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**SIMPLICITY OF THE AUTOMORPHISM GROUP
OF FIELDS WITH OPERATORS**

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*... A lot of things unsaid as well.
We shout and argue and fight,
and work it on out.
The sum is greater than the parts.
(Pink Floyd, "Things left unsaid")*

To Zoé, in memoriam.

We adapt a proof of Lascar in order to show the simplicity of the group of automorphisms fixing pointwise all nongeneric elements for a class of uncountable models of suitable theories, encompassing both strongly minimal theories as well as several theories of fields with operators.

1. Introduction

The study of the lattice of normal subgroups of the automorphism groups of classical structures is a recurrent topic in mathematics. Schreier and Ulam [31] classified all normal subgroups of the permutation group $\text{Sym}(\mathbb{N})$ of the integers. (The anonymous referee of a previous version of this article let us know that their result was already implicitly contained in Onofri's work [27]). In order to classify all normal subgroups of the permutation group of uncountable sets, Baer [1, Satz] shows that the group $\text{Sym}(\kappa)$ is simple modulo the normal subgroup of those automorphisms with support strictly smaller than κ . Rosenberg obtains a similar classification for the general linear group of a vector space of dimension κ over a fixed field and shows, among others, that this group is again simple, modulo the normal subgroup of those isomorphisms which differs from a homothecy by a linear transformation of range strictly less than κ [30, Proposition 3.6 and Lemma 3.7] (see also [14]).

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Infinite sets, infinite vector (or affine) spaces over a ground field as well as algebraically closed fields of some fixed characteristic constitute the three archetypes of strongly minimal sets. In [20] Lascar studied the structure of the automorphism group of a countable (almost) strongly minimal set \mathbb{M} of infinite dimension, more precisely, the normal subgroup $\text{Aut}(\mathbb{M}/\text{acl}^{\text{eq}}(\emptyset))$ of those automorphisms fixing the algebraic closure (in the imaginary expansion \mathbb{M}^{eq}) of the empty set. He showed in particular that this subgroup is simple modulo a certain class of automorphisms, which he referred to as *bounded* (see Definition 5.4). Under the continuum hypothesis, he concluded from his result in the countable case the simplicity of the group of field automorphisms of the complex numbers \mathbb{C} which fix pointwise the algebraic closure $\overline{\mathbb{Q}}$ of the prime field. Lascar himself wrote [20, p.249]: *J'ai peine à imaginer que ce fait n'est pas déjà connu* (I cannot imagine that this result is not already well known). Lascar's proof [20] strongly used that the group of automorphisms of a countable structure is naturally a Polish group, and hence many of his arguments have a topological flavour. His proof has two main ingredients: Firstly, the only bounded automorphism of the field \mathbb{C} is the identity. Secondly, he manages to fuse partial elementary maps defined on independent subsets, since algebraic independence coincides with nonforking independence for the stable theory of the field \mathbb{C} , which eliminates imaginaries, so types over algebraically closed subsets of \mathbb{C} are stationary. Lascar's proof was adapted by Evans, Ghadernezhad and Tent [12, Example 3.14] to show the simplicity of the group of automorphisms of a countable saturated differentially closed field fixing pointwise all differentially algebraic elements.

In [22] Lascar provided a more direct proof of the simplicity of $\text{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ in pure algebraic terms, circumventing both the use of topological arguments as well as the continuum hypothesis. His second proof holds for every uncountable almost strongly minimal set in a countable language (see [21, Partie 4]) and relies on a clever use via transfinite induction of the two main ingredients of his original proof [20].

Motivated by Lascar's latter approach in the uncountable case, we will present here a unifying approach to the proof given in [22] (see also the French version [21]) to study the automorphism group of several uncountable fields equipped with operators. More precisely, we show the simplicity of the group of automorphisms fixing the closure of the prime field with respect to a natural pregeometry, modulo the subgroup of bounded automorphisms (see Definition 5.4). The description of bounded automorphisms of several theories of fields with operators already appears in [4; 36]. Our work deals mainly with the case of stable theories: all but one example are stable. We retrieve the existing results (since they are all ω -stable), and get several new examples. Our method gives the blueprint to obtain many more. The unstable example is the theory ACFA_0 of existentially closed difference fields

of characteristic 0, which is unstable, but however supersimple, and therefore has a good notion of independence. It is likely that our methods will also work for other well-behaved simple theories.

Whilst uncountably saturated models exist for stable theories, this need no longer be the case for simple theories without assuming additional set-theoretical assumptions. A generalisation of saturation in its own cardinality $\kappa > \aleph_0$ is the notion of κ -primeness over a given set of parameters (see Definition 3.6). The existence and uniqueness of κ -prime models over a given subset of parameters is due to Shelah for stable theories [32, Theorem IV.4.18] under certain set-theoretic assumptions on the cardinal κ . An analogous result was obtained by Chatzidakis [6, Theorem 3.14] for the unstable theory ACFA_0 .

In this work, we will provide a general approach, encompassing both the case of uncountably saturated models (if they exist) as well as the case of κ -prime models, for suitable theories of fields with operators, to deduce the following results:

Theorem A (see Theorem 4.11 and Remark 5.6). *Consider a complete simple theory T in some language \mathcal{L} satisfying Properties 1-3 as in Section 2. Fix a cardinal $\kappa \geq |\mathcal{L}|^+$ as well as a κ -tame model \mathbb{U} over a subset Z of size strictly less than κ (see Definition 3.1). For every subset Z , we have its corresponding closure $\text{cl}(Z)$ as in Definition 2.3.*

Given an automorphism τ of \mathbb{U} fixing $\text{cl}(Z)$ pointwise which moves maximally (see Definition 4.6), every automorphism ν of \mathbb{U} fixing $\text{cl}(Z)$ pointwise can be written as the product of four conjugates of τ and τ^{-1} .

In particular, under the conditions of Remark 5.5, the group $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ of automorphisms of \mathbb{U} fixing $\text{cl}(Z)$ pointwise is simple modulo the normal subgroup of all bounded automorphisms fixing $\text{cl}(Z)$ pointwise (see Definition 5.4).

For some particular theories (listed below), an explicit description of the bounded automorphisms can be deduced. In the particular examples of uncountable sets and vector spaces over a field of uncountable dimension (both 1-based strongly minimal), we recover thus the results of Baer [1] and Rosenberg [30] mentioned at the beginning:

Theorem B (see Proposition 6.1 and Theorem 6.18). *For the two classical examples of 1-based strongly minimal theories, we have the following: given an uncountable cardinal κ and an infinite set \mathcal{M} of cardinality κ with no additional structure, the group of permutations of \mathcal{M} is simple modulo the subgroup of permutations of support strictly less than κ .*

Similarly, given a field K and an uncountable cardinal $\kappa > |K|$, the group of isomorphisms of a K -vector space V of cardinality κ is simple modulo the subgroup of those isomorphisms φ such that for some λ in K , the eigenspace $\text{Ker}(\varphi - \lambda \cdot \text{Id}_V)$ has codimension strictly less than κ .

Moreover, for each of the following countable theories of fields with operators:

- algebraically closed fields with the closure operator given by the field algebraic closure;
- differentially closed fields in characteristic 0 with finitely many commuting derivations with the closure operator given by the elements which are not differentially transcendental;
- differential fields in characteristic 0, maximal with the property of omitting a given strictly minimal type X with the same closure as above;
- difference closed fields in characteristic 0 with the closure operator given by the elements of transformal transcendence degree 0;
- proper pairs of algebraically closed fields (K, E) with the closure operator

$$\text{cl}(B) = E(B)^{\text{alg}};$$

- separably closed fields K in characteristic p and infinite imperfection degree with the closure operator

$$\text{cl}(B) = \bigcap_{n \in \mathbb{N}} (K^{p^n} \text{acl}(B));$$

given an uncountable model saturated \mathbb{U} in its uncountable cardinality κ (if such models exist) and a subset Z of parameters of size strictly less than κ , the group of automorphisms of \mathbb{U} fixing pointwise $\text{cl}(Z)$ is simple.

More generally, for any of the above theories, given an uncountable cardinal κ (with $\text{cof}(\kappa) \geq \aleph_1$ in the last example) and a κ -prime model \mathbb{U} over $\text{cl}(Z)$, where Z is a subset of size strictly less than κ , the automorphism group $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ is simple.

2. Preliminaries

Consider a first-order complete theory T in a language \mathcal{L} . We will work inside a κ -saturated model \mathbb{U} of T with $\kappa \geq |\mathcal{L}|^+$, so all subsets we consider will have size strictly less than κ , unless explicitly stated. Furthermore, we impose that the \mathcal{L} -theory T is simple.

In this section, we will list and discuss a series of properties, which will allow us to study the simplicity of the automorphism group.

Recall that a type p over \emptyset is *stationary* if for every subset A of \mathbb{U} , any two realisations of p which are both independent from A have the same type over A . Here, independence means nonforking independence in the sense of the simple theory T .

Property 1. *There exists a stationary (nonalgebraic) type p_0 over \emptyset such that every element of \mathbb{U} is algebraic over finitely many realisations of p_0 .*

Remark 2.1. We will mostly consider the two following classes of theories satisfying Property 1:

- (a) The theory T is almost strongly minimal, that is, there is a strongly minimal set X defined over \emptyset such that (every point in) our model \mathbb{U} is algebraic over $X(\mathbb{U})$. The unique nonalgebraic type containing the definable set X is our type p_0 .
- (b) The universe \mathbb{U} of T has the structure of a definable group G without parameters. We will restrict our attention to groups with a unique generic type p_0 , which is moreover stationary. In particular every element of G is the product of two realisations of p_0 .

Convention. From now on, we will refer to the above type p_0 as *the generic type*. Given a subset A of \mathbb{U} , we will say that the element b of \mathbb{U} is *generic over A* if b realises the unique nonforking extension of p_0 to A .

Definition 2.2. Two types p in $S(A_1)$ and q in $S(A_2)$ are *orthogonal* (denoted $p \perp q$) if whenever a set C contains $A_1 \cup A_2$, any realisation b_1 of a nonforking extension of p to C is independent over C from every realisation b_2 of a nonforking extension of q to C , that is, we have that $b_1 \perp_C b_2$ whenever $b_i \perp_{A_i} C$ for $i = 1, 2$.

In [35, Section 3.5], previous work of [15] on certain closure operators extending the (model-theoretic) algebraic closure was adapted in order to produce closure operators arising from an arbitrary collection of types. In our context, the closure is to be taken inside the ambient model \mathbb{U} with respect to the generic type p_0 (so the closure does depend on the model). By Property 1, the type of every tuple in \mathbb{U} is analysable with respect to the generic type, so that in our concrete case, the closure with respect to the generic type p_0 over \emptyset can be described as follows below.

Definition 2.3. Given a subset A of \mathbb{U} , an element b *belongs to the closure* $\text{cl}(A)$ *of A* if every extension of $\text{tp}(b/A)$ is orthogonal to p_0 , that is, for every subset C containing A and every generic element g over C , we have that g remains generic over $C \cup \{b\}$.

We say that the subset X is *cl-closed* if $\text{cl}(X) = X$. The set X is *cl-generated* by A if $X = \text{cl}(A)$.

Note that every element which is algebraic over A (in the model-theoretic sense) belongs to $\text{cl}(A)$. In contrast to the algebraic closure, in most examples of fields with operators we are interested in, the closure of a set will be rather large (e.g., of cardinality $\geq \kappa$ in a κ -saturated model), and may increase as the model changes. For example, in the case of differentially closed fields of characteristic 0, every constant element (that is, whose derivative is 0) belongs to $\text{cl}(\emptyset)$.

This notion of closure satisfies some important properties if every element is analysable with respect to p_0 , which justifies the following assumption: *henceforth,*

we will assume that the simple theory T satisfies Property 1 with respect to the fixed generic type p_0 .

Remark 2.4 (see [35, Lemmata 3.5.3 and 3.5.5]). (a) (usual properties of a closure operator) If $A \subset B$, then $\text{cl}(A) \subset \text{cl}(B)$ and $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

(b) If g is generic over A , then it remains so over $\text{cl}(A)$. In particular, no generic element over A lies in $\text{cl}(A)$.

(c) Given two subsets B and C of \mathbb{U} with a common subset A ,

$$B \downarrow_A C \implies \text{cl}(B) \downarrow_{\text{cl}(A)} \text{cl}(C).$$

It follows from local character and the above remark that the closure of a subset A of cardinality possibly larger than $|\mathcal{L}|$ is the union of all $\text{cl}(A_0)$, where A_0 runs among the subsets of A of cardinality at most $|\mathcal{L}|$.

The stationarity of the type p_0 will be crucial in many of our proofs. In a stable theory with weak elimination of imaginaries, types over algebraically closed sets are always stationary. As shown in [8, Theorem 5.3 and Corollary B.11] for difference closed fields (which are unstable), types of the form $\text{tp}(a/B)$ with $B = \text{cl}(B) \cap \text{acl}(B, a)$ are stationary, so this motivates the following property.

Property 2. *Types over relatively cl-closed subsets are stationary: given subsets A_1 , A_2 and B with a common algebraically closed subset C such that*

$$\text{acl}(A_1) \cap \text{cl}(C) = C = \text{acl}(A_2) \cap \text{cl}(C),$$

if $A_1 \equiv_C A_2$ and

$$A_1 \downarrow_C B \quad \text{and} \quad A_2 \downarrow_C B,$$

then $A_1 \equiv_B A_2$.

The following property will be fundamental in order to extend partial automorphisms to arbitrary subsets in terms of a small chain of extensions obtained by successively adding generic elements.

Property 3. *For every κ -saturated model \mathcal{M} and every subset Z of \mathcal{M} of cardinality strictly less than κ , we have that $\mathcal{M} = \text{cl}(Z \cup A)$, where A enumerates a sequence of independent realisations over Z of the generic type.*

Remark 2.5. (a) It follows from Remark 2.4(b) that the above sequence A in Property 3 is maximal with respect to being independent realisations of the generic type over Z . Moreover, notice that $|A| \geq \kappa$ by κ -saturation of \mathcal{M} .

(b) If Property 3 holds, then for every subset Z of the ambient model \mathbb{U} each element b of \mathbb{U} is contained in $\text{cl}(Z \cup A_0)$, where A_0 enumerates a sequence of length at most $|\mathcal{L}|$ of independent realisations of the generic type over Z . Indeed, we know that $\mathbb{U} = \text{cl}(Z \cup A)$, where A enumerates a sequence of independent

realisations over Z of the generic type. Applying local character to $\text{tp}(b/Z \cup A)$, we deduce from Remark 2.4(c) that b belongs to $\text{cl}(Z \cup A_0)$ for some $A_0 \subset A$ of cardinality at most $|\mathcal{L}|$, as desired.

Definition 2.6. Given a finite tuple c and a subset K of \mathbb{U} of cardinality possibly larger than κ , the type $\text{tp}(c/K)$ is κ -isolated if there is some subset E of K of cardinality strictly less than κ such that $\text{tp}(c/E) \vdash \text{tp}(c/K)$.

The model \mathbb{U} is κ -atomic over K if for every finite tuple c of \mathbb{U} , the type $\text{tp}(c/K)$ is κ -isolated.

Remark 2.7. If \mathbb{U} is κ -atomic over K , then for every c in \mathbb{U} and every partial elementary map $f : K \rightarrow \mathbb{U}$, there is an extension of f to a partial elementary map defined on $K \cup \{c\}$. Indeed, by κ -atomicity, we have that $\text{tp}(c/E) \vdash \text{tp}(c/K)$ for some subset E of K of cardinality strictly less than κ . In particular, the same holds for the images of these types under the map f . By κ -saturation of \mathbb{U} , we can find a realisation d of $f(\text{tp}(c/E))$, so we can extend f to $K \cup \{c\}$ mapping c to d .

The following property was shown by Konnerth for the theory DCF_m [18, Lemma 2.3]. It will be an important tool towards extending partial isomorphisms.

Definition 2.8. The theory T satisfies property (WH) (for *weak homogeneity*) if every κ -saturated model \mathbb{U} of T is κ -atomic over subsets of the form

$$K = \text{acl}(\text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B),$$

such that all A_i 's and B have cardinality strictly less than κ .

In particular, given some c in \mathbb{U} , every partial elementary map $f : K \rightarrow \mathbb{U}$ extends to a partial elementary map defined on $K \cup \{c\}$.

3. Tame models

In this section, we assume that $\kappa \geq |\mathcal{L}|^+$ and that the simple \mathcal{L} -theory T satisfies Properties 1–3.

In order to prove Theorem A (which corresponds to Theorem 4.11), we will need to restrict our focus to a particular class of κ -saturated models, which we call κ -tame.

Definition 3.1. A κ -saturated model \mathbb{U} of the theory T is κ -tame over the subset Z of size strictly less than κ if it satisfies the following conditions:

- (a) There is a subset A of \mathbb{U} of size κ enumerating an independent sequence over Z of realisations of the generic type p_0 (or rather of the nonforking extension of p_0 to Z) with $\mathbb{U} = \text{cl}(Z \cup A)$.
- (b) (strong homogeneity) Every partial elementary map $f : K \rightarrow \mathbb{U}$ fixing $\text{cl}(Z)$ pointwise with

$$K = \text{acl}(\text{cl}(Z) \cup \text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$$

and

$$f(K) = \text{acl}(\text{cl}(Z) \cup \text{cl}(f(A_1)) \cup \dots \cup \text{cl}(f(A_n)) \cup f(B)),$$

such that all A_i 's and B have cardinality strictly less than κ , extends to a global automorphism of \mathbb{U} fixing $\text{cl}(Z)$ pointwise.

Remark 3.2. (i) It follows directly from condition (a) that the model \mathbb{U} does not contain a sequence of length κ^+ of independent realisations over Z of the generic type.

(ii) We may replace condition (a) in Definition 3.1 by $\mathbb{U} = \text{cl}(Z \cup B)$ for some subset B of size κ (not necessarily consisting of realisations of p_0). Indeed, one applies Remark 2.5(b) to some enumeration of B of order type κ to obtain an independent set C of cardinality at most κ of realisations of the generic type over Z , such that $B \subset \text{cl}(Z \cup C)$, so $\text{cl}(Z \cup C) = \mathbb{U}$. Note that the set C has cardinality κ , by Remark 2.4(b) and κ -saturation of \mathbb{U} .

Given a subset $K = \text{acl}(\text{cl}(Z) \cup \text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$ as in Definition 3.1(b) and a sequence $(c_i)_{i < \lambda}$ with $\lambda < \kappa$, the set $\text{acl}(K, \{c_i\}_{i < \lambda})$ also satisfies the hypotheses of Definition 3.1(b). Thus, the next remark follows immediately from the stationarity of the generic type in Property 1:

Remark 3.3. Consider a κ -tame model \mathbb{U} over Z of the theory T and a partial elementary map $f : K \rightarrow f(K) \subset \mathbb{U}$ as in Definition 3.1(b). Given $\lambda < \kappa$ as well as two sequences $(c_i)_{i < \lambda}$ and $(d_i)_{i < \lambda}$ of realisations of the unique generic type which are, respectively, independent over K and over $f(K)$, there is an extension of f to a global automorphism mapping the sequence $(c_i)_{i < \lambda}$ to the sequence $(d_i)_{i < \lambda}$. In particular, there is a global automorphism of \mathbb{U} mapping $\text{cl}(K, \{c_i\}_{i < \lambda})$ onto $\text{cl}(f(K), \{d_i\}_{i < \lambda})$ and fixing $\text{cl}(Z)$ pointwise.

Lemma 3.4. *Let \mathbb{U} be a κ -tame model over Z of the theory T and consider three cl -closed subsets X, Y_1 and Y_2 of \mathbb{U} with $Z \subseteq X \subseteq Y_1 \cap Y_2$, each cl -generated over Z by subsets of size strictly less than κ . Given elementary automorphisms g_i of Y_i for $1 \leq i \leq 2$ fixing $\text{cl}(Z)$ pointwise which agree on X and are such that $g_1(X) = g_2(X) = X$, if $Y_1 \perp_X Y_2$, then there is a global automorphism of \mathbb{U} which extends both g_1 and g_2 (and hence fixes $\text{cl}(Z)$ pointwise).*

Proof. Set $Y_1 = \text{cl}(Z \cup A_1)$ for some subset A_1 of cardinality strictly less than κ . Note that $Y_1 = \text{cl}(Z \cup g_1(A_1))$, since g_1 is an elementary map of Y_1 onto itself. By κ -tameness, there exists a global automorphism \hat{g}_2 of \mathbb{U} extending g_2 fixing $\text{cl}(Z)$ pointwise. After fixing an enumeration of Y_1 , the subset $Y'_1 = \hat{g}_2^{-1}(g_1(Y_1))$ has the same type as Y_1 over X , since g_1 and g_2 (and thus \hat{g}_2) agree on X . Note that $Y'_1 \perp_X Y_2$. By Property 2, we have $Y'_1 \equiv_{Y_2} Y_1$, since X is cl -closed. Thus, there is an elementary map h on $\text{acl}(Y_1 \cup Y_2)$ fixing pointwise Y_2 whose restriction to Y_1 coincides with $\hat{g}_2^{-1} \circ g_1$. Note that $Y'_1 = \text{cl}(Z \cup h(A_1))$, so we obtain by tameness a

global automorphism \hat{h} of \mathbb{U} extending h . By construction, the automorphism $\hat{g}_2 \circ \hat{h}$ extends both g_1 and g_2 . □

Under the additional assumption that the simple theory T satisfies property (WH) (Definition 2.8), there are natural examples of κ -tame models: saturated models of cardinality κ and κ -prime models of T are κ -tame.

Lemma 3.5. *Assume the simple theory T satisfies Properties 1–3 and (WH). If \mathbb{U} is a saturated model of T of cardinality κ , then \mathbb{U} is κ -tame over every subset Z of size strictly less than κ .*

Proof. Definition 3.1(a) holds trivially, since the model \mathbb{U} has cardinality κ , by Remark 3.2(ii). Definition 3.1(b) holds clearly by a standard back-and-forth argument, using property (WH) (and Remark 2.7). □

Whenever T is stable, saturated models of cardinality κ exist if and only if T is stable at the cardinal κ [13; 32]. For a general theory T , if $\kappa = \lambda^+$ for some cardinal $\lambda \geq |\mathcal{L}|$ with $\lambda^+ = 2^\lambda$, or if κ is regular and strongly inaccessible, then there are saturated models of cardinality κ . However, these set-theoretic assumptions go beyond ZFC.

Shelah introduced the notion of κ -prime models [32, Chapter IV] (see also [19, Chapitre VI]) for an arbitrary stable theory, generalising Morley’s notion of prime models for an ω -stable theory.

Definition 3.6. A model \mathbb{U} of T is κ -prime over $A \subseteq \mathbb{U}$ if it is κ -saturated and elementarily embeds over A into every κ -saturated model of T containing A .

Remark 3.7. Shelah [32, Theorems IV.4.14 and IV.4.18] showed the existence and uniqueness (up to isomorphism) of κ -prime models over arbitrary subsets A (even if the size of A is greater than κ) whenever the theory is stable, as long as the cofinality of κ is at least $|\mathcal{L}|^+$. Indeed, Shelah showed that a κ -saturated model of the stable theory is κ -prime over A if and only if it is κ -atomic over A and contains no (nonconstant) A -indiscernible sequence of length κ^+ .

In the particular case that T is superstable, then the same holds for all $\kappa \geq |\mathcal{L}|^+$, regardless of the cofinality.

Whilst every completion T of the theory ACFA₀ is simple yet unstable, in [6] Chatzidakis obtained an analogous result to the existence and uniqueness of κ -prime models for ACFA₀ as well as their characterisation, with no restriction on $\kappa \geq \aleph_1$, but assuming that the base set A is algebraically closed and that the fixed field of A is a κ -saturated pseudofinite field.

The next result is motivated by the above description of κ -prime models.

Proposition 3.8. *Assume that the simple theory T satisfies Properties 1–3 and (WH). We suppose furthermore the following condition for every κ -saturated model \mathcal{M} :*

for every subset of \mathcal{M} of the form $K = \text{acl}(\text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$, such that all A_i 's and B have cardinality strictly less than κ , there exists a unique κ -prime model over K , which is characterised by being κ -atomic over K and containing no (nonconstant) K -indiscernible sequence of length κ^+ .

Then for every κ -saturated model \mathcal{M} of T and $Z \subseteq \mathcal{M}$ of cardinality strictly less than κ , the κ -prime model \mathbb{U} over $\text{cl}_{\mathcal{M}}(Z)$ is κ -tame over Z and $\text{cl}_{\mathbb{U}}(Z) = \text{cl}_{\mathcal{M}}(Z)$, where $\text{cl}_{\mathbb{U}}$ and $\text{cl}_{\mathcal{M}}$ denote the closures taken within the models \mathbb{U} and \mathcal{M} , respectively.

Proof. By assumption, the κ -prime model \mathbb{U} over $\text{cl}_{\mathcal{M}}(Z)$ exists. Since \mathbb{U} embeds into \mathcal{M} over $\text{cl}_{\mathcal{M}}(Z)$, it follows immediately that $\text{cl}_{\mathbb{U}}(Z) = \text{cl}_{\mathcal{M}}(Z)$. From now on, we will just write $\text{cl}(Z)$ without referring to a particular model. Let us now show that each of the conditions in Definition 3.1 of κ -tameness holds for the κ -prime model \mathbb{U} over $\text{cl}(Z)$. For the first condition, Property 3 yields that $\mathbb{U} = \text{cl}(Z \cup A)$, where A is a set enumerating a maximal sequence of independent realisations over Z of the generic type (note that $|A| \geq \kappa$ by saturation). Since the sequence determined by A is Z -indiscernible by the stationarity of the generic type p_0 in Property 1, we conclude from our assumption in the statement that A has cardinality exactly κ , as desired.

We now show the strong extension property for elementary maps $f : K \rightarrow \mathbb{U}$ such that there exists subsets A_i 's and B of \mathbb{U} of cardinality strictly less than κ with

- $K = \text{acl}(\text{cl}(Z) \cup \text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$;
- $f(K) = \text{acl}(\text{cl}(Z) \cup \text{cl}(f(A_1)) \cup \dots \cup \text{cl}(f(A_n)) \cup f(B))$;
- the map f is the identity on $\text{cl}(Z)$.

Now, property (WH) gives that the κ -saturated model \mathbb{U} is κ -atomic over K and over $f(K)$ as well. Our above characterisation of κ -prime models implies that \mathbb{U} contains no nonconstant indiscernible sequence of length κ^+ over $\text{cl}(Z)$, so \mathbb{U} does not have such indiscernible sequences over K nor over $f(K)$. Hence, by our hypothesis on the uniqueness and the characterisation of κ -prime models, the partial type $f : K \rightarrow f(K)$ extends to an automorphism of \mathbb{U} . \square

4. Automorphisms of κ -tame models

In this section we fix a cardinal $\kappa \geq |\mathcal{L}|^+$ and a simple \mathcal{L} -theory T , which satisfies Properties 1–3. We assume furthermore that T has a κ -tame model \mathbb{U} over a subset Z of cardinality strictly less than κ .

Definition 4.1. We denote by \mathcal{S} the collection of subsets of \mathbb{U} of the form $\text{cl}(Z \cup A)$, where A enumerates a sequence of length strictly less than κ of independent realisations of the generic type over Z .

Note that each member of \mathcal{S} is an algebraically closed substructure of \mathbb{U} . Moreover, if B enumerates a sequence of length strictly less than κ of independent realisations of the generic type over X (by Remark 2.4 it suffices that the sequence is independent over D with $X = \text{cl}(D)$), then $Y = \text{cl}(X \cup B)$ lies again in \mathcal{S} .

Definition 4.2. An extension $X \subseteq Y$ with X and Y in \mathcal{S} is *acceptable* if $Y = \text{cl}(X \cup A)$, where A enumerates a sequence of length $|\mathcal{L}|$ of independent realisations of the generic type over X .

Remark 4.3. For $\alpha < \kappa$, every increasing union $(X_\beta)_{\beta < \alpha}$ in \mathcal{S} with $X_{\beta+1}$ acceptable over X_β and $X_\gamma = \bigcup_{\beta < \gamma} X_\beta$ for $\gamma < \alpha$ a limit ordinal belongs again to \mathcal{S} .

Remark 3.3 yields immediately the following result:

Lemma 4.4. *Given an elementary automorphism f of X in \mathcal{S} and two acceptable extensions Y_1 and Y_2 of X , there is an extension of f to an elementary partial map sending Y_1 onto Y_2 . In particular, any two acceptable extensions Y_1 and Y_2 of X are conjugate by an automorphism of \mathbb{U} fixing X pointwise.* □

Remark 4.5. Given X in \mathcal{S} , an element b in \mathbb{U} as well as a countable collection \mathcal{F} of global automorphisms of \mathbb{U} which leaves X setwise invariant, there exists an acceptable extension Y of X in \mathcal{S} which contains b and is setwise stable under each automorphism of \mathcal{F} .

Proof. Without loss of generality, we may assume that \mathcal{F} is a group (under composition). By induction, successively applying Remark 2.5(b), we construct an increasing sequence $(A_n)_{n \in \mathbb{N}}$ of sets such that

- the set A_0 cl-generates X ;
- the element b belongs to $\text{cl}(A_1)$;
- for every $n \geq 1$ in \mathbb{N} , every τ in \mathcal{F} and every a in $A_n \setminus A_{n-1}$, the element $\tau(a)$ belongs to $\text{cl}(A_{n+1})$;
- the elements in $A_{n+1} \setminus A_n$ enumerate an independent sequence of realisations of length $|\mathcal{L}|$ of the generic type over A_n (and thus over $\text{cl}(A_n) \supset X$).

The set $Y = \text{cl}(\bigcup_{n \in \mathbb{N}} A_n)$ has all the desired properties. □

We recall now a modified version of when an automorphism *moves maximally*, according to the terminology of Tent and Ziegler [34, Definition 2.5] restricting our attention to the unique generic type p_0 .

Definition 4.6. An automorphism τ of a κ -saturated model \mathcal{M} of T *moves p_0 maximally* if for every subset A of \mathcal{M} of cardinality strictly less than κ which is stable under the action of τ , there exists a generic element b over A such that $\tau(b)$ is generic over $A \cup \{b\}$, or equivalently, such that $\tau(b) \perp_A b$.

Remark 4.7. If τ moves maximally, then for every subset A of cardinality strictly less than κ , which need not be stable under the action of τ , there is some generic element b over $\bigcup_{k \in \mathbb{Z}} \tau^k(A)$ with $\tau(b) \perp_A b$. In particular, both b and $\tau(b)$ are generic over A .

Lemma 4.8. *Let τ be an automorphism of \mathbb{U} moving p_0 maximally. If X in \mathcal{S} is stable setwise under the action of τ , then there is an acceptable extension Y of X such that Y and $\tau(Y)$ are independent over X .*

Notice that $\tau(Y)$ is again an acceptable extension of X .

Proof. Set $X = \text{cl}(D)$ for some $D \supset Z$ of cardinality strictly less than κ . By Remark 2.4(c), it suffices to inductively build an independent sequence $(a_\alpha)_{\alpha < |\mathcal{L}|}$ of realisations of the generic type over D such that for every $\alpha < |\mathcal{L}|$ we have

$$a_\alpha \perp_D (a_\beta, \tau(a_\beta))_{\beta < \alpha} \quad \text{and} \quad \tau(a_\alpha) \perp_D (a_\beta, \tau(a_\beta))_{\beta < \alpha} \cup \{a_\alpha\}.$$

Indeed, the set $Y = \text{cl}(X \cup \{a_\alpha\}_{\alpha < |\mathcal{L}|})$ belongs to \mathcal{S} and is an acceptable extension of X with the desired properties.

Suppose the sequence has been constructed for all $\beta < \alpha$. Now, the automorphism τ moves p_0 maximally over $D' = D \cup (a_\beta, \tau(a_\beta))_{\beta < \alpha}$, so by Remark 4.7 we find an element a_α in \mathbb{U} generic over D' such that

$$\tau(a_\alpha) \perp_{D'} a_\alpha.$$

In particular, the element a_α is generic over $D \cup \{a_\beta\}_{\beta < \alpha}$, as desired. □

We will now reproduce verbatim Lascar’s proof of the simplicity of the group of field automorphisms of \mathbb{C} fixing \mathbb{Q}^{alg} [22]. More precisely, we will show that every element in $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ is a product of four conjugates of τ and τ^{-1} , whenever τ moves p_0 maximally (in case such an automorphism exists).

We fix such an automorphism τ of \mathbb{U} fixing $\text{cl}(Z)$ pointwise. Define

$$\Psi(f, g) = \tau^f \circ (\tau^{-1})^{f \circ g} = f \circ \tau \circ g \circ \tau^{-1} \circ g^{-1} \circ f^{-1} = [\tau, g]^f$$

for f and g automorphisms of \mathbb{U} , where $\tau^f = f \circ \tau \circ f^{-1}$ and $[\tau, g] = \tau g \tau^{-1} g^{-1}$.

Whenever X in \mathcal{S} is stable under the action of τ , given f and g elementary automorphisms of X , we define

$$\Psi_X(f, g) = (\tau|_X)^f \circ (\tau|_X^{-1})^{f \circ g} = \Psi(f, g)|_X.$$

We will show in Theorem 4.11 that every automorphism ν of \mathbb{U} fixing $\text{cl}(Z)$ can be written as

$$\nu = \Psi(f_1, f_2) \circ \Psi(f_3, f_4)^{-1},$$

for suitable automorphisms f_1, \dots, f_4 . This will be done by a chain of approximations of ν to smaller substructures in the class \mathcal{S} .

If the partial isomorphism g extends the partial isomorphism f , we will denote it by $f \subset g$. For the back-and-forth process to describe ν as a product of conjugates of τ and τ^{-1} , we will need the following central result.

Proposition 4.9. *Consider X in \mathcal{S} stable under the action of the automorphism τ which moves p_0 maximally and fixes $\text{cl}(Z)$ pointwise, two elementary automorphisms f and g of X as well as an acceptable extension $Y \supseteq X$ in \mathcal{S} equipped with an elementary automorphism h of Y extending $\Psi_X(f, g)$. There exist two automorphisms f' and g' of \mathbb{U} extending f and g , respectively, such that*

$$\Psi(f', g') \supset h \supset \Psi_X(f, g),$$

so $\Psi(f', g')|_Y = h$.

Proof. By Lemma 4.8, there exists an acceptable extension Y_1 of X such that Y_1 and $Y_2 = \tau(Y_1)$ are independent over X . By Lemma 4.4, there is an automorphism f' of \mathbb{U} extending f such that f' maps Y_2 onto Y . Consider now the elementary automorphism

$$h_2 = f'^{-1} \circ h \circ f'$$

of Y_2 , which restricted to X equals

$$f^{-1} \circ h|_X \circ f = f^{-1} \circ \Psi_X(f, g) \circ f = \tau|_X \circ (\tau|_X^{-1})^g.$$

Again by Lemma 4.4, choose an elementary automorphism g_2 extending g which maps Y_2 onto itself. The elementary automorphism

$$g_1 = (h_2 \circ g_2)^{\tau^{-1}} = \tau^{-1} \circ h_2 \circ g_2 \circ \tau$$

of Y_1 restricted to X extends

$$\tau|_X^{-1} \circ h_2|_X \circ g \circ \tau|_X = (\tau|_X^{-1})^g \circ g \circ \tau|_X = g.$$

We include a diagram in Figure 1 to facilitate the arrow-chasing.

Lemma 3.4 yields now a common extension g' to \mathbb{U} of the elementary automorphisms g_1 of Y_1 and g_2 of Y_2 . We need only check now that the restriction $\Psi(f', g')|_Y$ of the global automorphism $\Psi(f', g')$ equals h , or equivalently, that $f'^{-1} \circ \Psi(f', g') \circ f'$ extends h_2 . By the definition of $\Psi(f', g')$, we have that the global automorphism

$$f'^{-1} \circ \Psi(f', g') \circ f' = \tau \circ (\tau^{-1})^{g'}$$

equals $h_2 = \tau \circ g_1 \circ \tau^{-1} \circ g_2^{-1}$ on Y_2 , as desired. □

Proposition 4.9 contains all the ingredients to tackle the successor stage of the back-and-forth construction required in the proof of Theorem 4.11

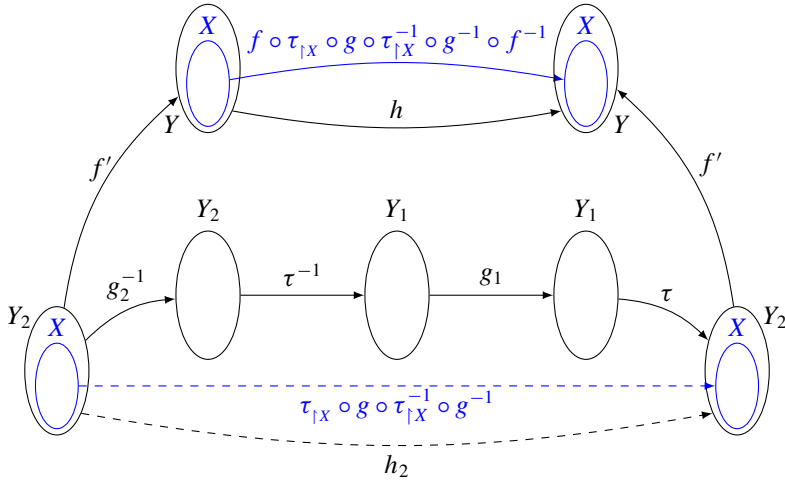


Figure 1. Arrow-chasing in the proof of Proposition 4.9.

Proposition 4.10. *Let v be an automorphism of \mathbb{U} fixing pointwise $\text{cl}(Z)$. Consider an automorphism τ which moves p_0 maximally and fixes $\text{cl}(Z)$ pointwise as well as a set X in S which is stable under the action of both τ and v such that*

$$v_{|X} = \Psi_X(f_1, f_2) \circ \Psi_X(f_3, f_4)^{-1}$$

for some elementary automorphisms f_i of X with $i = 1, \dots, 4$.

For every element a in \mathbb{U} , there are

- an acceptable extension $Y \supseteq X$ containing the element a which is stable under both τ and v ;
- elementary extensions f'_i of f_i , for $i = 1, \dots, 4$, to Y ,

such that

$$v_{|Y} = \Psi_Y(f'_1, f'_2) \circ \Psi_Y(f'_3, f'_4)^{-1}.$$

Proof. By Remark 4.5, there is an acceptable extension Y_1 of X containing a invariant under the action of both τ and v . By Lemma 4.4, choose two elementary automorphisms $f_{3,1}$ and $f_{4,1}$ of Y_1 extending, respectively, f_3 and f_4 . Set $h_1 = v_{|Y_1} \circ \Psi_{Y_1}(f_{3,1}, f_{4,1})$ and notice that

$$h_1 \supset \Psi_X(f_1, f_2).$$

By Proposition 4.9, there are two global automorphisms $f_{1,1}$ and $f_{2,1}$ of \mathbb{U} , extending f_1 and f_2 such that $\Psi(f_{1,1}, f_{2,1})$ extends h_1 .

Similarly, we find an acceptable extension Y_2 of Y_1 stable under the action of τ , v , $f_{1,1}$ and $f_{2,1}$. Then, we denote by $f_{1,2}$ and $f_{2,2}$ the restrictions of $f_{1,1}$ and $f_{2,1}$ to Y_2 .

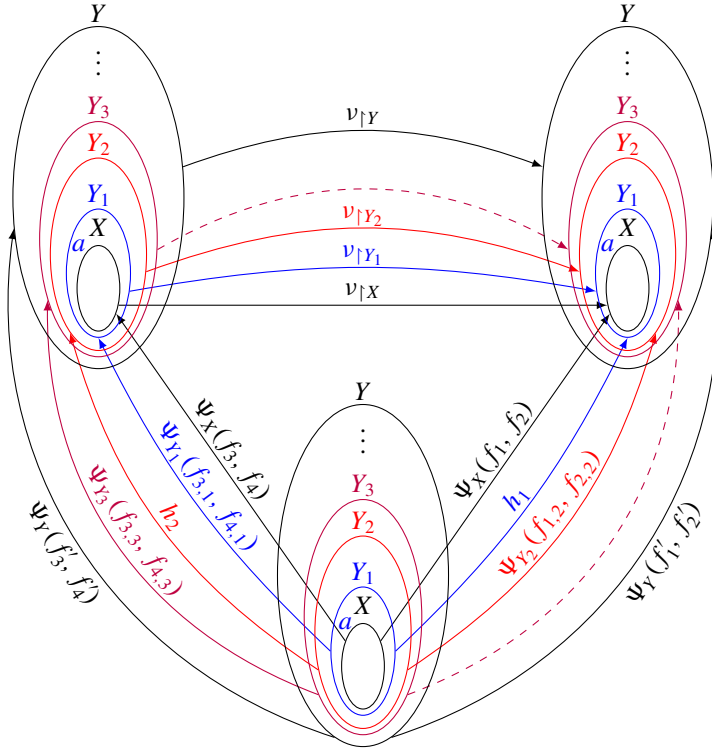


Figure 2. Illustration for the proof of Proposition 4.10.

Notice that

$$v_{|Y_2}^{-1} \circ \Psi_{Y_2}(f_{1,2}, f_{2,2}) \supset \Psi_{Y_1}(f_{3,1}, f_{4,1}).$$

Iterating the above argument countably many times, we construct an increasing chain Y_n of acceptable extensions (setting $Y_0 = X$) and compatible elementary extensions $f_{1,2n}$ and $f_{2,2n}$ of $f_1 = f_{1,0}$ and $f_2 = f_{2,0}$ to Y_{2n} as well as $f_{3,2n+1}$ and $f_{4,2n+1}$ of f_3 and f_4 to Y_{2n+1} such that for all n in \mathbb{N} ,

$$\begin{aligned} f_{1,2n+2} \supset f_{1,2n}, \quad f_{2,2n+2} \supset f_{2,2n}, \quad f_{3,2n+3} \supset f_{3,2n+1}, \quad f_{4,2n+3} \supset f_{4,2n+1}, \\ v_{|Y_{2n+1}} \circ \Psi_{Y_{2n+1}}(f_{3,2n+1}, f_{4,2n+1}) \supset \Psi_{Y_{2n}}(f_{1,2n}, f_{2,2n}), \\ v_{|Y_{2n+2}}^{-1} \circ \Psi_{Y_{2n+2}}(f_{1,2n+2}, f_{2,2n+2}) \supset \Psi_{Y_{2n+1}}(f_{3,2n+1}, f_{4,2n+1}). \end{aligned}$$

By construction of the chain, the subset $Y = \bigcup_{n \in \mathbb{N}} Y_n$ lies in S and is an acceptable extension of X . For $1 \leq i \leq 4$, denote by f'_i the corresponding elementary automorphism to Y given by the $f_{i,k}$'s. By construction, we have that

$$v_{|Y} = \Psi_Y(f'_1, f'_2) \circ \Psi_Y(f'_3, f'_4)^{-1},$$

as desired (see Figure 2). □

We have now all the ingredients to prove the simplicity, up to bounded automorphisms, of the automorphism group of \mathbb{U} which fix pointwise $\text{cl}(Z)$.

Theorem 4.11. *Consider a κ -tame model \mathbb{U} over Z of a simple theory T satisfying Properties 1–3. Assume that there is an automorphism τ of \mathbb{U} moving p_0 maximally and fixing $\text{cl}(Z)$ pointwise. Every automorphism ν of \mathbb{U} fixing $\text{cl}(Z)$ pointwise can be written as the product of four conjugates of τ and τ^{-1} .*

Proof. By κ -tameness over Z , write $\mathbb{U} = \text{cl}(Z \cup \{a_\alpha\}_{\alpha < \kappa})$, where $(a_\alpha)_{\alpha < \kappa}$ is an independent sequence of realisations of the generic type over Z . Given an automorphism ν of \mathbb{U} fixing $\text{cl}(Z)$, we construct recursively a increasing chain of subsets X_α , for $\alpha < \kappa$, in \mathcal{S} such that the extension $X_\alpha \subseteq X_{\alpha+1}$ is acceptable and each X_α is stable under the action of τ and ν and cl -generated by a independent sequence of length at most $\max(|\mathcal{L}|, |\alpha|)$ of realisations of the generic type, equipped with compatible elementary automorphisms $f_{i,\alpha}$ of X_α , for $1 \leq i \leq 4$, such that a_α lies in $X_{\alpha+1}$ and

$$\nu|_{X_\alpha} = \Psi_{X_\alpha}(f_{1,\alpha}, f_{2,\alpha}) \circ \Psi_{X_\alpha}(f_{3,\alpha}, f_{4,\alpha})^{-1}.$$

For the beginning of the recursion, set $X_0 = \text{cl}(Z)$ and $f_{i,0} = \text{Id}_{X_0}$. Assume now that X_β has already been constructed for $\beta < \alpha$. If α is a limit ordinal, the union $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$ belongs to \mathcal{S} by Remark 4.3 for it is cl -generated by an independent sequence of length $\max(|\mathcal{L}|, |\alpha|)$ over Z . Moreover, the automorphism ν restricted to X_α satisfies the above identity with $f_{i,\alpha} = \bigcup_{\beta < \alpha} f_{i,\beta}$ for $1 \leq i \leq 4$. If α is the successor of β , Proposition 4.10 applied to X_β yields an acceptable extension $X_{\beta+1}$ of X_β containing a_β and elementary extensions $f_{i,\beta+1}$, as desired.

Finally, the union $\bigcup_{\alpha < \kappa} X_\alpha$ is cl -closed and contains all a_α 's, so it must equal \mathbb{U} . By construction, the automorphism ν equals a product of four conjugates of τ and τ^{-1} globally on \mathbb{U} , since at every step the automorphisms are compatible. \square

Motivated by [4; 20; 36], we state the last property of interest for our purposes.

Property 4. *For every κ -saturated model \mathcal{M} , every nontrivial automorphism of \mathcal{M} fixing pointwise the closure $\text{cl}(\emptyset)$ moves p_0 maximally.*

Remark 4.12. If Property 4 holds, every nontrivial automorphism of the κ -tame model \mathbb{U} over Z fixing $\text{cl}(Z)$ pointwise moves p_0 maximally.

If our theory T satisfies Property 4, then we get a strengthening of Theorem 4.11.

Corollary 4.13. *For every κ -tame model \mathbb{U} over Z of a simple theory T satisfying Properties 1–4, the group $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ of automorphisms of \mathbb{U} fixing $\text{cl}(Z)$ pointwise is simple.*

In Remark 5.5, we will show that, under some mild conditions, the collection of automorphisms which do not move maximally the generic type coincide with the bounded automorphisms and thus build a normal subgroup. This will allow us to extend Corollary 4.13 in Remark 5.6.

5. General model-theoretic properties yield our properties

In this short section, we will show that most of our properties can be easily verified for suitable simple theories as long as they satisfy some general model-theoretic properties. This will be relevant in order to show that several of our examples of Theorem 6.18 (see Theorem B) fit into our framework.

As before, we fix a cardinal $\kappa \geq |\mathcal{L}|^+$ and a κ -saturated model \mathcal{M} of a simple \mathcal{L} -theory T .

Lemma 5.1 (see [18, Lemma 2.3]). *Suppose that T is a stable theory satisfying Property 1. Moreover, we assume that every type over an algebraically closed subset of \mathcal{M} is stationary. (The latter always holds if T has weak elimination of imaginaries.) Then T satisfies Properties 2 and (WH) (see Definition 2.8) with respect to the closure operator defined in Definition 2.3.*

Proof. Property 2 always holds in a theory for which types over algebraically closed subsets are stationary. Thus, we need only show that \mathcal{M} is κ -atomic over any subset K of the form $K = \text{acl}(\text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$, where all A_i 's and B have size strictly less than κ . In order to show that $\text{tp}(c/K)$ is κ -isolated, where c is a finite tuple of \mathcal{M} , we must find some subset E of K of size strictly less than κ such that $\text{tp}(c/E) \models \text{tp}(c/K)$. The local character of forking yields a subset C of K of size at most $|\mathcal{L}|$ such that $c \perp_C K$. Set now

$$E = \text{acl}(A_1, \dots, A_n, B, C).$$

Since $C \subseteq E$, it follows that $c \perp_E K$.

We now show that $\text{tp}(c/E) \models \text{tp}(c/K)$, or equivalently, that $\text{tp}(c/E) \models \text{tp}(c/E, \eta)$ for all finite tuples η in K . Since $\text{tp}(c/E)$ is stationary, we need only show that, for every realisation d in \mathcal{M} of $\text{tp}(c/E)$,

$$d \perp_E \eta.$$

By κ -saturation, there is a tuple η' in \mathcal{M} such that $c\eta' \equiv_E d\eta$. It suffices thus to show that $c \perp_E \eta'$. Now, the tuple η is algebraic over $E \cup \{\eta_i\}_{1 \leq i \leq n}$, where each η_i belongs to $\text{cl}(A_i)$ for $1 \leq i \leq n$. Hence, the tuple η' is algebraic over $E \cup \{\eta'_i\}_{1 \leq i \leq n}$, where each η'_i again belongs to $\text{cl}(A_i)$ for $1 \leq i \leq n$, since $\text{cl}(A)$ is invariant under all automorphisms of \mathcal{M} fixing A . We deduce that η' lies in K , as desired. \square

Recall that a stationary type q over \emptyset is *regular* if it is orthogonal (Definition 2.2) to every forking extension of q , that is, if a realises the unique nonforking extension of q to B and the realisation c of q is not independent from B , then $a \perp_B c$. A straightforward application of the inequalities of Lascar yields that types with Lascar rank of the form ω^γ for some ordinal γ are always regular. In particular, the generic type of a strongly minimal set is always regular.

Proposition 5.2. *Assume now that the simple theory T satisfies Property 1 with respect to the generic type p_0 , which we assume to be regular.*

- (a) *If the realisation b of p_0 is not generic over the subset A , then b belongs to $\text{cl}(A)$. (The converse is also true and is immediate from Remark 2.4(b).)*
- (b) *Property 3 holds.*
- (c) *The conclusion of Remark 2.5(b) holds with A_0 a finite sequence.*
- (d) *If T is almost strongly minimal, then the algebraic closure $\text{acl}(A)$ of a subset A of \mathcal{M} equals the closure $\text{cl}(A)$ of A .*
- (e) *If the universe of a model of T is a group with a single generic stationary type as in Remark 2.1, then*

$$\text{cl}(A) = \{b \in \mathcal{M} \mid b \text{ is not generic over } A\}.$$

Proof. For (a), consider $C \supset A$ and an element g generic over C . Now, the realisation b is not generic over C , so $g \perp_C b$ by regularity. Hence, we conclude that b belongs to $\text{cl}(A)$, as desired.

For (b), given a κ -saturated model \mathcal{M} of T and a subset Z of parameters of size strictly less than κ , we need to show that $\mathcal{M} = \text{cl}(Z \cup A)$, where A enumerates an independent sequence realisations of the generic type over Z . By Property 1, every b in \mathcal{M} is algebraic over finitely many realisations h_1, \dots, h_n of the generic type. It suffices hence to show that each h_i belongs to $\text{cl}(Z \cup A)$, which follows immediately from the above discussion, since h_i is not generic over $Z \cup A$, by maximality of A .

For (c), given a set of parameters Z of size strictly less than κ and an element b , choose by Property 1 a finite set of realisations h_1, \dots, h_n of p_0 algebraising b . Up to reordering, we may assume that h_1, \dots, h_m , with $m \leq n$, is a maximal subtuple independent over Z , so $h_r \not\perp_Z h_1, \dots, h_m$ for $m+1 \leq r \leq n$ (if $m=0$, then the latter independence means that no h_r is generic over Z). By (a), each h_r , and hence b , belongs to $\text{cl}(Z \cup A_0)$, where $A_0 = \{h_1, \dots, h_m\}$, as desired.

For (d), it suffices to show that every element b of $\text{cl}(A)$ is algebraic over A . By Property 1, the element b is algebraic over finitely many realisations h_1, \dots, h_n of the strongly minimal type p_0 . Choosing a maximal A -independent subtuple and relabelling, we may assume that b belongs to $\text{acl}(A, h_1, \dots, h_n)$ and choose n minimally. If $n \neq 0$, notice that h_n is generic over A, h_1, \dots, h_{n-1} , so

$$h_n \perp_{A, h_1, \dots, h_{n-1}} b,$$

for b belongs to $\text{cl}(A)$. We conclude that b belongs to $\text{acl}(A, h_1, \dots, h_{n-1})$, which gives the desired contradiction.

For (e), we need only show that if the element b of \mathcal{M} does not belong to $\text{cl}(A)$, then it must be generic over A . Choose now g generic over $A \cup \{b\}$ and write

$b = g \cdot h$, with $h = g^{-1} \cdot b$ generic over A . Now, the element h is a realisation of p_0 . If h were not generic over $A \cup \{g\}$, then by (a), the element h , and thus b , would belong to $\text{cl}(A \cup \{g\})$, and thus to $\text{cl}(A)$, by Remark 2.4(c), which is a contradiction. Hence, the element h must be generic over $A \cup \{g\}$, and thus so is b generic over A , as desired. \square

Lemma 5.1 and Proposition 5.2 yield the following consequence.

Corollary 5.3. *Every stable connected group with weak elimination of imaginaries whose generic type is regular satisfies Properties 1–3 as well as (WH). If the group is superstable of Lascar rank ω^α , then*

$$\text{cl}(A) = \{b \in \mathcal{M} \mid U(b/A) < \omega^\alpha\}.$$

Proof. Property 1 follows since the group is connected, by Remark 2.1. Properties 2 and (WH) follow from Lemma 5.1, whilst Property 3 follows from Proposition 5.2(b) and (c). The description of the closure is Proposition 5.2(e). \square

We recall now [4, Definition 2.14] (or a slightly modified version thereof; see also [36]). We will hence assume that the simple theory T satisfies Property 1 with respect to the generic type p_0 .

Definition 5.4. An \mathcal{L} -automorphism ρ of a κ -saturated model \mathcal{M} is *bounded* if there exists some subset A of cardinality strictly less than κ such that for every m in \mathcal{M} we have that $\rho(m)$ belongs to $\text{cl}(A \cup \{m\})$. We say that ρ is *unbounded* if it is not bounded.

We may assume that A is stable under the action of ρ in the above definition of bounded, since κ is uncountable. It is immediate to see that the collection of bounded automorphisms of an arbitrary model forms a normal subgroup of the automorphism group of \mathbb{U} .

Whenever we work with a theory of fields, the Frobenius map is a bounded automorphism if the underlying field has positive characteristic p and is perfect. However, no power of Frobenius is the identity on $\text{cl}(\emptyset)$, as long as $\text{cl}(\emptyset)$ contains $\overline{\mathbb{F}}_p$.

Remark 5.5. (a) No bounded automorphism of \mathcal{M} moves p_0 maximally.

(b) The converse is true whenever $\text{cl}(A) = \{b \in \mathcal{M} \mid b \text{ is not generic over } A\}$, which is always the case by Proposition 5.2 if T is strongly minimal or if the universe of every model of T is a group whose unique generic type is stationary and regular.

In particular, for both of the cases listed in (b), if there is no nontrivial bounded automorphism fixing $\text{cl}(\emptyset)$, then Property 4 holds.

Proof. For (a), assume that the automorphism ρ is bounded over the ρ -invariant subset A . For every generic element b over A , we have that $\rho(b)$ belongs to $\text{cl}(A \cup \{b\})$, so b and $\rho(b)$ cannot be independent over A by Remark 2.4(c). Thus, the automorphism ρ does not move maximally the generic type.

Assume now for (b) that the closure of every subset A is the collection of nongeneric elements of \mathcal{M} over A . If ρ is unbounded, then for every ρ -invariant subset A of cardinality strictly less than κ , there is some element m in \mathcal{M} such that $\rho(m)$ does not belong to $\text{cl}(A \cup \{m\})$ and thus $\rho(m)$ does not belong to $\text{cl}(A)$, so $\rho(m)$, and hence m , is generic over A , by our assumption. Moreover, the element $\rho(m)$ is generic over $A \cup \{m\}$, so ρ moves p_0 maximally. \square

Remark 5.6. If the bounded automorphisms coincide with those automorphisms which do not move maximally the generic type (as it is the case for the theories listed in Remark 5.5), then the proof of Theorem 4.11 shows that the quotient group $\text{Aut}(\mathbb{U}/\text{cl}(Z))/N$ is simple, where N is the normal subgroup of all automorphisms in $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ which are bounded (see [20, Théorème 2]).

6. Annex: the examples

Using the results of the previous sections, we will now show that all the theories listed in Theorem B satisfy the assumptions of Corollary 4.13. We will conclude the section with some remarks and open questions.

The two classical 1-based strongly minimal theories. Algebraic closure defines a natural pregeometry on a strongly minimal set, and thus each subset of the strongly minimal set X has a basis, and therefore a *dimension*. The \emptyset -definable strongly minimal set X is locally modular if the dimension formula

$$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$$

always holds for every two algebraically closed finite-dimensional subsets A and B with $\dim(A \cap B) > 0$. Archetypal examples of locally modular strongly minimal sets are infinite subsets with no additional structure as well as infinite-dimensional vector spaces over a fixed division ring. In the case of infinite sets, the dimension is just the cardinality of the set whilst for vector spaces, it is the linear dimension of the subspace generated.

If $\kappa > |\mathcal{L}|$, the two classical examples are κ -categorical and thus every model \mathbb{U} of cardinality κ is saturated. Remark 2.1 yields that these two examples satisfy Property 1. Now, the algebraic closure coincides with the definable closure, so it follows immediately that every type is stationary, so Property 2 and (WH) hold by Lemma 5.1. Moreover, Property 3 holds trivially. Lemma 3.5 yields now that the saturated model \mathbb{U} is κ -tame over \emptyset . By Proposition 5.2(d) and Remark 5.5, we conclude by Remark 5.6 that the quotient $\text{Aut}(\mathbb{U})/N$ is simple, where N is the normal subgroup of all automorphisms in $\text{Aut}(\mathbb{U})$ which are bounded.

In the particular case of an infinite set, an automorphism is a permutation. Since the closure is totally disintegrated, it follows that a permutation ρ is bounded if and only if its support $\{x \in \mathbb{U} \mid \rho(x) \neq x\}$ has cardinality strictly less than κ .

Automorphisms of vector spaces are linear isomorphisms. If for the isomorphism ρ there is some scalar λ such that $\mathbb{U}_\lambda = \text{Ker}(\rho - \lambda \cdot \text{Id}_{\mathbb{U}})$ has codimension strictly less than κ , it is easy to see that ρ is bounded over $A = W \cup \rho(W)$, where W is some linear complement of \mathbb{U}_λ . Conversely, if ρ is bounded over the subspace A , which we assume to be stable under ρ and ρ^{-1} , it is easy to show that there exists some scalar λ such that $\rho(b) - \lambda \cdot b = a$ belongs to A for every b in \mathbb{U} . Enlarging A by a set (and their images by ρ) of size strictly less than κ of representatives $(b_a)_{a \in A}$ for a suitable choice of such a 's, we conclude that $\text{Ker}(\rho - \lambda \cdot \text{Id}_{\mathbb{U}})$ has codimension strictly less than κ , as desired.

Since these two theories are uncountably categorical and $\kappa > |\mathcal{L}|$, the κ -prime model over $\text{acl}(Z)$, where Z is of cardinality strictly less than κ , is the unique saturated model of size κ containing Z , so for the sake of the presentation, we will assume that Z has been named as constants to the language. Hence, we recover the results of Baer [1] and Rosenberg [30] mentioned at the beginning of the Introduction.

Proposition 6.1. *Given an uncountable cardinal κ and an infinite set \mathcal{M} of cardinality $\kappa > |\mathcal{L}|$ with no additional structure (but possibly some additional constants being named), the group of permutations of \mathcal{M} is simple modulo the subgroup of permutations of support strictly less than κ .*

Similarly, given a field K and an uncountable cardinal $\kappa > |K|$, the group of isomorphisms of a K -vector space V of cardinality κ is simple modulo the subgroup of those isomorphisms φ such that for some λ in K , the eigenspace $\text{Ker}(\varphi - \lambda \cdot \text{Id}_V)$ has codimension strictly less than κ .

The almost strongly minimal case. As already introduced in Remark 2.1, a theory T is *almost strongly minimal* if there is a strongly minimal set X defined over \emptyset , such that every model \mathcal{M} of T equals $\text{acl}(X(\mathcal{M}))$. The unique nonalgebraic type of X is the *generic* type and is regular. From now on, we will assume that \mathcal{M} is κ -saturated with $\kappa \geq |\mathcal{L}|^+$.

Proposition 5.2(d) yields that $\text{cl}(A)$ equals $\text{acl}(A)$. Almost strongly minimal theories are again uncountably categorical, so the κ -prime model over $\text{acl}(Z)$, where Z is of cardinality strictly less than κ , is the unique saturated model of size κ containing Z .

If X is the strongly minimal \emptyset -definable set X associated to \mathcal{M} with $X \cap \text{acl}(\emptyset)$ infinite, a well-known result of Lascar and Pillay yields that every algebraically closed subset is an elementary substructure (which immediately gives Property 2). In particular, types over algebraically closed subsets will be stationary. The latter can always be achieved by working in T^{eq} , but we will prefer not to change our language: even if there is a natural isomorphism between the automorphism group of \mathcal{M} and the automorphism group of \mathcal{M}^{eq} , it need not be the case that the corresponding groups $\text{Aut}(\mathcal{M}/\text{acl}(\emptyset))$ and $\text{Aut}(\mathcal{M}^{\text{eq}}/\text{acl}^{\text{eq}}(\emptyset))$ are isomorphic.

Proposition 6.2. *Every almost strongly minimal theory T with respect to the strongly minimal \emptyset -definable set X such that $X \cap \text{acl}(\emptyset)$ is infinite satisfies Properties 1–3 as well as (WH).*

Therefore, for every subset Z of parameters of cardinality strictly less than κ , every (saturated) model \mathbb{U} of size κ is κ -tame over Z . In particular, the conjugacy class in the group $\text{Aut}(\mathbb{U}/\text{acl}(Z))$ of an automorphism which moves p_0 maximally generates the whole group in four steps.

Proof. Properties 1 and 3 follow from Remark 2.1 as well as Proposition 5.2(b) and (c). In order to show Properties 2 and (WH), we need only show that types over algebraically closed subsets are stationary by Lemma 5.1.

We will show that for almost strongly minimal theories with $X \cap \text{acl}(\emptyset)$ infinite, every algebraically closed subset $\text{acl}(A)$ is an elementary substructure. This is probably well known, but we could not find a suitable reference. It is a straightforward application of Tarski’s test: Consider a realisation b of a formula $\varphi(x, c_1, \dots, c_n)$, where each c_i belongs to $\text{acl}(A)$. Now, the element b is algebraic over a sequence d_1, \dots, d_m of realisations of the strongly minimal set X . Up to relabelling, we may assume that the d_i ’s are A -independent, so b belongs to $\text{acl}(A, d_1, \dots, d_m)$. Choose m minimally such that there exists such a realisation of φ . If $m = 0$, then we are done. Otherwise, choose a formula $\psi(x, d_1, \dots, d_m)$ with parameters in A which holds for b and such that $\psi(x, e_1, \dots, e_m)$ is always algebraic for all (e_1, \dots, e_m) in X^m .

Since $X \cap \text{acl}(\emptyset)$ is infinite, the generic type p_0 cannot be isolated. Thus, by the open mapping theorem, neither is the type $\text{tp}(d_m/\text{acl}(A), d_1, \dots, d_{m-1})$. The formula

$$X(y) \wedge \exists x(\varphi(x, c_1, \dots, c_n) \wedge \psi(x, d_1, \dots, d_{m-1}, y))$$

must therefore contain a realisation e which is not generic, and thus must be algebraic, over A, d_1, \dots, d_{m-1} , giving the desired contradiction.

Therefore, using Remark 3.7, Proposition 3.8 and Theorem 4.11, we deduce the desired conclusion of the statement. \square

Lascar showed in [20, Proposition 14] that, for a 1-based strongly minimal set \mathbb{U} , given a subset Z of \mathbb{U} of cardinality strictly less than κ , the group of automorphisms $\text{Aut}(\mathbb{U}/\text{acl}(Z))$ always contains (many) nontrivial bounded elements. However, this is not the case for the nonlocally modular strongly minimal theory ACF_p of algebraically closed fields of characteristic p , for p either 0 or a prime; see [4, Théorème 3.1; 22, Lemma 2].

Remark 5.5, with Proposition 6.2 and Corollary 4.13, yields the following:

Corollary 6.3. *Every strongly minimal theory ACF_p of algebraically closed fields of characteristic p , for p either 0 or a prime, satisfies that there are no nontrivial bounded automorphisms fixing $\text{cl}(\emptyset)$ pointwise, so Property 4 holds.*

Therefore, for every subset Z of cardinality at most κ inside an algebraically closed field \mathbb{U} of cardinality κ , the group $\text{Aut}(\mathbb{U}/\text{acl}(Z))$ is simple.

Differential fields. We fix some natural number $m \geq 1$ and consider first the theory $\text{DCF}_{0,m}$ of differentially closed fields of characteristic 0 with m commuting derivations (see [25] for the general results as well as [23, Chapter II] for $m = 1$). The theory $\text{DCF}_{0,m}$ is complete, eliminates quantifiers and imaginaries and is ω -stable. The generic type over \emptyset is the type of a differentially transcendental element, that is, satisfying no nontrivial differential equation over \mathbb{Q} . Note that this type is stationary and its Lascar rank equals ω^m , so the generic type is regular. Since the underlying additive group is divisible, it is connected. By Corollary 5.3, the theory $\text{DCF}_{0,m}$ satisfies Properties 1–3 as well as (WH). Moreover, if A is a subset of a κ -saturated model of $\text{DCF}_{0,m}$, then $\text{cl}(A)$ is the field of elements satisfying some nontrivial differential equation over the differential field generated by A .

It was shown in [4, Theorem 3.1] that the only bounded automorphism of a κ -saturated model of $\text{DCF}_{0,m}$ is the identity. By Remark 5.5, Property 4 holds for $\text{DCF}_{0,m}$.

Another theory of differential fields to consider was obtained by Hrushovski and Itai in [16]. The models of the theory $T(X)$ are the existentially closed models K of the class of differential fields of characteristic 0 in which the equations defining $X := \{x \in C \mid Dx = s(x)\}$ have no solution, where C is a curve of genus at least 1 and $s : C \rightarrow T(C)$ is a rational section of the tangent bundle, all defined over the field of constants of K . Each theory $T(X)$ is complete, eliminates quantifiers and imaginaries, and is ω -stable [16, Proposition 4.1 and Lemma 4.6]. As before, the generic type p_0 is the stationary type of a differentially transcendental element, and has Lascar rank ω . The underlying additive group is again connected and $\text{cl}(A)$ is the field of elements satisfying some nontrivial differential equation over the differential field generated by A . In particular, each theory $T(X)$ satisfies Properties 1–3 as well as (WH), by Corollary 5.3.

The proof of [4, Theorem 3.1] goes through verbatim for models of $T(X)$, so the only bounded automorphism of a κ -saturated model of $T(X)$ is the identity. Again by Remark 5.5, we have that Property 4 holds for $T(X)$.

As before, we deduce from Proposition 3.8 and Corollary 4.13 the next result:

Proposition 6.4. *Let T be either the theory $\text{DCF}_{0,m}$ of differentially closed fields of characteristic 0 with m commuting derivations or one of the theories $T(X)$ described above. For every subset Z of parameters of cardinality strictly less than κ , every saturated model \mathbb{U} of size κ of T as well as every κ -prime model over $\text{acl}(Z)$ is κ -tame over Z . For every κ -tame model \mathbb{U} over Z , $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ is simple, where*

$$\text{cl}(Z) = \{x \in \mathbb{U} \mid x \text{ satisfies some nontrivial differential equation over the differential field generated by } Z\}.$$

Difference fields of characteristic 0. Recall that ACFA_0 denotes the theory of existentially closed difference fields (K, σ) of characteristic 0. Every model \mathcal{M} of ACFA_0 is algebraically closed (as a field) and *inversive* (that is, the endomorphism σ is surjective). If $A \subset \mathbb{U}$, then $\text{acl}(A)$ is the smallest algebraically closed inversive difference field containing A . The type $\text{tp}(b/A)$ of a tuple b in \mathcal{M} is entirely determined by the isomorphism type over A of $\text{acl}(Ab)$ (see [7, Corollary 1.5]). Every completion of ACFA_0 eliminates imaginaries [7, (1.10)] and is simple. Moreover, the nonforking independence can be described in algebraic terms: given two inversive difference fields $A = \text{acl}(A)$ and $B = \text{acl}(B)$ with a common difference subfield $C = \text{acl}(C)$, we have that $A \perp_C B$ if and only if A and B are algebraically independent over C (see [7, Section (1.9)]).

Remark 6.5. Every completion T of ACFA_0 is simple and satisfies Properties 1–4. The closure operator can be described algebraically as

$$\text{cl}(B) = \{a \in \mathbb{U} \mid \text{tr.deg}(a, \sigma(a), \sigma^2(a), \dots / \text{acl}(B)) < \infty\}.$$

Proof. For Property 1, notice that there is a unique generic 1-type p_0 (generic in the sense of the field structure), which says that its realisation does not satisfy any nontrivial difference equation. This type is stationary [7, Proposition 2.10] and is the only 1-type of Lascar rank $\text{SU}(p_0) = \omega$, so it is regular. By Remark 2.1, the theory ACFA_0 satisfies the hypothesis of Proposition 5.2, hence Property 3 holds. Property 2 was already shown in [8, Theorem 5.3 and Corollary B.11]. Remark 5.5 yields Property 4 using [4, Theorem 3.1].

The description of the closure follows now directly from Proposition 5.2(e) and the above characterisation of the generic type. \square

The theory ACFA_0 is unstable, so we do not know whether there are saturated models in their cardinality as suitable candidates for κ -tame models in order to apply the results of Section 4. Now, given an uncountable cardinal κ , the existence and uniqueness of a κ -prime model \mathbb{U} over $\text{cl}_{\mathbb{U}}(Z)$ was shown by Chatzidakis [6]: Choose a κ -saturated model \mathcal{M} of ACFA of characteristic 0 containing a subset Z of cardinality strictly less than κ . Then κ -prime models over algebraically closed difference subfields A of \mathcal{M} containing $\text{cl}_{\mathcal{M}}(\emptyset)$ exist and are unique up to isomorphism over A (see Theorem 3.17 in [6]). As in the stable case, the κ -prime model \mathbb{U} of ACFA_0 over a subset A of \mathcal{M} containing $\text{cl}_{\mathcal{M}}(\emptyset)$ is characterised by being κ -saturated and containing A , κ -atomic over A and containing no (nonconstant) A -indiscernible sequence of length κ^+ (Theorem 3.14 in [6]). In particular, every completion of the theory ACFA_0 satisfies the condition of Proposition 3.8.

Therefore, the κ -prime \mathbb{U} over $\text{cl}_{\mathcal{M}}(Z)$ will be κ -tame over Z once we show that property (WH) holds for ACFA_0 , by Proposition 3.8. For the sake of the presentation, we have decided to provide two alternative proofs of this, one along

the lines of the proof of Lemma 5.1 and another one (see Remark 6.7) using the strength of the semiminimal analysis of types in ACFA₀.

Proposition 6.6. *Every completion T of ACFA₀ satisfies property (WH).*

Proof. To show that the κ -saturated model \mathcal{M} is κ -atomic over any subset K of the form $K = \text{acl}(\text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$, where all A_i 's and B have size strictly less than κ , we need to show, as in the proof of Lemma 5.1, that $\text{tp}(c/E) \models \text{tp}(c/E \cup \{\eta\})$ for all finite tuples c of \mathcal{M} and η of K , with $E = \text{acl}(A_1, \dots, A_n, B, D)$, where D is a subset of $K = \text{acl}(\text{cl}(A_1) \cup \dots \cup \text{cl}(A_n) \cup B)$ of cardinality bounded by $|\mathcal{L}| < \kappa$ such that $c \perp_D K$. Note that $c \perp_E K$. Possibly at the cost of enlarging η , we may also assume that $\eta = (\eta_0, \eta_1, \dots, \eta_n)$ with η_0 field algebraic over $E \cup \{\eta_i\}_{1 \leq i \leq n}$ and η_i in $\text{cl}(A_i)$ for $1 \leq i \leq n$.

By [7, Corollary 1.13], the field (\mathcal{M}, σ^k) is again a difference closed field for $k \neq 0$ in \mathbb{Z} . In particular, we will denote all throughout this proof by $\langle A \rangle_{\sigma^k}$, $\text{acl}_{\sigma^k}(A)$, $\text{cl}_{\sigma^k}(A)$ and $\text{tp}_{\sigma^k}(d)$ the corresponding notions in the reduct $\mathcal{M}[k] = (\mathcal{M}, \sigma^k)$.

Claim. *If $\langle E \cup \{c\} \rangle_{\sigma^k}$ and $\langle E \cup \{\eta'\} \rangle_{\sigma^k}$ are algebraically independent over E for every $k \neq 0$ in \mathbb{Z} and every finite tuple η' realising the quantifier-free type $\text{qftp}_{\sigma^k}(\eta)$, then the type $\text{tp}_{\sigma}(c/E)$ implies $\text{tp}_{\sigma}(c/E \cup \{\eta\})$.*

Proof of claim. Assume that c, η and E are as in the statement. We can apply [7, Proposition 4.9] and obtain that $\text{tp}_{\sigma}(c/E)$ and $\text{tp}_{\sigma}(\eta/E)$ are superficially costable: for every $d \perp_E \eta$ realising $\text{tp}_{\sigma}(c/E)$, setting $K = \text{acl}_{\sigma}(E \cup \{\eta\})$ and considering the unique extension σ_F of $\sigma_{\text{acl}_{\sigma}(E,d)} \otimes \sigma_{\text{cl}_{\sigma}(K)}$ to the compositum field F of $\text{acl}_{\sigma}(E, d)$ and K , we have that F has no proper finite Galois extension invariant under σ_F .

Moreover, setting $k = 1$ in our assumption, we deduce from the previous description of the nonforking independence in the simple theory ACFA that c is independent of every realisation η' in \cup of $\text{tp}_{\sigma}(\eta/E)$, or equivalently, that every realisation d of $\text{tp}_{\sigma}(c/E)$ is independent from η over E .

A realisation d of $\text{tp}_{\sigma}(c/E)$ yields an E -isomorphism of difference fields between $\text{acl}_{\sigma}(E \cup \{d\})$ and $\text{acl}_{\sigma}(E \cup \{c\})$ mapping d to c . We need to show that this isomorphism of difference fields extends to a K -isomorphism of difference fields between $\text{acl}_{\sigma}(K \cup \{d\})$ and $\text{acl}_{\sigma}(K \cup \{c\})$.

By superficial costability (since every d realising $\text{tp}_{\sigma}(c/E)$ is automatically independent from K over E), there is a unique extension of the field automorphism σ_F to the field algebraic closure of F , by [7, Lemma 2.8]. We conclude that there is an isomorphism of difference fields between the algebraic closure of F and the difference field $\text{acl}_{\sigma}(K \cup \{c\})$ over K mapping d to c , so $d \equiv_K c$ for every realisation d of $\text{tp}_{\sigma}(c/E)$, as desired. □

By the above claim, we need only show that for every integer k in \mathbb{N} and every η' in \cup realising the quantifier-free type $\text{qftp}_{\sigma^k}(\eta/E)$, the fields $\langle E, c \rangle_{\sigma^k}$ and $\langle E, \eta' \rangle_{\sigma^k}$

are algebraically independent over E . Note that $\text{cl}_\sigma(A_i) = \text{cl}_{\sigma^k}(A_i)$, so η_i belongs to $\text{cl}_{\sigma^k}(A_i)$ for $1 \leq i \leq n$. The isomorphism of difference fields in the structure (\mathbb{U}, σ^k) over E maps the tuple η to η' , with $\eta' = (\eta'_0, \eta'_1, \dots, \eta'_n)$, where η'_0 is field algebraic over $E \cup \{\eta'_i\}_{1 \leq i \leq n}$ and each η'_i lies in $\text{cl}_{\sigma^k}(A_i)$, since each of these properties is quantifier-free definable in the language of difference rings. Hence, the whole tuple η' is field algebraic over the field generated by $E \cup \bigcup_{1 \leq i \leq n} \text{cl}_\sigma(A_i) \subseteq K$. The difference field K is algebraically closed, so $\langle E, \bar{\eta}' \rangle_{\sigma^k} \subseteq K$. Since E was chosen so that $c \perp_E K$, we deduce the desired algebraic independence between $\langle E, c \rangle_{\sigma^k}$ and $\langle E, \eta' \rangle_{\sigma^k}$ over E . \square

Remark 6.7. We provide now an alternative proof that every completion T of ACFA_0 satisfies property (WH): Using the above notation, we want to show that $\text{tp}_\sigma(c/K)$ is κ -isolated. The proof is by induction on the Lascar rank $\text{SU}(c/K)$ of $\text{tp}(c/K)$, and we assume the property proved for any type of SU-rank less than $\text{SU}(c/K)$, over any set K' of the same kind. If $\text{SU}(c/K) = 0$, there is nothing to prove. The proof distinguishes two cases: whether or not $\text{tp}(c/K)$ is orthogonal to $\text{Fix}(\sigma)$.

If $\text{tp}(c/K)$ is orthogonal to $\text{Fix}(\sigma)$ then $\text{tp}(c/K)$ is stationary, by [5, Lemma 2], and the proof given in Lemma 5.1 goes through verbatim.

Suppose now that $\text{tp}(c/K)$ is nonorthogonal to $\text{Fix}(\sigma)$. Since K contains $\text{Fix}(\sigma)(\mathcal{M})$, every nonalgebraic type over K which is realised in \mathcal{M} is weakly orthogonal to $\text{Fix}(\sigma)$. Hence, the difference field $\text{acl}(Kc)$ has the same fixed field as K .

The proof of [7, Theorem 5.5] for the case q nonorthogonal to $\sigma(x) = x$ (page 3049) gives that there is b in $\text{acl}(Kc)$ such that $\text{tp}(b/K)$ is qf-internal to $\text{Fix}(\sigma)$. This means that there is some difference field L containing K and independent from b over K , such that b belongs to LF .

By [6, Lemma 3.4], there is countable subset $D = \text{acl}(D)$ of K such that $\text{tp}(b/D) \vdash \text{tp}(b/K)$, and we may impose that b belongs to $\text{acl}(Dc)$. Thus $\text{tp}(b/K)$ is κ -isolated. Applying the induction hypothesis to $\text{tp}(c/\text{acl}(Kb))$ yields that $\text{tp}(c/\text{acl}(Kb))$ is κ -isolated. We conclude that $\text{tp}(b, c/K)$, and thus $\text{tp}(c/K)$, is κ -isolated by [6, Remark 2.17(1)].

We deduce from Remark 6.5, Propositions 3.8 and 6.6 as well as Corollary 4.13 the following result:

Proposition 6.8. *For every subset Z of parameters of cardinality strictly less than κ , every κ -prime model over $\text{cl}(Z)$ is κ -tame over Z . For every κ -tame model \mathbb{U} over Z , the group $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ is simple.*

Proper pairs of algebraically closed fields. The theory ACFP of proper pairs of algebraically closed fields shares many traits with the theory DCF_0 (or rather

$\text{DCF}_{0,1}$) of differentially closed fields of characteristic 0, yet it is somewhat simpler to describe. Most of the results mentioned here appear in [3; 17; 24; 28].

Every completion of ACFP in the language $\mathcal{L}_P = \mathcal{L}_{\text{Rings}} \cup \{P\}$, where P denotes the distinguished proper algebraically closed subfield E of the model K of ACFP, is given by the characteristic of the field [29]. The type of a subfield k with k linearly disjoint from E over $k \cap E$ (which we will denote by $k \downarrow_{k \cap E}^{\text{ld}} E$) is uniquely determined by its quantifier-free \mathcal{L}_P -type, so ACFP has quantifier elimination after adding Delon’s λ -functions [10], which play a similar role to the λ -functions for separably closed fields (see next subsection). Given a tuple a_0, \dots, a_n of K , if a_1, \dots, a_n are E -linearly dependent, then $\lambda_n^i(-; a_1 \dots, a_n)$ are equal to zero for $i = 1, \dots, n$. If a_1, \dots, a_n are E -linearly independent, but a_0, a_1, \dots, a_n are not, then the values of the λ -functions are uniquely determined by $a_0 = \sum_{i=1}^n \lambda_n^i(a_0; a_1 \dots, a_n) a_i$. It follows from the above description of types that every completion of the theory ACFP is ω -stable and that nonforking independence can be characterised in terms of two field independences [3, Remark 7.2 and Proposition 7.3]: two \mathcal{L}_P -definably closed subfields L_1 and L_2 of a sufficiently saturated model \mathcal{M} of ACFP are independent over a common \mathcal{L}_P -definably closed subfield k if and only if

$$L_1 \downarrow_k^{\text{ACF}} L_2 \quad \text{and} \quad L_1 \downarrow_{Ek}^{\text{ACF}} L_2,$$

where Ek denotes the subfield generated by $E \cup k$ and \downarrow^{ACF} denotes independence in the sense of the reduct ACF.

Remark 6.9. Every completion T of ACFP is ω -stable and satisfies Properties 1–3 as well as (WH). The closure operator can be described algebraically as $\text{cl}(A) = (EA)^{\text{alg}}$.

Proof. For Property 1, notice that there is a unique generic 1-type p_0 (generic in the sense of the field structure) of (Morley and Lascar) rank ω , whose realisation is not a root of a nontrivial polynomial with coefficients in the subfield E . This type is stationary, by the above characterisation of independence in ACFP. Since $U(p_0) = \omega$, the type p_0 is regular. By Remark 2.1, the theory ACFP satisfies the hypothesis of Proposition 5.2, hence Property 3 holds. Now, types over algebraically closed subsets in the theory ACFP are stationary, as shown by Bartnick [2], so we deduce that Property 2 and (WH) hold by Lemma 5.1.

The description of the closure follows now directly from Proposition 5.2(e) and the above characterisation of the generic type. □

We will deduce now from Remarks 5.5 and 6.9 as well as the following lemma that every completion of ACFP satisfies Property 4.

Lemma 6.10. *Every bounded automorphism of a κ -saturated model \mathcal{M} of the theory ACFP of pairs of algebraically closed field is the identity or a power of Frobenius (in positive characteristic). In particular, there is no nontrivial bounded*

automorphism fixing $\text{cl}(\emptyset) = E$ and thus every completion of ACFP satisfies Property 4.

Proof. We will be concise, for the proof is a straightforward adaptation of [4, Theorem 3.1]. Assume that the automorphism ρ of the κ -saturated pair (\mathbb{U}, E) is bounded over the ρ -invariant subset A . By the description of the cl -closure, we know that $\rho(b)$ is algebraic over the subfield $EA(b)$ for every element b of \mathcal{M} , in particular it is so for every generic element b over A .

Choose now two generic independent elements b_1 and b_2 over A . By the description of the independence, the elements b_1 , b_2 and $b_1 + b_2$ are pairwise algebraically independent over EA , and thus so are the pairs $(b_1, \rho(b_1))$, $(b_2, \rho(b_2))$ and $(b_1 + b_2, \rho(b_1 + b_2))$. Analogously, the pairs $(b_1, \rho(b_1))$, $(b_2, \rho(b_2))$ and $(b_1 \cdot b_2, \rho(b_1 \cdot b_2))$ are algebraically independent over EA , since b_1 and b_2 are also independent multiplicative generics.

By Ziegler's lemma ([40, Theorem 1; see also Lemma 2] or [4, Lemme 2.5]), we deduce that there are two connected algebraic subgroups H_1 of \mathbb{G}_a^2 and H_2 of \mathbb{G}_m^2 , each defined over $(EA)^{\text{alg}}$, such that $(b_1, \rho(b_1))$ is a generic element of an additive translate of H_1 and of a multiplicative translate of H_2 , both translates defined over $(EA)^{\text{alg}}$. From here on, the rest of the proof of [4, Theorem 3.1] goes verbatim and yields that ρ is the identity or a power of Frobenius (in positive characteristic), as desired. \square

Remark 6.9 and Lemma 6.10 together with Proposition 6.2 and Corollary 4.13, together with Propositions 3.8 and 6.2 as well as Corollary 4.13 yield the following result.

Corollary 6.11. *For every subset Z of parameters of cardinality strictly less than κ , every saturated model \mathbb{U} of size κ of ACFP as well as every κ -prime model over $\text{cl}(Z) = (EZ)^{\text{alg}}$ is κ -tame over Z .*

For every κ -tame model \mathbb{U} of ACFP over Z , the group $\text{Aut}(\mathbb{U}/(EZ)^{\text{alg}})$ is simple.

The reader familiar with Lascar's original proof for the complex numbers will immediately notice that Corollary 6.11 follows from his proof, since an \mathcal{L}_p -automorphism of \mathbb{U} fixing $E = P^{\mathbb{U}}$ pointwise is just a field automorphism.

Separably closed fields of infinite degree of imperfection. Most of the references for this section appear in [9; 33], unless explicitly stated. From now on, we work inside an ambient field of positive characteristic $p > 0$.

Definition 6.12. (1) Given two subfields $k \subset K$, the extension $k \subset K$ is *separable* if $k \downarrow_{k^p} K^p$. Given a subring R of K , it is immediate to see that $k = \text{Quot}(R) \subset K$ is separable if and only if every tuple of elements of R which is R^p -linearly independent remains so over K^p .

(2) A subset A of K is p -independent (in K) if for every a in A , we have that a does not belong to $K^p[A \setminus \{a\}]$. If $k \subset K$ is separable, a subset A of k is p -independent in K if and only if it is p -independent in k .

(3) A subset A of K is a p -basis if it is maximal p -independent, or equivalently, if A is p -independent and $K = K^p[A]$. More generally, given a separable extension $k \subset K$, a subset B of K is p -independent over k if $A \cup B$ is p -independent in K , where A is a p -basis of k . This is equivalent to requiring that no b in B belongs to $K^p[k \cup B \setminus \{b\}]$.

The *degree of imperfection* of the field K is the unique element e of $\mathbb{N} \cup \{\infty\}$ with $[K : K^p] = p^e$. In particular, a field has infinite imperfection degree if and only if K contains an infinite p -independent subset.

The completions of the theory of separably closed fields of positive characteristic p are uniquely determined by the imperfection degree [11]. We will denote the corresponding completion by $\text{SCF}_{p,e}$ (notice that $\text{SCF}_{p,0}$ is the theory ACF_p). Ershov (see also Wood [38]) provides a description of types: if k and k' are isomorphic subfields of a model K of $\text{SCF}_{p,e}$ with both $k \subseteq K$ and $k' \subseteq K$ separable, then k and k' have the same type. In particular, the theory $\text{SCF}_{p,e}$ is stable [38, Theorem 3]. It follows implicitly from the above description of types that if $k \subseteq K$ is separable, then k is definably closed and its algebraic closure coincides with k^{sep} , the separable closure of k .

The above description of types yields an algebraic characterisation of nonforking independence: Given two definably closed subsets A and B of a model K of $\text{SCF}_{p,e}$ with $C = A \cap B$ algebraically closed, we have that

$$A \downarrow_C B \quad \text{if and only if} \quad A \downarrow_C^{\text{ACF}} B \quad \text{and} \quad AB \downarrow_{A^p B^p}^{\text{Id}} K^p,$$

so $AB \subseteq K$ is separable, where AB (resp. $A^p B^p$) denotes the field generated by $A \cup B$ (resp. $A^p \cup B^p$).

There is a simple way to expand the language in order to obtain quantifier elimination for $\text{SCF}_{p,e}$ after adding function symbols for the λ -functions [33, Remark 7], defined analogously as in the case of pairs of algebraically closed fields (historically the λ -functions were first introduced for $\text{SCF}_{p,e}$). Given a tuple a_0, \dots, a_n of a model K of $\text{SCF}_{p,e}$, if a_1, \dots, a_n are K^p -linearly dependent, then $\lambda_n^i(x; a_1 \dots, a_n)$ is always zero. If a_1, \dots, a_n are K^p -linearly independent, but a_0, a_1, \dots, a_n are not, the values of the λ -functions are uniquely determined by $a_0 = \sum_{i=1}^n \lambda_n^i(a_0; a_1 \dots, a_n)^p a_i$. Notice that every definably closed subset A of K is a subfield and the corresponding extension $A \subseteq K$ is separable. In particular, given a subset A of a model K of $\text{SCF}_{p,e}$, the definable closure $\text{dcl}(A)$ is the smallest subfield of K containing A and closed under the λ -functions [33, Lemma 0]. Moreover, the algebraic closure $\text{acl}(A)$ is $\text{dcl}(A)^{\text{sep}}$.

Remark 6.13. The theory $\text{SCF}_{p,\infty}$ of separably closed fields of characteristic $p > 0$ and infinite imperfection degree is stable and satisfies Properties 1 and 2 as well as (WH).

Proof. We work inside a sufficiently saturated model K of $\text{SCF}_{p,\infty}$. For Property 1, notice that for any definably closed subfield k and any element g which is p -independent over k , the field extension $k(g) \subset K$ is again separable. The above characterisation of types yields that any two p -independent g and g' elements over k have the same type over k . We denote by p_0 a generic type of the (additive group of the) field K . If a realisation g of p_0 were not p -independent over \mathbb{F}_p , then g would be contained in K^p . Now, a generic type in a stable group only contains formulae which are generic (or *syndetic*), so the subgroup K^p would have finite index, which is a contradiction. Hence p_0 is the unique generic type and the same argument yields that p_0 is stationary, since any two generic elements g and g' over the definably closed subfield k are p -independent over k and hence have the same type by the previous discussion. Remark 2.1 now yields Property 1. Moreover, an element g realises the nonforking extension of p_0 over k if and only if g is p -independent over k .

Now, types over algebraically closed subsets in the theory $\text{SCF}_{p,\infty}$ are stationary, as shown by Bartnick [2], so we deduce that Properties 2 and (WH) hold by Lemma 5.1. \square

Remark 6.14. It is easy to see that the generic type p_0 of the theory $\text{SCF}_{p,\infty}$ is not regular: Choose two independent realisations g and h of p_0 . By Remark 6.13, the element $g^p + h$ is again p -independent over the prime field, so it realises p_0 . Moreover, the type $\text{tp}(g/\mathbb{F}_p(h))$ is a nonforking extension of p_0 , whilst the type $\text{tp}(g^p + h/\mathbb{F}_p(h))$ is a forking extension, since the additive subgroup K^p does not have finite index in K . Now, the elements g and $g^p + h$ are dependent over $\mathbb{F}_p(h)$, since g is p -dependent over $\mathbb{F}_p(h, g^p + h)$.

The same proof works for the (unique) generic additive type of the field in any degree of imperfection $e \neq 0$.

Lemma 6.15. *Given a subset $A = \text{acl}(A)$ of a κ -saturated model K of $\text{SCF}_{p,\infty}$, the closure (with respect to p_0) is*

$$\text{cl}(A) = \{b \in K \mid \text{acl}(Ab) \text{ has the same } p\text{-basis as } A\} = \bigcap_{m \in \mathbb{N}} K^{p^m}[A].$$

In particular, if B is a p -basis of K over k , then $\text{cl}(k \cup B) = K$, so $\text{SCF}_{p,\infty}$ satisfies Property 3.

Proof. Clearly, if b belongs to $\text{cl}(A)$, then every generic element g over A remains so over $\text{cl}(A)$, and thus over $\text{acl}(Ab)$, by Remark 2.4(b). In particular, there is no realisation of p_0 in $\text{acl}(A \cup \{b\})$ generic over A , so no realisation of p_0 in

$\text{acl}(A \cup \{b\})$ is p -independent over the field generated by A . Thus, the subfield $\text{acl}(Ab)$ must have the same p -basis as A .

If b is such that $\text{acl}(Ab)$ has the same p -basis B as $A = \text{acl}(A)$, every iterated λ -function of b with respect to B belongs to $\text{acl}(A, b) \subset K^p[B]$, so b belongs to $K^{p^m}[A]$ for every m in \mathbb{N} and hence b lies in $\bigcap_{m \in \mathbb{N}} K^{p^m}[A]$.

For the last equality, suppose now that b belongs to $\bigcap_{m \in \mathbb{N}} K^{p^m}[A]$. In order to show that b belongs to $\text{cl}(A)$, choose some $D \supset A$ and a generic element g over D . By (the proof of) [33, Lemma 4] we have that $\text{dcl}(D \cup \{b\})$ is a subset of $\bigcap_{m \in \mathbb{N}} K^{p^m}[\text{acl}(D)]$. If g divides with $\text{dcl}(D \cup \{b\})$ over D , then g is not p -independent over $\text{dcl}(D \cup \{b\})$, so

$$g \in K^p[\text{dcl}(D \cup \{b\})] \subset K^p[\text{acl}(D)],$$

which is a contradiction since g is generic over D . □

Lemma 6.16. *Every nontrivial automorphism of a κ -saturated model K of the theory $\text{SCFP}_{p,\infty}$ moves p_0 maximally. In particular, the theory $\text{SCFP}_{p,\infty}$ satisfies Property 4.*

Proof. We will be concise, for the proof is a straightforward adaptation of [4, Theorem 3.1]. Assume that the automorphism ρ of the κ -saturated model K does not move p_0 maximally so there is a ρ -invariant subset $A = \text{acl}(A)$ such that for every generic element b over A , we have that $\rho(b)$ and b are not independent over A . As b is generic over A , the field $A(b)$ is separable and thus has some p -basis contained in $A \cup \{b\}$. Now, the element $\rho(b)$ is not p -independent over $A(b)$, so $\rho(b)$ belongs to $K^p[A(b)]$. Hence, we deduce that $\rho(b)$ belongs to $L[b]$, where $L = K^p(A)$.

Choose now two independent generic elements b and c over A . Our assumption implies that

$$\rho(b) \in L[b], \quad \rho(c) \in L[c] \quad \text{and} \quad \rho(b+c) \in L[b+c].$$

Write $\rho(b) = \sum_{i=0}^{p-1} d_i b^i$, $\rho(c) = \sum_{i=0}^{p-1} e_i c^i$ and $\rho(b+c) = \sum_{i=0}^{p-1} f_i (b+c)^i$. We deduce that

$$\begin{aligned} \sum_{i=0}^{p-1} d_i b^i + e_i c^i &= \rho(b) + \rho(c) = \rho(b+c) = \sum_{i=0}^{p-1} f_i (b+c)^i \\ &= \sum_{i=0}^{p-1} f_i \sum_{j=0}^i \binom{i}{j} b^j c^{i-j}. \end{aligned}$$

Note that all p -monomials in b and c which appear in the last sum are linearly independent over L , for b and c are independent generic elements. In particular, we

have that $f_2 = \dots = f_{p-1} = 0$, so $f_0 + f_1(b+c) = \rho(b+c) = (d_0+e_0) + d_1a + e_1b$ and thus

$$d_0 + e_0 = f_0 \quad \text{and} \quad d_1 = e_1 = f_1,$$

with $d_i = e_i = 0$ for $i \geq 2$. For the generic element $b \cdot c$ over A write $\rho(b \cdot c) = \sum_{i=0}^{p-1} h_i(b \cdot c)^i$ with h_i in A . We deduce from the above that

$$\rho(b \cdot c) = (d_0 + d_1b)(e_0 + e_1c) = d_0e_0 + d_1e_0b + e_0d_1c + d_1e_1b \cdot c,$$

so we deduce that $h_i = 0$ for $i \geq 2$ and

$$h_0 = d_0e_0, \quad h_1 = d_1e_1 \quad \text{as well as} \quad d_0e_1 = d_1e_0 = 0.$$

Now, since A is ρ -invariant, the generic element $\rho(b \cdot c)$ cannot lie in L , so $h_1 \neq 0$. Hence $d_1 \neq 0$, so $e_0 = 0$ and thus h_0 must be 0, which means that for every pair (b, c) of generic independent elements over A ,

$$\frac{\rho(b \cdot c)}{b \cdot c} = h_1 \in L.$$

Every generic element over A can be written as a product of two generic independent elements over A , so $\rho(x) = \lambda_x x$ for every generic element x over A where λ_x belongs to L . This yields the desired conclusion, since $x + 1$ is also generic over A and

$$\lambda_{x+1}x + \lambda_{x+1}1 = \lambda_{x+1}(x+1) = \rho(x+1) = \rho(x) + 1 = \lambda_x x + 1,$$

so $1 = \lambda_{x+1} = \lambda_x$. Whence ρ is the identity on K , for every element is a sum of two generic elements. \square

Remark 6.13, Lemmata 6.15 and 6.16 together with Lemma 3.5, Proposition 3.8 and Corollary 4.13 yield the following result, under the additional assumption that $\text{cof}(\kappa) \geq \aleph_1$, since $\text{SCFP}_{p,\infty}$ is stable yet not superstable (see Remark 3.7).

Corollary 6.17. *For every subset Z of parameters of cardinality strictly less than κ , every saturated model \mathbb{U} of size κ of $\text{SCFP}_{p,\infty}$ as well as every κ -prime model over $\text{cl}(Z) = \bigcap_{m \in \mathbb{N}} \mathbb{U}^{p^m}[Z]$ are κ -tame over Z (saturated models of $\text{SCFP}_{p,\infty}$ exist whenever $k \geq 2^{\aleph_0}$ whilst the existence and uniqueness of κ -prime models holds if $\text{cof}(\kappa) \geq \aleph_1$).*

For every κ -tame model \mathbb{U} of SCFP over Z , the group $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ is simple.

Corollaries 6.3, 6.11 and 6.17 as well as Propositions 6.4 and 6.8 yield now the following result:

Theorem 6.18. *For each of the following countable theories of fields with operators:*

- *algebraically closed fields with the closure operator given by the field algebraic closure;*

- differentially closed fields in characteristic 0 with finitely many commuting derivations with the closure operator given by the elements which are not differentially transcendental;
- differential fields in characteristic 0, maximal with the property of omitting a given strictly minimal type X ; same closure as above;
- difference closed fields in characteristic 0 with the closure operator given by the elements of transformal transcendence degree 0;
- proper pairs of algebraically closed fields (K, E) with the closure operator

$$\text{cl}(A) = E(A)^{\text{alg}};$$

- separably closed fields K in characteristic p and infinite imperfection degree with the closure operator

$$\text{cl}(A) = \bigcap_{n \in \mathbb{N}} K^{p^n}[\text{acl}(A)];$$

the group of automorphisms of every uncountable model saturated in its uncountable cardinality κ (if such models exist) fixing pointwise $\text{cl}(Z)$ is simple, where Z is a subset of parameters of size strictly less than κ .

More generally, for any of the above theories, given an uncountable cardinal κ (with $\text{cof}(\kappa) \geq \aleph_1$ in the last example) and a κ -prime model \mathbb{U} over $\text{cl}(Z)$, where Z is a subset of size strictly less than κ , the automorphism group $\text{Aut}(\mathbb{U}/\text{cl}(Z))$ is simple. □

We will finish this article with some questions we did not attempt to solve, and propose a list of possible examples for further research.

Question. The theory of separably closed fields of positive characteristic p and finite degree of imperfection e eliminates imaginaries [9, Proposition 43] in the language $\mathcal{L}_{\text{Rings}} \cup \{c_1, \dots, c_e\} \cup \{\lambda_n(x)\}_{0 \leq n < p^e}$, where $\{c_1, \dots, c_e\}$ denotes a p -basis and the λ -functions are taken with respect to the monomials in that basis. This theory satisfies Properties 1, 2 and (WH) with respect to the closure operator given by the unique generic type. We do not know whether this theory satisfies Property 3. By Remark 6.14, the unique generic type is not regular if $e \neq 0$.

Providing an explicit algebraic description of the closure $\text{cl}(A)$ in the finite degree of imperfection, even if we see such fields as fields equipped with (iterative) Hasse derivations [39], seems difficult, so we do not know whether Property 3 holds.

We have not attempted to list other natural examples of fields for which our methods could apply such as differentially closed fields of positive characteristic [37], DCFA [5], ACFE₀ [8] or \mathcal{D} -closed fields of characteristic 0 equipped with n free derivations [26].

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
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