the multiplicity sequence of z. Then $m_1 \geq m_2 \geq \cdots \geq m_n$. This means that $m_1 \sum(m_i - 1) \geq \sum m_i(m_i - 1)$, but in light of (3.2), this is precisely the statement of the proposition. □

4. The family K_n

For $n \geq 1$, our knot K_n is given by the surgery description shown in Figure 1. Let $L = K \cup C_1 \cup C_2$ be the oriented link as shown in Figure 1. If we perform (-1)-surgery on C_1 and $\left(-\frac{1}{n+1}\right)$-surgery on C_2, then K is changed into K_n. Thus K_n is the closure of the 4-braid:

$$[2, 1, 3, 2, (3, 2, 1)^4, 3^{2(n+1)}, 2],$$

where an integer k denotes the standard braid generator σ_k of the 4-string braid group. In particular, K_1 is $m211$, and K_2 is $t09284$ in the SnapPy census [11]. Since K_n is the closure of a positive braid, K_n is fibered and its genus is equal to $n+8$. The diagram of K_1 is given in Figure 1.

Theorem 4.1. *For $n \geq 1$, the knot K_n enjoys the following.*

(1) *K_n is hyperbolic.*

(2) *$(4n+24)$-surgery on K_n gives an L-space, so K_n is an L-space knot.*

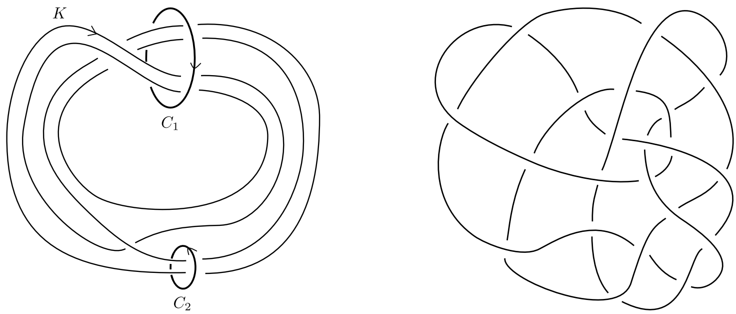

Figure 1. The surgery description of K_n (left) and the knot $K_1 = m211$ (right).

(3) *For the upsilon invariant* $\Upsilon_{K_n}(t)$ *of* K_n,

$$I = \int_0^2 \Upsilon_{K_n}(t)\,dt = -\big(n + \tfrac{34}{5}\big).$$

Thus $-3I = 3n + \frac{102}{5} \notin \mathbb{Z}$.

The proof of the second property is deferred to Section 5. Our first step towards proving Theorem 4.1 is to compute the Alexander polynomial. Once we know it, using the fact that K_n are L-space knots, we compute the integral proving (3). Moreover, the Alexander polynomial will allow us to prove that K_n is not hyperbolic.

Lemma 4.2. *The Alexander polynomial of* K_n *is given by*

$$\begin{aligned}\Delta_{K_n}(t) = (t^{2n+16} - t^{2n+15}) + (t^{2n+12} - t^{2n+11}) + (t^{2n+10} - t^{2n+9})\\ + (t^{2n+7} - t^{2n+6}) + (t^{2n+5} - t^{2n+4}) + \cdots + (t^{11} - t^{10})\\ + t^9 - t^7 + t^6 - t^5 + t^4 - t + 1.\end{aligned}$$

Proof. Let L be the link as in Figure 1. The multivariable Alexander polynomial of L can be readily computed using [11; 24]:

$$\begin{aligned}\Delta_L(x, y, z) = x^7y^3z - x^5y^3z + x^5y^2z + x^5y^2 - x^4y^2 - x^5y\\ - 2x^3y^2z + 2x^4y + x^2y^2z + x^3yz - x^2yz - x^2y + x^2 - 1,\end{aligned}$$

where the variables x, y, z correspond to the (oriented) meridians of K, C_1, C_2.

If we perform (-1)-surgery on C_1 and $\big(-\frac{1}{n+1}\big)$-surgery on C_2, then the link $K \cup C_1 \cup C_2$ is changed into $K_n \cup C_1^n \cup C_2^n$. Since these links have homeomorphic exteriors, the induced isomorphism on the homology groups relates their Alexander polynomials; compare [15; 29].

Let μ_K, μ_{C_1} and μ_{C_2} be the homology classes of meridians of K, C_1, C_2, respectively. Each meridian has linking number one with the corresponding component. Furthermore, let λ_{C_1} and λ_{C_2} be the homology classes of their oriented longitudes. We see that $\lambda_{C_1} = 4\mu_K$ and $\lambda_{C_2} = 2\mu_K$.

Next, let μ_{K_n}, $\mu_{C_1^n}$ and $\mu_{C_2^n}$ be the homology classes of meridians of K_n, C_1^n and C_2^n. Then we have that $\mu_{K_n}=\mu_K$, $\mu_{C_1^n}=-\mu_{C_1}+\lambda_{C_1}$, $\mu_{C_2^n}=-\mu_{C_2}+(n+1)\lambda_{C_2}$. Hence

$$\mu_K=\mu_{K_n},\quad \mu_{C_1}=-\mu_{C_1^n}+4\mu_{K_n},\quad \mu_{C_2}=-\mu_{C_2^n}+2(n+1)\mu_{K_n}.$$

Thus, we have the relation between the Alexander polynomials as

$$\Delta_{K_n\cup C_1^n\cup C_2^n}(x,y,z)=\Delta_L(x,x^4y^{-1},x^{2(n+1)}z^{-1}). \tag{4.3}$$

Since $\mathrm{lk}(K_n,C_2^n)=\mathrm{lk}(K,C_2)=2$ and $\mathrm{lk}(C_1^n,C_2^n)=\mathrm{lk}(C_1,C_2)=0$, the Torres condition [39] gives

$$\Delta_{K_n\cup C_1^n\cup C_2^n}(x,y,1)=(x^2y^0-1)\Delta_{K_n\cup C_1^n}(x,y)=(x^2-1)\Delta_{K_n\cup C_1^n}(x,y).$$

Furthermore, since $\mathrm{lk}(K_n,C_1^n)=\mathrm{lk}(K,C_1)=4$,

$$\Delta_{K_n\cup C_1^n}(x,1)=\frac{x^4-1}{x-1}\Delta_{K_n}(x).$$

Thus

$$\Delta_{K_n}(x)=\frac{x-1}{x^4-1}\Delta_{K_n\cup C_1^n}(x,1)=\frac{x-1}{(x^4-1)(x^2-1)}\Delta_{K_n\cup C_1^n\cup C_2^n}(x,1,1).$$

Then the relation (4.3) gives

$$\begin{aligned}
\Delta_{K_n}(t)&=\frac{t-1}{(t^4-1)(t^2-1)}\Delta_L(t,t^4,t^{2(n+1)})\\
&=\frac{1}{(t^4-1)(t+1)}(t^{2n+21}-t^{2n+19}+t^{2n+15}-2t^{2n+13}+t^{2n+12}+t^{2n+9}-t^{2n+8}\\
&\qquad\qquad +t^{13}-t^{12}-t^9+2t^8-t^6+t^2-1)\\
&=\frac{1}{(t^4-1)(t+1)}\big(t^{2n+13}(t^8-1)-t^{2n+15}(t^4-1)-t^{2n+9}(t^4-1)+t^{2n+8}(t^4-1)\\
&\qquad\qquad +t^9(t^4-1)-t^8(t^4-1)-t^2(t^4-1)+(t^8-1)\big)\\
&=\frac{1}{t+1}\big(t^{2n+13}(t^4+1)-t^{2n+15}-t^{2n+9}+t^{2n+8}+t^9-t^8-t^2+(t^4+1)\big)\\
&=\frac{1}{t+1}\big(t^{2n+15}(t^2-1)+t^{2n+9}(t^4-1)+t^9(t^{2n-1}+1)-t^4(t^4-1)-(t^2-1)\big)\\
&=t^{2n+15}(t-1)+t^{2n+9}(t^2+1)(t-1)+t^9\frac{t^{2n-1}+1}{t+1}-t^4(t^2+1)(t-1)-(t-1)\\
&=t^{2n+16}-t^{2n+15}+t^{2n+12}-t^{2n+11}+t^{2n+10}-t^{2n+9}\\
&\qquad\qquad +t^9\Big(\sum_{i=1}^{n-1}(t^{2i}-t^{2i-1})+1\Big)-t^7+t^6-t^5+t^4-t+1. \qquad \square
\end{aligned}$$

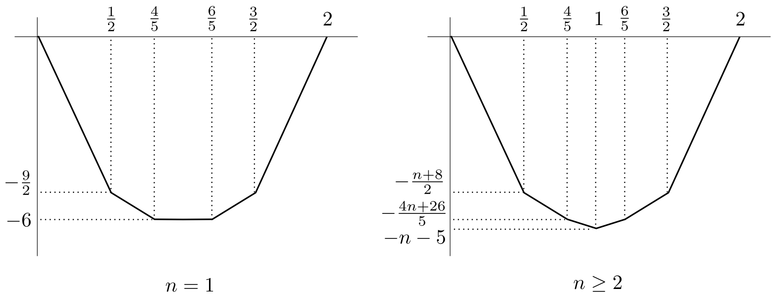

Figure 2. The upsilon function $\Upsilon_{K_n}(t)$: the case $n = 1$ (left) and $n \geq 2$ (right).

In Definition 2.1 we have recalled the notion of a formal semigroup of an L-space knot. While the proof that K_n are L-space knots is given only in Section 5, it is convenient to determine the Υ function of K_n from the Alexander polynomial we just computed.

First, we show that K_n is not algebraic.

Lemma 4.4. *The formal semigroup S_{K_n} of K_n is not closed under addition.*

Proof. By Lemma 4.2, the formal semigroup S_{K_n} of K_n starts:

$$0, 4, 6, 9, 11, 12, \dots, 2n+5, \dots.$$

Thus $4 \in S_{K_n}$, but $8 \notin S_{K_n}$. □

Lemma 4.5. *For $n \geq 1$, the upsilon function of K_n is given as*

$$\Upsilon_{K_n}(t) = \begin{cases} -(n+8)t & \left(0 \leq t \leq \frac{1}{2}\right), \\ -(n+4)t - 2 & \left(\frac{1}{2} \leq t \leq \frac{4}{5}\right), \\ -(n-1)t - 6 & \left(\frac{4}{5} \leq t \leq 1\right). \end{cases}$$

For $t \in [1, 2]$, we have $\Upsilon_{K_n}(t) = \Upsilon_{K_n}(2-t)$.

Figure 2 shows $\Upsilon_{K_n}(t)$ when $n = 1$ and $n \geq 2$.

Proof. By [6], the upsilon function is the Fenchel–Legendre transform of the gap function $G(x) = 2J(-x)$, in their notation, determined by the Alexander polynomial. In fact, there is a handy description of the graph of $G(x)$. Let us write $\Delta_{K_n}(t)$ as $t^{a_0} - t^{a_1} + t^{a_2} - \dots + t^{a_{2n}}$. Then the sequence of the jumps in the exponents is

$$a_1 - a_0, a_2 - a_1, \dots, a_{2n} - a_{2n-1}. \tag{4.6}$$

Consider the vectors $\boldsymbol{u} = (1, 2)$ and $\boldsymbol{h} = (1, 0)$ on $\mathbb{R}^2$. Then [38, Lemma 2.2] shows that the graph of $G(x)$ restricted on $[-g, g]$ has a form of staircase specified by (4.6). More precisely, we start at the point $(-g, 0)$, and move $a_1 - a_0$ times along $\boldsymbol{u}$, then $a_2 - a_1$ times along $\boldsymbol{h}$, and so on. Finally, we reach the point $(g, 2g)$. The function $G(x)$ is 0 for $x \leq -g$, and $2x$ for $x \geq g$.

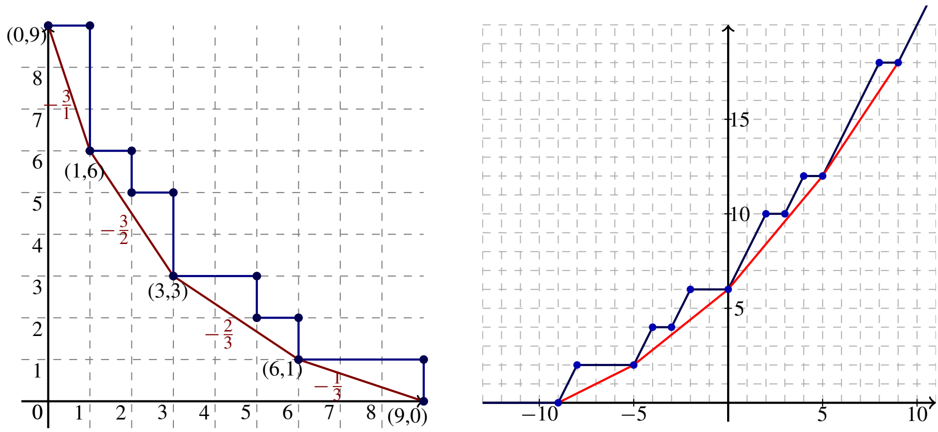

Figure 3. The knot K_1: the staircase (left) and the graph of the gap function (right). The red lines are piecewise-linear convex supporting functions, used to compute the Fenchel–Legendre transform.

By Lemma 4.2, the sequence of the jumps is

$$1, 3, 1, 1, 1, 2, \underbrace{1, 1, \dots, 1, 1}_{2n-2}, 2, 1, 1, 1, 3, 1.$$

Thus the graph of $G(x)$ goes through the points $(-g, 0) = (-n-8, 0)$, $(-n-4, 2)$, $(-n+1, 6)$, $(n-1, 2n+4)$, $(n+4, 2n+10)$ and $(g, 2g) = (n+8, 2n+16)$, which determine the convex hull. More precisely, the convex hull of $G(x)$ is determined by the following:

$$\begin{cases} y = 0 & (x \leq -n-8), \\ y = \frac{x+n+8}{2} & (-n-8 \leq x \leq -n-4), \\ y = \frac{4}{5}(x+n+4)+2 & (-n-4 \leq x \leq -n+1), \\ y = x+n+5 & (-n+1 \leq x \leq n-1), \\ y = \frac{6(x-n+1)}{5} + 2n+4 & (n-1 \leq x \leq n+4), \\ y = \frac{3(x-n-4)}{2} + 2n+10 & (n+4 \leq x \leq n+8), \\ y = 2x & (x \geq n+8). \end{cases}$$

See Figure 3 for the case $n = 1$. Then the Fenchel–Legendre transformation immediately gives the conclusion. □

In Section 5, we prove that $(4n+24)$-surgery on K_n yields an L-space by using the Montesinos trick. Once we admit it, it is easy to prove that K_n is hyperbolic.

Lemma 4.7. *K_n is hyperbolic.*

Proof. Recall that a torus knot of type (p,q) for $0<p<q$ has the formal semigroup $\langle p,q\rangle = \{ap+bq \mid a,b \geq 0\}$, which is closed under addition. Hence K_n is not a torus knot by Lemma 4.4.

By [25], K_n is prime. Assume that K_n is a satellite knot for a contradiction. Since the bridge number of K_n is at most four, the companion is a 2-bridge knot and the pattern has wrapping number two [35]. Since K_n is an L-space knot, the companion and the pattern knot are also L-space knots [2; 22]. Furthermore, the pattern is braided. Thus the companion is a 2-bridge torus knot by [31], and K_n is its 2-cable. In other words, K_n is an iterated torus L-space knot. Finally, Wang [40] shows that the formal semigroup of such a knot is closed under addition. This contradicts Lemma 4.4. □

Proof of Theorem 4.1. This immediately follows from Lemmas 4.5 and 4.7, and Proposition 5.1 below. □

5. Montesinos trick

We use the Montesinos trick [28] to prove that the $(4n+24)$-surgery on K_n yields an L-space. For a surgery diagram of a strongly invertible knot or link, the Montesinos trick describes the resulting closed 3-manifold as the double branched cover of another knot or link obtained from the tangle replacements corresponding to the surgery coefficients.

Figure 4 shows a surgery diagram in a strongly invertible position with the axis A. After performing (-1)-surgery, we have K_n with surgery coefficient $4n+24$.

Take the quotient of $K_n \cup A$ under the involution along A. Then we obtain a two-component link shown in Figure 5. The Montesinos trick claims that the resulting manifold of $(4n+24)$-surgery on K_n is the double branched cover of S^3 branched over this link. Figures 6 and 7 show the deformations of the link.

Let us denote this link by ℓ_n. We perform two resolutions as shown in Figure 8 at a crossing located in the box with n half twists. Let ℓ_∞ and ℓ_{n-1} be the resulting links. Then it is straightforward to calculate $\det \ell_n = 4n+24$ and $\det \ell_\infty = 4$. (For example, use the checkerboard pattern of the diagrams in Figures 7 and 9.) This shows the equation $\det \ell_n = \det \ell_{n-1} + \det \ell_\infty$. Thus if the double branched covers of ℓ_{n-1} and ℓ_∞ are L-spaces, then so is the double branched cover of ℓ_n [8; 31; 32].

As shown in Figure 9, the link ℓ_∞ is the Montesinos link $M\big(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\big)$. Thus the double branched cover of ℓ_∞ is a Seifert fibered space over the 2-sphere with three exceptional fibers of indices 2, 2, 2, which is elliptic. Then it is an L-space [31].

Inductively, it is enough to show that the double branched cover of ℓ_1 is an L-space. However, as shown in Figure 10, ℓ_1 is the Montesinos link $M\big(\frac{1}{2}, \frac{1}{2}, -\frac{4}{11}\big)$. The double branched cover is also an elliptic Seifert fibered space, which is an L-space.

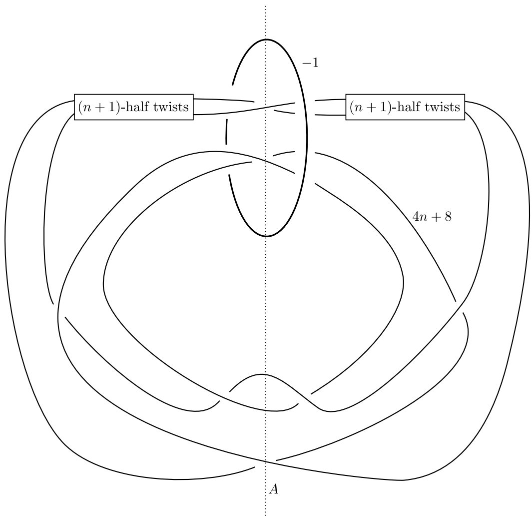

Figure 4. After (-1)-surgery, the surgery diagram gives $(4n+24)$-surgery on K_n. Each rectangle box contains right-handed $(n+1)$-half twists.

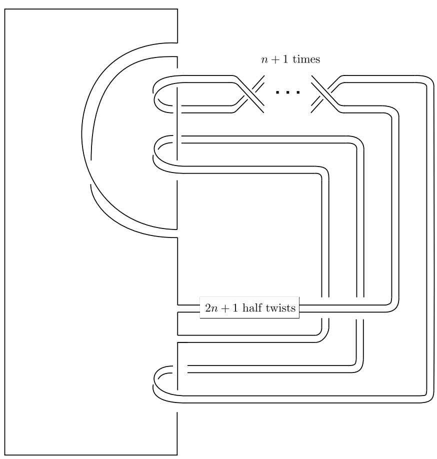

Figure 5. The double branched cover of S^3 branched over this link gives the resulting manifold of $(4n+24)$-surgery on K_n.

Thus, we have shown that:

Proposition 5.1. *For K_n, $(4n+24)$-surgery yields an L-space.*

6. Linear independence of K_n

Theorem 6.1. *There is an increasing sequence a_n such that the knots $K_{a_1}, K_{a_2}, \ldots$ are linearly independent in the topological concordance group.*

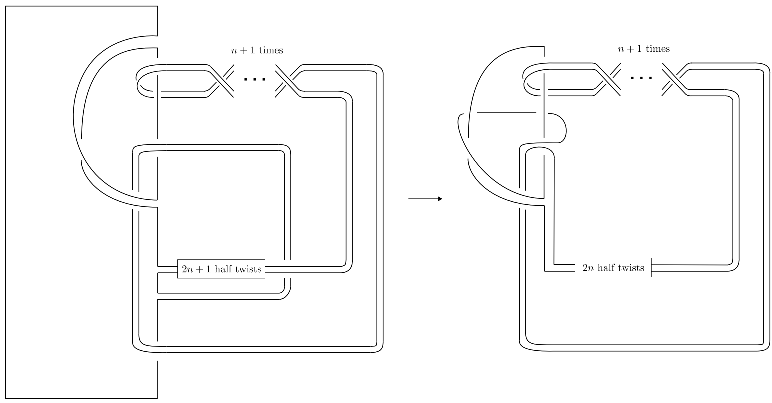

Figure 6. Deformation of the link.

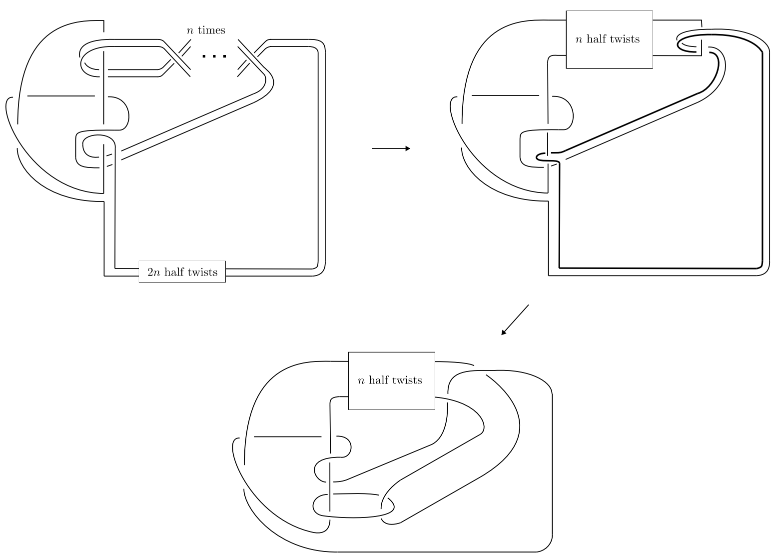

Figure 7. Deformation of the link (continued).

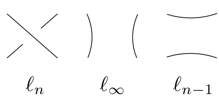

Figure 8. Two resolutions.

Proof. Recall that for any knot K, the Tristram–Levine signature function σ_K is a piecewise constant function from S^1 to $\mathbb{Z}$ with discontinuities only at the roots of the Alexander polynomial. We have the following classical result; see [23, Chapter 12] for a classical approach and [34] for the proof under topological concordance.

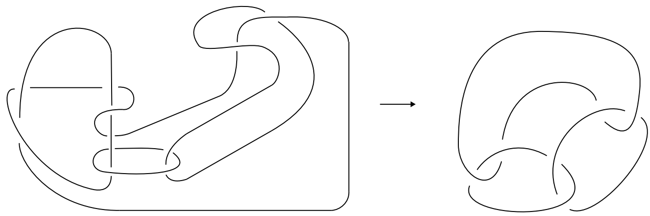

Figure 9. The link ℓ_∞ with $\det \ell_\infty = 4$ is the Montesinos link $M\big(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\big)$. The double branched cover over this link is a Seifert fibered space over the 2-sphere with three exceptional fibers of indices 2, 2, 2, which is elliptic.

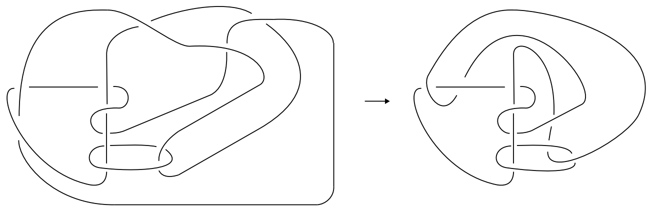

Figure 10. The link ℓ_1 is the Montesinos link $M\big(\frac{1}{2}, \frac{1}{2}, -\frac{4}{11}\big)$.

Proposition 6.2. *If K_1 and K_2 are topologically concordant, then $\sigma_{K_1}(t) = \sigma_{K_2}(t)$ for all but finitely many t in S^1.*

The next result is a consequence of the definition of the signature, it can be deduced from the dévissage of the Blanchfield form. We refer an interested reader to [7, Section 5].

Lemma 6.3. *Suppose $\Delta_K(t)$ has a zero at $t_0 \in S^1$ of odd multiplicity. Then, the signature function has jump at t_0, that is,*

$$\Big|\lim_{u\to 0^+}\big(\sigma(t_0\, e^{iu}) - \sigma(t_0\, e^{-iu})\big)\Big| \geq 2.$$

Our goal is to find, for all knots K_n with $n \geq 11$, a zero of Δ of odd multiplicity of Δ on S^1, and very close to 1. First, we symmetrize the Alexander polynomial for K_n computed in Lemma 4.2 as

$$\begin{aligned}\psi_n = t^{-n-8}\big((t^{2n+16} - t^{2n+15}) + (t^{2n+12} - t^{2n+11}) + (t^{2n+10} - t^{2n+9}) \\ + (t^{2n+7} - t^{2n+6}) + (t^{2n+5} - t^{2n+4}) + \cdots + (t^{11} - t^{10}) \\ + t^9 - t^7 + t^6 - t^5 + t^4 - t + 1\big).\end{aligned}$$

Decompose $\psi_n = \psi_n^1 + \psi_n^2$, where

$$\begin{aligned}\psi_n^1 = t^{n+8} + t^{-n-8} - (t^{n+7} + t^{-n-7}) + t^{n+4} + t^{-n-4} - (t^{n+3} + t^{-n-3}) \\ + (t^{n+2} + t^{-n-2}) - (t^{n+1} + t^{-n-1}) + (t^{n-1} + t^{-n+1}) - (t^{n-2} + t^{-n+2})\end{aligned}$$

and

$$\psi_n^2 = t^{-n-8}(t^{11} - t^{12} + \cdots - t^{2n+4} + t^{2n+5}).$$

It is now convenient to consider both functions on $[0, 2\pi]$. To this end, write $t = e^{iu}$, and set

$$\alpha_n(u) = \tfrac{1}{2}\psi_n^1(e^{iu}), \quad \beta_n(u) = \tfrac{1}{2}\psi_n^2(e^{iu}).$$

We first deal with α_n. We have

$$\alpha_n(u) = \cos(n+8)\,u - \cos(n+7)\,u + \cos(n+4)\,u - \cos(n+3)\,u \\ + \cos(n+2)\,u - \cos(n+1)\,u + \cos(n-1)\,u - \cos(n-2)\,u.$$

From the cosine sum formula we get

$$\alpha_n(u) = -2\sin\frac{u}{2}\Big(\sin\frac{2n+15}{2}u + \sin\frac{2n+7}{2}u + \sin\frac{2n+3}{2}u + \sin\frac{2n-3}{2}u\Big).$$

Applying the sine sum formula we obtain

$$\alpha_n(u) = -4\sin\frac{u}{2}\Big(\sin\big(n+\tfrac{11}{2}\big)\,u\cos 2u + \sin nu\cos\tfrac{3}{2}u\Big). \tag{6.4}$$

The sign of the function α_n near 0 can be examined using (6.4). Indeed, the sine function is positive on $(0, \pi)$, and hence $\sin\big(n+\frac{11}{2}\big)u$ is positive on $\big(0, \frac{2\pi}{2n+11}\big)$. On that interval, also $\cos 2u$, $\sin nu$ and $\cos\frac{3}{2}u$ are positive. That is,

$$\alpha_n(u) < 0 \quad \text{for } u \in \Big(0, \frac{2\pi}{2n+11}\Big).$$

As for the expression ψ_n^2, we note that

$$\psi_n^2 = t^{-n-8}\frac{t^{2n+6}+t^{11}}{t+1} = \frac{t^{n-5/2}+t^{-n+5/2}}{t^{1/2}+t^{-1/2}}.$$

Hence

$$\beta_n(u) = \frac{\cos(n-5/2)\,u}{2\cos(u/2)}.$$

Then, $\beta_n(0) = \frac{1}{2}$ and $\beta_n\big(\frac{\pi}{2n-5}\big) = 0$. We have assumed that $n \geq 11$. This implies that $\frac{\pi}{2n-5} \in \big(0, \frac{2\pi}{2n+11}\big)$. Define $\gamma_n = \alpha_n + \beta_n$, so that $\gamma_n(u) = \frac{1}{2}\psi_n(e^{iu})$. We have

$$\gamma_n(0) = \tfrac{1}{2}, \quad \gamma_n\Big(\frac{\pi}{2n-5}\Big) < 0.$$

Conversely, γ_n changes sign on the interval $\big(0, \frac{\pi}{2n-5}\big)$. Let u_n be the smallest positive zero of γ_n of odd multiplicity. Note that e^{iu_n} is a root of the Alexander polynomial.

Remark 6.5. Computer experiments suggest that there is only one zero of γ_n in that interval and that this zero is simple. We will not need this in the proof.

We can now define an infinite increasing sequence a_m such that K_{a_m} are linearly independent. To this end set $a_1 = 11$. Suppose $a_1, \dots, a_m$ are already defined. As the function $u \mapsto \gamma_{a_m}(u)$ is continuous, and $\gamma_{a_m}(0) = 1$, there is $\lambda_m > 0$ such that $\gamma_{a_m}(u) > 0$ for all $u \in (0, \lambda_m)$. We choose λ_m in such a way that λ_m form a decreasing sequence of real positive numbers.

Positivity of γ_{a_m} implies in particular that there are no jumps of the signature function of K_{a_m}, i.e., the signature jumps at all values of e^{iu} for $u \in [0, \lambda_m)$ are zero.

Choose a_{m+1} by the condition that $\frac{\pi}{2a_{m+1}-5} < \lambda_m$. This means that $u_{a_{m+1}} \in (0, \lambda_m)$. Then, $\gamma_{a_{m+1}}$ vanishes on $u_{a_{m+1}}$, but for all $j \leq m$, γ_{a_m} is positive near $u_{a_{m+1}}$. In particular, the signature jump at $e^{iu_{m+1}}$ for $K_{a_{m+1}}$ is not zero. That is, the map Ψ_{m+1} assigning to a knot half its signature jump at $e^{ia_{m+1}}$ is a homomorphism from the topological concordance group to $\mathbb{Z}$ that vanishes on the subgroup spanned by $K_{a_1}, \dots, K_{a_m}$, and is not zero on $K_{a_{m+1}}$.

The maps $\Psi_1, \dots,$ define an isomorphism between the subgroup spanned by $K_{a_1}, \dots,$ and $\mathbb{Z}^\infty$. □

We conclude the section by the following statement.

Lemma 6.6. *Suppose that Υ' is the Υ function for the positive trefoil. Then,* $\Upsilon_{K_{n+1}} = \Upsilon_{K_n} + \Upsilon'$.

Proof. Follows immediately from Lemma 4.5. □

Corollary 6.7. *The Υ function alone is not sufficient to show that K_n are independent in the concordance group modulo the subgroup generated by the algebraic knots.*

We remark that the signature jumps at $\zeta = e^{2\pi i/6}$ show that the trefoil does not belong to the subgroup spanned by all the K_{a_i}, not even to the subgroup spanned by all the K_n. Note that Lemma 4.2 computes the Alexander polynomial of K_n from the Alexander polynomial Δ_L, which is common for all n. It follows that the value of the symmetrized Alexander polynomial of K_n, $\psi_n(\zeta)$, depends only on n mod 6. For $n = 1, \dots, 6$, we can show that $\psi_n(\zeta) \neq 0$ by direct computations. Hence, none of the ψ_n vanishes on ζ. The signature jump at ζ is equal to zero for all the K_n, but it is not zero for the trefoil.

We do not continue this argument, because these methods alone are insufficient to prove independence of K_n modulo the subgroup generated by the algebraic knots. In fact, there exist linear combinations of algebraic knots with vanishing signature jumps; see [19].

7. Specific knots

In [1], Baker and Kegel shown a list of 632 hyperbolic L-space knots from the SnapPy census. For all of them, we have computed the Alexander polynomial using SnapPy [11], and by a simple algorithm we have determined the Υ function. The expression $-3\int\Upsilon$ turned out to be nonintegral for 96 knots, with the denominators in the set {3, 5, 7, 10, 11, 14, 15, 21, 30, 35, 42, 70, 105, 385} (see below).

$m211$	$\frac{117}{5}$	$s560$	$\frac{192}{5}$	$v0319$	$\frac{292}{5}$	$v0545$	$\frac{173}{5}$
$v0830$	$\frac{237}{5}$	$v1359$	$\frac{1874}{35}$	$v1423$	$\frac{326}{7}$	$v1565$	$\frac{267}{5}$
$v2900$	$\frac{389}{7}$	$v3070$	$\frac{445}{7}$	$v3335$	$\frac{188}{5}$	$t00621$	$\frac{467}{5}$
$t01966$	$\frac{357}{5}$	$t03106$	$\frac{999}{14}$	$t03710$	$\frac{342}{5}$	$t03843$	$\frac{293}{5}$
$t04927$	$\frac{571}{7}$	$t06246$	$\frac{3554}{35}$	$t06637$	$\frac{5987}{70}$	$t06957$	$\frac{2068}{21}$
$t08114$	$\frac{148}{5}$	$t08184$	$\frac{3099}{35}$	$t08936$	$\frac{1979}{35}$	$t09284$	$\frac{132}{5}$
$t09633$	$\frac{1566}{35}$	$t09882$	$\frac{108}{5}$	$t10177$	$\frac{690}{7}$	$t11887$	$\frac{308}{5}$
$t12288$	$\frac{634}{7}$	$t12533$	$\frac{157}{5}$	$o9\_01175$	$\frac{642}{5}$	$o9\_02383$	$\frac{413}{5}$
$o9\_02909$	$\frac{334}{7}$	$o9\_04054$	$\frac{348}{5}$	$o9\_04060$	$\frac{477}{5}$	$o9\_07044$	$\frac{4674}{35}$
$o9\_07152$	$\frac{687}{5}$	$o9\_07401$	$\frac{767}{7}$	$o9\_08402$	$\frac{417}{5}$	$o9\_09271$	$\frac{233}{3}$
$o9\_09731$	$\frac{867}{14}$	$o9\_10192$	$\frac{12346}{105}$	$o9\_10213$	$\frac{3727}{42}$	$o9\_11556$	$\frac{592}{5}$
$o9\_11658$	$\frac{816}{7}$	$o9\_12079$	$\frac{787}{5}$	$o9\_12253$	$\frac{413}{5}$	$o9\_12477$	$\frac{1881}{14}$
$o9\_13054$	$\frac{662}{7}$	$o9\_16431$	$\frac{5234}{35}$	$o9\_17382$	$\frac{637}{5}$	$o9\_19247$	$\frac{1067}{10}$
$o9\_19645$	$\frac{517}{5}$	$o9\_20029$	$\frac{1082}{7}$	$o9\_21620$	$\frac{1671}{14}$	$o9\_22252$	$\frac{1138}{7}$
$o9\_23032$	$\frac{1769}{35}$	$o9\_23461$	$\frac{6837}{70}$	$o9\_23723$	$\frac{4324}{35}$	$o9\_24069$	$\frac{9347}{70}$
$o9\_24126$	$\frac{59548}{385}$	$o9\_24407$	$\frac{3293}{30}$	$o9\_24946$	$\frac{268}{5}$	$o9\_25110$	$\frac{2005}{21}$
$o9\_27371$	$\frac{935}{7}$	$o9\_27767$	$\frac{15704}{105}$	$o9\_28751$	$\frac{3557}{30}$	$o9\_29551$	$\frac{1691}{15}$
$o9\_29648$	$\frac{725}{7}$	$o9\_30142$	$\frac{1570}{11}$	$o9\_31440$	$\frac{503}{5}$	$o9\_32065$	$\frac{7047}{70}$
$o9\_32314$	$\frac{1041}{14}$	$o9\_33380$	$\frac{252}{5}$	$o9\_33430$	$\frac{312}{5}$	$o9\_33486$	$\frac{3391}{21}$
$o9\_33801$	$\frac{3204}{35}$	$o9\_33959$	$\frac{6477}{70}$	$o9\_34689$	$\frac{428}{5}$	$o9\_35720$	$\frac{363}{5}$
$o9\_36380$	$\frac{1482}{11}$	$o9\_36544$	$\frac{2579}{35}$	$o9\_37482$	$\frac{849}{10}$	$o9\_37551$	$\frac{228}{5}$
$o9\_38287$	$\frac{963}{14}$	$o9\_38679$	$\frac{879}{7}$	$o9\_39162$	$\frac{821}{14}$	$o9\_39859$	$\frac{2159}{35}$
$o9\_40026$	$\frac{1656}{35}$	$o9\_40363$	$\frac{5427}{70}$	$o9\_40487$	$\frac{173}{5}$	$o9\_42493$	$\frac{2684}{35}$
$o9\_42675$	$\frac{203}{5}$	$o9\_42961$	$\frac{355}{7}$	$o9\_43750$	$\frac{357}{5}$	$o9\_43857$	$\frac{2269}{35}$

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WEIGHTED TOTAL VARIATION MINIMIZATION PROBLEM WITH MIXED DIRICHLET–NEUMANN BOUNDARY CONDITIONS

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We study the problem of minimizing the weighted total variation of a normalized BV function u plus a penalization on the weighted L^1 norm of the trace of u on the Neumann part Γ of the boundary, while assuming a Dirichlet condition $u = 0$ on the complement part $\Gamma^c \subset \partial\Omega$. We show that this problem is a relaxation of some shape optimization problem of type *Cheeger*, that is, both problems have the same minimum. Then, we prove that the level sets of minimizers are optimal sets. Finally, we study the regularity as well as some properties of these optimal sets.

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^N$ and Γ is an open subset of $\partial\Omega$. Let ϕ and ψ be two nonnegative functions over $\overline{\Omega}$ and w be defined on Γ. Then, we are interested in studying the following minimization problem:

$$\inf\left\{ \frac{\int_\Omega \psi|Du| - \int_\Gamma w|u|}{\int_\Omega \phi|u|} : u \neq 0 \in \mathrm{BV}(\Omega),\ u = 0 \text{ on } \Gamma^c := \partial\Omega\backslash\Gamma \right\}. \tag{1-1}$$

The case when $\Gamma = \varnothing$ has been already considered in [6]. Moreover, the authors in [16] have also studied problem (1-1) but in the case where $\Gamma \subset \partial\Omega$ and $w = 0$ on Γ. The interest in studying problem (1-1) is motivated by a landslide model (see [10]) in which ϕ and ψ represent the body forces and the (inhomogeneous) yield limit distribution, respectively. When $\phi = \psi = 1$ (which is not a relevant assumption in landslides modeling) and $\Gamma = \varnothing$, the infimum in (1-1) can be restricted to characteristic functions $u = \chi_A$ and so, we get

$$\min\left\{ \frac{\mathrm{Per}(A)}{|A|} : A \subset \Omega \right\}, \tag{1-2}$$

where $\mathrm{Per}(A)$ denotes the perimeter of the set A in $\mathbb{R}^N$ in the sense of De Giorgi (see [2]). This problem is known as Cheeger's problem [9], its value $\lambda(\Omega)$ is called

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the Cheeger constant of Ω and its minimizers are called Cheeger sets of Ω (see also [17; 18]). Moreover, $\lambda(\Omega)$ is the first eigenvalue of the 1-Laplacian on Ω [12; 13]. We note that the existence of an optimal set $A^\star$ in problem (1-2) is very simple and it follows from the direct method in calculus of variations. In the case where the densities ϕ and ψ are not uniform, but $w = 0$ on $\Gamma \subset \partial\Omega$, problem (1-1) will be the relaxation of the following problem:

$$\min\left\{\frac{\int_{\partial A\setminus\Gamma}\psi}{\int_A \phi} : A \subset \Omega\right\}. \tag{1-3}$$

This can be seen as a generalization of the Cheeger problem (1-2). However, up to our knowledge, the case when $w \neq 0$ has not been studied before in the literature. In this paper, we will show that problem (1-1) is equivalent to the following generalization of (1-3):

$$\min\left\{\frac{\int_{\partial A\setminus\Gamma}\psi - \int_{\partial A\cap\Gamma} w}{\int_A \phi} : A \subset \Omega\right\}. \tag{1-4}$$

We note that the existence of an optimal set to problem (1-4) is not guaranteed here for an arbitrary weight w on Γ. Indeed, the functional in (1-4) is not a priori lower semicontinuous with respect to the weak* convergence in BV. Thus, this additional difficulty imposed by the presence of a density $w \neq 0$ was the main motivation to write the present paper.

Inspired by [15], one can see that the variational formulation of the stationary antiplane flow of an inhomogeneous Bingham (rigid viscoplastic) fluid can be stated as follows: there is a function $u \in H^1(\Omega)$ with $u = 0$ on Γ^c such that

$$\int_\Omega \nabla u\cdot\nabla(v-u) + \int_\Omega \psi|\nabla v| - \int_\Omega \psi|\nabla u| \geq \int_\Omega \phi(v-u) + \int_\Gamma w(v-u) \tag{1-5}$$

for all $v \in H^1(\Omega)$ such that $v = 0$ on Γ^c. Assuming $N = 2$ the velocity field in the domain $D = \Omega\times\mathbb{R} \subset \mathbb{R}^3$ is given by $\boldsymbol{u} = (0, 0, u)$ with $u = u(x_1, x_2)$. The viscosity distribution is equal to 1; ψ is the yield limit distribution, ϕ is the body forces in the x_3 direction and, w is an additional force acting on the Neumann part Γ. A particular case of the Bingham model lies in the presence of rigid zones located in the interior of the flow of the Bingham solid/fluid. As the yield limit ψ increases, these rigid zones become larger and may completely block the flow so that $u = 0$ is the solution of (1-5). Conversely, the Bingham fluid is blocked if and only if

$$\int_\Omega \psi|\nabla v| - \int_\Gamma wv \geq \int_\Omega \phi v \quad \text{for all } v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma^c. \tag{1-6}$$

When considering oil transport in pipelines, in the process of oil drilling or in the case of metal forming, the blocking of the solid/fluid is a catastrophic event to be avoided. From (1-6), one can see the infimum in (1-1) as a safety coefficient.

In other words, the Bingham fluid is not blocked if and only if inf (1-1) < 1. In a completely opposite context, when modeling landslides, the solid is blocked in its natural configuration and the beginning of a flow can be seen as a disaster. Here, the $1/\inf$ (1-1) appears as a safety coefficient.

Notice that problem (1-1) can be seen as a study of the "eigenvalue problem" for the following degenerate inhomogeneous equation with mixed Dirichlet–Neumann boundary conditions (where the first eigenvalue is $\lambda^\star := \inf$ (1-1)):

$$\begin{cases} -\nabla\cdot\left[\psi\frac{\nabla u}{|\nabla u|}\right]=\lambda\phi & \text{in } \Omega,\\ \psi\left[\frac{\nabla u}{|\nabla u|}\cdot\boldsymbol{n}\right]=w & \text{on } \Gamma,\\ u=0 & \text{on } \Gamma^c.\end{cases}$$

On the other hand, the properties of Cheeger sets (i.e., optimal sets in (1-2)) have been studied in several papers (see [1; 8]). One of the very important results concerning the regularity of Cheeger sets, is that the internal boundary of Cheeger sets have constant curvature. In [16], the authors have also generalized some of these properties to optimal sets of the generalized Cheeger problem (1-3). More precisely, they show that the curvature of the boundary of any optimal set $A^\star$ at any point x in the interior of Ω is given by

$$\kappa(x)=\frac{\lambda^\star\phi(x)+\partial_{\boldsymbol{n}}\psi(x)}{\psi(x)},$$

where $\partial_{\boldsymbol{n}}\psi(x)$ is the inward normal derivative on $\partial A^\star$ at x (so, ψ should be at least of class C^1). Moreover, if $\partial A^\star$ crosses Γ at some point x where Γ is C^1 around x, then the tangent line to $\partial A^\star$ at x must be orthogonal to Γ.

This paper is organized as follows. In Section 2, we will show that problems (1-1) and (1-4) have the same minimal value and that each of these two problems has a solution. More precisely, we will show that from a minimizer of (1-1) one can construct an optimal set of (1-4) simply by considering its superlevel sets. Moreover, we will study in Section 3 the regularity properties of these optimal sets. Finally, we conclude the paper by some examples in Section 4.

2. Existence of solutions

Throughout this section, we assume that $\Omega\subset\mathbb{R}^N$ is an open bounded connected domain with Lipschitz boundary, $\phi(x)\geq\phi_0>0$ is a bounded function and, $\psi(x)\geq\psi_0>0$ is a continuous function on $\overline{\Omega}$ (where $\psi_0,\phi_0\in\mathbb{R}^+$ are fixed). Let Γ be a closed subset of $\partial\Omega$ and w be a bounded function on Γ. Then, we consider the minimization problem

$$\inf\left\{\frac{\int_\Omega\psi|Du|-\int_\Gamma w|u|}{\int_\Omega\phi|u|} : u\in\mathrm{BV}(\Omega),\ u=0 \text{ on } \Gamma^c\right\}. \tag{2-1}$$

We recall that proving existence of a minimizer for problem (2-1) is a difficult task due to different facts. First, we do not have a priori compactness: if $(u_n)_n$ is a minimizing sequence then it is not clear if one can extract a subsequence converging weakly* in $\mathrm{BV}(\Omega)$ and even so (i.e., assuming that $u_n \rightharpoonup^\star u$ in $\mathrm{BV}(\Omega)$), since the trace map is not lower semicontinuous with respect to this topology then it is not true in general that $u_n \rightharpoonup u$ in $L^1(\partial\Omega)$ and so, we do not know whether the limit function u satisfies the Dirichlet condition $u = 0$ on Γ^c or not. In particular, it is possible that a solution to this problem (2-1) does not exist! So, the idea is to relax the boundary condition $u = 0$ on Γ^c by adding a penalty term in the functional; this is a classical tool in the theory of Calculus of Variations and it has also been used to prove existence of a solution to the BV least gradient problem (see [20]).

Let $\tilde{\Omega}$ be an open bounded Lipschitz extension of Ω such that $\Gamma \subset \partial\tilde{\Omega}$ and $\Gamma^c \subset \tilde{\Omega}$. Then, we consider the following relaxation of (2-1):

$$\inf\left\{ \frac{\int_{\tilde{\Omega}} \psi|Du| - \int_\Gamma w|u|}{\int_{\tilde{\Omega}} \phi|u|} : u \in \mathrm{BV}(\tilde{\Omega}),\ u = 0 \text{ on } \tilde{\Omega}\backslash\Omega \right\}. \tag{2-2}$$

Note that $\int_{\tilde{\Omega}} \psi|Du| = \int_\Omega \psi|Du| + \int_{\Gamma^c} \psi|u|$. But again, it is not easy to show existence of a solution to the relaxed version (2-2) since in general the map $u \mapsto -\int_\Gamma w|u|$ is not lower semicontinuous with respect to the weak* convergence in $\mathrm{BV}(\tilde{\Omega})$. More precisely, we will show that the lower semicontinuity of the functional in (2-2) depends on the L^∞-bounds of ψ and w as well as the regularity of the Neumann part Γ. To motivate this fact, we consider the following examples.

Example 2.1. Let $\Omega =]0,1[^2$, $\Gamma = (\{0\}\times[0,1]) \cup ([0,1]\times\{0\})$, $\psi = \psi_0 > 0$ and $w = w_0 \in \mathbb{R}$. Set $u_n(x_1,x_2) = n \cdot \chi_{E_n}$ where $E_n := \left\{(x_1,x_2) \in \Omega : x_1 + x_2 \le \frac{1}{n}\right\}$. Then, it is clear that $u_n \rightharpoonup^\star 0$ in $\mathrm{BV}(\Omega)$. However,

$$\int_{\tilde{\Omega}} \psi|Du_n| - \int_\Gamma w|u_n| = \sqrt{2}\psi_0 - 2w_0 < 0$$

as soon as $\frac{w_0}{\psi_0} > \frac{\sqrt{2}}{2}$.

Example 2.2. Assume that $\Omega = B(0,1)$, Γ is a smooth arc of $\partial\Omega$, $\psi = \psi_0 > 0$ and $w = w_0 \in \mathbb{R}$. Take $u_n(x) = \min\{|x|, (n-1)(1-|x|)\}$. Then, it is clear that $u_n \rightharpoonup^\star u := |x|$ in $\mathrm{BV}(\Omega)$. But, we have

$$\int_{\tilde{\Omega}} \psi|Du_n| - \int_\Gamma w|u_n| = \pi\psi_0\left[\left(1-\frac{1}{n}\right)^2 + (n-1)\left(1-\left(1-\frac{1}{n}\right)^2\right)\right]$$
$$\longrightarrow\ 3\pi\psi_0 < \int_{\tilde{\Omega}} \psi|Du| - \int_\Gamma w|u| = \psi_0(\pi + \mathcal{H}^1(\Gamma^c)) - w_0\mathcal{H}^1(\Gamma)$$

as soon as $\frac{w_0}{\psi_0} < -1$.

We start by showing that problems (2-2) and (2-1) are completely equivalent.

Proposition 2.3. *Problems (2-1) and (2-2) have the same minimal value. If u is a solution for problem (2-1), then u solves problem (2-2). In addition, if u is a solution for problem (2-2) with $u = 0$ on Γ^c then u solves problem (2-1).*

Proof. It is obvious that $\inf(2\text{-}2) \le \inf(2\text{-}1)$. On the other hand, fix $u \in BV(\tilde{\Omega})$ such that $u = 0$ on $\tilde{\Omega} \backslash \Omega$. For every $n \in \mathbb{N}^\star$, let η_n be a cutoff function such that $0 \le \eta_n \le 1$, $\eta_n(x) = 0$ on Γ and, $\eta_n(x) = 1$ for all $x \in \Omega$ with $\operatorname{dist}(x, \Gamma^c) > 1/n$. Now, set $u_n = \eta_n u$. Then, we have $u_n = 0$ on Γ^c. In addition, it is clear that $u_n \to u$ in $L^1(\Omega)$ and so, $\int_\Omega \phi |u_n| \to \int_\Omega \phi |u|$. Moreover, one has

$$\int_\Omega \psi |Du_n| = \int_\Omega \psi |\eta_n Du + u D\eta_n| \le \int_\Omega \psi \eta_n |Du| + \int_\Omega \psi |D\eta_n|\,|u| \longrightarrow \int_\Omega \psi |Du| + \int_{\Gamma^c} \psi |u|\, d\mathcal{H}^{N-1}. \tag{2-3}$$

Yet, we also have

$$\int_\Gamma w |u_n| = \int_\Gamma w \eta_n |u| \longrightarrow \int_\Gamma w |u|.$$

Hence,

$$\lim_n \left[\int_\Omega \psi |Du_n| - \int_\Gamma w |u_n| \right] \le \int_\Omega \psi |Du| + \int_{\Gamma^c} \psi |u| - \int_\Gamma w |u|.$$

Finally, we get that

$$\inf(2\text{-}1) \le \lim_n \left[\frac{\int_\Omega \psi |Du_n| - \int_\Gamma w |u_n|}{\int_\Omega \phi |u_n|} \right] \le \frac{\int_{\tilde{\Omega}} \psi |Du| - \int_\Gamma w |u|}{\int_{\tilde{\Omega}} \phi |u|}.$$

Since u is arbitrary, then we infer that $\inf(2\text{-}1) \le \inf(2\text{-}2)$. Consequently, equality $\inf(2\text{-}1) = \inf(2\text{-}2)$ holds. The rest follows immediately from this equality. □

Remark 2.4. We clearly see that one can restrict problem (2-2) to nonnegative functions (just by replacing u with $|u|$) and so,

$$\min(2\text{-}2) = \min\left\{ \frac{\int_{\tilde{\Omega}} \psi |Du| - \int_\Gamma w u}{\int_{\tilde{\Omega}} \phi u} : u \ne 0 \in BV(\tilde{\Omega}),\ u \ge 0,\ u = 0 \text{ on } \tilde{\Omega} \backslash \Omega \right\}.$$

In the sequel, we will only consider nonnegative solutions to (2-2).

In order to prove existence of a solution to the relaxed problem (2-2), we need first to introduce the following constant:

$$\Lambda^\star := \sup\left\{ \frac{\int_\Gamma w |u|}{\int_\Omega \psi |Du|} : u \ne 0 \in BV(\Omega),\ u = 0 \text{ on } \Gamma^c \right\}. \tag{2-4}$$

Also, the analysis will be performed under the following geometric assumption.

Definition 2.5. Suppose that Γ is of class C^1. We say that Ω satisfies a C^1-extension property near Γ^c if there exists an open bounded set $\tilde{\Omega}$ with C^1 boundary such that $\Omega \subset \tilde{\Omega}$ and $\partial\Omega \cap \partial\tilde{\Omega} = \Gamma$.

Then, we have the following existence result.

Proposition 2.6. *Assume Γ is C^1, Ω satisfies a C^1-extension property near Γ^c, $\|w\|_\infty \le \psi_0$ and, $\Lambda^\star < 1$. Then, problem (2-2) reaches a minimum.*

Proof. Let $(u_n)_n$ be a minimizing sequence in problem (2-2). For every $n \in \mathbb{N}$, set $\tilde{u}_n := u_n / \int_{\tilde{\Omega}} \phi |u_n|$. So, it is clear that $(\tilde{u}_n)_n$ is also a minimizing sequence. In particular, there is a constant $C < \infty$ such that

$$\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w |\tilde{u}_n| \le C \quad \text{for all } n \in \mathbb{N}.$$

Hence,

$$(1-\Lambda^\star) \int_{\tilde{\Omega}} \psi |D\tilde{u}_n| \le \int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w |\tilde{u}_n| \le C \quad \text{for all } n \in \mathbb{N}.$$

Since $\Lambda^\star < 1$ and $\psi \ge \psi_0 > 0$, then we get that

$$\int_{\tilde{\Omega}} |D\tilde{u}_n| \le C \quad \text{for all } n \in \mathbb{N}. \tag{2-5}$$

Yet, we have $\|\phi \tilde{u}_n\|_{L^1} = 1$ and $\phi \ge \phi_0 > 0$. Hence, up to a subsequence, $\tilde{u}_n$ converges weakly* in $\mathrm{BV}(\tilde{\Omega})$ to some function $\tilde{u}$. In particular, $\tilde{u}_n \to \tilde{u}$ strongly in $L^1(\tilde{\Omega})$. This implies that $\|\phi \tilde{u}_n\|_{L^1} \to \|\phi \tilde{u}\|_{L^1} = 1$ and $\tilde{u} = 0$ on $\tilde{\Omega} \backslash \Omega$.

On the other side, inspired by [22, Proposition 1.2], we also claim that the functional in (2-2) is lower semicontinuous with respect to the weak* convergence in $\mathrm{BV}(\tilde{\Omega})$ and so, $\tilde{u}$ is a minimizer in (2-2). First, we clearly have

$$\begin{aligned} \int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| - \int_{\tilde{\Omega}} \psi |D\tilde{u}| + \int_\Gamma w |\tilde{u}| \\ \ge \int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_{\tilde{\Omega}} \psi |D\tilde{u}| - \|w\|_\infty \int_\Gamma |\tilde{u}_n - \tilde{u}|. \end{aligned}$$

Fix $\varepsilon > 0$. Then, we define $A_\varepsilon := \{x \in \tilde{\Omega} : d(x, \partial\tilde{\Omega}) \le \varepsilon\}$. Let $\eta_\varepsilon \in C_0^\infty(\tilde{\Omega})$ be a cutoff function such that $0 \le \eta_\varepsilon \le 1$ and $\eta_\varepsilon = 1$ on $\tilde{\Omega}_\varepsilon := \tilde{\Omega} \backslash A_\varepsilon$. Set $v_{\varepsilon,n} := (1-\eta_\varepsilon)(\tilde{u}_n - \tilde{u})$. By the trace inequality for BV functions (see [3]), there are two constants c_1 and c_2 such that

$$\int_{\partial\Omega} |v_{\varepsilon,n}| \le c_1 \int_{A_\varepsilon} |Dv_{\varepsilon,n}| + c_2 \int_{A_\varepsilon} |v_{\varepsilon,n}|.$$

Thus, we get that

$$\begin{aligned} \int_\Gamma |\tilde{u}_n - \tilde{u}| & \\ \le c_1 \int_{A_\varepsilon} (1-\eta_\varepsilon) |D(\tilde{u}_n - \tilde{u})| &+ c_1 \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}|\, |D\eta_\varepsilon| + c_2 \int_{A_\varepsilon} (1-\eta_\varepsilon) |\tilde{u}_n - \tilde{u}| \\ \le \frac{c_1}{\psi_0} \int_{A_\varepsilon} \psi |D(\tilde{u}_n - \tilde{u})| &+ \frac{C}{\varepsilon} \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}|, \end{aligned} \tag{2-6}$$

where the constant C depends on c_2. Hence, one has

$$\begin{aligned}
&\int_{\tilde\Omega}\psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n| - \int_{\tilde\Omega}\psi|D\tilde u| + \int_\Gamma w|\tilde u| \\
&\geq \int_{\tilde\Omega}\psi|D\tilde u_n| - \int_{\tilde\Omega}\psi|D\tilde u| - \frac{c_1\|w\|_\infty}{\psi_0}\int_{A_\varepsilon}\psi|D(\tilde u_n-\tilde u)| - \frac{C\|w\|_\infty}{\varepsilon}\int_{A_\varepsilon}|\tilde u_n-\tilde u| \\
&\geq \int_{\tilde\Omega}\psi|D\tilde u_n| - \int_{\tilde\Omega}\psi|D\tilde u| - \frac{c_1\|w\|_\infty}{\psi_0}\int_{A_\varepsilon}\psi|D\tilde u_n| - \frac{c_1\|w\|_\infty}{\psi_0}\int_{A_\varepsilon}\psi|D\tilde u| \\
&\qquad\qquad - \frac{C\|w\|_\infty}{\varepsilon}\int_{A_\varepsilon}|\tilde u_n-\tilde u| \\
&\geq \int_{\tilde\Omega_\varepsilon}\psi|D\tilde u_n| - \int_{\tilde\Omega_\varepsilon}\psi|D\tilde u| + \left(1-\frac{c_1\|w\|_\infty}{\psi_0}\right)\int_{A_\varepsilon}\psi|D\tilde u_n| - \left(1+\frac{c_1\|w\|_\infty}{\psi_0}\right)\int_{A_\varepsilon}\psi|D\tilde u| \\
&\qquad\qquad - \frac{C\|w\|_\infty}{\varepsilon}\int_{A_\varepsilon}|\tilde u_n-\tilde u|.
\end{aligned}$$

Since Γ is C^1 and Ω satisfies a C^1-extension property near Γ^c, then the boundary of $\tilde\Omega$ is of class C^1 and so, thanks to [3, Theorem 4], one can assume that in (2-6) the constants $c_1 = 1+\delta$ and $c_2 = c_2(\Omega, \delta)$, where $\delta > 0$ can be chosen sufficiently small. Let $\|w\|_\infty < \psi_0$. Hence, choosing $\delta > 0$ small enough, we infer that

$$\begin{aligned}
&\int_{\tilde\Omega}\psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n| - \int_{\tilde\Omega}\psi|D\tilde u| + \int_\Gamma w|\tilde u| \\
&\qquad \geq \int_{\tilde\Omega_\varepsilon}\psi|D\tilde u_n| - \int_{\tilde\Omega_\varepsilon}\psi|D\tilde u| - (2+\delta)\int_{A_\varepsilon}\psi|D\tilde u| - \frac{C\|w\|_\infty}{\varepsilon}\int_{A_\varepsilon}|\tilde u_n-\tilde u|.
\end{aligned}$$

Passing to the limit when $n\to\infty$ and using the lower semicontinuity of the weighted total variation (see [6, Corollary 1])

$$\liminf_n \int_{\tilde\Omega_\varepsilon}\psi|D\tilde u_n| \geq \int_{\tilde\Omega_\varepsilon}\psi|D\tilde u|$$

as well as the L^1 convergence, we get

$$\liminf_n\left[\int_{\tilde\Omega}\psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n|\right] - \int_{\tilde\Omega}\psi|D\tilde u| + \int_\Gamma w|\tilde u| \geq -(2+\delta)\int_{A_\varepsilon}\psi|D\tilde u|.$$

Let $\varepsilon\to 0^+$, this yields that

$$\liminf_n\left[\int_{\tilde\Omega}\psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n|\right] - \int_{\tilde\Omega}\psi|D\tilde u| + \int_\Gamma w|\tilde u| \geq 0.$$

Finally, assume that $\|w\|_\infty = \psi_0$. So, we will prove lower semicontinuity of the functional in (2-2) by approximation. More precisely, fix $\zeta > 0$ small enough. Then,

$$\int_{\tilde\Omega}\psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n| = \int_{\tilde\Omega}\psi|D\tilde u_n| - (1-\zeta)\int_\Gamma w|\tilde u_n| - \zeta\int_\Gamma w|\tilde u_n|.$$

Recalling (2-5) and the fact that $\Lambda^\star < 1$, we infer that

$$\liminf_n \left[\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n|\right] \geq \liminf_n \left[\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - (1-\zeta)\int_\Gamma w|\tilde{u}_n|\right] - C\zeta$$
$$\geq \int_{\tilde{\Omega}} \psi |D\tilde{u}| - (1-\zeta)\int_\Gamma w|\tilde{u}| - C\zeta.$$

Since $\zeta > 0$ is arbitrarily small, then this concludes the proof of our claim. □

Remark 2.7. If $w=0$ then the C^1 regularity of Γ is not needed and, the existence of a solution to problem (2-2) is trivial in this case. In Example 2.1, take $w_0 = \psi_0 = 1$, then the condition $\|w\|_\infty \leq \psi_0$ is well satisfied but, the functional

$$u \mapsto \int_{\tilde{\Omega}} \psi |Du| - \int_\Gamma w|u|$$

is not lower semicontinuous due to the lack of C^1-regularity of the arc Γ. However, in Example 2.2, the arc Γ is smooth but the functional is always not lower semicontinuous provided that $w_0 < -\psi_0$. This shows the necessity of the assumptions we made in Proposition 2.6.

Remark 2.8. Although problem (2-2) has a solution u but it is still not clear whether this solution solves problem (2-1), or equivalently if this solution u satisfies the Dirichlet condition ($u = 0$ on Γ^c). In fact, we will see that this is not necessarily the case and, a solution to (2-1) may not exist.

On the other hand, one can also study the summability of a solution u in (2-2). Inspired by the proof of [13, Proposition 7] (see also [6, Theorem 4]), one can show that any solution of (2-2) must be bounded. For this aim, we start by the following.

Proposition 2.9. *Let H be a Lipschitz nondecreasing function on $\mathbb{R}_+$ with $H(0)=0$. For any nonnegative solution u of problem* (2-2), *the function $H(u)$ is also a solution for* (2-2).

Proof. This proof follows the lines of the proof of [6, Proposition 1]. First, let us assume that H is smooth. Then, we consider the Cauchy problem:

$$\begin{cases} \partial_t y(t,v) = -H(y(t,v)), & t \geq 0, \\ y(0,v) = v. \end{cases} \tag{2-7}$$

Let $y(t,v)$ be the solution of (2-7). Thanks to our assumptions on H, $y(t,v)$ is smooth. For every $t \geq 0$, we define $u_t = y(t,u)$ (so, we have $u_0 = u$). Now, we consider the map

$$h(t) = \int_{\tilde{\Omega}} \psi |Du_t| - \int_\Gamma w u_t - \lambda^\star \int_{\tilde{\Omega}} \phi u_t,$$

where we recall that $\lambda^\star = \min$ (2-2). Since u_0 is a minimizer for problem (2-2) and $u_t = 0$ on $\tilde{\Omega}\backslash\Omega$ for every $t \geq 0$ (this follows from the fact that $y(t,0) = 0$

and the uniqueness of the solution in (2-7)), then h has a minimum at $t=0$. In particular, we have

$$\lim_{t\to 0^+}\frac{h(t)-h(0)}{t}\ge 0.$$

Yet,

$$\frac{h(t)-h(0)}{t}=\int_{\tilde\Omega}\psi\,\frac{|Du_t|-|Du_0|}{t}-\int_\Gamma w\,\frac{u_t-u_0}{t}-\lambda^\star\int_{\tilde\Omega}\phi\,\frac{u_t-u_0}{t}.$$

For every $x\in\tilde\Omega\cup\Gamma$, we have

$$\frac{u_t(x)-u_0(x)}{t}=\frac{y(t,u(x))-y(0,u(x))}{t}\quad\longrightarrow\quad -H(u(x)).$$

Taking the derivative with respect to v in (2-7), we get that

$$\begin{cases}\partial_t[\partial_v y(t,v)]=-H'(y(t,v))\,\partial_v y(t,v), & t\ge 0,\\ \partial_v y(0,v)=1.\end{cases}$$

Hence,

$$\partial_v y(t,v)=e^{-\int_0^t H'(y(s,v))\,\mathrm{d}s}\ge 0.$$

By the chain rule for BV functions (see [2]), we have

$$|Du_t|=\partial_v y(t,u)\,|\tilde Du|+[y(t,u^+)-y(t,u^-)]\cdot\mathcal{H}^{N-1}\llcorner J_u,$$

where u^+ and u^- are respectively the approximate upper and lower limits, J_u is the jump set of u, and the nonnegative measure $|\tilde Du|$ is the sum of the absolutely continuous part and the Cantor part of $|Du|$. Consequently, we have

$$\begin{aligned}\frac{|Du_t|-|Du_0|}{t}&=\frac{\partial_v y(t,u)-\partial_v y(0,u)}{t}|\tilde Du|\\ &\qquad+\frac{[y(t,u^+)-y(0,u^+)]-[y(t,u^-)-y(0,u^-)]}{t}\cdot\mathcal{H}^{N-1}\llcorner J_u\\ &\qquad\qquad\longrightarrow\quad -H'(u)|\tilde Du|-[H(u^+)-H(u^-)]\cdot\mathcal{H}^{N-1}\llcorner J_u.\end{aligned}$$

Therefore,

$$\int_{\tilde\Omega}\psi\,H'(u)|\tilde Du|+\int_{J_u}\psi[H(u^+)-H(u^-)]\,\mathcal{H}^{N-1}-\int_\Gamma wH(u)-\lambda^\star\int_{\tilde\Omega}\phi H(u)\le 0.$$

Since $H'\ge 0$ and $|D(H(u))|=H'(u)|\tilde Du|+[H(u^+)-H(u^-)]\cdot\mathcal{H}^{N-1}\llcorner J_u$, this yields that $H(u)$ also minimizes (2-2). Finally, it remains to extend the result to the case when H is not smooth; but this can be done by approximation. In fact, one can approximate H with a sequence of smooth Lipschitz increasing functions H_n with $H_n(0)=0$ such that $H_n(u)$ converges weakly* to $H(u)$ in $\mathrm{BV}(\tilde\Omega)$. Hence, $H_n(u)$ is a solution to (2-2) for every n. Yet, recalling the proof of Proposition 2.6, we know that the functional in (2-2) is lower semicontinuous with respect to the weak* convergence in $\mathrm{BV}(\tilde\Omega)$. This yields that $H(u)$ is also a solution. □

Under the assumptions of Proposition 2.6, we get the following summability result as a consequence of Proposition 2.9.

Proposition 2.10. *Let u be a solution for problem (2-2), then u belongs to $L^\infty(\Omega)$.*

Proof. Fix $M > 0$ large enough. Thanks to Proposition 2.9, $u_M := \min\{u, M\}$ is a solution for problem (2-2). Therefore, we have

$$\int_{\tilde{\Omega}} \psi |Du_M| - \int_\Gamma w u_M = \lambda^\star \int_{\tilde{\Omega}} \phi u_M \le \lambda^\star \|\phi\|_\infty \|u_M\|_1. \tag{2-8}$$

Since $\Lambda^\star < 1$ and $\psi \ge \psi_0 > 0$, then by (2-8) we get

$$\int_{\tilde{\Omega}} |Du_M| \le \frac{\lambda^\star \|\phi\|_\infty}{\psi_0(1-\Lambda^\star)} \|u_M\|_1. \tag{2-9}$$

Yet, one has

$$\|u_M\|_{\frac{N}{N-1}} \le C \int_{\tilde{\Omega}} |Du_M|.$$

Hence,

$$\|u_M\|_{\frac{N}{N-1}} \le C \|u_M\|_1. \tag{2-10}$$

But, it is clear that u_M^p is also a solution for problem (2-2) for all $p \ge 1$. Then, thanks to (2-10), we also have

$$\|u_M^p\|_{\frac{N}{N-1}} \le C \|u_M^p\|_1.$$

This yields that

$$\|u_M\|_{\frac{Np}{N-1}} \le C^{1/p} \|u_M\|_p.$$

Fix $n \in \mathbb{N}$. Then, by induction, we get that

$$\|u_M\|_{(\frac{N}{N-1})^n} \le C^{(\frac{N-1}{N})^{n-1}} \|u_M\|_{(\frac{N}{N-1})^{n-1}} \le C^{[(\frac{N-1}{N})^{n-1} + (\frac{N-1}{N})^{n-2} + \cdots + 1]} \|u_M\|_1.$$

Consequently,

$$\|u_M\|_{(\frac{N}{N-1})^n} \le C^{N[1-(\frac{N-1}{N})^n]} \|u\|_1 \quad \text{for all } n \in \mathbb{N}.$$

Passing to the limit when $n \to \infty$, this yields that

$$\|u_M\|_\infty \le C^N \|u\|_1. \tag{2-11}$$

Finally, letting $M \to \infty$ in (2-11), this concludes the proof that $u \in L^\infty(\Omega)$. □

In addition, one can show that problem (2-2) is also equivalent to a shape optimization problem of type *Cheeger* and that any superlevel set of a solution u is an optimal set (see [7; 16; 17] for similar level-sets approach for variational problems involving total variation minimization). More precisely, we introduce the following problem:

$$\min\left\{\frac{\mathrm{Per}_\psi(A)-\int_{\partial^\star A\cap\Gamma} w}{\int_A \phi} : A\subset\Omega\right\}, \tag{2-12}$$

where

$$\mathrm{Per}_\psi(A) := \int_{\tilde\Omega}\psi|D\chi_A| = \int_{\Omega\cup\Gamma^c}\psi|D\chi_A| = \int_{\partial^\star A\setminus\Gamma}\psi\,\mathrm{d}\mathcal{H}^{N-1}$$

is the weighted perimeter of A that is taken relative to $\tilde\Omega$ (or equivalently, relative to $\Omega\cup\Gamma^c$ since A is assumed to be a subset of Ω) and $\partial^\star A$ denotes the reduced boundary of A. Under the assumptions of Proposition 2.6, we have the following.

Proposition 2.11. *The values of problems (2-2) and (2-12) coincide (that is,* $\min$ (2-2) $=\min$ (2-12)*). In addition, a function u solves (2-2) if and only if the superlevel sets $A_t := \{u > t\}$ solve (2-12), for almost all $t\ge 0$. In particular, problem (2-12) admits an optimal set $A^\star$.*

Proof. By considering characteristic functions $u := \chi_A$ where $A\subset\Omega$ in (2-2), it is obvious that we get $\min$ (2-2) $\le\min$ (2-12). Now, let us show the reverse inequality. Fix $u\in\mathrm{BV}(\tilde\Omega)$ with $u\ge 0$ and $u = 0$ on $\tilde\Omega\setminus\Omega$. Using the coarea formula, we have

$$\begin{aligned}\int_{\tilde\Omega}\psi|Du| - \int_\Gamma wu &= \int_0^{+\infty}\int_{\partial^\star A_t\setminus\Gamma}\psi\,\mathrm{d}\mathcal{H}^{N-1} - \int_0^{+\infty}\int_{\partial^\star A_t\cap\Gamma} w\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t\\ &= \int_0^{+\infty}\frac{\mathrm{Per}_\psi(A_t)-\int_{\partial^\star A_t\cap\Gamma} w\,\mathrm{d}\mathcal{H}^{N-1}}{\int_{A_t}\phi}\left(\int_{A_t}\phi\right)\mathrm{d}t\\ &\ge \min\left\{\frac{\mathrm{Per}_\psi(A)-\int_{\partial^\star A\cap\Gamma} w}{\int_A\phi} : A\subset\Omega\right\}\int_0^{+\infty}\left(\int_{A_t}\phi\right)\mathrm{d}t \qquad (2\text{-}13)\\ &= \min\left\{\frac{\mathrm{Per}_\psi(A)-\int_{\partial^\star A\cap\Gamma} w}{\int_A\phi} : A\subset\Omega\right\}\int_{\tilde\Omega}\phi u\end{aligned}$$

Hence,

$$\min\,(2\text{-}2) \ge \min\left\{\frac{\mathrm{Per}_\psi(A)-\int_{\partial^\star A\cap\Gamma} w}{\int_A\phi} : A\subset\Omega\right\}.$$

This yields that $\min$ (2-2) $=\min$ (2-12). Moreover, if u is a minimizer in (2-2) then the inequality in (2-13) becomes equality. Yet, this means that for almost every $t\ge 0$, we have

$$\frac{\mathrm{Per}_\psi(A_t)-\int_{\partial^\star A_t\cap\Gamma} w}{\int_{A_t}\phi} = \min\left\{\frac{\mathrm{Per}_\psi(A)-\int_{\partial^\star A\cap\Gamma} w}{\int_A\phi} : A\subset\Omega\right\}.$$

Consequently, the superlevel sets $A_t = \{u > t\}$ solve (2-12), for almost every $t\ge 0$. The last statement follows directly from Proposition 2.6. □

Remark 2.12. In fact, one can show in Proposition 2.11 that for *every* $t \geq 0$, the superlevel set $A_t = \{u > t\}$ is optimal for problem (2-12). Indeed, let $(t_n)_n$ be a decreasing sequence such that $t_n \to t$ and A_{t_n} is optimal in (2-12) for all n. Recalling the estimate (2-9), we have

$$\mathrm{Per}_\psi(A_{t_n}) \leq \frac{\lambda^\star \|\phi\|_\infty}{\psi_0(1-\Lambda^\star)}|A_{t_n}| \leq \frac{\lambda^\star \|\phi\|_\infty}{\psi_0(1-\Lambda^\star)}|\Omega|.$$

Hence, $\chi_{A_{t_n}}$ is bounded in $\mathrm{BV}(\Omega)$ and so, up to a subsequence, $\chi_{A_{t_n}} \rightharpoonup^\star \chi_{A_t}$ in $\mathrm{BV}(\Omega)$. In particular, one has $\mathrm{Per}(A_t) < \infty$. Finally, the lower semicontinuity of the functional in (2-2) yields that χ_{A_t} is also a solution for problem (2-2).

Remark 2.13. Similarly to the proof of Proposition 2.11 about the equivalence between problems (2-2) and (2-12), one can show using the coarea formula that

$$\Lambda^\star = \sup\left\{\frac{\int_\Gamma w|u|}{\int_\Omega \psi|Du|} : u \neq 0 \in \mathrm{BV}(\Omega),\ u = 0 \text{ on } \Gamma^c\right\} = \sup\left\{\frac{\int_{\partial^\star A\cap\Gamma} w}{\mathrm{Per}_\psi(A)} : A \subset \Omega\right\}.$$

In particular, we have $\Lambda^\star < 1$ if and only if

$$\int_{\partial^\star A\cap\Gamma} w < \mathrm{Per}_\psi(A) \quad \text{for all } A \subset \Omega. \tag{2-14}$$

This condition is always satisfied as soon as $w \leq 0$. Otherwise, it holds obviously if for all $A \subset \Omega$, we have

$$\mathcal{H}^{N-1}(\partial^\star A \cap \Gamma) < \frac{\psi_0}{\|w^+\|_\infty}\mathrm{Per}(A).$$

For instance, if $\|w^+\|_\infty \leq \psi_0$ and Γ is a line segment, then the inequality above is clearly satisfied. Now, assume that Γ is not a line segment, the distance between the endpoints of Γ is D and the length of Γ is L. Then, we see that when the ratio L/D increases, the factor $\psi_0/\|w^+\|_\infty$ should be large enough in order to guarantee the existence of a solution to problem (2-12).

We conclude this section by showing that any solution u has a flat part $\{u = \|u\|_\infty\}$. This result has already been proven in [6, Theorem 5] but the proof here is completely different and we also consider it much simpler. More precisely, we have the following.

Proposition 2.14. *Let u be a solution of problem (2-2). Then,* $|\{u = \|u\|_\infty\}| > 0$.

Proof. Let $A_t := \{u \geq t\} \neq \varnothing$ be a superlevel set of u. Thanks to Proposition 2.11, we know that A_t is an optimal set in problem (2-12). Hence, one has

$$\int_{\tilde\Omega} \psi|D\chi_{A_t}| - \int_{\partial^\star A_t\cap\Gamma} w = \lambda^\star \int_{A_t} \phi. \tag{2-15}$$

From Remark 2.13 and since $\Lambda^\star < 1$, we get

$$\lambda^\star \|\phi\|_\infty |A_t| \geq \lambda^\star \int_{A_t} \phi = \int_{\tilde{\Omega}} \psi |D\chi_{A_t}| - \int_{\partial^\star A_t \cap \Gamma} w$$
$$\geq (1-\Lambda^\star) \int_{\tilde{\Omega}} \psi |D\chi_{A_t}| \geq c(1-\Lambda^\star)\, \psi_0 |A_t|^{\frac{N-1}{N}},$$

where $c > 0$ is a universal constant. Therefore, we infer the following estimate:

$$|A_t| \geq \left(\frac{c(1-\Lambda^\star)\, \psi_0}{\lambda^\star \|\phi\|_\infty} \right)^N .$$

In particular, this yields that

$$|\{u = \|u\|_\infty\}| \geq \left(\frac{c(1-\Lambda^\star)\, \psi_0}{\lambda^\star \|\phi\|_\infty} \right)^N > 0. \qquad \square$$

3. Regularity properties of optimal sets

In this section, we study the regularity of an optimal set $A^\star$ in problem (2-12).

In [16, Theorem 5], the authors have already studied the regularity of $\partial A^\star$ but in the particular case when $w = 0$ on Γ and $N = 2$. However, there is a gap in their proof since in order to prove regularity on $\partial A^\star$ they assume that $\partial A^\star$ is in $W^{1,1}$; but it is not clear why an arc of $\partial A^\star$ cannot be for instance the graph of a Cantor function. Fortunately, this is not the case as the results below show.

Proposition 3.1. *Let ψ be locally Lipschitz in Ω. Then, there exists a relatively closed set $\Sigma \subset \partial A^\star \cap \Omega$ such that $\mathcal{H}^{N-2}(\Sigma) = 0$ and for every $x \in (\partial A^\star \setminus \Sigma) \cap \Omega$, $\partial A^\star$ is of class $C^{1,1/2}$ around x.*

Proof. First of all, it is clear that if $A^\star$ minimizes (2-12), then $A^\star$ solves also the following problem:

$$\min \left\{ \mathrm{Per}_\psi(A) - \int_{\partial^\star A \cap \Gamma} w - \lambda^\star \int_A \phi : A \subset \Omega \right\}, \tag{3-1}$$

where $\lambda^\star = \min$ (2-2). Fix $x_0 \in \partial A^\star \cap \Omega$ and $0 < r_0 < d(x_0, \partial\Omega)$. Let $E \subset \mathbb{R}^N$ be a set with finite perimeter such that $A^\star \Delta E \subset B(x_0, r_0)$. In particular, we have $E \subset \Omega$. Thanks to the minimality of $A^\star$ in (3-1), we get that

$$\mathrm{Per}_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w - \lambda^\star \int_{A^\star} \phi \leq \mathrm{Per}_\psi(E) - \int_{\partial^\star E \cap \Gamma} w - \lambda^\star \int_E \phi.$$

Clearly, $\partial^\star A^\star \cap \Gamma = \partial^\star E \cap \Gamma$, since $A^\star \Delta E \subset B(x_0, r_0)$ and $r_0 < d(x_0, \partial\Omega)$. Hence, we infer that

$$\mathrm{Per}_\psi(A^\star) - \lambda^\star \int_{A^\star} \phi \leq \mathrm{Per}_\psi(E) - \lambda^\star \int_E \phi.$$

Consequently, we get that

$$\mathrm{Per}_\psi(A^\star) \le \mathrm{Per}_\psi(E) + \lambda^\star \|\phi\|_\infty |A^\star \Delta E|.$$

In other words, $A^\star$ is a (Λ, r_0)-minimizer of $\mathrm{Per}_\psi(E)$ in Ω with $\Lambda = \lambda^\star \|\phi\|_\infty$ (see [11]). Then, thanks to [11, Theorem 1.10], we infer that $A^\star$ has boundary of class $C^{1,1/2}$, out of a closed singular set $\Sigma \subset \partial A^\star$ of dimension $d < N-2$. □

Remark 3.2. In fact, we can reduce the dimension of the singular set Σ in Proposition 3.1 to $N-8$ but perhaps with less regularity on $\partial A^\star$. More precisely, thanks to [19, Theorem 3.2], one can show that $\partial A^\star$ is of class $C^{1,\frac{1}{4}}$ inside Ω, except at a singular set of dimension $N-8$. For this aim, we just need to show that $A^\star$ is an almost minimal set in $B(x_0, r_0)$, for every point $x_0 \in \partial A^\star$ and $r_0 > 0$ small enough such that $\overline{B(x_0, r_0)} \subset \Omega$. Indeed, let $x \in \partial A^\star \cap B(x_0, r_0)$ and $r > 0$ be small enough so that $B_r := B(x, r) \subset B(x_0, r_0)$. Recalling the proof of Proposition 3.1, for any subset $A \subset \Omega$ such that $A \Delta A^\star \subset B_r$, one has

$$\int_{\overline{B_r}} \psi |D\chi_{A^\star}| - \lambda^\star \int_{A^\star \cap B_r} \phi \le \int_{\overline{B_r}} \psi |D\chi_A| - \lambda^\star \int_{A \cap B_r} \phi.$$

In particular,

$$\int_{\overline{B_r}} |D\chi_{A^\star}| \le \frac{1}{\psi_0} \int_{\overline{B_r}} \psi |D\chi_{A^\star}| \le \frac{1}{\psi_0} \left[\int_{\partial B_r} \psi - \lambda^\star \int_{B_r} \phi + \lambda^\star \int_{A^\star \cap B_r} \phi \right] \le C r^{N-1}.$$

Yet, we have

$$\begin{aligned} \psi(x) \int_{\overline{B_r}} |D\chi_{A^\star}| + \int_{\overline{B_r}} [\psi - \psi(x)]\, |D\chi_{A^\star}| - \lambda^\star \int_{A^\star \cap B_r} \phi & \\ \le \psi(x) \int_{\overline{B_r}} |D\chi_A| + \int_{\overline{B_r}} [\psi - \psi(x)]\, |D\chi_A| - \lambda^\star \int_{A \cap B_r} \phi. & \end{aligned}$$

Since ψ is Lipschitz in $\overline{B(x_0, r_0)}$, this implies that $|\psi - \psi(x)| \le Cr$ on B_r and so, we get that

$$\int_{\overline{B_r}} |D\chi_{A^\star}| \le \int_{\overline{B_r}} |D\chi_A| + C r^N.$$

Proposition 3.3. *Assume that $\phi \in C(\Omega)$ and $\psi \in C^1(\Omega)$. Then, the boundary of $A^\star$, out of the singular set Σ, is of class $C^{1,\alpha}$, for all $\alpha < 1$. Moreover, $\partial A^\star \backslash \Sigma$ is $C^{2,\alpha}$ inside Ω as soon as $\phi \in C^{0,\alpha}(\Omega)$ and $\psi \in C^{1,\alpha}(\Omega)$. Moreover, the mean curvature $H_{A^\star}$ of $\partial A^\star$ at any point $x \notin \Sigma$ is given by the following formula* (*where $\partial_{\boldsymbol{n}} \psi$ denotes the interior normal derivative of ψ on $\partial A^\star$*):

$$(N-1) H_{A^\star}(x) = \frac{\lambda^\star \phi(x) + \partial_{\boldsymbol{n}} \psi(x)}{\psi(x)}.$$

Proof. First, we recall from Proposition 3.1 that there is a closed set $\Sigma \subset \partial A^\star$ such that $\partial A^\star \setminus \Sigma$ is $C^{1,1/2}$ inside Ω. Fix a point $x_0 \in (\partial A^\star \setminus \Sigma) \cap \Omega$. Without loss of generality, we assume that x_0 is the origin. We may also assume that near x_0, $\partial A^\star$ is the graph of a function $v^\star : B_\varepsilon \mapsto \mathbb{R}$, for some $\varepsilon > 0$ small enough. So, we already know that $v^\star \in C^{1,1/2}(B_\varepsilon)$. It is clear that $v^\star$ minimizes the following problem:

$$\min\Bigg\{\int_{B_\varepsilon} \psi(x, v(x))\sqrt{1+|\nabla v(x)|^2}\,\mathrm{d}x \\ + \lambda^\star \int_{B_\varepsilon}\int_0^{v(x)} \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x : v \in \mathrm{BV}(B_\varepsilon),\ v\,|\,\partial B_\varepsilon = v^\star\,|\,\partial B_\varepsilon\Bigg\}.$$

From the optimality conditions on $v^\star$, we have

$$\nabla\cdot\left[\psi(x, v^\star(x))\frac{\nabla v^\star(x)}{\sqrt{1+|\nabla v^\star(x)|^2}}\right] \\ = \partial_{x_N}\psi(x, v^\star(x))\sqrt{1+|\nabla v^\star(x)|^2} + \lambda^\star\phi(x, v^\star(x)). \tag{3-2}$$

Due to the regularity of $v^\star$, (3-2) can be written as

$$\nabla\cdot\left[\frac{\nabla v^\star(x)}{\sqrt{1+|\nabla v^\star(x)|^2}}\right] \\ = \frac{1}{\psi(x, v^\star(x))}\Bigg(-\frac{[\nabla_x\psi(x, v^\star(x)) + \partial_{x_N}\psi(x, v^\star(x))\nabla v^\star(x)]\cdot\nabla v^\star(x)}{\sqrt{1+|\nabla v^\star(x)|^2}} \\ + \partial_{x_N}\psi(x, v^\star(x))\sqrt{1+|\nabla v^\star(x)|^2} + \lambda^\star\,\phi(x, v^\star(x))\Bigg) \tag{3-3}$$

or equivalently,

$$\sum_{i.j} a_{ij}\, v^\star_{ij} = f,$$

where

$$a_{ij} = \frac{(1+|\nabla v^\star|^2)\,\delta_{ij} - v^\star_i v^\star_j}{(1+|\nabla v^\star|^2)^{3/2}} \tag{3-4}$$

and f is the right-hand side in (3-3), which is clearly bounded and, it is also Hölder continuous with exponent α as soon as $\phi \in C^{0,\alpha}$ and $\psi \in C^{1,\alpha}$. It is easy to check that there are two positive constants $0 < \lambda < \Lambda < \infty$ such that $\lambda|\xi|^2 \le a_{ij}\,\xi_i\,\xi_j \le \Lambda|\xi|^2$. Moreover, $a_{ij} \in C^{0,1/2}(B_\varepsilon)$ for all i, j. Thanks to the Calderón–Zygmund estimates, we infer that $v^\star$ is in $W^{2,p}(B_{\varepsilon/2})$ for any $p < \infty$, in particular $v^\star \in C^{1,\alpha}(B_{\varepsilon/2})$ for any $\alpha < 1$. Then, by the Schauder estimates (see also [5]), this implies that $v^\star$ is $C^{2,\alpha}$ in $B_{\varepsilon/2}$ provided that $\phi \in C^{0,\alpha}$ and $\psi \in C^{1,\alpha}$. In addition, the mean curvature $H_{A^\star}$ of $\partial A^\star$ at a point $(x, v^\star(x))$ is given by

$$(N-1)H_{A^\star} = \nabla\cdot\left[\frac{\nabla v^\star(x)}{\sqrt{1+|\nabla v^\star(x)|^2}}\right] = \frac{\lambda^\star\phi(x, v^\star(x)) + \partial_{\boldsymbol{n}}\psi(x, v^\star(x))}{\psi(x, v^\star(x))}. \qquad \square$$

In order to extend our regularity result on $\partial A^\star$ up to the boundary $\partial\Omega$, we need to introduce the following definition that generalizes the notion that the mean curvature of Γ^c is bounded from below in the case when $\psi = 1$ (see also [4, Definition 1] and [21]). Let us assume that ψ is extendable to a locally Lipschitz function in $\tilde{\Omega}$.

Definition 3.4. We say that Γ^c is a ψ-almost minimal set if for every $x_0 \in \Gamma^c$ there are constants $r_0 > 0$ small enough and $C < \infty$ such that

$$\mathrm{Per}_\psi(E \cap \Omega) \le \mathrm{Per}_\psi(E) + C|E\backslash\Omega|$$

for every set $E \subset \mathbb{R}^N$ where $E\backslash\Omega \subset B(x_0, r_0)$.

Remark 3.5. Assume that $N = 2$, $\psi = 1$ and, Γ^c is convex (as an arc of $\partial\Omega$). Fix a point $x_0 \in \Gamma^c$. Then, it is not difficult to check that

$$\mathrm{Per}(E \cap \Omega) \le \mathrm{Per}(E)$$

for every set $E \subset \mathbb{R}^2$ such that $E\backslash\Omega \subset B(x_0, r_0)$, where $r_0 > 0$ is small enough. In particular, this implies that Γ^c is an almost minimal set in the sense of Definition 3.4.

Proposition 3.6. *Assume that Γ^c is a ψ-almost minimal set. Then, there is a relatively closed singular set $\Sigma_b \subset \partial A^\star \cap \Gamma^c$ with dimension $d < N-2$ such that $\partial A^\star \cap \Gamma^c$ is of class $C^{1,1/2}$, outside Σ_b.*

Proof. Fix a point $x_0 \in \partial A^\star \cap \Gamma^c$ and $r_0 > 0$ small enough. We claim that $A^\star$ is a (Λ, r_0)-minimizer of $\mathrm{Per}_\psi(E)$ in $B(x_0, r_0)$. Let $E \subset \mathbb{R}^N$ be a set with finite perimeter such that $A^\star \Delta E \subset B(x_0, r_0)$. We note that here E is not necessarily contained in Ω. However, $E\cap\Omega$ is always admissible in (3-1). Hence, by minimality of $A^\star$ in (3-1), we infer that

$$\mathrm{Per}_\psi(A^\star) - \int_{\partial^\star A^\star\cap\Gamma} w - \lambda^\star \int_{A^\star} \phi \le \mathrm{Per}_\psi(E \cap \Omega) - \int_{\partial^\star(E\cap\Omega)\cap\Gamma} w - \lambda^\star \int_{E\cap\Omega} \phi.$$

Let us choose $r_0 > 0$ small enough so that $B(x_0, r_0) \cap \Gamma = \varnothing$. Then, one has $\partial^\star A^\star \cap \Gamma = \partial^\star(E \cap \Omega) \cap \Gamma$, since $A^\star \Delta E \subset B(x_0, r_0)$. Hence,

$$\mathrm{Per}_\psi(A^\star) - \lambda^\star \int_{A^\star} \phi \le \mathrm{Per}_\psi(E \cap \Omega) - \lambda^\star \int_{E\cap\Omega} \phi$$

and so,

$$\mathrm{Per}_\psi(A^\star) \le \mathrm{Per}_\psi(E \cap \Omega) + \lambda^\star \|\phi\|_\infty |A^\star\backslash E|.$$

Conversely, Γ^c is an almost minimizer of $\mathrm{Per}_\psi(E)$, $x_0 \in \Gamma^c$ and $E\backslash\Omega \subset B(x_0, r_0)$. Hence, one has

$$\mathrm{Per}_\psi(E \cap \Omega) \le \mathrm{Per}_\psi(E) + C|E\backslash\Omega|.$$

Therefore, we get that

$$\mathrm{Per}_\psi(A^\star) \le \mathrm{Per}_\psi(E) + C|E\backslash\Omega| + \lambda^\star \|\phi\|_\infty |A^\star \Delta E|.$$

Yet, $E\backslash\Omega \subset A^\star \Delta E$. Hence, $A^\star$ is a (Λ, r_0)-minimizer of $\mathrm{Per}_\psi(E)$ in $B(x_0, r_0)$. Thanks again to [11, Theorem 1.10], this yields that $\partial A^\star \cap \Gamma^c$ has boundary of class $C^{1,1/2}$, out of a closed singular set $\Sigma_b \subset \partial A^\star \cap \Gamma^c$ of dimension $d < N-2$. □

Proposition 3.7. *Assume that* Γ^c *is* $C^{1,1}$, $\phi \in C(\overline{\Omega})$ *and* $\psi \in C^1(\overline{\Omega})$. *Then,* $(\partial A^\star \cap \Gamma^c)\backslash\Sigma_b$ *is of class* $C^{1,1}$.

Proof. There is a relatively closed singular set $\Sigma_b \subset \partial A^\star \cap \Gamma^c$ (see Proposition 3.6) such that $(\partial A^\star \cap \Gamma^c)\backslash\Sigma_b$ is of class $C^{1,1/2}$. Fix $x_0 \in (\partial A^\star \cap \Gamma^c)\backslash\Sigma_b$. After rotation and translation of axes, we may assume that $x_0 = 0$ and that the tangent space to Γ^c at x_0 is the hyperplane $x_N = 0$. Assume that near x_0, Γ^c is the graph of $h : B_r \mapsto \mathbb{R}$ (where $r > 0$ is small enough) and $\partial A^\star$ is the graph of $v^\star : B_r \mapsto \mathbb{R}$. So, we have that $v^\star \in C^{1,1/2}(B_r)$, $h \in C^{1,1}(B_r)$, $h(0) = v^\star(0) = 0$ and $\nabla h(0) = \nabla v^\star(0) = 0$. Again, we see that $v^\star$ minimizes the following problem:

$$\min\left\{\int_{B_r} \psi(x, v(x))\sqrt{1+|\nabla v(x)|^2}\,\mathrm{d}x + \lambda^\star \int_{B_r}\int_0^{v(x)} \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x : v \in \mathrm{BV}(B_r),\ v \ge h,\ v|\partial B_r = v^\star|\partial B_r\right\}.$$

Taking into account the presence of the obstacle $v \ge h$ on B_r, we get instead of (3-3) the following inequality:

$$-\sum_{i.j} a_{ij}\, v^\star_{ij} + f \ge 0, \tag{3-5}$$

where a_{ij} and f are defined exactly as in the proof of Proposition 3.3 (see (3-3) and (3-4)). The equality in (3-5) holds inside the open set

$$O := \{x \in B_r : v^\star(x) > h(x)\}.$$

Since $v^\star \ge h$ on B_r and $h \in C^{1,1}(B_r)$, then we have

$$-Cr^2 \le h(x) \le v^\star(x) \quad \text{on } B_r.$$

In order to show that $v^\star$ is $C^{1,1}$ at the origin, we just need to show that

$$-Cr^2 \le v^\star(x) \le Cr^2 \quad \text{for all } x \in B_{r/2}. \tag{3-6}$$

The proof of (3-6) will follow the one in [8, Theorem 2] with some simplification (coming from the fact that one can always assume that the tangent space to Γ^c at x_0 is the hyperplane $x_N = 0$). Set $w^\star = v^\star + Cr^2 \ge 0$. Then, $w^\star$ satisfies the inequality

$$-\sum_{i.j} a_{ij}\, w^\star_{ij} + f \ge 0.$$

Let w_0 be the solution of

$$-\sum_{i.j} a_{ij}\, w_{0ij} + f = 0,$$

with $w_0 = w^\star \geq 0$ on ∂B_r. Then, by the comparison principle (see [5]), we get that $w_0 \leq w^\star$ on B_r. Let $x^\star \in B_r$ be such that

$$w^\star(x^\star) - w_0(x^\star) = \max_{B_r}(w^\star - w_0) \geq 0.$$

Then, we have two possibilities: either $x^\star \in O$ or $x^\star \in \partial O$. Assume the latter holds. Hence, we get that

$$\begin{aligned} w^\star(x) - w_0(x) &\leq w^\star(x^\star) - w_0(x^\star) \\ &= v^\star(x^\star) + Cr^2 - w_0(x^\star) = h(x^\star) + Cr^2 - w_0(x^\star). \end{aligned} \tag{3-7}$$

Now, assume that the maximum point $x^\star \in O$ and set $r^\star = \operatorname{dist}(x^\star, \partial O) > 0$. Then, we should have

$$-\sum_{i.j} a_{ij}(w^\star - w_0)_{ij} = 0 \quad \text{in } B(x^\star, r^\star).$$

But, by the maximum principle in [14, Theorem 9.6], $w^\star - w_0$ cannot achieve a (nonnegative) maximum in $B(x^\star, r^\star)$ unless it is a constant. Therefore,

$$w^\star(x^\star) - w_0(x^\star) = w^\star(y^\star) - w_0(y^\star),$$

where $y^\star \in \partial O \cap \partial B(x^\star, r^\star)$. Hence, we may always assume that $x^\star \in \partial O$ and so, (3-7) holds.

Now, consider the quadratic function $V(x) = \frac{\gamma}{2}(|x|^2 - r^2)$, where $\gamma > 0$ is to be chosen later. So, we have $V = 0$ on ∂B_r. Moreover, one can choose the constant γ large enough so that V solves

$$-\sum_{i.j} a_{ij}\, V_{ij} + f \leq 0.$$

Indeed,

$$-\sum_{i.j} a_{ij}\, V_{ij} + f = -\gamma \sum_{i,j} a_{ij}\, \delta_{ij} + f = -\gamma \sum_{i} a_{ii} + f \leq -N\lambda\gamma + \|f\|_\infty,$$

where in the last inequality we used that $a_{ii} \geq \lambda > 0$. Thanks again to the comparison principle and the fact that $V \leq w_0$ on ∂B_r, we get

$$V \leq w_0 \quad \text{on } B_r.$$

Recalling (3-7) and thanks to the fact that Γ^c is $C^{1,1}$ and $V(x) \geq -\frac{\gamma}{2} r^2$ for all $x \in B_r$, we get

$$w^\star(x) - w_0(x) \leq h(x^\star) + Cr^2 - V(x^\star) \leq Cr^2 \quad \text{for all } x \in B_r. \tag{3-8}$$

But, we have
$$-\sum_{i.j} a_{ij}(w_0 - V)_{ij} + f = \gamma \sum_i a_{ii}.$$

Recalling (3-4), we see that $a_{ii} \in C(\overline{B_r})$ for all i. Thanks to [14, Corollary 9.18], we infer that $w_0 - V \in W^{2,p}_{\text{loc}}(B_r) \cap C(\overline{B_r})$ for all $p < \infty$. By [14, Theorem 9.20], we have for any $p > 0$ that

$$\sup_{B_{r/2}}(w_0 - V) \le C\left(\left(\frac{1}{|B_r|}\int_{B_r}(w_0 - V)^p\right)^{1/p} + \frac{r}{\lambda}\|f\|_{L^N(B_r)}\right), \tag{3-9}$$

where the constant C does not depend on r. Yet, from [14, Theorem 9.22] it is seen that there are constants p and C depending only on N, λ and Λ such that

$$\left(\frac{1}{|B_r|}\int_{B_r}(w_0 - V)^p\right)^{1/p} \le C\left(\inf_{B_r}(w_0 - V) + \frac{r}{\lambda}\,\|f\|_{L^N(B_r)}\right). \tag{3-10}$$

Combining (3-9) and (3-10), we get the following Harnack inequality:

$$\sup_{B_{r/2}}(w_0 - V) \le C\left(\inf_{B_r}(w_0 - V) + \frac{r}{\lambda}\|f\|_{L^N(B_r)}\right) \le C(w_0(0) - V(0) + r^{N+1}),$$

since $f \in L^\infty(B_r)$. But, $w_0(0) \le w^\star(0) = v^\star(0) + Cr^2 = Cr^2$ and $V(0) = -\frac{\gamma}{2}r^2$. Hence, we infer that
$$\sup_{B_{r/2}}(w_0 - V) \le Cr^2.$$

Recalling (3-8), we infer that

$$w^\star(x) \le w_0(x) + Cr^2 \le V(x) + Cr^2 \le Cr^2 \quad \text{for all } x \in B_{r/2}.$$

But, this concludes our claim (3-6). □

Now, our aim is to study the shape of $\partial A^\star$ near Γ. For this, we need to restrict ourselves to dimension $N = 2$. The following proposition is a generalization of [16, Theorem 5] to the case when $w \neq 0$.

Proposition 3.8. *Assume that $\partial A^\star$ touches the interior of Γ at some point x and suppose that Γ is C^1 around x. Then, $\partial A^\star \backslash \{x\} \cap B(x, \delta)$ is composed of two arcs C_1 and C_2 such that $C_2 \subset \Gamma$. Let $\theta(x)$ be the angle between the tangent vectors to C_1 and C_2 at x. Then, we have the following estimate*:

$$\tan^{-1}\left(\frac{\psi(x)^2 - w(x)^2}{2\,\psi(x)\,w(x)}\right) \le \theta(x) \le \frac{\pi}{2} \quad \textit{if } w(x) \ge 0$$

and
$$\frac{\pi}{2} \le \theta(x) \le \tan^{-1}\left(\frac{\psi(x)^2 - w(x)^2}{2\,\psi(x)\,w(x)}\right) \quad \textit{if } w(x) \le 0.$$

Proof. First, we recall from Proposition 3.3 that $\partial A^\star$ is $C^{2,\alpha}$ inside Ω, for all $\alpha < 1$. Moreover, the curvature of $\partial A^\star$ is uniformly bounded in Ω:

$$\kappa = \frac{1}{\psi(x)}[\lambda^\star \phi(x) + \partial_{\boldsymbol{n}} \psi(x)].$$

Fix $x \in \partial A^\star$ in the interior of Γ such that $\partial A^\star \cap B(x, \delta) \cap \Omega \neq \varnothing$, for every $\delta > 0$ small enough. Assume that Γ is C^1 around x. Let C_1 and C_2 be two different arcs of $\partial A^\star \cap B(x, \delta)$ such that x is an endpoint of both C_1 and C_2.

Assume that C_1 and C_2 are contained in Ω (we note that the case when $C_1 \cap \Gamma \neq \varnothing$ or $C_2 \cap \Gamma \neq \varnothing$ can be treated similarly) with $0 \le \theta(x) < \pi$. After rotation and translation of axes, one can assume that x is the origin and $(s, \alpha(s))$ is a parametrization of C_1 ($s \in (0, \delta)$) and C_2 ($s \in (-\delta, 0)$) such that $\alpha(0) = 0$, $\alpha'(0^-) < 0$ and $\alpha'(0^+) > 0$. For $\varepsilon > 0$ small enough, let $s_\varepsilon < 0$ be such that $\alpha(s_\varepsilon) = \alpha(\varepsilon)$. Let us denote by $\mathcal{C}_\varepsilon := \{(s, \alpha(s)) : s \in (s_\varepsilon, \varepsilon)\}$ and by $\hat{\mathcal{C}}_\varepsilon \subset \Omega$ the line segment joining the points $(s_\varepsilon, \alpha(s_\varepsilon))$ and $(\varepsilon, \alpha(\varepsilon))$. Let A_ε be such that $\partial A_\varepsilon = (\partial A^\star \backslash \mathcal{C}_\varepsilon) \cup \hat{\mathcal{C}}_\varepsilon$. Thanks to the minimality of $A^\star$ in (3-1), we have

$$\mathrm{Per}_\psi(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w - \lambda^\star \int_{A_\varepsilon} \phi - \left[\mathrm{Per}_\psi(A^\star) - \int_{\partial A^\star \cap \Gamma} w - \lambda^\star \int_{A^\star} \phi\right] \ge 0.$$

Hence,

$$\int_{s_\varepsilon}^{\varepsilon} \psi(s, \alpha(\varepsilon))\, \mathrm{d}s - \int_{s_\varepsilon}^{\varepsilon} \psi(s, \alpha(s))\sqrt{1 + \alpha'(s)^2}\, \mathrm{d}s - \lambda^\star \int_{A_\varepsilon} \phi + \lambda^\star \int_{A^\star} \phi \ge 0. \tag{3-11}$$

Yet, we have

$$\left|\int_{A_\varepsilon} \phi - \int_{A^\star} \phi\right| = \left|\int_{s_\varepsilon}^{\varepsilon} \int_{\alpha(s)}^{\alpha(\varepsilon)} \phi(s, t)\, \mathrm{d}t\, \mathrm{d}s\right| \le \|\phi\|_\infty\, o(\varepsilon),$$

where the last inequality comes from the fact that the map $\varepsilon \mapsto s(\varepsilon) := s_\varepsilon$ is Lipschitz with $s(0) = 0$. Dividing (3-11) by ε and letting $\varepsilon \to 0^+$, we get

$$\alpha'(0^+)[1 - \sqrt{1 + \alpha'(0^-)^2}] - \alpha'(0^-)[1 - \sqrt{1 + \alpha'(0^+)^2}] \ge 0.$$

But, this is clearly a contradiction.

Let $C \subset \Omega$ be an arc of $\partial A^\star \cap B(x, \delta) \cap \Omega$ such that x is an endpoint of C. Let $(s, \beta(s))$, $s \in (-\delta, \delta)$, be a parametrization of Γ such that $\beta(0) = \beta'(0) = 0$. Assume that the angle between the tangent vector to C at x and $\langle 1, 0\rangle$ is less than $\frac{\pi}{2}$. Let $(s, \alpha(s))$, $s \in (0, \delta)$, be a parametrization of C with $\alpha(0) = 0$. For $\varepsilon > 0$ small enough, we define the set A_ε as

$$\partial A_\varepsilon = [\partial A^\star \backslash C_\varepsilon] \cup [(\varepsilon, \alpha(\varepsilon)), (\varepsilon, \beta(\varepsilon))] \cup \Gamma_\varepsilon,$$

where $C_\varepsilon := \{(s, \alpha(s)) : s \in (0, \varepsilon)\}$ and Γ_ε denotes the arc of Γ between $(0, 0)$ and $(\varepsilon, \beta(\varepsilon))$. Here, we assume that $\Gamma_\varepsilon \cap \partial A^\star = \varnothing$. So, one can see easily that

$$\begin{aligned}\mathrm{Per}_\psi(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w - \lambda^\star \int_{A_\varepsilon} \phi - \Big[\mathrm{Per}_\psi(A^\star) - \int_{\partial A^\star \cap \Gamma} w - \lambda^\star \int_{A^\star} \phi\Big] \\ = \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \psi(\varepsilon, t)\,\mathrm{d}t - \int_0^\varepsilon w(s, \beta(s))\sqrt{1+\beta'(s)^2}\,\mathrm{d}s \\ - \int_0^\varepsilon \psi(s, \alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s - \lambda^\star \int_{A_\varepsilon} \phi + \lambda^\star \int_{A^\star} \phi.\end{aligned}$$

But, one has

$$\lim_{\varepsilon \to 0^+} \frac{\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \psi(\varepsilon, t)\,\mathrm{d}t - \int_0^\varepsilon \psi(s, \alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s}{\varepsilon} = [\alpha'(0^+) - \sqrt{1+\alpha'(0^+)^2}]\,\psi(0, 0)$$

and

$$\left|\int_{A_\varepsilon} \phi - \int_{A^\star} \phi\right| = \left|\int_0^\varepsilon \int_{\beta(s)}^{\alpha(s)} \phi(s, t)\,\mathrm{d}t\,\mathrm{d}s\right| \le \|\phi\|_\infty\, o(\varepsilon).$$

Thanks to the optimality of $A^\star$ in (3-1), we get that

$$[\alpha'(0^+) - \sqrt{1+\alpha'(0^+)^2}]\,\psi(0, 0) - w(0, 0) \ge 0. \tag{3-12}$$

Hence, $w(0, 0) \le 0$. Using the above estimates, it is easy to check that if $w(0, 0) \le 0$ then Γ_ε cannot intersect $\partial A^\star$. Moreover, we have

$$\alpha'(0^+) \ge \frac{w(0, 0)^2 - \psi(0, 0)^2}{2w(0, 0)\,\psi(0, 0)}.$$

Now, let us assume that the angle between the tangent vector to C at x and $\langle 1, 0\rangle$ is greater than $\frac{\pi}{2}$. Let $(s, \alpha(s))$, $s \in (-\delta, 0)$, be a parametrization of C with $\alpha(0) = 0$. For $\varepsilon > 0$ small enough, we define the set A_ε as

$$\partial A_\varepsilon = [\partial A^\star \backslash C_\varepsilon] \cup [(-\varepsilon, \alpha(-\varepsilon)), (-\varepsilon, \beta(-\varepsilon))],$$

where $C_\varepsilon := \{(s, \alpha(s)) : s \in (-\varepsilon, 0)\} \cup \{(s, \beta(s)) : s \in (-\varepsilon, 0)\}$. Here, we assume that $C_\varepsilon \subset \partial A^\star$. Again from the optimality of $A^\star$ in (3-1), we must have the inequality

$$\mathrm{Per}_\psi(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w - \lambda^\star \int_{A_\varepsilon} \phi - \Big[\mathrm{Per}_\psi(A^\star) - \int_{\partial A^\star \cap \Gamma} w - \lambda^\star \int_{A^\star} \phi\Big] \ge 0.$$

Hence,

$$\begin{aligned}\int_{\beta(-\varepsilon)}^{\alpha(-\varepsilon)} \psi(-\varepsilon, t)\,\mathrm{d}t + \int_{-\varepsilon}^0 w(s, \beta(s))\sqrt{1+\beta'(s)^2}\,\mathrm{d}s \\ - \int_{-\varepsilon}^0 \psi(s, \alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s - \lambda^\star \int_{A_\varepsilon} \phi + \lambda^\star \int_{A^\star} \phi \ge 0.\end{aligned}$$

Dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0^+$, we get that

$$[-\alpha'(0^-) - \sqrt{1+\alpha'(0^-)^2}]\,\psi(0,0) + w(0,0) \geq 0. \tag{3-13}$$

In particular, this yields that $w(0,0) \geq 0$. If $w(0,0) \geq 0$ then one can also see that C_ε must be contained in $\partial A^\star$. In addition, we have

$$\alpha'(0^-) \leq \frac{w(0,0)^2 - \psi(0,0)^2}{2w(0,0)\,\psi(0,0)}. \qquad \square$$

We finish this section by some remarks on optimal sets.

Remark 3.9. An optimal set $A^\star$ is a priori not connected. For instance, this may happen when ψ has two minima or when Ω is not convex (like two disks connected by a tube). However, one can always show that there is an open connected set $A^\star$ that minimizes problem (2-12). Indeed, if $A^\star$ is an optimal set then it is not difficult to check that the interior of $A^\star$ is optimal too. So, let us assume that $A^\star$ is open. Now, let $\{A_i^\star\}_{i\in\mathbb{N}}$ be the family of disjoint open connected components of $A^\star$ (i.e., $A^\star = \bigcup_{i\in\mathbb{N}} A_i^\star$ and $A_i^\star \cap A_j^\star = \varnothing$, for all $i \neq j$). In fact, the optimality of $A^\star$ also implies that $\overline{A_i^\star} \cap \overline{A_j^\star} = \varnothing$, for all i, j. Yet, we have

$$\lambda^\star = \frac{\mathrm{Per}_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w}{\int_{A^\star} \phi} \leq \frac{\mathrm{Per}_\psi(A_i^\star) - \int_{\partial^\star A_i^\star \cap \Gamma} w}{\int_{A_i^\star} \phi} \quad \text{for all } i.$$

Hence,

$$\lambda^\star \int_{A_i^\star} \phi \leq \mathrm{Per}_\psi(A_i^\star) - \int_{\partial^\star A_i^\star \cap \Gamma} w. \tag{3-14}$$

Since the closures of these sets $A_i^\star$ are mutually disjoint, then taking the sum over i in (3-14), we get that

$$\begin{aligned} \lambda^\star \int_{A^\star} \phi = \lambda^\star \sum_i \int_{A_i^\star} \phi &\leq \sum_i \mathrm{Per}_\psi(A_i^\star) - \sum_i \int_{\partial^\star A_i^\star \cap \Gamma} w \\ &= \mathrm{Per}_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w. \end{aligned} \tag{3-15}$$

But, the inequality in (3-15) must be an equality. In particular, it implies that for all i, the inequality in (3-14) is an equality:

$$\lambda^\star \int_{A_i^\star} \phi = \mathrm{Per}_\psi(A_i^\star) - \int_{\partial^\star A_i^\star \cap \Gamma} w.$$

In other words, this means that $A_i^\star$ is an optimal set for problem (2-12) for all i.

In addition, one can show in two dimensions that any connected optimal set $A^\star$ is convex as soon as ψ is a constant function.

Remark 3.10. Assume that $\psi = 1$. For every point $x \in \partial A^\star \cap \Omega$, there is an $\varepsilon > 0$ such that $A^\star \cap B(x, \varepsilon)$ is convex. Moreover, if $x \in \partial A^\star \cap \partial\Omega$ and $\Omega \cap B(x, \varepsilon)$ is convex, then $A^\star \cap B(x, \varepsilon)$ is convex. To see this, assume that there are two points $x^\star, y^\star \in \partial A^\star \cap \Omega$ such that $]x^\star, y^\star[\subset \Omega \backslash A^\star$. Let E be the small region delimited by $[x^\star, y^\star]$ and $\partial A^\star$. Now, we define $\tilde{A} = A^\star \cup E$. Then, it is easy to see that $\operatorname{Per}(\tilde{A}) < \operatorname{Per}(A^\star)$. Yet, we also have $\int_{\tilde{A}} \phi > \int_{A^\star} \phi$ and $\partial \tilde{A} \cap \Gamma = \partial A^\star \cap \Gamma$. Thanks to (2-14), we infer that

$$0 < \frac{\operatorname{Per}(\tilde{A}) - \int_{\partial \tilde{A} \cap \Gamma} w}{\int_{\tilde{A}} \phi} < \frac{\operatorname{Per}(A^\star) - \int_{\partial A^\star \cap \Gamma} w}{\int_{A^\star} \phi}.$$

But, this contradicts the optimality of $A^\star$ in (2-12). We note that this argument does not work in higher dimension since it is not true when $N > 2$ that the perimeter of the convex hull of a set is less than the perimeter of the set itself.

Remark 3.11. Assume that $\phi = 1$. Then, one can show that any connected optimal set $A^\star$ in (2-12) has to intersect the boundary $\partial\Omega$. Indeed, assume that $A^\star$ is contained in the interior of Ω. Take $t > 1$ such that $tA^\star \subset \Omega$. Then, it is clear that

$$\operatorname{Per}(tA^\star) = t \operatorname{Per}(A^\star) \quad \text{and} \quad |tA^\star| = t^2 |A^\star|.$$

Hence, we have

$$\frac{\operatorname{Per}(tA^\star)}{|tA^\star|} < \frac{\operatorname{Per}(A^\star)}{|A^\star|},$$

which is a contradiction. Hence, $A^\star$ touches $\partial\Omega$. Moreover, if $\partial A^\star \cap \Gamma = \varnothing$ then $A^\star$ cannot be translated inside Ω, since if A is a translation of $A^\star$ inside Ω, then

$$\operatorname{Per}(A) = \operatorname{Per}(A^\star) \quad \text{and} \quad |A| = |A^\star|.$$

In addition, assume Ω is convex. Then, $A^\star$ must intersect Γ with

$$\mathcal{H}^1(\partial A^\star \cap \{x \in \Gamma : w(x) > -1\}) > 0.$$

Indeed, if $\mathcal{H}^1(\partial A^\star \cap \{x \in \Gamma : w(x) > -1\}) = 0$ then one can move $A^\star$ in $\overline{\Omega}$ until we obtain a new set A such that

$$\mathcal{H}^1(\partial A \cap \{x \in \Gamma : w(x) > -1\}) > 0.$$

Yet, we clearly have $|A| = |A^\star|$. Moreover, we have

$$\begin{aligned}\operatorname{Per}(A) - \int_{\partial A \cap \Gamma} w &= \operatorname{Per}(A) - \int_{\partial A \cap \{x \in \Gamma : w(x) = -1\}} w - \int_{\partial A \cap \{x \in \Gamma : w(x) > -1\}} w \\ &< \operatorname{Per}(A, \mathbb{R}^2) = \operatorname{Per}(A^\star, \mathbb{R}^2) = \operatorname{Per}(A^\star) - \int_{\partial A^\star \cap \{x \in \Gamma : w(x) = -1\}} w.\end{aligned}$$

Consequently, we get

$$\operatorname{Per}(A) - \int_{\partial A \cap \Gamma} w < \operatorname{Per}(A^\star) - \int_{\partial A^\star \cap \Gamma} w.$$

4. Examples

We conclude the paper by some examples in two dimensions where we can find explicitly the optimal set $A^\star$ in problem (2-12).

Example 4.1. Assume that $\Omega := [-1, 1] \times [0, 1]$, $\Gamma = [-1, 1] \times \{0\}$, $\psi = \phi = 1$ and $w = w_0 \in [0, 1]$. Thanks to Proposition 3.8, we know that any arc of $\partial A^\star$ inside Ω is an arc of circle with radius $R^\star = 1/\lambda^\star$ and, $\partial A^\star$ is also of class C^1 on Γ^c. Using Proposition 3.8, one can see that if $\partial A^\star$ touches Γ at some point x, then the tangent line to $\partial A^\star \cap \Omega$ at x must be orthogonal to Γ. Moreover, by Remark 3.11, one has $\mathcal{H}^1(\partial A^\star \cap \Gamma) > 0$. Thanks to Remark 3.10, $A^\star$ is also convex.

For every $\varepsilon \in]0, 1[$, let A_ε be the "rounded" rectangle Ω where the corners $(1, 1)$ and $(-1, 1)$ are cut off and replaced by arcs of circles with radius ε and centers $(1-\varepsilon, 1-\varepsilon)$ and $(-1+\varepsilon, 1-\varepsilon)$. We have

$$\mathrm{Per}(A_\varepsilon) = 4 + (\pi - 4)\,\varepsilon \quad \text{and} \quad |A_\varepsilon| = 2 - 2\Big(1 - \frac{\pi}{4}\Big)\,\varepsilon^2.$$

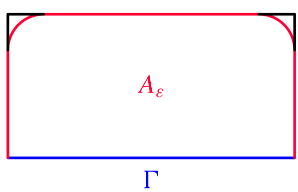
Hence,

$$\mathcal{J}(\varepsilon) := \frac{\mathrm{Per}(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w}{|A_\varepsilon|} = \frac{4 + (\pi - 4)\,\varepsilon - 2w_0}{2 - 2\big(1 - \frac{\pi}{4}\big)\,\varepsilon^2}.$$

Yet, this function $\mathcal{J}(\varepsilon)$ reaches a minimum at

$$\varepsilon^\star = \frac{4 - 2w_0 - 2\sqrt{\pi - 4 + (w_0 - 2)^2}}{4 - \pi}.$$

Then, we infer that the optimal set $A^\star$ is equal to $A_{\varepsilon^\star}$ and $\lambda^\star = 1/\varepsilon^\star$.



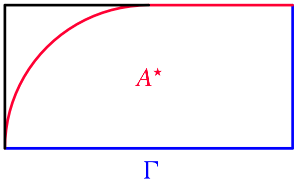
Example 4.2. Now, assume that $\Gamma = ([-1, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$ and $w_0 = 0$. For every $\varepsilon \in]0, 1[$, let us denote by A_ε the "rounded" rectangle Ω where the corner point $(-1, 1)$ is cut off and replaced by an arc of circle with center $(-1+\varepsilon, 1-\varepsilon)$ and radius ε. Again, it is clear that

$$\mathrm{Per}(A_\varepsilon) = 3 + \Big(\frac{\pi}{2} - 2\Big)\,\varepsilon \quad \text{and} \quad |A_\varepsilon| = 2 - \Big(1 - \frac{\pi}{4}\Big)\,\varepsilon^2.$$

Then, we get that

$$\frac{\mathrm{Per}(A_\varepsilon)}{|A_\varepsilon|} = \frac{3 + \big(\frac{\pi}{2} - 2\big)\,\varepsilon}{2 - \big(1 - \frac{\pi}{4}\big)\,\varepsilon^2},$$

attains a minimum at $\varepsilon = 1$. Consequently, this implies that the optimal set $A^\star$ in (2-12) is nothing else than A_1.



Example 4.3. In this example, we will see that the situation becomes much complicated when the penalization w on Γ is negative. Assume that $\Omega := [-1, 1] \times [0, 1]$, $\Gamma = [-1, 1] \times \{0\}$, $\psi = \phi = 1$ and $w = w_0$, where $-1 < w_0 < 0$. Let $A^\star$ be a convex optimal set in (2-12). We recall that any part of $\partial A^\star$ in the interior of Ω is an arc of circle with radius $R^\star = 1/\lambda^\star$ and, that $\partial A^\star$ is C^1 on Γ with $\mathcal{H}^1(\partial A^\star \cap \Gamma) > 0$. Moreover, we know that if $\partial A^\star$ touches Γ at a point x then the angle $\theta \in \left]\frac{\pi}{2}, \pi\right[$ between the tangent line to $\partial A^\star \cap \Omega$ at x and Γ should satisfy

$$\theta \le \tan^{-1}\left(\frac{1 - w_0^2}{2w_0}\right). \tag{4-1}$$

For all $\varepsilon \in]0, 1[$ and $\delta \in]0, \varepsilon[$, we define $A_{\varepsilon,\delta}$ as the "rounded" rectangle Ω where the corners $(1, 1)$ and $(-1, 1)$ are cut off and replaced by arcs of circles with radius ε and centers $(1-\varepsilon, 1-\varepsilon)$ and $(-1+\varepsilon, 1-\varepsilon)$, while the corners $(-1, 0)$ and $(1, 0)$ are cut off and replaced by arcs of the circles

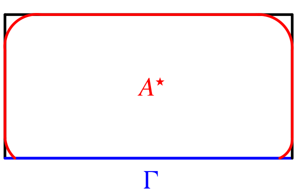
$$(x_1 - \varepsilon)^2 + (x_2 - \delta)^2 = \varepsilon^2 \quad \text{and} \quad (x_1 - 1 + \varepsilon)^2 + (x_2 - \delta)^2 = \varepsilon^2.$$

Then, it is not difficult to check that

$$\begin{aligned}\mathcal{J}(\varepsilon, \delta) &= \frac{\operatorname{Per}(A_{\varepsilon,\delta}) - \int_{\partial A_{\varepsilon,\delta} \cap \Gamma} w}{|A_{\varepsilon,\delta}|} \\ &= \frac{4 + (\pi - 4)\,\varepsilon + 2\left[\varepsilon \cos^{-1}\left(\frac{\sqrt{\varepsilon^2 - \delta^2}}{\varepsilon}\right) - \delta\right] - 2[1 - \varepsilon + \sqrt{\varepsilon^2 - \delta^2}]\, w_0}{2 - \frac{(4-\pi)}{2}\varepsilon^2 - 2\delta[\varepsilon - \sqrt{\varepsilon^2 - \delta^2}] + \varepsilon\delta\sqrt{\varepsilon^2 - \delta^2} - \varepsilon^2 \cos^{-1}\left(\frac{\sqrt{\varepsilon^2 - \delta^2}}{\varepsilon}\right)}.\end{aligned}$$

If $(\varepsilon^\star, \delta^\star)$ is a minimizer of $\mathcal{J}(\varepsilon, \delta)$, then the optimal set $A^\star$ will be $A_{\varepsilon^\star,\delta^\star}$. Notice that, thanks to (4-1), we must have the following estimate:

$$\delta^\star \le \frac{-2w_0}{\sqrt{(1 - w_0^2)^2 + 4w_0^2}}\, \varepsilon^\star.$$



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POSITIVE KNOTS AND RIBBON CONCORDANCE

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Ribbon concordances between knots generalize the notion of ribbon knots. Agol (2022) proved ribbon concordance which gives a partial order on knots in S^3, and Boninger and Greene (2024) conjectured that positive knots are minimal in this ordering. In this article we prove this conjecture for a large class of positive knots, and show that a positive knot cannot be expressed as a nontrivial band sum. Both results extend earlier theorems of Boninger and Greene for special alternating knots. In a related direction, we prove that if positive knots K and K' are concordant and $|\sigma(K)| \geq 2g(K) - 2$, then K and K' have isomorphic rational Alexander modules. This strengthens a result of Stoimenow, and gives evidence toward a conjecture that any concordance class contains at most one positive knot.

1. Introduction

A *smooth concordance* between knots K_0, $K_1 \subset S^3$ is a smooth, properly embedded cylinder $C \subset S^3 \times I$ such that $C \cap (S^3 \times \{i\}) = K_i$ for $i = 0, 1$. Here $I = [0, 1]$ is the unit interval. Perturbing C if necessary, we assume the height function $h : C \hookrightarrow S^3 \times I \to I$ is Morse, and we say C is a *ribbon concordance from K_1 to K_0* if h has no critical points of index two. If such a concordance exists, we say K_1 is *ribbon concordant* to K_0 and we write $K_0 \leq K_1$. This terminology generalizes the notion of a ribbon knot, since a knot is ribbon if and only if it is ribbon concordant to the unknot.

Gordon conjectured, in a now-classic paper [15], that ribbon concordance induces a partial ordering on the set of knots. This conjecture was settled in the affirmative by Agol [1], and many authors have shown that ribbon concordance places strong constraints on knot invariants. To give just a few examples, if $K_0 \leq K_1$, then:

- The Alexander polynomial Δ_{K_0} of K_0 divides Δ_{K_1} [12; 14].
- The genus $g(K_0)$ of K_0 is less than or equal to $g(K_1)$ [42].
- If K_1 is fibered, then K_0 is as well [25; 42].

MSC2020: 57K10.
Keywords: geometric topology, knot theory, concordance, ribbon concordance, positive knots.

Here, we consider the class of knots which are minimal under ribbon concordance. Gordon proved this class includes torus knots [15], and in a more recent paper Boninger and Greene [6] proved many special alternating knots are also ribbon concordance minimal. Torus knots and special alternating knots are both examples of *positive knots*, which are knots admitting a diagram in which all crossings are positive. This motivated Boninger and Greene to conjecture:

Conjecture 1.1 [6, Conjecture 1.6; 41, Question 1.3]. *If $K_1 \subset S^3$ is a positive knot and $K_0 \leq K_1$, then $K_0 \cong K_1$.*

Conjecture 1.1 was posed independently by Tagami, who proved it for positive two-bridge knots [41]. Boninger and Greene also observed that the conjecture holds for fibered positive knots [6, Proposition 1.7], adapting an argument used by Baker and Motegi [5, Theorem 1.1].

In this article we prove two theorems in support of Conjecture 1.1. First, we verify Conjecture 1.1 for a large class of positive knots.

Theorem 1.2. *Let $K \subset S^3$ be a positive knot. If the leading coefficient of Δ_K is a prime power, then K is ribbon concordance minimal.*

Second, given a two-component split link $K_0 \sqcup K_1 \subset S^3$ and an embedded band $b = I \times I \subset S^3$ satisfying $b \cap K_i = I \times \{i\}$ for $i = 0, 1$, we define the *band sum* $K_0 \#_b K_1 \subset S^3$ by

$$K_0 \#_b K_1 = (K_0 \cup K_1 - I \times \partial I) \cup \partial I \times I.$$

This band sum is *trivial* if there exists a sphere $\Sigma \subset S^3 - (K_0 \cup K_1)$ which intersects b in a single arc — in this case $K_0 \#_b K_1 \cong K_0 \# K_1$ — the ordinary connect sum. We say a knot is *band prime* if it cannot be written as a nontrivial band sum.

Miyazaki [24] proved any band sum $K_0 \#_b K_1$ is ribbon concordant to the connect sum $K_0 \# K_1$. Thus, band sums are a natural way in which ribbon concordances arise. Additionally, if $K' \leq K$ via a ribbon concordance with only two critical points, then Morse theory shows K is equivalent to a band sum of K' and an unknot. We prove:

Theorem 1.3. *Positive knots are band prime.*

The proof of Theorem 1.3 is somewhat more involved than that of Theorem 1.2.

Boninger and Greene [6] proved Theorems 1.2 and 1.3 for all special alternating knots, and our results here broadly extend that work. Additionally, while the proof that special alternating knots are band prime in [6] has a combinatorial flavor, our proof of Theorem 1.3 is more geometric. We use a theorem of Ozawa [28] stating that any incompressible Seifert surface of a positive knot is *free*, and we show that such a Seifert surface cannot witness a nontrivial, genus-preserving band sum.

In a related direction, we consider the following question of Gordon:

Question 1.4 [15, Question 6.1]. *Does every smooth concordance class contain a unique representative which is minimal with respect to ribbon concordance*?

Affirmative answers to this question and to [15, Question 6.2] would imply a generalization of the slice-ribbon conjecture. Question 1.4 also seems closely related to conjectures made independently by other authors: Rudolph [32] conjectured that each concordance class contains at most one algebraic knot and Baker [4] conjectured each concordance class contains at most one fibered knot supporting the tight contact structure. Most relevant to us, Stoimenow [38] conjectured each concordance class contains finitely many positive knots — this was verified by Baader, Dehornoy and Liechti [2]. Considering Conjecture 1.1 and Question 1.4, it is natural to posit:

Conjecture 1.5 [38]. *Every smooth concordance class contains at most one positive knot.*

We credit Stoimenow since Conjecture 1.5 seems implicit in his work. For any knot K, let $d(K)$ denote the degree of Δ_K when normalized to have no negative exponents. As evidence of Conjecture 1.5, we prove:

Theorem 1.6. *Let K and K' be* (*topologically or algebraically*) *concordant positive knots. If K satisfies $|\sigma(K)| \geq d(K) - 2$, where σ denotes the signature*, *then the rational Alexander modules of K and K' are isomorphic.*

By the *rational Alexander module* of K we mean the cohomology ring $H^*(\overline{X}; \mathbb{Q})$, where $\overline{X}$ denotes the infinite cyclic cover of the exterior of K, viewed as a module over the group ring of deck transformations. Theorem 1.6 strengthens a result of Stoimenow, which concluded under the above hypotheses that K and K' have the same Alexander polynomial [38, Theorem 4.5]. Additionally, although the hypothesis that

$$|\sigma(K)| \geq d(K) - 2 \tag{1}$$

is somewhat restrictive, it is known that the signatures of positive knots are linearly bounded from below by their genus [2]. In fact, (1) holds for all positive knots with genus less than or equal to four with the single exception of the knot 14_{45657} [8; 37; 38, Theorem 2.4; 39].

Corollary 1.7. *Let K and K' be* (*topologically or algebraically*) *concordant positive knots. If $g(K) \leq 4$, then the rational Alexander modules of K and K' are isomorphic.*

Condition (1) also includes all special alternating knots, since these satisfy $|\sigma(K)| = d(K)$. We prove Theorem 1.6 by showing that positive knots which satisfy (1) are $\mathbb{Q}$-*anisotropic* — for a definition of $\mathbb{Q}$-anisotropy, see Section 5 below. By a classical result of Kervaire and Gilmer, algebraically concordant knots which are $\mathbb{Q}$-anisotropic and admit nonsingular Seifert matrices have isomorphic rational Alexander modules [14, Proposition 4.2; 18]. Thus, we could remove the requirement (1) from Theorem 1.6 if we knew that:

Conjecture 1.8. *Positive knots are $\mathbb{Q}$-anisotropic.*

By work of Gilmer, Conjecture 1.8 may be thought of as the statement that positive knots are *algebraically* ribbon concordance minimal [14, Theorem 0.1]. Scharlemann proved Conjecture 1.8 for the case of torus knots [36, Proposition 2.3].

Further discussion. Section 5 below contains some results on roots of Alexander polynomials which may be of independent interest — for example, we show that Alexander polynomials of positive knots have no rational roots. We also remark that our proof of Theorem 1.6 extends to *almost positive* knots, knots which admit a diagram with one negative crossing, using results of Tagami [40] and Stoimenow [38, Theorem 2.3]. It may be interesting to consider whether Theorems 1.2 and 1.3 could also be extended to almost positive knots.

Outline. In Section 2 we recall relevant properties of positive knots, in Section 3 we prove Theorem 1.3 and in Section 4 we prove Theorem 1.2. In Section 5 we discuss $\mathbb{Q}$-anisotropy and prove Theorem 1.6 and Corollary 1.7.

2. Properties of positive knots

We gather some useful facts about positive knots. First, by results of Rudolph, the genus $g(K)$ and slice genus $g_4(K)$ of a positive knot K are equal [33; 34]. This motivates the following lemma, well known to experts:

Lemma 2.1. *Let $K_0, K_1 \subset S^3$ be such that $K_0 \leq K_1$ and K_1 satisfies $g(K_1) = g_4(K_1)$. Then $g(K_0) = g(K_1)$.*

In particular, the conclusion of Lemma 2.1 holds if $K_0 \leq K_1$ and K_1 is positive.

Proof. Since genus is nonincreasing under ribbon concordance [42] and slice genus is a concordance invariant, we have

$$g_4(K_1) = g_4(K_0) \leq g(K_0) \leq g(K_1) = g_4(K_1). \qquad \square$$

Second, we will need a theorem of Ozawa [28]. A Seifert surface $S \subset S^3$ is called *free* if $S^3 - \nu(S)$ is a handlebody, where ν denotes a regular open neighborhood; equivalently, S is free if $\pi_1(S^3 - S)$ is a free group [27, Lemma 2.2].

Theorem 2.2 [28, Corollary 1.2]. *If K is a positive knot, then every incompressible Seifert surface of K is free.*

Finally, we will use the fact that positive knots are *pseudoalternating* (not be confused with *quasialternating*!). The precise definition of pseudoalternating will not be important to us (see [21, Section 4]), but for experts we note that pseudoalternating links are those links which can be built from Murasugi sums of special alternating links. Positive knots are pseudoalternating because they are homogeneous [9].

3. Positive knots are band prime

In this section we prove Theorem 1.3, first recalling some standard definitions from three-manifold topology. Let Y be a three-manifold and $\Sigma \subset Y$ a properly embedded surface. A *compressing disk for* Σ is an embedded disk $D \subset Y$ with $D \cap \Sigma = \partial D$, such that ∂D does not bound a disk in Σ. Similarly, a *boundary-compressing disk for* Σ is a disk $D \subset Y$ such that:

- $D \cap \Sigma \subset \partial D$.
- ∂D consists of an arc in ∂Y and an arc in Σ which is not boundary-parallel in Σ.

The arc $\partial D \cap \Sigma$ is called a *boundary-compressing arc*. The surface Σ is called *compressible* (respectively *boundary-compressible*) if it admits a compressing disk (respectively boundary-compressing disk). Conversely, Σ is *incompressible* if it is not compressible, and is also not a two-sphere bounding a three-ball in Y. A nonboundary-compressible surface is called *boundary-incompressible*.

We will need a couple of lemmas on surfaces in handlebodies — the first is a classical fact.

Lemma 3.1 [16, Example III.13]. *Let H be a handlebody. If $\Sigma \subset H$ is a connected surface which is incompressible and boundary-incompressible, then Σ is a disk.*

In the next lemma, by a *planar surface* we mean a compact surface which can be embedded in $\mathbb{R}^2$.

Lemma 3.2. *Let H be a handlebody with boundary $F = \partial H$. Let $\Sigma \subset H$ be a properly embedded, connected planar surface such that:*

- *Σ is incompressible in H.*
- *The components of $\partial\Sigma$ are parallel to one another in F.*
- *Each component of $\partial\Sigma$ separates F.*

Then Σ is either a disk or a boundary-parallel annulus.

Proof. We will suppose Σ has more than one boundary component (i.e., Σ is not a disk) and show Σ is a boundary-parallel annulus. By Lemma 3.1, since Σ is incompressible it is boundary-compressible. Let D be a boundary-compressing disk for Σ, so that $\partial D = \alpha \cup \alpha'$ with $\alpha' \subset F$ and α a properly embedded arc in Σ which is not boundary-parallel.

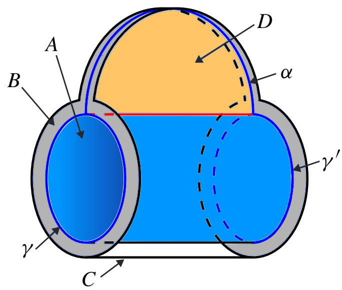

Figure 1. A boundary-compressing disk for Σ, intersecting distinct boundary components.

We first suppose the two boundary points $\partial\alpha = \partial\alpha'$ lie in distinct boundary components γ and γ' of $\partial\Sigma$. Since γ and γ' each separate F, $F - (\gamma \cup \gamma')$ has three components, and α' lies in the unique component with boundary $\gamma \cup \gamma'$. Since γ and γ' are parallel in F, this component is an annulus which we denote by A. Let N be a regular neighborhood of $A \cup D$ in $H - \Sigma$, and let

$$B = N \cap \Sigma = \partial N \cap \Sigma.$$

Equivalently, B is a regular neighborhood of $\gamma \cup \alpha \cup \gamma'$ in Σ. Let $C = \partial N - (A \cup B)$, as in Figure 1.

The neighborhood N is a thickened annulus, so ∂N is a torus. Since $A \cup B$ is a torus with a disk removed, as Figure 1 shows, it follows that C is a disk. Now

$$\partial C = \partial B \subset \Sigma,$$

and the incompressibility of Σ implies ∂C bounds a disk C' in Σ. Thus $\Sigma = B \cup C'$, and in this case Σ is an annulus with boundary $\gamma \cup \gamma'$. Additionally the union of the disks C and C' is a two-sphere, which bounds a three-ball in H by Lemma 3.1. It follows that C' and C are isotopic, so Σ is boundary-parallel.

We have shown that if Σ admits a boundary-compressing disk intersecting distinct components of $\partial\Sigma$, then Σ is a boundary-parallel annulus. We thus assume that each boundary-compressing disk intersects only one component of $\partial\Sigma$. The planarity of Σ then implies that, if D is such a disk with $\partial D = \alpha \cup \alpha'$ as above, then α separates Σ. We therefore choose a boundary-compressing disk D whose compressing arc $\alpha \subset \Sigma$ is "outermost", i.e., one of the two components of $\Sigma - \alpha$ does not contain any boundary-compressing arcs which are not isotopic to α. Let Σ' denote this component of $\Sigma - \alpha$.

Finally, we consider the surface $\Sigma'' = \Sigma' \cup_\alpha D$. Since D is a boundary-compressing disk α is not boundary-parallel in Σ, so neither Σ' nor Σ'' is a disk. Additionally, Σ'' is incompressible: suppose, toward a contradiction, that Σ'' admits a compressing disk D'. Then $\partial D'$ may be isotoped into $\Sigma' \subset \Sigma$, and the incompressibility of Σ implies $\partial D'$ bounds a disk in Σ. Since the component of $\Sigma - \partial D'$ not contained in Σ' has nonempty boundary, $\partial D'$ in fact bounds a disk in $\Sigma' \subset \Sigma''$, contradicting the compression assumption. A similar argument, using the fact that α is outermost in Σ, shows Σ'' is boundary-incompressible. Then Σ'' is incompressible, boundary-incompressible, and not a disk, contradicting Lemma 3.1. We conclude Σ must admit at least one boundary-compressing disk intersecting two boundary components, so Σ is a boundary-parallel annulus. □

Theorem 1.3 now follows from the more general Theorem 3.3 below. Positive knots satisfy the hypotheses of Theorem 3.3 by the discussion in Section 2.

Theorem 3.3. *Let $K \subset S^3$ be a knot such that*:

- $g(K) = g_4(K)$.
- *Every minimal genus Seifert surface of K is free.*

Then K is band prime.

Proof. Let K be a knot satisfying the given hypotheses, and suppose K can be written as a band sum of knots $K_0, K_1 \subset S^3$. We will show K is equivalent to the standard connect sum $K_0 \# K_1$; by a result of Miyazaki, this occurs if and only if the band sum is trivial [26] (compare [11, Theorem 2]).

As mentioned in the introduction, Miyazaki also showed K is ribbon concordant to the connect sum $K_0 \# K_1$ [24]. Thus, by Lemma 2.1,

$$g(K) = g(K_0 \# K_1) = g(K_0) + g(K_1).$$

Gabai proved that a band sum preserves genus in the above sense if and only if there exists a minimal genus Seifert surface S for K, such that S is a band sum

$$S = S_0 \#_b S_1$$

of Seifert surfaces S_i for K_i, $i = 0, 1$ [13]. In other words, S is the result of joining the split union $S_0 \sqcup S_1$ along a band b. The band b may be different from the band in our initial band sum representation of K, but this is not an issue since we will ultimately show $K = K_0 \# K_1$. By the discussion in the first paragraph, this will imply *any* representation of K as a band sum of K_0 and K_1 is trivial.

Fix such a surface $S = S_0 \#_b S_1$, and let $\Sigma \subset S^3$ be a sphere separating the component surfaces S_0 and S_1. We choose Σ transverse to S so that the number of intersection components $|\Sigma \cap S|$ is minimal among all such spheres; then $\Sigma \cap S$ consists of a set of parallel cocores of b, and $|\Sigma \cap S|$ is odd since Σ separates the feet of b. Let $\nu(S)$ be a regular neighborhood of S, and let $H = S^3 - \nu(S)$. We claim the planar surface

$$\Sigma \cap H = \Sigma - \nu(S)$$

is incompressible in H. Suppose not: then $\Sigma \cap H$ admits a compressing disk D. The curve ∂D separates Σ into components Σ_0 and Σ_1, and each component Σ_i contains some intersection with S since ∂D does not bound a disk in $\Sigma - \nu(S)$. Because Σ separates S_0 and S_1, one of the spheres $\Sigma_0 \cup D$ or $\Sigma_1 \cup D$ does as well. Assuming the former without loss of generality, we conclude that $\Sigma_0 \cup D$ is a sphere separating S_0 and S_1 with

$$|(\Sigma_0 \cup D) \cap S| < |\Sigma \cap S|.$$

This contradicts the minimality of $|\Sigma \cap S|$, proving the claim.

The surface S has minimal genus, so H is a handlebody by hypothesis. Now $\Sigma \cap H$ is an incompressible planar surface in H, and since $\Sigma \cap S$ consists of a set of parallel cocores of b, $\partial(\Sigma \cap H)$ consists of a set of parallel curves which are separating on ∂H. From Lemma 3.2, since $|\Sigma \cap S|$ is odd, we conclude that $|\Sigma \cap S| = 1$ and $\Sigma - S$ is a disk. Thus the band sum $K_0 \#_b K_1$ is trivial and K is band prime. □

Remark 3.4. It is not true that *strongly quasipositive* knots are band prime, even though a strongly quasipositive knot K satisfies $g(K) = g_4(K)$, see [5, Section 4.1]. It is true, however, that fibered strongly quasipositive knots are band prime by a result of Baker and Motegi [5, Theorem 1.1]. In fact, Baker and Motegi's argument can be modified to show that fibered strongly quasipositive knots — like fibered positive knots — are ribbon concordance minimal.

4. A condition for minimality

In this section we prove Theorem 1.2, which involves piecing together several existing results. First, the *lower central series* $\{\gamma_i\}_{i \geq 0}$ of a group G is defined recursively by

$$\gamma_0 = G, \qquad \gamma_i = [\gamma_{i-1}, G] \quad \text{for all } i > 0,$$

where $[*, *]$ indicates the commutator. The group G is *residually nilpotent* if $\bigcap_{i=0}^{\infty} \gamma_i = \{1\}$, and following Gordon [15] we say a knot $K \subset S^3$ is *residually nilpotent* if the commutator subgroup of the knot group is residually nilpotent. As in the introduction, let $d(K)$ denote the degree of Δ_K. Then Gordon proves:

Lemma 4.1 [15, Lemma 3.4]. *Let $K_0, K_1 \subset S^3$ be knots with $K_0 \leq K_1$. If K_1 is residually nilpotent and $d(K_0) = d(K_1)$, then $K_0 \cong K_1$.*

Fibered knots are examples of residually nilpotent knots, since their commutator subgroups are free (see [20, Chapter 5]), but little is known in general about which knots are residually nilpotent. Mayland and Murasugi [21] proved:

Theorem 4.2 [21]. *Let K be a pseudoalternating knot such that the leading coefficient of Δ_K is a prime power. Then K is residually nilpotent.*

Next, we recall some background on knot Floer homology [30; 31]. The hat version of knot Floer homology, $\widehat{HFK}$, associates a finitely generated, bigraded $\mathbb{F}_2$-vector space to any knot K:

$$\widehat{HFK}(K) = \bigoplus_{i,j\in\mathbb{Z}} \widehat{HFK}_i(K, j).$$

The i and j gradings are called the *Maslov* and *Alexander* gradings respectively, and the graded Euler characteristic of $\widehat{HFK}$ is the symmetrized Alexander polynomial:

$$\Delta_K(t) = \sum_{i,j} (-1)^i \dim(\widehat{HFK}_i(K, j))\, t^j. \tag{2}$$

Generalizing the classical fact that $d(K) \leq 2g(K)$, knot Floer homology detects knot genus in the following sense [29]:

$$g(K) = \max\{j \mid \widehat{HFK}(K, j) \neq 0\}.$$

Any concordance $C \subset S^3 \times I$ between knots $K_0 \subset S^3 \times \{0\}$ and $K_1 \subset S^3 \times \{1\}$ induces a bigrading-preserving homomorphism

$$C_* : \widehat{HFK}(K_0) \to \widehat{HFK}(K_1),$$

and Zemke [42] proved that if C is a ribbon concordance from K_1 to K_0, so $K_0 \leq K_1$, then the map C_* is injective. Finally, we require the following theorem of Cheng, Hedden and Sarkar [7]:

Theorem 4.3 [7, Corollary 1.6]. *If K is a pseudoalternating link, then the top Alexander grading $\widehat{HFK}(K, g(K))$ of $\widehat{HFK}(K)$ is supported in a single Maslov grading.*

Theorem 1.2 now follows easily from these results and the next proposition (compare [6, Propositon 1.4]).

Proposition 4.4. *Let K_1 be a pseudoalternating knot such that $g(K_1) = g_4(K_1)$, and suppose $K_0 \leq K_1$. Then $\Delta_{K_0} = \Delta_{K_1}$.*

Proof. By Lemma 2.1, $g(K_0) = g(K_1) = g$ for some $g \in \mathbb{N}$. Fix a ribbon concordance $C \subset S^3 \times I$ from K_1 to K_0, and let C_* denote the induced map

$$C_* : \widehat{HFK}(K_0, g) \to \widehat{HFK}(K_1, g).$$

This map is injective by Zemke's result, and both groups are nonzero since the knots have genus g. By Theorem 4.3 $\widehat{HFK}(K_1; g)$ is supported in a single Maslov grading, and thus $\widehat{HFK}(K_0; g)$ is as well. Consequently, (2) implies that

$$d(K_0) = 2g(K_0) = 2g(K_1) = d(K_1).$$

Since Δ_{K_0} divides Δ_{K_1}, we have $\Delta_{K_0} = m\Delta_{K_1}$ for some $m \in \mathbb{Z}$ [14]. But

$$\Delta_{K_0}(1) = \Delta_{K_1}(1) = 1,$$

so $\Delta_{K_0} = \Delta_{K_1}$. □

Proof of Theorem 1.2.. Suppose K_0 and K_1 are knots such that K_1 satisfies the hypotheses of the theorem and $K_0 \leq K_1$. By Proposition 4.4, $\Delta_{K_0} = \Delta_{K_1}$. Additionally K_1 is residually nilpotent by Theorem 4.2, so Lemma 4.1 implies $K_1 \cong K_0$. □

Remark 4.5. Two-bridge knots are residually nilpotent by a theorem of Johnson [17]. Thus our proof of Theorem 1.2 also gives an alternate proof of Tagami's theorem that positive two-bridge knots are ribbon concordance minimal, using [17, Corollary 1.3] in place of Theorem 4.2.

5. Alexander modules and $\mathbb{Q}$-anisotropy

We now work toward the proofs of Theorem 1.6 and Corollary 1.7. We expect that the following proposition is known to experts, but we have not been able to find it in the literature.

Proposition 5.1. *Let $K \subset S^3$ be a knot such that Δ_K has a rational root q. Then $q = (m-1)/m$ for some integer $m \notin \{0, 1\}$. In particular, q is positive.*

Proof. Let $q = a/b$ for $a, b \in \mathbb{Z}$. Fix an oriented Seifert surface S for K, and let

$$\iota_\pm : H_1(S) \to H_1(S^3 - S)$$

be the maps induced by pushing curves off S to the $\pm$-component of the unit normal bundle of S, with sign determined by the orientation. We represent these maps by Seifert matrices, which we also denote by ι_+ and ι_-.

Now

$$\Delta_K(t) = \det(\iota_+ - t\iota_-),$$

and $\Delta_K(q) = 0$ implies that

$$0 = \det(b\iota_+ - a\iota_-).$$

Therefore there exists a nonzero vector $v \in H_1(S)$ such that

$$b\iota_+(v) = a\iota_-(v).$$

Dividing by a scalar if necessary, we assume v is primitive, i.e., that v extends to a basis of $H_1(S)$.

The intersection pairing

$$\cdot : H_1(S) \times H_1(S) \to \mathbb{Z}$$

satisfies the identity

$$v_1 \cdot v_2 = \operatorname{lk}(v_1, (\iota_+ - \iota_-)(v_2)) \quad \text{for all } v_1, v_2 \in H_1(S),$$

where lk indicates the linking number. Since v is a primitive homology class on a once-punctured surface, v is representable by a simple, closed nonseparating curve on S [22; 35], and it follows that there exists $w \in H_1(S)$ such that $w \cdot v = 1$.

We have

$$1 = \mathrm{lk}(w, (\iota_+ - \iota_-)(v)) = \mathrm{lk}(w, \iota_+(v)) - \mathrm{lk}(w, \iota_-(v))$$

and therefore

$$a\,\mathrm{lk}(w, \iota_+(v)) = b\,\mathrm{lk}(w, \iota_-(v)) = b(\mathrm{lk}(w, \iota_+(v)) - 1).$$

Since a and b are both nonzero, $\mathrm{lk}(w, \iota_+(v)) \notin \{0, 1\}$. We conclude that

$$q = \frac{a}{b} = \frac{\mathrm{lk}(w, \iota_+(v)) - 1}{\mathrm{lk}(w, \iota_+(v))}$$

as desired. □

Corollary 5.2. *If K is a positive knot, then Δ_K has no rational roots.*

Proof. Let K be positive. The *Conway polynomial* of K, $\nabla_K(z) \in \mathbb{Z}[z^2]$, is the unique polynomial satisfying

$$\nabla_K(x - x^{-1}) = \Delta_K(x^2),$$

where Δ_K is the symmetrized Alexander polynomial. Cromwell proved that for a positive knot K, the coefficients of ∇_K are nonnegative [9, Corollary 2.1]. Also

$$\nabla_K(0) = \Delta_K(1) = 1,$$

so ∇_K has no real roots. It follows that Δ_K has no positive real roots, since a positive real root q of Δ_K would yield a real root $\sqrt{q} - 1/\sqrt{q}$ of ∇_K. Therefore, by Proposition 5.1, Δ_K has no rational roots. □

Let $\overline{X}$ denote the infinite cyclic cover of the exterior of a knot K. For any field $\mathbb{F}$, an *invariant $\mathbb{F}$-isotropic subspace* of $H^1(\overline{X}; \mathbb{F}) \cong H^1(\overline{X}, \partial\overline{X}; \mathbb{F})$ is one which is preserved by the action of deck transformations and self-annihilating with respect to the cup product

$$\smile : H^1(\overline{X}; \mathbb{F}) \times H^1(\overline{X}, \partial\overline{X}; \mathbb{F}) \to H^2(\overline{X}, \partial\overline{X}; \mathbb{F}) \cong \mathbb{F}.$$

The knot K is called $\mathbb{F}$-*anisotropic* if $H^1(\overline{X}; \mathbb{F})$ does not contain a nontrivial invariant $\mathbb{F}$-isotropic subspace, see [15] for more background. As the introduction discusses, $\mathbb{Q}$-anisotropy can be used to restrict changes to the Alexander module under concordance: Kervaire [18] and Gilmer [14, Proposition 4.2] proved that if (algebraically) concordant knots K and K' are $\mathbb{Q}$-anisotropic and admit Seifert matrices which are invertible over $\mathbb{Q}$, then the rational Alexander modules of K and K' are isomorphic.

Proposition 5.3. *If a positive knot K satisfies $|\sigma(K)| \geq d(K) - 2$, then K is $\mathbb{Q}$-anisotropic.*

Proof. Let $\overline{X}$ denote the infinite cyclic cover of the exterior of K, and let

$$t : H^1(\overline{X}; \mathbb{Q}) \to H^1(\overline{X}; \mathbb{Q})$$

be the map induced by a primitive deck transformation. Let $\Lambda \subset H^1(\overline{X}; \mathbb{Q})$ be a nontrivial invariant $\mathbb{Q}$-isotropic subspace of $H^1(\overline{X}; \mathbb{Q})$.

Up to a scalar in $\mathbb{Q}$, the characteristic polynomial of t coincides with Δ_K [19, Theorem 6.17]. Since the cup product is skew-symmetric, it is straightforward to check that $H^1(\overline{X}; \mathbb{Q})$ contains a one-dimensional invariant $\mathbb{Q}$-isotropic subspace if and only if Δ_K has a rational root, i.e., if and only if t has a rational eigenvalue (see [15, Proposition 4.3]). Thus, Corollary 5.2 implies $\dim(\Lambda) \geq 2$.

We now recall the Milnor form μ on $H^1(\overline{X}; \mathbb{Q})$, defined by

$$\mu(v, w) = t(v) \smile w + t(w) \smile v$$

for $v, w \in H^1(\overline{X}; \mathbb{Q})$. This is a nondegenerate symmetric bilinear form satisfying $\sigma(\mu) = \sigma(K)$, where σ denotes the signature [10; 23]. As Gordon observes and is easy to check, Λ is also a self-annihilating subspace for μ [15, Proposition 4.5]. Let $V_\pm$ denote a maximal subspace of $H^1(\overline{X}; \mathbb{Q})$ on which μ is $\pm$-definite. Then $V_\pm \cap \Lambda = \{0\}$, so

$$\dim V_\pm \leq \dim H^1(\overline{X}; \mathbb{Q}) - \dim \Lambda \leq \dim H^1(\overline{X}; \mathbb{Q}) - 2 = d(K) - 2.$$

For the last equality, we again use the fact that Δ_K is the characteristic polynomial of t. It follows that

$$|\sigma(K)| = |\sigma(\mu)| \leq d(K) - 4,$$

and since $d(K)$ and $\sigma(K)$ are even this implies the desired inequality. □

Remark 5.4. It is not true that positive knots are $\mathbb{R}$-anisotropic, even if they satisfy the hypothesis of Proposition 5.3: for example, the Alexander polynomial of the positive knot 10_{139} has a negative real root. It is also not true that strongly quasipositive knots are $\mathbb{Q}$-anisotropic, since there exist strongly quasipositive knots which are topologically slice, see, for example, [3].

Proof of Theorem 1.6. Suppose K and K' are concordant positive knots such that $|\sigma(K)| \geq d(K) - 2$. Then $\sigma(K) = \sigma(K')$, and since K and K' are positive we have

$$d(K) = g_4(K) = g_4(K') = d(K').$$

It follows that $|\sigma(K')| \geq d(K') - 2$, so K and K' are both $\mathbb{Q}$-anisotropic by Proposition 5.3. Since $g(K) = d(K)$ and $g(K') = d(K')$, any Seifert matrix of a minimal genus Seifert surface for K or K' is invertible over $\mathbb{Q}$. Thus, by the result of Kervaire and Gilmer discussed before Proposition 5.3, the rational Alexander modules of K and K' are isomorphic. □

Proof of Corollary 1.7. Since the knot 14_{45657} is only the positive knot satisfying $g(K)=4$ and $\sigma(K)=-4$, 14_{45657} is not concordant to any other positive knot. The corollary then follows from Theorem 1.6 and the discussion in the introduction. □

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EXISTENCE FOR SOME DEGENERATE HESSIAN-TYPE EQUATIONS ARISING IN CONFORMAL GEOMETRY

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We establish the a priori estimates for three kinds of degenerate Hessian-type equations arising in conformal geometry. Based on these a priori estimates, we obtain an existence result using the continuity method.

1. Introduction

Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$. Consider a symmetric polynomial of degree k

$$\sigma_k(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k,$$

where

$$\Gamma_k = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ 1 \leq j \leq k\}, \quad 1 \leq k \leq n.$$

For any $n\times n$ matrix U, $U \in \Gamma_k$ means the eigenvalues of U lie in Γ_k and $\sigma_k(U)$ means that σ_k is applied to the eigenvalues of U. In this paper, we derive the a priori estimates for three kinds of degenerate equations arising in conformal geometry. The existence of $C^{1,1}$ solutions, which is the central issue for these degenerate equations, has been obtained by using these estimates.

The degenerate Hessian equations

$$\sigma_k(D^2u) = f(x, u) \geq 0, \quad \Omega \subset \mathbb{R}^n, \tag{1-1}$$

have attracted much attention recently. For $k = n$, the degenerate Monge–Ampère-type equations, Guan and Li studied the degenerate Weyl problem and the degenerate Gauss curvature problem in [16; 17]. They found that the conditions

$$\Delta f^{\frac{1}{n-1}} \geq -A \quad \text{and} \quad |D(f^{1/(n-1)})| \leq A \tag{1-2}$$

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for some constant A are sufficient to get $C^{1,1}$ estimates. Using (1-2), Guan [14] established $C^{1,1}$ estimates for the Dirichlet problem of the degenerate Monge–Ampère equations with homogeneous boundary condition. Guan, Trudinger and Wang [20] established $C^{1,1}$ estimates for the Dirichlet problem with nonhomogeneous boundary condition given by

$$f^{\frac{1}{n-1}} \in C^{1,1}(\overline{\Omega}). \tag{1-3}$$

Here, (1-3) implies (1-2).

For general $1 < k < n$, Dong [9] obtained C^2 estimates for the homogeneous Dirichlet problem with homogeneous boundary by

$$|D(f^{\frac{1}{k-1}})| \leq C f^{\frac{1}{2k-2}} \quad \text{in } \overline{\Omega} \tag{1-4}$$

and

$$f^{\frac{1}{k-1}} \in C^{1,1}(\overline{\Omega}). \tag{1-5}$$

Jiao and Wang [28] generalized the results in [9], and obtained $C^{1,1}$ estimates for k-curvature problems with homogeneous boundary by $f^{\frac{1}{k-1}} \in C^{1,1}(\mathbb{R}^n)$. They also found convex solutions to the Dirichlet problem of degenerate k-Hessian equations with nonhomogeneous boundary condition by (1-5) in [29].

Motivated by [28], it is natural to study the degenerate equations in conformal geometry. One of the central issues in conformal geometry is the k-Yamabe problem. The k-Yamabe problem was raised in [41] in order to find a conformal metric g so that $\sigma_k(A_g) = \text{const.}$, where $A_g = \frac{1}{n-2}\big(\text{Ric}_g - \frac{R_g}{2(n-1)} g\big)$, Ric_g, R_g are the Schouten tensor, the Ricci curvature and scalar curvature of (M, g), respectively. The general prescribing curvature problem is to find a conformal metric such that $\sigma_k(A_g)$ equals a given function f,

$$\sigma_k(A_g) = f(x). \tag{1-6}$$

We say (1-6) is *nondegenerate* if $f > 0$. Fruitful results have been achieved on the k-Yamabe problem and the prescribing curvature problem. Here, we just mention some existence results of the k-Yamabe problem and the prescribing curvature problem. The research of the nondegenerate prescribing curvature problem on manifolds without boundary can be tracked back to Chang, Gursky and Yang [1; 2]. They proved that if the Yamabe constant and $\int_M \sigma_2(A_{g_0})$ are both positive, then there exists a conformal metric g such that $\sigma_2(A_g)$ is a positive constant; see also [22]. Later on, Guan and Wang [18] and Li and Li [31] concluded that on locally conformally flat manifolds, the k-Yamabe problem is solvable for $2 \leq k < n/2$; see also [38]. Gursky and Viaclovsky confirmed the existence of a solution and compactness of solution set to $\sigma_k(A_g) = f > 0$ in the case of $n \geq k > n/2$ when $A_{g_0} \in \Gamma_k$ and (M, g_0) is not conformally equivalent to a sphere. Li and Nguyen [34] obtained that if $A_{g_0} \in \Gamma_k$, $k = n/2$ and (M, g_0) is not conformally equivalent to a sphere, then there is a conformal metric $\tilde{g}$ such that $\sigma_k(A_{\tilde{g}}) = 1$, and the solution set is compact. Other relevant conclusions are included in [3; 4; 10; 11; 12; 32; 40].

In 2003, Gursky and Viaclovsky [23] introduced the modified Schouten tensor

$$A_g^t = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{tR_g}{2(n-1)} g\right).$$

This tensor is, in fact, a constant multiple of the tensor $sA_g - \frac{(1-s)R_g}{2(n-1)} g$ which is introduced in [31]. It is also a meaningful problem to find a conformal metric g with

$$\sigma_k(-A_g^t) = f(x). \tag{1-7}$$

When $t = 0$, $A_g^t = \frac{1}{n-2}\mathrm{Ric}_g$; when $t = 1$, A_g^t is just the Schouten tensor A_g; when $t = n-1$, A_g^t is the Einstein tensor. Gursky and Viaclovsky [23] and Li and Sheng [35] proved that when $t < 1$ every compact manifold with $-A_{g_0}^t \in \Gamma_k$ is conformal to a metric g with $\sigma_k(-A_g^t) = f(x) > 0$.

Another interesting problem is to study the boundary value problem for (1-6) and (1-7) on manifolds (M^n, g) with boundary ∂M. Guan [15] studied the existence problem for (1-6) under the Dirichlet boundary condition. Li and Sheng [36] considered the Dirichlet problem for $\sigma_k(\mathrm{Ric}_g - Rg) = f$. Under various conditions, the Neumann problem arising in conformal geometry is derived in [6; 7; 24; 30; 33; 37].

A key issue for the study of fully nonlinear equations in conformal geometry is the a priori estimates. Thus, there is much profound research on the a priori estimates for (1-6) and (1-7) and their further generalizations. We refer to [5; 18; 19; 25; 26; 27; 31; 38; 42; 43].

It is noticed that most of the results focus on nondegenerate cases in conformal geometry. For the degenerate case, Ge, Lin and Wang [13] obtained that if $f = 0$, $k = 2$, there exists a $C^{1,1}$ metric such that $\sigma_2(A_g) = 0$. A natural question is whether the existence is still valid if f is nonnegative.

We consider an existence for the solutions to degenerate equations in the negative cone $-\Gamma_k = \{\lambda \mid -\lambda \in \Gamma_k\}$. We raise the following problem,

Problem. Let f be a nonnegative function in M. Is it true that there exists a conformal metric g on M with $\sigma_k(-A_g^t) = f \geq 0$?

The question is partly answered in this paper. The results are as follows. Let $g = e^{2w} g_0$. Then

$$-A_g^t = \frac{1-t}{n-2}\Delta w + \nabla^2 w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0 - A_{g_0}^t.$$

Here, w is said to be *admissible* if $-A_g^t \in \Gamma_k$. Note that [38] provides a counterexample to show that the regularity for $\det(\nabla^2 w + |\nabla w|^2 + S) = f$ in $\mathbb{R}^2$ fails where $S(x)$ is some 2×2 matrix. We only consider $\sigma_k(-A_g^t) = f \geq 0$ with $t < 1$.

Theorem 1.1. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$, $-A^t_{g_0} \in \Gamma_k$, $2 \leq k \leq n$, $t < 1$. Suppose that $f^{\frac{1}{k-1}} \in C^{1,1}$ is a nonnegative function in M. Equation (1-7) has an admissible supersolution $\overline{w}$:*

$$\sigma_k(-A^t_{e^{2\overline{w}}g_0}) \leq f(x), \quad -A^t_{e^{2\overline{w}}g_0} \in \Gamma_k.$$

Then, there is a $C^{1,1}$ solution w of (1-7) satisfying $A_{e^{-2w}g_0} \in \bar{\Gamma}_k$.

Perturbation techniques are often used to deal with degenerate problems. First, we transform the original degenerate problem into a nondegenerate equation by adding a positive number ϵ to the right-hand function f. Next, we establish the a priori estimates, which are independent of ϵ. Finally, we find the existence of the solution u_ϵ for each ϵ and then let $\epsilon \to 0$ to derive the existence result for the degenerate problem (1-7).

Since the a priori estimates are the core of the existence theorems, we need the following C^1 and C^2 estimates.

Theorem 1.2. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$, $t < 1$. Suppose that f is a nonnegative function in M and $f^{\frac{1}{k-1}} \in C^{1,1}$. Let w be a C^4 solution to*

$$\sigma_k\left(-A^t_{g_0} + \nabla^2 w + \frac{1-t}{n-2}\Delta w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0\right) = (f^{\frac{1}{k-1}}(x) + \epsilon)^{k-1} e^{2kw}, \quad x \in M, \tag{1-8}$$

where $-A^t_{e^{2w}g_0} \in \Gamma_k$. Then, there is a positive constant C depending on g_0, n, k, t, $\|f^{\frac{1}{k-1}}\|_{C^1}$, $\|f\|_{C^0}$, $\|w\|_{C^0}$ but independent of ϵ, such that

$$\sup |\nabla w| \leq C. \tag{1-9}$$

Furthermore, there is a positive constant C depending on g_0, n, k, t, $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|\nabla w\|_{C^0}$ but independent of ϵ, such that

$$\sup |\nabla^2 w| \leq C(1 + e^{\frac{2k}{k-1}\sup w}). \tag{1-10}$$

The next theorems concern the estimates for $\sigma_k(A_g) = f \geq 0$. Before introducing the estimates, we give the following notation. Set $0 < \epsilon < 1$, and let $0 \leq \eta(t) \leq 1$ be a smooth function, which satisfies

$$\eta(t) = 1 \ \text{ for } t \leq \tfrac{1}{4}\theta; \quad \eta(t) = 0 \ \text{ for } t \geq \tfrac{1}{2}\theta; \quad \eta' \leq C_1\theta, \quad \eta'' \leq C_2\theta. \tag{1-11}$$

Here, $C_1, C_2, \theta > 0$ are constants.

Theorem 1.3. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n \geq 3$, $2 \leq k \leq n$. Suppose that f is a nonnegative function in M. Let u be a*

C^4 solution to

$$\begin{aligned}\text{(1-12)}\quad & \sigma_k\big(A_{g_0}+\nabla^2 u+\nabla u\otimes\nabla u-\tfrac{1}{2}|\nabla u|^2 g_0\big)\\ &\qquad = \big(f^{\frac{1}{k-1}}(x)+\epsilon\eta(f^{\frac{1}{k-1}}(x))\big)^{k-1}e^{-2ku}, \quad x\in M,\end{aligned}$$

where $A_{e^{-2u}g_0}\in\Gamma_k$. If $f^{\frac{1}{k-1}}\in C^{1,1}$, then

$$\sup|\nabla u|\le C, \tag{1-13}$$

where a positive constant C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|u\|_{C^0}$, g_0, n, k, θ, C_1, C_2 but is independent of ϵ.

Moreover,

$$\sup|\nabla^2 u|\le C(1+e^{-\frac{2k}{k-1}\inf u}), \tag{1-14}$$

where a positive constant C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, $\|\nabla u\|_{C^0}$, g_0, n, k, θ, C_1, C_2 but is independent of ϵ.

The next theorem concerns the case of $f^{\frac{1}{k}}\in C^2$.

Theorem 1.4. *Let (M, g_0) be an n-dimensional closed smooth Riemannian manifold, $n\ge 3$, $2\le k\le n$. Suppose that f is a nonnegative function in M. Let u be a C^4 solution to*

$$\begin{aligned}\text{(1-15)}\quad & \sigma_k^{\frac{1}{k}}\big(A_{g_0}+\nabla^2 u+\nabla u\otimes\nabla u-\tfrac{1}{2}|\nabla u|^2 g_0\big)\\ &\qquad = \big(f^{\frac{1}{k}}(x)+\epsilon\eta(f^{\frac{1}{k}}(x))\big)e^{-2u}, \quad x\in M,\end{aligned}$$

where $A_{e^{-2u}g_0}\in\Gamma_k$. If $f^{\frac{1}{k}}\in C^2$, then

$$\sup(|\nabla u|^2+|\nabla^2 u|)\le C(1+e^{-2\inf u}), \tag{1-16}$$

where a positive constant C depends on g_0, n, k, $\|f^{\frac{1}{k}}\|_{C^2}$, $\|f\|_{C^0}$, θ, C_1, C_2 but is independent of ϵ.

The paper is built up as follows. In Section 2, we introduce the notation and the necessary formulas. In Sections 3, 4, 5, we establish the crucial estimates and prove Theorems 1.3, 1.4 and 1.2, respectively. Finally, the existence result of Theorem 1.1 is proved in Section 6.

2. Preliminaries

Throughout this paper,

$$\sigma_{k-1}(\lambda\mid i)=\frac{\partial\sigma_k}{\partial\lambda_i}\quad\text{and}\quad\sigma_{k-2}(\lambda\mid ij)=\frac{\partial^2\sigma_k}{\partial\lambda_i\partial\lambda_j}.$$

We list some properties of σ_k, which will be used later.

Proposition 2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $1 \le k \le n$. Then we have*

(1) $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n$;

(2) $\sigma_{k-1}(\lambda \mid i) > 0$ *for* $\lambda \in \Gamma_k$ *and* $1 \le i \le n$;

(3) $\sigma_k(\lambda) = \sigma_k(\lambda \mid i) + \lambda_i \sigma_{k-1}(\lambda \mid i)$ *for* $1 \le i \le n$;

(4) *if* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *then* $\sigma_{k-1}(\lambda \mid 1) \le \sigma_{k-1}(\lambda \mid 2) \le \cdots \le \sigma_{k-1}(\lambda \mid n)$ *for* $\lambda \in \Gamma_k$;

(5) $\sum_{i=1}^n \sigma_{k-1}(\lambda \mid i) = (n-k+1)\sigma_{k-1}(\lambda)$.

The generalized Newton–MacLaurin inequality is as follows, which will be used later.

Proposition 2.2. *For $\lambda \in \Gamma_m$ and $m > l \ge 0$, $r > s \ge 0$, $m \ge r$, $l \ge s$, we have*

$$\left[\frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{m-l}} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}.$$

Proof. See [39]. □

The following lemma is the key in the proof of Theorems 1.3 and 1.2.

Lemma 2.3. *Let $\alpha = \frac{1}{k-1}$. If $U \in \Gamma_k$, then*

$$-\sigma_k^{ii,jj} U_{iip} U_{jjp} \ge \sigma_k \left[\frac{(\sigma_k)_p}{\sigma_k} - \frac{(\sigma_1)_p}{\sigma_1}\right]\left[(\alpha-1)\frac{(\sigma_k)_p}{\sigma_k} - (\alpha+1)\frac{(\sigma_1)_p}{\sigma_1}\right]. \tag{2-1}$$

Proof. See [21]. □

In this paper ∇ denotes the Levi-Civita connection on (M, g_0), and the curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let $e_1, e_2, \dots, e_n$ be local frames on M and define $g_{ij} = g_0(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, while the Christoffel symbols Γ_{ij}^k and curvature coefficients are given, respectively, by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ and

$$R_{ijkl} = g_0(R(e_k, e_l)e_j, e_i), \quad R^i_{jkl} = g^{im} R_{mjkl}.$$

We write $u_i = \nabla_i u = \nabla_{e_i} u$, $u_{ji} = [\nabla\nabla u](e_j, e_i) = [\nabla_i(\nabla u)](e_j) = \nabla_i(\nabla_j u) - \Gamma_{ij}^k u_k$, $u_{ijk} = [\nabla_k(\nabla\nabla u)](e_i, e_j)$, etc. Note that $u_{ij} = u_{ji}$,

$$u_{ijk} - u_{kij} = R^m_{ijk} u_m, \tag{2-2}$$

$$u_{kilj} - u_{klij} = R^m_{kil,j} u_m + R^m_{kil} u_{mj}, \tag{2-3}$$

$$u_{kijl} - u_{kilj} = R^m_{ijl} u_{km} + R^m_{kjl} u_{mi}. \tag{2-4}$$

From (2-2), (2-3) and (2-4), we obtain

(2-5) $$u_{ijkl}-u_{klij}=R^m_{ijk,l}u_m+R^m_{kil,j}u_m+R^m_{ijk}u_{ml}+R^m_{kil}u_{mj}+R^m_{ijl}u_{km}+R^m_{kjl}u_{mi}.$$

For convenience, we introduce the notation

$$F(U)=\sigma_k(U),\qquad F^{ij}=\frac{\partial F}{\partial U_{ij}},\qquad F^{ij,rs}=\frac{\partial^2 F}{\partial U_{ij}\partial U_{rs}}.$$

Lemma 2.4. (1) *If* $U=\nabla^2 u+\nabla u\otimes\nabla u-\frac{1}{2}|\nabla u|^2 g_0+A_{g_0}$, *then*

$$U_{ijp}=u_{ijp}+u_iu_{jp}+u_ju_{ip}-u_qu_{qp}g_{ij}+A_{ij,p},$$

and

$$U_{ijpp}=u_{ijpp}+2u_{ip}u_{jp}+u_iu_{jpp}+u_ju_{ipp}-u_{qp}u_{qp}g_{ij}-u_qu_{qpp}g_{ij}+A_{ij,pp},$$

where A_{ij} *are the components of* A_{g_0} *and* g_{ij} *are the components of* g_0.

(2) *If* $W=-A^t_{g_0}+\nabla^2 w+\frac{1-t}{n-2}\Delta w g_0-\nabla w\otimes\nabla w+\frac{2-t}{2}|\nabla w|^2 g_0$, *then*

$$W_{ijp}=-A^t_{ij,p}+w_{ijp}+\frac{1-t}{n-2}w_{qqp}g_{ij}-w_iw_{jp}-w_jw_{ip}+(2-t)w_qw_{qp}g_{ij},$$

and

$$\begin{aligned}W_{ijpp}=-A^t_{ij,pp}+w_{ijpp}+\frac{1-t}{n-2}w_{qqpp}g_{ij}-w_iw_{jpp}-w_jw_{ipp}\\ -2w_{ip}w_{jp}+(2-t)w_{qp}w_{qp}g_{ij}+(2-t)w_qw_{qpp}g_{ij},\end{aligned}$$

where A^t_{ij} *are the components of* $A^t_{g_0}$ *and* g_{ij} *are the components of* g_0.

The following two lemmas will be used in the proof of Theorems 1.3 and 1.2.

Lemma 2.5. *Assume that* $-\mu\le s\le\mu$. *Then, we may choose constants* a, b, *and* p *depending only on* $\mu>0$ *so that* $\gamma(s)=a(b+s)^p$ *satisfies*

(2-6) $$-\tfrac{1}{4}\gamma'(s)\ge\gamma''(s)+\gamma'(s)-\gamma'(s)^2>0.$$

Proof. We have

$$\gamma'(s)=pa(b+s)^{p-1},\qquad \gamma''(s)=p(p-1)a(b+s)^{p-2}.$$

Hence,

$$\begin{aligned}\frac{\gamma''(s)-\gamma'(s)^2}{-\gamma'(s)}&=\frac{p(p-1)a(b+s)^{p-2}-p^2a^2(b+s)^{2p-2}}{-pa(b+s)^{p-1}}\\&=\frac{1-p}{b+s}+(b+s)^{p-1}ap.\end{aligned}$$

Now choose $b = 17\mu + 16$, $p = -\frac{35}{34}(b+\mu)$, $a = \frac{(b-\mu)^{1-p}}{-34p}$. Then

$$0 < \frac{1}{b+\mu} \le \frac{1}{b+s} \le \frac{1}{16}.$$

Moreover,

$$\frac{35}{34} = \frac{-p}{b+\mu} \le \frac{-p}{b+s} \le \frac{35}{34}\frac{b+\mu}{b-\mu} < \frac{9}{8}\cdot\frac{35}{34},$$

and

$$|(b+s)^{p-1}ap| \le (b-\mu)^{p-1}|ap| = \frac{1}{34}.$$

Therefore,

$$1 < \frac{1-p}{b+s} + (b+s)^{p-1}ap \le \frac{5}{4}.$$

Then, (2-6) is proved. □

Lemma 2.6. *Assume that $-\mu \le s \le \mu$, $t < 1$. Then, we may choose constants a, b, and p depending on $\mu > 0$ and t so that $\gamma(s) = a(b+s)^p$ satisfies*

$$\frac{1-t}{2(n-2)}(\gamma''(s) - \gamma'(s)^2) \ge \gamma'(s) > 0. \tag{2-7}$$

Proof. It is easily seen that

$$\begin{aligned}\frac{\gamma''(s)-\gamma'(s)^2}{\gamma'(s)} &= \frac{p(p-1)a(b+s)^{p-2} - p^2a^2(b+s)^{2p-2}}{pa(b+s)^{p-1}} \\ &= \frac{p-1}{b+s} - (b+s)^{p-1}ap.\end{aligned}$$

Now choose $b = 2\mu + \frac{8(1-t)}{n-2}$, $p = \frac{9(n-2)}{4(1-t)}(b+\mu)$, $a = (b+\mu)^{1-p}\cdot\frac{n-2}{8p(1-t)}$. Similar to the calculation in Lemma 2.5, we have

$$\frac{2(n-2)}{1-t} \le \frac{p-1}{b+s} - (b+s)^{p-1}ap.$$

Then, (2-7) is proved. □

The following lemma will be used in the proof of Theorem 1.1.

Lemma 2.7. *Let*

$$\begin{aligned} W &= \frac{1-t}{n-2}\Delta w + \nabla^2 w - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0 - A^t_{g_0}, \\ \overline{W} &= \frac{1-t}{n-2}\Delta \overline{w} + \nabla^2 \overline{w} - \nabla \overline{w} \otimes \nabla \overline{w} + \frac{2-t}{2}|\nabla \overline{w}|^2 g_0 - A^t_{g_0}.\end{aligned}$$

Suppose $\sigma_k(\overline{W}) \le fe^{2k\overline{w}}$, $\sigma_k(\overline{W}) \ne 0$, $\sigma_k(W) = fe^{2kw}$. Then we have $w \le \overline{w}$.

Proof. Let $t < 1$, $v^{\frac{4}{n-2}} = e^{2w}$, $\bar{v}^{\frac{4}{n-2}} = e^{2\overline{w}}$,

$$V = \frac{1-t}{n-2}\Delta v + \nabla^2 v - \frac{n}{n-2}\frac{\nabla v \otimes \nabla v}{v} + \frac{1}{n-2}\frac{|\nabla v|^2}{v} g_0 - \frac{n-2}{2} v A^t_{g_0}, \tag{2-8}$$

$$\overline{V} = \frac{1-t}{n-2}\Delta \bar{v} + \nabla^2 \bar{v} - \frac{n}{n-2}\frac{\nabla \bar{v} \otimes \nabla \bar{v}}{\bar{v}} + \frac{1}{n-2}\frac{|\nabla \bar{v}|^2}{\bar{v}} g_0 - \frac{n-2}{2} \bar{v} A^t_{g_0}. \tag{2-9}$$

It is enough to prove that $v \le \bar{v}$. Suppose it is not true. Then by the positivity of v and $\bar{v}$, we find a number $\beta > 1$ such that $\beta\bar{v} \ge v$ and $\beta\bar{v}(\bar{x}) = v(\bar{x})$. Then

$$\nabla(\beta\bar{v})(\bar{x}) = \nabla v(\bar{x}), \quad [\nabla^2(\beta\bar{v}) - \nabla^2 v](\bar{x}) \ge 0. \tag{2-10}$$

Moreover,

$$\begin{aligned} \sigma_k(V)(\bar{x}) \le \sigma_k\bigg(&\frac{1-t}{n-2}\Delta(\beta\bar{v}) + \nabla^2(\beta\bar{v}) - \frac{n}{n-2}\frac{\nabla(\beta\bar{v}) \otimes \nabla(\beta\bar{v})}{\beta\bar{v}} \\ &+ \frac{1}{n-2}\frac{|\nabla(\beta\bar{v})|^2}{\beta\bar{v}} g_0 - \frac{n-2}{2}\beta\bar{v}A^t_{g_0}\bigg)\Big|_{\bar{x}} \\ = \beta^k \sigma_k(\overline{V})(\bar{x}).& \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_k(V)(\bar{x}) &= \left(\frac{n-2}{2} v^{\frac{n+2}{n-2}}\right)^k f\Big|_{\bar{x}} = \left(\frac{n-2}{2} (\beta\bar{v})^{\frac{n+2}{n-2}}\right)^k f\Big|_{\bar{x}} \\ &= \left(\frac{n-2}{2} \beta^{\frac{n+2}{n-2}} \bar{v}^{\frac{n+2}{n-2}}\right)^k f\Big|_{\bar{x}} \\ &\ge \beta^{k\frac{n+2}{n-2}} \sigma_k(\overline{V})(\bar{x}), \end{aligned}$$

which is a contradiction. □

3. The proof of Theorem 1.3

In this section, we obtain C^1 and C^2 estimates with the assumption that the maximum modulus estimation exists for (1-12). We define $\bar{f} = (f^\alpha + \epsilon\eta(f^\alpha))^{k-1}$, $\alpha = \frac{1}{k-1}$, $F = \sigma_k$, $U = A_{g_0} + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2}|\nabla u|^2 g_0$, where u is a solution to (1-12).

First of all, we consider C^1 estimates of u. Let $K = \big(1 + \frac{1}{2}|\nabla u|^2\big)e^{\gamma(u)}$. Here γ is a function of the form $\gamma(s) = a(b+s)^p$. The constants a, b, p are chosen as in Lemma 2.5. Assume that K achieves the maximal value at some point $\tilde{x}$, which implies that $K_i(\tilde{x}) = 0$. In the rest of the proof, all calculations will be performed at the maximum point $\tilde{x}$. We may assume U_{ij} and F^{ij} are diagonal. Then, in an orthonormal frame,

$$0 = K_i(\tilde{x}) = e^{\gamma(u)}\big(\big(1 + \tfrac{1}{2}u_l^2\big)\gamma' u_i + u_l u_{li}\big) \tag{3-1}$$

and

$$K_{ij}(\tilde{x}) = e^{\gamma(u)}\Big(\big(1+\tfrac{1}{2}u_l^2\big)((\gamma')^2 u_i u_j + \gamma' u_{ij} + \gamma'' u_i u_j) + u_l u_{lj}\gamma' u_i + u_l u_{li}\gamma' u_j + u_{lj}u_{li} + u_l u_{lij}\Big).$$

Differentiating (1-12) gives

$$F^{ij}U_{ijl} = (\bar{f}e^{-2ku})_l. \tag{3-2}$$

Using (3-1), (3-2) and (2-2), we have

$$\begin{aligned}
0 &\geq e^{-\gamma(u)}F^{ij}K_{ij}(\tilde{x}) \\
&= F^{ij}\big(u_l u_{lij} + \big(1+\tfrac{1}{2}u_l^2\big)\big(((\gamma')^2+\gamma'')u_i u_j + \gamma' u_{ij}\big) + 2\gamma' u_l u_{li} u_j + u_{li}u_{lj}\big) \\
&\geq F^{ij}\big(u_l u_{ijl} + \big(1+\tfrac{1}{2}u_l^2\big)\big(((\gamma')^2+\gamma'')u_i u_j + \gamma' u_{ij}\big) + 2\gamma' u_l u_{li} u_j\big) \\
&\qquad - C\sum F^{ii}(|\nabla u|^2+1) \\
&\geq u_l(\bar{f}e^{-2ku})_l + \gamma' k\bar{f}e^{-2ku}\big(1+\tfrac{1}{2}u_l^2\big) \\
&\qquad + F^{ij}\big(-2\big(-\tfrac{1}{2}\big)u_l u_s u_{sl}\delta_{ij} - 2u_l u_i u_{lj} + \gamma'\big(-\big(-\tfrac{1}{2}\big)u_s^2\delta_{ij} - u_i u_j\big)\big(1+\tfrac{1}{2}u_l^2\big)\big) \\
&\qquad + F^{ij}\big(\big(1+\tfrac{1}{2}u_l^2\big)(\gamma''-\gamma'^2)(u_i u_j)\big) - C\sum F^{ii}(|\nabla u|^2+1) \\
&\geq u_l(\bar{f}e^{-2ku})_l + \gamma' k\bar{f}e^{-2ku}\big(1+\tfrac{1}{2}u_l^2\big) \\
&\qquad + F^{ij}\big(\gamma'\big(1+\tfrac{1}{2}u_l^2\big)u_i u_j - \tfrac{1}{2}\gamma'\big(1+\tfrac{1}{2}u_l^2\big)u_s^2\delta_{ij} + \big(1+\tfrac{1}{2}u_l^2\big)(\gamma''-\gamma'^2)u_i u_j\big) \\
&\qquad - C\sum F^{ii}(|\nabla u|^2+1).
\end{aligned}$$

On the other hand, applying the Newton–MacLaurin inequality, we have

$$\sigma_{k-1} \geq C\sigma_k^{1-\frac{1}{k}}.$$

Thus

$$C\bar{f}^{1-\frac{1}{k}}e^{-2(k-1)u} \leq \sigma_{k-1} = C\sum F^{ii}. \tag{3-3}$$

Moreover, using $f^{\frac{1}{k-1}} \in C^{1,1}$, we have $\bar{f}^{\frac{1}{k-1}} \in C^{1,1}$, which implies

$$\bar{f}^{\frac{1}{k}} \in C^1.$$

Thus

$$|\bar{f}_i| \leq C\bar{f}^{1-\frac{1}{k}}. \tag{3-4}$$

Then, using (3-3) and (3-4), we obtain

$$\big|u_l(\bar{f}e^{-2ku})_l + \gamma' k\bar{f}e^{-2ku}\big(1+\tfrac{1}{2}u_l^2\big)\big| \leq C\bar{f}^{1-\frac{1}{k}}e^{-2ku}(|\nabla u|^2+1). \tag{3-5}$$

Let $\delta = \min\{-\frac{1}{8}\gamma'\}$. Combining (3-5) with (3-3), it follows that

$$\begin{aligned}
0 &\geq F^{ii}\big(-\tfrac{1}{2}\gamma' u_s^2 + (\gamma''+\gamma'-\gamma'^2)u_i^2\big)\big(1+\tfrac{1}{2}u_l^2\big) - C\bar{f}^{1-\frac{1}{k}}e^{-2ku}(|\nabla u|^2+1) \\
&\qquad - C\sum F^{ii}(|\nabla u|^2+1) \\
&\geq C_3\bar{f}^{1-\frac{1}{k}}e^{-2(k-1)u}\big(\delta|\nabla u|^4 - C(e^{-2u}+1)(|\nabla u|^2+1)\big).
\end{aligned} \tag{3-6}$$

To derive (1-13), we divide the proof into two cases.

(A) $\bar{f} > 0$. Then, (1-13) follows from (3-6).

(B) $\bar{f} = 0$. Then, (3-6) implies

$$\begin{aligned} 0 &\geq F^{ii}\big(-\tfrac{1}{2}\gamma' u_s^2 + (\gamma'' + \gamma' - \gamma'^2)u_i^2\big)\big(1 + \tfrac{1}{2}u_l^2\big) - C\sum F^{ii}(|\nabla u|^2 + 1) \\ &\geq \sum F^{ii}\big(\delta|\nabla u|^4 - C(|\nabla u|^2 + 1)\big) \end{aligned} \tag{3-7}$$

and (1-13) is proved.

Next, based on both maximum modulus estimation and gradient estimation, we discuss C^2 estimates of u. We may assume U is diagonal. It follows from $U \in \Gamma_1$,

$$0 \leq \operatorname{tr} U \leq \Delta u + \big(1 - \tfrac{1}{2}n\big)|\nabla u|^2 + C, \quad |\nabla u|^2 \leq C(\Delta u + 1). \tag{3-8}$$

Also, it follows from $U \in \Gamma_2$ that

$$|U_{ij}| \leq C\sigma_1, \quad |u_{ij}| \leq C(\Delta u + 1). \tag{3-9}$$

Thus, we may assume that Δu is sufficiently large and $\Delta u \geq \{1, 2\mu_1\}$, where $\mu_1 = \big(-1 + \tfrac{1}{2}n\big) \sup|\nabla u|^2 + \sup|\operatorname{tr} A_{g_0}|$. Then, we have

$$2\Delta u \geq \sigma_1 \geq \tfrac{1}{2}\Delta u. \tag{3-10}$$

Let $H = \Delta u + |\nabla u|^2$, and let $\breve{x}$ be the maximum point of H. By rotating the coordinates, we may assume $U_{ij}(\breve{x})$ is diagonal. Differentiating H at $\breve{x}$, we have

$$0 = H_i(\breve{x}) = u_{qqi} + 2u_{qi}u_q \tag{3-11}$$

and

$$0 \geq H_{ii}(\breve{x}) = u_{qqii} + 2u_q u_{qii} + 2u_{qi}^2. \tag{3-12}$$

Differentiating (1-12) gives

$$F^{ij}U_{ijl} = (\bar{f}e^{-2ku})_l. \tag{3-13}$$

By (3-11), (3-12), (3-13), (2-2), and (2-5), we obtain

$$\begin{aligned} 0 &\geq F^{ii}H_{ii}(\breve{x}) = F^{ii}(u_{qqii} + 2u_{qii}u_q + 2u_{qi}^2) \\ &\geq F^{ii}(u_{iiqq} + 2u_{iiq}u_q + 2u_{qi}^2) - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}u_{iiqq} + F^{ii}\big(2u_q\big(U_{iiq} - \big(-\tfrac{1}{2}u_l^2 + u_i^2\big)_q\big) + 2u_{qi}^2\big) - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}\big(2u_q u_l u_{lq} - 4u_q u_{iq} u_i + U_{iiqq} - \big(-\tfrac{1}{2}u_l^2 + u_i^2\big)_{qq} + 2u_{qi}^2\big) \\ &\qquad + 2u_q(\bar{f}e^{-2ku})_q - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}(2u_q u_l u_{lq} - 4u_q u_{iq} u_i + U_{iiqq} - (-u_{lq}^2 - u_l u_{lqq} + 2u_i u_{iqq})) \\ &\qquad + 2u_q(\bar{f}e^{-2ku})_q - C\sum F^{ii}(\Delta u + 1) \\ &\geq F^{ii}(U_{iiqq} + u_{ql}^2) + 2u_q(\bar{f}e^{-2ku})_q - C\sum F^{ii}(\Delta u + 1). \end{aligned} \tag{3-14}$$

Then, using the Newton–MacLaurin inequality, we have

$$\sigma_{k-1} \geq C\sigma_k^{1-\alpha}\sigma_1^{\alpha}.$$

Thus

$$\bar{f}^{1-\alpha} \leq Ce^{2k(1-\alpha)u}\sigma_1^{-\alpha}\sigma_{k-1} = Ce^{2k(1-\alpha)u}\sigma_1^{-\alpha}\sum F^{ii}. \tag{3-15}$$

Hence, combining (3-15), $|\bar{f}_i| \leq C\bar{f}^{1-\alpha}$ and $\sigma_1 > \frac{1}{2}$, we have

$$|u_q(\bar{f}e^{-2ku})_q| \leq Ce^{-2k\alpha u}\sum F^{ii}(\Delta u + 1).$$

Thus, (3-14) becomes

$$0 \geq F^{ii}(U_{iiqq} + u_{ql}^2) - C(1 + e^{-2k\alpha u})\sum F^{ii}(\Delta u + 1). \tag{3-16}$$

On the other hand, recall that

$$-\sigma_k^{ii,jj}U_{iip}U_{jjp} \geq \sigma_k\left[\frac{(\sigma_k)_p}{\sigma_k} - \frac{(\sigma_1)_p}{\sigma_1}\right]\left[(\alpha-1)\frac{(\sigma_k)_p}{\sigma_k} - (\alpha+1)\frac{(\sigma_1)_p}{\sigma_1}\right].$$

Using (3-9), (3-10), and (3-11), we have

$$\begin{aligned} |(\sigma_1)_p| &= \left|u_{qqp} + 2\left(1 - \frac{n}{2}\right)u_{qp}u_q + A_{qq,p}\right| \\ &= |-nu_{qp}u_q + A_{qq,p}| \leq C_0\sigma_1, \end{aligned} \tag{3-17}$$

where C_0 depends on $\|\nabla u\|_{C^0}$, n, k, g_0, μ_1, A_{ij} are the components of A_{g_0}. Here, we have used the fact that Δu is sufficiently large. From $|\bar{f}_i| \leq C\bar{f}^{1-\alpha}$, (3-15) and (3-17), noticing that $\sigma_k = \bar{f}e^{-2ku}$, we obtain

$$\begin{aligned} &-\sigma_k^{ii,jj}U_{iip}U_{jjp} \\ &\geq (\alpha-1)\left(e^{-2ku}\frac{\bar{f}_p^2 - 4ku_p\bar{f}\bar{f}_p + 4k^2\bar{f}^2u_p^2}{\bar{f}}\right) - Ce^{-2k\alpha u}\sum F^{ii}(\Delta u+1) \\ &\geq (\alpha-1)\frac{|\nabla\bar{f}|^2}{\bar{f}}e^{-2ku} - Ce^{-2k\alpha u}\sum F^{ii}(\Delta u+1). \end{aligned} \tag{3-18}$$

Moreover, it follows from $f^{\frac{1}{k-1}} \in C^{1,1}$ that

$$\bar{f}^{\frac{1}{k-1}} \in C^{1,1}, \quad (\bar{f}^{\frac{1}{k-1}})_{qq} \geq -C$$

and

$$\bar{f}_{qq} \geq (1-\alpha)\frac{|\nabla\bar{f}|^2}{\bar{f}} - C\bar{f}^{1-\alpha}. \tag{3-19}$$

Note that

$$F^{ii}U_{iiqq} \geq (\bar{f}e^{-2ku})_{qq} - \sigma_k^{ii,jj}U_{iip}U_{jjp}. \tag{3-20}$$

Thus, putting (3-19) and (3-18) into (3-20), we have

$$F^{ii}U_{iiqq} \geq -Ce^{-2ku}\bar{f}^{1-\alpha} - Ce^{-2k\alpha u}\sum F^{ii}(\Delta u+1).$$

Therefore, (3-16) becomes

$$0 \geq \sum F^{ii}\frac{1}{n^2}(\Delta u)^2 - Ce^{-2ku}\bar{f}^{1-\alpha} - C(1+e^{-2k\alpha u})\sum F^{ii}(\Delta u+1).$$

Then, using (3-15), we obtain

$$0 \geq \sum F^{ii}\left[\frac{1}{n^2}(\Delta u)^2 - Ce^{-2k\alpha u} - C(1+e^{-2k\alpha u})(\Delta u+1)\right].$$

According to $\sum F^{ii} > 0$, we have

$$0 \geq \frac{1}{n^2}(\Delta u)^2 - Ce^{-2k\alpha u} - C(1+e^{-2k\alpha u})(\Delta u+1),$$

hence (1-14) is proved.

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. To simplify notation, we define $\bar{f} = f^{1/k}(x)+\epsilon\eta(f^{1/k}(x))$, $F=\sigma_k^{1/k}$, $U = A_{g_0}+\nabla^2 u+\nabla u\otimes\nabla u-\frac{1}{2}|\nabla u|^2 g_0$, where u is a solution to (1-15). It follows from $U\in\Gamma_1$,

$$0 \leq \operatorname{tr} U \leq \Delta u + \left(1-\frac{n}{2}\right)|\nabla u|^2 + C, \quad |\nabla u|^2 \leq C(\Delta u+1). \tag{4-1}$$

Also, $U\in\Gamma_2$ implies

$$|u_{ij}| \leq C(\Delta u+1). \tag{4-2}$$

We may assume $\Delta u > 1$. Then let $H = \Delta u + |\nabla u|^2$, and let $\tilde{x}$ be the maximum point of H. So

$$0 = H_i(\tilde{x}) = u_{ppi} + 2u_{pi}u_p. \tag{4-3}$$

Let $U_{ij}(\tilde{x})$, $F^{ij}(\tilde{x})$ be diagonal. Differentiating (1-15) twice, we get

$$F^{ij}U_{ijp} = (\bar{f}e^{-2u})_p,$$
$$F^{ij}U_{ijpp} = (\bar{f}e^{-2u})_{pp} - F^{pq,rs}U_{pqp}U_{rsp} \geq (\bar{f}e^{-2u})_{pp}.$$

We have, by (4-1), (4-2), and Ricci identities,

$$
\begin{aligned}
0 &\geq F^{ii}H_{ii}(\tilde{x}) \geq F^{ii}(u_{ppii}+2u_{iip}u_p+2u_{pi}^2)-C\sum F^{ii}(\Delta u+1)\\
&= F^{ii}\big(u_{ppii}+2u_p\big(U_{iip}-\big(u_iu_i-\tfrac{1}{2}u_l^2+A_{ii}\big)_p\big)+2u_{pi}^2\big)-C\sum F^{ii}(\Delta u+1)\\
&\geq 2u_p(\bar{f}e^{-2u})_p+F^{ii}\big(U_{iipp}-\big(u_i^2-\tfrac{1}{2}u_l^2+A_{ii}\big)_{pp}-4u_{ip}u_iu_p+2u_{lp}u_lu_p+2u_{pi}^2\big)\\
&\qquad -C\sum F^{ii}(\Delta u+1)\\
&\geq 2u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}\\
&\qquad +F^{ii}(-2u_{ipp}u_i-2u_{ip}^2+(u_{lpp}u_l+u_{lp}^2)-4u_{ip}u_iu_p+2u_{lp}u_lu_p+2u_{pi}^2)\\
&\qquad -C\sum F^{ii}(\Delta u+1).
\end{aligned}
$$

Then, according to (4-3), we obtain

$$
\begin{aligned}
0 &\geq 2u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}\\
&\qquad +F^{ii}(4u_{ip}u_pu_i-2u_{lp}u_pu_l+u_{lp}^2-4u_{ip}u_iu_p+2u_{lp}u_lu_p)-C\sum F^{ii}(\Delta u+1)\\
&= 2u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}+\sum F^{ii}u_{lp}^2-C\sum F^{ii}(\Delta u+1).
\end{aligned}
$$

We notice that

$$
|u_p(\bar{f}e^{-2u})_p+(\bar{f}e^{-2u})_{pp}| \leq Ce^{-2u}(\Delta u+1),
$$

where C depends on $\|f^{\frac{1}{k}}\|_{C^2}$, $\|f\|_{C^0}$, n, k, g_0, θ, C_1, C_2. So

$$
\begin{aligned}
0 &\geq \sum F^{ii}u_{lp}^2 - C(1+e^{-2u})\sum F^{ii}(\Delta u+1)\\
&\geq \sum F^{ii}\cdot\Big(\frac{1}{n^2}(\Delta u)^2 - C(1+e^{-2u})(\Delta u+1)\Big).
\end{aligned}
$$

Hence

$$
\Delta u \leq C(1+e^{-2u}).
$$

Then, using (4-1) and (4-2), (1-16) is proved.

Remark. In this case, we do not need to discuss the gradient estimate alone by (4-1). Besides, in the proof of C^2 estimates, the third derivative term is treated by using the concavity of the operator F.

5. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2 under the assumption that the maximum modulus estimation holds. Since the proof is similar to the proof of Theorem 1.3, we present the outline for the proof and main differences here. Let

$\alpha = \frac{1}{k-1}$, $F = \sigma_k$,

$$W = -A_{g_0}^t + \nabla^2 w + \frac{1-t}{n-2}\Delta w g_0 - \nabla w \otimes \nabla w + \frac{2-t}{2}|\nabla w|^2 g_0,$$

where w is a solution to (1-8).

First, we consider C^1 estimates of w. Let $K = \left(1 + \frac{|\nabla w|^2}{2}\right)e^{\gamma(w)}$. Here γ is a function of the form $\gamma(s) = a(b+s)^p$. The constants a, b, p are chosen as in Lemma 2.6. Suppose $\max K = K(\tilde{x})$, W is diagonal at $\tilde{x}$. Differentiate K at $\tilde{x}$, by a direct calculation, we have

$$0 = K_i(\tilde{x}) = e^{\gamma(w)}\left(\left(1 + \frac{w_l^2}{2}\right)\gamma' w_i + w_l w_{li}\right) \tag{5-1}$$

and

$$K_{ij}(\tilde{x}) = e^{\gamma(w)}\left(\left(1 + \frac{w_l^2}{2}\right)((\gamma')^2 w_i w_j + \gamma' w_{ij} + \gamma'' w_i w_j) + w_l w_{lj}\gamma' w_i + w_l w_{li}\gamma' w_j + w_{lj} w_{li} + w_l w_{lij}\right).$$

Define

$$P^{ij} = F^{ij} + \frac{1-t}{n-2}\sum_k F^{kk}\delta^{ij}.$$

Combining (5-1) with (1-8), we obtain

$$\begin{aligned} 0 &\ge e^{-\gamma(w)} P^{ii} K_{ii}(\tilde{x}) \\ &= P^{ii}\left(w_l w_{lii} + \left(1 + \frac{w_l^2}{2}\right)\left(((\gamma')^2 + \gamma'')w_i^2 + \gamma' w_{ii}\right) + 2\gamma' w_l w_{li} w_i + w_{li}^2\right) \\ &\ge w_l(\bar{f}e^{2kw})_l + \gamma' k \bar{f} e^{2kw}\left(1 + \frac{w_l^2}{2}\right) \\ &\qquad + F^{ii}\left(-\gamma'\left(1 + \frac{w_l^2}{2}\right)w_i^2 + \left(1 + \frac{w_l^2}{2}\right)(\gamma'' - \gamma'^2)\frac{(1-t)w_s^2}{n-2}\right) \\ &\qquad\qquad - C\sum F^{ii}(|\nabla w|^2 + 1). \end{aligned}$$

Similarly, we have

$$\sum F^{ii} = C\sigma_{k-1} \ge C\sigma_k^{1-\frac{1}{k}} = C(\bar{f}e^{2kw})^{1-\frac{1}{k}} \tag{5-2}$$

and

$$\left|w_l(\bar{f}e^{2kw})_l + \gamma' k \bar{f} e^{2kw}\left(1 + \frac{w_l^2}{2}\right)\right| \le C\bar{f}^{1-\frac{1}{k}} e^{2kw}(|\nabla w|^2 + 1).$$

Then, letting $\delta=\min\{\frac{1-t}{4(n-2)}(\gamma''-\gamma'^2)\}$, we obtain

$$\begin{aligned}(5\text{-}3)\quad 0\ge & F^{ii}\Big((\gamma''-\gamma'^2)\frac{(1-t)w_s^2}{n-2}-\gamma' w_i^2\Big)\Big(1+\frac{w_l^2}{2}\Big)\\ &\qquad -C\bar{f}^{1-\frac{1}{k}}e^{2kw}(|\nabla w|^2+1)-C\sum F^{ii}(|\nabla w|^2+1)\\ \ge & C_4\bar{f}^{1-\frac{1}{k}}e^{2(k-1)w}\Big(\delta\frac{|\nabla w|^4}{2}-C(e^{2w}+1)(|\nabla w|^2+1)\Big).\end{aligned}$$

To obtain (1-9), we divide the proof into two different cases.

(A) $\bar{f}(\tilde{x})>0$. Using (5-2) and (5-3), we have

$$\delta\frac{|\nabla w|^4}{2}-C(e^{2w}+1)(|\nabla w|^2+1)\le 0.$$

Thus, (1-9) can be obtained.

(B) $\bar{f}(\tilde{x})=0$. Applying (5-3), we see that

$$\begin{aligned}(5\text{-}4)\quad 0\ge & F^{ii}\Big((\gamma''-\gamma'^2)\frac{(1-t)w_s^2}{n-2}-\gamma' w_i^2\Big)\Big(1+\frac{w_l^2}{2}\Big)-C\sum F^{ii}(|\nabla w|^2+1)\\ \ge & \sum F^{ii}\Big(\delta\frac{|\nabla w|^4}{2}-C(|\nabla w|^2+1)\Big).\end{aligned}$$

In view of $\sum F^{ii}>0$, (1-9) follows from (5-4).

Next, we discuss C^2 estimates of w. Let $H=\Delta w+a|\nabla w|^2$, and let $\breve{x}$ be the maximum point of H, $a\ge\frac{2-t}{1-t}(2n-4)$. Since

$$|W_{ii}|\le\sigma_1(W),$$

there exists C such that

$$(5\text{-}5)\qquad |w_{ij}|\le C\Delta w,$$

where C depends on n, k, g_0, $\|\nabla w\|_{C^0}$. Hence, we may assume $\Delta w\gg 1$ and

$$\tfrac{1}{2}\Delta w\le\sigma_1\le 2\Delta w.$$

We may further assume $W_{ij}(\breve{x})$ is diagonal. Differentiating H at $\breve{x}$, we obtain

$$(5\text{-}6)\qquad 0=H_i(\breve{x})=w_{qqi}+2aw_{qi}w_q$$

and

$$(5\text{-}7)\qquad 0\ge H_{ii}(\breve{x})=w_{qqii}+2aw_{qii}w_q+2aw_{qi}^2.$$

By (5-7), (5-5), (5-6), (2-2), (2-5) and (1-8), we have

$$\begin{aligned}
0 &\ge P^{ii}H_{ii}(\check{x}) = P^{ii}(u_{qqii}+2aw_{qii}w_q+2aw_{qi}^2)\\
&\ge F^{ii}\biggl(-4a\frac{2-t}{2}w_qw_lw_{lq}+4aw_qw_{iq}w_i\\
&\qquad\qquad +\frac{2a(1-t)}{n-2}w_{ql}^2+W_{iiqq}-\Bigl(\frac{2-t}{2}w_l^2-w_iw_i\Bigr)_{qq}\biggr)\\
&\qquad\qquad\qquad +2aw_q(\bar{f}e^{2kw})_q-C\sum F^{ii}(\Delta w+1)\\
&\ge F^{ii}\biggl(4a\Bigl(-\frac{2-t}{2}w_qw_lw_{lq}+w_qw_{iq}w_i\Bigr)+\frac{2a(1-t)}{n-2}w_{ql}^2\\
&\qquad\qquad -2\frac{2-t}{2}w_{lq}^2+2w_{iq}^2+4a\Bigl(\frac{2-t}{2}w_lw_{lq}w_q-w_iw_{iq}w_q\Bigr)\biggr)\\
&\qquad\qquad\qquad +F^{ii}W_{iiqq}+2aw_q(\bar{f}e^{2kw})_q-C\sum F^{ii}(\Delta w+1)\\
&\ge F^{ii}\Bigl(\frac{2a(1-t)}{n-2}w_{ql}^2-(2-t)w_{lq}^2\Bigr)+F^{ii}W_{iiqq}+2aw_q(\bar{f}e^{2kw})_q\\
&\qquad\qquad\qquad -C\sum F^{ii}(\Delta w+1).
\end{aligned}$$

We also have

$$C\bar{f}^{1-\alpha}e^{2k(1-\alpha)w}(\Delta w)^\alpha \le \sigma_{k-1}=C\sum F^{ii}, \tag{5-8}$$

$$|\bar{f}_i|\le C\bar{f}^{1-\alpha}\le Ce^{-2k(1-\alpha)w}\sum F^{ii}, \tag{5-9}$$

$$F^{ii}W_{iiqq}\ge -Ce^{2kw}\bar{f}^{1-\alpha}-Ce^{2k\alpha w}\sum F^{ii}(\Delta w+1). \tag{5-10}$$

Using (5-8)–(5-10), and the definition of a,

$$\begin{aligned}
0 &\ge F^{ii}\Bigl(\frac{2a(1-t)}{n-2}-(2-t)\Bigr)w_{lq}^2+F^{ii}W_{iiqq}-C(1+e^{2k\alpha w})\sum F^{ii}(\Delta w+1)\\
&\ge F^{ii}\Bigl(\frac{2a(1-t)}{n-2}-(2-t)\Bigr)w_{lq}^2-Ce^{2kw}\bar{f}^{1-\alpha}-C(1+e^{2k\alpha w})\sum F^{ii}(\Delta w+1)\\
&\ge F^{ii}\Bigl(\frac{a(1-t)}{(n-2)n^2}\Bigr)(\Delta w)^2-Ce^{2kw}\bar{f}^{1-\alpha}-C(1+e^{2k\alpha w})\sum F^{ii}(\Delta w+1)\\
&\ge F^{ii}\Bigl[\Bigl(\frac{a(1-t)}{(n-2)n^2}\Bigr)(\Delta w)^2-Ce^{2k\alpha w}-C(1+e^{2k\alpha w})(\Delta w+1)\Bigr].
\end{aligned}$$

Since $\sum F^{ii}>0$, we have $\bigl(\frac{a(1-t)}{(n-2)n^2}\bigr)(\Delta w)^2-Ce^{2k\alpha w}-C(1+e^{2k\alpha w})(\Delta w+1)\le 0$. Thus

$$\Delta w\le C(1+e^{2k\alpha w}),$$

and (1-10) is proved.

We remark that in this section $t < 1$ is the necessary condition since the Δw yields the dominating term in (5-3) and the penultimate display.

6. Proof of the existence result

In this section, we focus on dealing with the existence result for (1-7). We need to discuss the maximum modulus estimation under the assumption that the supersolution exists.

6.1. ***Proof of Theorem 1.1.*** Consider

$$\sigma_k(sW+(1-s)g_0)=(s\bar{f}+(1-s)C_n^k)e^{2kw}, \tag{6-1}$$

where

$$W=-A_{g_0}^t+\nabla^2 w+\frac{1-t}{n-2}\Delta w-dw\otimes dw+\frac{2-t}{2}|\nabla w|^2 g_0,$$
$$s\in[0,1],\ \bar{f}=(f^{\frac{1}{k-1}}+\epsilon)^{k-1}.$$

Claim: Equation (1-8) is solvable for each ϵ. In fact, define

$$\Phi_t[w]=\sigma_k(sW+(1-s)g_0)-(s\bar{f}+(1-s)C_n^k)e^{2kw}. \tag{6-2}$$

Note that the linearization of Φ_t is invertible, and $w=0$ is a solution to (6-1) at $s=0$. Thus, the problem is reduced to C^0, C^1, and C^2 estimates.

First, we consider C^0 estimates. Let $\bar{x}$ be the maximum point of w. From (6-1) we have

$$e^{2kw(\bar{x})}\le\frac{\max\sigma_k(-A_{g_0}^t+g_0)}{\min\{\bar{f},C_n^k\}}.$$

Thus, it provides an upper bound for w. Let $\hat{x}$ be the minimum point of w. Similarly, from (6-1) we have

$$e^{2kw(\hat{x})}\ge\frac{\min\sigma_k(-sA_{g_0}^t+(1-s)g_0)}{\max\bar{f}+C_n^k}.$$

Thus, it provides a lower bound for w.

Then, using Theorem 1.1 in [8], we get C^1 and C^2 estimates for solutions to (6-1). This proves the claim.

Next, we shall prove that (1-7) is solvable. Let us denote the solution derived in the claim by w_ϵ. If $\overline{w}$ is the supersolution of (1-7), then it is the supersolution of (1-8) as well. Thus, by Lemma 2.7,

$$w_\epsilon\le\overline{w}.$$

Moreover, from the equation, we have

$$e^{2k \min w_\epsilon} \geq \frac{\min \sigma_k(-A^t_{g_0})}{(\max f^{\frac{1}{k-1}} + 1)^{k-1}}.$$

Then, by Theorem 1.2, we have $|\nabla w_\epsilon| \leq C$ and $|\nabla^2 w_\epsilon| \leq C$. Here, C depends on $\|f^{\frac{1}{k-1}}\|_{C^{1,1}}$, $\|f\|_{C^0}$, g_0, n, k, $\overline{w}$. So, $\lim_{\epsilon\to 0} w_\epsilon$ is a solution to (1-6).

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TRANSPOSED POISSON SUPERALGEBRA STRUCTURES ON TWISTED $N = 1$ BLOCK–LIE SUPERALGEBRA

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We investigate the transposed Poisson superalgebra structures on the twisted $N = 1$ Block–Lie superalgebra $\mathcal{S}(p, q)$, where p and q are arbitrary complex numbers. We obtain that $\mathcal{S}(p, q)$ admits only trivial transposed Poisson superalgebra structure for $q \neq 0$ or $p \notin \mathbb{Z}$, while $\mathcal{S}(p, q)$ has nontrivial transposed Poisson superalgebra structures for $q = 0$ and $p \in \mathbb{Z}$, which are nonisomorphic with respect to $p \in \mathbb{Z}$.

1. Introduction

Poisson algebras have their roots in Hamiltonian mechanics and have become a significant research topic in mathematics and physics. In 1809, D. Poisson introduced the operation of Poisson brackets on smooth functions while studying Lagrangian mechanics, thus giving smooth functions a Poisson structure. Thereafter, many researchers realized the importance of Poisson algebras and conducted extensive research from different perspectives. From string theory, quantum groups, and differential geometry, to integrable systems, algebraic geometry and representation theory, especially with the rise of noncommutative geometry, Poisson algebras have become an important branch of algebraic research (see [2; 4; 12]). More precisely, every Poisson algebra $(\mathcal{L}, \cdot, [\cdot,\cdot])$ satisfies the *Leibniz rule*:

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z] \quad \text{for } x, y, z \in \mathcal{L},$$

Transposed Poisson algebras (see [1]) are a generalization of Poisson algebras. In the definition of Poisson algebra, the Leibniz rule regards the Lie bracket operation as a derivation in an associative algebra. In the definition of transposed Poisson algebra, the transposed Leibniz rule treats the associative operation as a $\frac{1}{2}$-derivation in a Lie algebra. The transposed Leibniz rule, the compatibility condition of the transposed Poisson algebra $(\mathcal{L}, \cdot, [\cdot,\cdot])$, is articulated by

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y] \quad \text{for } x, y, z \in \mathcal{L}.$$

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Therefore, the transposed Poisson algebra can be viewed as a dual structure of a Poisson algebra. In addition, the connection between the $\frac{1}{2}$-derivation of a Lie algebra and the transposed Poisson algebra is established in [6]. By applying this connection, all the transposed Poisson algebra structures can be obtained on the Witt algebra, Virasoro algebra, twisted Heisenberg–Virasoro algebra, Schrödinger–Virasoro algebra and extended Schrödinger–Virasoro algebra (see [6; 22]). The definition of the transposed Poisson superalgebra is also provided in [6]. A method for determining transposed Poisson algebra structures via the Kantor product is presented in [5]. The transposed Hom-Poisson algebras and the transposed BiHom-Poisson algebras are considered in [11; 13].

The study of antiderivation is generalized by Filippov (see [7]), in which the primary focus is on the δ-derivations of prime Lie algebras with a nondegenerated symmetric invariant bilinear form. Specifically for a fixed element δ in the ground field, a linear map φ on a Lie algebra $(\mathcal{L}, [\cdot\,,\cdot\,])$ is called a δ-*derivation* of $\mathcal{L}$ if

$$\varphi([x, y]) = \delta\big([\varphi(x), y] + [x, \varphi(y)]\big) \quad \text{for } x, y \in \mathcal{L}.$$

The usual derivations can be viewed as 1-derivations. The centroid of a Lie algebra $\mathcal{L}$, $\mathrm{Cent}(\mathcal{L})$, is the space of linear maps χ on $\mathcal{L}$ satisfying $\chi([x, y]) = [\chi(x), y] = [x, \chi(y)]$ for all $x, y \in \mathcal{L}$. It is easy to see that elements of the centroid are $\frac{1}{2}$-derivations. If $\delta = 0$ and $[\mathcal{L}, \mathcal{L}] \neq 0$, then every nonzero linear map is called a 0-*derivation*. If $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$, in particular, $\mathcal{L}$ is a simple Lie algebra, then $\mathcal{L}$ has no nonzero 0-derivations. If $\delta = -1$, then the linear map is called an *antiderivation*. The concept of the δ-superderivation on nonassociative superalgebras is introduced by Kaygorodov, and it is proved that simple finite-dimensional Lie superalgebras over an algebraically closed field of characteristic 0 do not have nontrivial δ-superderivations (see [8], [9]).

Block–Lie algebras are a class of infinite-dimensional simple Lie algebras introduced by Block in 1958 (see [3]). Since then, several generalizations of these algebras have been proposed (see [14; 15; 21]). The Block algebra $B(q)$ for a fixed complex number q is defined in [18]. The notion of the Block superalgebra $\mathfrak{K}(p)$ is introduced, and its finite-dimensional irreducible conformal modules are classified for any nonzero parameter p (see [17]). Based on the twisted rules from Ramond superalgebras to Neveu–Schwarz superalgebras, a twisted version of the $\mathbb{Z}$-graded conformal superalgebra $\mathfrak{T}(p)$ is introduced in [16], where p is a nonzero parameter. Precisely speaking, the subscripts of the odd generators of the original $\mathbb{Z}$-graded algebra are shifted by $\frac{1}{2}$. Motivated by [16; 19], the parameters of twisted $N = 1$ Block superalgebra are extended to include two nonzero parameters, p and q. Note that the special case of $p \in \mathbb{C}$ and $q = 1$ is considered in [16]. The nonweight modules over $N = 1$ Lie superalgebras of Block type is studied in [19]. In particular, transposed Poisson algebra structures on Block–Lie algebras $\mathcal{B}(q)$ and

Block–Lie superalgebras $\mathcal{S}(q)$ are described in [10]. This work propels us to delve into transposed Poisson superalgebra structures of the twisted $N=1$ Block–Lie superalgebra $\mathcal{S}(p,q)$ with two parameters.

This paper is organized as follows. In Section 2, we recall some basic definitions and the relation between transposed Poisson superalgebra structures and $\frac{1}{2}$-superderivations. In Section 3, we characterize all $\frac{1}{2}$-superderivations of the twisted $N=1$ Block–Lie superalgebra $\mathcal{S}(p,q)$. In Section 4, we present our main theorem about the transposed Poisson superalgebra structures on $\mathcal{S}(p,q)$; see Theorem 4.1. More precisely, we prove in Theorem 4.1 that there is only nontrivial transposed Poisson superalgebra structure for $q=0$ and $p\in\mathbb{Z}$, otherwise, such structures are trivial. Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}^*$ the complex numbers, nonzero complex numbers, rational numbers, integers and nonzero integers, respectively.

2. Preliminaries

In this section, we recall some definitions and notation for future convenience.

Motivated by the definitions of the Block-type algebra and the $N=1$ Lie superalgebra of Block type from [19; 20], we arrive at the subsequent definition.

Definition 2.1. Let p and q be fixed complex numbers. The *twisted $N=1$ Block–Lie superalgebra* $\mathcal{S}(p,q)$ is defined as an infinite-dimensional Lie superalgebra over $\mathbb{C}$ with the even part $\mathcal{S}(p,q)_{\bar{0}}=\{L_{m,i}\mid m,i\in\mathbb{Z}\}$ and the odd part $\mathcal{S}(p,q)_{\bar{1}}=\{G_{m+\frac{1}{2},i+\frac{1}{2}}\mid m,i\in\mathbb{Z}\}$ together with the relations

$$(2\text{-}1)\qquad [L_{m,i},L_{n,j}]=((n+p)(i+q)-(j+q)(m+p))L_{m+n,i+j}=\begin{vmatrix} n+p & j+q\\ m+p & i+q\end{vmatrix}L_{m+n,i+j},$$

$$[L_{m,i},G_{n+\frac{1}{2},j+\frac{1}{2}}]=\Big(\big(n+\tfrac{p}{2}+\tfrac{1}{2}\big)(i+q)-\big(j+\tfrac{q}{2}+\tfrac{1}{2}\big)(m+p)\Big)G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}=\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2}\\ m+p & i+q\end{vmatrix}G_{m+n+\frac{1}{2},i+j+\frac{1}{2}},$$

$$[G_{m+\frac{1}{2},i+\frac{1}{2}},G_{n+\frac{1}{2},j+\frac{1}{2}}]=2qL_{m+n+1,i+j+1}$$

for $m,n,i,j\in\mathbb{Z}$.

Let $\mathcal{L}=\mathcal{L}_{\bar{0}}\oplus\mathcal{L}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. For $i\in\mathbb{Z}_2=\{\bar{0},\bar{1}\}$, if $x\in\mathcal{L}_i$, then $|x|$ denotes the parity of x, that is, $|x|=0$ if $x\in\mathcal{L}_{\bar{0}}$ or $|x|=1$ if $x\in\mathcal{L}_{\bar{1}}$.

We will briefly recall some definitions from [10].

Definition 2.2. Let $\mathcal{L}=\mathcal{L}_{\bar{0}}\oplus\mathcal{L}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space equipped with two nonzero bilinear superoperations $\cdot$ and $[\cdot,\cdot]$. The triple $(\mathcal{L},\cdot,[\cdot,\cdot])$ is called a *transposed Poisson superalgebra* if $(\mathcal{L}, \cdot)$ is a supercommutative associative superalgebra and $(\mathcal{L}, [\cdot, \cdot])$ is a Lie superalgebra that satisfies the compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{|x||z|}[x, z \cdot y] \quad \text{for } x, z \in \mathcal{L}_i,\ y \in \mathcal{L},\ i \in \mathbb{Z}_2. \tag{2-2}$$

Definition 2.3. Let $(\mathcal{L}, [\cdot, \cdot])$ be a Lie superalgebra. A transposed Poisson superalgebra structure on $(\mathcal{L}, [\cdot, \cdot])$ is a supercommutative associative multiplication $\cdot$ on $\mathcal{L}$ which makes $(\mathcal{L}, \cdot, [\cdot, \cdot])$ a transposed Poisson superalgebra.

If φ is a homogeneous linear map $\mathcal{L} \to \mathcal{L}$, then $|\varphi| = 0$ for $\varphi(\mathcal{L}_i) \subseteq \mathcal{L}_i$ or $|\varphi| = 1$ for $\varphi(\mathcal{L}_i) \subseteq \mathcal{L}_{\bar{1}-i}$, $i \in \mathbb{Z}_2$.

Definition 2.4. Let $(\mathcal{L}, [\cdot, \cdot])$ be a superalgebra and φ a homogeneous linear map $\mathcal{L} \to \mathcal{L}$. Then φ is called $\frac{1}{2}$-*superderivation* if it satisfies

$$\varphi([x, y]) = \tfrac{1}{2}\big([\varphi(x), y] + (-1)^{|\varphi||x|}[x, \varphi(y)]\big) \quad \text{for } x, y \in \mathcal{L}_i,\ i \in \mathbb{Z}_2. \tag{2-3}$$

We will use the notation $\Delta(\mathcal{L})$ for the space of $\frac{1}{2}$-superderivations of Lie superalgebra $\mathcal{L}$.

Let L_z denote the operator of the left multiplication by an element $z \in \mathcal{L}$, that is,

$$L_z(x) = z \cdot x \quad \text{for } x \in \mathcal{L}.$$

Definitions 2.2 and 2.4 immediately imply the following key lemma.

Lemma 2.5. *Let $(\mathcal{L}, \cdot, [\cdot, \cdot])$ be a transposed Poisson superalgebra and $z \in \mathcal{L}_i$, $i \in \mathbb{Z}_2$. Then the left multiplication L_z of $(\mathcal{L}, \cdot)$ is a $\frac{1}{2}$-superderivation of $(\mathcal{L}, [\cdot, \cdot])$ and $|L_z| = |z|$.*

The basic example of a $\frac{1}{2}$-superderivation is the multiplication by a field element. Such $\frac{1}{2}$-superderivations will be called *trivial*.

Theorem 2.6. *Let $\mathcal{L}$ be a Lie superalgebra without nontrivial $\frac{1}{2}$-superderivations. Then all transposed Poisson superalgebra structures on $\mathcal{L}$ are trivial.*

Let $\cdot$ be a transposed Poisson superalgebra structure on a Lie superalgebra $(\mathcal{L}, [\cdot, \cdot])$. Then any automorphism ϕ of $(\mathcal{L}, [\cdot, \cdot])$ induces another transposed Poisson superalgebra structure $*$ on $(\mathcal{L}, [\cdot, \cdot])$ given by

$$x * y = \phi(\phi^{-1}(x) \cdot \phi^{-1}(y)) \quad \text{for } x, y \in \mathcal{L}.$$

Clearly, ϕ is an isomorphism of transposed Poisson superalgebras $(\mathcal{L}, \cdot, [\cdot, \cdot])$ and $(\mathcal{L}, *, [\cdot, \cdot])$.

3. $\frac{1}{2}$-superderivations of the twisted $N = 1$ Block–Lie superalgebra

In this section, we will investigate and describe all the $\frac{1}{2}$-superderivations of the twisted $N = 1$ Block–Lie superalgebra $\mathcal{S}(p, q)$.

3.1. ***Even $\frac{1}{2}$-superderivations of $\mathcal{S}(p,q)$.*** In this subsection we consider only *even* linear maps $\varphi:\mathcal{S}(p,q)\to\mathcal{S}(p,q)$, i.e., those which satisfy $\varphi(\mathcal{S}(p,q)_i)\subseteq\mathcal{S}(p,q)_i$ for $i\in\mathbb{Z}_2$. We thus have $|\varphi|=0$, so φ is a $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$ if and only if φ is a usual $\frac{1}{2}$-derivation of $\mathcal{S}(p,q)$. We now write

$$\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s},$$

where

$$\varphi_{r,s}(L_{m,i})=d^{\bar{0}}_{r,s}(m,i)L_{m+r,i+s},\tag{3-1}$$

$$\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}})=d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)G_{m+r+\frac{1}{2},i+s+\frac{1}{2}}\tag{3-2}$$

for some $d^{\bar{0}}_{r,s}(m,i)$, $d^{\bar{1}}_{r,s}\big(m+\frac{1}{2},i+\frac{1}{2}\big)\in\mathbb{C}$, $m,i,r,s\in\mathbb{Z}$. Then we have $\varphi\in\Delta^{\bar{0}}(\mathcal{S}(p,q))$ if and only if $\varphi_{r,s}\in\Delta^{\bar{0}}(\mathcal{S}(p,q))$ for all $r,s\in\mathbb{Z}$.

Lemma 3.1. *Let $\varphi_{r,s}:\mathcal{S}(p,q)\to\mathcal{S}(p,q)$, $r,s\in\mathbb{Z}$, be a linear map satisfying (3-1) and (3-2). Then $\varphi_{r,s}\in\Delta^{\bar{0}}(\mathcal{S}(p,q))$ if and only if the these three conditions hold*:

$$2\begin{vmatrix}n+p & j+q\\ m+p & i+q\end{vmatrix}d^{\bar{0}}_{r,s}(m+n,i+j)=\begin{vmatrix}n+p & j+q\\ m+r+p & i+s+q\end{vmatrix}d^{\bar{0}}_{r,s}(m,i)+\begin{vmatrix}n+r+p & j+s+q\\ m+p & i+q\end{vmatrix}d^{\bar{0}}_{r,s}(n,j),\tag{3-3}$$

$$\begin{aligned}2\begin{vmatrix}n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2}\\ m+p & i+q\end{vmatrix}d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\big)&=\begin{vmatrix}n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2}\\ m+r+p & i+s+q\end{vmatrix}d^{\bar{0}}_{r,s}(m,i)\\&\quad+\begin{vmatrix}n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2}\\ m+p & i+q\end{vmatrix}d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big),\end{aligned}\tag{3-4}$$

$$2qd^{\bar{0}}_{r,s}(m+n+1,i+j+1)=q\big(d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)+d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)\big).\tag{3-5}$$

Proof. By Definition 2.1, $\varphi_{r,s}$ is an even $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$ if and only if (2-3) hold on the basis $\{L_{m,i},G_{m+\frac{1}{2},i+\frac{1}{2}}\mid m,i\in\mathbb{Z}\}$. Namely, the subsequent three equations are satisfied:

$$2\begin{vmatrix}n+p & j+q\\ m+p & i+q\end{vmatrix}\varphi_{r,s}(L_{m+n,i+j})=[\varphi_{r,s}(L_{m,i}),L_{n,j}]+[L_{m,i},\varphi_{r,s}(L_{n,j})],\tag{3-6}$$

$$\begin{aligned}2\begin{vmatrix}n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2}\\ m+p & i+q\end{vmatrix}\varphi_{r,s}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}})&=[\varphi_{r,s}(L_{m,i}),G_{n+\frac{1}{2},j+\frac{1}{2}}]\\&\quad+[L_{m,i},\varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})],\end{aligned}\tag{3-7}$$

$$\begin{aligned}4q\varphi_{r,s}(L_{m+n+1,i+j+1})&=[\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}),G_{n+\frac{1}{2},j+\frac{1}{2}}]\\&\quad+[G_{m+\frac{1}{2},i+\frac{1}{2}},\varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})].\end{aligned}\tag{3-8}$$

In view of (3-6), we can see that

$$2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r,s}(m+n, i+j)L_{m+n+r,i+j+s}$$
$$= 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r,s}(L_{m+n,i+j})$$
$$= [\varphi_{r,s}(L_{m,i}), L_{n,j}] + [L_{m,i}, \varphi_{r,s}(L_{n,j})]$$
$$= [d^{\bar{0}}_{r,s}(m, i)L_{m+r,i+s}, L_{n,j}] + [L_{m,i}, d^{\bar{0}}_{r,s}(n, j)L_{n+r,j+s}]$$
$$= \begin{vmatrix} n+p & j+q \\ m+r+p & i+s+q \end{vmatrix} d^{\bar{0}}_{r,s}(m, i)L_{m+n+r,i+j+s} + \begin{vmatrix} n+r+p & j+s+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r,s}(n, j)L_{m+n+r,i+j+s}.$$

Thus, we come to (3-3). By (3-7), we observe

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big)G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}}$$
$$= 2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r,s}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}})$$
$$= [\varphi_{r,s}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [L_{m,i}, \varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})]$$
$$= [d^{\bar{0}}_{r,s}(m, i)L_{m+r,i+s}, G_{n+\frac{1}{2},j+\frac{1}{2}}] + \big[L_{m,i}, d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)G_{n+r+\frac{1}{2},j+s+\frac{1}{2}}\big]$$
$$= \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+r+p & i+s+q \end{vmatrix} d^{\bar{0}}_{r,s}(m, i)G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}}.$$

Accordingly, we arrive at (3-4). With the use of (3-8), we have

$$4q\, d^{\bar{0}}_{r,s}(m+n+1, i+j+1)L_{m+n+r+1,i+j+s+1}$$
$$= 4q\,\varphi_{r,s}(L_{m+n+1,i+j+1})$$
$$= [\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [G_{m+\frac{1}{2},i+\frac{1}{2}}, \varphi_{r,s}(G_{n+\frac{1}{2},j+\frac{1}{2}})]$$
$$= \big[d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big)(G_{m+r+\frac{1}{2},i+s+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}\big] + \big[G_{m+\frac{1}{2},i+\frac{1}{2}}, d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)(G_{n+r+\frac{1}{2},j+s+\frac{1}{2}})\big]$$
$$= 2q\big(d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\big)L_{m+n+r+1,i+j+s+1}.$$

So, we get (3-5). □

Inspired by [10], we proceed to analyze the $\frac{1}{2}$-superderivatives of $\mathcal{S}(p,q)$, considering whether p and q are zero or not.

3.1.1. *The case $p, q \neq 0$.*

Lemma 3.2. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be an even $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$ and $(r,s) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. Then*

$$\varphi_{r,s} = 0.$$

Proof. Taking $n = j = 0$ in (3-3), we obtain

$$(p(i-s)-q(m-r))d^{\bar{0}}_{r,s}(m,i) = ((r+p)(i+q)-(s+q)(m+p))d^{\bar{0}}_{r,s}(0,0). \tag{3-9}$$

(1) For $\frac{p}{q} \in \mathbb{Q}$, there exists some $k \in \mathbb{C}^*$ satisfying $kp, kq \in \mathbb{Z}$. Putting $m = kp + r$ and $i = kq + s$ in (3-9), it follows that

$$(qr - ps)d^{\bar{0}}_{r,s}(0,0) = 0.$$

Clearly, we will discuss $qr - ps$ being zero or nonzero.

Case 1: $qr - ps \neq 0$. We have $d^{\bar{0}}_{r,s}(0,0) = 0$. And hence by (3-9), we can obtain

$$d^{\bar{0}}_{r,s}(m,i) = 0 \quad \text{if } i \neq \frac{q(m-r)}{p} + s. \tag{3-10}$$

If $\frac{q(m-r)}{p} + s \in \mathbb{Z}$, we set $i = \frac{q(m-r)}{p} + s$, $n = -m$, $j = -i$. Since

$$j = -i = -\frac{q(m-r)}{p} - s = \frac{q(n+r)}{p} - s \neq \frac{q(n-r)}{p} + s,$$

we can apply (3-10). It follows that

$$d^{\bar{0}}_{r,s}(n,j) = d^{\bar{0}}_{r,s}(-m,-i) = 0.$$

Substituting it into (3-3), we get

$$(3p + r - m)d^{\bar{0}}_{r,s}\left(m, \frac{q(m-r)}{p} + s\right) = 0.$$

Therefore, it leads to

$$d^{\bar{0}}_{r,s}\left(m, \frac{q(m-r)}{p} + s\right) = 0 \quad \text{if } m \neq 3p + r.$$

Taking $m = 3p + r$, $i = q(m-r)/p + s = 3q + s$ and $n \notin \{0, 3p + r\}$ in (3-3), for $d^{\bar{0}}_{r,s}(m+n, i+j) = 0$ and $d^{\bar{0}}_{r,s}(n,j) = 0$, we obtain

$$((n+p)(2q+s) - (j+q)(2p+r))d^{\bar{0}}_{r,s}(3p+r, 3q+s) = 0. \tag{3-11}$$

If $r+2p \neq 0$, then we take $j \neq \frac{(n+p)(2q+s)}{r+2p} - q$. So, we obtain

$$d^{\bar{0}}_{r,s}(3p+r, 3q+s) = 0.$$

If $r+2p = 0$, it follows from (3-11) that

$$(n+p)(2q+s)d^{\bar{0}}_{-2p,s}(3p+r, 3q+s) = 0.$$

Due to $qr - ps \neq 0$ and $r = -2p$, then we have $2q+s \neq 0$. It only needs to take $n \neq -p$, then we get

$$d^{\bar{0}}_{-2p,s}(p, 3q+s) = 0.$$

Case 2: $qr - ps = 0$. Since $\frac{p}{q} = \frac{r}{s} \in \mathbb{Q}$, there exists $t \in \mathbb{C}^*$ satisfying $tp, tq \in \mathbb{Z}$. Taking $r = tp$, $s = tq$ in (3-3), we have

$$2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{tp,tq}(m+n, i+j) = \begin{vmatrix} n+p & j+q \\ m+(t+1)p & i+(t+1)q \end{vmatrix} d^{\bar{0}}_{tp,tq}(m,i) + \begin{vmatrix} n+(t+1)p & j+(t+1)q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{tp,tq}(n,j). \tag{3-12}$$

Considering $n = j = 0$, $m \neq 0$ in (3-12), we get

$$(pi - qm)d^{\bar{0}}_{tp,tq}(m,i) = (t+1)(pi - qm)d^{\bar{0}}_{tp,tq}(0,0).$$

Clearly, our discussion will proceed based on whether $pi - qm$ is zero or not.

Subcase 1: $pi - qm \neq 0$. We have

$$d^{\bar{0}}_{tp,tq}(m,i) = (t+1)d^{\bar{0}}_{tp,tq}(0,0). \tag{3-13}$$

Taking $m = -n \neq 0$, $j = -i$ and $pi + qn \neq 0$ in (3-12), we obtain

$$4d^{\bar{0}}_{tp,tq}(0,0) = (t+2)d^{\bar{0}}_{tp,tq}(-n,i) + (t+2)d^{\bar{0}}_{tp,tq}(n,-i). \tag{3-14}$$

Then applying (3-13), we have $d^{\bar{0}}_{tp,tq}(n,-i) = d^{\bar{0}}_{tp,tq}(-n,i) = (t+1)d^{\bar{0}}_{tp,tq}(0,0)$. As a result, (3-14) can be simplified as

$$(t+3)d^{\bar{0}}_{tp,tq}(0,0) = 0.$$

If $t \neq -3$, then we arrive at $d^{\bar{0}}_{tp,tq}(0,0) = 0$. And hence by (3-13), it follows that

$$d^{\bar{0}}_{tp,tq}(m,i) = 0.$$

If $t = -3$, by (3-13), we derive

$$d^{\bar{0}}_{-3p,-3q}(m,i) = -2d^{\bar{0}}_{-3p,-3q}(0,0). \tag{3-15}$$

Letting $i = j = 0$ and $m,\ n,\ m \pm n \neq 0$ in (3-12), using (3-15), we are able to conclude that

$$2(n-m)d^{\bar{0}}_{-3p,-3q}(m+n,0) = -(m+2n)d^{\bar{0}}_{-3p,-3q}(m,0)+(2m+n)d^{\bar{0}}_{-3p,-3q}(-n,0),$$
$$d^{\bar{0}}_{-3p,-3q}(0,0) = 0.$$

Substituting it into (3-15), we see that

$$d^{\bar{0}}_{-3p,-3q}(m,i) = 0.$$

Hence, we can deduce that

$$d^{\bar{0}}_{tp,tq}(m,i) = 0 \quad \text{for } pi - qm \neq 0. \tag{3-16}$$

Subcase 2: $pi - qm = 0$. Since $\frac{p}{q} = \frac{m}{i} \in \mathbb{Q}$, there exists $l \in \mathbb{C}^*$ satisfying $lp, lq \in \mathbb{Z}$. By substituting $m = j = 0$, $n = lp$ and $i = lq$ into (3-12), since $pi - qm \neq 0$, $pj - qn \neq 0$ and (3-16), we arrive at $d^{\bar{0}}_{tp,tq}(lp,0) = d^{\bar{0}}_{tp,tq}(0,lq) = 0$. So (3-12) can be written as

$$(l+2)d^{\bar{0}}_{tp,tq}(lp,lq) = 0.$$

Then, it yields

$$d^{\bar{0}}_{tp,tq}(lp,lq) = 0,\ \text{if } l \neq -2.$$

If $l = -2$, then taking $m = -2p$, $n = 2p$, $i = -2q$ and $j = 0$ in (3-12) and considering (3-16), it follows that

$$(t-1)pq\,d^{\bar{0}}_{tp,tq}(-2p,-2q) = 0.$$

Since $p, q \neq 0$, then $d^{\bar{0}}_{tp,tq}(-2p,-2q) = 0$ for $t \neq 1$ is derived.

Therefore, we can conclude that $d^{\bar{0}}_{r,s}(m,i) = 0$ for all $(m,i) \in \mathbb{Z}\times\mathbb{Z}$, $(r,s) \notin \{(0,0),(p,q)\}$ and $d^{\bar{0}}_{p,q}(m,i) = 0$ unless $(m,i) = (-2p,-2q)$.

(2) For $\frac{p}{q} \notin \mathbb{Q}$, we can obtain, from (3-9),

$$d^{\bar{0}}_{r,s}(m,i) = \frac{(r+p)(i+q)-(s+q)(m+p)}{p(i-s)-q(m-r)}d^{\bar{0}}_{r,s}(0,0) \quad \text{for } (m,i) \neq (r,s). \tag{3-17}$$

In the following analysis, we will proceed by the conditions whether r and s are zero or not.

Case 1: $r, s \neq 0$. Taking $m = -r$ and $j = -s$ into (3-9), we can see that

$$d^{\bar{0}}_{r,s}(-r,-s) = d^{\bar{0}}_{r,s}(0,0).$$

Given the conditions $m = 0$, $n = -r$, $i = -s$ and $j = 0$ in (3-3), it follows that

$$2((p-r)(q-s)-pq)d^{\bar{0}}_{r,s}(-r,-s) = -2qr\,d^{\bar{0}}_{r,s}(0,-s) - 2ps\,d^{\bar{0}}_{r,s}(-r,0).$$

By combining

$$(qr-2ps)d^{\bar{0}}_{r,s}(0,-s)=(qr-2ps-rs)d^{\bar{0}}_{r,s}(0,0),$$
$$(ps-2qr)d^{\bar{0}}_{r,s}(-r,0)=(ps-2qr-rs)d^{\bar{0}}_{r,s}(0,0),$$

we can deduce that

$$d^{\bar{0}}_{r,s}(0,0)=0.$$

So it leads to

$$d^{\bar{0}}_{r,s}(m,i)=0 \quad \text{for } (m,i)\neq(r,s).$$

Taking $m=r$, $i=s$, $j\notin\{0,s\}$ and $n\notin\{0,r,2r\}$ into (3-3), we can get

$$((n+p)(2s+q)-(j+q)(2r+p))d^{\bar{0}}_{r,s}(r,s)=0.$$

Since $\frac{p}{q}\neq\mathbb{Q}$, we have $d^{\bar{0}}_{r,s}(r,s)=0$.

Case 2: $r=0$, $s\neq 0$. In this case, (3-17) becomes

$$d^{\bar{0}}_{0,s}(m,i)=\frac{p(i+q)-(s+q)(m+p)}{p(i-s)-qr}d^{\bar{0}}_{0,s}(0,0) \quad \text{for } (m,i)\neq(0,s). \tag{3-18}$$

Substituting $i=s$ and $m\neq 0$ in (3-18), we have

$$d^{\bar{0}}_{0,s}(m,s)=\frac{s+q}{q}d^{\bar{0}}_{0,s}(0,0) \quad \text{for } m\neq 0.$$

Putting $m=0$ and $i=2s$ into (3-18), we obtain

$$d^{\bar{0}}_{0,s}(0,2s)=d^{\bar{0}}_{0,s}(0,0).$$

Furthermore, considering $m+n=0$, $m,n\neq 0$ and $i=j=0$ in (3-3), we can deduce that

$$\begin{aligned}4n(s+q)d^{\bar{0}}_{0,s}(0,0)=((n+p)(2s+q)-(-n+p)(s+q))d^{\bar{0}}_{0,s}(m,s)\\ +((n+p)(s+q)-(-n+p)(2s+q))d^{\bar{0}}_{0,s}(n,s).\end{aligned}$$

Therefore, it can be checked that

$$d^{\bar{0}}_{0,s}(m,i)=0 \quad \text{for } (m,i)\neq(0,s).$$

Taking $m=0$, $i=s$, $j\notin\{0,s\}$ and $n\neq 0$ into (3-3), we can get

$$((n+p)(2s+q)-p(j+q))d^{\bar{0}}_{0,s}(0,s)=0.$$

Since $\frac{p}{q}\neq\mathbb{Q}$, we have $d^{\bar{0}}_{0,s}(0,s)=0$.

Case 3: $r \neq 0$, $s = 0$. Similarly, it can be shown that $d^{\bar{0}}_{r,0}(m, i) = 0$ for $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.

Subsequently, we consider the value of $d^{\bar{1}}_{r,s}\big(m+\frac{1}{2}, i+\frac{1}{2}\big)$. If $(r, s) \notin \{(0, 0), (p, q)\}$, (3-5) becomes

$$d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0.$$

Taking $m = n$ and $i = j$, we get

$$d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (m, i) \in \mathbb{Z} \times \mathbb{Z} \text{ and } (r, s) \notin \{(0, 0), (p, q)\}.$$

If $(r, s) = (p, q)$, it implies $p, q \in \mathbb{Z}$. Then taking $m = n$ and $i = j$ in (3-5), we get

$$d^{\bar{1}}_{p,q}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = d^{\bar{0}}_{r,s}(2m+1, 2i+1) = 0 \quad \text{for all } (m, i) \in \mathbb{Z} \times \mathbb{Z}.$$

Considering $m = 0$, $i = 0$, $n = -2p-1$ and $j = -2q-1$ in (3-5), we get

$$2d^{\bar{0}}_{p,q}(-2p, -2q) = d^{\bar{1}}_{p,q}\big(\tfrac{1}{2}, \tfrac{1}{2}\big) + d^{\bar{1}}_{p,q}\big(-2p-\tfrac{1}{2}, -2q-\tfrac{1}{2}\big) = 0.$$

So we obtain $d^{\bar{0}}_{r,s}(m, i) = 0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$ and $(r, s) \neq (0, 0)$. □

Lemma 3.3. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, q)$ and $(r, s) = (0, 0)$. Then*

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{1}}_{0,0}\big(m'+\tfrac{1}{2}, i'+\tfrac{1}{2}\big)$$

for all (m, i), $(m', i') \in \mathbb{Z} \times \mathbb{Z}$.

Proof. Writing (3-3) with $(r, s) = (0, 0)$, we have

$$((n+p)(i+q) - (m+p)(j+q)) \times (2d^{\bar{0}}_{0,0}(m+n, i+j) - d^{\bar{0}}_{0,0}(m, i) - d^{\bar{0}}_{0,0}(n, j)) = 0. \tag{3-19}$$

Taking $n = j = 0$ in (3-19), we obtain

$$(pi - qm)(d^{\bar{0}}_{0,0}(m, i) - d^{\bar{0}}_{0,0}(0, 0)) = 0.$$

Therefore, we derive

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{0}}_{0,0}(0, 0) \quad \text{if } pi - qm \neq 0. \tag{3-20}$$

Now, if $pi - qm = 0$, then we can assume $m = kp$ and $i = kq$, $k \in \mathbb{C}^*$. We choose $pj - qn \neq 0$ in (3-19), it becomes

$$(k+1)((n+p)q - (j+q)p) \times (2d^{\bar{0}}_{0,0}(kp+n, kq+j) - d^{\bar{0}}_{0,0}(kp, kq) - d^{\bar{0}}_{0,0}(n, j)) = 0. \tag{3-21}$$

Since $p(kq+j) - q(kp+n) \neq 0$, by (3-20) we obtain

$$d^{\bar{0}}_{0,0}(kp+n, kq+j) = d^{\bar{0}}_{0,0}(n, j) = d^{\bar{0}}_{0,0}(0, 0).$$

Substituting it into (3-21), we have

$$(k+1)(d^{\bar{0}}_{0,0}(kp,kq)-d^{\bar{0}}_{0,0}(0,0))=0.$$

Hence, it follows that

$$d^{\bar{0}}_{0,0}(kp,kq)=d^{\bar{0}}_{0,0}(0,0) \quad \text{if } k\neq -1.$$

We substitute $m=0$, $n=-p$, $i=-q$ and $j=0$ into (3-19). By (3-20), it leads to

$$d^{\bar{0}}_{0,0}(-p,-q)=d^{\bar{0}}_{0,0}(0,0). \tag{3-22}$$

Consequently, we can conclude that $d^{\bar{0}}_{0,0}(m,i)=d^{\bar{0}}_{0,0}(0,0)$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$.

Therefore, (3-5) becomes

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)+d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)=2d^{\bar{0}}_{0,0}(0,0).$$

Taking $m=n$ and $i=j$, we get

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(0,0) \quad \text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}. \tag{3-23}$$

Combining with (3-22) and (3-23), we conclude that $\varphi_{0,0}=d^{\bar{0}}_{0,0}(0,0)id$. □

From Lemmas 3.2 and 3.3, the subsequent proposition is directly derived.

Proposition 3.4. *Let $p,q\in\mathbb{C}^*$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(p,q))=\langle id\rangle.$$

3.1.2. *The case $p=0$, $q\neq 0$.*

Lemma 3.5. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,q)$, $r\in\mathbb{Z}^*$ and $s\in\mathbb{Z}$. Then*

$$\varphi_{r,s}=0.$$

Proof. According to Lemma 2.3 from [10], we can obtain $d^{\bar{0}}_{r,s}(m,i)=0$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$. So, (3-5) becomes

$$d^{\bar{1}}_{r,s}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)+d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)=0.$$

Taking $m=n$ and $i=j$, we get $d^{\bar{1}}_{r,s}\big(m+\frac{1}{2},i+\frac{1}{2}\big)=0$ for all $(m,i)\in\mathbb{Z}\times\mathbb{Z}$. □

Lemma 3.6. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,q)$, $r=0$ and $s\in\mathbb{Z}^*$. Then*

$$\varphi_{0,s}=0.$$

Proof. If $s \neq q$, then we can obtain that $d^{\bar{0}}_{0,s}(m,i) = 0$ for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$ is established based on Lemma 2.4 in [10]. Hence, (3-5) becomes

$$d^{\bar{1}}_{0,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{0,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0.$$

Taking $m = n$ and $i = j$, we get

$$d^{\bar{1}}_{0,s}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (m,i) \in \mathbb{Z} \times \mathbb{Z}.$$

If $s = q$, then we arrive at $d^{\bar{0}}_{0,q}(m,i) = 0$ unless $(m,i) = (0,-2q)$, which has been proved in Lemma 2.4 of [10]. Therefore (3-5) becomes

$$d^{\bar{1}}_{0,q}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{0,q}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \quad \text{for } (m+n+1, i+j+1) \neq (0,-2q), \tag{3-24}$$

$$d^{\bar{1}}_{0,q}\big(-n-\tfrac{1}{2}, -j-2q-\tfrac{1}{2}\big) + d^{\bar{1}}_{0,q}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 2d^{\bar{0}}_{0,q}(0,-2q). \tag{3-25}$$

By setting $m = n = i = j$ into (3-24) and observing $(m+n+1, i+j+1) = (2m+1, 2i+1) \neq (0,-2q)$, we deduce, from (3-24),

$$d^{\bar{1}}_{0,q}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (m,i) \in \mathbb{Z} \times \mathbb{Z}.$$

In the end, with the use of (3-25), we can obtain $d^{\bar{0}}_{0,q}(0,-2q) = 0$. □

Lemma 3.7. *Let $\varphi = \sum_{r,s \in \mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,q)$ and $(r,s) = (0,0)$. Then*

$$d^{\bar{0}}_{0,0}(m,i) = d^{\bar{1}}_{0,0}\big(m'+\tfrac{1}{2}, i'+\tfrac{1}{2}\big)$$

for all $(m,i), (m',i') \in \mathbb{Z} \times \mathbb{Z}$.

Proof. As stated in Lemma 2.7 of [10], we can deduce $d^{\bar{0}}_{0,0}(m,i) = d^{\bar{0}}_{0,0}(0,0)$ for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$. Then, (3-5) becomes

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) + d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 2d^{\bar{0}}_{0,0}(0,0).$$

Taking $m = n = i = j$, we have

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = d^{\bar{0}}_{0,0}(0,0)$$

for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$. □

Lemmas 3.5, 3.6 and 3.7 jointly yield the following proposition.

Proposition 3.8. *Let $p = 0$ and $q \in \mathbb{C}^*$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(0,q)) = \langle id \rangle.$$

3.1.3. *The case $p \neq 0$, $q = 0$.* For this case, (3-3) and (3-4) are expressed as, respectively,

$$2\begin{vmatrix} n+p & j \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r,s}(m+n, i+j) = \begin{vmatrix} n+p & j \\ m+r+p & i+s \end{vmatrix} d^{\bar{0}}_{r,s}(m,i) + \begin{vmatrix} n+r+p & j+s \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r,s}(n,j), \tag{3-26}$$

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+r+p & i+s \end{vmatrix} d^{\bar{0}}_{r,s}(m,i) + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big). \tag{3-27}$$

Lemma 3.9. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p,0)$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}^*$. Then*

$$\varphi_{r,s} = 0.$$

Proof. Taking $n = j = 0$ in (3-26), we can see that

$$p(i-s)d^{\bar{0}}_{r,s}(m,i) = (i(r+p) - s(m+p))d^{\bar{0}}_{r,s}(0,0). \tag{3-28}$$

Letting $i = s$ and $m \neq r$ in (3-28), we obtain

$$d^{\bar{0}}_{r,s}(0,0) = 0.$$

Furthermore, substituting it into (3-28), it leads to

$$(i-s)d^{\bar{0}}_{r,s}(m,i) = 0 \quad \text{and} \quad d^{\bar{0}}_{r,s}(m,i) = 0 \qquad \text{for } i \neq s.$$

Putting $i = s$, $j \notin \{0, s\}$ and $n \neq \frac{j(m+r+p)}{2s} - p$ in (3-26), then we have

$$d^{\bar{0}}_{r,s}(m+n, i+j) = 0 \quad \text{and} \quad d^{\bar{0}}_{r,s}(n,j) = 0.$$

Hence we obtain

$$(2s(n+p) - j(m+r+p))d^{\bar{0}}_{r,s}(m,s) = 0 \quad \text{and} \quad d^{\bar{0}}_{r,s}(m,s) = 0.$$

Therefore, we derive

$$d^{\bar{0}}_{r,s}(m,i) = 0 \quad \text{for all } (m,i) \in \mathbb{Z}\times\mathbb{Z}. \tag{3-29}$$

Taking into account both (3-27) and (3-29), then we infer

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big).$$

Substituting $m = i = 0$, we get

$$\left(j - s + \tfrac{1}{2}\right) d^{\bar{1}}_{r,s}\left(n + \tfrac{1}{2}, j + \tfrac{1}{2}\right) = 0.$$

Since $j, s \in \mathbb{Z}$, then we have $j - s + \frac{1}{2} \neq 0$. Hence it follows that

$$d^{\bar{1}}_{r,s}\left(n + \tfrac{1}{2}, j + \tfrac{1}{2}\right) = 0$$

for all $(n, j) \in \mathbb{Z} \times \mathbb{Z}$. □

Lemma 3.10. *Let $\varphi = \sum_{r,s \in \mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, 0)$, $r \in \mathbb{Z}^*$ and $s = 0$.*

(1) *If $r \neq p$, then $\varphi_{r,0} = 0$.*

(2) *If $r = p$, then $d^{\bar{0}}_{p,0}(m, i) = 0$ for all $(m, i) \neq (-2p, 0)$ and $d^{\bar{1}}_{p,0}\left(m + \frac{1}{2}, i + \frac{1}{2}\right) = 0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.*

Proof. With $s = 0$, (3-26) can be expressed as

$$2 \begin{vmatrix} n+p & j \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r,0}(m+n, i+j) = \begin{vmatrix} n+p & j \\ m+r+p & i \end{vmatrix} d^{\bar{0}}_{r,0}(m, i) + \begin{vmatrix} n+r+p & j \\ m+p & i \end{vmatrix} d^{\bar{0}}_{r,0}(n, j). \tag{3-30}$$

Taking $n = j = 0$ in (3-30), we get

$$p i d^{\bar{0}}_{r,0}(m, i) = (p + r) i d^{\bar{0}}_{r,0}(0, 0).$$

Thus, we have

$$d^{\bar{0}}_{r,0}(m, i) = (1 + r p^{-1}) d^{\bar{0}}_{r,0}(0, 0) \text{ for } i \neq 0. \tag{3-31}$$

We proceed with a classification discussion based on whether or not $p \in \mathbb{Z}$.

Case 1: $p \in \mathbb{Z}$. Choosing $m = -n$ and $j = -i \neq 0$ in (3-30) and applying (3-31), then we obtain

$$\begin{aligned} 4 p d^{\bar{0}}_{r,0}(0, 0) &= (r + 2p)\left(d^{\bar{0}}_{r,0}(-n, i) + d^{\bar{0}}_{r,0}(n, -i)\right) \\ &= 2(r + 2p)(1 + r p^{-1}) d^{\bar{0}}_{r,0}(0, 0), \end{aligned}$$

$$r(r + 3p) d^{\bar{0}}_{r,0}(0, 0) = 0.$$

Thus, we get

$$d^{\bar{0}}_{r,0}(0, 0) = 0 \quad \text{for } r \neq -3p.$$

We take $r = -3p$ and use (3-31). Hence it follows that

$$d^{\bar{0}}_{r,0}(m, i) = -2 d^{\bar{0}}_{r,0}(0, 0) \quad \text{for } i \neq 0. \tag{3-32}$$

Putting $m=n=0$, i, j, $i\pm j\neq 0$ in (3-30) and applying (3-32), then we can derive

$$\begin{aligned}2p(i-j)d^{\bar{0}}_{r,0}(0,i+j)&=p(i+2j)d^{\bar{0}}_{r,0}(0,i)-p(2i+j)d^{\bar{0}}_{r,0}(0,j),\\ 2p(i-j)d^{\bar{0}}_{r,0}(0,0)&=p(i+2j)d^{\bar{0}}_{r,0}(0,0)-p(2i+j)d^{\bar{0}}_{r,0}(0,0),\\ d^{\bar{0}}_{r,0}(0,0)&=0.\end{aligned}$$

Case 2: $p\notin\mathbb{Z}$. Observing that $1+rp^{-1}\neq 0$ in this case, we take $m=n=0$ in (3-30), then we can obtain

$$2p(i-j)d^{\bar{0}}_{r,0}(0,i+j)=(p(i-j)-rj)d^{\bar{0}}_{r,0}(0,i)+(p(i-j)+ri)d^{\bar{0}}_{r,0}(0,j).$$

We choose i, j, $i\pm j\neq 0$. Then owing to (3-31) and $1+rp^{-1}\neq 0$, we have

$$d^{\bar{0}}_{r,0}(0,0)=0.$$

Hence by (3-31), we can deduce that

$$d^{\bar{0}}_{r,0}(m,i)=0\quad\text{for } i\neq 0. \tag{3-33}$$

It remains to analyze $d^{\bar{0}}_{r,0}(m,0)$. To this end, putting $m=0$ and $j=-i\neq 0$ in (3-30) and applying (3-33), hence we have

$$\begin{aligned}2(2p+n)id^{\bar{0}}_{r,0}(n,0)&=(2p+n+r)i(d^{\bar{0}}_{r,0}(0,i)+d^{\bar{0}}_{r,0}(n,-i)),\\ (2p+n)d^{\bar{0}}_{r,0}(n,0)&=0.\end{aligned}$$

Therefore, we can conclude

$$d^{\bar{0}}_{r,0}(n,0)=0\quad\text{for } n\neq -2p.$$

Taking $m=-2p$, $i=0$ and $j\neq 0$ in (3-30) and using (3-33), it follows that

$$\begin{aligned}2pjd^{\bar{0}}_{r,0}(-2p+n,j)&=(p-r)jd^{\bar{0}}_{r,0}(-2p,0)+pjd^{\bar{0}}_{r,0}(n,j),\\ (p-r)d^{\bar{0}}_{r,0}(-2p,0)&=0.\end{aligned}$$

So, $d^{\bar{0}}_{r,0}(-2p,0)=0$ for $r\neq p$ is derived.

If $r\neq p$, then (3-27) becomes

$$2\begin{vmatrix}n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2}\\ m+p & i\end{vmatrix}d^{\bar{1}}_{r,0}\big(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\big)=\begin{vmatrix}n+r+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2}\\ m+p & i\end{vmatrix}d^{\bar{1}}_{r,0}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big). \tag{3-34}$$

By taking $m=i=0$ into (3-34), we get

$$\big(j+\tfrac{1}{2}\big)d^{\bar{1}}_{r,0}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)=0.$$

Since $j \in \mathbb{Z}$, then $j+\frac{1}{2} \neq 0$ is inferred. Furthermore, we have

$$d^{\bar{1}}_{r,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \quad \text{for all } (n, j) \in \mathbb{Z}\times\mathbb{Z} \text{ and } r \neq p.$$

If $r = p$, the term in (3-27) that includes $d^{\bar{0}}_{p,0}(m, i)$ is

$$\begin{aligned}\big(\big(n+\tfrac{p}{2}+\tfrac{1}{2}\big)i - \big(j+\tfrac{1}{2}\big)(m+2p)\big)d^{\bar{0}}_{p,0}(m, i) &= 0 \quad \text{for } (m, i) \neq (-2p, 0),\\ \big(j+\tfrac{1}{2}\big)(-2p+p+p)d^{\bar{0}}_{p,0}(-2p, 0) &= 0 \quad \text{for } (m, i) = (-2p, 0).\end{aligned}$$

Therefore, the term with $d^{\bar{0}}_{p,0}(m, i)$ in (3-27) vanishes. Equation (3-27) becomes

$$2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{p,0}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+\frac{3p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+p & i \end{vmatrix} d^{\bar{1}}_{p,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big). \tag{3-35}$$

Taking $m = i = 0$ into (3-35), so we obtain

$$d^{\bar{1}}_{p,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0$$

for all $(n, j) \in \mathbb{Z}\times\mathbb{Z}$. □

According to this conclusion, we can get the following lemma.

Lemma 3.11. *Let $p \in \mathbb{Z}^*$. Then the linear map $\alpha : \mathcal{S}(p, 0) \to \mathcal{S}(p, 0)$ is a $\frac{1}{2}$-derivation of $\mathcal{S}(p, 0)$ such that*

$$\alpha(L_{m,i}) = \begin{cases} 0, & (m, i) \neq (-2p, 0), \\ L_{-p,0}, & (m, i) = (-2p, 0), \end{cases} \qquad \alpha(G_{m+\frac{1}{2},i+\frac{1}{2}}) = 0$$

for $(m, i) \in \mathbb{Z}\times\mathbb{Z}$.

Proof. We observe that $\alpha = \sum_{r,s\in\mathbb{Z}} \alpha_{r,s} = \alpha_{p,0}$. In view of Lemma 3.1 we need to check (3-3) and (3-4) for $(r, s) = (p, 0)$ and

$$d^{\bar{0}}_{p,0}(m, i) = \begin{cases} 0, & (m, i) \neq (-2p, 0), \\ 1, & (m, i) = (-2p, 0), \end{cases} \tag{3-36}$$

$$d^{\bar{1}}_{p,0}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0, \quad (m, i) \in \mathbb{Z}\times\mathbb{Z}. \tag{3-37}$$

Firstly, we prove that (3-3) holds.

Case 1: $(m, i), (n, j), (m+n, i+j) \neq (-2p, 0)$. Then both sides of (3-3) are zero.

Case 2: $(m, i) = (-2p, 0)$. Then (3-3) becomes

$$2pjd^{\bar{0}}_{p,0}(-2p+n, j) = pjd^{\bar{0}}_{p,0}(n, j).$$

If $j = 0$, then it is trivially satisfied, otherwise both sides are zero by (3-36).

Case 3: $(n, j) = (-2p, 0)$. Then (3-3) becomes

$$2pid^{\bar{0}}_{p,0}(-2p+m, i) = pid^{\bar{0}}_{p,0}(m, i),$$

so this case is similar to Case 2.

Case 4: $(m+n, i+j) = (-2p, 0)$. Then (3-3) becomes

$$0 = pid^{\bar{0}}_{p,0}(-2p-n, i) + pid^{\bar{0}}_{p,0}(n, -i),$$

and again this holds by (3-36).

Next, we proceed to prove that (3-4) holds. Due to (3-37), (3-4) becomes

$$\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{1}{2} \\ m+2p & i \end{vmatrix} d^{\bar{0}}_{p,0}(m, i) = 0.$$

Given (3-36), it holds. □

The corollary is an immediate consequence of Lemmas 3.9 and 3.11.

Corollary 3.12. *Let φ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, 0)$, $r, s \in \mathbb{Z}$.*

(1) *If $p \notin \mathbb{Z}$, then $\varphi_{r,s} = 0$ for $(r, s) \neq (0, 0)$.*

(2) *If $p \in \mathbb{Z}$, then $\varphi_{r,s} = 0$ for all $(r, s) \notin \{(0, 0), (p, 0)\}$ and $\varphi_{p,0} \in \langle\alpha\rangle$.*

Finally we proceed to prove the case of $r = s = 0$.

Lemma 3.13. *Let φ be a $\frac{1}{2}$-derivation of $\mathcal{S}(p, 0)$ and $r = s = 0$. Then*

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{1}}_{0,0}\big(m' + \tfrac{1}{2}, i' + \tfrac{1}{2}\big)$$

for all $(m, i), (m', i') \in \mathbb{Z} \times \mathbb{Z}$.

Proof. In this situation, (3-3) and (3-4) are presented as, respectively,

$$(i(n+p) - j(m+p))(2d^{\bar{0}}_{0,0}(m+n, i+j) - d^{\bar{0}}_{0,0}(m, i) - d^{\bar{0}}_{0,0}(n, j) = 0, \tag{3-38}$$

$$\big(i\big(n+\tfrac{p}{2}+\tfrac{1}{2}\big) - \big(j+\tfrac{1}{2}\big)(m+p)\big) \times \big(2d^{\bar{1}}_{0,0}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) - d^{\bar{0}}_{0,0}(m, i) - d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\big) = 0. \tag{3-39}$$

Taking $n = j = 0$ and $i \neq 0$ in (3-38), we obtain

$$d^{\bar{0}}_{0,0}(m, i) = d^{\bar{0}}_{0,0}(0, 0) \quad \text{for } i \neq 0. \tag{3-40}$$

We choose $j \neq 0$ and $n = -m$ in (3-38), then we arrive at

$$(m+p)(2d^{\bar{0}}_{0,0}(0, j) - d^{\bar{0}}_{0,0}(m, 0) - d^{\bar{0}}_{0,0}(-m, j)) = 0.$$

Due to (3-40), it follows that

$$d^{\bar{0}}_{0,0}(m, 0) = d^{\bar{0}}_{0,0}(0, 0) \quad \text{for } m \neq -p.$$

Substituting $m = -p$, $n = 0$ and $j = -i \neq 0$ into (3-38) and using (3-40), we can derive that

$$2d^{\bar{0}}_{0,0}(-p,0) - d^{\bar{0}}_{0,0}(-p,i) - d^{\bar{0}}_{0,0}(0,-i) = 0,$$
$$d^{\bar{0}}_{0,0}(-p,0) = d^{\bar{0}}_{0,0}(0,0).$$

Taking $m = i = 0$ in (3-39), we can see that

$$p\big(j+\tfrac{1}{2}\big)\big(2d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) - d^{\bar{0}}_{0,0}(0,0) - d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big)\big) = 0.$$

Since $p \neq 0$ and $j \in \mathbb{Z}$, then we have $p\big(j+\frac{1}{2}\big) \neq 0$. Furthermore, we are able to conclude that

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = d^{\bar{0}}_{0,0}(0,0)$$

for all $(n, j) \in \mathbb{Z} \times \mathbb{Z}$. □

Directly from Lemmas 3.9, 3.13 and Corollary 3.12, we deduce the following proposition.

Proposition 3.14. *Let $p \in \mathbb{C}^*$ and $q = 0$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(p,0)) = \begin{cases} \langle id \rangle, & p \notin \mathbb{Z}, \\ \langle id, \alpha \rangle, & p \in \mathbb{Z}^*, \end{cases}$$

where α is as in Lemma 3.11.

3.1.4. *The case $p = q = 0$.* Rewriting (3-4) with $p = q = 0$, we obtain

$$(3\text{-}41)\quad 2\begin{vmatrix} n+\frac{1}{2} & j+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+\frac{1}{2} & j+\frac{1}{2} \\ m+r & i+s \end{vmatrix} d^{\bar{0}}_{r,s}(m,i) + \begin{vmatrix} n+r+\frac{1}{2} & j+s+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big).$$

Lemma 3.15. *Let $\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,0)$ and let $(r,s) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$. Then*

$$\varphi_{r,s} = 0.$$

Proof. $d^{\bar{0}}_{r,s}(m,i) = 0$ for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$ is derived from Lemma 2.9 in [10]. Then, (3-41) becomes

$$(3\text{-}42)\quad 2\begin{vmatrix} n+\frac{1}{2} & j+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = \begin{vmatrix} n+r+\frac{1}{2} & j+s+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big).$$

Taking $m=j=0$ in (3-42), we obtain

$$2i\big(n+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},i+\tfrac{1}{2}\big)=i\big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big),$$

$$2\big(n+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},i+\tfrac{1}{2}\big)=\big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } i\neq 0. \tag{3-43}$$

It implies that $d^{\bar{1}}_{r,s}\big(n+\frac{1}{2},i+\frac{1}{2}\big)$ is independent of i for $i\neq 0$. Substitution $m=0$, $j=-i\neq 0$ into (3-42), it leads to

$$\begin{aligned}(2n+1)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)&=\big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},-i+\tfrac{1}{2}\big)\\&=\big(n+r+\tfrac{1}{2}\big)d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},i+\tfrac{1}{2}\big).\end{aligned}\tag{3-44}$$

Multiplying (3-44) by $2n+1$ and using (3-43), we get

$$(2n+1)^2d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=\big(n+r+\tfrac{1}{2}\big)^2d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big).$$

Assuming $d^{\bar{1}}_{r,s}\big(n+\frac{1}{2},\frac{1}{2}\big)\neq 0$, we obtain $(2n+1)^2=\big(n+r+\frac{1}{2}\big)^2$, whence $2n+1=\pm\big(n+r+\frac{1}{2}\big)$. It follows that $n=r-\frac{1}{2}$ or $n=-\frac{r}{3}-\frac{1}{2}$, which contradicts $n\in\mathbb{Z}$.

Then utilizing (3-44) and $n+r+\frac{1}{2}\neq 0$, we have

$$d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=\frac{2n+1}{n+r+\frac{1}{2}}d^{\bar{1}}_{r,s}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=0$$

for all $(n,i)\in\mathbb{Z}\times\mathbb{Z}$. □

Lemma 3.16. *Let $\varphi=\sum_{r,s\in\mathbb{Z}}\varphi_{r,s}$ be a $\frac{1}{2}$-derivation of $\mathcal{S}(0,0)$ and $(r,s)=(0,0)$. Then*

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(m',i')$$

for $(m,i)\in\mathbb{Z}\times\mathbb{Z}$, $(m',i')\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$.

Proof. According to Lemma 2.10 from [10], we can obtain

$$d^{\bar{0}}_{0,0}(m,i)=d^{\bar{0}}_{0,0}(m',i')\quad\text{for all } (m,i),(m',i')\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}. \tag{3-45}$$

By substituting $(r,s)=(0,0)$ into (3-41), we get

$$\begin{vmatrix}n+\frac{1}{2} & j+\frac{1}{2}\\ m & i\end{vmatrix}\Big(2d^{\bar{1}}_{0,0}\big(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\big)-d^{\bar{0}}_{0,0}(m,i)-d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)\Big)=0. \tag{3-46}$$

We take $m=j=0$ and $i\neq 0$ in (3-46). And since $n+\frac{1}{2}\neq 0$, we obtain

$$2d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(0,i)+d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } i\neq 0. \tag{3-47}$$

Letting $n=i=0$ and $m\neq 0$ in (3-46), and because of $j+\frac{1}{2}\neq 0$, we have

$$2d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},j+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(m,0)+d^{\bar{1}}_{0,0}\big(12,j+\tfrac{1}{2}\big)\quad\text{for } m\neq 0.$$

Hence, due to (3-45), we arrive at

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)\quad\text{for } n,i\neq 0. \tag{3-48}$$

Substituting $m=-n\neq 0$, $i\neq 0$ and $j=0$ into (3-46), it leads to

$$\big(ni+\tfrac{1}{2}(n+i)\big)\big(2d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)-d^{\bar{0}}_{0,0}(-n,i)-d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)\big)=0.$$

If $(n,i)\notin\{(0,0),(-1,-1)\}$, then we have $ni+\frac{1}{2}(n+i)\neq 0$. Therefore, it follows that

$$2d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(-n,i)+d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } (n,i)\neq(0,0),(-1,-1). \tag{3-49}$$

Now, combining (3-47), (3-48) and (3-49), we get

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(-n,i)\quad\text{for } n,i\neq 0 \text{ and } (n,i)\neq(-1,-1),$$

$$d^{\bar{1}}_{0,0}\big(n+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(-n,i)=d^{\bar{0}}_{0,0}(n',i')$$
$$\text{for } (n,i)\neq(0,0),(-1,-1) \text{ and } (n',i')\neq(0,0).$$

In particular, taking $n=i=-1$ in (3-47), we have

$$2d^{\bar{1}}_{0,0}\big(-\tfrac{1}{2},-\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(0,-1)+d^{\bar{1}}_{0,0}\big(-\tfrac{1}{2},\tfrac{1}{2}\big)=2d^{\bar{0}}_{0,0}(0,-1)=2d^{\bar{0}}_{0,0}(n',i')$$

for $(n',i')\neq(0,0)$. Therefore, we can conclude

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{0}}_{0,0}(m',i')\quad\text{for } (m,i),(m',i')\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\} \tag{3-50}$$

Taking $n=j=0$ in (3-46), we can deduce

$$(i-m)\big(2d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)-d^{\bar{0}}_{0,0}(m,i)-d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\big)=0. \tag{3-51}$$

If $m\neq i$, then we get $(m,i)\neq(0,0)$. Consequently, it follows that $d^{\bar{0}}_{0,0}(m,i)=d^{\bar{1}}_{0,0}\big(m+\frac{1}{2},i+\frac{1}{2}\big)$. Hence, from (3-51) we obtain that

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } m\neq i.$$

Using (3-50), we have

$$d^{\bar{1}}_{0,0}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=d^{\bar{1}}_{0,0}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\quad\text{for } (m,i)\in\mathbb{Z}\times\mathbb{Z}. \qquad\square$$

According to this conclusion, we are able to deduce the subsequent lemma.

Lemma 3.17. *Let $p=q=0$. Then the linear map $\beta:\mathcal{S}(0,0)\to\mathcal{S}(0,0)$ is a $\frac{1}{2}$-derivation of $\mathcal{S}(0,0)$ such that*

$$\beta(L_{m,i})=\begin{cases}0, & (m,i)\neq(0,0),\\ L_{0,0}, & (m,i)=(0,0),\end{cases}\qquad \beta(G_{m+\frac{1}{2},i+\frac{1}{2}})=0$$

for $(m,i)\in\mathbb{Z}\times\mathbb{Z}$.

Proof. We observe that $\beta = \sum_{r,s\in\mathbb{Z}} \beta_{r,s} = \beta_{0,0}$. In view of Lemma 3.1, we need to check (3-3) and (3-4) for $(r, s) = (0, 0)$ and

$$d^{\bar{0}}_{0,0}(m, i) = \begin{cases} 0, & (m, i) \neq (0, 0), \\ 1, & (m, i) = (0, 0), \end{cases} \tag{3-52}$$

$$d^{\bar{1}}_{0,0}\big(m + \tfrac{1}{2}, i + \tfrac{1}{2}\big) = 0 \tag{3-53}$$

for $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.

Firstly, we prove (3-3). In fact, we can split this proof into four cases:

(1) $(m, i), (n, j), (m+n, i+j) = (0, 0)$,

(2) $(m, i) = (0, 0)$,

(3) $(n, j) = (0, 0)$,

(4) $(m+n, i+j) \neq (0, 0)$.

Obviously, under all these different cases, (3-3) is trivially satisfied.

Subsequently, we prove (3-4). Due to (3-53), (3-4) becomes

$$\begin{vmatrix} n+\frac{1}{2} & j+\frac{1}{2} \\ m & i \end{vmatrix} d^{\bar{0}}_{0,0}(m, i) = 0.$$

It clearly holds by (3-52). □

With Lemmas 3.15 and 3.17, we can directly infer the subsequent proposition.

Proposition 3.18. *Let $p = q = 0$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(0, 0)) = \langle id, \beta\rangle,$$

where β is as in Lemma 3.17.

By integrating Propositions 3.4, 3.8, 3.14 and 3.18, we can deduce the following corollary.

Corollary 3.19. *Let $p, q \in \mathbb{C}$. Then*

$$\Delta^{\bar{0}}(\mathcal{S}(p, q)) = \begin{cases} \langle id\rangle, & q \neq 0, \\ \langle id, \alpha\rangle, & p \neq 0, q = 0, \\ \langle id, \beta\rangle, & p = q = 0. \end{cases}$$

where α and β are as in Lemmas 3.17 and 3.11.

3.2. ***Odd $\frac{1}{2}$-derivations of $\mathcal{S}(p, q)$.*** In this subsection we consider that a linear map $\varphi : \mathcal{S}(p, q) \to \mathcal{S}(p, q)$ is *odd*, if $\varphi(\mathcal{S}(p, q)_i) \subseteq \mathcal{S}(p, q)_{\bar{1}-i}$ for $i \in \mathbb{Z}_2$. In this case $|\varphi| = 1$, so φ is a $\frac{1}{2}$-superderivation of $\mathcal{S}(p, q)$ if and only if

$$\varphi([x, y]) = \tfrac{1}{2}\big([\varphi(x), y] + [x, \varphi(y)]\big), \quad x \in \mathcal{S}(p, q)_{\bar{0}},$$
$$\varphi([x, y]) = \tfrac{1}{2}\big([\varphi(x), y] - [x, \varphi(y)]\big), \quad x \in \mathcal{S}(p, q)_{\bar{1}}.$$

Denote by $\Delta^{\bar{1}}(\mathcal{S}(p,q))$ the space of odd $\frac{1}{2}$-superderivations of $\mathcal{S}(p,q)$. As usual, for any $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$, we write

$$\varphi = \sum_{r,s\in\mathbb{Z}} \varphi_{r+\frac{1}{2},s+\frac{1}{2}},$$

where

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}) = d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i)G_{m+r+\frac{1}{2},i+s+\frac{1}{2}}, \tag{3-54}$$

$$\varphi_{r,s}(G_{m+\frac{1}{2},i+\frac{1}{2}}) = d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big)L_{m+r+1,i+s+1} \tag{3-55}$$

for some $d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i)$, $d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\frac{1}{2}, i+\frac{1}{2}\big) \in \mathbb{C}$, $m,i,r,s\in\mathbb{Z}$. We have

$$\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p,q)) \quad \text{if and only if} \quad \varphi_{r+\frac{1}{2},s+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q)) \ \text{ for all } r,s\in\mathbb{Z}.$$

Lemma 3.20. *Let*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} : \mathcal{S}(p,q) \to \mathcal{S}(p,q)$$

be a linear map satisfying (3-54) and (3-55). Then

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$$

if and only if the following three conditions hold:

$$\begin{aligned} 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n,i+j) &= \begin{vmatrix} n+p & j+q \\ m+r+\frac{p}{2}+\frac{1}{2} & i+s+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) \\ &\quad + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j), \end{aligned} \tag{3-56}$$

$$\begin{aligned} 2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) \\ = 2q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) + \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big), \end{aligned} \tag{3-57}$$

$$\begin{aligned} 4q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1,i+j+1) &= \begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+r+p+1 & i+s+q+1 \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) \\ &\quad - \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big). \end{aligned} \tag{3-58}$$

Proof. By Definition 2.1, $\varphi_{r+\frac{1}{2},s+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p,q))$ if and only if the following three expressions are valid:

$$(3\text{-}59)\quad 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n,i+j}) = [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), L_{n,j}] + [L_{m,i} + \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{n,j})],$$

$$(3\text{-}60)\quad 2\begin{vmatrix} n+\frac{1}{2}+\frac{p}{2} & j+\frac{1}{2}+\frac{q}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}) = [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [L_{m,i}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})],$$

$$(3\text{-}61)\quad 4q\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n+1,i+j+1}) = [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}), G_{n+\frac{1}{2},j+\frac{1}{2}}] - [G_{m+\frac{1}{2},i+\frac{1}{2}}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})].$$

Due to (3-59), we can see that

$$\begin{aligned}
2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} & d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n, i+j) G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} \\
&= 2\begin{vmatrix} n+p & j+q \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n,i+j}) \\
&= [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), L_{n,j}] + [L_{m,i}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{n,j})] \\
&= [d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) G_{m+\frac{1}{2},i+\frac{1}{2}}, L_{n,j}] + [L_{m,i}, d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j) G_{n+r+\frac{1}{2},j+r+\frac{1}{2}}] \\
&= \begin{vmatrix} n+p & j+q \\ m+r+\frac{p}{2}+\frac{1}{2} & i+s+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}} \\
&\qquad + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j) G_{m+n+r+\frac{1}{2},i+j+s+\frac{1}{2}}.
\end{aligned}$$

Thus, (3-56) is obtained. By (3-60), we have

$$\begin{aligned}
2\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2} \\ m+p & i+q \end{vmatrix} & d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) L_{m+n+r+1,i+j+s+1} \\
&= 2\begin{vmatrix} n+\frac{1}{2}+\frac{p}{2} & j+\frac{1}{2}+\frac{q}{2} \\ m+p & i+q \end{vmatrix} \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+n+\frac{1}{2},i+j+\frac{1}{2}}) \\
&= [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m,i}), G_{n+\frac{1}{2},j+\frac{1}{2}}] + [L_{m,i}, \varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})] \\
&= [d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) G_{m+\frac{1}{2},i+\frac{1}{2}}, G_{n+\frac{1}{2},j+\frac{1}{2}}] \\
&\qquad + [L_{m,i}, d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) L_{n+r+1,j+s+1}] \\
&= 2q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) L_{m+n+r+1,i+j+s+1} \\
&\qquad + \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+p & i+q \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) L_{m+n+r+1,i+j+s+1}.
\end{aligned}$$

Consequently, we come to (3-57). Because of (3-61), we observe

$$
\begin{aligned}
&4qd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1,i+j+1)G_{m+n+r+\frac{3}{2},i+j+s+\frac{3}{2}}\\
&\quad=4q\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{m+n+1,i+j+1})\\
&\quad=[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}),G_{n+\frac{1}{2},j+\frac{1}{2}}]-[G_{m+\frac{1}{2},i+\frac{1}{2}},\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})]\\
&\quad=[d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}),G_{n+\frac{1}{2},j+\frac{1}{2}}]+[G_{m+\frac{1}{2},i+\frac{1}{2}},d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})]\\
&\quad=\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2}\\ m+r+p+1 & i+s+q+1\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)G_{m+n+r+\frac{3}{2},i+j+s+\frac{3}{2}}\\
&\qquad-\begin{vmatrix} n+r+p+1 & j+s+q+1\\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)G_{m+n+r+\frac{3}{2},i+j+s+\frac{3}{2}}.
\end{aligned}
$$

Hence, it follows that (3-58) holds truly. □

Let $\varphi\in\Delta^{\bar{1}}(\mathcal{S}(p,q))$ and $\psi_{m+\frac{1}{2},i+\frac{1}{2}}$ be the left multiplication by $G_{m+\frac{1}{2},i+\frac{1}{2}}$ in $\mathcal{S}(p,q)$, $m,i,r,s\in\mathbb{Z}$. Since $|\psi_{m+\frac{1}{2},i+\frac{1}{2}}|=1$ and we have, for all $x,y\in\{L_{n,j},G_{n+\frac{1}{2},j+\frac{1}{2}}\mid n,j\in\mathbb{Z}\}$,

$$
\begin{aligned}
\psi_{m+\frac{1}{2},i+\frac{1}{2}}([x,y])&=[G_{m+\frac{1}{2},i+\frac{1}{2}},[x,y]]\\
&=[[G_{m+\frac{1}{2},i+\frac{1}{2}},x],y]+(-1)^{|x|}[x,[G_{m+\frac{1}{2},i+\frac{1}{2}},y]]\\
&=[\psi_{m+\frac{1}{2},i+\frac{1}{2}}(x),y]+(-1)^{|x|}[x,\psi_{m+\frac{1}{2},i+\frac{1}{2}}(y)],
\end{aligned}
$$

$\psi_{m+\frac{1}{2},i+\frac{1}{2}}$ is an odd superderivation of $\mathcal{S}(p,q)$.

Since the supercommutator

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}},\psi_{m+\frac{1}{2},i+\frac{1}{2}}]=\varphi_{r+\frac{1}{2},s+\frac{1}{2}}\psi_{m+\frac{1}{2},i+\frac{1}{2}}+\psi_{m+\frac{1}{2},i+\frac{1}{2}}\varphi_{r+\frac{1}{2},s+\frac{1}{2}}$$

satisfies

$$
\begin{aligned}
&[\varphi_{r+\frac{1}{2},s+\frac{1}{2}},\psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j})\\
&\quad=\varphi_{r+\frac{1}{2},s+\frac{1}{2}}([G_{m+\frac{1}{2},i+\frac{1}{2}},L_{n,j}])+[G_{m+\frac{1}{2},i+\frac{1}{2}},\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(L_{n,j})]\\
&\quad=\begin{vmatrix} n+p & j+q\\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\big)L_{m+n+r+1,i+j+s+1}\\
&\qquad\qquad+2qd^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j)L_{m+n+r+1,i+j+s+1},
\end{aligned}
\tag{3-62}
$$

$$
\begin{aligned}
&[\varphi_{r+\frac{1}{2},s+\frac{1}{2}},\psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}})\\
&\quad=\varphi_{r+\frac{1}{2},s+\frac{1}{2}}([G_{m+\frac{1}{2},i+\frac{1}{2}},G_{n+\frac{1}{2},j+\frac{1}{2}}])+[G_{m+\frac{1}{2},i+\frac{1}{2}},\varphi_{r+\frac{1}{2},s+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}})]\\
&\quad=2qd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1,i+j+1)G_{m+n+r+\frac{3}{2},i+j+s+\frac{3}{2}}\\
&\qquad+\begin{vmatrix} n+r+p+1 & j+s+q+1\\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2},j+\tfrac{1}{2}\big)G_{m+n+r+\frac{3}{2},i+j+s+\frac{3}{2}},
\end{aligned}
\tag{3-63}
$$

for all $x, y \in \{L_{n,j}, G_{n+\frac{1}{2},j+\frac{1}{2}} \mid n, j \in \mathbb{Z}\}$, it follows that

$$\begin{aligned}
&[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}]([x,y]) \\
&\qquad = \varphi_{r+\frac{1}{2},s+\frac{1}{2}}\big(\psi_{m+\frac{1}{2},i+\frac{1}{2}}([x,y])\big) + \psi_{m+\frac{1}{2},i+\frac{1}{2}}\big(\varphi_{r+\frac{1}{2},s+\frac{1}{2}}([x,y])\big) \\
&\qquad = \tfrac{1}{2}\big([[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](x), y] + [x, [\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](y)]\big).
\end{aligned}$$

Therefore, we can conclude that $[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}]$ is an even $\frac{1}{2}$-superderivation of $\mathcal{S}(p,q)$, which was given in Corollary 3.19.

So, if $q \neq 0$, then we arrive at

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ cL_{n,j}, & (m+r+1, i+s+1) = (0,0), \end{cases} \tag{3-64}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ cL_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-65}$$

for some constant $c \in \mathbb{C}$.

If $q = 0$ and $p \notin \mathbb{Z}$, then we get

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_1L_{n,j}, & (m+r+1, i+s+1) = (0,0), \end{cases} \tag{3-66}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_1L_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-67}$$

for some constant $c_1 \in \mathbb{C}$.

If $q = 0$ and $p \in \mathbb{Z}^*$, then we have

$$\begin{aligned}
&[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) \\
&\quad = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), (0,p), \\ c_2L_{n,j}, & (m+r+1, i+s+1) = (0,0), \\ 0, & (m+r+1, i+s+1) = (0,p) \text{ and } (n,j) \neq (-2p,0), \\ c_3L_{-p,0}, & (m+r+1, i+s+1) = (0,p) \text{ and } (n,j) = (-2p,0), \end{cases}
\end{aligned} \tag{3-68}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_2L_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-69}$$

for some constants $c_2, c_3 \in \mathbb{C}$.

If $p = q = 0$, then we obtain

$$\begin{aligned}
&[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](L_{n,j}) \\
&\quad = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_4L_{n,j}, & (m+r+1, i+s+1) = (0,0) \text{ and } (n,j) \neq (0,0), \\ c_5L_{n,j}, & (m+r+1, i+s+1) = (0,0) \text{ and } (n,j) = (0,0), \end{cases}
\end{aligned} \tag{3-70}$$

$$[\varphi_{r+\frac{1}{2},s+\frac{1}{2}}, \psi_{m+\frac{1}{2},i+\frac{1}{2}}](G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (m+r+1, i+s+1) \neq (0,0), \\ c_4L_{n,j}, & (m+r+1, i+s+1) = (0,0) \end{cases} \tag{3-71}$$

for some constants $c_4, c_5 \in \mathbb{C}$.

3.2.1. *The case $p, q \neq 0$.* By (3-62)–(3-65), we have

$$\begin{vmatrix} n+p & j+q \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) + 2q d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(n, j) = 0, \tag{3-72}$$

$$2q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1, i+j+1) + \begin{vmatrix} n+r+p+1 & j+s+q+1 \\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0 \tag{3-73}$$

for all $(m, i) \neq (-r-1, -s-1)$.

Lemma 3.21. *Let $r, s \in \mathbb{Z}$, and $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(p, q))$.*

(1) *If $p, q \in 2\mathbb{Z}+1$, then*

$$\begin{aligned} \varphi_{r+\frac{1}{2},s+\frac{1}{2}} &= 0 \quad \text{for } (r, s) \neq \big(\tfrac{p}{2}-\tfrac{1}{2}, \tfrac{q}{2}-\tfrac{1}{2}\big), \\ \varphi_{\frac{p}{2},\frac{q}{2}}(L_{m,i}) &= 0 \quad \text{for all } (m, i) \in \mathbb{Z}\times\mathbb{Z}, \\ \varphi_{\frac{p}{2},\frac{q}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}) &= 0 \quad \text{for } (m, i) \neq \big(-\tfrac{3p}{2}-\tfrac{1}{2}, -\tfrac{3q}{2}-\tfrac{1}{2}\big). \end{aligned}$$

(2) *Otherwise, $\varphi_{r+\frac{1}{2},s+\frac{1}{2}} = 0$.*

Proof. Taking $m = i = 0$ in (3-56), we obtain

$$(q(n-r) - p(j-s)) d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 2q d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0, 0). \tag{3-74}$$

Then taking $n = r$ and $j = s$ in (3-74), it gives $d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0, 0) = 0$. Hence, (3-74) can be given by

$$(p(i-s)-q(m-r)) d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (r, s), (m, i) \in \mathbb{Z}\times\mathbb{Z}. \tag{3-75}$$

Taking $n = j = 0$ in (3-72), it yields

$$\big(p\big(i+\tfrac{1}{2}\big) - q\big(m+\tfrac{1}{2}\big)\big) d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \tag{3-76}$$

for all $(r, s) \in \mathbb{Z}\times\mathbb{Z}$ and $(m, i) \neq (-r-1, -s-1)$. When $(m, i) = (-r-1, -s-1)$, (3-75) can be written

$$\big(p\big(s+\tfrac{1}{2}\big) - q\big(r+\tfrac{1}{2}\big)\big) d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(-r-\tfrac{1}{2}, -s-\tfrac{1}{2}\big) = 0.$$

In addition, subtracting (3-75) from (3-76), we get

$$\big(p\big(s+\tfrac{1}{2}\big) - q\big(r+\tfrac{1}{2}\big)\big) d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0$$

for all $(r, s), (m, i) \in \mathbb{Z}\times\mathbb{Z}$. Clearly, we will analyze whether $p\big(s+\frac{1}{2}\big) - q\big(r+\frac{1}{2}\big)$ is zero or not.

Case 1: $p\big(s+\frac{1}{2}\big)-q\big(r+\frac{1}{2}\big)\neq 0$. We have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0 \quad \text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}.$$

So (3-58) becomes

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1,i+j+1)=0 \quad \text{for all } (m,i),(n,j)\in\mathbb{Z}\times\mathbb{Z}.$$

Letting $u=m+n+1$ and $v=i+j+1$, we obtain

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(u,v)=0 \quad \text{for all } (u,v)\in\mathbb{Z}\times\mathbb{Z}.$$

Case 2: $p\big(s+\frac{1}{2}\big)-q\big(r+\frac{1}{2}\big)=0$. There exists some $k\in\mathbb{C}$ satisfying $r=kp-\frac{1}{2}$, $s=kq-\frac{1}{2}\in\mathbb{Z}$. Now taking into (3-75), we have

$$\big(p\big(i+\tfrac{1}{2}\big)-q\big(m+\tfrac{1}{2}\big)\big)d^{\bar{1}}_{kp,kq}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0.$$

Accordingly, it follows that

$$d^{\bar{1}}_{kp,kq}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0 \quad \text{for } (m,i)\in\mathbb{Z}\times\mathbb{Z} \text{ and } \frac{m+\frac{1}{2}}{i+\frac{1}{2}}\neq\frac{p}{q}. \tag{3-77}$$

Letting $u=m+n+1$ and $v=i+j+1$ with $\frac{m+\frac{1}{2}}{i+\frac{1}{2}}\neq\frac{p}{q}$ and $\frac{n+\frac{1}{2}}{j+\frac{1}{2}}\neq\frac{p}{q}$, we can obtain, by (3-58) and (3-77),

$$d^{\bar{0}}_{kp,kq}(u,v)=0 \quad \text{for all } (u,v)\in\mathbb{Z}\times\mathbb{Z}.$$

In conclusion, we obtain

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(u,v)=0 \quad \text{for all } (u,v),(r,s)\in\mathbb{Z}\times\mathbb{Z}. \tag{3-78}$$

Next, we will calculate

$$d^{\bar{1}}_{kp,kq}\big(m+\tfrac{1}{2},i+\tfrac{1}{2}\big)=0 \quad \text{with } \frac{m+\frac{1}{2}}{i+\frac{1}{2}}=\frac{p}{q}.$$

We choose $(m,i)\in\mathbb{Z}\times\mathbb{Z}$ with $\frac{m+\frac{1}{2}}{i+\frac{1}{2}}=\frac{p}{q}$, then there exists some $l\in\mathbb{C}$ satisfying $m=lp-\frac{1}{2}$, $i=lq-\frac{1}{2}\in\mathbb{Z}$. When $p\neq q$, taking $n=lp-\frac{1}{2}$, $j=lq-\frac{1}{2}$, $(m,i)\in\mathbb{Z}\times\mathbb{Z}$ with $\frac{m}{i}=\frac{p}{q}$ and $m\neq -r-1$ in (3-73), we have

$$(k+l+1)(p-q)d^{\bar{1}}_{kp,kq}(lp,lq)=0.$$

So, we get

$$d^{\bar{1}}_{kp,kq}(lp,lq)=0 \quad \text{for } k+l+1\neq 0 \text{ and } p\neq q.$$

When $p=q$, taking $n=lp-\frac{1}{2}$, $j=lq-\frac{1}{2}$ and $m\neq i\in\mathbb{Z}$ in (3-73), we have

$$(k+l+1)d^{\bar{1}}_{kp,kp}(lp,lp)=0.$$

Therefore, we can conclude

$$d^{\bar{1}}_{kp,kp}(lp, lp) = 0 \quad \text{for } k+l+1 \neq 0.$$

If k, l satisfy $k+l+1=0$, i.e., $(r, s) = \left(kp-\frac{1}{2}, kq-\frac{1}{2}\right)$ and $(m, i) = \left(lp-\frac{1}{2}, lq-\frac{1}{2}\right) = \left(-kp-p-\frac{1}{2}, -kq-q-\frac{1}{2}\right)$, then we can deduce $p, q \in \mathbb{Z}$. Furthermore, taking $n=-p$, $m \neq -\frac{p+1}{2}$, $j \neq -q$ and $i \neq -s-1$ in (3-72), we have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(m-p+\tfrac{1}{2}, i+j+\tfrac{1}{2}\right) = 0.$$

Letting $u = m-p$ and $v = i+j$, we get $d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(u+\frac{1}{2}, v+\frac{1}{2}\right) = 0$ for $u \neq -\frac{3p}{2}-\frac{1}{2}$. And hence, we can arrive at

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = 0$$

for $(r, s) \neq \left(\frac{p}{2}-\frac{1}{2}, \frac{q}{2}-\frac{1}{2}\right)$ and $(u, v) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right)$. □

According to this conclusion, we can obtain the following lemma.

Lemma 3.22. *Let $p, q \in 2\mathbb{Z}+1$. Then the linear map $\gamma : \mathcal{S}(p, q) \to \mathcal{S}(p, q)$ is a $\frac{1}{2}$-derivation of $\mathcal{S}(p, q)$ such that*

$$\gamma(L_{m,i}) = 0,$$
$$\gamma(G_{m+\frac{1}{2},i+\frac{1}{2}}) = \begin{cases} 0, & (m, i) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right), \\ L_{-p,-q}, & (m, i) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right) \end{cases}$$

for $(m, i) \in \mathbb{Z} \times \mathbb{Z}$.

Proof. We observe that

$$\gamma = \sum_{r,s\in\mathbb{Z}} \gamma_{r+\frac{1}{2},s+\frac{1}{2}} = \gamma_{\frac{p}{2},\frac{q}{2}}.$$

Given Lemma 3.20, we need to check (3-56)–(3-58) for $(r, s) = \left(\frac{p}{2}-\frac{1}{2}, \frac{q}{2}-\frac{1}{2}\right)$ and

(3-79) $$d^{\bar{0}}_{\frac{p}{2},\frac{q}{2}}(m, i) = 0, \quad (m, i) \in \mathbb{Z} \times \mathbb{Z},$$

(3-80) $$d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(m+\tfrac{1}{2}, i+\tfrac{1}{2}\right) = \begin{cases} 0, & (m, i) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right), \\ 1, & (m, i) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right). \end{cases}$$

Evidently, (3-56) is satisfied. The next step is to check (3-57).

Case 1: $(n, j), (m+n, i+j) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right)$. Clearly, both sides of (3-57) are zero by (3-80).

Case 2: $(n, j) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right)$. Then (3-57) becomes

$$(pi-qm)d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(m-\tfrac{3p}{2}, i-\tfrac{3q}{2}\right) = 0.$$

If $(m,i)=(0,0)$, then it is trivially satisfied, otherwise both sides are zero by (3-80).

Case 3: $(m+n,i+j)=\left(-\frac{3p}{2}-\frac{1}{2},-\frac{3q}{2}-\frac{1}{2}\right)$, and $(n,j)\neq\left(-\frac{3p}{2}-\frac{1}{2},-\frac{3q}{2}-\frac{1}{2}\right)$. Then (3-57) becomes

$$\left(\left(n+\tfrac{p}{2}+\tfrac{1}{2}\right)\left(-j-\tfrac{q}{2}-\tfrac{1}{2}\right)-\left(-n-\tfrac{p}{2}-\tfrac{1}{2}\right)\left(j+\tfrac{q}{2}+\tfrac{1}{2}\right)\right)d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(-\tfrac{3p}{2},-\tfrac{3q}{2}\right)=0.$$

Both sides of (3-57) are zero by (3-80).

It remains to check (3-58). Due to (3-79) and (3-80), (3-58) becomes

$$\begin{vmatrix} n+\frac{p}{2}+\frac{1}{2} & j+\frac{q}{2}+\frac{1}{2}\\ m+\frac{3p}{2}+\frac{1}{2} & i+\frac{3q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(m+\tfrac{1}{2},i+\tfrac{1}{2}\right) - \begin{vmatrix} n+\frac{3p}{2}+\frac{1}{2} & j+\frac{3q}{2}+\frac{1}{2}\\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{\frac{p}{2},\frac{q}{2}}\left(n+\tfrac{1}{2},j+\tfrac{1}{2}\right)=0.$$

It evidently holds by (3-80). □

With Lemma 3.22, we can directly infer the proposition below.

Proposition 3.23. *Let $p,q\in\mathbb{C}^*$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,q))=\begin{cases}\langle\gamma\rangle, & p,q\in 2\mathbb{Z}+1,\\ \{0\}, & \textit{otherwise},\end{cases}$$

where γ is as in Lemma 3.22.

3.2.2. *The case $p\neq 0$, $q=0$.* In this case, (3-56) becomes

$$2\begin{vmatrix} n+p & j\\ m+p & i\end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n,i+j)=\begin{vmatrix} n+p & j\\ m+r+\frac{p}{2}+\frac{1}{2} & i+s+\frac{1}{2}\end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) + \begin{vmatrix} n+r+\frac{p}{2}+\frac{1}{2} & j+s+\frac{1}{2}\\ m+p & i\end{vmatrix} d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j). \tag{3-81}$$

On the other hand, by using (3-62)–(3-63) and (3-66)–(3-69), respectively, we have

$$\begin{vmatrix} n+p & j\\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(m+n+\tfrac{1}{2},i+j+\tfrac{1}{2}\right)=0 \tag{3-82}$$

for $(m,i)\neq(-r-1,-s-1),(-p-r-1,-s-1)$, and

$$\begin{vmatrix} n+r+p+1 & j+s+1\\ m+\frac{p}{2}+\frac{1}{2} & i+\frac{1}{2}\end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\left(n+\tfrac{1}{2},j+\tfrac{1}{2}\right)=0$$

for $(m,i)\neq(-r-1,-s-1)$.

Lemma 3.24. *Let $r,s\in\mathbb{Z}$ and $\varphi\in\Delta^{\bar{1}}(\mathcal{S}(p,0))$. Then*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}}=0.$$

Proof. Taking $n \notin \{-p, 0\}$, $j = 0$ and $m \notin \{-p-r-1, -r-1\}$ in (3-82), we have

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0.$$

Since any $k \in \mathbb{Z}$ can be written as $m+n$ with $m \notin \{-p-r-1, -r-1\}$ and $n \notin \{-p, 0\}$ (by choosing $n \notin \{-p, 0, k+p+r+1, k+r+1\}$), we obtain

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(k+\tfrac{1}{2}, i+\tfrac{1}{2}\big) = 0 \quad \text{for all } (k,i) \in \mathbb{Z}\times\mathbb{Z}.$$

Choosing $n = j = 0$ in (3-81), we obtain

$$p\big(i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) = \big(i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)-(m+p)\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0). \tag{3-83}$$

Substituting $j = -i \neq 0$ and $m = n = 0$ into (3-81), it leads to

$$\begin{aligned}4pid^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0) = \big(p\big(i+s+\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)\\ + \big(p\big(i-s-\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-i).\end{aligned}$$

Given (3-83), then we get

$$\begin{aligned}&4pip\big(i-s-\tfrac{1}{2}\big)\big(-i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0)\\ &= \Big(\big(p\big(i+s+\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)\big(i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)-p\big(s+\tfrac{1}{2}\big)\big)\big(-i-s-\tfrac{1}{2}\big)\\ &\quad+\big(p\big(i-s-\tfrac{1}{2}\big)+i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)\big)\big(-i\big(r+\tfrac{p}{2}+\tfrac{1}{2}\big)-p\big(s+\tfrac{1}{2}\big)\big)\big(i-s-\tfrac{1}{2}\big)\Big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0).\end{aligned}$$

Thus, we have

$$\big(r+\tfrac{5p}{2}+\tfrac{1}{2}\big)\big(r-\tfrac{p}{2}+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0) = 0.$$

We will discuss whether r is an element of $\big\{-\frac{5p}{2}-\frac{1}{2}, \frac{p}{2}-\frac{1}{2}\big\}$.

Case 1: $r \neq \{-\frac{5p}{2}-\frac{1}{2}, \frac{p}{2}-\frac{1}{2}\}$. It follows that

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0) = 0 \quad \text{for } r \notin \big\{-\tfrac{5p}{2}-\tfrac{1}{2}, \tfrac{p}{2}-\tfrac{1}{2}\big\}.$$

From (3-83), we can get

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) = 0 \quad \text{for } r \notin \big\{-\tfrac{5p}{2}-\tfrac{1}{2}, \tfrac{p}{2}-\tfrac{1}{2}\big\}.$$

Case 2: $r = -\frac{5p}{2}-\frac{1}{2}$. Then (3-81) becomes

$$\begin{aligned}2(i(n+p)-j(m+p))d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m+n,i+j)\\ = \big((n+p)\big(i+s+\tfrac{1}{2}\big)-j(m-2p)\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m,i)\\ + \big(i(n-2p)-(m+p)\big(j+s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(n,j).\end{aligned} \tag{3-84}$$

And (3-83) becomes

$$p\big(i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m,i)=-\big(2pi+(m+p)\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,0). \tag{3-85}$$

Taking $m=n=0$ and $i,\ j,\ i\pm j\neq 0$ in (3-84), we have

$$2(i-j)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,i+j)=\big(i+2j+s+\tfrac{1}{2}\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,i)-\big(2i+j+s+\tfrac{1}{2}\big)d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,j).$$

Because of (3-85), we can obtain

$$d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(0,0)=0.$$

Then we use (3-85), it follows that

$$d^{\bar{0}}_{-\frac{5p}{2},s+\frac{1}{2}}(m,i)=0 \quad \text{for all } (m,i)\in\mathbb{Z}\times\mathbb{Z}.$$

Case 3: $r=\frac{p}{2}-\frac{1}{2}$. In this case, (3-81) becomes

$$\begin{aligned}2(i(n+p)-j(m+p))d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m+n,i+j)&=\big((n+p)\big(i+s+\tfrac{1}{2}\big)-j(m+p)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m,i)\\&\quad+\big(i(n+p)-(m+p)\big(j+s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,j).\end{aligned} \tag{3-86}$$

And (3-83) can be expressed as

$$p\big(i-s-\tfrac{1}{2}\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m,i)=\big(pi-(m+p)\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0). \tag{3-87}$$

Choosing $m=0$ in (3-87) and utilizing $i-s-\frac{1}{2}\neq 0$, we can see

$$d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,i)=d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0) \quad \text{for all } i\in\mathbb{Z}. \tag{3-88}$$

Substituting $m=j=0$ in (3-86), we have

$$\begin{aligned}2i(n+p)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i)&=(n+p)\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,i)\\&\quad+\big(i(n+p)-p\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,0).\end{aligned} \tag{3-89}$$

Furthermore, taking $i=0$, it follows that

$$pd^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,0)=(n+p)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0). \tag{3-90}$$

Multiplying (3-89) by p and using (3-88) and (3-90), we get

$$2pi(n+p)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i) = p(n+p)\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0) + (n+p)\big(i(n+p)-p\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0).$$

Additionally, we arrive at

$$2pid^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i) = i(2p+n)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0) \quad \text{for } n \neq -p.$$

And then, we can deduce that

$$2pd^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(n,i) = (2p+n)d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0) \quad \text{for } n \neq -p \text{ and } i \neq 0. \tag{3-91}$$

Taking $n \notin \{-p, 0\}$ and $i \neq 0$ and combining (3-87) and (3-91), we have

$$d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(0,0) = 0.$$

Due to (3-87), we obtain

$$d^{\bar{0}}_{\frac{p}{2},s+\frac{1}{2}}(m,i) = 0$$

for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$. □

By Lemma 3.24, we directly obtain the following proposition.

Proposition 3.25. *Let $p \in \mathbb{C}^*$ and $q = 0$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,0)) = 0.$$

3.2.3. *The case $p = 0$, $q \neq 0$.* By (3-62)–(3-63) and (3-64)–(3-65), respectively, we have

$$\begin{vmatrix} n & j+q \\ m+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) + 2qd^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,j) = 0, \tag{3-92}$$

$$2qd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m+n+1, i+j+1) + \begin{vmatrix} n+r+1 & j+s+q+1 \\ m+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2}, j+\tfrac{1}{2}\big) = 0$$

for all $(m,i) \neq (-r-1, -s-1)$.

Lemma 3.26. *Let $r, s \in \mathbb{Z}$ and $\varphi \in \Delta^{\bar{1}}(\mathcal{S}(0,q))$. Then*

$$\varphi_{r+\frac{1}{2},s+\frac{1}{2}} = 0.$$

Proof. Applying the proof of Lemma 3.24 concerning $d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+\frac{1}{2}, i+\frac{1}{2}\big)$, we can conclude that

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(m,i) = 0 \quad \text{for all } (m,i) \in \mathbb{Z} \times \mathbb{Z}.$$

Then (3-92) becomes

$$\begin{vmatrix} n & j+q \\ m+\frac{1}{2} & i+\frac{q}{2}+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(m+n+\tfrac{1}{2}, i+j+\tfrac{1}{2}\big) = 0.$$

Taking $n=0$, $j\neq -q$ and $i\neq -s-1$, we have

$$d^{\bar{1}}_{r+\frac12,s+\frac12}\big(m+\tfrac12,\, i+j+\tfrac12\big)=0.$$

Since any $k\in\mathbb{Z}$ can be written as $i+j$ with $i\neq -s-1$ and $j\neq -q$, by choosing $j\notin\{-q,\, k+s+1\}$, we obtain

$$d^{\bar{1}}_{r+\frac12,s+\frac12}\big(m+\tfrac12,\, k+\tfrac12\big)=0$$

for all $(m,k)\in\mathbb{Z}\times\mathbb{Z}$. □

A direct result is the subsequent proposition.

Proposition 3.27. *Let $p=0$ and $q\in\mathbb{C}^*$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,0))=0.$$

3.2.4. *The case $p=q=0$.* In this case, (3-56) becomes

$$2\begin{vmatrix} n & j\\ m & i\end{vmatrix} d^{\bar{0}}_{r+\frac12,s+\frac12}(m+n,\, i+j)=\begin{vmatrix} n & j\\ m+r+\frac12 & i+s+\frac12\end{vmatrix} d^{\bar{0}}_{r+\frac12,s+\frac12}(m,\, i) + \begin{vmatrix} n+r+\frac12 & j+s+\frac12\\ m & i\end{vmatrix} d^{\bar{0}}_{r+\frac12,s+\frac12}(n,\, j). \tag{3-93}$$

On the other hand, by using (3-62)–(3-63) and (3-70)–(3-71), respectively, we have

$$\begin{vmatrix} n & j\\ m+\frac12 & i+\frac12\end{vmatrix} d^{\bar{1}}_{r+\frac12,s+\frac12}\big(m+n+\tfrac12,\, i+j+\tfrac12\big)=0,\quad (m,i)\neq(-r-1,-s-1), \tag{3-94}$$

$$\begin{vmatrix} n+r+1 & j+s+1\\ m+\frac12 & i+\frac12\end{vmatrix} d^{\bar{1}}_{r+\frac12,s+\frac12}\big(n+\tfrac12,\, j+\tfrac12\big)=0,\quad (m,i)\neq(-r-1,-s-1). \tag{3-95}$$

Lemma 3.28. *Let $r,s\in\mathbb{Z}$ and $\varphi\in\Delta^{\bar{1}}(\mathcal{S}(0,0))$. Then*

$$\varphi_{r+\frac12,s+\frac12}=0.$$

Proof. If $j\neq -s-1$, then we take $m\notin\Big\{-r-1,\, \frac{(n+r+1)(i+\frac12)}{j+s+1}-\frac12\Big\}$ in (3-95). It follows that

$$d^{\bar{1}}_{r+\frac12,s+\frac12}\big(n+\tfrac12,\, j+\tfrac12\big)=0\quad \text{for all } n\in\mathbb{Z} \text{ and } j\neq -s-1.$$

If $n\neq -r-1$, then we choose $i\notin\Big\{-s-1,\, \frac{(j+s+1)(m+\frac12)}{n+r+1}-\frac12\Big\}$ in (3-95). We can arrive at

$$d^{\bar{1}}_{r+\frac12,s+\frac12}\big(n+\tfrac12,\, j+\tfrac12\big)=0\quad \text{for all } j\in\mathbb{Z} \text{ and } n\neq -r-1.$$

Hence, we can conclude

$$d^{\bar{1}}_{r+\frac12,s+\frac12}\big(n+\tfrac12,\, j+\tfrac12\big)=0\quad \text{for } (n,j)\neq(-r-1,-s-1).$$

Substituting $m=-n-r-1$, $j=-i-s-1$ and $i\neq -s-1$ into (3-94), it leads to

$$\begin{vmatrix} -r-\frac{1}{2} & -s-\frac{1}{2} \\ -n-r-\frac{1}{2} & i+\frac{1}{2} \end{vmatrix} d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(-r-\tfrac{1}{2},-s-\tfrac{1}{2}\big)=0.$$

Since $r\in\mathbb{Z}$ and $r+\frac{1}{2}\neq 0$, then we take $i\notin\big\{-\frac{n(s+\frac{1}{2})}{r+\frac{1}{2}}-s-1,\,-s-1\big\}$. It follows that

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(-r-\tfrac{1}{2},-s-\tfrac{1}{2}\big)=0.$$

Therefore, we arrive at

$$d^{\bar{1}}_{r+\frac{1}{2},s+\frac{1}{2}}\big(n+\tfrac{1}{2},\,j+\tfrac{1}{2}\big)=0 \quad \text{for all } (n,j),(r,s)\in\mathbb{Z}\times\mathbb{Z}.$$

Choosing $n=j=0$ in (3-93), we have

$$\big(i\big(r+\tfrac{1}{2}\big)-m\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0)=0.$$

Since $s+\frac{1}{2}\neq 0$, then taking $m\neq\frac{i(r+\frac{1}{2})}{s+\frac{1}{2}}$, we have

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,0)=0.$$

Substituting $m=j=0$ into (3-93), we obtain

$$2ni\,d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,i)=n\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)+i\big(n+r+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0). \tag{3-96}$$

Putting $m=-n$ and $j=0$ into (3-93), we get

$$\begin{aligned}2ni\,d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)&=n\big(i+s+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-n,i)\\&\quad+\big(i\big(n+r+\tfrac{1}{2}\big)+n\big(s+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0).\end{aligned} \tag{3-97}$$

Taking $i=0$ in (3-97), we can conclude

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-n,0)=-d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0) \quad \text{for all } n\in\mathbb{Z}. \tag{3-98}$$

Substituting $j=-i$ and $m=0$ into (3-93), it follows that

$$\begin{aligned}2ni\,d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)&=\big(n\big(i+s+\tfrac{1}{2}\big)+i\big(r+\tfrac{1}{2}\big)\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)\\&\quad+i\big(n+r+\tfrac{1}{2}\big)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-i).\end{aligned} \tag{3-99}$$

Letting $n=0$ in (3-99), we have

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-i)=-d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i) \quad \text{for all } i\in\mathbb{Z}. \tag{3-100}$$

Choosing $n=-(2r+1)$ in (3-96), we have

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) = (2i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)+id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0). \tag{3-101}$$

Then we let $n=2r+1$ in (3-97) together with (3-98). It follows that

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i) = (2i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) + (3i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(2r+1,0),$$

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i) = (2i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) - (3i+2s+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0). \tag{3-102}$$

Replacing i by $-i$ and taking $n=-2r-1$ in (3-99) together with (3-100), it gives

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0) = -(2s-i+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-i)+id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i),$$

$$4id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0) = (2s-i+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)+id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i). \tag{3-103}$$

Combining (3-101) and (3-103), we can deduce

$$15id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,0) = (10s-2i+5)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i), \tag{3-104}$$

$$15id^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(-2r-1,i) = (10s+7i+5)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i). \tag{3-105}$$

If $i\neq 0$, taking (3-104) and (3-105) in (3-102), we have

$$(10i-2s-1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)=0.$$

Since $s\in\mathbb{Z}$, then we obtain $\frac{1}{5}\big(s+\frac{1}{2}\big)\notin\mathbb{Z}$ and $i\neq\frac{1}{5}\big(s+\frac{1}{2}\big)$, i.e., $10i-2s-1\neq 0$. Hence, we arrive at

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,i)=0 \quad \text{for all } i\in\mathbb{Z}.$$

We take $i=-2s-1$ in (3-96). It follows that

$$4nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-2s-1) = nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-2s-1)+(2n+2r+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0). \tag{3-106}$$

Replacing n by $-n$ and taking $i=-2s-1$ in (3-97) together with (3-98), it gives

$$(3\text{-}107)\quad 4nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-2s-1) = nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-2s-1)+(2r-n+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0).$$

Letting $i=2s+1$ in (3-99), we obtain

$$(3\text{-}108)\quad 4nd^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)=(2n+2r+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,-2s-1) - (3n+2r+1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(0,-2s-1).$$

Combining (3-106)–(3-108), we have

$$(10n-2r-1)d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)=0.$$

As the similar reason above, we can get

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,0)=0 \quad \text{for all } n\in\mathbb{Z}.$$

In this end, by (3-96), we obtain

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,i)=0 \quad \text{for } n,i\neq 0.$$

Therefore, we can conclude

$$d^{\bar{0}}_{r+\frac{1}{2},s+\frac{1}{2}}(n,i)=0$$

for $(n,j)\in\mathbb{Z}\times\mathbb{Z}$. □

It leads directly to the following proposition.

Proposition 3.29. *Let $p=q=0$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(0,0))=0.$$

By integrating Propositions 3.23, 3.25, 3.27 and 3.29, we can deduce the following corollary.

Corollary 3.30. *Let $p,q\in\mathbb{C}$. Then*

$$\Delta^{\bar{1}}(\mathcal{S}(p,q))=\begin{cases}\langle\gamma\rangle, & p,q\in 2\mathbb{Z}+1,\\ \{0\}, & \textit{otherwise},\end{cases}$$

where γ is as in Lemma 3.22.

4. Transposed Poisson superalgebra structures on $\mathcal{S}(p, q)$

Theorem 4.1. *Let p and q be fixed complex numbers.*

(1) *If $p \notin \mathbb{Z}$ or $q \neq 0$, then all the transposed Poisson superalgebra structures on $(\mathcal{S}(p, q), [\cdot, \cdot])$ are trivial.*

(2) *If $p \in \mathbb{Z}$ and $q = 0$, then the nontrivial transposed Poisson superalgebra structure $(\mathcal{S}(p, 0), \cdot, [\cdot, \cdot])$ on $(\mathcal{S}(p, 0), [\cdot, \cdot])$ is, up to an isomorphism,*

$$L_{-2p,0} \cdot L_{-2p,0} = L_{-p,0}. \tag{4-1}$$

Proof. Let $(\mathcal{S}(p, q), \cdot, [\cdot, \cdot])$ be a transposed Poisson superalgebra, i.e., $(\mathcal{S}(p, q), \cdot)$ is supercommutative and (2-2) holds. Given $(m, i) \in \mathbb{Z} \times \mathbb{Z}$, we denote by $\varphi^{m,i}$ and $\psi^{m+\frac{1}{2},i+\frac{1}{2}}$ the left multiplication by $L_{m,i}$ and $G_{m+\frac{1}{2},i+\frac{1}{2}}$, respectively, in $(\mathcal{S}(p, q), \cdot)$, that is,

$$\begin{aligned} L_{m,i} \cdot L_{n,j} &= \varphi^{m,i}(L_{n,j}), \qquad (4\text{-}2)\\ L_{m,i} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= \varphi^{m,i}(G_{n+\frac{1}{2},j+\frac{1}{2}}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot L_{n,j} &= \psi^{m+\frac{1}{2},i+\frac{1}{2}}(L_{n,j}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= \psi^{m+\frac{1}{2},i+\frac{1}{2}}(G_{n+\frac{1}{2},j+\frac{1}{2}}). \qquad (4\text{-}3)\end{aligned}$$

In view of supercommutativity of $(\mathcal{S}(p, q), \cdot)$, we have

$$\begin{aligned} L_{m,i} \cdot L_{n,j} &= \varphi^{n,j}(L_{m,i}), \qquad (4\text{-}4)\\ L_{m,i} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= \psi^{n+\frac{1}{2},j+\frac{1}{2}}(L_{m,i}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot L_{n,j} &= \varphi^{n,j}(G_{m+\frac{1}{2},i+\frac{1}{2}}),\\ G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} &= -\psi^{n+\frac{1}{2},j+\frac{1}{2}}(G_{m+\frac{1}{2},i+\frac{1}{2}}). \qquad (4\text{-}5)\end{aligned}$$

By using (2-2), we have $\varphi^{m,i} \in \Delta^{\bar{0}}(\mathcal{S}(p, q))$ and $\psi^{m+\frac{1}{2},i+\frac{1}{2}} \in \Delta^{\bar{1}}(\mathcal{S}(p, q))$. Furthermore, based on the Corollaries 3.19 and 3.30, we proceed to the following discussion.

Case 1: $p, q \neq 0$ and $p, q \notin 2\mathbb{Z}+1$. It is clear that $\varphi^{m,i} = a^{m,i} id$ for some $a^{m,i} \in \mathbb{C}$ by Proposition 3.4 and $\psi^{m+\frac{1}{2},i+\frac{1}{2}} = 0$ by Proposition 3.23 . It follows from (4-2) and (4-4) that $a^{m,i} = 0$ for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. So, $\cdot$ is trivial whenever $p, q \neq 0$ and $p, q \notin 2\mathbb{Z} + 1$.

Case 2: $p, q \neq 0$ and $p, q \in 2\mathbb{Z} + 1$. We have $\varphi^{m,i} = a^{m,i} id$ for some $a^{m,i} \in \mathbb{C}$ by Proposition 3.4. Equations (4-2) and (4-4) imply $a^{m,i} = 0$, whence $\varphi^{m,i} = 0$. We arrive at $L_{m,i} \cdot L_{n,j} = L_{m,i} \cdot G_{n,j} = 0$. From Proposition 3.23, it is easy to observe that

$$\psi^{m+\frac{1}{2},i+\frac{1}{2}} = b^{m+\frac{1}{2},i+\frac{1}{2}}\gamma,$$

with $b^{m+\frac{1}{2},i+\frac{1}{2}} \in \mathbb{C}$. On the one hand, it follows from (3-80) and (4-3) that

$$G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} = b^{m+\frac{1}{2},i+\frac{1}{2}}\gamma(G_{n+\frac{1}{2},j+\frac{1}{2}}) = \begin{cases} 0, & (n,j) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right), \\ b^{m+\frac{1}{2},i+\frac{1}{2}}L_{-p,-q}, & (n,j) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right). \end{cases}$$

On the other hand, by using (3-80) and (4-5), we arrive at

$$G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} = -b^{n+\frac{1}{2},j+\frac{1}{2}}\gamma(G_{m+\frac{1}{2},i+\frac{1}{2}}) = \begin{cases} 0, & (m,i) \neq \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right), \\ -b^{n+\frac{1}{2},j+\frac{1}{2}}L_{-p,-q}, & (m,i) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right). \end{cases}$$

Thus, the product $G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}}$ is zero unless

$$(m,i) = (n,j) = \left(-\frac{3p}{2}-\frac{1}{2}, -\frac{3q}{2}-\frac{1}{2}\right).$$

But $G_{-\frac{3p}{2},-\frac{3q}{2}} \cdot G_{-\frac{3p}{2},-\frac{3q}{2}} = -G_{-\frac{3p}{2},-\frac{3q}{2}} \cdot G_{-\frac{3p}{2},-\frac{3q}{2}}$, then $G_{-\frac{3p}{2},-\frac{3q}{2}} \cdot G_{-\frac{3p}{2},-\frac{3q}{2}} = 0$. So, $\cdot$ is trivial.

Case 3: $p \notin \mathbb{Z}$ and $q = 0$. We obtain that $\varphi^{m,i} = a^{m,i}id$ for some $a^{m,i} \in \mathbb{C}$ by Proposition 3.14 and $\psi^{m+\frac{1}{2},i+\frac{1}{2}} = 0$ by Proposition 3.25 . It follows from (4-2) and (4-4) that $a^{m,i} = 0$ for all $(m,i) \in \mathbb{Z} \times \mathbb{Z}$. So, $\cdot$ is trivial.

Case 4: $p \in \mathbb{Z}^*$ and $q = 0$. We have $\psi^{m+\frac{1}{2},i+\frac{1}{2}} = 0$ by Proposition 3.25. It leads to $G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot L_{n,j} = 0$ and $G_{m+\frac{1}{2},i+\frac{1}{2}} \cdot G_{n+\frac{1}{2},j+\frac{1}{2}} = 0$.

Because of Proposition 3.14, it is clear to show that $\varphi^{m,i} = f^{m,i}id + g^{m,i}\alpha$ for $f^{m,i}, g^{m,i} \in \mathbb{C}$. We can get from (3-36) and (4-2) that

$$L_{m,i} \cdot L_{n,j} = f^{m,i}L_{n,j} + g^{m,i}\alpha(L_{n,j}) = \begin{cases} f^{m,i}L_{n,j}, & (n,j) \neq (-2p,0), \\ f^{m,i}L_{-2p,0} + g^{m,i}L_{-p,0}, & (n,j) = (-2p,0). \end{cases}$$

On the other hand, by using (3-36) and (4-4), we have

$$L_{m,i} \cdot L_{n,j} = f^{m,i}L_{n,j} + g^{m,i}\alpha(L_{n,j}) = \begin{cases} f^{n,i}L_{m,i}, & (m,i) \neq (-2p,0), \\ f^{n,j}L_{-2p,0} + g^{n,j}L_{-p,0}, & (m,i) = (-2p,0). \end{cases}$$

Thus, we can discuss it into the following cases.

Subcase 1: $(m,i), (n,j) \neq (-2p,0)$. We get $f^{m,i}L_{n,j} = f^{n,j}L_{m,i}$. So taking $(m,i) \neq (n,j)$ we conclude that $f^{m,i} = f^{n,j} = 0$. Thus, $L_{m,i} \cdot L_{n,j} = 0$.

Subcase 2: $(m,i) = (-2p,0)$, $(n,j) \neq (-2p,0)$. We have

$$f^{-2p,0}L_{n,j} = f^{n,j}L_{-2p,0} + g^{n,j}L_{-p,0} = g^{n,j}L_{-p,0},$$

because of $f^{n,j}=0$ for $(n, j)\neq(-2p, 0)$. So, taking $(n, j)\neq(-p, 0)$, we conclude that $f^{-2p,0}=0$, whence $L_{-2p,0}\cdot L_{n,j}=0$.

Subcase 3: $(m, i)\neq(-2p, 0)$, $(n, j)=(-2p, 0)$. It implies $L_{m,i}\cdot L_{-2p,0}=0$.

Subcase 4: $(m, i)=(n, j)=(-2p, 0)$. It leads to

$$L_{-2p,0}\cdot L_{-2p,0}=f^{-2p,0}L_{-2p,0}+g^{-2p,0}L_{-p,0}=g^{-2p,0}L_{-p,0},$$

because of $f^{-2p,0}=0$.

Thus, we can conclude that

$$L_{m,i}\cdot L_{n,j}=\begin{cases}0, & (m, i), (n, j)\neq(-2p, 0),\\ g^{-2p,0}L_{-p,0}, & (m, i)=(n, j)=(-2p, 0).\end{cases}$$

Therefore, the product $(\mathcal{S}(p, 0), \cdot)$ is of the form

$$L_{-2p,0}\cdot L_{-2p,0}=cL_{-p,0}, \tag{4-6}$$

where $c\in\mathbb{C}$. Assume that $c\neq 0$, otherwise the transposed Poisson superalgebra structure is trivial. We observe that $L_{-p,0}\in Z(\mathcal{S}(p, 0))$, where $Z(\mathcal{S}(p, 0))$ is the center of the Lie superalgebra $(\mathcal{S}(p, 0), [\cdot,\cdot])$. Indeed,

$$[L_{m,i}, L_{-p,0}]=((-p+p)\cdot 0-(m+p)\cdot 0)L_{m+n,i+j}=0$$

for all $(m, i)\in\mathbb{Z}\times\mathbb{Z}$. Hence the linear map ϕ such that $\phi(L_{m,i})=L_{m,i}$ for $(m, i)\neq(-p, 0)$ and $\phi(L_{-p,0})=kL_{-p,0}$ is an automorphism of $(\mathcal{S}(p, 0), [\cdot,\cdot])$ for any $k\in\mathbb{C}^*$. If $p\neq 0$, then we take $k=c^{-1}$. Furthermore, we obtain an isomorphic transposed Poisson superalgebra structure $*$ on $(\mathcal{S}(p, 0), [\cdot,\cdot])$ in which the only nonzero product is

$$\begin{aligned}L_{-2p,0}*L_{-2p,0}&=\phi(L_{-2p,0})*\phi(L_{-2p,0})=\phi(L_{-2p,0}\cdot L_{-2p,0})\\&=\phi(cL_{-2p,0})=c^{-1}\cdot cL_{-p,0}=L_{-p,0}.\end{aligned}$$

So, up to an isomorphism, we may consider $c=1$ in (4-6).

Case 5: $p=q=0$. We have $\psi^{m+\frac{1}{2},i+\frac{1}{2}}=0$ by Proposition 3.29. It implies

$$G_{m+\frac{1}{2},i+\frac{1}{2}}\cdot L_{n,j}=0 \quad\text{and}\quad G_{m+\frac{1}{2},i+\frac{1}{2}}\cdot G_{n+\frac{1}{2},j+\frac{1}{2}}=0.$$

It is easy to show that $\varphi^{m,i}=x^{m,i}id+y^{m,i}\beta$ for $x^{m,i}, y^{m,i}\in\mathbb{C}$ by Proposition 3.18. We can get from (3-52) and (4-2) that

$$L_{m,i}\cdot L_{n,j}=x^{m,i}L_{n,j}+y^{m,i}\alpha(L_{n,j})=\begin{cases}x^{m,i}L_{n,j}, & (n, j)\neq(0, 0),\\ (x^{m,i}+y^{m,i})L_{0,0}, & (n, j)=(0, 0).\end{cases}$$

On the other hand, by using (3-52) and (4-4), we obtain

$$L_{m,i}\cdot L_{n,j}=x^{m,i}L_{n,j}+y^{m,i}\alpha(L_{n,j})=\begin{cases}x^{n,i}L_{m,i}, & (m, i)\neq(0, 0),\\ (x^{n,j}+y^{n,j})L_{0,0}, & (m, i)=(0, 0).\end{cases}$$

Thus, we can discuss it into the following cases.

Subcase 1: $(m, i), (n, j) \neq (0,0)$. We get $x^{m,i} L_{n,j} = x^{n,j} L_{m,i}$. So taking $(m, i) \neq (n, j)$ we conclude that $x^{m,i} = x^{n,j} = 0$. Thus, we arrive at $L_{m,i} \cdot L_{n,j} = 0$.

Subcase 2: $(m, i) = (0,0)$, $(n, j) \neq (0,0)$. We have $x^{0,0} L_{n,j} = (x^{n,j} + y^{n,j}) L_{0,0} = y^{n,j} L_{0,0}$. So, we obtain $x^{0,0} = y^{n,j} = 0$, whence $L_{0,0} \cdot L_{n,j} = 0$.

Subcase 3: $(m, i) \neq (0,0)$, $(n, j) = (0,0)$. It implies $L_{m,i} \cdot L_{0,0} = 0$.

Subcase 4: $(m, i) = (n, j) = (0,0)$. It leads to $L_{0,0} \cdot L_{0,0} = (x^{0,0} + y^{0,0}) L_{0,0} = y^{0,0} L_{0,0}$, because of $x^{0,0} = 0$.

Thus, we have

$$L_{m,i} \cdot L_{n,j} = \begin{cases} 0, & (m,i), (n,j) \neq (0,0), \\ y^{0,0} L_{0,0}, & (m,i), (n,j) = (0,0). \end{cases}$$

Therefore, the product $(\mathcal{S}(0,0), \cdot)$ is of the form

$$L_{0,0} \cdot L_{0,0} = c L_{0,0}, \tag{4-7}$$

where $c \in \mathbb{C}$. Assume that $c \neq 0$, otherwise the transposed Poisson superalgebra structure is trivial. Observe that $L_{0,0} \in Z(\mathcal{S}(0,0))$, where $Z(\mathcal{S}(0,0))$ is the center of the Lie superalgebra $(\mathcal{S}(0,0), [\cdot,\cdot])$. Indeed,

$$[L_{m,i}, L_{0,0}] = 0$$

for all $(m, i) \in \mathbb{Z} \times \mathbb{Z}$. Hence the linear map ϕ such that $\phi(L_{m,i}) = L_{m,i}$ for $(m, i) \neq (0,0)$ and $\phi(L_{0,0}) = k L_{-p,0}$ is an automorphism of $(\mathcal{S}(0,0), [\cdot,\cdot])$ for any $k \in \mathbb{C}^*$. Then taking $k = c$, we obtain an isomorphic transposed Poisson superalgebra structure $*$ on $(\mathcal{S}(0,0), [\cdot,\cdot])$ in which the only nonzero product is

$$\begin{aligned} L_{0,0} * L_{0,0} &= c^{-1}\phi(L_{0,0}) * c^{-1}\phi(L_{0,0}) = c^{-2}\phi(L_{0,0} \cdot L_{0,0}) \\ &= c^{-2}\phi(c L_{0,0}) = c^{-2} \cdot c^2 L_{0,0} = L_{0,0}. \end{aligned}$$

So, up to an isomorphism, we may consider $c = 1$ in (4-7).

Conversely, each of two associative and supercommutative multiplication (4-1) defines a transposed Poisson superalgebra structure on $\mathcal{S}(p, 0)$, $p \in \mathbb{Z}$. If $p \in \mathbb{Z}^*$, we can observe that $\mathcal{S}(p,0) \cdot \mathcal{S}(p,0) \subseteq \langle L_{-p,0} \rangle \subseteq \mathbb{Z}(\mathcal{S}(p,0))$. Hence, the right-hand side of (2-2) is always zero. In fact, the left-hand side of (2-2) is zero as well, because of $[\mathcal{S}(p,0), \mathcal{S}(p,0)] \subseteq \mathrm{Ann}(\mathcal{S}(p,0))$, where $\mathrm{Ann}(\mathcal{S}(p,0))$ is the annihilator of $(\mathcal{S}(p,0), \cdot)$. Assuming $[L_{m,i}, L_{n,j}] \in \langle L_{-2p,0} \rangle$, we obtain from (2-1) that $m + n = -2p$ and $i + j = 0$. But it leads to

$$-j(n+p) - j(m+p) = -j(n+p) - j(-2p - n + p) = 0,$$

so we have $[L_{m,i}, L_{n,j}] = 0$. Thus, we have $[L_{m,i}, L_{n,j}] \subseteq \mathrm{Ann}(\mathcal{S}(p,0))$ for all $(m, i), (n, j) \in \mathbb{Z} \times \mathbb{Z}$, as needed. If $p = 0$, since $L_{0,0} \in \mathbb{Z}(\mathcal{S}(0,0))$, then it leads to

$\mathcal{S}(0,0)\cdot\mathcal{S}(0,0)\subseteq\mathbb{Z}(\mathcal{S}(0,0))$. Hence the right-hand side of (2-2) is zero. Assuming $[L_{m,i}, L_{n,j}]\in\langle L_{0,0}\rangle$, we obtain from (2-1) that $m=-n$ and $j=-i$. But then we get

$$ni-(-n)(-i)=ni-ni=0.$$

So $[L_{m,i}, L_{n,j}]=0$. We get $[L_{m,i}, L_{n,j}]\subseteq \mathrm{Ann}(\mathcal{S}(0,0))$ for all $(m,i),(n,j)\in\mathbb{Z}\times\mathbb{Z}$. Then the left-hand side of (2-2) is zero as needed. □

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THE CR YAMABE CONSTANT AND INEQUIVALENT CR STRUCTURES

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The CR Yamabe constant is an invariant of a compact strongly pseudoconvex CR manifold and plays an important role in CR geometry. We show some integral formulas of the CR Yamabe constant. We also construct an infinite-dimensional family of strongly pseudoconvex CR structures with varying CR Yamabe constants and a compact simply connected manifold admitting two strongly pseudoconvex CR structures with different signs of the CR Yamabe constant.

1. Introduction

The Yamabe problem, which is one of the most important problems in conformal geometry, asks whether there exists a Riemannian metric in a given conformal class minimizing the Yamabe functional. The infimum of this functional defines a conformal invariant called the Yamabe constant. The Yamabe constant is a fundamental invariant in conformal geometry, and there is intensive research on this invariant. It is known that every compact manifold of dimension greater than 2 has a continuous family of conformal structures with all different Yamabe constants. Moreover, every compact manifold of dimension greater than 2 admits a conformal structure with negative Yamabe constant. Furthermore, there exist some integral formulas of this invariant, which have been useful for computing not only the Yamabe constants but also other curvature parts of 3- and 4-manifolds [LeBrun 1999; Sung 2012; 2021].

Jerison and Lee [1987] have considered a CR analog of the Yamabe problem, known as the *CR Yamabe problem*. The CR Yamabe problem asks whether on any compact strongly pseudoconvex CR manifold (X, H, J) of dimension $2n+1$, there exists a contact form θ minimizing the functional

$$\mathfrak{F}(\theta) := \frac{\int_X R_\theta \, d\mu_\theta}{\left(\int_X d\mu_\theta\right)^{n/(n+1)}},$$

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where R_θ is the pseudohermitian scalar curvature for θ and $d\mu_\theta = \theta \wedge (d\theta)^n$. The infimum $\lambda(X, H, J)$ of the above functional is an invariant of the CR manifold (X, H, J) and called the *CR Yamabe constant* of (X, H, J); we will simply write $\lambda(X)$ when the CR structure on X is clear from the context. Just as in the Yamabe problem, one has

$$\lambda(X) \leq \lambda(S^{2n+1}) = 2n(n+1)\pi$$

for the standard CR structure on S^{2n+1}, and the CR Yamabe problem is solvable when $\lambda(X) < \lambda(S^{2n+1})$ [Jerison and Lee 1987]. There is intensive research on conditions when $\lambda(X) < \lambda(S^{2n+1})$ holds for X not CR equivalent to S^{2n+1}; see [Jerison and Lee 1988; 1989; Cheng et al. 2014; 2017; 2023; Takeuchi 2020] for related results. Every minimizer θ has constant R_θ, and when $\lambda(X) \leq 0$, any θ with constant R_θ is a CR Yamabe minimizer, which is unique up to a constant. Moreover, an Einstein contact form is also a CR Yamabe minimizer; see Lemma 2.1. In a similar way to the Yamabe constant, the CR Yamabe constant can be written as various integral formulas; see Theorem 3.1. This may be useful for estimating norms of parts of pseudohermitian curvature tensor as in the Riemannian case [Sung 2021].

It is natural to ask whether a manifold admitting one CR structure has abundant other CR structures with all different CR Yamabe constants or CR structures with negative CR Yamabe constant. In comparison to the Riemannian case, the difficulty lies in imposing the integrability condition to an almost CR structure, which obstructs generic deformations of almost CR structures. In this paper, we will construct an infinite-dimensional family of strongly pseudoconvex CR structures with varying CR Yamabe constants. To this end, we consider an n-dimensional compact Hodge manifold (M, J, ω) with constant scalar curvature. Denote the space of Kähler potentials in the class $[\omega]$ by

$$\mathcal{K} := \{\varphi \in C^\infty(M) \mid \omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0\}$$

endowed with the C^4-topology, and write $\mathcal{F}$ for the subset of $\varphi \in \mathcal{K}$ such that ω_φ has constant scalar curvature. Let $p : P_M \to M$ be a principal S^1-bundle over M whose Euler class is $-[\omega]$. For any $\varphi \in \mathcal{K}$, there exists a principal connection θ_φ on P_M such that $d\theta_\varphi/2\pi = p^*\omega_\varphi$. The complex structure J on M induces a strongly pseudoconvex CR structure p^*J on $H_\varphi := \operatorname{Ker}\theta_\varphi$; see Proposition 2.2. This gives an infinite-dimensional family of pseudohermitian manifolds $(P_M, H_\varphi, p^*J, \theta_\varphi/2\pi)$ underlying the same manifold P_M.

Theorem 1.1. *Let (M, J, ω) be an n-dimensional compact Hodge manifold. Then the map*

$$\mathcal{K} \to \mathbb{R}; \quad \varphi \mapsto \lambda(P_M, H_\varphi, p^*J)$$

is continuous. Moreover if $\theta_0/2\pi$ is a CR Yamabe minimizer, then

$$\lambda(P_M, H_\varphi, p^*J) < \lambda(P_M, H_0, p^*J) = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{([\omega]^n)^{n/(n+1)}}$$

for any $\varphi \in \mathcal{K} \setminus \mathcal{F}$. The assumption holds if ω has nonpositive constant scalar curvature or it defines a Kähler–Einstein metric.

We will also show the existence of a compact simply connected manifold admitting two strongly pseudoconvex CR structures with different signs of the CR Yamabe constant.

Theorem 1.2. *For each $n \geq 3$, there exists a compact simply connected $(2n+1)$-manifold X admitting two strongly pseudoconvex CR structures (H, J) and $(\widetilde{H}, \tilde{J})$ such that they have different signs of the CR Yamabe constants, and (M, H) and $(M, \widetilde{H})$ are not isomorphic as cooriented contact manifolds.*

We remark that the existence of CR structures with different signs of the CR Yamabe constant on a fixed contact structure remains unsolved.

This paper is organized as follows. In Section 2, we recall basic facts on CR manifolds and show that a principal S^1-bundle over a Hodge manifold has a canonical strongly pseudoconvex CR structure. Some integral formulas of the CR Yamabe constant are given in Section 3. In Section 4, we prove the continuity of the CR Yamabe constant under suitable deformations of CR structures, which will be used for the proof of Theorem 1.1. Section 5 is devoted to constructions of deformations of strongly pseudoconvex CR structures with varying CR Yamabe constants. In Section 6, we give a proof of Theorem 1.2.

2. CR manifolds

An almost CR structure on a smooth $(2n+1)$-manifold X is a pair (H, J) where $H \subset TX$ is a codimension 1 smooth subbundle with an almost complex structure J. An almost CR structure is called integrable or a CR structure if

$$[\Gamma(T^{1,0}X), \Gamma(T^{1,0}X)] \subset \Gamma(T^{1,0}X)$$

for

$$T^{1,0}X := \{v - iJv \mid v \in H\} \subset H \otimes \mathbb{C}.$$

We shall consider only an orientable CR manifold. Then one can choose a smooth real-valued 1-form θ annihilating exactly H, which is determined up to multiplication by a nowhere vanishing real-valued function on X. By the integrability condition, $d\theta$ is J-invariant; i.e., a (1, 1)-form, and hence one can introduce the symmetric bilinear form

$$L_\theta := d\theta(\cdot, J\cdot)$$

defined on H, called the *Levi form*.

A CR manifold (X, H, J) is called *strongly pseudoconvex* if L_θ is definite for some (and hence all) θ, and a strongly pseudoconvex CR manifold (X, H, J) with a choice of θ is called a *pseudohermitian* manifold. We shall always assume that X is strongly pseudoconvex and θ is chosen so that L_θ is positive definite, unless otherwise specified. In this case, the distribution H is a contact structure on X with a contact form θ. Let T be its Reeb vector field; i.e., the unique vector field satisfying $\theta(T) = 1$ and $\iota(T)\, d\theta = 0$. The Levi form induces a Hermitian metric L_θ^* on H^*. The *sublaplacian* Δ_b is defined by

$$\int_X (\Delta_b u) v \, d\mu_\theta = \int_X L_\theta^*(du|_H, dv|_H)\, d\mu_\theta.$$

A set of local 1-forms $\{\theta^1, \dots, \theta^n\}$ of type $(1, 0)$ is called *admissible*, if its restriction to $T^{1,0}X$ forms a basis of $(T^{1,0}X)^*$ at each point and $\theta^\alpha(T) = 0$ for all α. For an admissible coframe, we have

$$d\theta = i h_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta}, \tag{2-1}$$

where $(h_{\alpha\bar\beta})$ is a positive-definite hermitian matrix of functions and $\theta^{\bar\beta} = \overline{\theta^\beta}$. We shall always adopt the Einstein convention and use the matrix $(h_{\alpha\bar\beta})$ and its inverse $(h^{\alpha\bar\beta})$ to raise and lower indices. The integrability condition of J can be rephrased as

$$d\theta^\alpha \equiv 0 \mod \theta, \theta^\gamma$$

along with (2-1).

A pseudohermitian manifold carries a canonical linear connection, the *Tanaka–Webster connection* [Tanaka 1975; Webster 1978], whose connection 1-forms ${\omega_\alpha}^\beta$ and torsion forms τ^α of type (0, 1) are uniquely determined by the relations

$$d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta \quad \text{and} \quad \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha} = dh_{\alpha\bar\beta}$$

together with (2-1). We call τ^α the *pseudohermitian torsion*. The whole torsion tensor is composed of $\theta \wedge \tau^\gamma$ and $i h_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta}$, and so it is nowhere vanishing.

The covariant differentiation with respect to this connection is given by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = {\omega_{\bar\alpha}}^{\bar\beta} \otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where a local frame $\{Z_\alpha\}$ of $T^{1,0}X$ is dual to $\{\theta^\alpha\}$. Its curvature 2-forms

$${\Omega_\alpha}^\beta := d{\omega_\alpha}^\beta - {\omega_\alpha}^\gamma \wedge {\omega_\gamma}^\beta$$

may have several types, but one considers only its (1, 1)-part:

$${R_\alpha}^\beta{}_{\rho\bar\sigma}\, \theta^\rho \wedge \theta^{\bar\sigma}$$

to get its *pseudohermitian Ricci tensor*

$$R_{\rho\bar\sigma} := {R_\alpha}^\alpha{}_{\rho\bar\sigma}$$

by taking its contraction. Finally its *pseudohermitian scalar curvature* is the metric contraction $R_\rho{}^\rho = h^{\rho\bar\sigma} R_{\rho\bar\sigma}$. We say θ to be *Einstein* if its pseudohermitian torsion is identically zero and its pseudohermitian Ricci curvature is a constant multiple of the Levi form.

Lemma 2.1. *Any Einstein contact form is a CR Yamabe minimizer.*

Proof. Let (X, H, J) be a compact strongly pseudoconvex CR manifold of dimension $2n+1$ and θ be an Einstein contact form on X. If the pseudohermitian scalar curvature is nonpositive, then it must be a CR Yamabe minimizer. If the pseudohermitian scalar curvature is positive, we may assume that it is equal to $n(n+1)$ by homothety. Consider the Riemannian metric g_θ on X given by

$$g_\theta(U, V) := \tfrac{1}{2} d\theta(U, JV) + \theta(U)\,\theta(V), \quad U, V \in TX.$$

Here we extend J to an endomorphism on TM by $JT = 0$. Note that the volume form of g_θ coincides with $(2^n n!)^{-1}\, d\mu_\theta$. This g_θ satisfies $\mathrm{Ric}_{g_\theta} = 2n g_\theta$; see [Takeuchi 2018, Proposition 2.9] for example. The Bishop inequality implies that

$$\begin{aligned}\mathfrak{F}(\theta) &= n(n+1)(2^n n!\,\mathrm{Vol}_{g_\theta}(X))^{\frac{1}{n+1}} \\ &\le n(n+1)(2^n n!\,\mathrm{Vol}_{g_0}(S^{2n+1}))^{\frac{1}{n+1}} = 2n(n+1)\pi = \lambda(S^{2n+1}),\end{aligned}$$

where g_0 is the standard Riemannian metric on S^{2n+1}. Moreover, the equality holds if and only if (X, g_θ) is isometric to (S^{2n+1}, g_0). In this case, (X, H, J) is CR isomorphic to the standard CR sphere and θ is a CR Yamabe minimizer; see the paragraph after the proof of Proposition 4 in [Wang 2015]. If $\mathfrak{F}(\theta) < \lambda(S^{2n+1})$, then (X, H, J) has a CR Yamabe minimizer by [Jerison and Lee 1987, Theorem 3.4(c)], and θ is also a CR Yamabe minimizer by [Wang 2015, Theorem 3]. □

An important example of a strongly pseudoconvex CR manifold is a principal S^1-bundle over a Hodge manifold. Given a Hodge manifold (M, J, ω); that is, its Kähler class $[\omega]$ is an integral cohomology class, we consider a principal S^1-bundle $p : P_M \to M$ whose Euler class is $-[\omega]$. Recall that for any $\mathbb{R}$-valued principal connection θ on P_M, $d\theta/2\pi$ descends to M and its cohomology class coincides with $[\omega]$. We take a principal connection θ satisfying $d\theta/2\pi = p^*\omega$, and consider the lifted almost complex structure

$$p^*J : H := \mathrm{Ker}\,\theta \to H.$$

Proposition 2.2. *The triple (P_M, H, p^*J) is a strongly pseudoconvex CR manifold. Moreover, the pseudohermitian scalar curvature of $(P_M, H, p^*J, \theta/2\pi)$ is equal to $p^*S(\omega)$, where $S(\omega)$ is the scalar curvature of (M, J, ω).*

Proof. This result is essentially well known; see [Webster 1978, Section 3] or [Wang 2019, Section 5] for example. However, we give a proof for the reader's convenience. Let $(z^1, \dots, z^n)$ be a holomorphic local coordinate of M. Then $\theta^\alpha := p^* dz^\alpha$ defines an admissible coframe. Since $d\theta^\alpha = p^* ddz^\alpha = 0$, the almost CR structure (H, p^*J) is integrable. Moreover,

$$L_\theta(X, X) = d\theta(X, (p^*J)X) = 2\pi\omega(p_*X, J(p_*X)) > 0$$

for any nonzero $X \in H$, which implies that (M, H, p^*J) is strongly pseudoconvex.

Consider the Tanaka–Webster connection with respect to $\theta/2\pi$. The Kähler form ω is written as $\omega = i g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$, where $(g_{\alpha\bar\beta})$ is a positive definite Hermitian matrix. Since $d\theta/2\pi = p^*\omega$, we have

$$d\theta/2\pi = i(p^* g_{\alpha\bar\beta})\, \theta^\alpha \wedge \theta^{\bar\beta},$$

which implies $h_{\alpha\bar\beta} = p^* g_{\alpha\bar\beta}$. The structure equation on the Kähler manifold (M, J, ω) says that

$$0 = d(dz^\beta) = dz^\alpha \wedge \phi_\alpha{}^\beta, \quad dg_{\alpha\bar\beta} = \phi_{\alpha\bar\beta} + \phi_{\bar\beta\alpha},$$

where $\phi_\alpha{}^\beta$ is the Levi-Civita connection 1-form, and so

$$d\theta^\beta = \theta^\alpha \wedge (p^*\phi_\alpha{}^\beta), \quad d(p^* g_{\alpha\bar\beta}) = p^*\phi_{\alpha\bar\beta} + p^*\phi_{\bar\beta\alpha}.$$

Hence the pull-back connection given by $p^*\phi_\alpha{}^\beta$ coincides with the Tanaka–Webster connection. The pseudohermitian torsion is equal to zero and P_M is Sasakian.

The curvature 2-forms of the Levi-Civita connection on (M, J, ω) are given by

$$\Phi_\alpha{}^\beta := d\phi_\alpha{}^\beta - \phi_\alpha{}^\gamma \wedge \phi_\gamma{}^\beta = R_\alpha{}^\beta{}_{\rho\bar\sigma}\, dz^\rho \wedge dz^{\bar\sigma}.$$

Its pull-back to P_M yields

$$p^*\Phi_\alpha{}^\beta = d(p^*\phi_\alpha{}^\beta) - (p^*\phi_\alpha{}^\gamma) \wedge (p^*\phi_\gamma{}^\beta) = (p^* R_\alpha{}^\beta{}_{\rho\bar\sigma})\, \theta^\rho \wedge \theta^{\bar\sigma},$$

which are the curvature 2-forms of the Tanaka–Webster connection. Hence the pseudohermitian Ricci tensor $R_{\rho\bar\sigma}$ is given by

$$p^* R_\alpha{}^\alpha{}_{\rho\bar\sigma} = p^* \operatorname{Ric}_{\rho\bar\sigma},$$

where Ric is the Ricci tensor of (M, J, ω), and the pseudohermitian scalar curvature is given by

$$R_\rho{}^\rho = p^* \operatorname{Ric}_\rho{}^\rho = p^* S(\omega). \qquad \square$$

3. Formulas for the CR Yamabe constant

As with the Riemannian Yamabe problem, the functional $\mathfrak{F}(\theta)$ can be rewritten as a functional on $C^\infty(X, \mathbb{R}_+)$:

$$\mathfrak{F}(u^{2/n}\theta) = \frac{\int_X ((2 + 2/n)\,|du|_\theta^2 + R_\theta u^2)\, d\mu_\theta}{\left(\int_X u^{2+2/n}\, d\mu_\theta\right)^{n/(n+1)}}, \tag{3-1}$$

where θ is a fixed contact form and $|du|_\theta^2 = L_\theta^*(du|_H, du|_H)$ [Jerison and Lee 1987]. It follows from this that $\lambda(X) \geq 0$ if $R_\theta \geq 0$. Moreover, consider the case $R_\theta > 0$. Suppose to the contrary that $\lambda(X) = 0$. Then there exists a CR Yamabe contact form $\tilde{\theta} = u^{2/n}\theta$ satisfying $R_{\tilde{\theta}} = 0$, which contradicts (3-1). Therefore $\lambda(X) > 0$ if $R_\theta > 0$; see [Wang 2003, Proposition 3.1] for another proof.

Theorem 3.1. *Let (X, H, J) be a compact strongly pseudoconvex CR manifold of dimension $2n+1$. If $\lambda(X) \geq 0$, then for any $r \in [1, \infty]$*

$$\lambda(X) \leq \inf_{\tilde{\theta}} \|R_{\tilde{\theta}}\|_{L^r} \operatorname{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}},$$

where $\operatorname{Vol}_\theta(X)$ is the volume of X with respect to $d\mu_\theta$. If the CR Yamabe problem is solvable on X, then the equality holds. If $\lambda(X) \leq 0$, then for any $r \in [n+1, \infty]$

$$\lambda(X) = -\inf_{\tilde{\theta}} \|R_{\tilde{\theta}}\|_{L^r} \operatorname{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}}, \tag{3-2}$$

$$= -\inf_{\tilde{\theta}} \|R_{\tilde{\theta}}^-\|_{L^r} \operatorname{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}}, \tag{3-3}$$

where $R_{\tilde{\theta}}^- := \min(R_{\tilde{\theta}}, 0)$, and the two infima are realized only by a CR Yamabe minimizer.

Proof. When $\lambda(X) \geq 0$, the Hölder inequality implies

$$\lambda(X) \leq \frac{\int_X R_{\tilde{\theta}}\, d\mu_{\tilde{\theta}}}{\operatorname{Vol}_{\tilde{\theta}}(X)^{n/(n+1)}} \leq \|R_{\tilde{\theta}}\|_{L^r} \operatorname{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}},$$

and the equality holds if $\tilde{\theta}$ is a CR Yamabe minimizer.

Now in the case of $\lambda(X) \leq 0$, we use the technique of Besson, Courtois, and Gallot [Besson et al. 1991]. Let θ be a CR Yamabe minimizer, which is unique up to a constant in this case. Consider another contact form $\tilde{\theta} = u^{2/n}\theta$, where u is a positive smooth function. The Hölder inequality yields

$$\begin{aligned}
\|R_{\tilde{\theta}}\|_{L^r} \operatorname{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}} &= \left(\int_X |R_{\tilde{\theta}}|^r\, d\mu_{\tilde{\theta}}\right)^{\frac{1}{r}} \operatorname{Vol}_{\tilde{\theta}}(X)^{\frac{1}{n+1}-\frac{1}{r}} \\
&\geq \left(\int_X |R_{\tilde{\theta}}^-|^r u^{2+2/n}\, d\mu_\theta\right)^{\frac{1}{r}} \left(\int_X u^{2+2/n}\, d\mu_\theta\right)^{\frac{1}{n+1}-\frac{1}{r}} \\
&\geq \left(\int_X (-R_{\tilde{\theta}}^-)\, u^{2/n}\, d\mu_\theta\right) \operatorname{Vol}_\theta(X)^{-\frac{n}{n+1}} \\
&\geq \left(\int_X (-R_{\tilde{\theta}})\, u^{2/n}\, d\mu_\theta\right) \operatorname{Vol}_\theta(X)^{-\frac{n}{n+1}}.
\end{aligned}$$

Here recall that

$$R_{\tilde{\theta}} = u^{-1-2/n}(R_\theta + (2+2/n)\Delta_b)\, u,$$

where Δ_b is the sublaplacian [Jerison and Lee 1987]. Therefore

$$\begin{aligned}\|R_{\tilde\theta}\|_{L^r}\operatorname{Vol}_{\tilde\theta}(X)^{\frac{1}{n+1}-\frac{1}{r}} &\geq \left(\int_X(-R_\theta-(2+2/n)\,u^{-1}\Delta_b\,u)\,d\mu_\theta\right)\operatorname{Vol}_\theta(X)^{-\frac{n}{n+1}}\\ &= \left(\int_X(-R_\theta+(2+2/n)\,u^{-2}|du|^2_\theta)\,d\mu_\theta\right)\operatorname{Vol}_\theta(X)^{-\frac{n}{n+1}}\\ &\geq -\left(\int_X R_\theta\,d\mu_\theta\right)\operatorname{Vol}_\theta(X)^{-\frac{n}{n+1}}\\ &= -\lambda(X).\end{aligned}$$

This proves

$$\lambda(X)\geq -\inf_{\tilde\theta}\|R^-_{\tilde\theta}\|_{L^r}\operatorname{Vol}_{\tilde\theta}(X)^{\frac{1}{n+1}-\frac{1}{r}}\geq -\inf_{\tilde\theta}\|R_{\tilde\theta}\|_{L^r}\operatorname{Vol}_{\tilde\theta}(X)^{\frac{1}{n+1}-\frac{1}{r}}.$$

On the other hand,

$$\lambda(X)=-\|R^-_\theta\|_{L^r}\operatorname{Vol}_\theta(X)^{\frac{1}{n+1}-\frac{1}{r}}=-\|R_\theta\|_{L^r}\operatorname{Vol}_\theta(X)^{\frac{1}{n+1}-\frac{1}{r}}$$

since R_θ is a nonpositive constant, which proves the two desired formulas together. It remains to decide when the infima are realized. The infimum of (3-2) or (3-3) is realized by $\tilde\theta$ if and only if the above all inequalities are attained by equalities, which holds if and only if $R^-_{\tilde\theta}\equiv R_{\tilde\theta}$ is a nonpositive constant (and hence u is a positive constant); i.e., $\tilde\theta$ is a CR Yamabe minimizer. □

4. Continuity of the CR Yamabe constant

In this section, we prove the continuity of the CR Yamabe constant under suitable deformations of CR structures. Remark that Lemma 4.2 below is a generalization of [Dietrich 2021, Lemma 5.5].

Proposition 4.1. *Let (X, H, J, θ) be a compact pseudohermitian manifold of dimension $2n+1$. Assume that $(X, H_i, J_i, \theta_i)_{i\in\mathbb{N}}$ is a sequence of pseudohermitian structures on X such that $\theta_i\to\theta$ in the C^2-topology, and $J_i\to J$ and $R_{\theta_i}\to R_\theta$ in the C^0-topology, where J_i and J extend to endomorphisms on TX in an obvious way. Then one has $\lambda(X, H_i, J_i)\to\lambda(X, H, J)$.*

Proof. Since $\theta_i\to\theta$ in the C^2-topology, we may assume that $\theta_i^t := t\theta_i+(1-t)\theta$ for $t\in[0,1]$ is a smooth family of contact forms on X. Let T_i^t be the Reeb vector field of θ_i^t, which is determined by

$$\theta_i^t(T_i^t)=1,\quad \iota(T_i^t)\,d\theta_i^t=0.$$

Since $\theta_i \to \theta$ in the C^2-topology, $\sup_{t\in[0,1]}\|T_i^t - T\|_{C^1} \to 0$. Take the time-dependent vector field $V_i^t \in \Gamma(\operatorname{Ker}\theta_i^t)$ satisfying

$$\iota(V_i^t)\,d\theta_i^t = (\theta_i - \theta)(T_i^t)\,\theta_i^t - (\theta_i - \theta).$$

It follows from $\sup_{t\in[0,1]}\|\theta_i^t - \theta\|_{C^2} \to 0$ and $\sup_{t\in[0,1]}\|T_i^t - T\|_{C^1} \to 0$ that

$$\sup_{t\in[0,1]} \|V_i^t\|_{C^1} \to 0. \tag{4-1}$$

The isotopy $\psi_i^t : X \to X$ generated by V_i^t satisfies $(\psi_i^t)^* H_i^t = H$ for any $t \in [0, 1]$; see the proof of the Gray stability theorem [Geiges 2008, Theorem 2.2.2]. Equation (4-1) yields that $\psi_i^1 \to \mathrm{id}_X$ in the C^1-topology; see, e.g., [Zhang 2022] for a modern treatment of time-dependent vector fields with parameters and a proof of this fact. In particular,

$$\tilde\theta_i := (\psi_i^1)^*\theta_i \to \theta, \quad d\tilde\theta_i = (\psi_i^1)^* d\theta_i \to d\theta,$$
$$\tilde J_i := (\psi_i^1)^* J_i \to J, \quad R_{\tilde\theta_i} = (\psi_i^1)^* R_{\theta_i} \to R_\theta$$

in the C^0-topology. Since $\lambda(X, H_i, J_i) = \lambda(X, H, \tilde J_i)$, the statement follows from the lemma below. □

Lemma 4.2. *Let (X, H, J, θ) be a compact pseudohermitian manifold of dimension $2n+1$. Assume that $(X, H, J_i, \theta_i)_{i\in\mathbb{N}}$ is a sequence of pseudohermitian structures on X such that $\theta_i \to \theta$, $d\theta_i \to d\theta$, $J_i \to J$, and $R_{\theta_i} \to R_\theta$ in the C^0-topology. Then one has $\lambda(X, H, J_i) \to \lambda(X, H, J)$.*

Proof. Without loss of generality, we may assume that $\mathrm{Vol}_{\theta_i}(X) = \mathrm{Vol}_\theta(X) = 1$. Since $R_{\theta_i} \to R_\theta$ in the C^0-topology, we can find $K > 0$ such that $|R_\theta| < K$ and $|R_{\theta_i}| < K$ for any i. For each $\varepsilon \in (0, 1)$, take $N(\varepsilon) \in \mathbb{Z}_+$ such that $i \ge N(\varepsilon)$ implies

$$(1+\varepsilon)^{-1} L_\theta < L_{\theta_i} < (1+\varepsilon) L_\theta, \quad (1+\varepsilon)^{-1} d\mu_\theta < d\mu_{\theta_i} < (1+\varepsilon)\, d\mu_\theta, \quad |R_{\theta_i} - R_\theta| < \varepsilon.$$

For any $f \in C^\infty(X, \mathbb{R})$, we write $f^+ := \max(f, 0)$ and $f^- := \min(f, 0)$. Let $u \in C^\infty(X, \mathbb{R}_+)$. The Hölder inequality yields

$$\int_X u^2\, d\mu_\theta \le \left(\int_X u^{2+2/n}\, d\mu_\theta\right)^{\frac{n}{n+1}} \mathrm{Vol}_\theta(X)^{\frac{1}{n+1}} = \left(\int_X u^{2+2/n}\, d\mu_\theta\right)^{\frac{n}{n+1}}$$

and

$$\left|\int_X R_\theta^\pm u^2\, d\mu_\theta\right| \le K \int_X u^2\, d\mu_\theta \le K \left(\int_X u^{2+2/n}\, d\mu_\theta\right)^{\frac{n}{n+1}}.$$

It follows from the above inequalities that

$$\begin{aligned}\int_X R_{\theta_i} u^2\, d\mu_{\theta_i} &\le \int_X (R_\theta+\varepsilon)\, u^2\, d\mu_{\theta_i}\\ &= \int_X (R_\theta^+ +\varepsilon)\, u^2\, d\mu_{\theta_i} + \int_X R_\theta^- u^2\, d\mu_{\theta_i}\\ &\le (1+\varepsilon)\int_X (R_\theta^+ +\varepsilon)\, u^2\, d\mu_\theta + (1+\varepsilon)^{-1}\int_X R_\theta^- u^2\, d\mu_\theta,\end{aligned}$$

and

$$(1+\varepsilon)^{-1}\int_X u^{2+2/n} d\mu_\theta \le \int_X u^{2+2/n} d\mu_{\theta_i} \le (1+\varepsilon)\int_X u^{2+2/n} d\mu_\theta.$$

Thus we have

$$\begin{aligned}\mathfrak{F}(u^{2/n}\theta_i) &= \left[\int_X \left(\left(2+\frac{2}{n}\right)|du|^2_{\theta_i} + R_{\theta_i}u^2\right) d\mu_{\theta_i}\right]\left(\int_X u^{2+\frac{2}{n}}\, d\mu_{\theta_i}\right)^{-\frac{n}{n+1}}\\ &\le \Bigg[(1+\varepsilon)^{2+\frac{n}{n+1}}\int_X\left(2+\frac{2}{n}\right)|du|^2_\theta\, d\mu_\theta + (1+\varepsilon)^{1+\frac{n}{n+1}}\int_X (R_\theta^+ +\varepsilon)\, u^2\, d\mu_\theta\\ &\qquad + (1+\varepsilon)^{-1-\frac{n}{n+1}}\int_X R_\theta^- u^2\, d\mu_\theta\Bigg]\left(\int_X u^{2+\frac{2}{n}}\, d\mu_\theta\right)^{-\frac{n}{n+1}}\\ &\le (1+\varepsilon)^{2+\frac{n}{n+1}}\,\mathfrak{F}(u^{2/n}\theta) + C\varepsilon,\end{aligned}$$

where C is a positive constant independent of u and ε. Taking the infimum yields

$$\lambda(X, H, J_i) \le (1+\varepsilon)^{2+\frac{n}{n+1}}\,\lambda(X, H, J) + C\varepsilon.$$

Since (J, θ) and (J_i, θ_i) are symmetric, we also obtain

$$\lambda(X, H, J) \le (1+\varepsilon)^{2+\frac{n}{n+1}}\,\lambda(X, H, J_i) + C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{i\to\infty}\lambda(X, H, J_i) \le \lambda(X, H, J) \le \liminf_{i\to\infty}\lambda(X, H, J_i),$$

which implies $\lambda(X, H, J_i) \to \lambda(X, H, J)$. □

5. Deformations of CR structures with varying CR Yamabe constants

In this section, we construct a family of strongly pseudoconvex CR structures with varying CR Yamabe constants. Let (M, J, ω) be an n-dimensional compact Hodge manifold with constant scalar curvature. Let us write

$$\mathcal{K} := \{\varphi \in C^\infty(M) \mid \omega_\varphi := \omega + i\partial\bar\partial\varphi > 0\}$$

for the space of Kähler potentials in the class $[\omega]$ endowed with the C^4-topology. For any $\varphi \in \mathcal{K}$, we have

$$\int_M \omega_\varphi^n = [\omega]^n, \quad \int_M S(\omega_\varphi)\, \omega_\varphi^n = 2n\pi\, c_1(M) \cup [\omega]^{n-1},$$

which are independent of the choice of φ. In particular if ω_φ is a constant scalar curvature Kähler metric, then

$$S(\omega_\varphi) = \frac{2n\pi\, c_1(M) \cup [\omega]^{n-1}}{[\omega]^n} =: \hat{S}.$$

Set

$$\mathcal{F} := \{\varphi \in \mathcal{K} \mid S(\omega_\varphi) = \hat{S}\}$$

so that ω_φ is not a constant scalar curvature Kähler metric for any $\varphi \in \mathcal{K} \setminus \mathcal{F}$. It is known that ω_φ is a constant scalar curvature Kähler metric if and only if there exists $F \in \mathrm{Aut}^0(M)$ such that $\omega_\varphi = F^*\omega$ [Berman and Berndtsson 2017, Theorem 1.3]. In particular if $\mathrm{Aut}(M)$ is discrete, or equivalently, M admits no nontrivial holomorphic vector fields, then $\mathcal{F} = \mathbb{R}$. More generally, if any holomorphic vector field is parallel, then $\mathcal{F} = \mathbb{R}$. This is because a parallel and holomorphic vector field preserves the Kähler form ω.

For each $\varphi \in \mathcal{K}$, take a principal connection θ_φ on P_M satisfying $d\theta_\varphi/2\pi = p^*\omega_\varphi$, which is given by

$$\theta_\varphi := \theta + \pi p^* d^c \varphi,$$

where $d^c := i(\bar{\partial} - \partial)$. This gives an infinite-dimensional family of pseudohermitian manifolds

$$X_\varphi := (P_M,\, H_\varphi := \operatorname{Ker}\theta_\varphi,\, p^*J,\, \theta_\varphi/2\pi)$$

underlying the same manifold P_M. Integration along fibers yields that

$$\begin{aligned}\mathfrak{F}(\theta_\varphi/2\pi) &= \frac{\int_{P_M} p^* S(\omega_\varphi)(\theta_\varphi/2\pi) \wedge (p^*\omega_\varphi)^n}{\left(\int_{P_M} (\theta_\varphi/2\pi) \wedge (p^*\omega_\varphi)^n\right)^{n/(n+1)}} \\ &= \frac{\int_M S(\omega_\varphi)\, \omega_\varphi^n}{\left(\int_M \omega_\varphi^n\right)^{n/(n+1)}} = \frac{2n\pi\, c_1(M) \cup [\omega]^{n-1}}{([\omega]^n)^{n/(n+1)}},\end{aligned}$$

which is independent of φ.

Proof of Theorem 1.1. The continuity follows from Proposition 4.1. If $\theta_0 = \theta$ is a CR Yamabe minimizer, then $\lambda(P_M, H, p^*J) = \mathfrak{F}(\theta_0/2\pi)$. On the other hand, $p^*S(\omega_\varphi)$ is not constant for any $\varphi \in \mathcal{K} \setminus \mathcal{F}$, and so

$$\lambda(P_M, H_\varphi, p^*J) < \mathfrak{F}(\theta_\varphi/2\pi) = \mathfrak{F}(\theta_0/2\pi) = \lambda(P_M, H, p^*J),$$

which completes the first assertion. If ω has nonpositive constant scalar curvature, then so does $\theta/2\pi$, which must be a CR Yamabe minimizer. If ω is a Kähler–Einstein metric, then $\theta/2\pi$ is an Einstein contact form. It follows from Lemma 2.1 that $\theta/2\pi$ is a CR Yamabe minimizer. □

In general, it may be cumbersome to have nontrivial $\mathcal{F}$, since it can be singular and not easy to locate. In fact, we do not have many examples with nontrivial $\mathcal{F}$. We give some examples of Hodge manifolds with $\mathcal{F} = \mathbb{R}$.

Example 5.1 (Kähler–Einstein manifolds with nonpositive scalar curvature). First, a compact complex manifold M with $c_1(M) < 0$ admits a Kähler–Einstein metric in the Kähler class $-c_1(M)$ by the Aubin–Yau theorem [Aubin 1976; Yau 1978]. Second, the celebrated Calabi–Yau theorem [Yau 1978] implies that any Kähler class on a compact complex manifold M with $c_1(M) = 0$ in $H^2(M; \mathbb{R})$ is represented by a unique Ricci-flat Kähler metric. Thus we can take any integral Kähler class for our purpose. In these cases, a constant scalar curvature Kähler metric in any Kähler class, if it exists, must be unique [Chen 2000, Theorem 7] and hence $\mathcal{F} = \mathbb{R}$.

Example 5.2 (Fano manifolds). Let M be a Fano manifold; that is, a compact complex manifold with $c_1(M) > 0$. If M admits a Kähler–Einstein metric ω in the Kähler class $c_1(M)$, then $\theta/2\pi$ gives a CR Yamabe minimizer. For example, a complex surface given by a blow-up of $\mathbb{C}P^2$ at m points in general position with $3 \le m \le 8$ admits a Kähler–Einstein metric of positive scalar curvature [Tian and Yau 1987; Tian 1990]. Note that the automorphism groups of these surfaces are discrete. As a higher-dimensional example, consider the Fermat hypersurface $F_{n,d} \subset \mathbb{C}P^{n+1}$ of degree $3 \le d \le n+1$, that is,

$$F_{n,d} := \left\{ [z_0 : \cdots : z_{n+1}] \in \mathbb{C}P^{n+1} \,\middle|\, \sum_{k=0}^{n+1} z_k^d = 0 \right\}.$$

This $F_{n,d}$ admits a Kähler–Einstein metric [Tian 2000, Section 6.3]. Note that $F_{n,d}$ has no nontrivial holomorphic vector field [Kodaira and Spencer 1958, Lemma 14.2]; see [Matsumura and Monsky 1964] for another proof.

Example 5.3 (scalar-flat but not Ricci-flat surfaces). Take a complex surface S_m obtained by blowing up $\mathbb{C}P^1 \times \Sigma$ at generic m points $p_1, \ldots, p_m$, where Σ is a compact Riemann surface of genus $g \ge 2$ and $m \ge 3$. It is known that S_m admits a scalar-flat Kähler form $\widetilde{\omega}$ [LeBrun and Singer 1993, Theorem 3.11]. Note that S_m does not admit a Ricci-flat Kähler metric since $c_1(S_m)^2 = 8(1-g) - m < 0$. Assume that the projection of $\{p_1, \ldots, p_m\}$ to $\mathbb{C}P^1$ consists of at least 3 points. Since Σ has no nontrivial holomorphic vector fields and any holomorphic vector field on $\mathbb{C}P^1$ vanishing at least 3 points must be trivial, S_m admits no nontrivial holomorphic vector fields also.

It still remains to show that S_m has a scalar-flat integral Kähler class. Since $H^{2,0}(S_m) = 0$, we have $H^{1,1}(S_m;\mathbb{R}) \cong H^2(S_m;\mathbb{Z}) \otimes \mathbb{R}$. The linear operator

$$c_1(S_m)\cup : H^{1,1}(S_m;\mathbb{R}) \to \mathbb{R}$$

has integer coefficients and satisfies $c_1(S_m) \cup [\widetilde{\omega}] = 0$. Hence there exists a rational Kähler class μ close enough to $[\widetilde{\omega}]$ satisfying $c_1(S_m) \cup \mu = 0$. [LeBrun and Simanca 1995, Theorem 1] implies that μ contains a scalar-flat Kähler form ω. Therefore we obtain a scalar-flat integral Kähler class by homothety.

6. Construction of a manifold with different signs of CR Yamabe constants

In this section, we construct a manifold admitting two strongly pseudoconvex CR structures with different signs of CR Yamabe constants. Our construction is based on a exotic smooth structure of a certain complex surface and an adaptation of the technique originated by Ruan [1994] and used by Kim and Sung [2016] to show the existence of inequivalent symplectic structures on certain 6-manifolds.

Let B be the Barlow surface [Barlow 1985; Kotschick 1989] and R_8 be a complex surface given by a blow-up of $\mathbb{C}P^2$ at 8 points in general position. The Barlow surface is a simply connected minimal surface of general type with $q = p_g = 0$ and $c_1(B)^2 = 1$, and contains (-2)-curves so that its canonical line bundle is not ample. But as shown in [Catanese and LeBrun 1997, Theorem 7], it has a small deformation with ample canonical line bundle and hence admits a Kähler–Einstein metric of negative scalar curvature by the celebrated Aubin–Yau theorem. By the results in [Tian and Yau 1987; Tian 1990], R_8 admits a Kähler–Einstein metric of positive scalar curvature. It is well-known that B and R_8 are homeomorphic by Freedman's classification [Freedman 1982, Theorem 1.5] while they are not diffeomorphic by Kotschick's theorem [Kotschick 1989, Theorem 1].

Remark 6.1. An easier way of proving Kotschick's theorem by using Seiberg–Witten invariant runs as follows. Since R_8 and B have $b_2^+ = 1$, their Seiberg–Witten invariants for a Spin^c structure ξ with $c_1(\xi)^2 > 0$ are well-defined for any small perturbation. The complex surface R_8 admits a metric of positive scalar curvature, so its Seiberg–Witten invariants all vanish. However, the Seiberg–Witten invariant of B for the canonical Spin^c structure determined by the complex structure is ± 1 [Morgan 1996].

Since the intersection forms of both B and R_8 are indefinite and odd, they are isomorphic to $(1) \oplus 8(-1)$. Wall [1962, p. 336] has proved that all characteristic vectors with square 1 in $(1) \oplus 8(-1)$ are equivalent. Since the first Chern class of B and R_8 are characteristic with square 1 by Wu's formula, there is an isomorphism from $H^2(R_8;\mathbb{Z})$ to $H^2(B;\mathbb{Z})$ preserving the intersection form and the first

Chern class. This induces an isomorphism

$$\Psi : H^*(R_8 \times \mathbb{C}P^1; \mathbb{Z}) \to H^*(B \times \mathbb{C}P^1; \mathbb{Z})$$

preserving $H^*(\mathbb{C}P^1; \mathbb{Z})$ in the obvious way. We claim that Ψ satisfies the conditions of the following theorem.

Theorem 6.2 [Jupp 1973, Theorem 1]. *Let X and Y be smooth closed simply connected 6-manifolds with torsion-free homology. Suppose that there is an isomorphism from $H^*(X; \mathbb{Z})$ to $H^*(Y; \mathbb{Z})$ preserving the triple cup product structure $\mu : H^2 \otimes H^2 \otimes H^2 \to \mathbb{Z}$, the second Stiefel–Whitney class, and the first Pontryagin class. Then there exists an orientation-preserving diffeomorphism from X to Y realizing this algebraic isomorphism.*

It is enough to check that Ψ preserves the specified characteristic classes. By the product formula,

$$\begin{aligned} w_2(R_8 \times \mathbb{C}P^1) &= w_2(R_8) + w_1(R_8)\, w_1(\mathbb{C}P^1) + w_2(\mathbb{C}P^1) \\ &= c_1(R_8) + 0 + c_1(\mathbb{C}P^1) \mod 2, \end{aligned}$$

and likewise for $B \times \mathbb{C}P^1$. Since

$$\Psi\big(c_1(R_8) + c_1(\mathbb{C}P^1)\big) = c_1(B) + c_1(\mathbb{C}P^1),$$

Ψ preserves the second Stiefel–Whitney class. Using the fact that $p_1 = c_1^2 - 2c_2$ and the product formula, we have

$$\begin{aligned} p_1(R_8 \times \mathbb{C}P^1) &= c_1(R_8 \times \mathbb{C}P^1)^2 - 2c_2(R_8 \times \mathbb{C}P^1) \\ &= \big(c_1(R_8) + c_1(\mathbb{C}P^1)\big)^2 - 2\big(c_2(R_8) + c_1(R_8)\, c_1(\mathbb{C}P^1)\big), \end{aligned}$$

and likewise for $B \times \mathbb{C}P^1$. Since Ψ preserves the Euler characteristic; i.e., the alternating sum of Betti numbers, Ψ maps $e(R_8) = c_2(R_8)$ to $e(B) = c_2(B)$. Therefore Ψ preserves the first Pontryagin class too, and we have an orientation-preserving diffeomorphism

$$\psi : R_8 \times \mathbb{C}P^1 \to B \times \mathbb{C}P^1$$

satisfying $\psi^*(c_1(B)) = c_1(R_8)$ and $\psi^*(c_1(\mathbb{C}P^1)) = c_1(\mathbb{C}P^1)$.

Let $n \geq 3$. Take Kähler forms $\omega_1 \in c_1(R_8)$, $\omega_2 \in c_1(\mathbb{C}P^1)$, and $\omega_3 \in nc_1(\mathbb{C}P^{n-3})$ such that

$$\mathrm{Ric}(\omega_1) = 2\pi\omega_1, \quad \mathrm{Ric}(\omega_2) = 2\pi\omega_2, \quad \mathrm{Ric}(\omega_3) = \frac{2\pi}{n}\omega_3.$$

Then

$$(M := R_8 \times \mathbb{C}P^1 \times \mathbb{C}P^{n-3},\ \omega := \omega_1 + \omega_2 + \omega_3)$$

is a Kähler manifold with positive Ricci curvature. Let P_M be the principal S^1-bundle over M whose Euler class is $-[\omega]$. This P_M admits a connection one-form θ

and the lifted CR structure J such that $d\theta/2\pi$ projects down to ω on M. The CR Yamabe constant of $(P_M, H := \operatorname{Ker}\theta, J)$ must be positive by the argument after formula (3-1).

Lemma 6.3. *The manifold P_M is simply connected.*

Proof. Let $E \cong \mathbb{C}P^1$ be an exceptional divisor in R_8 and consider it as a complex curve in M. Then the restriction of $-[\omega]$ to E coincides with the first Chern class of the tautological line bundle over E. Hence $(P_M)|_E \to E$ is a Hopf fibration. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \pi_2((P_M)|_E) & \longrightarrow & \pi_2(E) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1((P_M)|_E) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \pi_2(M) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(P_M) \longrightarrow \pi_1(M) = 0 \end{array}$$

Since $(P_M)|_E \to E$ is a Hopf fibration, $\pi_1((P_M)|_E) = 0$ and $\pi_2((P_M)|_E) = 0$, and so the map $\pi_2(E) \to \pi_1(S^1)$ is an isomorphism. Thus we have $\pi_2(M) \to \pi_1(S^1)$ is surjective and $\pi_1(P_M) = 0$. □

On the other hand, let $\tilde{J}_1$ and $-\widetilde{\omega}_1$ be the complex structure and a Kähler form in the class $-c_1(B)$ giving an Einstein metric of negative scalar curvature on B. Denote by $\widetilde{M}'$ the complex 3-manifold $(B \times \mathbb{C}P^1, (-\tilde{J}_1) \times J_2)$. The two-form $\widetilde{\omega}' = \widetilde{\omega}_1 + \omega_2$ defines a Kähler form on $\widetilde{M}'$ with constant scalar curvature -2π. Consider the Kähler manifold

$$(\widetilde{M} := \widetilde{M}' \times \mathbb{C}P^{n-3},\ \widetilde{\omega} := \widetilde{\omega}' + \omega_3).$$

The scalar curvature of this manifold is given by $-2\pi + \frac{2(n-3)\pi}{n} < 0$. Denote by $\widetilde{\psi}$ the diffeomorphism

$$\psi \times \mathrm{id}_{\mathbb{C}P^{n-3}} : (R_8 \times \mathbb{C}P^1) \times \mathbb{C}P^{n-3} \to (B \times \mathbb{C}P^1) \times \mathbb{C}P^{n-3}.$$

Then $\widetilde{\psi}^*([\widetilde{\omega}]) = [\omega]$ since $\psi^*([\widetilde{\omega}']) = [\omega_1] + [\omega_2]$. Hence there exists a connection form $\tilde{\theta}$ of P_M and the lifted CR structure $\tilde{J}$ such that $d\tilde{\theta}/2\pi = p^*\widetilde{\psi}^*\widetilde{\omega}$. We derive from Proposition 2.2 that $(P_M, \widetilde{H} := \operatorname{Ker}\tilde{\theta}, \tilde{J})$ has negative CR Yamabe constant.

Before the proof of Theorem 1.2, we recall some facts on contact geometry. Two cooriented contact manifolds (X, H) and (X', H') are isomorphic if there exists a diffeomorphism $\psi : X \to X'$ preserving contact structures and coorientation. Moreover, the first Chern class of a strongly pseudoconvex CR manifold is an invariant of the underlying cooriented contact structure; see [Geiges 2008, Section 2.4] for example.

Proof of Theorem 1.2. It remains to show that (P_M, H) is not isomorphic to $(P_M, \widetilde{H})$ as cooriented contact manifolds. Denote by $p : P_M \to M$ the projection from P_M

to M. Then

$$c_1(P_M, H, J) = p^* c_1(M), \quad c_1(P_M, \widetilde{H}, \tilde{J}) = p^* \widetilde{\psi}^* c_1(\widetilde{M}).$$

Now consider the Gysin exact sequence

$$H^0(M;\mathbb{Z}) = \mathbb{Z} \xrightarrow{-[\omega]} H^2(M;\mathbb{Z}) \xrightarrow{p^*} H^2(P_M;\mathbb{Z}) \to H^1(M;\mathbb{Z}) = 0.$$

We first consider the case $n = 3$. Since $c_1(R_8)$ and $[\omega] = c_1(R_8) + c_1(\mathbb{C}P^1)$ are linearly independent in $H^2(M;\mathbb{Z})$, we have

$$c_1(P_M, H, J) = p^* c_1(M) = 0, \quad c_1(P_M, \widetilde{H}, \tilde{J}) = p^* \widetilde{\psi}^* c_1(\widetilde{M}) = -2p^* c_1(R_8) \neq 0.$$

Hence (P_M, H) is not isomorphic to $(P_M, \widetilde{H})$.

In the remainder of the proof, we assume that $n \geq 4$. In this case,

$$c_1(P_M, H, J) = p^* c_1(M) = -(n-1)(n-2) p^* c_1(\mathcal{O}_{\mathbb{C}P^{n-3}}(1)).$$

In particular, $[c_1(P_M, H, J)] = 0$ in $H^2(P_M;\mathbb{Z})/(n-1)H^2(P_M;\mathbb{Z})$. It suffices to show that $[c_1(P_M, \widetilde{H}, \tilde{J})] \neq 0$ in $H^2(P_M;\mathbb{Z})/(n-1)H^2(P_M;\mathbb{Z})$. Suppose to the contrary that $[c_1(P_M, \widetilde{H}, \tilde{J})] = [p^* \widetilde{\psi}^* c_1(\widetilde{M})] = 0$ in $H^2(P_M;\mathbb{Z})/(n-1)H^2(P_M;\mathbb{Z})$. Consider the following exact sequence:

$$H^0(M;\mathbb{Z}) = \mathbb{Z} \xrightarrow{-[\omega]} \frac{H^2(M;\mathbb{Z})}{(n-1)H^2(M;\mathbb{Z})} \xrightarrow{p^*} \frac{H^2(P_M;\mathbb{Z})}{(n-1)H^2(P_M;\mathbb{Z})}.$$

This yields that there exists $k \in \mathbb{Z}$ such that

$$\widetilde{\psi}^* c_1(\widetilde{M}) + k[\omega] = -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega] \in (n-1)H^2(M;\mathbb{Z})$$

Hence

$$\langle -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega], a\rangle \equiv 0 \mod n-1$$

for any $a \in H_2(M;\mathbb{Z})$. Let $E \cong \mathbb{C}P^1$ be an exceptional divisor in R_8 and consider it as a complex curve in M. Taking $a = [E]$ gives that

$$0 \equiv \langle -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega], [E]\rangle = k - 1 \mod n-1.$$

Consider also a projective line $L \subset \mathbb{C}P^{n-3}$, which is seen as a complex curve in M. Then

$$\begin{aligned} 0 &\equiv \langle -c_1(R_8) + c_1(\mathbb{C}P^1) + c_1(\mathbb{C}P^{n-3}) + k[\omega], [L]\rangle \\ &= (n-2) + kn(n-2) \equiv n-3 \mod n-1, \end{aligned}$$

which is a contradiction. □

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WEIGHTED LOW-LYING ZEROS OF L-FUNCTIONS ATTACHED TO SIEGEL MODULAR FORMS

SHIFAN ZHAO

We study weighted low-lying zeros of spinor and standard L-functions attached to degree 2 Siegel modular forms. We show that the symmetry type of weighted low-lying zeros of spinor L-functions is symplectic, for test functions whose Fourier transform have support in $(-1, 1)$, extending the previous range $\left(-\frac{4}{15}, \frac{4}{15}\right)$. We then show that the symmetry type of weighted low-lying zeros of standard L-functions is also symplectic. We further extend the range of support by performing an average over weight. As an application, we discuss nonvanishing of central values of those L-functions.

1. Introduction

D. Hilbert and G. Pólya suggested that nontrivial zeros of the Riemann zeta function $\zeta(s)$ correspond to eigenvalues of a self-adjoint operator on some Hilbert space. The first evidence of such a connection was found by H. L. Montgomery [1973], who investigated the pair correlation of nontrivial zeros of $\zeta(s)$ and conjectured that it is, as pointed out by F. J. Dyson, the same as the pair correlation of eigenvalues of random Hermitian or unitary matrices of large order, also known as the gaussian unitary ensemble (GUE) model. This conjecture of Montgomery was later supported by numerical results by A. M. Odlyzko [1987], based on values for the first 10^5 zeros and for zeros number $10^{12}+1$ to $10^{12}+10^5$. The local spacing between these sample zeros matches the prediction by the GUE model quite well.

Z. Rudnick and P. Sarnak [1996] extended Montgomery's work by computing the general n-level correlation function of zeros of any principal L-function $L(s,\pi)$ attached to a cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ (in a restricted range). Their answer is universal and is precisely the one predicted by the GUE model. Numerical evidence was found by R. Rumely [1993] for primitive Dirichlet L-functions, and by M. O. Rubinstein [1998] for Hasse–Weil L-functions of three distinct elliptic curves and for the Hecke L-function associated to Ramanujan's τ-function.

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Although the n-level correlation statistic of zeros of any fixed principal automorphic L-function obeys the universal GUE law, there is another statistic, called the n-level density of low-lying zeros, that is sensitive to families. N. Katz and Sarnak [1999a] studied low-lying zeros of zeta functions of varieties over finite fields (the "function field" analogue). For these they indicated that a spectral interpretation exists in terms of eigenvalues of Frobenius on cohomology groups. On the number field side, although many results concerning low-lying zeros have been proved, it is still not clear where their spectral nature comes from. See also [Katz and Sarnak 1999b] for a nice survey on these topics.

Before stating our results, we first describe the problem in general terms. Let $\mathcal{F}_Q$ be a family of automorphic forms, ordered by conductor $Q \geq 1$. To each $f \in \mathcal{F}_Q$ one associates an L-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \tag{1-1}$$

which converges absolutely for $s \in \mathbb{C}$ in some right half-plane. We assume that $L(s, f)$ admits meromorphic continuation to the whole complex plane $\mathbb{C}$. We also assume that $L(s, f)$ satisfies a functional equation

$$\Lambda(s, f) = L_\infty(s, f) L(s, f) = \varepsilon_f \Lambda(1-s, f), \tag{1-2}$$

where $\varepsilon_f = \pm 1$ is the root number.

We assume the generalized Riemann hypothesis (GRH) for $L(s, f)$. That is, nontrivial zeros of $L(s, f)$ all lie on the critical line. We may denote those zeros by

$$\rho_f = \tfrac{1}{2} + i\gamma_f, \quad \gamma_f \in \mathbb{R}. \tag{1-3}$$

Let $\Phi \in \mathcal{S}(\mathbb{R})$ be an even Schwartz function (called "test function" throughout) whose Fourier transform $\hat{\Phi}$ has compact support. To this end we define the 1-level density of low-lying zeros of $L(s, f)$, with respect to the test function Φ, to be

$$D(f; \Phi) = \sum_{\rho_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right), \tag{1-4}$$

where ρ_f runs through nontrivial zeros of $L(s, f)$, counted with multiplicity, and c_f is a parameter associated with $f \in \mathcal{F}_Q$, comparable to the analytic conductor of f (specified later). The density conjecture for low-lying zeros of $L(s, f)$ asserts that:

Conjecture 1.1 (density conjecture). *For any even Schwartz function Φ whose Fourier transform $\hat{\Phi}$ has compact support, we have*

$$\lim_{Q \to \infty} \frac{1}{|\mathcal{F}_Q|} \sum_{f \in \mathcal{F}_Q} D(f; \Phi) = \int_{-\infty}^{\infty} \Phi(x) W(\mathcal{F})(x)\, dx \tag{1-5}$$

for some distribution $W(\mathcal{F})$ depending only on $\mathcal{F}$.

Many observations and results in [Katz and Sarnak 1999a] suggest that the distribution $W(\mathcal{F})$ depends on the family $\mathcal{F}$ through a symmetry group $G(\mathcal{F})$. Possible symmetry types are orthogonal O, special orthogonal even SO(even), special orthogonal odd SO(odd), symplectic Sp and unitary U. The corresponding distributions and their Fourier transforms are

$$
\begin{aligned}
W(\mathrm{O})(x) &= 1+\tfrac{1}{2}\delta_0(x), & \hat{W}(\mathrm{O})(y) &= \delta_0(y)+\tfrac{1}{2},\\
W(\mathrm{SO(even)})(x) &= 1+\frac{\sin 2\pi x}{2\pi x}, & \hat{W}(\mathrm{SO(even)})(y) &= \delta_0(y)+\tfrac{1}{2}\eta(y),\\
W(\mathrm{SO(odd)})(x) &= 1-\frac{\sin 2\pi x}{2\pi x}+\delta_0(x), & \hat{W}(\mathrm{SO(odd)})(y) &= \delta_0(y)-\tfrac{1}{2}\eta(y)+1,\\
W(\mathrm{Sp})(x) &= 1-\frac{\sin 2\pi x}{2\pi x}, & \hat{W}(\mathrm{Sp})(y) &= \delta_0(y)-\tfrac{1}{2}\eta(y),\\
W(\mathrm{U})(x) &= 1, & \hat{W}(\mathrm{U})(y) &= \delta_0(y),
\end{aligned}
\tag{1-6}
$$

where δ_0 is the Dirac distribution at 0, and $\eta(y)=1, \frac{1}{2}, 0$ for $|y|<1$, $|y|=1$ and $|y|>1$ respectively. The first three distributions of different orthogonal symmetry type have indistinguishable Fourier transforms within $(-1, 1)$, while the symplectic and unitary symmetry types are distinguishable from the orthogonal ones.

The density conjecture (Conjecture 1.1) has been verified for many families (in restricted ranges). See [Iwaniec et al. 2000; Rubinstein 2001; Fouvry and Iwaniec 2003; Guloglu 2005; Young 2006; Dueñez and Miller 2006; Gao and Zhao 2011; Cho and Kim 2015; Shin and Templier 2016; Liu and Miller 2017; Kim et al. 2020], to name a few. In all results towards this direction, the support of Fourier transform of the test function Φ is restricted within certain range. One important question in this topic is how to extend the range as large as possible, for the full density Conjecture 1.1 does not require any condition on the compact support of $\hat{\Phi}$.

One can also consider "weighted" distribution of low-lying zeros by allowing certain weights ω_f. The weighted average density under consideration is

$$
\Bigl(\sum_{f\in\mathcal{F}_Q}\omega_f\Bigr)^{-1}\sum_{f\in\mathcal{F}_Q}\omega_f D(f;\Phi). \tag{1-7}
$$

Often these weights ω_f contain important arithmetic information such as central values of L-functions, and including them may possibly change the symmetry type. Recent results in this direction include [Kowalski et al. 2012; Knightly and Reno 2019; Sugiyama and Suriajaya 2022; Fazzari 2024].

In this article we study weighted low-lying zeros of spinor and standard L-functions attached to degree 2 Siegel modular forms. For a general introduction on Siegel modular forms, we refer readers to [Klingen 1990; Pitale 2019].

We proceed to describe our results. Let $k \geq 6$ be an even integer. Let $S_k(\Gamma_2)$ be the space of degree 2 holomorphic Siegel cusp forms of weight k for the symplectic group $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$. Each form $F \in S_k(\Gamma_2)$ is a holomorphic function on the Siegel upper half-plane

$$\mathbb{H}_2 = \{Z = X + iY \in M_2(\mathbb{C}) : Z = Z^T, Y > 0\}, \tag{1-8}$$

which satisfies the automorphy condition

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2,\ Z \in \mathbb{H}_2. \tag{1-9}$$

Here and after we use $M_n(R)$ to denote the ring of $n \times n$ matrices over a ring R.

The Fourier expansion of F is

$$F(Z) = \sum_{T \in \mathcal{T}} a_F(T)(\det T)^{\frac{k}{2}-\frac{3}{4}}\, e(\mathrm{Tr}(TZ)), \quad Z \in \mathbb{H}_2, \tag{1-10}$$

where the summation is taken over the set

$$\mathcal{T} = \{T = (t_{ij}) \in M_2(\mathbb{R}) : T > 0,\ t_{11} \in \mathbb{Z},\ t_{22} \in \mathbb{Z},\ 2t_{12} = 2t_{21} \in \mathbb{Z}\}. \tag{1-11}$$

We call $a_F(T)$ the (normalized) Fourier coefficient of F at T. It is known that $a_F(T) \in \mathbb{R}$.

We use I to denote the 2×2 identity matrix. For $F \in S_k(\Gamma_2)$ we set

$$\omega_F = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\,\Gamma\big(k - \tfrac{3}{2}\big)\,\Gamma(k-2)\frac{a_F(I)^2}{\|F\|^2} \tag{1-12}$$

to be the "harmonic" weight attached to F, where $\|F\|$ is the Petersson norm of F defined by

$$\|F\| = \left(\int_{\Gamma_2\backslash\mathbb{H}_2} |F(Z)|^2 (\det Y)^k \frac{dX\,dY}{(\det Y)^3}\right)^{1/2}. \tag{1-13}$$

We now choose a basis $H_k(\Gamma_2)$ of $S_k(\Gamma_2)$ consisting of eigenforms for all Hecke operators (we call such a form a Hecke eigenform). It is known (see, e.g., (1.8) in [Blomer 2019]) that

$$\sum_{F \in H_k(\Gamma_2)} \omega_F = 1 + O(e^{-k}). \tag{1-14}$$

Note that the above sum is independent of the choice of basis $H_k(\Gamma_2)$.

To each form $F \in H_k(\Gamma_2)$ we can attach a degree 4 spinor L-function $L(s, F; \text{spin})$ and a degree 5 standard L-function $L(s, F; \text{std})$, both normalized so that the central point is $s = \frac{1}{2}$. The analytic conductors of those L-functions are of size k^2 and k^4, respectively. Further properties of these L-functions are discussed in Section 2.

We assume GRH for both spinor and standard L-functions, and denote their nontrivial zeros on the critical line by

$$\rho_{F,\mathrm{spin}} = \tfrac{1}{2} + i\gamma_{F,\mathrm{spin}}, \quad \rho_{F,\mathrm{std}} = \tfrac{1}{2} + i\gamma_{F,\mathrm{std}}. \tag{1-15}$$

The corresponding density functions with respect to a test function Φ are

$$D(F;\Phi;\mathrm{spin}) = \sum_{\rho_{F,\mathrm{spin}}} \Phi\left(\frac{\gamma_{F,\mathrm{spin}}}{2\pi}\log c_{F;\mathrm{spin}}\right), \tag{1-16}$$

$$D(F;\Phi;\mathrm{std}) = \sum_{\rho_{F,\mathrm{std}}} \Phi\left(\frac{\gamma_{F,\mathrm{std}}}{2\pi}\log c_{F;\mathrm{std}}\right). \tag{1-17}$$

Our first result concerning low-lying zeros of spinor L-functions is as follows:

Theorem 1.2. *Let Φ be an even Schwartz function whose Fourier transform has support in $(-1,1)$. For $F \in H_k(\Gamma_2)$, define $D(F;\Phi;\mathrm{spin})$ as in (1-16) with $c_{F;\mathrm{spin}} = k^2$ and ω_F as in (1-12). Assume GRH for $L(s,F;\mathrm{spin})$. Then we have*

$$\lim_{k\to\infty} \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F;\Phi;\mathrm{spin}) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\,dx. \tag{1-18}$$

Remark 1.3. The result above has been obtained from [Kowalski et al. 2012], but only for test functions Φ with $\mathrm{supp}(\hat{\Phi}) \subset \left(-\frac{4}{15},\frac{4}{15}\right)$, as an application of their quantitative local equidistribution result. Here we extend the range of support to $(-1,1)$. This improvement is crucial in application to nonvanishing problems, as we will explain in Section 5.

Let $H_k^*(\Gamma_2) \subset H_k(\Gamma_2)$ denote a Hecke basis of the space of Saito–Kurokawa lifts (these concepts will be discussed in Section 2). As a direct corollary of Theorem 1.2 we can establish the following nonvanishing result:

Corollary 1.4. *Assume GRH for $L(s,F;\mathrm{spin})$. Then we have*

$$\liminf_{k\to\infty} \sum_{\substack{F\in H_k(\Gamma_2)\setminus H_k^*(\Gamma_2)\\ L(1/2,F;\mathrm{spin})\neq 0}} \omega_F \geq \tfrac{3}{4}. \tag{1-19}$$

Remark 1.5. For comparison, it is shown in [Blomer 2019] that

$$\sum_{\substack{F\in H_k(\Gamma_2)\setminus H_k^*(\Gamma_2)\\ L(1/2,F;\mathrm{spin})\neq 0}} \omega_F \gg \frac{1}{\log k}, \tag{1-20}$$

unconditionally for large k. This follows from asymptotic formulas for the first and second moments of central values. Although it is not surprising that GRH would yield much stronger result, one still needs the range of support in Theorem 1.2 not to be too small to carry out the argument. It was also pointed out in [Blomer 2019, page 1756, Remark (b)] that it is possible to use the mollifier technique to obtain a positive proportional result unconditionally. The exact (unconditional) proportion would not be as large as our (conditional) proportion $\frac{3}{4}$ though.

For low-lying zeros of standard L-functions, we have the following result:

Theorem 1.6. *Let Φ be an even Schwartz function whose Fourier transform has support in $\left(-\frac{1}{4}, \frac{1}{4}\right)$. For $F \in H_k(\Gamma_2)$, define $D(F; \Phi; \mathrm{std})$ as in (1-17) with $c_{F;\mathrm{std}} = k^4$ and ω_F as in (1-12). Assume GRH for $L(s, F; \mathrm{std})$. Then we have*

$$\lim_{k\to\infty} \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F; \Phi; \mathrm{std}) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\, dx. \tag{1-21}$$

Remark 1.7. An unweighted version of Theorem 1.6 was established in [Kim et al. 2020], for test functions Φ whose Fourier transforms have sufficiently small support (for a precise range of support, see Proposition 9.3 in [Kim et al. 2020]). The (unweighted) symmetry type is also symplectic. For comparison, the symmetry type of low-lying zeros of spinor L-functions changes from orthogonal to symplectic when weighted by ω_F.

We may further extend the range of support in Theorem 1.6 from $\left(-\frac{1}{4}, \frac{1}{4}\right)$ to $\left(-\frac{5}{18}, \frac{5}{18}\right)$ by performing an extra (smooth) average over weight k. Our result is:

Theorem 1.8. *Let $\Omega \in C_c^\infty(0, \infty)$ be such that $\Omega \geq 0$, not identically 0. Let Φ be an even Schwartz function whose Fourier transform has support in $\left(-\frac{5}{18}, \frac{5}{18}\right)$. For $F \in H_k(\Gamma_2)$ and large parameter $K > 0$, define $D(F; \Phi; \mathrm{std})$ as in (1-17) with $c_{F;\mathrm{std}} = K^4$ and ω_F as in (1-12). Assume GRH for $L(s, F; \mathrm{std})$. Then we have*

$$\lim_{K\to\infty} \left(\sum_k \Omega\Big(\frac{k}{K}\Big)\right)^{-1} \sum_k \Omega\Big(\frac{k}{K}\Big) \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F; \Phi; \mathrm{std}) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(\mathrm{Sp})(x)\, dx. \tag{1-22}$$

where the summation in k is over even integers.

This article is organized as follows: In Section 2, we first review some facts about spinor and standard L-functions. We then work out the combinatorial relations between certain functions in Satake parameters of a form $F \in H_k(\Gamma_2)$ and its Fourier coefficients at scalar matrices. These relations allow us to apply Kitaoka's formula, which we state in Section 3. In Section 3 we also take average over weight k in Kitaoka's formula and give an upper bound for the off-diagonal term. In Section 4 we apply the results established in previous sections, as well as the explicit formula to prove Theorems 1.2–1.8. In Section 5 we prove Corollary 1.4 and discuss some other issues concerning nonvanishing of central L-values.

2. Spinor and standard L-functions

Let $F \in H_k(\Gamma_2)$ be a Hecke eigenform. It is known that for each prime p there are three complex numbers $\alpha_{F,0}(p)$, $\alpha_{F,1}(p)$, $\alpha_{F,2}(p)$, called the Satake parameters of F at p, with certain prescribed properties. See Chapter 3 in [Pitale 2019] for a detailed discussion. In particular, these Satake parameters satisfy the relation

$$\alpha_{F,0}(p)^2 \alpha_{F,1}(p)\, \alpha_{F,2}(p) = 1. \tag{2-1}$$

Let $S_{2k-2}(\Gamma_1)$ denote the space of holomorphic cusp forms of weight $2k-2$ for the full modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$. There is an injective Hecke-equivariant linear map

$$SK : S_{2k-2}(\Gamma_1) \to S_k(\Gamma_2), \quad f \mapsto F_f, \tag{2-2}$$

called the Saito–Kurokawa lifting. We denote the image of SK by $S_k^*(\Gamma_2)$ and call forms in $S_k^*(\Gamma_2)$ Saito–Kurokawa lifts. We also use $H_k^*(\Gamma_2)$ for a basis of $S_k^*(\Gamma_2)$ consisting of Hecke eigenforms. There are various ways to construct such a lifting map. For a construction using half-integral weight modular forms, see Section 2.1.3 in [Pitale 2019].

For $F \in H_k(\Gamma_2)$ that is not a Saito–Kurokawa lift, it is known that $|\alpha_{F,i}(p)| = 1$ for all prime p, by a result in [Weissauer 2009]. However, this is not true for Saito–Kurokawa lifts $F_f \in H_k^*(\Gamma_2)$. We will see this in Andrianov's explicit formula (2-8) stated below.

2.1. ***The spinor L-function.*** The spinor L-function attached to a Hecke eigenform $F \in H_k(\Gamma_2)$ is defined by a degree 4 Euler product

$$\begin{aligned} L(s, F; \mathrm{spin}) &= \prod_p \left(1 - \frac{\alpha_{F,0}(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{F,0}(p)\,\alpha_{F,1}(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{F,0}(p)\,\alpha_{F,2}(p)}{p^s}\right)^{-1} \\ &\qquad \times \left(1 - \frac{\alpha_{F,0}(p)\,\alpha_{F,1}(p)\,\alpha_{F,2}(p)}{p^s}\right)^{-1}, \end{aligned} \tag{2-3}$$

which converges absolutely in some right half-plane. By setting

$$\alpha_F(p) = \alpha_{F,0}(p), \quad \beta_F(p) = \alpha_{F,0}(p)\,\alpha_{F,1}(p), \tag{2-4}$$

we may rewrite the above Euler product as

$$\begin{aligned} L(s, F; \mathrm{spin}) &= \prod_p \left(1 - \frac{\alpha_F(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_F(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_F(p)^{-1}}{p^s}\right)^{-1} \left(1 - \frac{\beta_F(p)^{-1}}{p^s}\right)^{-1}, \end{aligned} \tag{2-5}$$

in view of the relation (2-1).

It is proved by A. N. Andrianov [1974] that $L(s, F; \text{spin})$ extends to a meromorphic function on $\mathbb{C}$, which has a simple pole at $s = \frac{3}{2}$ if F is a Saito–Kurokawa lift, and is entire otherwise. Its functional equation takes the form

$$\Lambda(s, F; \text{spin}) = \Gamma_{\mathbb{C}}\big(s+\tfrac{1}{2}\big)\Gamma_{\mathbb{C}}\big(s+k-\tfrac{3}{2}\big)L(s, F; \text{spin}) = \Lambda(1-s, F; \text{spin}), \tag{2-6}$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. For $F_f \in H_k^*(\Gamma_2)$ a Saito–Kurokawa lift, its spinor L-function decomposes as

$$L(s, F_f; \text{spin}) = \zeta\big(s+\tfrac{1}{2}\big)\zeta\big(s-\tfrac{1}{2}\big)L(s, f), \tag{2-7}$$

where $L(s, f)$ is the Hecke L-function of the elliptic cusp form f.

For $F \in H_k(\Gamma_2)$ we have Andrianov's explicit formula [Andrianov 1974]:

$$a_F(I)L(s, F; \text{spin}) = \zeta\big(s+\tfrac{1}{2}\big)L\big(s+\tfrac{1}{2}, \chi_{-4}\big)\sum_{n=1}^{\infty}\frac{a_F(nI)}{n^s}, \tag{2-8}$$

where χ_{-4} is the nontrivial Dirichlet character modulo 4. From this formula it follows

$$a_F(I) = 0 \implies a_F(nI) = 0, \quad n \geq 1. \tag{2-9}$$

2.2. ***The standard L-function.*** The standard L-function attached to a Hecke eigenform $F \in H_k(\Gamma_2)$ is defined by a degree 5 Euler product

$$\begin{aligned} L(s, F; \text{std}) &= \prod_p \Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_{F,1}(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_{F,1}(p)^{-1}}{p^s}\Big)^{-1} \\ &\qquad\times\Big(1-\frac{\alpha_{F,2}(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_{F,2}(p)^{-1}}{p^s}\Big)^{-1}, \end{aligned} \tag{2-10}$$

which converges absolutely in some right half-plane. Using (2-1), we rewrite this Euler product as

$$\begin{aligned} L(s, F; \text{std}) &= \prod_p \Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_F(p)\beta_F(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_F(p)^{-1}\beta_F(p)}{p^s}\Big)^{-1} \\ &\qquad\times\Big(1-\frac{\alpha_F(p)\beta_F(p)^{-1}}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_F(p)^{-1}\beta_F(p)^{-1}}{p^s}\Big)^{-1}. \end{aligned} \tag{2-11}$$

The analytic continuation and functional equation of standard L-functions were worked out by S. Böcherer [1985]. He proved that $L(s, F; \text{std})$ extends to an entire function and satisfies a functional equation

$$\Lambda(s,F;\mathrm{std}) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+k-1)\Gamma_{\mathbb{C}}(s+k-2)L(s,F;\mathrm{std}) = \Lambda(1-s,F;\mathrm{std}), \tag{2-12}$$

where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

2.3. ***Combinatorial relations.*** For $F \in H_k(\Gamma_2)$, $m \geq 1$ and prime p, we set

$$c_m(p;F) = \alpha_F(p)^m + \alpha_F(p)^{-m} + \beta_F(p)^m + \beta_F(p)^{-m}, \tag{2-13}$$

$$\begin{aligned}\tau_{2m}(p;F) = 1 &+ \alpha_F(p)^m\beta_F(p)^m + \alpha_F(p)^m\beta_F(p)^{-m}\\ &+ \alpha_F(p)^{-m}\beta_F(p)^m + \alpha_F(p)^{-m}\beta_F(p)^{-m}\end{aligned} \tag{2-14}$$

to be the m-th power sum of local parameters of $L(s,F;\mathrm{spin})$ and $L(s,F;\mathrm{std})$ at p respectively.

The main goal of this section is to find expressions of these power sums in terms of Fourier coefficients of F at scalar matrices, for $m = 1, 2$, under the assumption that $a_F(I) \neq 0$. Note that the condition $a_F(I) \neq 0$ is not a direct consequence of $F \neq 0$, unlike in the elliptic case, where a primitive form f vanishes if and only if its first Fourier coefficient vanishes. In fact, determining whether $a_F(I) = 0$ or not is a difficult problem because ω_F is intimately connected to central values of spinor L-functions (Böcherer's conjecture, now a theorem proved by M. Furusawa and K. Morimoto [2021]). However, as we shall see later in Section 4, making this assumption here does no harm to our argument. Our result is as follows:

Lemma 2.1. *Let $F \in H_k(\Gamma_2)$ be a Hecke eigenform. For any prime p and $m \geq 1$, define $c_m(p;F)$ and $\tau_{2m}(p;F)$ as in (2-13) and (2-14). Assume that $a_F(I) \neq 0$, and set $U_m(p;F) = a_F(p^m I)/a_F(I)$. Also set $\lambda_p = 1 + \chi_{-4}(p)$ and $\mu_p = \chi_{-4}(p)$, where χ_{-4} is the nontrivial Dirichlet character modulo 4. Then we have*

$$c_1(p;F) = U_1(p;F) + \frac{\lambda_p}{\sqrt{p}},$$

$$c_2(p;F) = -U_1(p;F)^2 + 2U_2(p;F) + \frac{\lambda_p^2 - 2\mu_p}{p},$$

$$\tau_2(p;F) = U_1(p;F)^2 - U_2(p;F) + \frac{\lambda_p}{\sqrt{p}}U_1(p;F) + \frac{\mu_p}{p} - 1,$$

$$\begin{aligned}\tau_4(p;F) = &-U_3(p;F)U_1(p;F) + U_2(p;F)^2 + \frac{\lambda_p}{\sqrt{p}}U_2(p;F)U_1(p;F)\\ &+ \left(\frac{\mu_p}{p} - 1\right)U_1(p;F)^2 - \frac{\lambda_p}{\sqrt{p}}U_3(p;F) + \left(\frac{\lambda_p^2 - 2\mu_p}{p}\right)U_2(p;F)\\ &+ \left(\frac{\lambda_p\mu_p}{p^{3/2}} - 2\frac{\lambda_p}{\sqrt{p}}\right)U_1(p;F) + \frac{\mu_p^2}{p^2} - \frac{\lambda_p^2}{p} + 1.\end{aligned}$$

Remark 2.2. The key feature of Lemma 2.1 is that we are able to express $\tau_4(p; F)$ using polynomials of $U_m(p; F)$ of degree 2 (and not of higher degree). This is essential when we deal with weighted low-lying zeros of standard L-functions using Kitaoka's formula.

Proof. Throughout the proof, F and p are fixed. To save notation we use c_m, τ_{2m}, U_m, α_p, β_p to denote $c_m(p; F)$, $\tau_{2m}(p; F)$, $U_m(p; F)$, $\alpha_F(p)$, $\beta_F(p)$ respectively, with the understanding that they depend on F and p.

We start with Andrianov's explicit formula (2-8):

$$a_F(I)\,L(s, F; \text{spin}) = \zeta\left(s+\tfrac{1}{2}\right)L\left(s+\tfrac{1}{2}, \chi_{-4}\right)\sum_{n=1}^{\infty}\frac{a_F(nI)}{n^s}.$$

Using Euler product expansions for the L-functions involved, we see that the two Dirichlet series

$$\left(\prod_p\left(1-\frac{\alpha_p}{p^s}\right)\left(1-\frac{\beta_p}{p^s}\right)\left(1-\frac{\alpha_p^{-1}}{p^s}\right)\left(1-\frac{\beta_p^{-1}}{p^s}\right)\right)\cdot\left(\sum_{n=1}^{\infty}\frac{a_F(nI)\,a_F(I)^{-1}}{n^s}\right) \tag{2-15}$$

and

$$\prod_p\left(1-\frac{1}{p^{s+1/2}}\right)\left(1-\frac{\chi_{-4}(p)}{p^{s+1/2}}\right) \tag{2-16}$$

both converge absolutely in some right half-plane and are equal. Comparing coefficients of p^{-as}, $a = 1, 2, 3, 4$, we obtain

$$-\frac{\lambda_p}{\sqrt{p}} = U_1 - c_1, \tag{2-17}$$

$$\frac{\mu_p}{p} = U_2 - U_1 c_1 + \tau_2 + 1, \tag{2-18}$$

$$0 = U_3 - U_2 c_1 + U_1(\tau_2+1) - c_1, \tag{2-19}$$

$$0 = U_4 - U_3 c_1 + U_2(\tau_2+1) - U_1 c_1 + 1. \tag{2-20}$$

We also have elementary relations

$$c_1^2 = c_2 + 2(\tau_2+1), \tag{2-21}$$

$$(\tau_2+1)^2 = 3 + \tau_4 + 4\tau_2 + 2c_2. \tag{2-22}$$

From (2-17) and (2-18) we obtain directly

$$c_1 = U_1 + \frac{\lambda_p}{\sqrt{p}}, \tag{2-23}$$

$$\tau_2 = U_1^2 - U_2 + \frac{\lambda_p}{\sqrt{p}}U_1 + \frac{\mu_p}{p} - 1. \tag{2-24}$$

Combining (2-21), (2-23) and (2-24) we have

$$c_2 = -U_1^2 + 2U_2 + \frac{\lambda_p^2 - 2\mu_p}{p}. \tag{2-25}$$

Using (2-22), (2-24) and (2-25) we express τ_4 as

$$\begin{aligned}\tau_4 = U_1^4 - 2U_2U_1^2 + 2\frac{\lambda_p}{\sqrt{p}}U_1^3 + U_2^2 - 2\frac{\lambda_p}{\sqrt{p}}U_2U_1 + \left(\frac{\lambda_p^2 + 2\mu_p}{p} - 2\right)U_1^2 \\ - 2\frac{\mu_p}{p}U_2 + \left(2\frac{\lambda_p\mu_p}{p^{3/2}} - 4\frac{\lambda_p}{\sqrt{p}}\right)U_1 + \frac{\mu_p^2}{p^2} - 2\frac{\lambda_p^2}{p} + 1.\end{aligned} \tag{2-26}$$

However, to get the final form of τ_4, we must express U_1^4, $U_2U_1^2$ and U_1^3 using degree 2 polynomials in U_a ($a = 1, 2, 3, 4$). Combining (2-19), (2-23) and (2-24), we have

$$U_1^3 = 2U_2U_1 - \frac{\lambda_p}{\sqrt{p}}U_1^2 - U_3 + \frac{\lambda_p}{\sqrt{p}}U_2 + \left(1 - \frac{\mu_p}{p}\right)U_1 + \frac{\lambda_p}{\sqrt{p}}. \tag{2-27}$$

Likely, equations (2-20), (2-23) and (2-24) give us

$$U_2U_1^2 = U_3U_1 + U_2^2 - \frac{\lambda_p}{\sqrt{p}}U_2U_1 + U_1^2 - U_4 + \frac{\lambda_p}{\sqrt{p}}U_3 - \frac{\mu_p}{p}U_2 + \frac{\lambda_p}{\sqrt{p}}U_1 - 1. \tag{2-28}$$

Further, we multiply (2-27) by U_1 and apply (2-27), (2-28) to get

$$\begin{aligned}U_1^4 = U_3U_1 + 2U_2^2 - 3\frac{\lambda_p}{\sqrt{p}}U_2U_1 + \left(3 + \frac{\lambda_p^2 - \mu_p}{p}\right)U_1^2 - 2U_4 + 3\frac{\lambda_p}{\sqrt{p}}U_3 \\ - \left(\frac{\lambda_p^2 + 2\mu_p}{p}\right)U_2 + \left(2\frac{\lambda_p}{\sqrt{p}} + \frac{\lambda_p\mu_p}{p^{3/2}}\right)U_1 - \left(2 + \frac{\lambda_p^2}{p}\right).\end{aligned} \tag{2-29}$$

Finally, we insert (2-27), (2-28) and (2-29) into (2-26) to get

$$\begin{aligned}\tau_4 = -U_3U_1 + U_2^2 + \frac{\lambda_p}{\sqrt{p}}U_2U_1 + \left(\frac{\mu_p}{p} - 1\right)U_1^2 - \frac{\lambda_p}{\sqrt{p}}U_3 + \left(\frac{\lambda_p^2 - 2\mu_p}{p}\right)U_2 \\ + \left(\frac{\lambda_p\mu_p}{p^{3/2}} - 2\frac{\lambda_p}{\sqrt{p}}\right)U_1 + \frac{\mu_p^2}{p^2} - \frac{\lambda_p^2}{p} + 1. \quad \square\end{aligned} \tag{2-30}$$

3. Kitaoka's formula

The main tool used in this paper is a spectral summation formula of Petersson type. This formula was first proved by Y. Kitaoka [1984] by computing Fourier coefficients of Siegel Poincaré series. In this section we introduce this formula and consider an averaged (over weight) version of it.

We begin by introducing some notations. For $k \geq 6$ even, we set

$$c_k = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\,\Gamma\Big(k-\frac{3}{2}\Big)\Gamma(k-2). \tag{3-1}$$

For $T, Q \in \mathcal{T}$ we define

$$\Delta_k(T,Q) = \sum_{F \in H_k(\Gamma_2)} c_k \frac{a_F(T)\,a_F(Q)}{\|F\|^2}. \tag{3-2}$$

For a matrix $C \in M_2(\mathbb{Z})$ with $\det C \neq 0$ (we denote the set of such matrices by $\mathcal{C}$) and $Q, T \in \mathcal{T}$, define the symplectic Kloosterman sum to be

$$K(Q,T;C) = \sum_D e\big(\mathrm{Tr}(AC^{-1}Q + C^{-1}DT)\big), \tag{3-3}$$

where D runs through the set

$$\left\{ D \in M_2(\mathbb{Z}) \bmod C\Lambda : \begin{pmatrix} A & * \\ C & D \end{pmatrix} \in \Gamma_2 \right\}, \tag{3-4}$$

and Λ is the set of 2×2 symmetric integral matrices. By elementary divisor theory and Weil's bound for classical Kloosterman sums one has [Kitaoka 1984]:

$$|K(Q,T;C)| \leq |\det C|^{3/2}. \tag{3-5}$$

Remark 3.1. Optimal bounds for these symplectic Kloosterman sums were obtained in [Tóth 2013]. However, since applying Tóth's optimal bound does not improve our result, Kitaoka's bound suffices for our purpose.

For $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix}$, $S = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4 \end{pmatrix} \in \mathcal{T}$ and $c \geq 1$, we define another exponential sum:

$$\begin{aligned} &H^{\pm}(P,S;c) \\ &= \delta_{s_4=p_4} \sideset{}{^*}\sum_{d_1 \bmod c} \sum_{d_2 \bmod c} e\left(\frac{\bar{d}_1 s_4 d_2^2 \mp \bar{d}_1 p_2 d_2 + s_2 d_2 + \bar{d}_1 p_1 + d_1 s_1}{c} \mp \frac{p_2 s_2}{2 c s_4} \right). \end{aligned} \tag{3-6}$$

For these we have the trivial bound

$$|H^{\pm}(P,S;c)| \leq c^2. \tag{3-7}$$

For $P \in M_2(\mathbb{R})$ with positive eigenvalues $\lambda_1, \lambda_2 > 0$ we set

$$\mathcal{J}_{k-3/2}(P) = \int_0^{\pi/2} J_{k-3/2}(4\pi\sqrt{\lambda_1}\sin\theta)\,J_{k-3/2}(4\pi\sqrt{\lambda_2}\sin\theta)\sin\theta\,d\theta, \tag{3-8}$$

where $J_{k-3/2}$ is the usual J-Bessel function of half-integral order $k - \frac{3}{2}$. With these notation, we can now state Kitaoka's formula.

Lemma 3.2. *For $T, Q \in \mathcal{T}$ and $k \geq 6$ even, define $\Delta_k(T, Q)$ as in (3-2). Then*

$$\Delta_k(T, Q) = \frac{1}{8}|\mathrm{Aut}(T)|\left(\frac{\det Q}{\det T}\right)^{\frac{k}{2}-\frac{3}{4}} \delta_{Q\sim T} + \frac{\sqrt{2}\pi}{8} G_{1,k}(T, Q) + \pi^2 G_{2,k}(T, Q), \tag{3-9}$$

where $\mathrm{Aut}(T)$, $G_{1,k}(T, Q)$ *and* $G_{2,k}(T, Q)$ *are defined by*

$$\mathrm{Aut}(T) = \{U \in \mathrm{GL}_2(\mathbb{Z}) : U^T T U = T\}, \tag{3-10}$$

$$G_{1,k}(T, Q) = \sum_{\pm} \sum_{s=1}^{\infty} \sum_{c=1}^{\infty} \sum_{U,V} \frac{(-1)^{k/2}}{c^{3/2} s^{1/2}} \times H^{\pm}(UQU^T, V^{-1}TV^{-T}; c)\, J_{k-3/2}\left(\frac{4\pi\sqrt{\det(TQ)}}{cs}\right), \tag{3-11}$$

$$G_{2,k}(T, Q) = \sum_{C \in \mathcal{C}} \frac{K(Q, T; C)}{|\det C|^{3/2}} \mathcal{J}_{k-3/2}(TC^{-1}QC^{-T}). \tag{3-12}$$

Here $\sum_{U,V}$ in (3-11) is over $U = (u_{ij})/\{\pm I\}$, $V = (v_{ij}) \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$(u_{21}, u_{22})\, Q (u_{21}, u_{22})^T = (-v_{21}, v_{11})\, T (-v_{21}, v_{11})^T = s. \tag{3-13}$$

The delta symbol $\delta_{Q\sim T}$ is equal to 1 if Q and T are equivalent in the sense of quadratic forms, and is equal to 0 otherwise.

Remark 3.3. Following Kitaoka [1984], we call the three terms in (3-9) containing $\delta_{Q\sim T}$, $G_{1,k}(T, Q)$ and $G_{2,k}(T, Q)$ the diagonal term, the rank 1 term and the rank 2 term respectively. Note that the classical Petersson formula for elliptic modular forms contains only a diagonal term and an off-diagonal term.

Remark 3.4. As pointed out by V. Blomer (see Remark 1 in [Blomer 2019]), there are some numerical errors in Kitaoka's original derivation of Kitaoka's formula. The version that we present here is based on Lemma 1 in [Blomer 2019]. However, our results do not depend on exact values of those constants.

The main purpose of this section is to establish the following averaged Kitaoka's formula, which is asymptotic in nature.

Lemma 3.5. *Let $m, n \geq 1$ be positive integers such that $m \mid n$. For $k \geq 6$ even, define $\Delta_k(mI, nI)$ as in (3-2). Let $\Omega \in C_c^\infty(0, \infty)$ be such that $\Omega \geq 0$, not identically 0. Then for large $K > 0$ we have*

$$\left(\sum_k \Omega\left(\frac{k}{K}\right)\right)^{-1} \sum_k \Omega\left(\frac{k}{K}\right) \Delta_k(mI, nI) = \delta_{m=n} + O_{j,\epsilon,\Omega}\left(\left(\frac{m^{(3/2)-\epsilon} n^{-(1/2)+\epsilon}}{K^4}\right) + \left(\frac{(mn)^{2+\epsilon}}{K^{5+2\epsilon}}\right) + \left(\frac{(mn)^{(j/2)+1}}{K^{2j+3}}\right)\right) \tag{3-14}$$

for any $j \geq 3$ and $\epsilon > 0$ small. Here $\sum_k$ is over positive even integers $k \geq 6$.

Remark 3.6. Our Lemma 3.5 can be viewed as a GSp_4 analogue of the classical averaged Petersson formula on GL_2; see (5.81) in [Iwaniec 1997]. The main difficulty is the presence of a product of two Bessel functions (instead of a single Bessel function), each of half-integral order (instead of integral order). As we shall see in the proof below, this can be overcome by applying an integral representation (3-33) of a product of two Bessel functions. A related averaged Kitaoka's formula was also discussed in recent work of G. Felber [2023].

Proof. After applying Kitaoka's formula (3-9), we divide the left side of (3-14) into three terms. We also set $g(x) = \Omega(x/K)$ and $\ell = k - \frac{3}{2}$ to save notation.

We denote the contribution of the diagonal term by R_0. Thus

$$R_0 = \frac{1}{8}\Big(\sum_k g(k)\Big)^{-1} \sum_k g(k)|\mathrm{Aut}(mI)|\Big(\frac{n}{m}\Big)^{\ell} \delta_{mI\sim nI}. \tag{3-15}$$

Note that mI and nI define the same quadratic form if and only if $m = n$, and that $|\mathrm{Aut}(mI)| = 8$. Thus the above expression reduces to $R_0 = \delta_{m=n}$.

Denote by R_1 the sum of the rank 1 term over k. We have

$$R_1 = \sum_k g(k) \sum_{\pm} \sum_{s=1}^{\infty} \sum_{c=1}^{\infty} \sum_{U,V} \frac{(-1)^{k/2}}{c^{3/2} s^{1/2}} \times H^{\pm}(nUU^T, mV^{-1}V^{-T}; c)\, J_\ell\Big(\frac{4\pi mn}{cs}\Big), \tag{3-16}$$

where the sum $\sum_{U,V}$ is over

$$n(u_{21}^2 + u_{22}^2) = m(v_{11}^2 + v_{21}^2) = s. \tag{3-17}$$

So in particular $n \mid s$. Making change of variable $s \mapsto ns$, we may rewrite R_1 as

$$R_1 = \sum_k g(k) \sum_{\pm} \sum_{s=1}^{\infty} \sum_{c=1}^{\infty} \sum_{U,V} \frac{(-1)^{k/2}}{c^{3/2} (ns)^{1/2}} \times H^{\pm}(nUU^T, mV^{-1}V^{-T}; c)\, J_\ell\Big(\frac{4\pi m}{cs}\Big), \tag{3-18}$$

where $\sum_{U,V}$ is over

$$u_{21}^2 + u_{22}^2 = s, \quad v_{11}^2 + v_{22}^2 = \frac{n}{m} s. \tag{3-19}$$

These equations have $O(s^\epsilon)$ and $O\big((\frac{n}{m}s)^\epsilon\big)$ integral solutions, respectively, for any $\epsilon > 0$, by the fact

$$|\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = s\}| = O(s^\epsilon). \tag{3-20}$$

In view of the estimate

$$J_\ell(x) \ll \left(\frac{x}{\ell}\right)^\ell, \quad x > 0,\ \ell > \tfrac{1}{2}, \tag{3-21}$$

which follows immediately from the integral representation [Gradshteyn and Ryzhik 2015, (8.411.4)], we may cut-off the sum in (3-18) by $sc \ll m/K$ up to a negligible error. In this range we change summation order and deal with the inner sum

$$\sum_k g(k)(-1)^{k/2} J_\ell\left(\frac{4\pi m}{cs}\right) \tag{3-22}$$

by applying Lemma 20 of [Blomer and Corbett 2022] to obtain

$$\begin{aligned}\sum_k g(k)(-1)^{k/2} J_\ell\left(\frac{4\pi m}{cs}\right) \\ = \omega_0\left(\frac{4\pi m}{cs}\right) + e^{\frac{4\pi i m}{cs}}\omega_+\left(\frac{4\pi m}{cs}\right) + e^{-\frac{4\pi i m}{cs}}\omega_-\left(\frac{4\pi m}{cs}\right),\end{aligned} \tag{3-23}$$

where $\omega_0(x)$, $\omega_\pm(x)$ are some smooth functions on $(0, \infty)$ satisfying

$$\omega_0(x) \ll_A K^{-A}, \tag{3-24}$$

$$\omega_\pm(x) \ll_A \left(1 + \frac{K^2}{x}\right)^{-A} \tag{3-25}$$

for any $A > 0$. The contribution of the ω_0 term is negligible, while the contribution of $\omega_\pm$ term depends on the size of $x = 4\pi m/cs$. For example, for $x \le K^2$ (i.e., $cs \gg m/K^2$), we have

$$\omega_+(x) \ll_A \left(1 + \frac{K^2}{x}\right)^{-A} \le \left(\frac{K^2}{x}\right)^{-A} \ll_A K^{-2A} m^A (cs)^{-A} \tag{3-26}$$

for any $A > 0$. This estimate, together with (3-7) and (3-20), give rise to

$$\begin{aligned} R_1^{+, cs \gg \frac{m}{K^2}} \\ &= \sum_\pm \sum_{\frac{m}{K^2} \ll sc \ll \frac{m}{K}} \sum_{U,V} \frac{1}{c^{3/2}(ns)^{1/2}} H^\pm(nUU^T, mV^{-1}V^{-T}; c)\, \omega_+\left(\frac{4\pi m}{cs}\right) \\ &\ll_{\epsilon, A} \sum_{\frac{m}{K^2} \ll sc \ll \frac{m}{K}} c^{-3/2}(ns)^{-1/2} s^\epsilon \left(\frac{n}{m}s\right)^\epsilon c^2 K^{-2A} m^A c^{-A} s^{-A} \\ &\ll_\epsilon m^{(3/2)-\epsilon} n^{-(1/2)+\epsilon} K^{-3} \end{aligned} \tag{3-27}$$

for any small $\epsilon > 0$ if one fixes some $A > \frac{3}{2}$. The case where $cs \ll m/K^2$ is analyzed similarly, and its contribution to R_1 is again at most $m^{\frac{3}{2}-\epsilon} n^{-\frac{1}{2}+\epsilon} K^{-3}$. Therefore, we have obtained that

$$R_1 \ll_\epsilon m^{\frac{3}{2}-\epsilon} n^{-\frac{1}{2}+\epsilon} K^{-3} \tag{3-28}$$

for any small $\epsilon > 0$.

Denote by R_2 the sum of the rank 2 term over k. Explicitly,

$$R_2 = \sum_k g(k) \sum_{C \in \mathcal{C}} \frac{K(nI, mI; C)}{|\det C|^{3/2}} \int_0^{\pi/2} J_\ell(4\pi\sqrt{\lambda_1}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_2}\sin\theta) \sin\theta \, d\theta, \tag{3-29}$$

where λ_1, λ_2 are eigenvalues of the matrix $mnC^{-1}C^{-T}$. We set $\lambda_{\min}$ and $\lambda_{\max}$ to be the smaller and the larger eigenvalue of $mnC^{-1}C^{-T}$ respectively. Denote by $\|\cdot\|_F$ the Frobenius matrix norm. Then by Lemma 2 in [Blomer 2019] we have

$$\lambda_{\min} \ll \frac{mn}{\|C\|_F^2}. \tag{3-30}$$

Applying this estimate and (3-21) to $J_\ell(4\pi\sqrt{\lambda_{\min}}\sin\theta)$, and applying the estimate

$$J_\ell(x) \ll 1, \quad x > 0,\ \ell > \tfrac{1}{2} \tag{3-31}$$

that follows from [Gradshteyn and Ryzhik 2015, (8.411.13)] to $J_\ell(4\pi\sqrt{\lambda_{\max}}\sin\theta)$, we may cut-off the sum in R_2 by $\|C\|_F \ll \sqrt{mn}/K$ up to an negligible error. In this range we change the summation order and deal with the inner sum

$$\sum_k g(k)\, J_\ell(4\pi\sqrt{\lambda_1}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_2}\sin\theta) \tag{3-32}$$

by making use of the following integral representation of product of two Bessel functions [Erdélyi et al. 1981, page 47, (8)]:

$$J_v(z)\, J_v(\zeta) = \frac{2}{\pi} \int_0^{\pi/2} \cos((z-\zeta)\cos\alpha)\, J_{2v}(2\sqrt{z\zeta}\sin\alpha)\, d\alpha, \tag{3-33}$$

when $\Re(v) > -\frac{1}{2}$, $z > 0$, $\zeta > 0$. Choosing $v = \ell$, $z = 4\pi\sqrt{\lambda_1}\sin\theta$, $\zeta = 4\pi\sqrt{\lambda_2}\sin\theta$, and setting

$$\xi = 2\sqrt{z\zeta}\sin\alpha = \frac{8\pi\sqrt{mn}}{\sqrt{|\det C|}} \sin\theta \sin\alpha, \tag{3-34}$$

we obtain

$$\sum_k g(k)\, J_\ell(z)\, J_\ell(\zeta) = \frac{2}{\pi} \int_0^{\pi/2} \cos((z-\zeta)\cos\alpha) \Big(\sum_k g(k)\, J_{2k-3}(\xi) \Big)\, d\alpha. \tag{3-35}$$

Let $r = 2k-3$. We have $r \equiv 1 \bmod 4$, since k is even. Setting $g_1(x) = g\left(\frac{x+3}{2}\right)$, we have

$$\sum_k g(k)\, J_{2k-3}(\xi) = \sum_{r\equiv 1 \bmod 4} g_1(r)\, J_r(\xi). \tag{3-36}$$

From here the method of Neumann series can be applied, in view of the following integral representation of Bessel functions of integral order [Gradshteyn and Ryzhik 2015, (8.411.1)]:

$$J_r(x) = \int_{-1/2}^{1/2} e(rt)\, e^{-ix\sin 2\pi t}\, dt. \tag{3-37}$$

We quote the following result (Lemma 5.8 in [Iwaniec 1997]):

$$4 \sum_{r\equiv 1 \bmod 4} g_1(r)\, J_r(\xi) = g_1(\xi) + h(\xi) + O(\xi c_3(g_1)), \tag{3-38}$$

where $h(\xi)$ and $c_3(g_1)$ are defined by

$$h(\xi) = \int_0^\infty g_1(\sqrt{2\xi y})\sin\left(\xi + y - \frac{\pi}{4}\right)(\pi y)^{-1/2}\, dy, \tag{3-39}$$

$$c_3(g_1) = \int_{-\infty}^\infty |\hat{g_1}(t)t^3|\, dt. \tag{3-40}$$

We refer readers to Section 5.5 in [Iwaniec 1997] for a proof of (3-38). Recall that for g_1 we have

$$g_1^{(j)}(x) \ll_j K^{-j} \tag{3-41}$$

for any $j \geq 0$. Thus by repeated partial integration we have

$$h(\xi) \ll_j (\xi K^{-2})^j, \tag{3-42}$$

$$c_3(g_1) \ll K^{-3}. \tag{3-43}$$

See also (5.73) and (5.74) in [Iwaniec 1997].

The contribution of $g_1(\xi)$ to R_2 is

$$R_2^{g_1(\xi)} = \sum_{\|C\|_F \ll \frac{\sqrt{mn}}{K}} \frac{K(nI, mI; C)}{|\det C|^{3/2}} \int_0^{\pi/2}\int_0^{\pi/2} \cos((z-\zeta)\cos\alpha)g_1(\xi)\, d\alpha \sin\theta\, d\theta. \tag{3-44}$$

In view of the support of g_1, the sum in (3-44) is confined in the range

$$\xi = \frac{8\pi\sqrt{mn}}{\sqrt{|\det C|}}\sin\theta\sin\alpha \gg K. \tag{3-45}$$

Thus we have $\det(C) \ll mn/K^2$. By the estimate (3-5), we obtain

$$R_2^{g_1(\xi)} \ll \sum_{\substack{0\neq|\det C|\ll \frac{mn}{K^2} \\ \|C\|_F \ll \frac{\sqrt{mn}}{K}}} 1 = \sum_{0\neq|d|\ll \frac{mn}{K^2}} \sum_{\substack{\det C=d \\ \|C\|\ll \frac{\sqrt{mn}}{K}}} 1 = \sum_{0\neq|d|\ll \frac{mn}{K^2}} P_d\Big(C\cdot\frac{mn}{K^2}\Big), \tag{3-46}$$

where $P_d(X)$ is the hyperbolic lattice counting function

$$P_d(X) = |\{(\alpha,\beta,\gamma,\delta)\in\mathbb{Z}^4 : \alpha\delta-\beta\gamma = d,\ \alpha^2+\beta^2+\gamma^2+\delta^2 \le X\}| \tag{3-47}$$

and $C>0$ is some constant. For $1\le d\le X$ we have the following asymptotic formula (see Theorem 12.4 in [Iwaniec 2002]):

$$P_d(X) = 6\Big(\sum_{\tau|d}\tau^{-1}\Big)(X + O(d^{\frac{1}{3}}X^{\frac{2}{3}})) \ll X\log|d|. \tag{3-48}$$

This estimate also applies to $-X\le d\le -1$ by symmetry. Thus we have

$$R_2^{g_1(\xi)} \le \frac{mn}{K^2}\sum_{0\neq|d|\ll\frac{mn}{K^2}} \log|d| \ll \frac{m^2n^2}{K^4}\log\frac{mn}{K^2} \ll_\epsilon \Big(\frac{(mn)^{2+\epsilon}}{K^{4+2\epsilon}}\Big). \tag{3-49}$$

The contributions of $h(\xi)$ and $O(\xi c_3(g_1))$ are analyzed similarly, making use of the bounds (3-42) and (3-43). We have

$$R_2^{h(\xi)} \ll_j \frac{(mn)^{(j/2)+1}}{K^{2j+2}}, \tag{3-50}$$

$$R_2^{O(\xi c_3(g_1))} \ll_\epsilon \frac{(mn)^{2+\epsilon}}{K^{6+2\epsilon}} \tag{3-51}$$

for any $j\ge 3$ and small $\epsilon>0$. Thus we obtain

$$R_2 \ll_{j,\epsilon} \frac{(mn)^{2+\epsilon}}{K^{4+2\epsilon}} + \frac{(mn)^{(j/2)+1}}{K^{2j+2}}. \tag{3-52}$$

Combining the estimates of R_1 and R_2 above, and that

$$\sum_k g(k) = \sum_k \Omega\Big(\frac{k}{K}\Big) \gg K, \tag{3-53}$$

by our choice of Ω, the proof is now complete. □

4. Proof of main theorems

In this section we prove Theorems 1.2–1.8. We assume $\hat{\Phi}$ is supported in $(-\alpha,\alpha)$. We also set $\ell = k-\frac{3}{2}$ to save notation.

4.1. *Proof of Theorem 1.2.* By standard argument using the explicit formula [Iwaniec et al. 2000, Section 4], we write the density function $D(F;\Phi;\text{spin})$ as

$$D(F;\Phi;\text{spin}) = \frac{2}{2\pi i}\int_{(2)} \Phi\Big(\frac{s-1/2}{2\pi i}\log k^2\Big)\frac{\Lambda'}{\Lambda}(s,F;\text{spin})\,ds + 2\Phi\Big(\frac{\log k^2}{2\pi i}\Big)\delta_{F\in H_k^*(\Gamma_2)}, \tag{4-1}$$

where $\delta_{F\in H_k^*(\Gamma_2)}=1$ if $F\in H_k^*(\Gamma_2)$ is a Saito–Kurokawa lift (in which case $L(s,F;\text{spin})$ has a pole at $s=\frac{3}{2}$) and is 0 otherwise. By (2-6) and (2-5) we may further write

$$\begin{aligned} D(F;\Phi;\text{spin}) &= \frac{2}{\log k^2}\int_{\mathbb{R}}\Phi(x)\Big(-\log(2\pi)^2+\frac{\Gamma'}{\Gamma}\Big(1+\frac{2\pi i x}{\log k^2}\Big)+\frac{\Gamma'}{\Gamma}\Big(k-1+\frac{2\pi i x}{\log k^2}\Big)\Big)dx \\ &\quad -\frac{2}{\log k^2}\sum_{m=1}^{\infty}\sum_p c_m(p;F)\frac{\log p}{p^{m/2}}\hat{\Phi}\Big(\frac{m\log p}{\log k^2}\Big)+2\Phi\Big(\frac{\log k^2}{2\pi i}\Big)\delta_{F\in H_k^*(\Gamma_2)} \end{aligned} \tag{4-2}$$

by shifting contour from $\sigma=2$ to $\sigma=\frac{1}{2}$.

For the integral involving gamma factors, we use the following estimate [Gradshteyn and Ryzhik 2015, (8.363.4)]:

$$\frac{\Gamma'}{\Gamma}(a+bi)+\frac{\Gamma'}{\Gamma}(a-bi)=2\frac{\Gamma'}{\Gamma}(a)+O\Big(\frac{b^2}{a^2}\Big),\quad a>0,\ b\in\mathbb{R} \tag{4-3}$$

and the fact that $\Phi(x)$ is even to get

$$\frac{2}{\log k^2}\int_{\mathbb{R}}\Phi(x)\Big(-\log(2\pi)^2+\frac{\Gamma'}{\Gamma}\Big(1+\frac{2\pi i x}{\log k^2}\Big)+\frac{\Gamma'}{\Gamma}\Big(k-1+\frac{2\pi i x}{\log k^2}\Big)\Big)dx = \hat{\Phi}(0)+o(1). \tag{4-4}$$

This is done by splitting the integral over $\mathbb{R}$ to two integrals on $(-\infty,0)$ and $(0,\infty)$. Then we use the fact that $\Phi(x)$ is even, estimate (4-3), and the estimate $\frac{\Gamma'}{\Gamma}(k)=\log k+O(1)$.

For $c_1(p;F)$ and $c_2(p;F)$ we sum over F against the weight ω_F. Using (1-12), Lemma 2.1 and equation (2-9) we obtain

$$\sum_{F\in H_k(\Gamma_2)}\omega_F\, c_1(p;F)=\Delta_k(pI,I)+\frac{\lambda_p}{\sqrt{p}}\Delta_k(I,I), \tag{4-5}$$

$$\sum_{F\in H_k(\Gamma_2)}\omega_F\, c_2(p;F)=-\Delta_k(pI,pI)+2\Delta_k(p^2I,I)+O\Big(\frac{1}{p}\Big)\Delta_k(I,I). \tag{4-6}$$

Collecting these we have the “explicit formula”:

$$\begin{aligned}
(4\text{-}7)\quad & \sum_{F\in H_k(\Gamma_2)} \omega_F\, D(F;\Phi;\text{spin}) \\
&= \hat{\Phi}(0)+o(1)-\frac{2}{\log k^2}\sum_p\Big(\Delta_k(pI,I)+\frac{\lambda_p}{\sqrt{p}}\Delta_k(I,I)\Big)\frac{\log p}{\sqrt{p}}\hat{\Phi}\Big(\frac{\log p}{\log k^2}\Big) \\
&\quad -\frac{2}{\log k^2}\sum_p\Big(-\Delta_k(pI,pI)+2\Delta_k(p^2I,I)+O\Big(\frac{1}{p}\Big)\Delta_k(I,I)\Big) \\
&\qquad \times\frac{\log p}{p}\hat{\Phi}\Big(\frac{2\log p}{\log k^2}\Big) \\
&\quad -\frac{2}{\log k^2}\sum_{m=3}^{\infty}\sum_p\Big(\sum_{F\in H_k(\Gamma_2)}\omega_F\, c_m(p;F)\Big)\frac{\log p}{p^{m/2}}\hat{\Phi}\Big(\frac{m\log p}{\log k^2}\Big) \\
&\quad +2\Phi\Big(\frac{\log k^2}{2\pi i}\Big)\sum_{F_f\in H_k^*(\Gamma_2)}\omega_{F_f}.
\end{aligned}$$

We treat the terms $\Delta_k(pI,I)$, $\Delta_k(pI,pI)$ and $\Delta_k(p^2I,I)$ using Kitaoka’s formula (3-9). Take the term $\Delta_k(pI,I)$ for example:

$$(4\text{-}8)\qquad \Delta_k(pI,I)=\frac{\sqrt{2}\pi}{8}G_{1,k}(pI,I)+\pi^2G_{2,k}(pI,I).$$

The rank 1 term $G_{1,k}(pI,I)$ is

$$G_{1,k}(pI,I)=\sum_{\pm}\sum_{s=1}^{\infty}\sum_{c=1}^{\infty}\sum_{U,V}\frac{(-1)^{k/2}}{c^{3/2}(ps)^{1/2}}H^{\pm}(UU^T,pV^{-1}V^{-T};c)\,J_\ell\Big(\frac{4\pi}{cs}\Big),$$

after a change of variable $s\mapsto ps$, where the summation $\sum_{U,V}$ is over

$$(4\text{-}9)\qquad u_{21}^2+u_{22}^2=ps,\qquad v_{11}^2+v_{21}^2=s.$$

By the estimates (3-7), (3-21) and (3-20), we bound $G_{1,k}(pI,I)$ as

$$(4\text{-}10)\qquad G_{1,k}(pI,I)\ll\sum_{s=1}^{\infty}\sum_{c=1}^{\infty}(ps)^{\epsilon}s^{\epsilon}c^{-\frac{3}{2}}p^{-\frac{1}{2}}s^{-\frac{1}{2}}c^2\Big(\frac{4\pi}{cs\ell}\Big)^{\ell}\ll p^{-\frac{1}{2}+\epsilon}\Big(\frac{4\pi}{\ell}\Big)^{\ell}$$

for k sufficiently large. Thus its contribution to (4-7) is at most

$$\begin{aligned}
(4\text{-}11)\quad & \frac{1}{\log k}\sum_p p^{-\frac{1}{2}+\epsilon}\Big(\frac{4\pi}{\ell}\Big)^{\ell}\frac{\log p}{\sqrt{p}}\hat{\Phi}\Big(\frac{\log p}{\log k^2}\Big) \\
&\ll\frac{1}{\log k}\Big(\frac{4\pi}{\ell}\Big)^{\ell}\sum_{p\le k^{2\alpha}}p^{-1+\epsilon}\log p\ll\frac{1}{\log k}\Big(\frac{4\pi}{\ell}\Big)^{\ell}k^{2\alpha\epsilon}=o(1)
\end{aligned}$$

for any $\alpha>0$ as $k\to\infty$.

The rank 2 term $G_{2,k}(pI, I)$ is

$$G_{2,k}(pI, I) = \sum_{C\in\mathcal{C}} \frac{K(I, pI; C)}{|\det C|^{3/2}} \int_0^{\pi/2} J_\ell(4\pi\sqrt{\lambda_{\min}}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_{\max}}\sin\theta)\sin\theta\, d\theta, \tag{4-12}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smaller and larger eigenvalues of $pC^{-1}C^{-T}$ respectively. By estimates (3-5), (3-21), (3-31) and (3-30), we have

$$G_{2,k}(pI, I) \ll \sum_{C\in\mathcal{C}} \left(\frac{4\pi\sqrt{p}\sin\theta}{\ell\|C\|_F}\right)^{\ell} \ll p^{\ell/2}\left(\frac{4\pi}{\ell}\right)^{\ell} \tag{4-13}$$

for k sufficiently large. Thus its contribution to (4-7) is at most

$$\begin{aligned}\frac{1}{\log k}\sum_p p^{\ell/2}\left(\frac{4\pi}{\ell}\right)^{\ell}\frac{\log p}{\sqrt{p}}\hat{\Phi}\left(\frac{\log p}{\log k^2}\right) &\ll \frac{1}{\log k}\left(\frac{4\pi}{\ell}\right)^{\ell}\sum_{p\le k^{2\alpha}} p^{\frac{\ell-1}{2}}\log p \\ &\ll k^{\alpha}\left(\frac{4\pi k^{\alpha}}{\ell}\right)^{\ell},\end{aligned} \tag{4-14}$$

which goes to 0 as $k\to\infty$ when $\alpha<1$. Thus we have proved the contribution of $\Delta_k(I, pI)$ to (4-7) is small when $\alpha<1$. Other off-diagonal contributions are estimated similarly, and are all small when $\alpha<1$. We skip the details here.

The diagonal contribution of $\frac{\lambda_p}{\sqrt{p}}\Delta_k(I, I)$ to (4-7) from the $m=1$ term is

$$\begin{aligned} -\frac{2}{\log k^2}\sum_p \frac{\lambda_p\log p}{p}\hat{\Phi}\left(\frac{\log p}{\log k^2}\right) &= -\frac{2}{\log k^2}\int_1^\infty \frac{\log x}{x}\hat{\Phi}\left(\frac{\log x}{\log k^2}\right) d\pi(x) + o(1) \\ &= -\frac{2}{\log k^2}\int_1^\infty \frac{\log x}{x}\hat{\Phi}\left(\frac{\log x}{\log k^2}\right)\frac{1}{\log x}dx + o(1) \\ &= -2\int_0^\infty \hat{\Phi}(y)\,dy + o(1) \\ &= -\Phi(0) + o(1).\end{aligned} \tag{4-15}$$

Here we have used the prime number theorem (PNT) for the prime counting function $\pi(x)$ and the fact the $\lambda_p = 1+\chi_{-4}(p)$ takes values 0 and 2 for primes p with density $\frac{1}{2}$ each.

The diagonal contribution of $-\Delta_k(pI, pI)$ from the $m=2$ term is

$$\frac{2}{\log k^2}\sum_p \frac{\log p}{p}\hat{\Phi}\left(\frac{2\log p}{\log k^2}\right) = \frac{\Phi(0)}{2} + o(1). \tag{4-16}$$

It is shown in [Kowalski et al. 2012] that the $m \geq 3$ terms in (4-7) contribute at most $O\left(\frac{1}{\log k}\right)$. For non-Saito–Kurokawa lifts F this follows from the Ramanujan bound $|c_m(p; F)| \leq 4$ and the fact

$$\sum_{m=3}^{\infty} \sum_{p} \frac{\log p}{p^{m/2}} < \infty. \tag{4-17}$$

For a treatment of Saito–Kurokawa lifts, we refer readers to Section 5 in [Kowalski et al. 2012].

For the last term in (4-7) we use the fact that (see, for example, page 1754 in [Blomer 2019])

$$\omega_{F_f} \ll \frac{1}{k^3} \frac{L\left(\frac{1}{2}, f \times \chi_{-4}\right)}{L(1, \operatorname{sym}^2 f)}. \tag{4-18}$$

This, combined with the convexity bound for $L\left(\frac{1}{2}, f \times \chi_{-4}\right)$ and the lower bound [Hoffstein and Lockhart 1994]

$$L(1, \operatorname{sym}^2 f) \gg k^{-\epsilon} \tag{4-19}$$

give us

$$\sum_{F_f \in H_k^*(\Gamma_2)} \omega_{F_f} = o(1). \tag{4-20}$$

Combining all results above, we finally have

$$\begin{aligned} \sum_{F \in H_k(\Gamma_2)} \omega_F \, D(F; \Phi; \text{spin}) &= \hat{\Phi}(0) - \Phi(0) + \frac{\Phi(0)}{2} + o(1) \\ &= \hat{\Phi}(0) - \frac{\Phi(0)}{2} + o(1) \end{aligned} \tag{4-21}$$

for $\alpha < 1$, as $k \to \infty$. This completes the proof of Theorem 1.2.

4.2. *Proof of Theorem 1.6.* The proof is similar to that of Theorem 1.2. The explicit formula for $D(F; \Phi; \text{std})$ is

$$\begin{aligned} &D(F; \Phi; \text{std}) \\ &= \frac{2}{\log k^4} \int_{\mathbb{R}} \Phi(x) \left(-2 \log(2\pi) - \tfrac{1}{2} \log \pi + \tfrac{1}{2} \frac{\Gamma'}{\Gamma} \left(\tfrac{1}{4} + \frac{\pi i x}{\log k^4} \right) \right) dx \\ &\quad + \frac{2}{\log k^4} \int_{\mathbb{R}} \Phi(x) \left(\frac{\Gamma'}{\Gamma} \left(k - \tfrac{1}{2} + \frac{2\pi i x}{\log k^4} \right) + \frac{\Gamma'}{\Gamma} \left(k - \tfrac{3}{2} + \frac{2\pi i x}{\log k^4} \right) \right) dx \\ &\quad - \frac{2}{\log k^4} \sum_{m=1}^{\infty} \sum_{p} \tau_{2m}(p; F) \frac{\log p}{p^{m/2}} \hat{\Phi}\left(\frac{m \log p}{\log k^4} \right). \end{aligned} \tag{4-22}$$

The integrals of gamma factors and the sum over $m \geq 3$ can be computed similarly as before. We thus have the explicit formula for the weighted sum

$$\begin{aligned}(4\text{-}23)\quad &\sum_{F\in H_k(\Gamma_2)} \omega_F D(F;\Phi;\mathrm{std})\\ &= \hat{\Phi}(0) - \frac{2}{\log k^4}\sum_{m=1}^{\infty}\sum_{p}\sum_{F\in H_k(\Gamma_2)} \omega_F \tau_{2m}(p;F)\frac{\log p}{p^{m/2}}\hat{\Phi}\left(\frac{m\log p}{\log k^4}\right) + o(1).\end{aligned}$$

We have seen in the proof of Theorem 1.2 that the symmetry type is determined by diagonal contributions. So we shall concentrate on those terms and be brief about the rest.

By Lemma 2.1, the $m = 1$ term is

$$\begin{aligned}(4\text{-}24)\quad &\sum_{F\in H_k(\Gamma_2)} \omega_F \tau_2(p;F)\\ &= \Delta_k(pI,pI) - \Delta_k(p^2I,I) + \frac{\lambda_p}{\sqrt{p}}\Delta_k(pI,I) + \left(\frac{\mu_p}{p}-1\right)\Delta_k(I,I),\end{aligned}$$

in which the diagonal contribution of $\Delta_k(pI, pI)$ and $-\Delta_k(I, I)$ cancel each other.

The $m = 2$ term is

$$\begin{aligned}(4\text{-}25)\quad \sum_{F\in H_k(\Gamma_2)} \omega_F \tau_4(p;F) &= -\Delta_k(p^3I,pI) + \Delta_k(p^2I,p^2I)\\ &\quad + \frac{\lambda_p}{\sqrt{p}}\Delta_k(p^2I,pI) + \left(\frac{\mu_p}{p}-1\right)\Delta_k(pI,pI)\\ &\quad - \frac{\lambda_p}{\sqrt{p}}\Delta_k(p^3I,I) + O\left(\frac{1}{p}\right)\Delta_k(p^2I,I)\\ &\quad + O\left(\frac{1}{\sqrt{p}}\right)\Delta_k(pI,I) + \left(O\left(\frac{1}{p}\right)+1\right)\Delta_k(I,I).\end{aligned}$$

The diagonal contribution from $\Delta_k(p^2I, p^2I)$, $-\Delta_k(pI, pI)$ and $\Delta_k(I, I)$ combined is

$$(4\text{-}26)\qquad -\frac{2}{\log k^4}\sum_{p}\frac{\log p}{p}\hat{\Phi}\left(\frac{2\log p}{\log k^4}\right) = -\frac{\Phi(0)}{2} + o(1).$$

To illustrate why the range of support is restricted to $\left(-\frac{1}{4},\frac{1}{4}\right)$, we analyze the contribution of the rank 2 term $G_{2,k}(pI, pI)$:

$$\begin{aligned}(4\text{-}27)\quad &G_{2,k}(pI,pI)\\ &= \sum_{C\in\mathcal{C}}\frac{K(pI,pI;C)}{|\det C|^{3/2}}\int_0^{\pi/2} J_\ell(4\pi\sqrt{\lambda_{\min}}\sin\theta)\, J_\ell(4\pi\sqrt{\lambda_{\max}}\sin\theta)\sin\theta\, d\theta,\end{aligned}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are eigenvalues of $p^2C^{-1}C^{-T}$. We estimate as before to get

$$G_{2,k}(pI, pI) \ll \sum_{C\in\mathcal{C}} \left(\frac{4\pi p\sin\theta}{\ell\|C\|_F}\right)^{\ell} \ll p^{\ell}\left(\frac{4\pi}{\ell}\right)^{\ell}. \tag{4-28}$$

Thus it contributes at most

$$\frac{1}{\log k}\sum_p p^{\ell}\left(\frac{4\pi}{\ell}\right)^{\ell}\frac{\log p}{\sqrt{p}}\hat{\Phi}\left(\frac{\log p}{\log k^4}\right) \ll \frac{1}{\log k}\left(\frac{4\pi}{\ell}\right)^{\ell}\sum_{p\le k^{4\alpha}} p^{\ell-\frac{1}{2}}\log p \ll k^{2\alpha}\left(\frac{4\pi k^{4\alpha}}{\ell}\right)^{\ell}, \tag{4-29}$$

which is $o(1)$ as $k\to\infty$ if $\alpha < \frac{1}{4}$. Other off-diagonal terms are estimated similarly.

4.3. ***Proof of Theorem 1.8.*** The contribution of gamma factors and the diagonal contribution do not change upon averaging over k with respect to Ω. To illustrate how we may extend the range of support form $\left(-\frac{1}{4}, \frac{1}{4}\right)$ to $\left(-\frac{5}{18}, \frac{5}{18}\right)$, we take the term $\Delta_k(pI, pI)$ for example.

By Lemma 3.5, the off-diagonal part of

$$\left(\sum_k \Omega\left(\frac{k}{K}\right)\right)^{-1}\sum_k \Omega\left(\frac{k}{K}\right)\Delta_k(pI, pI) \tag{4-30}$$

is at most

$$\frac{p}{K^4} + \frac{p^{4+2\epsilon}}{K^{5+2\epsilon}} + \frac{p^{j+2}}{K^{2j+3}} \tag{4-31}$$

for any $j \ge 3$ and small $\epsilon > 0$. It contributes at most

$$\frac{1}{\log K}\sum_{p\le K^{4\alpha}}\left(\frac{p}{K^4} + \frac{p^{4+2\epsilon}}{K^{5+2\epsilon}} + \frac{p^{j+2}}{K^{2j+3}}\right)\frac{\log p}{\sqrt{p}} \ll K^{6\alpha-4} + K^{18\alpha-5+6\epsilon} + K^{(4j+10)\alpha-(2j+3)}, \tag{4-32}$$

which is $o(1)$ if $\alpha < \frac{5}{18}$, by taking j sufficiently large.

Finally, to see

$$\hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty}\Phi(x)W(\mathrm{Sp})(x)\,dx, \tag{4-33}$$

we use the Plancherel formula

$$\int_{-\infty}^{\infty}\Phi(x)W(\mathrm{Sp})(x)\,dx = \int_{-\infty}^{\infty}\hat{\Phi}(y)\hat{W}(\mathrm{Sp})(y)\,dy \tag{4-34}$$

and the Fourier pair (1-6). The proof is now complete.

5. Application to nonvanishing of central values

5.1. ***Proof of Corollary 1.4.*** By Theorem 1.2 we have

$$\sum_{F\in H_k(\Gamma_2)} \omega_F \sum_{\rho_{F,\mathrm{spin}}} \Phi\left(\frac{\gamma_{F,\mathrm{spin}}}{2\pi}\log k^2\right) < \hat{\Phi}(0) - \frac{\Phi(0)}{2} + \epsilon \tag{5-1}$$

for any $\epsilon > 0$ and k large enough. We further assume

$$\Phi(x) \geq 0, \quad \Phi(0) = 1. \tag{5-2}$$

By these conditions we may pick up only the zeros $\rho_{F,\mathrm{spin}} = \frac{1}{2}$ to get

$$\begin{aligned}\sum_{F\in H_k(\Gamma_2)} \omega_F \sum_{\rho_{F,\mathrm{spin}}} \Phi\left(\frac{\gamma_{F,\mathrm{spin}}}{2\pi}\log k^2\right) &\geq \sum_{F\in H_k(\Gamma_2)} \omega_F \cdot \mathrm{ord}_{s=1/2} L(s, F; \mathrm{spin}) \\ &= \sum_{m=2}^{\infty} m \sum_{\mathrm{ord}_{s=1/2} L(s,F;\mathrm{spin})=m} \omega_F \\ &\geq 2 \sum_{\mathrm{ord}_{s=1/2} L(s,F;\mathrm{spin})\geq 2} \omega_F.\end{aligned} \tag{5-3}$$

Here we have used the fact that the root number of $L(s, F; \mathrm{spin})$ is always $+1$. Thus the vanishing order of $L(s, F; \mathrm{spin})$ at $s = \frac{1}{2}$ is even. These inequalities, together with (1-14), give us

$$\sum_{L(1/2,F;\mathrm{spin})\neq 0} \omega_F > 1 - \tfrac{1}{2}\left(\hat{\Phi}(0) - \frac{\Phi(0)}{2}\right) - \epsilon. \tag{5-4}$$

It is discussed in [Iwaniec et al. 2000, Appendix A] that the Fourier pair

$$\Phi(x) = \left(\frac{\sin \pi v x}{\pi v x}\right)^2, \qquad \hat{\Phi}(y) = \frac{1}{v}\left(1 - \frac{|y|}{v}\right), \quad |y| < v \ \ (v > 0). \tag{5-5}$$

gives essentially the optimal bound. With this choice we have

$$\sum_{L(1/2,F;\mathrm{spin})\neq 0} \omega_F > \frac{5}{4} - \frac{1}{2v} - \epsilon \tag{5-6}$$

for any $0 < v < 1$. Taking liminf in k and $v \to 1$, we have

$$\liminf_{k\to\infty} \sum_{L(1/2,F;\mathrm{spin})\neq 0} \omega_F \geq \frac{3}{4}. \tag{5-7}$$

We can further ignore the contribution of Saito–Kurokawa lifts, in view of (4-20). This completes the proof of Corollary 1.4.

5.2. ***Further discussion.*** From the proof of Corollary 1.4 we see that in order to obtain any result on nonvanishing of central values of $L(s, F; \text{spin})$ or $L(s, F; \text{std})$, the range of support in the corresponding Density Theorem must go beyond $\left(-\frac{2}{5}, \frac{2}{5}\right)$. This range is by setting

$$\frac{5}{4} - \frac{1}{2v} = 0. \tag{5-8}$$

The previous range of support $\left(-\frac{4}{15}, \frac{4}{15}\right)$ obtained in [Kowalski et al. 2012] for spinor L-functions is not large enough, for $\frac{4}{15} < \frac{2}{5}$. Thus our extension to $(-1, 1)$ is significant for the purpose of nonvanishing.

Unfortunately, for standard L-functions, our range of support is still not large enough to obtain a nonvanishing result, even after performing an average over weight $\left(\frac{5}{18} < \frac{2}{5}\right)$. The author would like to address this problem by establishing a more refined version of Lemma 3.5 in the future.

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