Pacific Journal of Mathematics

REPRESENTATIONS OF $\mathrm{SL}_2(F)$

Guy Henniart and Marie-France Vignéras

Volume 335 No. 2 April 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 335, No. 2, 2025
https://doi.org/10.2140/pjm.2025.335.229

REPRESENTATIONS OF $\mathrm{SL}_2(F)$

GUY HENNIART AND MARIE-FRANCE VIGNÉRAS

Let p be a prime number, F a nonarchimedean local field with residue field k_F of characteristic p, and R an algebraically closed field of characteristic different from p. We investigate the irreducible smooth R-representations of $\mathrm{SL}_2(F)$. The components of an irreducible smooth R-representation Π of $\mathrm{GL}_2(F)$ restricted to $\mathrm{SL}_2(F)$ form an L-packet $L(\Pi)$. We use the classification of such Π to determine the cardinality of $L(\Pi)$, which is 1, 2 or 4. When $p=2$ we have to use the Langlands correspondence for $\mathrm{GL}_2(F)$. When ℓ is a prime number distinct from p and $R=\mathbb{Q}_\ell^{\mathrm{ac}}$, we determine the behaviour of an integral L-packet under reduction modulo ℓ. We prove a Langlands correspondence for $\mathrm{SL}_2(F)$, and an enhanced one when the characteristic of R is not 2. Finally, pursuing a theme of Henniart and Vignéras (2024), which studied the case of inner forms of $\mathrm{GL}_n(F)$, we show that near identity a nontrivial irreducible smooth R-representation π of $\mathrm{SL}_2(F)$ is, up to a finite-dimensional representation, isomorphic to a sum of 1, 2 or 4 representations in an L-packet of size 4 (when p is odd there is only one such L-packet). We show that for π in an L-packet of size r_π and a sufficiently large integer j, the dimension of the invariants of π by the j-th congruence subgroup of an Iwahori or a pro-p Iwahori subgroup of $\mathrm{SL}_2(F)$ is equal to $a_\pi+2r_\pi^{-1}|k_F|^j$, with $a_\pi=-\frac{1}{2}$ if p is odd and $r_\pi=4$, otherwise a_π is an integer. We also study the fixed points by the j-th congruence subgroups of the maximal compact subgroups of $\mathrm{SL}_2(F)$ where the answer depends on the parity of j.

MSC2020: primary 22E50; secondary 11F70.
Keywords: modular irreducible representations, L-packets, Whittaker spaces, local Langlands correspondence.

1. Introduction

1.1. Let F be a locally compact nonarchimedean field with residue characteristic p and R an algebraically closed field of characteristic $\mathrm{char}_R \neq p$. We investigate the irreducible smooth R-representations of $\mathrm{SL}_2(F)$. Although when $R = \mathbb{C}$ and p is odd the first investigations appeared in the 1960s, in work of Gelfand–Graev and Shalika, the study of the modular case (i.e., when $\mathrm{char}_R > 0$) started only recently [Cui 2023; Cui et al. 2024] when $\mathrm{char}_F \neq 2$ and $\mathrm{char}_R \neq 2$. Here we give a complete treatment and we make no assumption on p, char_F, char_R, apart from $\mathrm{char}_R \neq p$.

As Labesse and Langlands did in the 1970s when $R = \mathbb{C}$ and $\mathrm{char}_F = 0$, we use the restriction of smooth R-representations from $G = \mathrm{GL}_2(F)$ to $G' = \mathrm{SL}_2(F)$. We prove that an irreducible smooth R-representation of G' extends to a smooth representation of an open subgroup H of G containing ZG' where Z is the centre of G, and appears in the restriction to G' of a smooth irreducible R-representation of G, unique up to isomorphism and twist by smooth R-characters of G/G'. When $\mathrm{char}_F \neq 2$ we can simply take $H = ZG'$, but not when $\mathrm{char}_F = 2$ because the compact quotient G/ZG' is infinite. Those results follow from general facts about R-representations, which appear in Section 2. They apply to more general reductive groups over F, as we show in Section 3.

In Section 4, using Whittaker models, we show that the restriction to G' of an irreducible smooth R-representation Π of G is semisimple and has finite length and multiplicity one. Its irreducible components form an L-packet $L(\Pi)$. An L-packet $L(\Pi)$ is called cuspidal when Π is cuspidal, supercuspidal when Π is supercuspidal, of level 0 if Π can be chosen to have level 0 (that is, having nonzero fixed vectors under $1 + M_2(P_F)$), and of positive level otherwise.

Theorem 1.1. *The size of an L-packet is* 1, 2 *or* 4.

When p is odd that follows rather easily from $|G/ZG'| = 4$, but it is also true when $p = 2$, in which case the proof is completed only in Proposition 4.22, and uses the Langlands R-correspondence for G, which we recall in Section 4.4.

Proposition 1.2 (Corollary 4.29, Proposition 4.22). *The L-packets of size* 4 *are cuspidal and in bijection with the biquadratic separable extensions of F.*

The bijection is described in the proof. When $p \neq 2$ there is just one L-packet of size 4 and it has level 0. When $p = 2$ the L-packets of size 4 have positive level, their number is finite if $\mathrm{char}_F = 0$, but there are infinitely many if $\mathrm{char}_F = 2$.

Proposition 1.3 (Proposition 4.7). *When p is odd, the cuspidal L-packets are not singletons. When $p = 2$, the cuspidal L-packets of level* 0 *have size* 2.

Proposition 1.4 (Proposition 4.28). *There is a cuspidal nonsupercuspidal L-packet if and only if $q + 1 = 0$ in R. It is unique of level* 0, *and size* 4 *when* $\mathrm{char}_R = 2$, *and size* 2 *when* $\mathrm{char}_R \neq 2$.

From the Langlands R-correspondence for $\mathrm{GL}_2(F)$, we get a bijection from the set of L-packets to the set of conjugacy classes of Deligne morphisms of W_F into $\mathrm{PGL}_2(R)$, the dual group of SL_2 over R. When $\mathrm{char}_R \neq 2$, we even get an enhanced Langlands correspondence, in that we parametrize the elements in an L-packet $L(\Pi)$ by the characters of the group S_Π of connected components of the centralizer C_Π of the image of the corresponding Deligne morphism in $\mathrm{PGL}_2(R)$. When $\mathrm{char}_R = 2$, C_Π is always connected and the supercuspidal L-packets are not singletons. We will determine explicitly C_Π for each Π.

Theorem 1.5 (Theorem 5.2[1]). *Let Π be an irreducible smooth R-representation of* $\mathrm{GL}_2(F)$.

When $\mathrm{char}_R \neq 2$, *the component group S_Π of C_Π is isomorphic to* $\{1\}, \mathbb{Z}/2\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

When $\mathrm{char}_R = 2$, *C_Π is connected for each Π, but the cardinality of the L-packet $L(\Pi)$ is*

- 1 *if Π is not cuspidal*,
- 2 *if Π is supercuspidal*,
- 4 *if Π is cuspidal not supercuspidal.*

When $L(\Pi)$ is not a singleton, we take as a base point the element having a nonzero Whittaker model with respect to a nontrivial smooth R-character of F. When $\mathrm{char}_R \neq 2$, the theorem gives a bijection

$$\iota : L(\Pi) \to \mathrm{Irr}_R(S_\Pi)$$

respecting the base points (the trivial representation in $\mathrm{Irr}_R(S_\Pi)$). It is unique when $|L(\Pi)| = 2$. There are partial results on the uniqueness of ι when $|L(\Pi)| = 4$. Under the restriction $p = 2$, $\mathrm{char}_F = 0$, for the complex L-packet of size 4 (unique, of level 0), there is a unique bijection compatible with the endoscopic character identities [Aubert and Plymen 2024].

When $\mathrm{char}_R = 2$, a "linkage" between irreducible smooth R-representations of G and G' is introduced in [Treumann and Venkatesh 2016]. In §5.0.3 we interpret this notion in terms of dual groups, thus proving their conjectures in a special case.

Let $\ell \neq p$ be a prime number, and $\mathbb{Q}_\ell^{\mathrm{ac}}$ an algebraic closure of $\mathbb{Q}_\ell$ with residue field $\mathbb{F}_\ell^{\mathrm{ac}}$. Each irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation of $\mathrm{GL}_2(F)$ lifts to a smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation. We show that this remains true for $\mathrm{SL}_2(F)$.

Proposition 1.6 (Corollary 4.24, Proposition 4.30). *Each irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation π of* $\mathrm{SL}_2(F)$ *is the reduction modulo ℓ of an integral irreducible smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\tilde{\pi}$ of* $\mathrm{SL}_2(F)$.

[1]When $R = \mathbb{C}$ this was already established by Gelbart and Knapp [1982, §4] assuming that it could be done for $\mathrm{GL}_n(F)$.

An equivalent formulation is that each irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation Π of $\mathrm{GL}_2(F)$ is the reduction modulo ℓ of an integral irreducible smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\widetilde{\Pi}$ of $\mathrm{GL}_2(F)$ such that

$$|L(\Pi)| = |L(\widetilde{\Pi})|.$$

The reduction modulo ℓ of each integral supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation of $\mathrm{GL}_2(F)$ is irreducible, but this is not true for $\mathrm{SL}_2(F)$. Each supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\tilde{\pi}$ of $\mathrm{SL}_2(F)$ is integral and we determine all the cases of reducibility. We choose a supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\widetilde{\Pi}$ of $\mathrm{GL}_2(F)$ such that $\tilde{\pi} \in L(\widetilde{\Pi})$ and denote by $\sigma_{\widetilde{\Pi}}$ the irreducible 2-dimensional $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation of W_F image of $\widetilde{\Pi}$ by the local Langlands correspondence.

Proposition 1.7 (Corollary 4.24). *The reduction modulo ℓ of $\tilde{\pi}$ has length ≤ 2. The length is 2 if and only if*

$$p = 2, \quad \sigma_{\widetilde{\Pi}} = \mathrm{ind}_{W_E}^{W_F} \tilde{\xi}, \quad \tilde{\xi}(b) \neq 1, \quad \tilde{\xi}(b)^{\ell^s} = 1,$$
$$\ell^s \text{ divides } q+1, \quad \text{the order of } (\tilde{\xi}^\tau/\tilde{\xi})|_{1+P_E} \text{ is } 2,$$

where b is a root of unity of order $q+1$ in a quadratic unramified extension E/F, $\tilde{\xi}$ is a smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-character of E^* (of W_E via class field theory), and $\tau \in \mathrm{Gal}(E/F)$ is not trivial.

Finally we study for G' the problem that we treated in [Henniart and Vignéras 2024] for inner forms of $\mathrm{GL}_n(F)$. An infinite-dimensional irreducible smooth R-representation Π of $G = \mathrm{GL}_2(F)$ is isomorphic near the identity to $a_\Pi 1 + \mathrm{ind}_B^G 1$ where a_Π is an integer (its value is given in Proposition 7.5) and $\mathrm{ind}_B^G 1$ is the usual principal series. For an infinite-dimensional irreducible smooth R-representation π of G', we show that up to finitely many trivial R-characters, π is isomorphic near the identity to the sum of 1, 2 or 4 elements of an L-packet of size 4.

Theorem 1.8 (Theorem 6.17). *Let π be an infinite-dimensional irreducible and smooth R-representation of G'. There are irreducible smooth R-representations $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ of G' forming an L-packet, and an integer a_0, such that on a small enough compact open subgroup K of G' we have*

$$\pi \simeq a_0 1 + \sum_{i=1}^{4/r} \tau_i,$$

where r is the size of the L-packet containing π.

For $R = \mathbb{C}$ and p odd, Monica Nevins has similar results which are more precise in that the subgroup K is large. We show that her results carry over to any R (§6.2.8).

As in [Henniart and Vignéras 2024] we first deal with the case where $R=\mathbb{C}$, using a germ expansion near the identity à la Harish-Chandra, in terms of nilpotent orbital integrals. However, when $\mathrm{char}_F=2$, such an expansion is not available, so we work instead with a complex representation π of an open subgroup H of G containing ZG'. For such a group a germ expansion has been obtained by Lemaire [2004]. Adapting [Mœglin and Waldspurger 1987] and [Varma 2014] (who assumed $\mathrm{char}_F=0$) we compute the germ expansion in terms of the dimensions of the different Whittaker models of π, and express it in terms of L-packets of size 4. Theorem 1.8 easily transfers to any R with $\mathrm{char}_R=0$, in particular $R=\mathbb{Q}_\ell^{\mathrm{ac}}$. From our complete classification of irreducible smooth R-representations of G', and in particular that the $\mathbb{F}_\ell^{\mathrm{ac}}$-representations of G' lift to characteristic 0 when $\ell\neq p$ (Proposition 1.6), we get Theorem 1.8 for $R=\mathbb{F}_\ell^{\mathrm{ac}}$ and transfer it to any R with $\mathrm{char}_R=\ell$.

We think that Theorem 1.8 will extend in the same way to inner forms of SL_n, using the work of [Hiraga and Saito 2012]. We expect that if $\mathrm{char}_F=0$ and $R=\mathbb{C}$, a variant of the theorem is true for any connected reductive F-group $\underline{H}$, because of the Harish-Chandra germ expansion and of the work of Mœglin–Waldspurger and Varma. But when $\ell\neq p$, it is not known in general if virtual finite length $\mathbb{F}_\ell^{\mathrm{ac}}$-representations lift to characteristic 0 and it is unlikely that cuspidal irreducible $\mathbb{F}_\ell^{\mathrm{ac}}$-representations lift. The reason is that the first point has a positive answer when G is a finite group and the answer to the second is negative in general for finite reductive groups. When $\mathrm{char}_F=p$ and $R=\mathbb{C}$, we have to face the problem that a germ expansion in terms of nilpotent orbital integrals might not exist. It is not clear how to define such integrals for bad primes, and sometimes the number of unipotent orbits in H and of nilpotent orbits in $\mathrm{Lie}(H)$ are not the same, even over an algebraic closure of F. Given our investigation of the case $\mathrm{SL}_2(F)$, which uses L-indistinguishability, one may wonder about the role of endoscopy and stability in analogous results for a general H.

The dimension of the invariants by the j-th congruence subgroup of a Moy–Prasad group of an infinite-dimensional irreducible smooth R-representation of G for j large, is the value at q^j of a polynomial of degree 1 and integral coefficients. We will prove a similar result for G' but the coefficients of the polynomial are not always integral and the polynomial may depend on the parity of j.

Let Π be an infinite-dimensional irreducible smooth R-representation of G and π be an element of $L(\Pi)$. Around the identity,

$$\Pi\simeq a_\Pi 1+\mathrm{ind}_B^G 1$$

for an integer a_Π and the usual principal series $\mathrm{ind}_B^G 1$. Let O_F denote the ring of integers of F, $K'=\mathrm{SL}_2(O_F)$, I' its Iwahori subgroup, $I'_{1/2}$ its pro-p Iwahori, and K'_j, I'_j, $I'_{1/2+j}$ their j-th congruence subgroups.

Theorem 1.9 (Theorem 7.6). *For a sufficiently large j,*

$$\dim_R \pi^{I'_j} = \dim_R \pi^{I'_{1/2+j}} = |L(\Pi)|^{-1}(a_\Pi + 2q^j),$$
$$\dim_R \pi^{K'_j} = |L(\Pi)|^{-1}(a_\Pi + (q+1)q^{j-1}) \quad \text{if } \Pi|_{ZKG'} \text{ is irreducible.}$$

When p is odd and $|L(\Pi)| = 4$, we have $|L(\Pi)|^{-1} a_\Pi = -\frac{1}{2}$.

When $\Pi|_{ZKG'}$ is reducible, it has length 2. The two irreducible components Π^+ and Π^- are distinguished by their Whittaker models.

Theorem 1.10 (Corollary 7.10). *If $\Pi|_{ZKG'}$ is reducible, for a sufficiently large j,*

$$\dim_R \pi^{K'_j} = \begin{cases} |L(\Pi)|^{-1}(a_\Pi + 2q^j) & \text{for } j \text{ odd and } \pi \subset \Pi^+|_{G'} \text{ or } j \text{ even and } \pi \subset \Pi^-|_{G'}, \\ |L(\Pi)|^{-1}(a_\Pi + 2q^{j-1}) & \text{otherwise.} \end{cases}$$

By G-conjugation, we have similar asymptotics for all Moy–Prasad subgroups of G'.

The study of R-representations of G' has a long history, especially when $R = \mathbb{C}$. Even for odd p and $R = \mathbb{C}$, there is current research on GL_2 and SL_2 [Luo and Chau 2024]. Inevitably some of our proofs are adapted from previous papers. However, because we make only the assumption that $\mathrm{char}_R \neq p$, we have usually preferred to give complete proofs in that general setting. We refer essentially only to papers that we are using.

2. Generalities

2.1. Let R be a field, G a group, H a subgroup of G, V an R-representation of G. We denote char_R the characteristic of R, and $V|_H$ the restriction of V to H.

2.1.1. When H has *finite index* in G, any irreducible R-representation of H is contained in the restriction to H of an irreducible R-representation of G [Henniart 2001, proposition 2.2].

2.1.2. If H is *normal of finite index* in G and V is irreducible, then $V|_H$ is semisimple of finite length [loc. cit., proposition 2.1].

2.1.3. If H is *normal* in G, V is irreducible and $V|_H$ contains an irreducible subrepresentation, then $V|_H$ is semisimple and its isotypic components are G-conjugate with the same multiplicity.

Proof. Let W be an irreducible subrepresentation of $V|_H$. Since H is normal in G, for $g \in G$, H acts irreducibly on gW by $(h, gw) \mapsto hgh^{-1}hw$. The subspace $\sum_{g \in G} gW$ is a nonzero subrepresentation of V. Since V is irreducible, it is equal to V. Since a representation which is a sum of irreducible subrepresentations is semisimple [Bourbaki 2012, §4.1, corollaire 1, p. 52], $V|_H$ is semisimple. The last assertion follows in the same way. □

2.1.4. Assume H *normal of finite index* in G and let π be an irreducible R-representation of H. We saw that there is an irreducible R-representation Π of G whose restriction to H (which is semisimple of finite length) contains π. Clearly if χ is a R-character of G trivial on H then the restriction of $\Pi \otimes \chi$ to H contains π.

Lemma 2.1. *Assume R algebraically closed and G/H* ***abelian****. Any irreducible R-representation Π' of G containing π is isomorphic to $\Pi \otimes \chi$ for some R-character χ of G trivial on H.*

Proof. [2]We have $\mathrm{Hom}_H(\Pi'|_H, \Pi|_H) \neq 0$. The right adjoint of the restriction from G to H is the induction Ind_H^G from H to G, therefore Π' is isomorphic to an irreducible subrepresentation of $\mathrm{Ind}_H^G(\Pi|_H)$. We have $\mathrm{Ind}_H^G(\Pi|_H) \simeq (\mathrm{Ind}_H^G 1) \otimes \Pi$ because G/H is finite, and the irreducible subquotients of $\mathrm{Ind}_H^G 1$ are the characters χ of G trivial on H because R is algebraically closed. Therefore, there exists χ such that $\Pi' \simeq \Pi \otimes \chi$. □

2.2. We suppose that H is a *closed* subgroup of a *locally profinite* group G and V is an R-representation of G.

If the index of H in G is finite, then H is open. Conversely, if H is open cocompact in G, then the index of H in G is finite. If V is smooth (i.e., the G-stabilizer of any vector is open), then $V|_H$ is smooth. Conversely, if H is open in G and $V|_H$ is smooth (resp. admissible: smooth and the dimension of the space V^K of K-fixed vectors of V is finite, for any open compact subgroup $K \subset H$), then V is smooth (resp. admissible).

We suppose also from now on that H is *normal in G with a compact quotient G/H and that V is smooth* (so $V|_H$ is smooth).

2.2.1. If V is *finitely generated* then $V|_H$ is finitely generated [Henniart 2001, lemme 4.1].

2.2.2. If V is *irreducible*, any irreducible subrepresentation of $V|_H$ (when there exists one) extends to a (smooth and irreducible) representation of an open subgroup of G of finite index which is admissible if V is as well [loc. cit., proposition 4.4].

2.2.3. If V is *irreducible and $V|_H$ contains an irreducible subrepresentation or is noetherian* (any subrepresentation is finitely generated), then $V|_H$ is semisimple of finite length [loc. cit., théorème 4.2].

We introduce the two properties:

(2-1) Any finitely generated admissible R-representation of G has finite length.

(2-2) Any finitely generated smooth R-representation of H is noetherian.

[2]This proof was suggested by Peiyi Cui [2023, Proposition 2.6], and replaces a more complicated argument of ours.

2.2.4. Let W be an admissible irreducible R-representation of H.

(1) If (2-1) *and* (2-2) *are true*, then W is contained in some irreducible admissible R-representation of G restricted to H [Henniart 2001, corollaire 4.6].

(2) If (2-1) *is true*, then W is a quotient of some irreducible admissible R-representation of G restricted to H [loc. cit., théorème 4.5].

We give a simple proof of (2) adapted from [Tadić 1992, Proposition 2.2]. The smooth induction $\mathrm{Ind}_H^G W$ of W to G is admissible since W is as well and G/H is compact [Vignéras 1996, chapitre I, §5.6]. A finitely generated subrepresentation of $\mathrm{Ind}_H^G W$ is admissible, hence of finite length by (2-1). So $\mathrm{Ind}_H^G W$ contains an irreducible admissible representation U. The restriction to H is the left adjoint of the induction Ind_H^G hence W is a quotient of $U|_H$.

2.2.5. Let X_V be the group of R-characters χ of G trivial on H such that $V\otimes\chi\simeq V$. The characters in X_V are smooth by the following lemma.

Lemma 2.2. *$V\otimes\chi$ is smooth if and only if χ is smooth.*

Proof. Let $v\in V$ a nonzero element. An open subgroup $K\subset G$ fixing v in V, fixes v in $V\otimes\chi$ if and only if χ is trivial on K. The lemma follows because V is smooth. □

2.2.6. Assume also that V is *irreducible and $V|_H$ has finite length* (semisimple by §2.2.3 and its isotypic components are G-conjugate).[3]

Let W be an irreducible component of $V|_H$, π its isomorphism class, G_π the G-stabilizer of π. Let V_π be the π-isotypic component of $V|_H$. The G-stabilizer of V_π is G_π. The G-stabilizer of W is open in G (because it contains the G-stabilizer of $v\in W$ nonzero and V is smooth) and is contained in G_π. Both have finite index in G (G/H is compact) and

$$V=\mathrm{Ind}_{G_\pi}^G(V_\pi)$$

by Clifford's theory. The representation of G_π on V_π is irreducible and the length of $V|_H$ is

$$\mathrm{lg}(V|_H)=[G:G_\pi]\,\mathrm{lg}(V_\pi|_H).$$

Lemma 2.3. *Assume that G/H is abelian. Then*:

(1) *G_π is normal in G and does not depend on the choice of π in $V|_H$. The smooth R-characters of G trivial on G_π are in X_V.*

(2) *Assume R algebraically closed.*

[3] This subsection generalizes [Cui 2023, Corollary 3.8.3; Tadić 1992, Corollary 2.5; Bushnell and Kutzko 1994, Corollary 1.6(iii)].

(a) *Any irreducible subquotient of the smooth induction* $\mathrm{Ind}_H^G 1$ *is a smooth R-character χ of G trivial on H.*

(b) *Any irreducible R-representation of G containing π is a twist $V\otimes\chi$ of V by some smooth R-character χ of G trivial on H.*

(3) *When $V|_H$ has multiplicity* 1, *then $W = V_\pi$, for a smooth R-character χ of G trivial on H, $V\otimes\chi\simeq V$ if and only if χ is trivial on G_π, and G_π is the largest subgroup I of G containing H such that* $\lg(V|_I) = \lg(V|_H)$.

(4) *When R is algebraically closed and $V|_H$ has multiplicity* 1, *then*

$$|X_V| = \begin{cases} [G:G_\pi] & \text{if } \mathrm{char}_R = 0,\\ [G:G_{\pi,\ell}] & \text{if } \mathrm{char}_R = \ell > 0,\end{cases}$$

where $G_{\pi,\ell}$ is the smallest subgroup of G containing G_π such that $[G:G_{\pi,\ell}]$ is relatively prime to ℓ.

Proof. (1) The isotypic components of $\Pi|_H$ are G-conjugate, their G-stabilizers are G-conjugate and contain H hence they are equal because G/H is abelian.

Since $V\otimes\chi\simeq\mathrm{Ind}_{G_\pi}^G(\chi|_{G_\pi}\otimes V_\pi)$ for any smooth R-character χ of G, the smooth R-characters of G trivial on $G(\pi)$ are in X_V.

(2) (a) For any closed subgroup Q of G and a smooth R-representation X of Q, the representation $\mathrm{Ind}_Q^G X$ is the space of functions $f: G\to X$ with the property $f(qgk) = qf(g)$ for $q\in Q$, $g\in G$, $k\in K_f$ for some open subgroup K_f of G, with the action of G by right translation, and where $\mathrm{ind}_Q^G 1$ is the subrepresentation on the subspace of functions of compact support modulo Q. When G/Q is compact, $\mathrm{Ind}_Q^G X = \mathrm{ind}_Q^G X$.

Let $V\supset U$ be G-stable subspaces with V/U irreducible. We can suppose V generated by an element f (indeed $V'/U'\simeq V/U$ for the G-stable space V' generated by $f\in V\setminus U$ and the kernel U' of the map $V'\to V/U$). There is an open subgroup K of G which fixes f. We have $U\subset V\subset\mathrm{ind}_K^G 1$ and one is reduced to the case where G/H is finite.

(b) The proof of Lemma 2.1 remains valid with the smooth induction Ind_H^G, which is the smooth compact induction $\mathrm{ind}_H^G 1$, because G/H is compact, so that $\mathrm{ind}_H^G(\Pi|_H) = \Pi\otimes\mathrm{ind}_H^G 1$.

(3) Any smooth character χ of G trivial on H with $\mathrm{ind}_{G_\pi}^G(V_\pi)\simeq\mathrm{ind}_{G_\pi}^G(V_\pi\otimes\chi|_{G_\pi})$ is trivial on G_π. Indeed, restricting to G_π we see that $V_\pi\otimes\chi|_{G_\pi}$ is conjugate to V_π by some $g\in G$. Restricting to H gives that $\pi\simeq\pi^g$, so $g\in G_\pi$, hence $V_\pi\otimes\chi|_{G_\pi}\simeq V_\pi$. As $\mathrm{Ker}(\chi)$ is open in G and G/H is compact, $J = \mathrm{Ker}(\chi)\cap G_\pi$ has finite index in G_π. If χ is not trivial on G_π then the action of J on V_π is reducible. Indeed, $\mathrm{ind}_J^{G_\pi}(1)$ contains subrepresentations 1 and $\chi|_{G_\pi}$, and by Frobenius reciprocity $\mathrm{End}_J(V_\pi|_J)$ is equal to $\mathrm{Hom}_{G_\pi}(V_\pi, \mathrm{ind}_J^{G_\pi}(V_\pi|_J)) = \mathrm{Hom}_{G_\pi}(V_\pi, V_\pi\otimes\mathrm{ind}_J^{G_\pi}(1))$.

Hence $\dim(\mathrm{End}_{J_\pi}(V_\pi|_J)) \geq 2$ and $V_\pi|_J$ is reducible. By the hypothesis of multiplicity 1, $V_\pi|_H$ is irreducible, hence $V_\pi|_J$ is irreducible as $H \subset J$. So χ is trivial on G_π.

The group G_π is a subgroup I of G containing H with $\lg(V|_I) = \lg(V|_H)$. If I has this property, the restriction to H of any irreducible component on $V|_I$ is irreducible, hence I is contained in G_π.

(4) follows from (3). □

Remark 2.4. Assume that $V|_H$ has multiplicity 1. The G-stabilizer of any irreducible component of V is G_π. Denote $G_\pi = G_V$. Let I be a subgroup of G containing H. The number of orbits of I in the irreducible components of $V|_{G_V}$ is $\lg(V|_I)$. This number is the same for I and IG_V, hence $\lg(V|_I) = \lg(V|_{IG_V})$. We deduce that $G_V \subset I$ if $V|_I$ is reducible and $|G/I|$ is a prime number.

Let θ be a smooth R-representation of a closed subgroup $U \subset H$. We consider the property:

(2-3) The functor $\mathrm{Hom}_U(-, \theta)$ is exact on smooth R-representations of H.

Lemma 2.5. *If* (2-3) *is true and* $\dim \mathrm{Hom}_U(V, \theta) = 1$, *then* $V|_H$ *has multiplicity* 1.

Proof. We denote by $m_V(\pi)$ the multiplicity of any irreducible smooth R-representation π of H in $V|_H$. By (2-3),

$$\sum_\pi m_V(\pi) \dim \mathrm{Hom}_U(\pi, \theta) = \dim \mathrm{Hom}_U(V, \theta) = 1.$$

There is a single π with $m_V(\pi) = \dim \mathrm{Hom}_U(V, \theta) = 1$. □

3. *p*-adic reductive group

Suppose now that G is a p-adic reductive group, that is, the group of rational points $\underline{G}(F)$ of a reductive connected F-group $\underline{G}$. Here F is a local nonarchimedean field of residual characteristic p, ring of integers O_F, uniformizer p_F, maximal ideal P_F, residue field $k_F = O_F/P_F$ with q elements, and absolute value $|x|_F = q^{-\mathrm{val}(x)}$, $|p_F|_F = q^{-1}$ (we do not suppose that the characteristic of F is 0).

For an algebraic group $\underline{X}$ over F, we denote by the corresponding unadorned letter $X = \underline{X}(F)$ the group of its F-points.

Let R be a field of characteristic $\mathrm{char}_R \neq p$. Any irreducible smooth R-representation of G is admissible [Henniart and Vignéras 2019], and the properties (2-1) and (2-2) hold for G. For (2-1) see [Vignéras 1996, chapitre II, §5.10; 2023, §5], and for (2-2) see [Dat 2009; Dat et al. 2024].

Lemma 3.1. *Let* $f : \underline{H} \to \underline{G}$ *be an* F*-morphism of reductive connected* F*-groups. Then the subgroup* $f(H)$ *of* G *is closed.*

Proof. The morphism f induces a constructible action of H on G [Bernstein and Zelevinsky 1976, §6.15, Theorem A]; in particular the group $f(H)$, which is the H-orbit of the unit of G, is locally closed [loc. cit., Proposition 6.8], $f(H)$ is equal to its closure in G (the closure of $f(H)$ in G is a subgroup containing $f(H)$ as an open, hence closed, subgroup). Note that $f(H)$ is open in G when $\mathrm{char}_F = 0$ [Platonov and Rapinchuk 1994, §3.1, Corollary 1]. □

Theorem 3.2. *Let $f : \underline{H} \to \underline{G}$ be an F-morphism of reductive connected F-groups such that $f(H)$ is a normal subgroup of G of compact quotient $G/f(H)$. Then, the restriction to $f(H)$ of any irreducible admissible R-representation of G is semisimple of finite length. Any irreducible admissible R-representation of $f(H)$ is contained in some irreducible admissible R-representation of G restricted to $f(H)$, and extends to an irreducible admissible representation of some open subgroup of G of finite index.*

Proof. G satisfies (2-1) and $f(H)$ satisfies the property (2-2) because H does. Apply the results of Section 2.2. □

We now give two examples where we can apply Theorem 3.2.

Proposition 3.3. *Let $f : \underline{H} \to \underline{G}$ be a surjective central F-morphism of connected reductive F-groups. Then, the subgroup $f(H)$ of G is normal of abelian compact quotient $G/f(H)$.*

Proof. There is an F-morphism $\kappa : \underline{G} \times \underline{G} \to \underline{H}$ such that $\kappa(f(x), f(y)) = xhx^{-1}y^{-1}$ for all $x, y \in \underline{H}$ [Borel and Tits 1972, définition 2.2]. So for all $u, v \in G$ we have $uvu^{-1}v^{-1} = f \circ \kappa(u, v) \in f(H)$. The subgroup $f(H)$ of H is closed (Lemma 3.1) and normal with abelian quotient $G/f(H)$ [loc. cit., proposition 2.7].

The compactness of G/H is stated in [Silberger 1979] without proof and in [Labesse and Schwermer 2019, Proposition A.2.1] with indications for the proof. The idea is to reduce to a connected reductive F-anisotropic modulo the centre F-group.

Let $\underline{S}$ be a maximal F-split subtorus of $\underline{G}$, and $\underline{B}$ a parabolic F-subgroup of $\underline{G}$ containing $\underline{S}$. The $\underline{G}$-centralizer $\underline{M}$ of $\underline{S}$ is compact modulo its centre and is a Levi component of $\underline{B}$. Let $\underline{U}$ be the unipotent radical of $\underline{B}$. By [Borel 1991, Theorem 22.6], the inverse image $\underline{S}'$ of $\underline{S}$ in $\underline{H}$ is a maximal F-split torus in $\underline{H}$, and the inverse image $\underline{B}'$ of $\underline{B}$ is a parabolic F-subgroup of $\underline{H}$ Put $\underline{M}'$ for the $\underline{H}$-centralizer of $\underline{S}'$ and $\underline{U}'$ for the unipotent radical of $\underline{B}'$. From [loc. cit.], f induces a surjective central F-morphism $\underline{M}' \to \underline{M}$ and an F-isomorphism $\underline{U}' \to \underline{U}$. On the other hand, we have the Iwasawa decomposition $G = KB$ for an open compact subgroup K of G. The product map $K \times B \to G$ gives a surjective map $K \times B/f(B') \to G/f(H)$. We have $B/f(B') = M/f(M')$, so we just need to prove the compactness of $M/f(M')$.

Let $X^*(\underline{S})$ denote the group of algebraic characters of $\underline{S}$, and $\underline{S}(p_F)$ denote $\mathrm{Hom}(X^*(\underline{S}), p_F^{\mathbb{Z}})$. The subgroup $\underline{S}(p_F)$ of S is free abelian of finite rank with a compact quotient $S/\underline{S}(p_F)$. On the other hand, f induces a surjective F-morphism $\underline{S}' \to \underline{S}$ sending $\underline{S}'(p_F)$ onto a sublattice of $\underline{S}(p_F)$. Hence $S/f(S')$ is finite. So $M/f(S')$ is compact since M/S is compact, a fortiori $M/f(M')$ is compact. □

Remark 3.4. The condition that f is central in Proposition 3.3 is necessary. Indeed, assume $\mathrm{char}_F = 2$ and $f : \underline{\mathrm{GL}}_2 \to \underline{\mathrm{SL}}_2$, $f(g) = \varphi(g)/\det g$ where $\varphi(x) = x^2$ for $x \in F$ is the Frobenius.[4] The F-morphism f is surjective but not central. Let $G = \mathrm{GL}_2(F)$, $G' = \mathrm{SL}_2(F)$, T' the diagonal torus of G' and U the group of unipotent upper triangular matrices in G'. Then $f(G) = T'\varphi(G')$ is closed but not normal and not cocompact in G' (since $\varphi(U) = U \cap T'\varphi(G')$ and $U/\varphi(U)$ homeomorphic to F/F^2 is not compact).

Corollary 3.5. *Assume R algebraically closed. Let $f : \underline{H} \to \underline{G}$ be an F-morphism of connected reductive F-groups which induces a central F-isogeny $\underline{H}^{\mathrm{der}} \to \underline{G}^{\mathrm{der}}$ between the derived groups. Then the conclusions of Theorem 3.2 apply to $f(H)$.*

Proof. The F-isogeny $\underline{H}^{\mathrm{der}} \to \underline{G}^{\mathrm{der}}$ is surjective with finite kernel contained in the centre of $\underline{H}^{\mathrm{der}}$ [Springer 1998, §12.2.6]. If $\underline{Z}$ is the connected centre of $\underline{G}$, the natural map $\underline{Z} \times \underline{G}^{\mathrm{der}} \to \underline{G}$ is surjective [Springer 1998, Corollary 8.1.6]. Hence the obvious map $\underline{Z} \times \underline{H} \to \underline{G}$ is surjective and central. Proposition 3.3 applies to $Zf(H)$. But R being algebraically closed, Z acts by a character in any irreducible smooth R-representations of G, and we get the corollary. □

Remark 3.6. There is a more elementary proof that the restriction to $f(H)$ of any irreducible admissible R-representation of G is semisimple of finite length in [Silberger 1979].

4. Restriction to $\mathrm{SL}_2(F)$ of representations of $\mathrm{GL}_2(F)$

Let F be a local nonarchimedean field of residue field k_F of characteristic p as in Section 3, and R an algebraically closed field of characteristic different from p.

Let $G = \mathrm{GL}_2(F)$, and let B (resp. B^-) denote the subgroup of upper (resp. lower) triangular matrices, T the subgroup of diagonal matrices, U (resp. U^-) the subgroup of upper (resp. lower) triangular unipotent matrices, and Z the centre of G.

Let $G' = \mathrm{SL}_2(F)$. The subgroup $H = ZG'$ of G is closed normal of compact abelian quotient G/ZG' isomorphic via the determinant to $F^*/(F^*)^2$, which (see [Neukirch 1999, Chapter II, Corollary 5.8]) is a $\mathbb{F}_2$-vector space of dimension

$$\dim_{\mathbb{F}_2} F^*/(F^*)^2 = \begin{cases} 2+e & \text{if } \mathrm{char}_F \neq 2, \\ \infty & \text{if } \mathrm{char}_F \neq 2, \end{cases} \quad \text{where } 2O_F = P_F^e. \tag{4-1}$$

[4]The map f will also appear in §5.0.3.

Note that ZG' is open in G if and only if $\mathrm{char}_F \neq 2$.

For a subset $X \subset G$, put $X' = X \cap G'$. Write $x = (x_{i,j})$ a matrix in G or $\mathrm{Lie}\, G = M_2(F)$.

We fix a separable closure F^{sc} of F and will consider only extensions of F contained in F^{sc}. We write W_F for the Weil group of F^{sc}/F and Gal_F for the Galois group of F^{sc}/F. For a field k, we denote by k^{ac} an algebraic closure of k, and if $k \subset R$ we suppose $k^{\mathrm{ac}} \subset R$.

We fix an additive R-character ψ of F trivial on O_F but not on P_F^{-1}.

4.1. ***Whittaker spaces.*** The smooth R-characters of U have the form

$$\theta_Y(u) = \psi \circ \mathrm{tr}(Y(u-1)) = \psi(Y_{2,1}u_{1,2}), \quad u \in U, \tag{4-2}$$

for a lower triangular nilpotent matrix Y in $M_2(F)$. The case $Y = 0$ gives the trivial character of U, the cases with $Y \neq 0$ give the *nondegenerate* characters of U.

Notation 4.1. When $Y_{2,1} = 1$ we denote $\theta_Y = \theta$.

The normalizer of U in G is TU. By conjugation, U acts trivially on U and its characters, and a diagonal matrix $t = \mathrm{diag}(t_1, t_2)$ acts on $u \in U$ by $(tut^{-1})_{1,2} = (t_1/t_2)u_{1,2}$. Also, t acts on a lower triangular nilpotent matrix Y by $(tYt^{-1})_{2,1} = (t_2/t_1)Y_{2,1}$. It follows that T acts transitively on the nondegenerate characters of U, the quotient T/Z acting simply transitively. By the same formulas, two nontrivial characters θ_Y and $\theta_{Y'}$ of U are conjugate in G' if and only if they are conjugate by an element of T' if and only if $Y_{1,2}$ and $Y'_{1,2}$ differ by a square in F^*.

The T-normalizer of θ_Y is equal to Z if $Y \neq 0$ and to T if $Y = 0$. The θ_Y-coinvariant functor $\tau \mapsto W_Y(\tau)$ from the smooth R-representations τ of U to the smooth R-representations of the T-normalizer of θ_Y is exact. A smooth R-representation τ of U is called *degenerate* when $W_Y(\tau) = 0$ for all $Y \neq 0$, and *nondegenerate* otherwise. A smooth R-representation of G or of G' is called degenerate (or nondegenerate) if its restriction to U is as well.

The finite-dimensional irreducible smooth R-representations of G are of the form $\chi \circ \det$ for a smooth R-character χ of F^* and are degenerate. If Π is an infinite-dimensional irreducible smooth R-representation of G, then $\dim W_Y(\Pi) = 1$ for all $Y \neq 0$ by the uniqueness of Whittaker models [Vignéras 1996, chapitre III, §5.10] when $\mathrm{char}_R > 0$.

4.2. ***L-packets.*** We will classify the irreducible smooth R-representations of G' by restricting to G' the irreducible smooth R-representations Π of G. The set $L(\Pi)$ of (isomorphism classes of) irreducible components of $\Pi|_{G'}$ is called an L-packet. A parametrization along these lines was obtained when $\mathrm{char}_F = 0$ and $\mathrm{char}_R = \mathbb{C}$ in [Labesse and Langlands 1979]. When $\mathrm{char}_F \neq 2$ and $\mathrm{char}_R \neq 2$, this question is studied for supercuspidal representations in the recent work [Cui et al. 2024, §6.2 and §6.3].

Applying Lemma 2.3, Remark 2.4, Lemma 2.5, Theorem 3.2 and Corollary 3.5, we have:

(4-3) Any irreducible smooth R-representation of G' belongs to a unique L-packet.

For two irreducible smooth R-representations Π_1, Π_2 of G,

$$L(\Pi_1) = L(\Pi_2) \Longleftrightarrow \Pi_1 = (\chi \circ \det) \otimes \Pi_2 \tag{4-4}$$

for some R-character $\chi \circ \det$ of G.

The trivial character of G' is the unique finite-dimensional irreducible smooth R-representation of G', it is degenerate and forms an *L-packet* $L(1) = L(\chi \circ \det)$ for any smooth R-character χ of F^*.

If Π is an irreducible smooth R-representation of G,[5]

(4-5) the restriction of Π to G' is semisimple of finite length and multiplicity 1.

The irreducible constituents of $\Pi|_{G'}$ are G-conjugate (even B-conjugate as $G = BG'$), and form an *L-packet* $L(\Pi)$ whose cardinality is the length of $\Pi|_{G'}$. The G-stabilizer of $\pi \in L(\Pi)$ does not depend on the choice of π in $L(\Pi)$ and is denoted G_Π. By §2.2.6, G_Π is an open normal subgroup of G containing $G'Z$, the subgroup $\det G_\Pi$ of F^* is open and contains $(F^*)^2$. The order of the quotient $G/G_\Pi \simeq F^*/\det G_\Pi$ is a power of 2. When $\mathrm{char}_F \neq 2$, $|G/G_\Pi|$ divides $|F^*/(F^*)^2| = 2^{2+e}$ with e defined in (4-1).

(4-6) G_Π is the largest subgroup I of G such that $\lg(\Pi|_I) = \lg(\Pi|_{G'})$.

(4-7) $\Pi = \mathrm{ind}_{G_\Pi}^G V_\pi$ where V_π is the space of π.

(4-8) $L(\Pi)$ is a principal homogeneous space for G/G_Π.

(4-9) $|L(\Pi)|$ is a power of 2, and $|L(\Pi)|$ divides 2^{2+e} when $\mathrm{char}_F \neq 2$.

When p is odd, since $|F^*/(F^*)^2| = 4$ we deduce:

Proposition 4.2. *When p is odd, the cardinality of an L-packet is* 1, 2 *or* 4.

When $p = 2$ we will prove that this remains true using the local Langlands correspondence.

By class field theory, any open subgroup of F^* of index 2 is equal to $N_{E/F}(E^*)$ for a unique quadratic separable extension E/F of relative norm $N_{E/F} : E^* \to F^*$, and conversely. Any open subgroup of F^* of index 4 containing $(F^*)^2$ is equal to $N_{K/F}(K^*)$ for a unique biquadratic separable extension K/F of relative norm $N_{K/F} : K^* \to F^*$, and conversely.

[5] For cuspidal representations this is proved in [Cui 2023, Proposition 2.37 and Corollary 2.38].

When p is odd, each quadratic extension of F is separable and tamely ramified, and there is a unique biquadratic separable extension of F.

When $p=2$, if $\mathrm{char}_F=0$, there are finitely many quadratic separable extensions of F and finitely many biquadratic separable extensions of F; see (4-1). If $\mathrm{char}_F=2$, there are infinitely many quadratic, resp. biquadratic, separable extensions of F.

Definition 4.3. When Π is an irreducible smooth R-representation of G, we denote by E_Π the separable extension of F such that $N_{E_\Pi/F}(E_\Pi^*)=\det G_\Pi$.

(4-10) We denote by X_Π the group of characters $\chi\circ\det$ of G such that

$$\Pi\otimes(\chi\circ\det)\simeq\Pi.$$

A character of X_Π is smooth (Lemma 2.2) of trivial square. So $X_\Pi=\{1\}$ if $\mathrm{char}_R=2$.

Notation 4.4. When $\mathrm{char}_R\neq 2$, the nontrivial smooth R-characters of F^* of trivial square are the R-characters η_E of F^* of kernel $N_{E/F}(E^*)$, for quadratic separable extensions E/F. The modulus $q^{\pm\mathrm{val}}$ of F^* is equal to η_E if and only if E/F is unramified and $q+1=0$ in R.

By Lemma 2.3 and (4-8):

(4-11) X_Π is the group of R-characters of G trivial on G_Π.

(4-12) When $\mathrm{char}_R\neq 2$, the cardinality of $L(\Pi)$ is $|X_\Pi|$.

It is known that $|X_\Pi|=1,2$ or 4 when:

(a) $R=\mathbb{C}$ and $\mathrm{char}_F=0$ [Labesse and Langlands 1979; Shelstad 1979].

(b) $\mathrm{char}_F\neq 2$ and $\mathrm{char}_R\neq 2$ [Cui et al. 2024, Proposition 6.6].

When $\mathrm{char}_R\neq 2$ we will prove that $|X_\Pi|=1,2$ or 4 using the local Langlands correspondence, therefore $|L_\Pi|=1,2$ or 4 when $p=2$.

For a lower triangular matrix $Y\neq 0$, we have

$$\sum_{\pi\in L(\Pi)}\dim_R W_Y(\pi)=\dim_R W_Y(\Pi).$$

Since $\dim_R W_Y(\Pi)=1$, we have $\dim_R W_Y(\pi)=0$ or 1, and there is a single $\pi\in L(\Pi)$ with $W_Y(\pi)\neq 0$.

4.3. ***Representations.*** We denote by $\mathrm{Gr}_R^\infty(G)$ the Grothendieck group of finite length smooth R-representations of G and by $[\tau]$ the image in $\mathrm{Gr}_R^\infty(G)$ of a finite length smooth R-representation τ of G. Similarly for G'.

4.3.1. *Parabolic induction.* The smooth *parabolic induction* $\mathrm{ind}_B^G(\sigma)$ of a smooth R-representation (σ, V) of T is the space of functions $f : G \to V$ such that $f(tugk) = \sigma(t)f(g)$ for $t \in T$, $u \in U$, $g \in G$ and an open compact subgroup $K_f \subset G$, with the action of G by right translation. The functor ind_B^G is exact with the U-coinvariant functor $(-)_U$ as left adjoint, and $(-)_{\overline{U}} \otimes \delta$ as right adjoint where δ is the homomorphism of T:

$$\delta(\mathrm{diag}(a, d)) = q^{-\mathrm{val}(a/d)} : T \to q^{\mathbb{Z}} \quad (a, d \in F^*),$$

[Dat et al. 2024, Corollary 1.3]. The modulus $|\cdot|_F$ of F^* is $q^{-\mathrm{val}}$ and the modulus of B is the inflation of δ. *We choose a square root $q^{1/2}$ of q in R^* to define the* square root of δ,

$$\nu(\mathrm{diag}(a, d)) = (q^{1/2})^{-\mathrm{val}(a/d)} : T \to (q^{1/2})^{\mathbb{Z}} \quad (a, d \in F^*), \tag{4-13}$$

and the *normalized parabolic induction* $i_B^G(\sigma) = \mathrm{ind}_B^G(\sigma\nu)$. For a smooth R-character $\chi \circ \det$ of G we have

$$(\mathrm{ind}_B^G \sigma) \otimes (\chi \circ \det) \simeq \mathrm{ind}_B^G(\sigma \otimes (\chi \circ \det)), \quad (i_B^G \sigma) \otimes (\chi \circ \det) \simeq i_B^G(\sigma \otimes (\chi \circ \det)).$$

Similarly for G', we define the parabolic induction $\mathrm{ind}_{B'}^{G'}$ from the smooth R-representation σ of T' to those of G' and the normalized parabolic induction $i_{B'}^{G'}$,

$$i_{B'}^{G'}(\sigma) = \mathrm{ind}_{B'}^{G'}(\nu'\sigma), \quad \nu'(\mathrm{diag}(a, a^{-1})) = q^{-\mathrm{val}(a)} : T' \to q^{\mathbb{Z}} \quad (a \in F^*).$$

As $G = BG'$ and G/B is compact, the restriction map $f \mapsto f|_{G'}$ gives isomorphisms

$$(\mathrm{ind}_B^G(\sigma))|_{G'} \mapsto \mathrm{ind}_{B'}^{G'}(\sigma|_{T'}), \quad (i_B^G(\sigma))|_{G'} \mapsto i_{B'}^{G'}(\sigma|_{T'}). \tag{4-14}$$

4.3.2. *Cuspidal representations of* $\mathrm{GL}_2(F)$. When χ is a smooth R-character of T, $\mathrm{ind}_B^G(\chi)$ is called a *principal series* of G. An irreducible smooth R-representation of G which is not a subquotient of a principal series, is called *supercuspidal*. It is called *cuspidal* when its U-coinvariants are 0. A supercuspidal representation is cuspidal (the converse is true only when $q + 1 \neq 0$ in R). The principal series and the cuspidal R-representations are infinite-dimensional. Similarly for G'.

Let Π be an irreducible smooth R-representation of G and $\pi \in L(\Pi)$. Then

(4-15) Π is cuspidal if and only if π is cuspidal (similarly for supercuspidal).

Indeed, $L(\Pi)$ is the B-orbit of π, the U-coinvariant functor is exact and commutes with the restriction to G'. We say that $L(\Pi)$ is cuspidal if Π is. Similarly for supercuspidal using the formula (4-14).

Let Π be a cuspidal R-representation of G. It is the compact induction of an extended maximal simple type (J, Λ),

$$\Pi = \mathrm{ind}_J^G(\Lambda);$$

see [Bushnell and Kutzko 1994; Bushnell and Henniart 2002] when $R=\mathbb{C}$ and [Vignéras 1996, chapitre III, §3.4] for general R. The group J contains Z and a unique maximal open compact subgroup J^0. Let J^1 be the pro-p radical of J^0. The representation $\Lambda|_{J^0}$ is irreducible, equal to $\lambda=\kappa\otimes\bar{\sigma}$ where $\kappa|_{J^1}$ is irreducible and $\bar{\sigma}$ is inflated from an irreducible R-representation σ of J^0/J^1. The type (J,Λ) is unique modulo G-conjugacy; see [Bushnell and Henniart 2006, Chapter 4, §15.5, Induction theorem] when $R=\mathbb{C}$ and [Vignéras 1996, chapitre III, §5.3] for general R.[6]

The open normal subgroup JG' of G has index $|F^*/\det J|$, and by Mackey theory,

$$\Pi|_{JG'}=\bigoplus_{g\in G/JG'}\operatorname{ind}_{J^g}^{JG'}\lambda^g. \tag{4-16}$$

Denote J', $(J^0)'$, $(J^1)'$ the intersections of J, J^0, J^1 with G'. We have $J'=(J^0)'$ and the length of

$$(\operatorname{ind}_{J^g}^{JG'}\lambda^g)|_{G'}\simeq\operatorname{ind}_{J'^g}^{G'}(\lambda^g|_{J'^g})$$

is independent of g. By transitivity of the restriction $\Pi|_{G'}=\bigoplus_{g\in G/JG'}\operatorname{ind}_{J'^g}^{G'}(\lambda^g|_{J'^g})$, and

$$|L(\Pi)|=|F^*/\det J|\,\lg(\operatorname{ind}_{J'}^{G'}(\lambda|_{J'})), \tag{4-17}$$

it follows from Lemma 2.3(3), Remark 2.4 and the formula (4-16) that:

Lemma 4.5. *If* $|F^*/\det J|=2$ *then* $\det G_\Pi\subset\det J$.

Remark 4.6. We have $\det G_\Pi=\det J\iff G_\Pi=JG'$. If $|F^*/\det J|=2$, the group J determines a quadratic separable extension E/F such that $\det J=N_{E/F}(E^*)$. The representation $\operatorname{ind}_{J'}^{G'}(\lambda|_{J'})$ is irreducible if and only if $|L(\Pi)|=|F^*/\det J|$.

If there is a smooth R-character χ of F^* such that $\Lambda\simeq\Lambda_0\otimes(\chi\circ\det)$ and (J,Λ_0) is of level 0, we say that the L-packet $L(\Pi)$ and its elements are of level 0. Otherwise we say that $L(\Pi)$ and its elements are of positive level.

Level 0. $J=Z\,\mathrm{GL}_2(O_F)$, $J^0=\mathrm{GL}_2(O_F)$, $J^0/J^1\simeq\mathrm{GL}_2(k_F)$, $\kappa=1$, σ is a cuspidal R-representation of $\mathrm{GL}_2(k_F)$, $\lambda=\bar{\sigma}$. We have $\det J=\mathrm{val}^{-1}(2\mathbb{Z})$, and by (4-17),

$$|L(\Pi)|=2\lg(\lambda|_{J'})=2\lg(\sigma|_{\mathrm{SL}_2(k_F)}), \tag{4-18}$$

because $\lambda|_{J'}$ is semisimple with length $\lg(\sigma|_{\mathrm{SL}_2(\mathbb{F}_q)})$, and for any irreducible component $\lambda'\subset\lambda|_{J'}$, the compact induction $\operatorname{ind}_{J'}^{G'}(\lambda')$ is irreducible [Henniart and Vignéras 2022, Corollary 4.29].

[6] It is proved only that (J^0,λ) is unique modulo G-conjugacy, but J is the normalizer of (J^0,λ) and Λ is the λ-isotypic part of Π.

The cardinality of the cuspidal L-packet $L(\Pi)$ of level 0 can be computed via (4-17), (4-18), and Remark A.4(b) given in the Appendix on the classification of the irreducible R-representations of $\mathrm{GL}_2(k)$ and of $\mathrm{SL}_2(k)$ for a finite field k with $\mathrm{char}_k \neq \mathrm{char}_R$. We have two cases:

(i) $|F^*/\det G_\Pi| = 2$ and E_Π/F is the unramified quadratic extension.

(ii) p is odd, $\det G_\Pi = (F^*)^2$ and E_Π/F is the unique biquadratic extension. This case occurs for a unique packet $L(\Pi)$.

We deduce:

Proposition 4.7. *When $p = 2$, each level 0 cuspidal L-packet has size 2.*

When p is odd, there is a unique level 0 cuspidal L-packet of size 4, the other level 0 cuspidal L-packets have size 2.

These results can be deduced from [Kutzko and Pantoja 1991, §2] and the size 4 depth zero L-packet has been obtained in [Cui 2023, Example 3.11, Method 2].

Positive Level. $J = E^* J^0$ for a quadratic separable[7] extension E/F, $J^0 = O_E^* J^1$, $J^0/J^1 \simeq k_E^*$, σ is an R-character of k_E^*, $\lambda = \kappa \otimes \sigma$ and $\lambda|_{J'}$ is irreducible. The representation $\lambda_1 = \lambda|_{J^1}$ is irreducible of G-intertwining equal to J, because J normalizes λ_1 and the G-intertwining of σ is already J [Bushnell and Henniart 2006, Chapter 4, §15.1]. We have $N_{E/F}(E^*) \subset \det J$. If the quadratic extension E/F is tamely ramified, then $\det J = N_{E/F}(E^*)$, because $J = E^* J^1$, $J^1 = (1 + P_F)(J^1)'$ and $1 + P_F \subset \det E^* = N_{E/F}(E^*)$.

If $p = 2$ a tamely ramified quadratic extension of F is unramified, and E/F is unramified if and only if $\det J = \mathrm{Ker}((-1)^{\mathrm{val}})$.

If p is odd, each quadratic extension of F is tamely ramified.

Proposition 4.8. *If p is odd, each positive level cuspidal L-packet $L(\Pi)$ has size 2 and $E = E_\Pi$ (Definition 4.3).*

Proof. [8]The central subgroup $1 + P_F$ of $J^1 = (1 + P_F)(J^1)'$ acts by scalars, the representation $\lambda_1' = \lambda|_{(J^1)'}$ is still irreducible of G-intertwining J, so its G'-intertwining is J'. The isotypic component of $\Pi|_{J^1}$ of type λ_1 is the space of λ, so the isotypic component of $\Pi|_{(J^1)'}$ of type λ_1' is still the space of λ. As in the proof of [Henniart and Vignéras 2022, Corollary 4.29], we deduce that $\mathrm{ind}_{J'}^{G'}(\lambda|_{J'})$ is irreducible. Apply Lemma 4.5. □

Remark 4.9. When $p = 2$ and E/F is ramified, then $J^0 \cap G'$ is a pro-2-group. Indeed, the determinant induces a morphism $J^0/J^1 \to k_F^*$ equal via the natural

[7]When $\mathrm{char}_F = 2$ the quadratic extension appearing in the construction [Bushnell and Henniart 2006] is not necessarily separable. It is generated by an element $x \in G$, determined up to some open subgroup of G, so that modifying x slightly yields a separable extension.

[8]This can also be obtained using [Cui 2023].

isomorphism $J^0/J^1 \to k_E = k_F^*$ to the automorphism $x \mapsto x^2$ on k_F^*. Hence $(J^0)' = (J^1)'$ is a pro-2-group. Note also that Λ is a character [Bushnell and Henniart 2006, §15].

Corollary 4.10 (Propositions 4.7 and 4.8). *When p is odd, there is a unique cuspidal L-packet of size 4, and it is of level 0. The other cuspidal L-packets have size 2.*

4.3.3. *Principal series of* $\mathrm{GL}_2(F)$. We recall the description of the normalized principal series $i_B^G(\chi)$ of G for a smooth R-character χ of T.

Denote by χ_1, χ_2 the smooth R-characters of F^* such that

$$\chi(\mathrm{diag}(a,d)) = \chi_1(a)\chi_2(d) \quad (a, d \in F^*), \tag{4-19}$$

and by χ^w the character $\chi^w(\mathrm{diag}(a,d)) = \chi(\mathrm{diag}(d,a))$ of T. In particular in (4-13), $\nu^w = \nu^{-1}$ and $\nu/\nu^w = \delta$.

Proposition 4.11. (i) *For two smooth R-characters χ, χ' of T, $[i_B^G(\chi)]$ and $[i_B^G(\chi')]$ are disjoint or equal, with equality if and only if $\chi' = \chi$ or χ^w.*

(ii) *The smooth dual of $i_{B'}^{G'}(\chi)$ is $i_{B'}^{G'}(\chi^{-1})$.*

(iii) *$(i_B^G(\chi))_U$ has dimension 2, contains χ^w and has quotient χ.*

(iv) $\dim W_Y(i_B^G(\chi)) = 1$ *when $Y \neq 0$* [Vignéras 1996, chapitre III, §5.10].

(v) *$i_B^G(\chi)$ is reducible if and only if $\chi_1\chi_2^{-1} = q^{\pm \mathrm{val}}$.*

(vi) $\mathrm{ind}_B^G(1) = i_B^G(\nu^{-1})$ *contains the trivial representation* 1 *and*:

- *If $q+1 \neq 0$ in R, $\lg(\mathrm{ind}_B^G(1)) = 2$, in particular* $\mathrm{St} = (\mathrm{ind}_B^G 1)/1$ *is irreducible (the Steinberg R-representation). The representation $\mathrm{ind}_B^G 1$ is semisimple if and only if $q = 1$ in R (and $\mathrm{char}_R \neq 2$).*
- *If $q+1 = 0$ in R, $\lg(\mathrm{ind}_B^G(1)) = 3$, $\mathrm{ind}_B^G(1)$ is indecomposable of quotient $(-1)^{\mathrm{val}} \circ \det$, and $\mathrm{ind}_B^G(1)/1$ contains a cuspidal representation*

$$\Pi_0 = \mathrm{ind}_{Z\,\mathrm{GL}_2(O_F)}^G \tilde{\sigma}_0$$

where $\tilde{\sigma}_0$ is the inflation to $Z\,\mathrm{GL}(2, O_F)$ of the cuspidal subquotient σ_0 of $\mathrm{ind}_{B(k_F)}^{\mathrm{GL}_2(k_F)} 1$ (Appendix).

This is [Vignéras 1989, théorème 3] but the proof of (i) is incomplete. What is missing is the proof that Π_0 occurs only in $i_B^G(\nu)$ and $i_B^G(\nu^{-1})$ when $q+1=0$ in R. This is equivalent to $X_{\Pi_0} = \{1, (-1)^{\mathrm{val}} \circ \det\}$ with the notation (4-10). This follows from Remark A.4(a) given in the Appendix.

Remark 4.12. (1) The Steinberg representation St is infinite-dimensional and not cuspidal.

(2) When $\mathrm{char}_R \neq 2$, the principal series $[i_B^G(\chi)]$ are multiplicity free.

When $\mathrm{char}_R = 2$, then q is odd, $\mathrm{ind}_B^G(1)$ has length 3, of subquotients Π_0 and the trivial representation 1 as a subrepresentation and a quotient.

Corollary 4.13. *The nonsupercuspidal irreducible smooth R-representations of G are*

- *the characters $\chi \circ \det$ for the smooth R-characters χ of F^*,*
- *the principal series $i_B^G(\chi)$ for the smooth R-characters χ of T with $\chi_1\chi_2^{-1} \neq q^{\pm \mathrm{val}}$.*
- *the twists $(\chi \circ \det) \otimes \mathrm{St}$ of the Steinberg representation for the smooth R-characters χ of F^* if $q+1 \neq 0$ in R,*
- *the twists $(\chi \circ \det) \otimes \Pi_0$ of the cuspidal nonsupercuspidal representation Π_0 for the smooth R-characters χ of F^* if $q+1 = 0$ in R.*

The only isomorphisms between those representations are $i_B^G(\chi) \simeq i_B^G(\chi^w)$ for the irreducible principal series and $(\chi \circ \det) \otimes \Pi_0 \simeq ((-1)^{\mathrm{val}}\chi \circ \det) \otimes \Pi_0$.

4.3.4. Let ℓ be a prime number different from p. An irreducible smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation τ of G or G' is integral if it preserves a lattice. It then gives by reduction modulo ℓ and semisimplification a finite length semisimple smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation, of isomorphism class (not depending of the lattice) which we write $r_\ell(\tau)$. The restriction from G to G' from irreducible smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representations $\widetilde{\Pi}$ of G to finite length semisimple smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representations of G' respects integrality and commutes with the reduction modulo ℓ. When $\widetilde{\Pi}$ is integral, then any irreducible representation $\tilde{\pi} \subset \widetilde{\Pi}|_{G'}$ is integral, the length of the reduction $r_\ell(\tilde{\pi})$ modulo ℓ of $\tilde{\pi}$ does not depend on the choice of $\tilde{\pi}$. If $\Pi = r_\ell(\widetilde{\Pi})$ is irreducible, we have

$$|L(\Pi)| = |L(\widetilde{\Pi})| \lg(r_\ell(\tilde{\pi})), \tag{4-20}$$

and by (4-11),

$$\lg(r_\ell(\tilde{\pi})) = |X_\Pi / X_{\widetilde{\Pi}}| \quad \text{when } \mathrm{char}_R \neq 2. \tag{4-21}$$

Proposition 4.14. *Each irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation Π of G is the reduction modulo ℓ of some integral irreducible smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\widetilde{\Pi}$ of G.*

Proof. Corollary 4.13 for Π not cuspidal, [Vignéras 2001] for Π cuspidal. □

A supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\widetilde{\Pi} = \mathrm{ind}_J^G \widetilde{\Lambda}$ of G is integral if and only if $\widetilde{\Lambda}$ is integral. Then, its reduction modulo ℓ is irreducible [Vignéras 1989], equal to $\Pi = \mathrm{ind}_J^G \Lambda$ where $\Lambda = r_\ell(\widetilde{\Lambda})$. The reduction modulo ℓ of the L-packet $L(\widetilde{\Pi})$ is $L(\Pi)$. The reduction modulo ℓ respects level 0 and positive level. Conversely, any cuspidal $\mathbb{F}_\ell^{\mathrm{ac}}$-representation $\Pi = \mathrm{ind}_J^G \Lambda$ of G is the reduction modulo ℓ of an integral cuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\widetilde{\Pi} = \mathrm{ind}_J^G \widetilde{\Lambda}$ of G where $\Lambda = r_\ell(\widetilde{\Lambda})$ [Vignéras 2001]. By the uniqueness of the extended maximal simple type (J, Λ) modulo G (see Section 4.3.2), two supercuspidal integral $\mathbb{Q}_\ell^{\mathrm{ac}}$-representations have isomorphic reduction modulo ℓ if and only if the reduction modulo ℓ of their extended maximal simple types are G-conjugate.

Any supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\tilde\pi$ of G' is integral, as $\tilde\pi \in L(\widetilde\Pi)$ where $\widetilde\Pi$ is a supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation of G, and some twist of $\widetilde\Pi$ by a character is integral. Suppose that $\widetilde\Pi$ has level 0. With the notations of the formula (4-18), the formula (4-21) implies

$$\lg(r_\ell(\tilde\pi)) = \lg(\sigma|_{\mathrm{SL}_2(k_F)})/\lg(\tilde\sigma|_{\mathrm{SL}_2(k_F)}). \tag{4-22}$$

Proposition 4.15. *When $\tilde\pi$ is supercuspidal of level 0, the length of $r_\ell(\tilde\pi)$ is ≤ 2.*

When $\tilde\pi$ is supercuspidal and p is odd, $r_\ell(\tilde\pi)$ is irreducible if $\tilde\pi$ is minimal of positive level or if $\ell = 2$.

Any cuspidal $\mathbb{F}_\ell^{\mathrm{ac}}$-representation π of G' is the reduction modulo ℓ of a supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation of G', except maybe when $p = 2$ and π is in an L-packet $L(\Pi)$ with Π minimal of positive level with E_Π/F unramified (Definition 4.3).

Proof. • For $\widetilde\Pi$ of level 0, we show in the Appendix the computation of the integer $\lg(\sigma|_{\mathrm{SL}_2(k_F)})/\lg(\tilde\sigma|_{\mathrm{SL}_2(k_F)})$, and one sees that it is equal to 1 or 2 and that there exists $\tilde\sigma$ such that it is 1.

• For p odd, if the level of $\tilde\pi$ is positive then $\lg(\Pi|_{G'}) = \lg(\widetilde\Pi|_{G'})$ by Proposition 4.8, hence $r_\ell(\tilde\pi)$ is irreducible.

• For $\ell = 2$ (so p is odd), if the level of $\tilde\pi$ is 0, then $r_\ell(\tilde\pi)$ is also irreducible by (4-22) and Lemma A.3 in the Appendix.

• For $p = 2$ (so ℓ is odd), π is in a cuspidal L-packet $L(\Pi)$ with Π minimal of positive level with E_Π/F ramified. Let $\widetilde\Pi$ a $\mathbb{Q}_\ell^{\mathrm{ac}}$-lift of Π. The reduction modulo ℓ from $X_{\widetilde\Pi}$ onto X_Π is injective. The proposition follows from the next lemma. ☐

Lemma 4.16. *The reduction modulo ℓ from $X_{\widetilde\Pi}$ onto X_Π is a bijection.*

Proof. Let $\chi \in X_\Pi$, $\chi \neq 1$, and $\tilde\chi$ the unique $\mathbb{Q}_\ell^{\mathrm{ac}}$ lift of χ of order 2. We have $\widetilde\Pi = \mathrm{ind}_J^G \widetilde\Lambda$ where $\widetilde\Lambda$ is a character (Remark 4.9). We have $\Pi = \mathrm{ind}_J^G \Lambda$ where $\Lambda = r_\ell(\widetilde\Lambda)$ and $(J, \chi\Lambda) = (J, {}^g\Lambda)$ for $g \in G$ normalizing J. So $\tilde\chi\widetilde\Lambda = \epsilon\, {}^g\widetilde\Lambda$ for a $\mathbb{Q}_\ell^{\mathrm{ac}}$-character ϵ of J of order a power of ℓ. So, $\epsilon|_{J_1} = 1$ and $\epsilon|_Z = 1$. Since E_Π/F is ramified, the index of ZJ^1 in J is 2, hence $\epsilon = 1$ and $\tilde\chi \in X_{\widetilde\Pi}$. ☐

When $\mathrm{char}_F \neq 2$ and $\mathrm{char}_R \neq 2$, compare with [Cui et al. 2024, Proposition 6.7]. When $p = 2$, we shall complete the proposition in Corollary 4.24: if $\tilde\pi$ has positive level then $r_\ell(\tilde\pi)$ has length ≤ 2, if π is in an L-packet $L(\Pi)$ of positive level with E_Π/F unramified then π lifts to $\mathbb{Q}_\ell^{\mathrm{ac}}$.

4.4. *Local Langlands R-correspondence for* $\mathbf{GL_2}(F)$.

4.4.1. By local class field theory, the smooth R-characters χ of F^* identify with the smooth R-characters $\chi \circ \alpha_F$ of W_F where $\alpha_F : W_F \to F^*$ is the Artin reciprocity map sending a arithmetic Frobenius Fr to p_F^{-1} [Bushnell and Henniart 2002, §29]. This is the local Langlands R-correspondence for $\mathrm{GL}_1(F)$.

A two-dimensional Deligne R-representation of the Weil group W_F is a pair (σ, N) where σ is a two-dimensional semisimple smooth R-representation of the Weil group W_F and N a nilpotent R-endomorphism of the space of σ with the usual requirement:

$$\sigma(w)N = N|\alpha_F(w)|_F\,\sigma(w) \quad \text{for } w \in W_F. \tag{4-23}$$

Two two-dimensional Deligne R-representations (σ, N) and (σ', N') of W_F are isomorphic if there exists a linear isomorphism $f : V \to V'$ from the space V of σ to the space V' of σ' such that $\sigma'(w) \circ f = f \circ \sigma(w)$ for $w \in W_F$ and $N' \circ f = f \circ N$.

For a smooth R-character χ of F^*, the twist $(\sigma, N) \otimes (\chi \circ \alpha_F)$ of (σ, N) by $\chi \circ \alpha_F$ is $(\sigma \otimes (\chi \circ \alpha_F), N)$.

When $R = \mathbb{Q}_\ell^{\mathrm{ac}}$, (σ, N) is called integral if σ is integral.

Remark 4.17. • When σ is irreducible we have $N = 0$.

• When $\sigma = (\chi_1 \oplus \chi_2) \circ \alpha_F$, if $\chi_1\chi_2^{-1} \neq q^{\pm \mathrm{val}}$ then $N = 0$. When $N \neq 0$, we have $\{\chi_1, \chi_2\} = \{\chi_i, q^{-\mathrm{val}}\chi_i\}$ for some i and N sends the $(\chi_i \circ \alpha_F)$-eigenspace to the $(q^{-\mathrm{val}}\chi_i \circ \alpha_F)$-eigenspace or 0. Therefore when $\chi_1\chi_2^{-1} = q^{\mathrm{val}}$:

• If $q - 1 \neq 0$ and $q + 1 \neq 0$ in R, then $N = 0$ or the kernel of N is the $(\chi_2 \circ \alpha_F)$-eigenline.

• If $q - 1 \neq 0$ and $q + 1 = 0$ in R, then $N = 0$, or the kernel of N is the $(\chi_2 \circ \alpha_F)$-eigenline, or the kernel of N is the $(\chi_1 \circ \alpha_F)$-eigenline.

• If $q - 1 = 0$, then N is any nilpotent.

The local Langlands R-correspondence for $G = \mathrm{GL}_2(F)$ is a canonical bijection

$$\mathrm{LL}_R : \Pi \mapsto (\sigma_\Pi, N_\Pi) \tag{4-24}$$

from the isomorphism classes of the irreducible smooth R-representations Π of G onto the equivalence classes of the two-dimensional Weil–Deligne R-representations of W_F.[9] It identifies supercuspidal R-representations of G and irreducible two-dimensional R-representations of W_F, commutes with the automorphisms of R respecting a chosen square root of q, with the twist by smooth R-characters χ of F^*:

$$\mathrm{LL}_R(\Pi \otimes (\chi \circ \det)) = \mathrm{LL}_R(\Pi) \otimes (\chi \circ \alpha_F). \tag{4-25}$$

The local Langlands complex correspondence was proved by Kutzko [Bushnell and Henniart 2002, §33]. An isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{\mathrm{ac}}$ and the choice of a square root of q in $\mathbb{Q}_\ell^{\mathrm{ac}}$ transfers $\mathrm{LL}_{\mathbb{C}}$ to a local Langlands $\mathbb{Q}_\ell^{\mathrm{ac}}$-correspondence $\mathrm{LL}_{\mathbb{Q}_\ell^{\mathrm{ac}}}$ respecting integrality. Any irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation Π of G lifts to a $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\widetilde{\Pi}$ of G (Proposition 4.14) and $\mathrm{LL}_{\mathbb{Q}_\ell^{\mathrm{ac}}}$ descends to a local

[9] (σ_Π, N_Π) is called the L-parameter of Π.

Langlands $\mathbb{F}_\ell^{\mathrm{ac}}$-correspondence $\mathrm{LL}_{\mathbb{F}_\ell^{\mathrm{ac}}}$ compatible with reduction modulo ℓ in the sense of [Vignéras 2001, § 1.8.5]. The nilpotent part N_Π is subtle but the semisimple part σ_Π is simply the reduction modulo ℓ of $\sigma_{\tilde{\Pi}}$,

$$\sigma_\Pi = r_\ell(\sigma_{\tilde{\Pi}}). \tag{4-26}$$

The local Langlands correspondence LL_R of G over R is deduced from $\mathrm{LL}_{\mathbb{Q}_\ell^{\mathrm{ac}}}$ when $\mathrm{char}_R = 0$ and from $\mathrm{LL}_{\mathbb{F}_\ell^{\mathrm{ac}}}$ when $\mathrm{char}_R = \ell$ [Vignéras 1997, § 3.3; 2001, § 1.7 and § 1.8]. We recall from the latter paper a representative (σ_Π, N_Π) of $\mathrm{LL}_R(\Pi)$ for an irreducible smooth R-representation Π of G.

Proposition 4.18. (A) *Let Π be an irreducible subquotient of the unnormalized R-principal series $\mathrm{ind}_B^G(1)$ of G. Then, $\sigma_\Pi = ((q^{1/2})^{-\mathrm{val}} \oplus (q^{1/2})^{\mathrm{val}}) \circ \alpha_F$. We have $N_\Pi = 0$ if $\Pi = 1$ (the trivial character) when $q+1 \neq 0$ in R, and $\Pi = \Pi_0$ cuspidal when $q+1=0$ in R. Otherwise $N_\Pi \neq 0$. When $q-1 \neq 0$ in R, the kernel of N_Π is*

- *the $((q^{1/2})^{-\mathrm{val}} \circ \alpha_F)$-eigenline if $q+1=0$ in R and $\Pi = 1$,*
- *the $((q^{1/2})^{\mathrm{val}} \circ \alpha_F)$-eigenline if $q+1=0$ in R and $\Pi = q^{\mathrm{val}} \circ \det$,*
- *the $((q^{1/2})^{-\mathrm{val}} \circ \alpha_F)$-eigenline if $q+1 \neq 0$ in R and $\Pi = \mathrm{St}$ the Steinberg representation.*

(B) *Let Π be the irreducible normalized principal series $i_B^G(\eta)$, i.e., $\eta \neq q^{\pm\mathrm{val}}$, with the notation of (4-29). Then $\sigma_\Pi = (\eta \oplus 1) \circ \alpha_F$ and $N_\Pi = 0$.*

(C) *Let Π be a supercuspidal R-representation of G. Then σ_Π is irreducible and $N_\Pi = 0$.*

4.4.2. For a two-dimensional semisimple smooth R-representation σ of W_F, put

$$X_\sigma = \{\text{smooth } R\text{-characters } \chi \text{ of } F^* \text{ such that } (\chi \circ \alpha_F) \otimes \sigma \simeq \sigma\}.$$

The square of each $\chi \in X_\sigma$ is trivial because $\dim_R \sigma = 2$. We shall compute X_σ when $\mathrm{char}_R \neq 2$. When $\mathrm{char}_R = 2$, $X_\sigma = \{1\}$.

To a pair (E, ξ) where E is a quadratic separable extension of F and ξ is a smooth R-character of E^* different from its conjugate ξ^τ by a generator τ of $\mathrm{Gal}(E/F)$ (i.e., ξ is not trivial on $\mathrm{Ker}\, N_{E/F} = \{x/x^\tau \mid x \in E^*\}$), is associated a 2-dimensional irreducible smooth R-representation of W_F

$$\sigma(E, \xi) = \mathrm{ind}_{W_E}^{W_F}(\xi \circ \alpha_E).$$

The character ξ is unique modulo $\mathrm{Gal}(E/F)$-conjugation.

When $\mathrm{char}_R \neq 2$, let σ be a two-dimensional irreducible smooth R-representation of W_F and E/F a quadratic separable extension. By Clifford's theory [Bushnell and Henniart 2006, Chapter 10, § 41.3, Lemma] with Notation 4.4,

$$\eta_E \in X_\sigma \Longleftrightarrow \sigma \simeq \sigma(E, \xi) \quad \text{for some } \xi.$$

Proposition 4.19. *Assume* $\mathrm{char}_R \neq 2$. *For a pair* (E,ξ) *as above,*

$$X_{\sigma(E,\xi)} = \begin{cases} \{1,\eta_E\} & \text{if } (\xi/\xi^\tau)^2 \neq 1, \\ \{1,\eta_E,\eta_{E'},\eta_E\eta_{E'}\} & \text{if } (\xi/\xi^\tau)^2 = 1,\ \xi/\xi^\tau = \eta_{E'}\circ N_{E/F}. \end{cases}$$

For each biquadratic separable extension K/F, *there exists a two-dimensional irreducible smooth* R*-representation* σ *of* W_F, *unique modulo twist by a character, with*

$$X_\sigma = \{1,\eta_E,\eta_{E'},\eta_{E''}\}$$

for the three quadratic extensions E, E', E'' *of* F *contained in* K.

Proof. • We have

$$\chi \in X_{\sigma(E,\xi)} \Longleftrightarrow (\chi\circ\alpha_F)\otimes \mathrm{ind}_{W_E}^{W_F}(\xi\circ\alpha_E) \simeq \mathrm{ind}_{W_E}^{W_F}(\xi\circ\alpha_E) \Longleftrightarrow \xi(\chi\circ N_{E/F}) = \xi \text{ or } \xi^\tau.$$

• $\xi(\chi\circ N_{E/F}) = \xi \Longleftrightarrow \chi$ is trivial on $N_{E/F}(E^*)$, so $\chi = 1$ or η_E.

• $\xi(\chi\circ N_{E/F}) = \xi^\tau \Longleftrightarrow \chi = \eta_{E'}$ for a quadratic separable extension $E' \neq E$ of F, as $\chi^2 = 1$.

If χ satisfies $\xi(\chi\circ N_{E/F}) = \xi^\tau$, the order of ξ^τ/ξ is 2, ξ^τ/ξ is fixed by τ and determines χ up to multiplication by η_E. Let K/F be the biquadratic extension generated by E and E' and E''/F the third quadratic extension contained in K/F. We have $\eta_E\eta_{E'} = \eta_{E''}$. Hence the first assertion.

The uniqueness in the second assertion follows from the fact that for two smooth R-characters ξ_1,ξ_2 of E^*, $\xi_1^\tau/\xi_1 = \xi_2^\tau/\xi_2 \Longleftrightarrow \xi_1 = \xi_2(\chi\circ N_{E/F})$ for a smooth R-character χ of F^*.

The existence in the second assertion is as follows. When p is odd, there is a unique biquadratic extension K/F of F. Let E/F be the unramified quadratic extension. We take $\sigma = \sigma(E,\xi)$ where ξ is the character of E^* trivial on $1+p_F O_E$, $\xi(p_F) = -1$ and $\xi(x) = x^{\frac{1}{2}(q+1)}$ if $x^{q^2-1} = 1$, satisfies $\xi^\tau/\xi \neq 1$ and $(\xi^\tau/\xi)^2 = 1$ hence $\xi^\tau/\xi = \eta_{E'}\circ N_{E/F} = \eta_E\eta_{E'}\circ N_{E/F}$ for E'/F ramified. When $p=2$, given two different quadratic separable extensions E'/F and E/F, there exists a smooth R-character ξ of E^* such that $\xi^\tau/\xi = \eta_{E'}\circ N_{E/F} = \eta_E\eta_{E'}\circ N_{E/F}$, because $\mathrm{char}_R \neq 2$, and this is known when $R = \mathbb{C}$ ([Bushnell and Henniart 2006, Chapter 10, §41] when $p \neq 2$, but the proof does not use $p \neq 2$).[10,11] □

Remark 4.20. Let Π be a supercuspidal R-representation of G. Then Π has level 0 (resp. $L(\Pi)$ has level 0), if and only if $\sigma_\Pi = \mathrm{ind}_{W_E}^{W_F}(\xi\circ\alpha_E)$ where E/F is quadratic unramified and ξ is a tame character of E^* (resp. ξ^τ/ξ is a tame character of E^* where τ is the nontrivial element of $\mathrm{Gal}(E/F)$).

[10] We gave a direct proof when p is odd, this was unnecessary.

[11] When p is odd and $\mathrm{char}_R = 2$, there is no ξ such that $\sigma(E,\xi)$ is induced from a character of $W_{E'}$ for a quadratic extension E'/F distinct from E/F.

Remark 4.21. Assume $\mathrm{char}_R \neq 2$. Let $\sigma = \chi_1 \circ \alpha_F \oplus \chi_2 \circ \alpha_F$ be a reducible two-dimensional semisimple smooth R-representation of W_F. Then

$$\chi \circ \alpha_F \in X_\sigma \iff \{\chi\chi_1, \chi\chi_2\} = \{\chi_1, \chi_2\} \iff \chi = 1 \text{ or } \chi\chi_1 = \chi_2, \chi\chi_2 = \chi_1$$
$$\iff \chi = 1 \text{ or } \chi = \chi_2\chi_1^{-1}, \chi^2 = 1.$$

If $\chi_1\chi_2^{-1} = \eta_E$ for a quadratic separable extension E/F, then $X_\sigma = \{1, \eta_E\}$. Otherwise, $X_\sigma = \{1\}$.

4.4.3. *Application to the cuspidal L-packets.* For a two-dimensional Weil–Deligne R-representation (σ, N) of W_F, put $X_{(\sigma,N)}$ for the group of $\chi \in X_\sigma$ such that there exists an isomorphism of $\chi \otimes \sigma$ onto σ preserving N. For any irreducible R-representation Π of G, applying the formulas (4-24), (4-25) and (4-11) we obtain:

(4-27) $X_\Pi = \{\chi \circ \det \mid \chi \in X_{(\sigma_\Pi, N_\Pi)}\}$.

(4-28) When $\mathrm{char}_R \neq 2$, the cardinality of the L-packet $L(\Pi)$ is $|X_{\sigma_\Pi}|$.

Proposition 4.22. (1) *When* $\mathrm{char}_R \neq 2$, *we have*:

- *The cardinality of a cuspidal L-packet is* 1, 2 *or* 4.
- *The map $L(\Pi) \mapsto E_\Pi$ is a bijection from the cuspidal L-packets of size* 4 *to the biquadratic separable extensions of F.*

(2) *There is a bijection from the cuspidal L-packets of size* 4 *to the biquadratic separable extensions of F, sending the unique cuspidal L-packet of size* 4 *to the unique biquadratic separable extension of F when* $\mathrm{char}_R = 2$, *and equal to the map $L(\Pi) \mapsto E_\Pi$ when* $\mathrm{char}_R \neq 2$.

Proof. (a) Assume $\mathrm{char}_R \neq 2$. If Π is cuspidal and $X_\Pi \neq \{1\}$ then $\eta_E \in X_\Pi$ for some quadratic separable extension E/F, $\sigma_\Pi = \sigma(E, \xi)$ for some ξ and $|X_{\sigma(E,\xi)}| = 2$ or 4 by Proposition 4.19. When $p = 2$ then the map is a bijection by Proposition 4.19 via the local Langlands correspondence.

(b) Assume p is odd (and $\mathrm{char}_R \neq p$). There is a unique biquadratic separable extension of F and a unique cuspidal L-packet of size 4 (Corollary 4.10).

(c) As p is odd when $\mathrm{char}_R = 2$, the proposition follows from (a) and (b). □

When $R = \mathbb{F}_\ell^{ac}$ and $\ell \neq p$, it is well known that an irreducible smooth $\mathbb{F}_\ell^{ac}$-representation σ of W_F of dimension 2 lifts to an integral irreducible smooth $\mathbb{Q}_\ell^{ac}$-representation $\tilde{\sigma}$ of W_F.[12] The order of $X_{\tilde{\sigma}}$ is at most to the order of X_σ. We give now all the cases where the orders are different.

Theorem 4.23. *Assume* $\ell \neq 2$.

[12] σ extends to a $\mathbb{F}_\ell^{ac}$-representation of the Galois group Gal_F. As Gal_F is solvable this representation lifts to a $\mathbb{Q}_\ell^{ac}$-representation of Gal_F that one restricts to W_F to get $\tilde{\sigma}$.

(1) *Let $\tilde\sigma$ be a lift to $\mathbb{Q}_\ell^{\mathrm{ac}}$ of a two-dimensional irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation σ of W_F. The cardinalities of X_σ and of $X_{\tilde\sigma}$ are different if and only if $|X_\sigma|=4$, $|X_{\tilde\sigma}|=2$, and this happens if and only if*

$$p=2, \quad \ell \textit{ divides } q+1, \quad \tilde\sigma=\mathrm{ind}_{W_E}^{W_F}(\tilde\xi\circ\alpha_E),$$

where E/F is a quadratic unramified extension, $\tilde\xi$ a smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-character of E^* such that:

(i) *The order of $\tilde\xi^\tau/\tilde\xi$ on $1+P_E$ is 2 where* $\mathrm{Gal}(E/F)=\{1,\tau\}$.

(ii) $\tilde\xi(b)\neq 1,\ \tilde\xi(b)^{\ell^s}=1$ *for a root of unity $b\in E^*$ of order $q+1$, and s is a positive integer such that ℓ^s divides $q+1$.*

(2) *Each irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation σ of W_F of dimension 2 admits a lift $\tilde\sigma$ to $\mathbb{Q}_\ell^{\mathrm{ac}}$ such that $|X_{\tilde\sigma}|=|X_\sigma|$.*

Proof. (1) Let Π be the supercuspidal smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representation of G and $\widetilde\Pi$ the integral supercuspidal smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation of G lifting Π such that $\sigma=\sigma_\Pi$, $\tilde\sigma=\sigma_{\widetilde\Pi}$ by the Langlands correspondence (4-24). We have $|X_\Pi|=|X_\sigma|$, $|X_{\widetilde\Pi}|=|X_{\tilde\sigma}|$ (4-27). By Proposition 4.15, $|X_\sigma|=|X_{\tilde\sigma}|$ or $2|X_{\tilde\sigma}|$, except maybe when $p=2$ and $\widetilde\Pi$ has positive level. In this exceptional case, $\eta_E\in X_{\widetilde\Pi}$. By Remark 4.21, $|X_\sigma|$ and $|X_{\tilde\sigma}|$ are equal to $1,2$ or 4. Therefore, $|X_\sigma|\neq|X_{\tilde\sigma}|$ is equivalent to $|X_\sigma|=4$ and $|X_{\tilde\sigma}|=2$.

When $|X_\sigma|=4$ and $|X_{\tilde\sigma}|=2$, $\sigma=\mathrm{ind}_{W_E}^{W_F}\xi$, $\tilde\sigma=\mathrm{ind}_{W_E}^{W_F}\tilde\xi$ for a quadratic unramified extension E/F, an integral smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-character $\tilde\xi$ of E^*, of reduction ξ modulo ℓ, with $\xi/\xi^\tau\neq 1$ where τ is the generator τ of $\mathrm{Gal}(E/F)$, and $(\xi/\xi^\tau)^2=1$. This implies $(\tilde\xi/\tilde\xi^\tau)^2=1$ on $p_F^{\mathbb{Z}}(1+P_E)$ because $\ell\neq p$. We have $E^*=p_F^{\mathbb{Z}}(1+P_E)\mu_E$ where $\mu_E=\{x\in E^*\mid x^{q^2-1}=1\}$. We have $\tau(x)=x^q$ if $x\in\mu_E$. The group $\{x^{q-1}\mid x\in\mu_E\}$ is generated by an arbitrary root of unity $b\in E^*$ of order $q+1$. So

$$(\tilde\xi/\tilde\xi^\tau)^2=1\iff\tilde\xi(b)^2=1\iff|X_{\tilde\sigma}|=4,\quad(\tilde\xi/\tilde\xi^\tau)^2\neq1\iff\tilde\xi(b)^2\neq1\iff|X_{\tilde\sigma}|=2.$$

In the exceptional case, $p=2$ hence ℓ is odd and $\xi(b)^2=1$ implies $\xi(b)=1$ (and conversely), or equivalently, the order of $\tilde\xi(b)$ is a power of ℓ dividing $q+1$. There exists a lift $\tilde\xi$ of ξ such that $\tilde\xi(b)\neq1$ if and only if ℓ divides $q+1$.

(2) Given a positive integer s, each element $x\in(\mathbb{F}_\ell^{\mathrm{ac}})^*$, $x\neq1$, is the reduction modulo ℓ of an element $\tilde x\in(\mathbb{Z}_\ell^{\mathrm{ac}})^*$ such that $\tilde x^{\ell^s}\neq1$. □

Corollary 4.24. (1) *The reduction modulo ℓ of a supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\tilde\pi$ of G' has length ≤ 2. It has length 2 if and only if*

$$p=2, \quad \ell \textit{ divides } q+1, \quad \sigma_{\widetilde\Pi}=\mathrm{ind}_{W_E}^{W_F}(\tilde\xi\circ\alpha_E),$$

where $\tilde\pi\in L(\widetilde\Pi)$, E/F is unramified, and $\tilde\xi$ is a smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-character of E^* such that:

(i) *The order of* $\tilde{\xi}^{\tau}/\tilde{\xi}$ *on* $1+P_E$ *is* 2 *where* $\mathrm{Gal}(E/F)=\{1,\tau\}$.

(ii) $\tilde{\xi}(b)\neq 1$, $\tilde{\xi}(b)^{\ell^s}=1$ *for a root of unity* $b\in E^*$ *of order* $q+1$, *and* ℓ^s *divides* $q+1$.

(2) *Each cuspidal* $\mathbb{F}_\ell^{\mathrm{ac}}$*-representation* π *of* G' *is the reduction modulo* ℓ *of an integral supercuspidal* $\mathbb{Q}_\ell^{\mathrm{ac}}$*-representation of* G'.

Proof. (1) This follows from

- Theorem 4.23(1), (4-21), and the local Langlands correspondence if $\ell\neq 2$,
- Proposition 4.15(1) if $\ell=2$.

(2) This follows from

- the fact that π lifts to $\mathbb{Q}_\ell^{\mathrm{ac}}$ by Theorem 4.23(2), (4-21), and the local Langlands correspondence if $p=2$ and π is in an L-packet $L(\Pi)$ with Π minimal of positive level (hence π is supercuspidal, see Corollary 4.27) with E_Π/F unramified,
- Proposition 4.15(2) otherwise. □

Remark 4.25. Assume $p\neq 2$. A pair (E,θ) where E/F is a quadratic extension of F and θ is a smooth R-character of E^*, is called *admissible* [Bushnell and Henniart 2006, Chapter 5, §18.2] if either:

(1) θ does not factorize through $N_{E/F}$ (equivalently is regular with respect to $\mathrm{Gal}(E/F)$).

(2) E/F is unramified whenever $\theta|_{1+P_E}$ does factorize through $N_{E/F}$ (equivalently is invariant under $\mathrm{Gal}(E/F)$).

To an admissible pair (E,θ) is associated the two-dimensional irreducible R-representation $\sigma(E,\theta)=\mathrm{ind}_{W_E}^{W_F}(\theta\circ\alpha_E)$ of W_F, and when $R=\mathbb{C}$ an explicitly constructed supercuspidal representation $\pi(E,\theta)$ of G [loc. cit., Chapter 5, §19]. Isomorphism classes of supercuspidal complex representations of G, are parametrized by isomorphism classes of admissible pairs (E,θ) [loc. cit., Chapter 5, §20.2]. The Langlands local correspondence sends $\pi(E,\theta)$ to $\sigma(E,\mu\theta)$ where the explicit "rectifier" μ is a tame character of E^* depending only on $\theta|_{1+P_E}$. As the Langlands correspondence is compatible with automorphisms of $\mathbb{C}$ preserving $\sqrt{q}$, the previous classification in terms of admissible pairs transfers to R-representations where R is an algebraically closed field of characteristic 0 (given a choice of square root of q in R). The classification and correspondence for $R=\mathbb{Q}_\ell^{\mathrm{ac}}$ reduce modulo $\ell\neq p$ (the integrality property for a pair (E,θ) is that θ takes integral values) to get a similar classification of supercuspidal $\mathbb{F}_\ell^{\mathrm{ac}}$-representations in terms of admissible pairs. The integral admissible pairs over $\mathbb{Q}_\ell^{\mathrm{ac}}$ that do not reduce to admissible pairs over $\mathbb{F}_\ell^{\mathrm{ac}}$, yield under reduction cuspidal but not supercuspidal $\mathbb{F}_\ell^{\mathrm{ac}}$-representations.

4.5. ***Principal series.*** We use the notations of Section 4. We identify a smooth R-character η of T' with a R-character of F^* and of T by

$$\eta(\mathrm{diag}(a,d)) = \eta(\mathrm{diag}(a,a^{-1})) = \eta(a) \quad (a,d \in F^*). \tag{4-29}$$

Proposition 4.11 describes $i_B^G(\eta)$. The transfer of the properties (i) to (iv) to

$$i_{B'}^{G'}(\eta) = (i_B^G(\eta))|_{G'}$$

is easy and gives:

(i) For smooth R-characters η, η' of F^*, $[i_{B'}^{G'}(\eta)]$ and $[i_{B'}^{G'}(\eta')]$ are disjoint if $\eta' \neq \eta^{\pm 1}$, and equal if $\eta' = \eta^{\pm 1}$.

(ii) The smooth dual of $i_{B'}^{G'}(\eta)$ is $i_{B'}^{G'}(\eta^{-1})$.

(iii) $(i_{B'}^{G'}(\eta))_U$ has dimension 2, contains η^{-1} and η is a quotient.

(iv) $\dim W_Y(i_{B'}^{G'}(\eta)) = 1$ for all $Y \neq 0$.

The transfer of the properties (v) and (vi) is harder.

Proposition 4.26. (i) $i_{B'}^{G'}(\eta)$ *is reducible if and only if* $\eta = q^{\pm \mathrm{val}}$, *or* $\eta \neq 1$ *and* $\eta^2 = 1$.

(ii) *When* $\mathrm{char}_R \neq 2$, $i_{B'}^{G'}(\eta_E)$ *is semisimple of length* 2, *when* E/F *is a quadratic separable extension*, *which is ramified if* $q+1=0$ *in* R.

(iii) *When* $\mathrm{char}_R = 2$, *the only reducible principal series is* $i_{B'}^{G'}(1) = \mathrm{ind}_{B'}^{G'}(1)$.

(iv) *The length of* $i_{B'}^{G'}(q^{-\mathrm{val}})$ *and of* $i_{B'}^{G'}(q^{\mathrm{val}}) = \mathrm{ind}_{B'}^{G'}(1)$ *is*

$$\lg(\mathrm{ind}_{B'}^{G'} 1) = \begin{cases} 2 & \text{if } q+1 \neq 0 \text{ in } R, \\ 4 & \text{if } q+1 = 0 \text{ in } R \text{ and } \mathrm{char}_R \neq 2, \\ 6 & \text{if } \mathrm{char}_R = 2. \end{cases}$$

Note that $\mathrm{char}_R = 2$ implies $q+1=0$ in R.

Proof. We show (i), (ii) and (iii).

If $i_B^G(\eta)$ is reducible, then its restriction $i_{B'}^{G'}(\eta)$ to G' is reducible. By Proposition 4.11, $i_B^G(\eta)$ is reducible if and only if $\eta = q^{\pm \mathrm{val}}$.

Assume $i_B^G(\eta)$ irreducible, i.e., $\eta \neq q^{\pm\mathrm{val}}$. If $\mathrm{char}_R \neq 2$, we have $X_{i_B^G(\eta)} = 2$ if and only if $\eta \neq 1$ and $\eta^2 = 1$ by the Langlands correspondence and Remark 4.21.[13] We have $\eta \neq 1$, $\eta^2 = 1$ if and only if $\eta = \eta_E$ for a quadratic separable extension E/F, which is ramified if $q+1=0$ in R (Notation 4.4) as $\eta \neq q^{\pm \mathrm{val}}$. If $\mathrm{char}_R = 2$, then p is odd, $\eta \neq 1$, and $i_{B'}^{G'}(\eta)$ is irreducible. Indeed, the irreducible components of $i_{B'}^{G'}(\eta)$ are B-conjugate (§6.2.1). They give a partition of the set of irreducible

[13] It can also be done directly because for a smooth R-character χ of F^*, Proposition 4.11(i) implies $(\chi \circ \det) \otimes i_B^G(\eta) \simeq i_B^G(\eta) \iff \chi\eta = \eta$ or $\eta^{-1} \iff \chi = 1$ or $\chi = \eta$ and $\eta^2 = 1$.

components of $(i_{B'}^{G'}(\eta))|_{B'}$. The character η appears with multiplicity 1 as $\eta \neq \eta^{-1}$, but as it is fixed by B, the partition is trivial, i.e., $i_{B'}^{G'}(\eta)$ is irreducible.

(iv) [Cui 2023, Example 3.11, Method 2] We give a proof for the convenience of the reader. When $q+1 \neq 0$ in R, the restriction to G' of the Steinberg representation St of G is irreducible, otherwise it would contain a cuspidal representation as $\dim_R \mathrm{St}_U = 1$ which is impossible by (4-15). When $q+1=0$ in R, the cuspidal R-representation Π_0 (see Proposition 4.11) is induced from the inflation to $Z\,\mathrm{GL}_2(O_F)$ of a cuspidal R-representation σ_0 of $\mathrm{GL}_2(k_F)$. By (4-18), $\lg(\Pi_0|_{G'}) = 2\lg(\sigma_0|_{\mathrm{SL}_2(k_F)})$. The representation $\sigma_0|_{\mathrm{SL}_2(k_F)}$ is irreducible if $\mathrm{char}_R \neq 2$, and has length 2 if $\mathrm{char}_R = 2$ (Appendix). □

Corollary 4.27. *The nonsupercuspidal smooth R-representations of G' are:*

- *The trivial character.*
- *If $q+1 \neq 0$ in R, the Steinberg R-representation* $\mathrm{st} = \mathrm{St}|_{G'}$.
- *The principal series $i_{B'}^{G'}(\eta)$ for the smooth R-characters η of F^* with $\eta \neq q^{\pm \mathrm{val}}$ and $\eta \neq \eta_E$ for any quadratic separable extension E/F.*
- *If* $\mathrm{char}_R \neq 2$, *the two irreducible components $\pi_E^{\pm}$ of $i_{B'}^{G'}(\eta_E)$ for a quadratic separable extension E/F, which is ramified if $q+1=0$ in R.*
- *If* $\mathrm{char}_R \neq 2$ *and $q+1=0$ in R, the two irreducible components of* $\Pi_0|_{G'}$.
- *If* $\mathrm{char}_R = 2$, *the four irreducible components of* $\Pi_0|_{G'}$.

The only isomorphisms between those representations are $i_{B'}^{G'}(\eta) \simeq i_{B'}^{G'}(\eta^{-1})$ for the irreducible principal series.

We get for nonsupercuspidal L-packets:

Proposition 4.28. *When $q+1=0$ in R, there is a unique cuspidal nonsupercuspidal L-packet. Its size is 2 if* $\mathrm{char}_R \neq 2$ *and 4 if* $\mathrm{char}_R = 2$.

- *When* $\mathrm{char}_R = 2$, *every noncuspidal L-packet is a singleton.*
- *When* $\mathrm{char}_R \neq 2$, *the noncuspidal L-packets are singletons or of size 2. Those of size 2 are in bijection with the isomorphism classes of the quadratic separable extensions of F.*

This proposition and Corollary 4.10 imply:

Corollary 4.29. *The L-packets of size 4 are cuspidal.*

We consider now the reduction modulo a prime number $\ell \neq p$. A noncuspidal irreducible $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\tilde{\pi}$ of G' is integral except when $\tilde{\pi} \simeq i_{B'}^{G'}(\tilde{\eta})$ for a nonintegral smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-character $\tilde{\eta}$ of F^*. When $\tilde{\pi}$ is integral, we deduce from Corollary 4.27 the length of the reduction $r_\ell(\tilde{\pi})$ modulo ℓ of $\tilde{\pi}$.

Proposition 4.30. (1) *The reduction $r_\ell(\tilde\pi)$ modulo ℓ of $\tilde\pi$ irreducible noncuspidal and integral is irreducible with the exceptions*:

- *If* $\ell = 2$, *then* $\lg(r_\ell(\tilde{s}t)) = 5$, $\lg(r_\ell(\tilde\pi_E^{\pm})) = 3$, $\lg\big(r_\ell(i_{B'}^{G'}(\tilde\eta))\big) = 6$ *for* $\tilde\eta$ *of order a finite power of* ℓ.
- *If* $\ell \neq 2$ *and* ℓ *divides* $q+1$, *then* $\lg(r_\ell(\tilde{s}t)) = 3$, $\lg\big(r_\ell(i_{B'}^{G'}(\tilde\eta))\big) = 4$ *for* $\tilde\eta$ *of order a finite power of* ℓ, $\lg\big(r_\ell(i_{B'}^{G'}(\tilde\eta))\big) = 2$ *if* $\tilde\eta = \tilde\eta_E\tilde\xi$, *for a ramified quadratic separable extension* E/F *and a character* $\tilde\xi$ *of order a power of* ℓ.

(2) *Each noncuspidal irreducible* $\mathbb{F}_\ell^{\mathrm{ac}}$*-representation of* G' *is the reduction modulo* ℓ *of an integral noncuspidal irreducible* $\mathbb{Q}_\ell^{\mathrm{ac}}$*-representation of* G'.

5. Local Langlands R-correspondence for $\mathrm{SL}_2(F)$

5.0.1. If (σ, N) is a two-dimensional Deligne R-representation of the Weil group W_F (§4.4.1), a choice of a basis of the space of σ gives a Deligne morphism of W_F into $\mathrm{GL}_2(R)$.[14] In this way equivalence classes of two-dimensional Deligne R-representations of W_F identify with Deligne morphisms of W_F into $\mathrm{GL}_2(R)$, up to $\mathrm{GL}_2(R)$-conjugacy.

By a Deligne morphism of W_F into $\mathrm{PGL}_2(R)$, we mean a pair (σ, N) where $\sigma : W_F \to \mathrm{PGL}_2(R)$ is a smooth morphism, semisimple in the sense that if $\sigma(W_F)$ is in a parabolic subgroup P then it is in a Levi of P, and N is a nilpotent[15] element in $\mathrm{Lie}(\mathrm{PGL}_2(R))$ with the usual requirement (4-23). We say that (σ, N) is irreducible if $\sigma(W_F)$ is not contained in a proper parabolic subgroup (meaning that $N = 0$ and the inverse image of $\sigma(W_F)$ in $\mathrm{GL}_2(R)$ acts irreducibly on R^2). The question arises whether a Deligne morphism (σ, N) of W_F into $\mathrm{PGL}_2(R)$ lifts to a two-dimensional Weil–Deligne R-representation.

When (σ, N) is reducible, we may assume that σ takes value in the diagonal torus of $\mathrm{PGL}_2(R)$, and that N is upper triangular. The map $x \mapsto \mathrm{diag}(x, 1)$ modulo scalars is an isomorphism from R^* to this torus, so σ comes from an R-character χ of W_F, and σ lifts to the two-dimensional $\chi \oplus 1$. That deals with the case where $N = 0$. When $N \neq 0$, then (σ, N) lifts to $(q^{-\mathrm{val}} \oplus 1, N)$.

The following lemma answers the question more generally for irreducible Deligne morphisms of W_F into $\mathrm{PGL}_n(R)$ for integers $n \geq 2$ (the definitions above for $n = 2$ generalize to $n > 2$).

Lemma 5.1. *Any irreducible smooth morphism* $\rho : W_F \to \mathrm{PGL}_n(R)$ *has finite image and its natural extension to* Gal_F *lifts to an irreducible smooth* R*-representation of* Gal_F *of dimension* n.

[14] We use the same notation (σ, N) for the Deligne morphism of W_F into $\mathrm{GL}_2(R)$.

[15] N is nilpotent in $\mathrm{Lie}(\mathrm{PGL}_2(R))$ if the Zariski closure of the $\mathrm{PGL}_2(R)$-orbit of N contains 0.

Proof. Because the inertia group I_F of W_F is profinite and ρ is smooth, $\rho(I_F)$ is finite. Let φ be a Frobenius element in W_F. If the order of $\rho(\varphi)$ is finite, then $\rho(W_F)$ is finite, so ρ extends by continuity to a smooth R-representation ρ' of Gal_F. The proof of Tate's theorem [Serre 1977, §6.5] applies with R instead of $\mathbb{C}$ and that shows that ρ' lifts to a smooth R-representation of Gal_F. Let us show that $\rho(\varphi)$ has finite order. Since $\rho(\varphi)$ acts by conjugation on $\rho(I_F)$ which is finite, a power $\rho(\varphi^d)$ for some positive d acts trivially on $\rho(I_F)$. But it also acts trivially on $\rho(\varphi)$, hence on all of $\rho(W_F)$. Let $A \in \mathrm{GL}_n(R)$ be a lift of $\rho(\varphi^d)$. For $B \in \mathrm{GL}_n(R)$, the commutator (A, B) depends only on the image of B in $\mathrm{PGL}_n(R)$, and if B has image $\rho(i)$ for $i \in I_F$, then (A, B) is a scalar $\mu(i)$. If $B' \in \mathrm{GL}_n(R)$ has image $\rho(i')$ for $i' \in I_F$, then $A(BB')A^{-1} = ABA^{-1}AB'A^{-1}$, giving $\mu(ii') = \mu(i)\mu(i')$, so conjugation by A induces a morphism $\mu : I_F \to R^*$. Since $\rho(I_F)$ is finite, a power A^e for some positive e commutes with the inverse image J in $\mathrm{GL}_n(R)$ of $\rho(W_F)$. Let V be an eigenspace of A^e. It is stable under J. If $V \neq R^n$, that yields a proper parabolic subgroup P (the image in $\mathrm{PGL}_n(R)$ of the stabilizer of V) of $\mathrm{PGL}_n(R)$ which contains $\rho(W_F)$, contrary to the hypothesis. So A^e is scalar, which implies that $\rho(\varphi)$ has finite order dividing de. □

Two 2-dimensional Deligne R-representations of W_F in $\mathrm{GL}_2(R)$ are twists of each other by a smooth R-character of W_F if and only if they give the same Deligne morphism of W_F in $\mathrm{PGL}_2(R)$. This happens if and only if the two corresponding irreducible smooth R-representations Π, Π' of G are twists of each other by a smooth R-character of G (4-25), that is, if and only if Π and Π' define the same L-packet $L(\Pi) = L(\Pi')$ of irreducible smooth R-representations of G' (4-4).

5.0.2. From the above the local Langlands correspondence for G induces a bijection between L-packets of irreducible smooth R-representations of G' and Deligne morphisms of W_F in $\mathrm{PGL}_2(R)$ up to $\mathrm{PGL}_2(R)$-conjugacy. We would like to understand the internal structure of a given packet in terms of an associated Deligne morphism $W_F \to \mathrm{PGL}_2(R)$ (called its L-parameter).

Let Π be an irreducible smooth R-representation of G. The L-packet $L(\Pi)$ is a principal homogeneous space of G/G_Π. The packet containing the trivial representation of G' is a singleton, so the parametrization is trivial. When $L(\Pi)$ is a packet of infinite-dimensional representations of G' we take as a base point in $L(\Pi)$ the element with nonzero Whittaker model with respect to the character ψ of F (that is, θ_0 of U) fixed in Section 4.1. Let C_Π denote the centralizer of the image in $\mathrm{PGL}_2(R)$ of a Deligne morphism (σ_Π, N_Π) of W_F in $\mathrm{GL}_2(R)$ associated to Π, and S_Π the component group of C_Π. We shall compute C_Π and S_Π, and when $\mathrm{char}_R \neq 2$ we shall construct a canonical isomorphism from G/G_π onto the R-characters of S_Π. In this way we get an enhanced local Langlands correspondence for $\mathrm{SL}_2(F)$ in the sense of [Aubert et al. 2016; 2017] if $\mathrm{char}_R \neq 2$ but not if $\mathrm{char}_R = 2$. J.-F. Dat tells us that our results for $\mathrm{char}_R = 2$ should still be compatible with the stacky approach of Fargues and Scholze to the semisimple Langlands correspondence. For example, for a supercuspidal R-representation Π of G, the two components of $\Pi|_{G'}$ should be indexed by the two irreducible R-representations of the group scheme μ_2.

The group of R-characters of G/G_Π is X_Π, and $X_\Pi = \{\chi \circ \det \mid \chi \in X_{(\sigma_\Pi, N_\Pi)}\}$ (4-27). We now construct a homomorphism $\varphi : X_{(\sigma_\Pi, N_\Pi)} \to S_\Pi$. Let $\chi \in X_{(\sigma_\Pi, N_\Pi)}$. By definition, there exists $A \in \mathrm{GL}_2(R)$ such that $AN_\Pi = N_\Pi$ and for $w \in W_F$, $A\sigma_\Pi(w)A^{-1} = \chi(w)\sigma_\Pi(w)$. The image $\bar{A}$ of A in $\mathrm{PGL}_2(R)$ belongs to C_Π and we shall show that its image $\varphi(\chi)$ in S_Π does not depend on the choice of A.

Theorem 5.2. *The map* $\varphi : X_{(\sigma_\Pi, N_\Pi)} \to S_\Pi$ *is a group isomorphism, and* $S_\Pi = \{1\}$, $\mathbb{Z}/2\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

When $\mathrm{char}_R = 2$, $S_\Pi = \{1\}$ *for each* Π, *but the length of* $\Pi|_{G'}$ *is*

- 1 *if* Π *is not cuspidal*,
- 2 *if* Π *is supercuspidal*,
- 4 *if* Π *is cuspidal not supercuspidal.*

Proof. (A) Let Π be a supercuspidal R-representation of G. Then σ_Π is irreducible and $N_\Pi = 0$ (Proposition 4.18).

When $\mathrm{char}_R \neq 2$, the authors of [Cui et al. 2024, Proposition 6.4] construct an isomorphism $\varphi : X_{\sigma_\Pi} \to C_\Pi$ when $\mathrm{char}_F \neq 2$, but their proof does not use this hypothesis. This implies $C_\Pi = S_\Pi$. One checks that $\varphi(\chi) = \varphi(\chi)$ for $\chi \in X_{\sigma_\Pi}$, an isomorphism.

When $\mathrm{char}_R = 2$, we have that p is odd, the cardinality of $L(\Pi)$ is 2 or 4 (Propositions 4.7 and 4.8), and $\sigma_\Pi = \mathrm{ind}_{W_E}^{W_F}(\theta)$ where E/F is a quadratic separable extension and θ a smooth R-character of W_E (or equivalently of E^*) different from its conjugate θ^τ by a generator τ of $\mathrm{Gal}(E/F)$. The character θ^τ/θ has finite odd order, say m, and $\sigma_\Pi(W_F) \subset \mathrm{GL}_2(R)$ is a dihedral group of order $2m$, generated by a matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ of order m and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ modulo conjugation in $\mathrm{GL}_2(R)$. So $C_\Pi = \{1\}$ and there is no enhanced correspondence.

(B) Let $\Pi = i_B^G(\eta)$ be an irreducible normalized principal series with the notation of (4-29), with $\eta \neq q^{\pm \mathrm{val}}$. The cardinality of $L(\Pi)$ is 2 if $\eta \neq 1$, $\eta^2 = 1$, and $L(\Pi)$ is a singleton otherwise. We have $\sigma_\Pi = (\eta \oplus 1) \circ \alpha_F$, $N_\Pi = 0$ (Proposition 4.18) and we easily see that C_Π is

- $\mathrm{PGL}_2(R)$ when $\eta = 1$, so $S_\Pi = \{1\}$,
- the diagonal torus when $\eta \neq 1$, $\eta^2 \neq 1$, $S_\Pi = \{1\}$,
- the normalizer of the trivial torus when $\eta \neq 1$, $\eta^2 = 1$, so $\mathrm{char}_R \neq 2$ and $S_\Pi = \mathbb{Z}/2\mathbb{Z}$. We have $X_\Pi = \{1, \eta \circ \det\}$ (Remark 4.21) and $\varphi(\eta)$ is not trivial, so $\varphi : X_\Pi \to S_\Pi$ is an isomorphism.

(C) If Π is an irreducible subquotient of $\mathrm{ind}_B^G 1$, the length of $\Pi|_{G'}$ (Section 4.5) is

- 1 when $\Pi = 1$, $q^{\mathrm{val}} \circ \det$ or St,
- 2 when $\Pi = \Pi_0$ if $\mathrm{char}_R \neq 2$ and $q+1=0$ in R,
- 4 when $\Pi = \Pi_0$ if $\mathrm{char}_R = 2$.

We have $\sigma_\Pi = ((q^{1/2})^{\mathrm{val}} \oplus (q^{-1/2})^{\mathrm{val}}) \circ \alpha_F$ ((4-24), Proposition 4.18). The centralizer C'_Π of the image of $\sigma_\Pi(W_F)$ in $\mathrm{PGL}_2(R)$ is the image in $\mathrm{PGL}_2(R)$ of

$$\{A \in \mathrm{GL}_2(R) \mid A \operatorname{diag}(q,1) A^{-1} \in R^* \operatorname{diag}(q,1)\}$$
$$= \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{GL}_2(R) \;\middle|\; \begin{pmatrix} xq & y \\ zq & t \end{pmatrix} = u \begin{pmatrix} xq & yq \\ z & t \end{pmatrix} \text{ for some } u \in R^* \right\}.$$

If $x \neq 0$ or $t \neq 0$ then $u = 1$, and if $y \neq 0$ then $qu = 1$. If $z \neq 0$ then $u = q$. So, C'_Π is

- $\mathrm{PGL}_2(R)$ if $q-1=0$ in R,
- the diagonal torus when $q-1 \neq 0$ in R and $q+1 \neq 0$ in R,
- the centralizer of the diagonal torus if $q-1 \neq 0$ in R and $q+1=0$ in R.

We have $N_\Pi = 0$, hence $C_\Pi = C'_\Pi$ when:

- $\Pi = 1$ when $q+1 \neq 0$ in R, hence $C_1 = \mathrm{PGL}_2(R)$ if $q+1 \neq 0$, $q-1=0$ in R (so $\mathrm{char}_R \neq 2$) and C_1 is the diagonal torus if $q+1 \neq 0$, $q-1 \neq 0$ in R. In both cases $S_1 = \{1\}$.
- $\Pi = \Pi_0$ cuspidal when $q+1=0$ in R. Recalling Section 4.5, when $\mathrm{char}_R \neq 2$, $\lg(\Pi_0|_{G'}) = 2$ and C_{Π_0} is the normalizer of the diagonal torus and $S_\Pi = \mathbb{Z}/2\mathbb{Z}$. We have $X_{\sigma_{\Pi_0}} = \{1, (-1)^{\mathrm{val}}\}$ (Corollary 4.13). As in (B), $\varphi((-1)^{\mathrm{val}})$ is not trivial, so $\varphi : X_\Pi \to S_\Pi$ is an isomorphism.

But when $\mathrm{char}_R = 2$, then $q-1=0$ in R and $C_{\Pi_0} = \mathrm{PGL}_2(R)$. As $S_{\Pi_0} = \{1\}$ and $\lg(\Pi_0|_{G'}) = 4$, there is no enhanced correspondence.

We suppose now $N_\Pi \neq 0$. Then (Proposition 4.18) $\Pi = \mathrm{St}$ when $q+1 \neq 0$ in R and Π is a character when $q+1=0$ in R. In both cases $\Pi|_{G'}$ is irreducible (Corollary 4.27). We can suppose that N_Π is a nontrivial upper triangular matrix. A similar analysis gives that C_Π is

- the diagonal torus if $q-1 \neq 0$ in R,
- the upper triangular subgroup if $q-1=0$ in R.

In both cases $S_\Pi = \{1\}$. □

Remark 5.3. We computed the centralizer $C_\Pi \subset \mathrm{PGL}_2(R)$:

- C_Π is finite if and only if Π is supercuspidal.

- When C_Π is connected, it is isomorphic to $\mathrm{PGL}_2(R)$, the upper triangular subgroup, the diagonal subgroup, or $\{1\}$.
- When C_Π has two connected components it is isomorphic to the normalizer of the diagonal subgroup or to $\mathbb{Z}/2\mathbb{Z}$.
- When C_Π has four connected components, it is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

5.0.3. Assume $\mathrm{char}_R = 2$. A kind of lifting has been introduced by [Treumann and Venkatesh 2016] and generalized in [Feng 2023]. They consider a (connected) split reductive F-group $\underline{H}$, equipped with an involution ι such that the group of fixed points $\underline{H}^\iota$ is (connected) split reductive. They set up a correspondence, called linkage, between ι-invariant irreducible smooth R-representations Π of $H = \underline{H}(F)$ and irreducible smooth R-representations of $H^\iota = \underline{H}^\iota(F)$. More precisely they show that there is a unique isomorphism ι_Π from Π to its conjugate Π^ι by ι, which has trivial square. They say that an irreducible smooth R-representation π of H^ι is linked with Π if the Frobenius twist of π occurs as a subquotient of the representation $T(\Pi) = \mathrm{Ker}(1+\iota_\Pi)/\,\mathrm{Im}(1+\iota_\Pi)$ of H^ι. They ask for an interpretation of linkage in terms of dual groups.

Let us consider the special case where $\underline{H} = \mathrm{GL}_2$ and $\iota(g) = g/\det g$.[16] Then $\underline{H}^\iota = \mathrm{SL}_2$, so $H = G,\ H^\iota = G'$. Let Π be an irreducible smooth R-representation of G of central character ω_Π. It is invariant under ι if and only if $\Pi \simeq \Pi \otimes (\omega_\Pi \circ \det)$. This implies that ω_Π has trivial square, so is trivial because $\mathrm{char}_R = 2$. In other words, Π is ι-invariant if and only if Π factors to a representation of $\mathrm{PGL}_2(F)$. It follows that then ι_Π is the identity, and $T(\Pi)$ is simply the restriction of Π to G', which we have thoroughly investigated. In particular $T(\Pi)$ has finite length, as expected. The dual group of $\underline{H}$ over R is $\mathrm{GL}_2(R)$, that of $\underline{H}^\iota$ is $\mathrm{PGL}_2(R)$. Treumann and Venkatesh ask for an interpretation of linkage in terms of a natural homomorphism from $\mathrm{PGL}_2(R)$ to $\mathrm{GL}_2(R)$.

Let $\sigma_\Pi : W_F \to \mathrm{GL}_2(R)$ be the semisimple L-parameter of Π. The map $\varphi^{-1}(\sigma_\Pi) : W_F \to \mathrm{GL}_2(R)$, followed by the quotient map $\mathrm{GL}_2(R) \to \mathrm{PGL}_2(R)$, is the semisimple L-parameter $\rho_\Pi : W_F \to \mathrm{PGL}_2(R)$ of the Frobenius twist of any constituent π of $\Pi|_{G'}$.

The map $\Psi(g) = \varphi(g)/\det g$ for $g \in \mathrm{GL}_2(R)$ where $\varphi : x \to x^2$ is the Frobenius map of R, is trivial on scalar matrices, hence factors through a homomorphism $\Psi : \mathrm{PGL}_2(R) \to \mathrm{GL}_2(R)$. The homomorphism Ψ is injective of image $\mathrm{SL}_2(R)$. Now if Π is ι-invariant, the determinant of σ_Π is trivial so $\sigma_\Pi = \Psi \circ \rho_\Pi$ and the conjectures of [Treumann and Venkatesh 2016, §6.3] are indeed true in our special case.

[16] $\iota(g)$ is conjugate to the transpose of the inverse of g.

6. Representations of $\mathbf{SL_2}(F)$ near the identity

6.1. Assume $\mathrm{char}_F = 0$ and $R = \mathbb{C}$. Let H be the group of F-points of a connected reductive group over F. We denote by $C_c^\infty(X; \mathbb{C})$ the space of smooth complex functions with compact support on a locally profinite space X. The exponential map exp from $\mathrm{Lie}(H)$ to H induces an H-equivariant bijection between a neighbourhood of 0 in $\mathrm{Lie}(H)$ and a neighbourhood of 1 in H. So a function $f \in C_c^\infty(H; \mathbb{C})$ with support small enough around 1 gives a smooth function $f \circ \exp$ around 0 in $C_c^\infty(\mathrm{Lie}(H); \mathbb{C})$. Also there are only finitely many nilpotent orbits of H in $\mathrm{Lie}(H)$, for the adjoint action. For each such orbit $\mathfrak{O}$, there is an H-invariant measure on $\mathfrak{O}$, and a function $\varphi \in C_c^\infty(\mathrm{Lie}(H); \mathbb{C})$ can be integrated along $\mathfrak{O}$ with respect to that measure, yielding an orbital integral $I_{\mathfrak{O}}(\varphi)$. Choosing a nondegenerate invariant bilinear form on $\mathrm{Lie}(H)$, a nontrivial character of $\mathrm{Lie}(H)$ and a Haar measure on $\mathrm{Lie}(H)$ yields a Fourier transform $\hat{\varphi}$ for a function $\varphi \in C_c^\infty(\mathrm{Lie}(H); \mathbb{C})$. Fix also a Haar measure dh on H.

Theorem 6.1. *Let Π be a smooth complex representation of H with finite length. Then there is an open neighbourhood $V(\Pi)$ of 1 in H and for each nilpotent orbit $\mathfrak{O}$ a unique complex number $c_{\mathfrak{O}} = c_{\mathfrak{O}}(\Pi)$ such that if $f \in C_c^\infty(H; \mathbb{C})$ has compact support in $V(\Pi)$ then the trace $\mathrm{tr}_\Pi(f)$ of the linear endomorphism $\int_H f(h)\Pi(h)\,dh$ is equal to*

$$\mathrm{tr}_\Pi(f) = \sum_{\mathfrak{O}} c_{\mathfrak{O}}(\Pi) I_{\mathfrak{O}}(\hat{\varphi}) \quad \textit{where } \varphi = f \circ \exp. \tag{6-1}$$

This was first proved by Roger Howe when $H = \mathrm{GL}_n(F)$, and the general case is due to Harish-Chandra.

As is usual, we say that a nilpotent orbit $\mathfrak{O}'$ is smaller than a nilpotent orbit $\mathfrak{O}$ if $\mathfrak{O}'$ is contained in the closure of $\mathfrak{O}$. With the normalizations as in [Varma 2014] we have:

Theorem 6.2. *Let Π be a smooth complex representation of H with finite length. When $\mathfrak{O}$ is maximal among the orbits with $c_{\mathfrak{O}}(\Pi) \neq 0$, then $c_{\mathfrak{O}}(\Pi)$ is equal to the dimension of generalized Whittaker spaces for Π attached to $\mathfrak{O}$.*

The result when p is odd due to [Mœglin and Waldspurger 1987] is extended to $p = 2$ in [Varma 2014] in general. When $\mathfrak{O}$ is a regular nilpotent orbit, the generalized Whittaker model is the usual one, and the result then goes back to Rodier [1975]. Varma actually proves that with that normalization all coefficients $c_{\mathfrak{O}}(\Pi)$ are rational [2014].

6.2. Assume $R = \mathbb{C}$. For any F, when H is an open normal subgroup of $\mathrm{GL}_r(D)$ where D is a finite-dimensional central division F-algebra, Theorem 6.1 still holds, with the exponential map replaced by the map $X \mapsto 1 + X$ [Lemaire 2004]. In the special case where $H = \mathrm{GL}_r(D)$, Theorem 6.2 also holds, at least for the natural generalized Whittaker space attached to each nilpotent orbit [Henniart and Vignéras 2024].

6.2.1. We use the notations and definitions introduced in Section 4.1. Let H be an open normal subgroup of $G = \mathrm{GL}_2(F)$ containing ZG'. The index of H in G is finite as H/ZG' is open in the compact group G/ZG'. Put

$$V_H = F^*/\det H, \quad \dim_{\mathbb{F}_2} V_H = d, \quad |G/H| = 2^d. \tag{6-2}$$

A nilpotent matrix can be conjugated in a lower triangular nilpotent matrix Y by an element of G'. Two such matrices Y and Y' are H-conjugate if and only if their bottom left coefficients differ by multiplication by an element of $\det H$.

(6-3) The number of H-orbits in the nilpotent matrices in $M_2(F)$ is $1 + 2^d$.

The 0-matrix forms the smallest nilpotent H-orbit (the "trivial" one). The nontrivial nilpotent H-orbits are maximal, and parametrized by V_H via their bottom left coefficient.

With the same arguments as those given for ZG' in Section 4.1, any irreducible smooth R-representation π of H appears in the restriction to H of an irreducible smooth representation Π of G, unique modulo torsion by a smooth R-character of G. The irreducible components π of $\Pi|_H$ are G-conjugate (even B-conjugate) and the G-stabilizer of π does not depend on the choice of π in $\Pi|_H$, and denoted by $G_{\Pi|_H}$. The representation $\Pi|_H$ is semisimple of multiplicity 1 with length

$$\lg(\Pi|_H) = |G/G_{\Pi|_H}| \quad \text{dividing } \lg(\Pi|_{ZG'}) = |G/G_\Pi| = |L(\Pi)|, \tag{6-4}$$

hence equal to 1, 2 or 4 by Theorem 1.1. The representation $\pi|_{G'}$ is semisimple of multiplicity 1 with length $\lg(\pi|_{G'}) = \lg(\Pi|_{G'})/\lg(\Pi|_H) = |G_{\Pi|_H}/G_\Pi|$.

For a lower triangular matrix $Y \neq 0$, we have

$$\sum_{\pi \subset \Pi|_H} \dim_R W_Y(\pi) = \dim_R W_Y(\Pi) = 1.$$

There is a single irreducible π in $\Pi|_H$ with $W_Y(\pi) \neq 0$, and $\dim_R W_Y(\pi) \neq 0 \Longleftrightarrow \dim_R W_Y(\pi) = 1$. If $W_Y(\pi) \neq 0$ then $W_{Y'}(\pi) \neq 0$ when Y' and Y are H-conjugate. We consider $\dim_R W_Y(\pi)$ as a function m_π on V_H. Because π extends to $G_{\Pi|_H}$, m_π is invariant under translations by

$$W_{\Pi|_H} = \det G_{\Pi|_H}/\det H.$$

It follows that m_π is the characteristic function of an affine subspace A_π of V_H with direction $W_{\Pi|_H}$, each such affine subspace being obtained exactly for one $\pi \subset \Pi|_H$. For $g \in G$ we denote $\pi^g(x) = \pi(gxg^{-1})$ for $g \in G$, $x \in H$, so $\pi^{gh} = (\pi^g)^h$

for $g, h \in G$. We have $A_{\pi^g} = \det g\, A_\pi$. We have a disjoint union (the Whittaker decomposition):

$$V_H = \bigsqcup_{\pi \subset \Pi|_H} A_\pi. \tag{6-5}$$

If $\lg(\Pi|_H) = 1$, m_π is the constant function on V_H with value 1. If $\lg(\Pi|_H) = 2$, the two irreducible components of $\Pi|_H$ yield the characteristic functions of two affine hyperplanes of V_H with the same direction. Finally for $\lg(\Pi|_H) = 4$, we get the characteristic functions of four affine subspaces of codimension 2 in V_H with the same direction. In particular when p is odd and $\lg(\Pi|_H) = 4$, we have $H = ZG'$ and m_π is a nonzero delta function on $V_H = F^*/(F^*)^2$.

Let $C(V_H; \mathbb{Z})$ denote the $\mathbb{Z}$-module of functions $f : V_H \to \mathbb{Z}$. For an integer $0 \le r < d$, let I_r denote the $\mathbb{Z}$-submodule of $C(V_H; \mathbb{Z})$ generated by the characteristic functions of the r-dimensional affine subspaces of V_H. We have $I_0 = C(V_H; \mathbb{Z})$.

Lemma 6.3. *When $0 < r < d$, $2I_{r-1}$ is included in I_r and the exponent of I_0/I_r is 2^r.*

Proof. Let W be a $(r-1)$-dimensional vector subspace of V_H and $\{0, e, f, e+f\}$ a supplementary plane. For an affine subspace A of V_H of direction W, the affine subspaces $A_e = A \cup A + e$, $A_f = A \cup A + f$ and $B = A + e \cup A + f$ of V_H are r-dimensional, and $\chi_{A_e} + \chi_{A_f} - \chi_B = 2\chi_A$ by taking their characteristic functions χ. Thus $2I_{r-1} \subset I_r$. By induction $2^r I_0 \subset I_r$. The map $s_r : C(V_H; \mathbb{Z}) \to \mathbb{Z}/2^r\mathbb{Z}$ given by the sum of coordinates is surjective and vanishes on I_r but not on I_{r-1}. So the exponent of I_0/I_r is 2^r. □

6.2.2. Let us make Theorem 6.1 more precise for an open normal subgroup H of $G = \mathrm{GL}_2(F)$ as in §6.2.1.

Notation 6.4. On G (hence on H) we put a Haar measure dg, and on $\operatorname{Lie} G = \operatorname{Lie} H = M_2(F)$ we put the Haar measure dX such that $X \mapsto 1 + X$ preserves measures near 0. The invariant bilinear map $(X, X') \mapsto \operatorname{tr}(XX')$ on $\operatorname{Lie}(H)$ is nondegenerate. The Fourier transform $\varphi \mapsto \hat\varphi$ on $C_c^\infty(\operatorname{Lie}(H); \mathbb{C})$ is taken with respect to the nontrivial character $\psi \circ \operatorname{tr}$ on $\operatorname{Lie}(H)$. For each nilpotent H-orbit $\mathfrak{O}$ in $\operatorname{Lie}(H)$, we normalize the nilpotent orbital integral $I_{\mathfrak{O}}(\hat\varphi)$ [Lemaire 2005, proposition 1.5] in the same way as [Varma 2014, §3]; that normalization is valid even when $\mathrm{char}_F > 0$. By [loc. cit., Remark 2], for large enough i, $K_i = 1 + M_2(P_F^i)$ and a lower triangular nilpotent matrix Y, the measure of $\operatorname{Ad}(K_i)(Y)$ is 0 if $Y = 0$ and q^{-2i} otherwise. In particular $I_0(\hat\varphi) = \varphi(0)$ for the nilpotent trivial orbit $0 \in \operatorname{Lie} H$.

Theorem 6.5. *Let π be a smooth complex representation of H with finite length. There is an open neighbourhood $V(\pi)$ of 1 in H and for each nilpotent H-orbit $\mathfrak{O}$ a unique complex number $c_{\mathfrak{O}} = c_{\mathfrak{O}}(\pi)$ such that if $f \in C_c^\infty(H; \mathbb{C})$ has compact* *support in* $V(\pi)$ *then*

(6-6) $$\mathrm{tr}_\pi(f) = c_0(\pi) f(1) + \sum_{\mathfrak{O}\neq 0} c_{\mathfrak{O}}(\pi) I_{\mathfrak{O}}(\hat{\varphi})$$

where $\varphi(X) = f(1+X)$ *for* $1+X \in V(\pi)$.

We call (6-6) the germ expansion and $c_0(\pi)$ the constant coefficient of the trace of π around 1. A character twist of π does not change $c_0(\pi)$. For π irreducible, $c_{\mathfrak{O}}(\pi) = 0$ for all $\mathfrak{O} \neq 0$ if and only if π is degenerate (by Theorem 6.2) if and only if $\dim_{\mathbb{C}} \pi = 1$. In this case $c_0(\pi) = 1$.

We can determine that constant coefficient $c_0(\pi)$ for any irreducible smooth representation π of H from the case of G, because π appears in the restriction to H of an irreducible smooth complex representation Π of G. The irreducible components of $\Pi|_H$ being G-conjugate to π have the same constant coefficient,[17] and

(6-7) $$c_0(\Pi) = \lg(\Pi|_H) c_0(\pi).$$

By [Henniart and Vignéras 2024], we have $c_0(1_G) = 1$. When Π is parabolically induced, for example when Π is tempered and not a discrete series,

$$c_0(\Pi) = 0.$$

When Π is a discrete series representation of formal degree $d(\Pi)$,

$$c_0(\Pi) = -d(\Pi)/d(\mathrm{St}).$$

When Π is a supercuspidal complex smooth representation of G of minimal level f_Π (the minimal level[18] of the character twists of Π),

(6-8) $$c_0(\Pi) = \begin{cases} -2q^{f_\Pi} & \text{if } f_\Pi \text{ is an integer,} \\ -(q+1)q^{f_\Pi - \frac{1}{2}} & \text{if } f_\Pi \text{ is a half-integer (not an integer).} \end{cases}$$

When f_Π is a half-integer (not an integer), Π has positive level (Section 4.3.2), $\Pi = \mathrm{ind}_J^G \Lambda$ where $J = E^*(1 + Q^{f_\Pi + \frac{1}{2}})$, where E/F is ramified, Q is the Jacobson radical of an Iwahori order in $M_2(F)$, and Λ is trivial on $1 + Q^{2f_\Pi + 1}$ [Bushnell and Henniart 2006, Chapter 4, §15]. Let $\chi \in X_\Pi \setminus \{1\}$. Then χ is ramified [Bushnell and Henniart 2006, Chapter 5, §20.3, Lemma]. The level r_χ of χ is the largest positive integer r such that χ is nontrivial on $1 + P_F^r$ when χ is ramified. We have

(6-9) $$1 \leq r_\chi < f_\Pi.$$

[17] By the linear independence of nilpotent orbital integrals.

[18] The level is the normalized level of [Bushnell and Henniart 2006, Chapter 4, §12.6] and the depth is in the sense of Moy–Prasad.

Indeed, if $r_\chi > f_\Pi$ then $\chi \circ \det$ is nontrivial on $1+Q^{2r_\chi}$ (as $\det(1+Q^{2r_\chi}) = 1+P_F^{r_\chi}$), and $(\chi \circ \det) \otimes \Lambda$ would be nontrivial on $1+Q^{2r_\chi}$ implying that the level of $(\chi \circ \det) \otimes \Lambda$ is at least r_χ. By [Bushnell and Henniart 2006, §15.6, Proposition 1], this contradicts the assumption that $\chi \in X_\Pi$. So $f_\Pi < r_\chi$ as r_χ is an integer but not f_Π.

Lemma 6.6. *If $f_\Pi = \frac{1}{2}$ then $X_\Pi = \{1\}$. If $q = 2$ and $f_\Pi = \frac{3}{2}$ then X_Π cannot have four elements.*

Proof. If $f_\Pi = \frac{1}{2}$, then X_Π is trivial by the formula (6-9). If $f_\Pi = \frac{3}{2}$, then $r_\chi = 1$, and if $q = 2$ there are only two quadratic characters of level 1. That implies that X_Π cannot have four elements. □

Proposition 6.7. *Let Π be an irreducible complex smooth representation of G and π an irreducible representation of H contained in $\Pi|_H$. Then*:

- $c_0(\pi) = -\frac{1}{2}$ *if p is odd, Π is cuspidal of minimal level* 0 *and $L(\Pi)$ has four elements.*
- $c_0(\pi)$ *is an integer otherwise.*
- $c_0(\pi) = 0$ *if π is a principal series, and $c_0(\pi) < 0$ if π is infinite-dimensional and not a principal series.*

Proof. By formulas (6-4), (6-7), (6-8), we have:

- $c_0(1_G) = 1$, so $c_0(1_H) = 1$.
- $c_0(\mathrm{St}) = -1$ so $c_0(\mathrm{st}_H) = -1$, since the restriction st_H of St to H is irreducible as $\mathrm{st} = \mathrm{St}\,|_{G'}$ is irreducible.
- $c_0(\Pi) = 0$ so $c_0(\pi) = 0$, when Π is an irreducible principal series.
- $c_0(\Pi) < 0$ so $c_0(\pi) < 0$, when Π supercuspidal of level f_Π (the minimal level).

If p is odd, then $c_0(\Pi)$ is an even integer by (6-8), so that $c_0(\pi)$ is an integer if $L(\Pi)$ has one or two elements by (6-7); if $L(\Pi)$ has four elements, then $f_\Pi = 0$ by Proposition 4.8 and $c_0(\Pi) = -2$, so $c_0(\pi) = -\frac{1}{2}$. If $p = 2$, then $c_0(\Pi)$ is a multiple of 4 (so $c_0(\pi)$ is an integer) by (6-8) except when:

(i) $f_\Pi = 0$, where $c_0(\Pi) = -2$. But $L(\Pi)$ has size 2 by Proposition 4.7, so $c_0(\pi) = -1$.

(ii) $f_\Pi = \frac{1}{2}$, where $c_0(\Pi) = -(q+1)$. But $L(\Pi)$ has size 1 by Lemma 6.6, so $c_0(\pi) = -(q+1)$.

(iii) $f_\Pi = \frac{3}{2}$ and $q = 2$, where $c_0(\Pi) = -6$. But $L(\Pi)$ has size 1 or 2 by Lemma 6.6, so $c_0(\pi) = -6$ or -3. □

Theorem 6.8. *Let π be a finite length complex representation of H, $Y \neq 0$ a lower triangular matrix in $M_2(F)$ and $\mathfrak{O}$ its H-orbit. Then $c_{\mathfrak{O}}(\pi) = \dim_{\mathbb{C}} W_Y(\pi)$.*

Proof. We use the same idea as [Rodier 1975]. Remarking that the lower triangular group B^- of G acts transitively on lower triangular nilpotent matrices Y, and that for $g \in B^-$ we have $c_{\mathfrak{O}}(\pi) = c_{\mathfrak{O}^g}(\pi^g)$, $\dim_{\mathbb{C}}(W_Y(\pi)) = \dim_{\mathbb{C}}(W_{Y^g}(\pi^g))$, it suffices to consider the case where $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. We stick to that Y (so $\theta_Y = \theta$ with Notation 4.1).

For each positive integer i, we define a character χ_i of the pro-p group $K_i = 1 + M_2(P_F^i)$ by the formula

$$\chi_i(1+X) = \psi \circ \mathrm{tr}(p_F^{-2i} YX) = \psi(p_F^{-2i} X_{1,2}), \quad X = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \in M_2(P_F^i).$$

The character χ_i is trivial on K_{2i}. When conjugating by the diagonal matrix $d_i = \mathrm{diag}(p_F^i, p_F^{-i})$ we get a character θ_i on

$$H_i = d_i^{-1} K_i d_i = 1 + \begin{pmatrix} P_F^i & P_F^{-i} \\ P_F^{3i} & P_F^i \end{pmatrix} \tag{6-10}$$

such that $\theta_i(1+X) = \psi(X_{1,2})$. The limit of the groups H_i as $i \to \infty$ is the group U. We will prove that the θ_i approximate the character θ_Y of U in the sense that

$$\lim_{i \to \infty} \dim_{\mathbb{C}} \mathrm{Hom}_{H_i}(\theta_i, \pi) = \dim_{\mathbb{C}} W_Y(\pi). \tag{6-11}$$

On the other hand we will also prove in §6.2.3, following [Varma 2014], that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_i}(\chi_i, \pi) = c_{\mathfrak{O}}(\pi) \quad \text{for large } i. \tag{6-12}$$

Since $\dim_{\mathbb{C}} \mathrm{Hom}_{H_i}(\theta_i, \pi) = \dim_{\mathbb{C}} \mathrm{Hom}_{K_i}(\chi_i, \pi)$, we shall get the result. □

6.2.3. Let us proceed to the proof of the formulas (6-11) and (6-12), through a sequence of lemmas that are rather easy compared to the analogous statements in the more general cases treated by [Rodier 1975; Mœglin and Waldspurger 1987; Varma 2014] when $\mathrm{char}_F = 0$, and [Henniart and Vignéras 2024] for arbitrary char_F.

For $X \in M_2(F)$, put $\delta_i(X) = \chi_i^{-1}(1+X)$ if $X \in M_2(P_F^i)$ and $\delta_i(X) = 0$ otherwise. Using Notation 6.4, the Fourier transform $\hat{\delta}_i$ of δ_i is

$$\hat{\delta}_i(X) = \begin{cases} q^{-4i} \,\mathrm{vol}(M_2(O_F), dX) & \text{if } X \in p_F^{-2i} Y + M_2(P_F^{-i}), \\ 0 & \text{otherwise.} \end{cases} \tag{6-13}$$

Lemma 6.9. *The K_1-normalizer of χ_i is $(ZU^- \cap K_1)K_i$.*

Proof. For a positive integer $j \le i$, we prove that the K_1-normalizer of the restriction of χ_i to K_{2i-j} is $(ZU^- \cap K_1)K_j$ by induction on j. This is clear for $j = 1$ and the case $j = i$ gives what we want. Assume that the claim is true for $j < i$ and let us prove it for $j+1$. Let $g \in K_1$, normalizing the restriction of χ_i to K_{2i-j-1}. By induction $g \in (ZU^- \cap K_1)K_j$ and we may assume $g \in K_j$. Write $g = 1 + X$ with $X \in M_2(P_F^j)$. Then $g^{-1}Yg \equiv Y + YX - XY$ modulo $M_2(P_F^{j+1})$ and the hypothesis on g means that $YX - XY \equiv 0$ modulo $M_2(P_F^{j+1})$, which gives that $p_F^{-j} X$ commutes with Y modulo P_F. But the commutant of Y modulo P_F in

$M_2(k_F)$ is made out of lower triangular matrices with the same diagonal elements. Consequently $g \in (ZU^- \cap K_1)K_{j+1}$ as claimed. □

Lemma 6.10. *The K_i-orbit of Y is the set of nilpotent matrices in $Y + M_2(P_F^i)$.*

Proof. Clearly, gYg^{-1} is a nilpotent element in $Y + M_2(P_F^i)$ for $g \in K_i$. Conversely, let $Y + p_F^i Z$ nilpotent (hence of trace 0) with $Z \in M_2(O_F)$. If $g = 1 + p_F^i X$ with $X \in M_2(O_F)$, then $g(Y + p_F^i Z)g^{-1} \equiv Y + p_F^i(YX - XY + Z)$ modulo $M_2(P_F^{i+1})$. We choose X, as we can, so that $YX - XY + Z \equiv 0$ modulo P_F. So $g(Y + p_F^i Z)g^{-1} \in Y + M_2(P_F^{i+1})$. The K_i-orbit of Y is closed in $M_2(F)$. We finish the proof by successive approximations. □

Let π be a smooth representation of H on a complex vector space V, and $\varphi : V \to V_\theta$ be the quotient map from V to the θ-coinvariants V_θ of V. For large enough i such that $H_i \subset H$ let V_i be the θ_i-isotypic component of V.

Lemma 6.11. *For large enough i, $\varphi(V_i) = V_\theta$.*

Proof. It is the same as that of Lemma 8.7 in [Henniart and Vignéras 2024]. □

We have

$$H_{i+1} = (H_{i+1} \cap H_i)(H_{i+1} \cap U), \quad [H_{i+1} : (H_{i+1} \cap H_i)] = [(H_{i+1} \cap U) : (H_i \cap U)] = q^{-1},$$

and $\theta_{i+1} = \theta_i$ on $H_{i+1} \cap H_i$. Let $e_i = f_i dg$ where dg is the Haar measure on H giving the volume 1 to H_i and f_i is the function on G with support H_i and value θ_i^{-1} on H_i.

Lemma 6.12. *We have $e_i e_{i+1} e_i = q^{-1} e_i$ when $i > 1$ and $H_i \subset H$. In particular, the map $v \to \pi(e_{i+1})v : V_i \to V_{i+1}$ is injective.*

Proof. The lemma is equivalent to $\pi(e_i e_{i+1} e_i)v = q^{-1}\pi(e_i)v$ for all $v \in V$ and (π, V) as above. The projector $V \to V_i$ is $\pi(e_i)$ and

$$\pi(e_i e_{i+1} e_i)v = q^{-1} \sum_{u \in (H_{i+1} \cap U)/(H_i \cap U)} \pi(e_i \theta_{i+1}(u)^{-1} u e_i)v.$$

If $\pi(e_i u e_i)v \neq 0$ for $u \in H_{i+1} \cap U$, then u intertwines θ_i. To interpret that condition we conjugate θ_i back to χ_i. Then H_i is sent to K_i and H_{i+1} is sent to $d_1^{-1} K_{i+1} d_1$ which, we remark, is contained in K_{i-1}. By Lemma 6.9, $u \in H_{i+1} \cap U$ conjugates to an element in $(ZU^- \cap K_1)K_i$, so that $u \in H_i \cap U$. We then deduce that $\pi(e_i e_{i+1} e_i)v = q^{-1}\pi(e_i)v$ as claimed. □

Proof of formula (6-11). Fix a large integer i such that the lemmas apply. The projector $\pi(e_i) : V \to V_i$ can be obtained by first projecting onto $V^{H_i \cap B^-}$, and then applying the projector $\pi(e_{i,U})$ where $e_{i,U} = f_i|_{H_i \cap U}\, du$ for the Haar measure on $H \cap U$ giving the volume 1 to $H_i \cap U$. Since $V_i \subset V^{H_{i+1} \cap B^-}$, we have that $\pi(e_{i+1}) = \pi(e_{i+1,U})$ on V_i. It follows that for $v \in V_i$ and $v_1 = \pi(e_{i+1})v = \pi(e_{i+1,U})v$ have the same image $\varphi(v_1) = \varphi(v)$ in V_θ. Iterating the process, we get for positive integers k, vectors $v_k = \pi(e_{j+k})v_{k-1} = \pi(e_{j+k,U})v_{k-1}$ with $\varphi(v_k) = \varphi(v)$. As $e_{i+1,U}e_{i,U} = e_{i+1,U}$ we have $v_k = \pi(e_{i+k,U})v$. But $\varphi(v) = 0$ is equivalent to $\pi(e_{i+k,U})v = 0$ for large k. As $v_k = 0$ implies $v_{k-1} = 0$ by Lemma 6.12, we get that φ is injective on V_i. Since it is also surjective by Lemma 6.11, we deduce that it gives an isomorphism $V_i \simeq V_\theta$. □

Proof of formula (6-12). Fix an integer i such that $K_i \subset H$. We have that $\dim_{\mathbb{C}}(\mathrm{Hom}_{K_i}\,\chi_i, \pi) = \mathrm{tr}\,\pi(e'_i)$ where $e'_i = f'_i\,dg$ where dg is the Haar measure on H giving the volume 1 to K_i and f'_i is the function on G with support K_i and value χ_i^{-1} on K_i. We have that $f'_i(1+X) = \delta_i(X)$. To prove (6-12), it suffices to apply the germ expansion (6-6) to tr_π and to show that for large i, $I_{\mathfrak{O}}(\hat{\delta}_i) = 1$, whereas $I_{\mathfrak{O}'}(\hat{\delta}_i) = 0$ for any nilpotent orbit $\mathfrak{O}' \neq \mathfrak{O}$. From the formula (6-13), $\hat{\delta}_i$ is a multiple of the characteristic function of $-p_F^{-2i}Y + M_2(P_F^{-i})$ and from Lemma 6.10 the nilpotent elements there form the K_i-orbit of $p_F^{-2i}Y$. It follows that $I_{\mathfrak{O}'}(\hat{\delta}_i) = 0$ if $\mathfrak{O}' \neq \mathfrak{O}$. That $I_{\mathfrak{O}}(\hat{\delta}_i) = 1$ is proved exactly as in the proof of Lemma 7 in [Varma 2014]. □

6.2.4. For a locally profinite space X, $x \in X$, and a field C, two linear forms f, f' on $C_c^\infty(V; C)$ for some open neighbourhood V of x in X are called equivalent if their restrictions to $C_c^\infty(W; C)$ for some open neighbourhood W of x contained in V are equal. The equivalence class of f is called its germ $\tilde{f}$ at x. Denote $\mathfrak{G}_x(X)$ the space of the germs at x.

For a locally profinite space X', an open subset W in X and an open subset W' in X', a homeomorphism $j : W \to W'$ gives by functoriality an isomorphism $C_c^\infty(W'; C) \to C_c^\infty(W; C)$ and an isomorphism $\mathfrak{G}_{j(x)}(X') \to \mathfrak{G}_x(X)$ from the space of the germs of X' at $j(x)$ to the space of the germs of X at $x \in W$.

The nilpotent orbital integrals $\mathcal{F}_{\mathfrak{O}} : \varphi \mapsto I_{\mathfrak{O}}(\hat{\varphi})$ for $\varphi \in C_c^\infty(\mathrm{Lie}\,H; \mathbb{C})$ and the nilpotent H-orbits $\mathfrak{O}$ in $\mathrm{Lie}(H)$ are linearly independent H-equivariant linear forms on $C_c^\infty(\mathrm{Lie}\,H; \mathbb{C})$ [Lemaire 2005, page 79]. They form a basis of a $\mathbb{Z}$-module I_H with rank $1 + 2^d$ (6-3). For each H-equivariant open neighbourhood V of 0 in $\mathrm{Lie}\,H$, the $\mathcal{F}_{\mathfrak{O}}$ remain independent as linear forms on $C_c^\infty(V; \mathbb{C})$. The germs $\widetilde{\mathcal{F}}_{\mathfrak{O}}$ form a basis of the $\mathbb{Z}$-module $\tilde{I}_H$ of germs of elements of I_H. Denote by I_H^{Wh} the $\mathbb{Z}$-submodule of I_H of basis $\mathcal{F}_{\mathfrak{O}}$ for $\mathfrak{O} \neq 0$.

Theorems 6.5 and 6.8 say that the germ at 1 of the trace of an irreducible complex smooth representation π of H identifies via the map $X \mapsto 1 + X$ with the germ at 0 of a unique element $T_\pi = c_0(\pi)\mathcal{F}_0 + T_\pi^{\mathrm{Wh}}$ where $c_0(\pi) \in \mathbb{Q}$, and $T_\pi^{\mathrm{Wh}} \in I_H^{\mathrm{Wh}}$ is determined by the nondegenerate Whittaker models of π. Note that $T_\pi^{\mathrm{Wh}} = 0$ if and only if $\dim_{\mathbb{C}} \pi = 1$.

Denote by T_H^{Wh} the $\mathbb{Z}$-submodule of I_H^{Wh} generated by the T_π^{Wh}, for all irreducible complex smooth representations π of H. Write $\tilde{I}_H^{\mathrm{Wh}}$, $\tilde{T}_H^{\mathrm{Wh}}$ for the space of germs at 0 of I_H^{Wh}, T_H^{Wh}.

Theorem 6.13. *We have $\tilde{T}_H = \tilde{I}_H$ when $d = 0, 1$.*

The $\mathbb{Z}$-submodule $\tilde{T}_H^{\mathrm{Wh}}$ is a submodule of $\tilde{I}_H^{\mathrm{Wh}}$ of finite index. The exponent of $\tilde{I}_H^{\mathrm{Wh}}/\tilde{T}_H^{\mathrm{Wh}}$ is 2^{d-2} when $d \geq 2$.

Proof. When $d = 0$, I_H has $\mathbb{Z}$-rank 2, and the germs of the traces of the trivial representation 1 and of the Steinberg representation st_H form a $\mathbb{Z}$-basis $\{\widetilde{\mathrm{tr}}_1, \widetilde{\mathrm{tr}}_{\mathrm{st}_H}\}$ of $\tilde{I}_H$.

When $d = 1$, I_H has $\mathbb{Z}$-rank 3, $\det H = N_{E/F}(E^*)$ for a quadratic separable extension E/F, the principal series $(i_B^G \eta_E)|_H$ is semisimple of length 2 and multiplicity free (Lemma 2.3 and footnote in the proof of Proposition 4.26), and the germs of the traces of the trivial representation 1 and of the two components π_E^+, π_E^- of $(i_B^G \eta_E)|_H$ form a $\mathbb{Z}$-basis $\{\widetilde{\mathrm{tr}}_1, \widetilde{\mathrm{tr}}_{\pi_E^+}, \widetilde{\mathrm{tr}}_{\pi_E^-}\}$ of $\tilde{I}_H$.

When $d \geq 2$, the theorem follows from Lemma 6.3. □

Theorem 6.13 can be equally well expressed in terms of the Grothendieck group $\mathrm{Gr}_R(H)$. It is under this form that the theorem extends to R-representations. For an open compact subgroup K of H, and π a finite length smooth complex representation π of H, $\pi|_K$ is semisimple with finite multiplicities, and is determined by the restriction of the trace of π to $C_c^\infty(K, \mathbb{C})$.

Corollary 6.14. *There are 2^d virtual finite length smooth complex representations $\pi_1, \dots, \pi_{2^d}$ of H with the following property: for any finite length smooth complex representation π of H, there are unique integers $a_0(\pi), a_1(\pi), \dots, a_{2^d}(\pi)$, such that on some compact open subgroup $K = K(\pi)$ of H,*

$$\pi \simeq a_0(\pi)1 + \sum_{i=1}^{2^d} a_i(\pi)\pi_i.$$

Proof. By Theorem 6.13, the $\mathbb{Z}$-module $\tilde{T}_H^{\mathrm{Wh}}$ has a basis $\{\tilde{T}_{\pi_1}^{\mathrm{Wh}}, \dots, \tilde{T}_{\pi_{2^d}}^{\mathrm{Wh}}\}$ where $\pi_1, \dots, \pi_{2^d}$ are virtual finite length smooth representations of H. By Theorem 6.5, for any finite length smooth representation π of H there exist a unique rational number $a_0(\pi)$ and unique integers $a_1(\pi), \dots, a_{2^d}(\pi)$, such that

$$\mathrm{tr}_\pi = a_0(\pi)\,\mathrm{tr}_1 + \sum_{i=1}^{2^d} a_i(\pi)\,\mathrm{tr}_{\pi_i}$$

on restriction to $C_c^\infty(K(\pi), \mathbb{C})$ for some compact open subgroup $K(\pi)$ of H. As $a_0(\pi) = \dim_{\mathbb{C}} \pi^{K(\pi)} - \sum_{i=1}^{2^d} a_i(\pi) \dim_{\mathbb{C}} \pi_i^{K(\pi)}$, we see that $a_0(\pi)$ is an integer. Equivalently, on restriction to $K(\pi)$,

$$\pi \simeq a_0(\pi)1 + \sum_{i=1}^{2^d} a_i(\pi)\pi_i. \qquad \square$$

6.2.5. This has consequences for the representations of G'.

An irreducible complex representation of G' extends to ZG', and we can apply Theorem 6.5 to $H = ZG'$ when $\mathrm{char}_F \neq 2$. When p is odd, there is a unique L-packet $\tau_1, \tau_2, \tau_3, \tau_4$ of G' with four elements (Proposition 4.22). One can enumerate the four nontrivial nilpotent G'-orbits $\mathfrak{O}_1, \ldots, \mathfrak{O}_4$ such that $c_{\mathfrak{O}_i}(\tau_j) = 1$ if $i = j$, and 0 if $i \neq j$. For $i = 1, \ldots, 4$ we choose a lower triangular element $Y_i \in \mathfrak{O}_i$.

Theorem 6.15 (p odd, $R = \mathbb{C}$). *Let π be a finite length smooth complex representation of G'. On restriction to a small enough compact open subgroup $K(\pi)$ of G', we have*

$$\pi \simeq a_0(\pi)1 + \sum_{i=1}^{4} c_{\mathfrak{O}_i}(\pi)\tau_i, \quad c_{\mathfrak{O}_i}(\pi) = \dim_{\mathbb{C}} W_{Y_i}(\pi), \tag{6-14}$$

where $a_0(\pi) = \dim_{\mathbb{C}} \pi^{K(\pi)} - \sum_{i=1}^{4} c_{\mathfrak{O}_i}(\pi) \dim_{\mathbb{C}} \tau_i^{K(\pi)}$. The constant term in Theorem 6.5 is

$$c_0(\pi) = a_0(\pi) - \frac{1}{2}\bigg(\sum_{i=1}^{4} c_{\mathfrak{O}_i}(\pi)\bigg).$$

The constant term $c_0(\pi)$ can be computed using (6-7) and (6-8).

Remark 6.16. When $\mathrm{char}_F = 0$, p is odd and $R = \mathbb{C}$, the theorem was already known; see [Assem 1994] and the last section of [Nevins 2024].

6.2.6. For any p, let π be an irreducible smooth complex representation of G' and r the cardinality of the L-packet of π.

For any L-packet $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ of size 4, there exist integers a_0, b_0 such that on a small enough compact open subgroup of G' we have

$$\mathrm{ind}_{B'}^{G'} 1 \simeq b_0 T_1 + \sum_{i=1}^{4} \tau_i \quad \text{and} \quad \text{if } r = 1,\ \pi \simeq a_0 T_1 + \sum_{i=1}^{4} \tau_i. \tag{6-15}$$

If $r = 2$, then $\det G_\pi = N_{E/F}(E^*/F)$ for a quadratic separable extension E/F. Choose a biquadratic separable extension of F containing E. There exist τ_1 and τ_2 in the associated L-packet of size 4 (Proposition 4.22) and an integer a_0 such that on a small enough compact open subgroup K of G' we have

$$\pi \simeq a_0 T_1 + \sum_{i=1}^{2} \tau_i. \tag{6-16}$$

Therefore, when $R = \mathbb{C}$ we have:

Theorem 6.17. *Let π be an irreducible smooth R-representation of G'. There are an integer a_0 and irreducible smooth R-representations $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ of G'*

forming an L-packet, such that on a small enough compact open subgroup K of G' we have

$$\pi \simeq a_0 1 + \sum_{i=1}^{4/r} \tau_i,$$

where r is the cardinality of the L-packet containing π.

6.2.7. Let us prove Theorem 6.17 for any R.

Let R_c be the algebraic closure in R of the prime field of R. Write $R_c = \mathbb{Q}^{ac}$ when $\mathrm{char}_R = 0$ and $R_c = \mathbb{F}_\ell^{ac}$ when $\mathrm{char}_R = \ell > 0$.

(a) We show first that Theorem 6.17 for R_c extends to R. A cuspidal R-representation of G' is the scalar extension $\pi_R = R \otimes_{R_c} \pi$ to R of a cuspidal R_c-representation π of G' [Vignéras 1996] and the L-packets of size 4 are cuspidal. The scalar extension from R_c to R respects irreducibility, identifies the L-packets of size 4 over R_c with those over R and sends the L-packets of size r over R_c to L-packets of size r over R. Theorem 6.17 for R_c-representations imply Theorem 6.17 extends for R-representations which are scalar extensions of R_c-representations:

$$\pi \simeq a_0 1 + \sum_{i=1}^{4/r} \tau_i \quad \text{implies by scalar extension } \pi_R \simeq a_0 1 + \sum_{i=1}^{4/r} \tau_{i,R}.$$

The only irreducible smooth R-representations ot G' which are not scalar extensions of R_c-representations, are principal series $i_{B'}^{G'}(\eta)$. But

$$i_{B'}^{G'}(\eta) \simeq \mathrm{ind}_{B'}^{G'}(1) \quad \text{on some small open compact subgroup } K \text{ of } G', \tag{6-17}$$

and we have (6-15) for the R_c-representation $\mathrm{ind}_{B'}^{G'}(1)$.

Therefore, for any L-packet $\{\tau_{1,R}, \tau_{2,R}, \tau_{3,R}, \tau_{4,R}\}$ of size 4, there is an integer a_0 such that

$$\mathrm{ind}_{B'}^{G'}(1) \simeq a_0 1 + \sum_{i=1}^{4} \tau_{i,R} \quad \text{on some small open compact subgroup } K \text{ of } G'.$$

(b) Theorem 6.17 for $\mathbb{C}$ extends to $\mathbb{Q}^{ac}$ because the scalar extension from $\mathbb{Q}^{ac}$ to $\mathbb{C}$ respects irreducibility, representations in an L-packet of size 4 are cuspidal, and complex cuspidal representations of G' are defined over $\mathbb{Q}^{ac}$.

(c) Via an isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{ac}$, Theorem 6.17 for $\mathbb{C}$ extends to $\mathbb{Q}_\ell^{ac}$. Theorem 6.17 for $\mathbb{Q}_\ell^{ac}$ extends to $\mathbb{F}_\ell^{ac}$-representations. Indeed, from Proposition 4.30 an irreducible smooth $\mathbb{F}_\ell^{ac}$-representation π of G' in an L-packet of size r lifts to an integral irreducible smooth $\mathbb{Q}_\ell^{ac}$-representation $\tilde{\pi}$ of G' in an L-packet of size r (Proposition 1.6). From Theorem 6.17 for $\mathbb{Q}_\ell^{ac}$, there is an L-packet $\{\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4\}$ of irreducible smooth $\mathbb{Q}_\ell^{\mathrm{ac}}$-representations of G' and an integer a_0, such that on a small enough compact open subgroup K of G', we have

$$\tilde{\pi} \simeq a_0 1 + \sum_{i=1}^{4/r} \tilde{\tau}_i \implies \pi \simeq a_0 1 + \sum_{i=1}^{4/r} \tau_i$$

by reduction modulo ℓ of $\{\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4\}$ to $\{\tau_1, \tau_2, \tau_3, \tau_4\}$, reduction which forms an L-packet of irreducible smooth $\mathbb{F}_\ell^{\mathrm{ac}}$-representations of G'. This ends the proof of Theorem 6.17.

Remark 6.18. The formulas (6-7), (6-15) and (6-16) remain valid for R.

6.2.8. For an irreducible infinite-dimensional complex representation Π of G with conductor c, Casselman had already described the restriction of Π to K_0 as the direct sum of the fixed points under K_{c-1} and a complement depending only on the central character of Π.

Similarly, when p is odd, and π is an irreducible infinite-dimensional complex representation of G', Nevins [2005; 2013] described explicitly the restriction of π to K_0' as a finite-dimensional part specific to π, and a complement depending only on the central character of π. More recently, Nevins [2024] defined for any vertex x of the Bruhat–Tits building of G', admissible complex representations $\tau_{x,1}, \dots \tau_{x,5}$ of the maximal open compact subgroup G_x' fixing x such that the following is true. Let δ_π be the depth of π in the sense of Moy–Prasad. Then, there are integers $a_{\pi,1}, \dots, a_{\pi,5}$ such that on restriction to $G'_{x,\delta_\pi +}$,

$$\pi \simeq \sum_{i=1}^{5} a_{\pi,i} \tau_{x,i}.$$

Now allow any R with $\mathrm{char}_R \neq p$ (still assuming p odd). The representations $\tau_{x,i}$ of Nevins transferred to $\mathbb{Q}_\ell^{\mathrm{ac}}$ are integral, defined over $\mathbb{Q}^{\mathrm{ac}}$ and can be transferred to R-representations $\tau_{x,i,R}$. The proof in §6.2.7 applies and shows that the above result is also valid over R with $\tau_{x,1,R}, \dots, \tau_{x,5,R}$.

7. Asymptotics of invariant vectors by Moy–Prasad subgroups

We use notations as in Sections 3 and 4. The Moy–Prasad subgroups of $G' = \mathrm{SL}_2(F)$ are the intersections of the Moy–Prasad subgroups of $G = \mathrm{GL}_2(F)$ with G' because the Bruhat–Tits of G' and of $\mathrm{PGL}_2(F)$ are the same. We write $K' = G' \cap K$ for a subgroup K of G.

Let $\mathrm{red} : K_0 = \mathrm{GL}_2(O_F) \to \mathrm{GL}_2(k_F)$ and $\mathrm{red}' : K_0' = \mathrm{SL}_2(O_F) \to \mathrm{SL}_2(k_F)$ denote the usual quotient maps. The parahoric subgroups of G are the G-conjugates of the maximal open compact subgroup K_0 or of its Iwahori subgroup $I_0 = \mathrm{red}^{-1}(B(k_F))$. Those of G' are the G'-conjugates of the maximal open compact subgroup K_0'

or its Iwahori subgroup $I'_0 = \mathrm{red}'^{-1}(B'(k_F))$, or of the maximal open subgroup $dK'_0 d^{-1} = (dK_0 d^{-1})'$ where $d = \left(\begin{smallmatrix}1 & 0\\ 0 & p_F\end{smallmatrix}\right)$ [Abdellatif 2011, §3].

The Moy–Prasad subgroups of G are the G-conjugates of the j-th congruence subgroups $K_j, I_j, I_{1/2+j}$ of K_0, I_0, the pro-p Iwahori subgroup $I_{1/2} = \mathrm{red}^{-1}(U(k_F))$ of I_0, for any integer $j \geq 0$ [Henniart and Vignéras 2024, §12]. The Moy–Prasad subgroups of G' are the G'-conjugates of the j-th congruence subgroups K'_j, $dK'_j d^{-1}$, I'_j, $I'_{1/2+j}$ for $j \geq 0$.

Let $\mathfrak{j}$ denote the O_F-lattice of matrices $(x_{i,j}) \in M_2(O_F)$ with $x_{1,2} \in P_F$, and $\mathfrak{j}_{1/2}$ the O_F-lattice of matrices $(x_{i,j}) \in \mathfrak{j}$ with $x_{1,1}, x_{2,2} \in P_F$. We have

$$\begin{gathered} K_0 = M_2(O_F)^*, \quad I_0 = \mathfrak{j}^*, \\ I_{1/2+j} = 1 + p_F^j \mathfrak{j}_{1/2}, \quad K_{1+j} = 1 + p_F^j M_2(P_F), \quad I_{1+j} = 1 + P_F^j\, \mathfrak{j} \end{gathered} \tag{7-1}$$

for $j \geq 0$. Note that $I_0 = K_0 \cap dK_0 d^{-1}$, and consider the decreasing sequence for $H_j = K_j$ or $dK_j d^{-1}$,

$$H_0 \supset I_0 \supset I_{1/2} \supset \cdots \supset H_j \supset I_j \supset I_{1/2+j} \supset H_{1+j} \supset I_{1+j} \supset \cdots .$$

The G-normalizer ZK_0 of the maximal compact subgroup K_0 normalizes all subgroups K_j for $j \geq 0$. The G-normalizer of the Iwahori group I is generated by I and $\left(\begin{smallmatrix}0 & 1\\ p_F & 0\end{smallmatrix}\right)$; it normalizes all subgroups $I_{1/2+j}$, I_j for $j \geq 0$. Let

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \beta' = \begin{pmatrix} 0 & -p_F^{-1} \\ p_F & 0 \end{pmatrix}.$$

The Iwasawa decomposition of G with respect to (B, K_0) and the decomposition of G in double cosets modulo (B, I_0) or $(B, I_{1/2})$ are

$$G = BK_0 = BI_0 \sqcup BsI_0 = BI_{1/2} \sqcup BsI_{1/2}; \tag{7-2}$$

see [Henniart and Vignéras 2024, §12]. Note that $BsI_{1/2} = B\beta' I_{1/2}$. The Iwasawa decomposition of G' with respect to (B', K'_0) or $(B', dK'_0 d^{-1})$ and the decomposition of G' in double classes modulo (B', I'_0) or $(B', I'_{1/2})$ are

$$G' = B'K'_0 = B'dK'_0 d^{-1} = B'I'_0 \sqcup B'\beta' I'_0 = B'I'_{1/2} \sqcup B'\beta' I'_{1/2}; \tag{7-3}$$

see [Abdellatif 2011, lemme 3.2.2, lemme 3.2.8].

Proposition 7.1. *The map $B' \backslash G'/H'_j \to B\backslash G/H_j$ induced by the inclusion $G' \subset G$ is bijective, for any j-th congruence subgroup $H_j = K_j, dK_j d^{-1}, I_j, I_{1/2+j}$ and $j \geq 0$.*

Proof. The map $B' \backslash G'/H'_j \to B\backslash G/H_j$ is surjective as $G = BG'$. When $j = 0$, the map is bijective because the two sets have the same cardinality (7-2), (7-3).

Take $j > 0$ and g', g'' in G' such that $bg'h = g''$ with $b \in B$, $h \in H_j$. We want to prove that $b'g'h' = g''$ with $b' \in B'$, $h' \in H'_j$. Multiplying g' on the left by an element of B', we reduce to $g' \in H_0'$ if $H_0 = K_0, dH_0'd^{-1}$, and $g \in H_0' \cup \beta' H_0'$ if $H_0 = I_0, I_{1/2}$ (7-3). We have $\det b \det h = 1$. There exists $c \in B \cap H_j$ such that $\det c = \det h$ by the Iwahori decomposition of the j-th congruence subgroup $H_j = (B \cap H_j)(H_j \cap U^-)$ when $j > 0$. Three cases occur:

(1) $g' \in H_0'$. Write $(bc)g'(g'^{-1}c^{-1}g')h = g''$ with $b' = bc \in B'$, $g'^{-1}c^{-1}g' \in H_j$ and $h' = (g'^{-1}c^{-1}g')h \in H_j'$.

(2) $g' \in \beta' H_0'$ and $g'' \in H_0'$. Apply the same argument to g''.

(3) g' and g'' are in $\beta' H_0'$. Changing notations we want to prove that for g' and g'' in H_0' such that $b\beta' g' h = \beta g''$ with $b \in B$, $h \in H_j$, we have $b'\beta g' h' = \beta g''$ with $b' \in B'$, $h' \in H_j'$. Multiply on the left by β^{-1}. Noting that $\beta^{-1} B \beta = B^-$, we still need to prove that for $g', g'' \in H_0'$ such that $bg'h = g''$ with $b \in B^-$, $h \in H_j$, we have $b'g'h' = g''$ with $b' \in (B^-)'$, $h' \in H_j'$. The argument used before with B works also for B^-, because we have the Iwahori decomposition $H_j = (B^- \cap H_j)(H_j \cap U)$ when $j > 0$. There exists $c \in B^- \cap H_j$ such that $\det c = \det h$. Proceeding as in (1), we write $(bc)g'(g'^{-1}c^{-1}g')h = g''$ with $b' = bc \in (B^-)'$, $g'^{-1}c^{-1}g' \in H_j$ and $h' = (g'^{-1}c^{-1}g')h \in H_j'$. □

Proposition 7.1 has important applications. The cardinality of $B\backslash G/H_j$ is computed in [Henniart and Vignéras 2024, Proposition 11.2] for $j \geq 0$. By Proposition 7.1, $|B\backslash G/H_j| = |B'\backslash G'/H_j|$.

Corollary 7.2. *The cardinality of $B'\backslash G'/H_j'$ for $H_j' = K_j', dK_j'd^{-1}, I_j', I_{1/2+j}'$ and $j \geq 0$, is*

$$|B'\backslash G'/K_0'| = |B'\backslash G'/dK_0'd^{-1}| = |B\backslash G/K_0| = 1,$$
$$|B'\backslash G'/K_{1+j}'| = |B'\backslash G'/dK_{1+j}'d^{-1}| = |B\backslash G/K_{1+j}| = (q+1)q^j,$$
$$|B'\backslash G'/I_j'| = |B'\backslash G'/I_{1/2+j}'| = |B\backslash G/I_j| = |B\backslash G/I_{1/2+j}| = 2q^j.$$

Over any coefficient ring, the restriction to G' of $\operatorname{ind}_B^G 1$ is $\operatorname{ind}_{B'}^{G'} 1$. The vector spaces $(\operatorname{ind}_{B'}^{G'} 1)^{H_j'} \supset (\operatorname{ind}_B^G 1)^{H_j}$ have the same dimension by Proposition 7.1, hence are equal.

Corollary 7.3. *Over any coefficient ring, any element in $\operatorname{ind}_B^G 1$ fixed by H_j' is also fixed by H_j for $j \geq 0$.*

It is known that any infinite-dimensional irreducible smooth R-representation Π of G near the identity is isomorphic to $\operatorname{ind}_B^G 1$ modulo a multiple of the trivial representation [Henniart and Vignéras 2024]. There exist integers a_Π and $j_\Pi \geq 0$ such that for $j \geq j_\Pi$,

$$\Pi \simeq a_\Pi 1 + \operatorname{ind}_B^G 1 \quad \text{on } I_j. \tag{7-4}$$

Corollary 7.4. *For $j \geq j_\Pi$, any element in Π fixed by H_j' is also fixed by H_j.*

Proposition 7.5. *$a_\Pi = 0$ if Π is a principal series, $a_\Pi = -1$ when $q+1 \neq 0$ in R and Π is the twist of the Steinberg representation by a character, and when Π is cuspidal with minimal depth δ_Π under torsion by characters,*

$$a_\Pi = \begin{cases} -2q^{\delta_\Pi} & \text{if } \delta_\Pi \text{ is an integer,} \\ -(q+1)q^{\delta_\Pi - 1/2} & \text{otherwise.} \end{cases}$$

If $|L(\Pi)| = 4$, then $a_\Pi = -2$ for p odd and a_Π is a multiple of 4 if $p = 2$.

Proof. When $R = \mathbb{C}$, then a_Π is the constant term $c_0(\Pi)$ of the germ expansion for Π because the constant term $c_0(\mathrm{ind}_B^G 1)$ of the germ expansion of the trace of $\mathrm{ind}_B^G 1$ around 1 (6-6) is 0.

When $R = \mathbb{F}_\ell^{\mathrm{ac}}$ and $\widetilde{\Pi}$ is a $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation lifting Π, $a_\Pi = a_{\widetilde{\Pi}}$. When Π is cuspidal, $\widetilde{\Pi}$ is supercuspidal and the formula for a_Π follows from (6-8). If $|L(\Pi)| = 4$ the assertion on a_Π follows from the proof of Proposition 6.7 □

In the particular case where $\Pi|_{G'} = \pi$ is irreducible, we deduce that for $j \geq j_\Pi$,

$$\pi \simeq a_\Pi 1 + \mathrm{ind}_{B'}^{G'} 1 \quad \text{on } I'_j.$$

For example, an irreducible principal series π of G' is the restriction to G' of a principal series Π of G, and on $I'_{1/2+j}$ for $j \geq j_\Pi$ we have $\pi \simeq \mathrm{ind}_{B'}^{G'} 1$.

By (7-4) if $j \geq j_\Pi$,

$$\dim_{\mathbb{C}} \Pi^{H_j} = a_\Pi + |B\backslash G/H_0| q^j. \tag{7-5}$$

By Proposition 7.1, $\Pi^{H_j} = \sum_{\pi \in L(\Pi)} \pi^{H'_j}$ for $H_j = I_{1/2+j}, K_{1+j}, I_{1+j}$ and $j \geq 0$.

In particular, if $\Pi|_{G'} = \pi$ is irreducible, then if $j \geq j_\Pi$,

$$\dim \pi^{H'_j} = a_\Pi + |B\backslash G/H_0| q^j.$$

In general, by Corollary 7.2 [Henniart and Vignéras 2024, § 12.2], for j large,[19]

$$\dim_{\mathbb{C}} \Pi^{I_j} = \dim_{\mathbb{C}} \Pi^{I_{1/2+j}} = a_\Pi + 2q^j, \quad \dim_{\mathbb{C}} \Pi^{K_{1+j}} = a_\Pi + (q+1)q^j. \tag{7-6}$$

Let π be an infinite-dimensional irreducible smooth R-representation of G' contained in $\Pi|_{G'}$. The Moy–Prasad filtration of the Iwahori subgroup I' of G' is

$$I' = I'_0 \supset I'_{1/2} \supset I'_1 \supset \cdots \supset I'_j \supset I'_{1/2+j} \supset I_{j+1} \supset \cdots.$$

Theorem 7.6. *With a_Π as in (7-4) and Proposition 7.5, we have for j large,*[20]

$$\dim_R \pi^{I'_j} = \dim_R \pi^{I'_{1/2+j}} = |L(\Pi)|^{-1} (a_\Pi + 2q^j).$$

$|L(\Pi)|^{-1} a_\Pi = -\frac{1}{2}$ if $|L(\Pi)| = 4$ and p is odd, otherwise $|L(\Pi)|^{-1} a_\Pi$ is an integer.

[19] $j \geq j_\Pi + 1$ for I_j, H_j and $j \geq j_\Pi$ for $I_{1/2+j}$.

[20] $j \geq j_\Pi + 1$ for I_j and $j \geq j_\Pi$ for $I_{1/2+j}$.

Proof. The determinant of the G-normalizer $N_G(I)$ of the Iwahori group I is equal to F^* (first part of Section 7). Thus, $N_G(I)$ acts transitively on $L(\Pi)$ and as $N_G(I)$ normalizes the Moy–Prasad filtration of I, the dimension of the invariants of π by $I'_{1/2+j}$ and I'_j of G' for $j \geq 0$, does not depend on the choice of π in the L-packet $L(\Pi)$. For these two groups H'_j we have $\dim_R \pi^{H'_j} = |L(\Pi)|^{-1} \dim_R \Pi^{H_j}$ for $j \geq j_\Pi$, by Proposition 7.1. Apply now (7-6). The assertion on $|L(\Pi)|^{-1} a_\Pi$ follows from Proposition 7.5. □

Let us now turn to the asymptotics for fixed points under congruence subgroups K'_j of $K'_0 = \mathrm{SL}_2(O_F)$. The G-normalizer ZK_0 of $K_0 = \mathrm{GL}_2(O_F)$ normalizes the K'_j. The subgroup $H = ZK_0G'$ of G has index 2 as $\det H = (F^*)^2 O_F^*$ has index 2 in F^*. The restriction of Π to H has length 1 or 2. All the elements π of $L(\Pi)$ in the same H-orbit share the same dimension $\dim_R \pi^{K'_j}$. With a_Π, j_Π as in (7-4), we deduce from (7-6):

Theorem 7.7. *When $\Pi|_H$ is irreducible, we have, for $j \geq j_\Pi$,*

$$\dim_R \pi^{K'_{j+1}} = |L(\Pi)|^{-1}(a_\Pi + (q+1)q^j).$$

Proposition 7.8. *The representation $\Pi|_H$ is reducible if and only if Π is cuspidal induced from ZK_0 or $\mathrm{char}_R \neq 2$ and Π is a principal series $\mathrm{ind}_B^G \chi$ where $\chi_1\chi_2^{-1} = (-1)^{\mathrm{val}}$.*

Proof. When $\Pi|_{G'}$ is irreducible, then $\Pi|_H$ is irreducible. When $\Pi = i_B^G(\chi)$ is a principal series of reducible restriction to G', then $\mathrm{char}_R \neq 2$, and $i_B^G(\chi)|_H$ is reducible if and only if $(-1)^{\mathrm{val}} \circ \det \otimes i_B^G(\chi) \simeq i_B^G(\chi)$ if and only if $\chi_1\chi_2^{-1} = (-1)^{\mathrm{val}}$ (notations of Section 4.3.1 and $\chi = \chi_1 \otimes \chi_2$).

When Π is cuspidal, if $\Pi = \mathrm{ind}_{ZK_0}^G \lambda$ is induced from ZK_0, then $\Pi|_H$ is reducible because $ZK_0 \subset H$ and $(\mathrm{ind}_H^G(\mathrm{ind}_{ZK_0}^H \lambda))|_H$ contains $\mathrm{ind}_{ZK_0}^G \lambda$ but is different from it. If Π is not induced from ZK_0, then with the notations of Section 4.3.2, $\Pi = \mathrm{ind}_J^G \lambda$ has positive level, E/F is ramified, and $G = JH$. As $J^1 \subset H$ and the intertwining of $\lambda_1 = \lambda|_{J^1}$ in G is J, then the intertwining of λ_1 in H is $J \cap H$. The vectors λ_1-equivariant in Π are the functions supported in J. Applying [Henniart and Vignéras 2022, Proposition 6.5 and Corollary 6.6], $\Pi|_H = \mathrm{ind}_{J\cap H}^H \lambda|_{J\cap H}$ is irreducible. □

Assume now that $\Pi|_H$ is reducible. Let Π^+ be the component having a Whittaker model with respect to a character ψ nontrivial on O_F but trivial on P_F, and Π^- the other one.

Theorem 7.9. *When $\Pi|_H$ is reducible, we have for large j,*

$$\dim_R(\Pi^+)^{K'_j} = \tfrac{1}{2}a_\Pi + q^{2m+1} \quad \textit{when } j = 2m+1,\ 2m+2,$$
$$\dim_R(\Pi^-)^{K'_j} = \tfrac{1}{2}a_\Pi + q^{2m} \quad \textit{when } j = 2m,\ 2m+1.$$

Proof. When $R=\mathbb{C}$, the constant term in the germ expansion of the trace of Π^+ around the identity is $\frac{1}{2}a_\Pi$ by (6-7) and Remark 6.18, and $\dim_R(\Pi^+)^{K'_j}-\frac{1}{2}a_\Pi$ for large j, which depends only on the characters of F for which Π^+ has a Whittaker model. This set does not depend on the choice of Π, as Π^+ has a Whittaker model only with respect to the characters $\psi_{t_1t_2^{-1}}$ for $\mathrm{diag}(t_1,t_2)\in T\cap H$, that is, ψ_a for $a\in\det H$ where $\psi_a(x)=\psi(ax)$ for $x\in F$. By the usual arguments, the same is true for any R. It suffices to prove the theorem for $\Pi=\mathrm{ind}_{ZK_0}^G\lambda$ where $\lambda|_{K_0}$ is the inflation of a cuspidal representation λ_0 of $\mathrm{GL}_2(k_F)$ (Proposition 7.8). In this special case we will show

$$\dim_R(\Pi^+)^{K'_j}=-1+q^{2m+1}\qquad\text{for } j=2m+1,\ 2m+2,\ j\geq 1,\tag{7-7}$$

$$\dim_R(\Pi^-)^{K'_j}=-1+q^{2m}\qquad\text{for } j=2m,\ 2m+1,\ j\geq 1.\tag{7-8}$$

Note that $a_\Pi=-2$ (Proposition 7.5) and that (7-7) implies (7-8) for $j\geq j_\Pi+1$, as

$$\dim_R(\Pi^+)^{K'_j}+\dim_R(\Pi^-)^{K'_j}=a_\Pi+(q+1)q^{j-1}\quad\text{for } j\geq j_\Pi+1.$$

The representation λ_0 is generic, and it follows that $\Pi^+=\mathrm{ind}_{ZK_0}^H\lambda$ [Bushnell and Henniart 1998, Proposition 1.6]. Let $t=\begin{pmatrix}p_F&0\\0&p_F^{-1}\end{pmatrix}$. The group $H=ZK_0G'$ is the disjoint union

$$H=\bigsqcup_{i\geq 0} ZK_0t^iK'_0.$$

For $i\geq 0$, $j>0$ and $k\in K'_0$, consider the representation of K'_j on the functions in $\mathrm{ind}_{ZK_0}^H\lambda$ supported on the coset $ZK_0t^ikK'_j$. That it contains nonzero K'_j-fixed vectors does not depend on the choice of $k\in K'_0$, and it happens if and only if $t^iK'_jt^{-i}\cap ZK_0$ has nonzero fixed vectors in λ. For $j\leq 2i$, $t^iK'_jt^{-i}\cap ZK_0$ contains the lower unipotent subgroup of K_0 and fixes no nonzero vector in λ_0 which is cuspidal. For $j>2i$, $t^iK'_jt^{-i}\subset K_1$ and K_1 acts trivially on λ_0. So the space of functions in $\mathrm{ind}_{ZK_0}^H$ supported in $ZK_0t^ikK'_j$ and fixed by K'_j has dimension 0 if $j\leq 2i$ and $q-1=\dim_R\lambda_0$ if $j>2i$. The number of cosets $ZK_0t^ikK'_j$ in $ZK_0t^iK_0$ is the index in K'_0/K'_j of the image of $t^{-i}ZK_0t^i\cap K'_0$ in K'_0/K'_j. As $K'_{2i}\subset t^{-i}ZK_0t^i\cap K'_0$, this index does not depend on j when $j>2i$. It is the index in K'_0 of $t^{-i}ZK_0t^i\cap K'_0=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in K'_0,\ c\in P_F^{2i}\right\}$. One computes its value to be 1 if $i=0$ and $(q+1)q^{2i-1}$ if $i>0$. Consequently for $j>0$,

$$\dim_R(\Pi^+)^{K'_j}=(q-1)\left(1+\sum_{0<i<\frac{1}{2}j}(q+1)q^{2i-1}\right).$$

This is equal to $q-1$ for $j=1,2$, to $(q-1)(q^2+q+1)=-1+q^3$ for $j=3,4$, and by induction to $-1+q^{2m+1}$ for $j=2m+1,\ 2m+2$, implying (7-7), hence the theorem.

To prove (7-8) for $j \geq 1$, one can work in the same manner as above using that Π^- is the conjugate of Π^+ by $\left(\begin{smallmatrix} p_F & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We find that $\dim_R(\Pi^-)^{K'_j}$ is equal to 0 for $j = 1$, to $-1 + q^2$ for $j = 2, 3$, and to $-1 + q^{2m}$ for $j = 2m,\ 2m+1$, implying (7-8). □

Corollary 7.10. *When $\Pi|_H$ is reducible, we have for large j,*

$$\dim_R \pi^{K'_j} = \begin{cases} |L(\Pi)|^{-1}(a_\Pi + 2q^j) & \text{for } j \text{ odd and } \pi \subset \Pi^+|_{G'} \text{ or } j \text{ even and } \pi \subset \Pi^-|_{G'}, \\ |L(\Pi)|^{-1}(a_\Pi + 2q^{j-1}) & \text{otherwise.} \end{cases}$$

For the maximal compact group dK_0d^{-1} of G', the two asymptotics are interchanged.

We find remarkable that the regularity is obtained when increasing the index j by 2, and not by 1 as was the case for the Iwahori or the pro-p Iwahori subgroups. But that could have been anticipated, given the homogeneity properties of the nilpotent orbital integrals in H.

Remark 7.11. The asymptotics (Theorems 7.6 and 7.7, Corollary 7.10) are likely valid when $2j \geq c$ where c is the conductor of Π. When $R = \mathbb{C}$ and Π is cuspidal, this is actually true for $\dim_{\mathbb{C}} \Pi^{K_j}$ and can be derived from the formulas in [Miyauchi and Yamauchi 2022]. When p is odd, Nevins has completely analyzed the restriction to K'_0 of the irreducible smooth complex representations of G', and we presume that the asymptotics (and for which j it is valid) can be derived from her results [Nevins 2005; 2013].

Appendix: The finite group $\mathrm{SL}_2(\mathbb{F}_q)$

Let k be a finite field of characteristic p with q elements. In this Appendix we classify irreducible representations of $G = \mathrm{GL}_2(k)$ and of $G' = \mathrm{SL}_2(k)$ over an algebraically closed field R of characteristic 0 or $\ell > 0$, $\ell \neq p$. We could use [Bonnafé 2011] for $\mathrm{char}_R \neq 2$ and [Kleshchev and Tiep 2009] for any R, but we prefer using the same methods as in the main text.

Note that the irreducible R-representations of the finite groups G and G' are defined over the algebraic closure of the prime field, and we can freely pass from R to any other algebraically closed field of the same characteristic. Thus it is enough to consider the cases where $R = \mathbb{C}$ or $R = \mathbb{F}_\ell^{\mathrm{ac}}$. We also aim to prove the following theorem.

Theorem A.1. *Any irreducible $\mathbb{F}_\ell^{\mathrm{ac}}$ representation σ of $\mathrm{GL}_2(k)$ is the reduction modulo ℓ of a $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\tilde\sigma$ of $\mathrm{GL}_2(k)$ such that $\tilde\sigma|_{\mathrm{SL}_2(k)}$ and $\sigma|_{\mathrm{SL}_2(k)}$ have the same length.*

Any irreducible $\mathbb{F}_\ell^{\mathrm{ac}}$-representation of $\mathrm{SL}_2(k)$ is the reduction modulo ℓ of a $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation of $\mathrm{SL}_2(k)$.

Write Z for the centre of G, B for the upper triangular subgroup of G, and U for its unipotent radical. Let us first recall the known classification of the R-representations of G; see [Bushnell and Henniart 2002] for $R=\mathbb{C}$ and [Vignéras 1988] for $R=\mathbb{F}_\ell^{\mathrm{ac}}$.

The parabolically induced representation $\mathrm{ind}_B^G(1)$ realized by the space of constant functions on $B\backslash G$ contains the trivial character. It also has the trivial character as a quotient, given by the functional λ which sums the values of functions on $B\backslash G$. The map from the trivial subrepresentation to the trivial quotient is multiplication by $q+1$, so is an isomorphism if ℓ does not divide $q+1$, and is 0 otherwise. In the first case the quotient $\mathrm{St}=\mathrm{ind}_B^G(1)/1$ is irreducible, in the second case $\mathrm{Ker}(\lambda)/1$ is a cuspidal but not supercuspidal representation σ_0 of G.

The irreducible (classes of) R-representations σ of G are:

(1) The characters $\chi\circ\det$ where χ is an R-character of k^*.

(2) When $q+1\neq 0$ in R, the twists $(\chi\circ\det)\otimes\mathrm{St}$ of St by the R-characters $\chi\circ\det$ of G.

(2′) When $q+1=0$ in R, the twists $(\chi\circ\det)\otimes\sigma_0$ of σ_0 by the R-characters $\chi\circ\det$ of G.

(3) The irreducible principal series $\mathrm{ind}_B^G(\chi_1\otimes\chi_2)$, where χ_1 and χ_2 are two distinct R-characters of k^*.

(4) The supercuspidal representations $\sigma(\theta)$, where θ is an R-character of k_2^*, $\theta\neq\theta^q$, where k_2/k is a quadratic extension.

The only isomorphisms between those representations are given by exchanging χ_1 and χ_2 in (3), as well as θ and θ^q in (4).

Twisting by an R-character $\chi\circ\det$ of G has the obvious effect, for example sending θ to $(\chi\circ N)\theta$ where $N(x)=x^{q+1}$ for $x\in k_2^*$ in (4).

Any irreducible R-representation τ of G' is contained in the restriction $\sigma|_{G'}$ to G' of an irreducible R-representation σ of G. The representation $\sigma|_{G'}$ is semisimple of multiplicity 1 and its irreducible components are G-conjugate. The stabilizer of τ contains ZG' and G/ZG' is isomorphic to $k^*/(k^*)^2$. We have $|k^*/(k^*)^2|=1$ when $p=2$ and $|k^*/(k^*)^2|=2$ when p is odd. Therefore $\sigma|_{G'}$ is irreducible when $p=2$ and $\sigma|_{G'}$ has length 1 or 2 when p is odd.

When $\mathrm{char}_R\neq 2$, the length $\mathrm{lg}(\sigma|_{G'})$ of $\sigma|_{G'}$ is the number of R-characters χ of k^* such that $(\chi\circ\det)\otimes\sigma\simeq\sigma$, so

$$\text{(A-1)}\quad \mathrm{lg}(\sigma|_{G'})=\begin{cases}2 & \text{in case (3) if } (\chi_1/\chi_2)^2=1 \text{ and in case (4) if } (\theta^{q-1})^2=1,\\ 1 & \text{otherwise.}\end{cases}$$

The restrictions $\sigma_1|_{G'},\sigma_2|_{G'}$ of two irreducible representations σ_1,σ_2 of G are isomorphic if and only if σ_1, σ_2 are twists of each other by an R-character of G.

Otherwise $\sigma|_{G'}$, $\sigma_2|_{G'}$ are disjoint. So, we have a classification of the (isomorphism classes of) irreducible representations of G' when $\mathrm{char}_R \neq 2$.

Remark A.2. The restriction to B of a cuspidal representation of G is the Kirillov representation κ of B (the irreducible R-representation of B induced by any nontrivial R-character of U). The restriction of κ to U is the direct sum of all nontrivial R-characters of U. The group B acts transitively on such characters, whereas $B' = B \cap G'$ acts with two orbits. It follows that the restriction of κ to B' has two inequivalent irreducible components. Consequently a cuspidal representation of G restricts to G' with length 1 or 2.

Let ℓ be an odd prime number different from p. Let us consider the reduction modulo ℓ of the previous irreducibles σ over $\mathbb{Q}_\ell^{\mathrm{ac}}$ (since G is finite, they are integral). For an integral $\mathbb{Q}_\ell^{\mathrm{ac}}$-character χ (with values in $\mathbb{Z}_\ell^{\mathrm{ac}}$), let $\bar{\chi}$ denote its reduction modulo ℓ. Reduction modulo ℓ is compatible with twisting by a $\mathbb{Q}_\ell^{\mathrm{ac}}$-character $\chi \circ \det$ in the sense that the reduction of $(\chi \circ \det) \otimes \sigma$ is the twist by $\bar{\chi} \circ \det$ of the reduction of σ.

(1) The trivial $\mathbb{Q}_\ell^{\mathrm{ac}}$-character of G reduces to the trivial $\mathbb{F}_\ell^{\mathrm{ac}}$-character.

(2) When ℓ does not divide $q+1$, the Steinberg $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation reduces to the Steinberg $\mathbb{F}_\ell^{\mathrm{ac}}$-representation.

(2′) When ℓ divides $q+1$, the Steinberg $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation reduces to a length 2 representation with subrepresentation σ_0 and trivial quotient (for the natural integral structure).

(3) The irreducible principal series $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ reduces to the irreducible principal series $\mathrm{ind}_B^G(\bar{\chi}_1 \otimes \bar{\chi}_2)$ when $\bar{\chi}_1 \neq \bar{\chi}_2$, and to $(\bar{\chi}_1 \circ \det) \otimes \mathrm{ind}_B^G(1)$ (of length 2 when ℓ does not divide $q+1$, and length 3 otherwise) when $\bar{\chi}_1 = \bar{\chi}_2$ (for the natural integral structure).

(4) The supercuspidal $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation $\sigma(\theta)$ reduces to the supercuspidal $\mathbb{F}_\ell^{\mathrm{ac}}$-representation $\sigma(\bar{\theta})$ if $\bar{\theta} \neq (\bar{\theta})^q = \overline{\theta^q}$, and otherwise (which can happen only if ℓ divides $q+1$) to $(\eta \circ \det) \otimes \sigma_0$ where η is the $\mathbb{F}_\ell^{\mathrm{ac}}$-character of $\mathbb{F}_q^*$ such that $\eta \circ N = \bar{\theta}$.

A given $\mathbb{F}_\ell^{\mathrm{ac}}$-character of k^* or k_2^* has a unique lift to a $\mathbb{Z}_\ell^{\mathrm{ac}}$-character of the same order, and from the above it is clear that any irreducible $\mathbb{F}_\ell^{\mathrm{ac}}$-representation σ of G lifts to a $\mathbb{Q}_\ell^{\mathrm{ac}}$-representation. Moreover, one can choose a lift of σ with the same length on restriction to G', thus proving the theorem when ℓ is odd.

Let us finally assume $\mathrm{char}_R = 2$. Then p is odd and $q+1 = 0$ in R. Write $q-1 = 2^s m$ with a positive integer s and an odd integer m. The number of irreducible R-representations of G (resp. ZG') is the number of conjugacy classes in G (resp. ZG') of elements of odd order. Let $g \in G$ be of odd order. Then $\det g \in k^*$ has odd order so $\det g \in (k^*)^2$ and $g \in ZG'$. The G-conjugacy class of g is equal to its ZG'-conjugacy class unless the G-centralizer of g is entirely in ZG'. In that exceptional case, the G-equivalence class of g is the union of two ZG'-equivalence classes. This happens only when $g = zu$ where $z \in Z$ (of odd order) and $u \neq 1$ is unipotent. That shows that m is the number of ZG'-conjugacy classes of elements of odd order minus the number of G-conjugacies of such elements. Consequently m is the number of irreducible R-representations of ZG' minus the number of irreducible R-representations of G.

Consider first $\sigma(\theta)$ for a $\mathbb{Q}_2^{\mathrm{ac}}$-character θ of k_2^* of order 2^{s+1}. Certainly $\bar{\theta}$ is trivial so that the reduction of $\sigma(\theta)$ modulo 2 is σ_0. But $\ell(\sigma(\theta)|_{G'}) = 2$ by (A-1), from which it follows that $\ell(\sigma_0|_{G'}) \geq 2$. We have seen however that $\ell(\sigma_0|_{G'}) \leq 2$ (Remark A.2), so $\ell(\sigma_0|_{G'}) = 2$, and each irreducible component of $\sigma_0|_{G'}$ lifts to an irreducible component of $\sigma(\theta)|_{G'}$. The $\mathbb{F}_2^{\mathrm{ac}}$-characters χ of k^* have odd order, their number is m, and the representations $(\chi \circ \det) \otimes \sigma_0$ are not equivalent (the order of χ is odd). We deduce:

Lemma A.3. *All irreducible $\mathbb{F}_2^{\mathrm{ac}}$-representations of G restrict irreducibly to G' except the twists of σ_0 by characters.*

The reduction modulo 2 of any supercuspidal $\mathbb{Q}_2^{\mathrm{ac}}$-representation of G' is irreducible.

We deduce the classification of irreducible R-representations of G' when $\mathrm{char}_R = 2$ and Theorem A.1 when $\ell = 2$.

Remark A.4. For use in the main text we summarize:

(a) When $q+1 = 0$ in R, $\sigma_0|_{\mathrm{SL}_2(k)}$ is irreducible if $\mathrm{char}_R \neq 2$, and has length 2 if $\mathrm{char}_R = 2$.

(b) In (4), let $b \in k_2$ be an element of order $q+1$. We have $\theta \neq \theta^q \iff \theta(b) \neq 1$ and $\sigma(\theta)|_{\mathrm{SL}_2(k)}$ is irreducible if $\theta^2(b) \neq 1$, and has length 2 if $\theta^2(b) = 1$.

When $\mathrm{char}_R = 2$, or when $p = 2$, hence $(2, q+1) = 1$, we have $\theta(b) \neq 1 \iff \theta(b^2) \neq 1$, hence $\sigma(\theta)|_{\mathrm{SL}_2(k)}$ is irreducible for all $\theta \neq \theta^q$.

When $\mathrm{char}_R \neq 2$ and p is odd, there exists θ such that $\theta(b) \neq 1$, $\theta(b)^2 = 1$, unique modulo the twist by a character χ such that $\chi(b) = 1$. The corresponding representations $\sigma(\theta)$ of G are twists of each other by a character of G. Their restrictions to $\mathrm{SL}_2(k)$ are isomorphic and reducible of length 2.

Acknowledgements

We thank Anne-Marie Aubert, Don Blasius, Cédric Bonnafé, Jean-François Dat, Jean-Pierre Labesse, Bertrand Lemaire, Monica Nevins, Dipendra Prasad, and Akshay Venkatesh, for helpful communications. Special thanks are due to Peiyi Cui, for conversations and correspondences. Some arguments using types in Section 4.3.2 are due to her, she called our attention to the study of the quotient in (4-21), and

shared her results on principal series. She also kindly gave us references to her works [Cui 2023; Cui et al. 2024].

References

[Abdellatif 2011] R. Abdellatif, *Autour des représentations modulo p des groupes réductifs p-adiques de rang* 1, Ph.D. Thesis, Université Paris-Sud XI, 2011, available at https://theses.hal.science/tel-00651063.

[Assem 1994] M. Assem, "The Fourier transform and some character formulae for p-adic SL_ℓ, ℓ a prime", *Amer. J. Math.* **116**:6 (1994), 1433–1467. MR Zbl

[Aubert and Plymen 2024] A.-M. Aubert and R. Plymen, "Endoscopic character identities for depth-zero supercuspidal representations of SL_2", preprint, 2024. arXiv 2410.20183v2

[Aubert et al. 2016] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld, "The local Langlands correspondence for inner forms of SL_n", *Res. Math. Sci.* **3** (2016), art. id. 32. MR Zbl

[Aubert et al. 2017] A.-M. Aubert, S. Mendes, R. Plymen, and M. Solleveld, "On L-packets and depth for $\mathrm{SL}_2(K)$ and its inner form", *Int. J. Number Theory* **13**:10 (2017), 2545–2568. MR Zbl

[Bernstein and Zelevinsky 1976] I. N. Bernstein and A. V. Zelevinsky, "Representations of the group $\mathrm{GL}(n, F)$, where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3(189) (1976), 5–70. In Russian; translated in *Russ. Math. Surv.* **31**:3 (1976), 1–68. MR Zbl

[Bonnafé 2011] C. Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$, Algebra and Applications **13**, Springer, London, 2011. MR Zbl

[Borel 1991] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, New York, 1991. MR Zbl

[Borel and Tits 1972] A. Borel and J. Tits, "Compléments à l'article "Groupes réductifs"", *Inst. Hautes Études Sci. Publ. Math.* 41 (1972), 253–276. MR Zbl

[Bourbaki 2012] N. Bourbaki, *Algèbre, chapitre* 8*: modules et anneaux semi-simples*, revised 2nd ed., Springer, Berlin, 2012. MR Zbl

[Bushnell and Henniart 1998] C. J. Bushnell and G. Henniart, "Supercuspidal representations of GL_n: explicit Whittaker functions", *J. Algebra* **209**:1 (1998), 270–287. MR Zbl

[Bushnell and Henniart 2002] C. J. Bushnell and G. Henniart, "On the derived subgroups of certain unipotent subgroups of reductive groups over infinite fields", *Transform. Groups* **7**:3 (2002), 211–230. MR Zbl

[Bushnell and Henniart 2006] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), Grundl. Math. Wissen. **335**, Springer, Berlin, 2006. MR Zbl

[Bushnell and Kutzko 1994] C. J. Bushnell and P. C. Kutzko, "The admissible dual of $\mathrm{SL}(N)$, II", *Proc. London Math. Soc.* (3) **68**:2 (1994), 317–379. MR Zbl

[Cui 2023] P. Cui, "ℓ-modular representations of p-adic groups $\mathrm{SL}_n(F)$: maximal simple k-types", preprint, 2023. arXiv 2012.07492v2

[Cui et al. 2024] P. Cui, T. Lanard, and H. Lu, "Modulo ℓ distinction problems", *Compos. Math.* **160**:10 (2024), 2285–2321. MR Zbl

[Dat 2009] J.-F. Dat, "Finitude pour les représentations lisses de groupes p-adiques", *J. Inst. Math. Jussieu* **8**:2 (2009), 261–333. MR Zbl

[Dat et al. 2024] J.-F. Dat, D. Helm, R. Kurinczuk, and G. Moss, "Finiteness for Hecke algebras of p-adic groups", *J. Amer. Math. Soc.* **37**:3 (2024), 929–949. MR Zbl

[Feng 2023] T. Feng, "Modular functoriality in the Local Langlands Correspondence", preprint, 2023. arXiv 2312.12542

[Gelbart and Knapp 1982] S. S. Gelbart and A. W. Knapp, "L-indistinguishability and R groups for the special linear group", *Adv. in Math.* **43**:2 (1982), 101–121. MR Zbl

[Henniart 2001] G. Henniart, "Représentations des groupes réductifs p-adiques et de leurs sous-groupes distingués cocompacts", *J. Algebra* **236**:1 (2001), 236–245. MR Zbl

[Henniart and Vignéras 2019] G. Henniart and M.-F. Vignéras, "Representations of a p-adic group in characteristic p", pp. 171–210 in *Representations of reductive groups*, edited by M. Nevins and P. E. Trapa, Proc. Sympos. Pure Math. **101**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Henniart and Vignéras 2022] G. Henniart and M.-F. Vignéras, "Representations of a reductive p-adic group in characteristic distinct from p", *Tunis. J. Math.* **4**:2 (2022), 249–305. MR Zbl

[Henniart and Vignéras 2024] G. Henniart and M.-F. Vignéras, "Representations of $GL_n(D)$ near the identity", *Proc. Lond. Math. Soc.* (3) **129**:6 (2024), art. id. e70000. MR Zbl

[Hiraga and Saito 2012] K. Hiraga and H. Saito, *On L-packets for inner forms of* SL_n, Mem. Amer. Math. Soc. **1013**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Kleshchev and Tiep 2009] A. S. Kleshchev and P. H. Tiep, "Representations of finite special linear groups in non-defining characteristic", *Adv. Math.* **220**:2 (2009), 478–504. MR Zbl

[Kutzko and Pantoja 1991] P. Kutzko and J. Pantoja, "The restriction to SL_2 of a supercuspidal representation of GL_2", *Compositio Math.* **79**:2 (1991), 139–155. MR Zbl

[Labesse and Langlands 1979] J.-P. Labesse and R. P. Langlands, "L-indistinguishability for SL(2)", *Canadian J. Math.* **31**:4 (1979), 726–785. MR Zbl

[Labesse and Schwermer 2019] J.-P. Labesse and J. Schwermer, "Central morphisms and cuspidal automorphic representations", *J. Number Theory* **205** (2019), 170–193. MR Zbl

[Lemaire 2004] B. Lemaire, "Intégrabilité locale des caractères tordus de $GL_n(D)$", *J. Reine Angew. Math.* **566** (2004), 1–39. MR Zbl

[Lemaire 2005] B. Lemaire, "Intégrabilité locale des caractères de $SL_n(D)$", *Pacific J. Math.* **222**:1 (2005), 69–131. MR Zbl

[Luo and Chau 2024] Z. Luo and N. B. Chau, "Nonabelian Fourier kernels on SL_2 and GL_2", preprint, 2024. arXiv 2409.14696

[Miyauchi and Yamauchi 2022] M. Miyauchi and T. Yamauchi, "A remark on conductor, depth and principal congruence subgroups", *J. Algebra* **592** (2022), 424–434. MR Zbl

[Mœglin and Waldspurger 1987] C. Mœglin and J.-L. Waldspurger, "Modèles de Whittaker dégénérés pour des groupes p-adiques", *Math. Z.* **196**:3 (1987), 427–452. MR Zbl

[Neukirch 1999] J. Neukirch, *Algebraic number theory*, Grundl. Math. Wissen. **322**, Springer, Berlin, 1999. MR Zbl

[Nevins 2005] M. Nevins, "Branching rules for principal series representations of SL(2) over a p-adic field", *Canad. J. Math.* **57**:3 (2005), 648–672. MR Zbl

[Nevins 2013] M. Nevins, "Branching rules for supercuspidal representations of $SL_2(k)$, for k a p-adic field", *J. Algebra* **377** (2013), 204–231. MR Zbl

[Nevins 2024] M. Nevins, "The local character expansion as branching rules: nilpotent cones and the case of SL(2)", *Pacific J. Math.* **329**:2 (2024), 259–301. MR Zbl

[Platonov and Rapinchuk 1994] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics **139**, Academic Press, Boston, MA, 1994. MR Zbl

[Rodier 1975] F. Rodier, "Modèle de Whittaker et caractères de représentations", pp. 151–171 in *Non-commutative harmonic analysis*, edited by J. Carmona et al., Lecture Notes in Mathematics **466**, Springer, 1975. Zbl

[Serre 1977] J.-P. Serre, "Modular forms of weight one and Galois representations", pp. 193–268 in *Algebraic number fields*: *L-functions and Galois properties* (Durham, 1975), edited by A. Fröhlich, Academic Press, 1977. MR Zbl

[Shelstad 1979] D. Shelstad, "Notes on L-indistinguishability", pp. 193–203 in *Automorphic forms, representations and L-functions* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **XXXIII**, Amer. Math. Soc., Providence, RI, 1979. MR Zbl

[Silberger 1979] A. J. Silberger, "Isogeny restrictions of irreducible admissible representations are finite direct sums of irreducible admissible representations", *Proc. Amer. Math. Soc.* **73**:2 (1979), 263–264. MR Zbl

[Springer 1998] T. A. Springer, *Linear algebraic groups*, 2nd ed., Progress in Mathematics **9**, Birkhäuser, Boston, MA, 1998. MR Zbl

[Tadić 1992] M. Tadić, "Notes on representations of non-archimedean $\mathrm{SL}(n)$", *Pacific J. Math.* **152**:2 (1992), 375–396. MR Zbl

[Treumann and Venkatesh 2016] D. Treumann and A. Venkatesh, "Functoriality, Smith theory, and the Brauer homomorphism", *Ann. of Math.* (2) **183**:1 (2016), 177–228. MR Zbl

[Varma 2014] S. Varma, "On a result of Moeglin and Waldspurger in residual characteristic 2", *Math. Z.* **277**:3–4 (2014), 1027–1048. MR Zbl

[Vignéras 1988] M.-F. Vignéras, "Représentations modulaires de $\mathrm{GL}(2, \mathbb{F})$ en caractéristique ℓ, $\mathbb{F}$ corps fini de caractéristique $p \neq \ell$", *C. R. Acad. Sci. Paris Sér. I Math.* **306**:11 (1988), 451–454. MR Zbl

[Vignéras 1989] M.-F. Vignéras, "Représentations modulaires de $\mathrm{GL}(2, F)$ en caractéristique ℓ, F corps p-adique, $p \neq \ell$", *Compositio Math.* **72**:1 (1989), 33–66. MR Zbl

[Vignéras 1996] M.-F. Vignéras, *Représentations ℓ-modulaires d'un groupe réductif p-adique avec $\ell \neq p$*, Progress in Mathematics **137**, Birkhäuser, Boston, MA, 1996. MR Zbl

[Vignéras 1997] M.-F. Vignéras, "À propos d'une conjecture de Langlands modulaire", pp. 415–452 in *Finite reductive groups*: *related structures and representations* (Luminy, 1994), edited by M. Cabanes, Progr. Math. **141**, Birkhäuser, Boston, MA, 1997. MR Zbl

[Vignéras 2001] M.-F. Vignéras, "Correspondance de Langlands semi-simple pour $\mathrm{GL}(n, F)$ modulo $\ell \neq p$", *Invent. Math.* **144**:1 (2001), 177–223. MR Zbl

[Vignéras 2023] M.-F. Vignéras, "Representations of p-adic groups over commutative rings", pp. 332–374 in *International Congress of Mathematicians*, *Vol.* 1: *Prize lectures*, edited by D. Beliaev and S. Smirnov, EMS Press, Berlin, 2023. MR Zbl

Received April 9, 2024. Revised January 3, 2025.

GUY HENNIART
LABORATOIRE DE MATHÉMATIQUES D'ORSAY
UNIVERSITÉ PARIS-SACLAY
ORSAY
FRANCE
guy.henniart@math.u-psud.fr

MARIE-FRANCE VIGNÉRAS
INSTITUT DE MATHÉMATIQUES DE JUSSIEU
UNIVERSITÉ PARIS CITÉ
PARIS
FRANCE
marie-france.vigneras@imj-prg.fr

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 335 No. 2 April 2025