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REALIZING TREES OF CONFIGURATIONS IN THIN SETS

ALLAN GREENLEAF, ALEX IOSEVICH AND KRYSTAL TAYLOR

Let $\phi(x, y)$ be a continuous function, smooth away from the diagonal, such that, for some $\alpha > 0$, the associated generalized Radon transforms

$$R_t^{\phi} f(x) = \int_{\phi(x,y)=t} f(y)\psi(y)\, d\sigma_{x,t}(y)$$

map $L^2(\mathbb{R}^d) \to L^2_{\alpha}(\mathbb{R}^d)$ for all $t > 0$. Let E be a compact subset of $\mathbb{R}^d$ for some $d \geq 2$, and suppose that the Hausdorff dimension of E is greater than $d - \alpha$. We show that any tree graph T on $k+1$ ($k \geq 1$) vertices is stably realizable in E, in the sense that for each t in some open interval there exist distinct $x^1, x^2, \dots, x^{k+1} \in E$ such that the ϕ-distance $\phi(x^i, x^j)$ equals t for all pairs (i, j) corresponding to the edges of T.

We extend this result to trees whose edges are prescribed by more complicated point configurations, such as congruence classes of triangles.

1. Introduction

The celebrated Falconer distance conjecture (see, e.g., [6; 21; 22]) states that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the distance set $\Delta(E) = \{|x - y| : x, y \in E\}$ is positive. Until recently, the best results known were due to Wolff [31] in two dimensions and Erdoğan [4] in higher dimensions. They proved that Lebesgue measure of the distance set is positive if the Hausdorff dimension of E satisfies $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}$. When $d = 2$, Orponen [27] proved that, under the additional assumption that $E \subset \mathbb{R}^2$ is Ahlfors–David regular, if $\dim_{\mathcal{H}}(E) > 1$, then the packing dimension of $\Delta(E)$ is 1.

Currently, the best known exponent threshold for the Falconer distance problem in two dimensions is $\frac{5}{4}$, due to Guth, Iosevich, Ou and Wang [14]. In higher dimensions, the best exponent in odd dimensions, recently established by Du, Ou, Ren and Zhang [3], is $\frac{d}{2} + \frac{1}{4} - \frac{1}{8d+4}$; see [2] for even dimensions.

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The Falconer problem has many variations, where distance is replaced by more general k-point configurations, which need not be scalar-valued. For $p \in \mathbb{N}$ and $k \geq 2$, let $\Phi : (\mathbb{R}^d)^k \to \mathbb{R}^p$ be a continuous function which is smooth (except possibly on a lower-dimensional set). For a compact $E \subset \mathbb{R}^d$, the Φ-*configuration set* of E [9] is the compact set

$$\Delta_\Phi(E) := \{\Phi(x^1, \dots, x^k) : x^1, \dots, x^k \in E\} \subset \mathbb{R}^p,$$

and one can look for lower bounds on $\dim_{\mathcal{H}}(E)$ ensuring that $\Delta_\Phi(E)$ has positive Lebesgue measure in $\mathbb{R}^p$.

A further variation on the Falconer problem, originating in Mattila and Sjölin's result for the distance set [23], seeks to determine values s_Φ so that $\dim_{\mathcal{H}}(E) > s_\Phi$ guarantees that $\Delta_\Phi(E)$ has *nonempty interior* in $\mathbb{R}^p$, in which case Φ is said to be a *Mattila–Sjölin function*. See [7; 11; 12; 13; 16; 17; 20; 23; 28; 29] for results of this type.

A particularly interesting example arises when the Euclidean distance $|x - y|$ is replaced by a more general function $\phi(x, y)$. For a compact $E \subseteq \mathbb{R}^d$, for some $d \geq 2$, and a $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, continuous and smooth away from the diagonal, we define the *generalized distance set*

$$\Delta_\phi(E) = \{\phi(x, y) : x, y \in E\} \subset \mathbb{R}. \tag{1-1}$$

Eswarathasan, Iosevich and Taylor proved in [5] that if ϕ satisfies the nonvanishing Monge–Ampère determinant condition,

$$\det\begin{pmatrix} 0 & \nabla_x \phi \\ -(\nabla_y \phi)^T & \frac{\partial^2 \phi}{dx_i dy_j} \end{pmatrix} \neq 0, \tag{1-2}$$

on the set $\{(x, y) : \phi(x, y) = t\}$, and if $E \subset \mathbb{R}^d$, $d \geq 2$, is a compact set with $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, then the Lebesgue measure of $\Delta_\phi(E)$ is positive. A particularly compelling case arises when E is a subset of a compact Riemannian manifold without boundary or conjugate points and ϕ is the induced distance function. Iosevich, Liu and Xi proved in two dimensions [19] that if $\dim_{\mathcal{H}}(E) > \frac{5}{4}$ then the Lebesgue measure of $\Delta_\phi(E)$ is positive, matching the exponent obtained in the Euclidean case of [14].

The main thrust of this paper is to develop a general technique to study finite point configurations of a graph-theoretic nature in Euclidean space and Riemannian manifolds, and apply it to resolve several open problems. We let $G = (\mathbb{V}, \mathbb{E})$ denote an undirected graph on k vertices. The *edge map* of G is $\mathcal{E}_G : \mathbb{V} \times \mathbb{V} \to \{0, 1\}$, $\mathcal{E}_G(i, j) = 1$ if $i \neq j$ and the i-th and j-th vertices are connected by an edge in $\mathbb{E}$, and 0 otherwise.

Definition 1.1 (generalized distance graph). A continuous $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, smooth away from the diagonal and such that $\phi(x, y) = \phi(y, x)$, a compact $E \subset \mathbb{R}^d$, and a $t > 0$ define the *generalized distance graph* $G_{\phi,t}(E)$, whose vertices are the points in E, and for which two vertices $x, y \in E$, $x \neq y$, are connected by an edge if and only if $\phi(x, y) = t$.

We say that an (abstract) connected finite graph G can be *realized* in E if there exists $t > 0$ such that G is isomorphic to a subgraph of $G_{\phi,t}(E)$; furthermore, G is said to be *stably realized* in E if the set of such t has nonempty interior.

Bennett, Iosevich and Taylor proved in [1] that if $E \subset \mathbb{R}^d$, $d \geq 2$, such that $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, $\phi(x, y) = |x - y|$, and G is a *path* (or *chain*), then G can be stably realized in E. In [15], Iosevich and Taylor extended this result to the more general case when G is a *tree*.

We note that trees and chains have also been considered for notions of size other than Hausdorff dimension, see, for instance, [24; 25], where McDonald and Taylor proved that chains and trees are stably realized in product sets of sufficient Newhouse thickness, and see [26] for a topological variant, where the same authors showed that all countably infinite bounded point configurations, including infinite trees, are stably realized in second category Baire sets [26] and linked this area to the Erdős similarity conjecture.

In this paper, we begin by extending these types of results to generalized distance graphs, showing that arbitrary trees are stably realized in sets E with $\dim_{\mathcal{H}}(E)$ sufficiently high (with threshold depending on ϕ but not on the tree G), under the assumption that the generalized Radon transform associated with ϕ satisfies a suitable Sobolev mapping property.

We shall also show that our method allows one to prove such results for trees G composed of elements of a fixed configuration; for brevity, we illustrate this just for a tree of triangles.

Example 1.2. Let $\phi(u, v) = |u - v|$ denote the standard Euclidean distance and the tree G be a path on three vertices, $\mathbb{V} = \{x^1, x^2, x^3\}$. In Section 2.2 below, we show that the configurations consisting of G, or in fact any tree, 'decorated' with congruent triangles can be stably realized in any compact $E \subset \mathbb{R}^d$, $d \geq 4$, with $\dim_{\mathcal{H}}(E) > (2d + 3)/3$. See Figure 1 below, the discussion in the next section, and further details in Section 2.2.

1.1. ***Structure of this paper.*** We begin by proving a result about tree structures in sets of a given Hausdorff dimension based on a general scheme that applies to a wide variety of situations. We then show that, if a (sufficiently symmetric) configuration can be embedded in the distance graph of a set E and $\dim_{\mathcal{H}}(E)$ is sufficiently high, then a *tree* of such configurations, where the edges of the tree are suitable subsets of the hyperedge defined by this configuration, is also guaranteed to

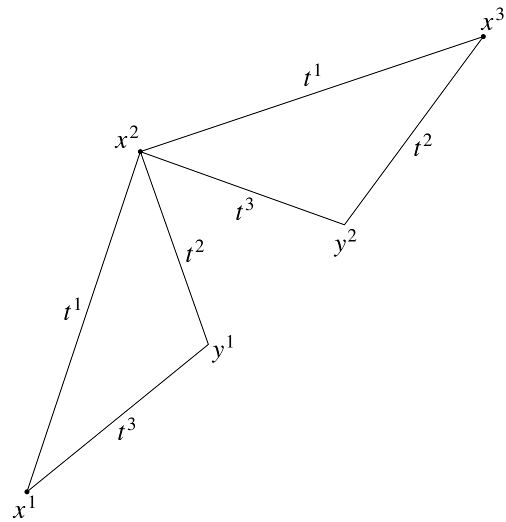

Figure 1. Chain of two congruent triangles.

be stably realizable under a suitable dimensional threshold on E. We then provide concrete applications based on Fourier integral operator bounds for associated Radon transforms.

1.2. ***Trees in general distance graphs.*** An important aspect of our approach is to formulate the realizability of trees (and certain other configurations) in a general setting. Let μ be a compactly supported nonnegative probability Borel measure on $\mathbb{R}^d$, and

$$K : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$$

a symmetric, $(\mu\times\mu)$-integrable continuous function. Consider the graph, denoted by G_K, whose vertices are the points of $E \subset \mathbb{R}^d$, with two vertices $x,\ y \in E$ being connected by an edge if and only if $K(x, y) > 0$.

The following result will allow us to reduce a variety of configuration problems to a series of concrete operator bounds.

Theorem 1.3 (tree building criteria). *Let μ and K be as above. Define*

$$U_K f(x) = \int K(x, y) f(y)\, d\mu(y) \quad \textit{for all } f \in C_0(\mathbb{R}^d),$$

and suppose that

$$\iint K(x, y)\, d\mu(x)\, d\mu(y) > 0, \tag{1-3}$$

and

$$U_K : L^2(\mu) \to L^2(\mu) \textit{ continuously.} \tag{1-4}$$

Then for any $k \geq 1$,

$$\int \cdots \int K(x^1, x^2) K(x^2, x^3) \cdots K(x^k, x^{k+1})\, d\mu(x^1)\, d\mu(x^2) \cdots d\mu(x^{k+1}) > 0. \tag{1-5}$$

More generally, let T *be a tree graph on* n *vertices,* $n \geq 2$, *with edge map* $\mathcal{E}_T$. *Define* $K^* : (\mathbb{R}^d)^n \to [0, \infty)$ *by*

$$K^*(x^1, \ldots, x^n) = \prod_{\substack{1 \leq i < j \leq n \\ \mathcal{E}_T(i,j)=1}} K(x^i, x^j).$$

Then

$$\int \cdots \int K^*(x^1, \ldots, x^n)\, d\mu(x^1)\, d\mu(x^2) \cdots d\mu(x^n) > 0. \tag{1-6}$$

In other words, the existence of trees even in this generalized setting is simply a consequence of assumptions (1-3) and (1-4). For example, in order to handle the Euclidean distance graph, where the vertices are points in a compact set E and two vertices are connected by an edge if and only if the distance between them is equal to some fixed $t > 0$, one takes $K = \sigma_t(x - y)$, where σ is the surface measure on the sphere of radius t. In this case, U_K is the corresponding translation-invariant spherical averaging operator. As a technical point, even though σ is a measure and not an L^1 function, the proof is accomplished by convolving σ with the approximation to the identity at scale ϵ and checking that the estimates (upper and lower bounds (see, e.g., [1]) do not depend on ϵ. The existence of arbitrary trees in the case $K = \sigma_t(x - y)$ was previously established by Iosevich and Taylor in [15]. Further, observe that nondegeneracy of the point configurations in question is guaranteed by the positivity of the integrals in (1-5) and (1-6), since degenerate configurations form lower-dimensional sets, which are of measure 0 with respect to $\bigotimes^n \mu$.

The view point afforded by Theorem 1.3 also proves useful in the context of trees of hypergraphs, significantly expanding the scope of configurations that can be handled. For example, suppose that we want to show that a set E contains many 2-chains of congruent triangles (see Figure 1). We are led to considering, for a Borel measure μ supported on E, the expression

$$\int^{(5)} F_1(x^1, x^2) F_2(x^2, y^1) F_3(y^1, x^1) F_1(x^2, x^3) F_2(x^3, y^2) F_3(y^2, x^2) \\ d\mu(x^1)\, d\mu(y^1)\, d\mu(x^2)\, d\mu(y^2)\, d\mu(x^3), \tag{1-7}$$

where, for $j = 1, 2, 3$, the $F_j(x, y)$ are nonnegative L^1 functions which will be smoothed out versions of $\delta(|x - y| - t_j)$, where the vector $\vec{t} = (t_1, t_2, t_3)$ can range over a set S of side length vectors with nonempty interior in $\mathbb{R}^3_+$. Then we may rewrite (1-7) in the form

$$\iiint K(x^1, x^2) K(x^2, x^3)\, d\mu(x^1)\, d\mu(x^2)\, d\mu(x^3),$$

where

$$K(x, y) := F_1(x, y) \int F_2(x, z) F_3(y, z)\, d\mu(z) = K(y, x) \tag{1-8}$$

is symmetric and satisfies (1-3) and (1-4). Thus, Theorem 1.3 applies, establishing the existence in E of chains of two congruent triangles, for all vectors $\vec{t} \in S$. More details can be found in Section 2.2 below, together with the extension from 2-chains to arbitrary trees of triangles and certain other configurations.

2. Consequences of Theorem 1.3

We begin with corollaries in the setting of trees, followed by hypergraphs, showing that Theorem 1.3 reduces the existence of a wide variety of configuration to the verification of conditions (1-3) and (1-4). These conditions amount to certain function space estimates which may, depending on the particular result, be of greater or lesser difficulty to establish. We will now proceed to work out a variety of such examples.

2.1. *Generalized Radon transforms.*

Corollary 2.1 (realizing trees in sets of sufficient Hausdorff dimension). *Let T be a tree graph on n vertices, and $\mathcal{E}_T$ its edge map. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be continuous. Suppose that for all $t > 0$, ϕ is smooth near $\Sigma_t := \{(x, y) : \phi(x, y) = t\}$, with $\nabla_x \phi(x, y), \nabla_y \phi(x, y) \neq \vec{0}$, so that $\Sigma_t \subset \mathbb{R}^d \times \mathbb{R}^d$ is smooth and for each $x \in \mathbb{R}^d$, $\Sigma_t^x := \{y \in \mathbb{R}^d : \phi(x, y) = t\} \subset \mathbb{R}^d$ is smooth. Further assume that, if ψ is a smooth cut-off and $\sigma_{x,t}$ is the surface measure on Σ_t^x, there is some $\alpha > 0$ such that the generalized Radon transform*

$$R_t^\phi f(x) := \int_{\Sigma_t^x} f(y)\, \psi(y)\, d\sigma_{x,t}(y)$$

is continuous $L^2(\mathbb{R}^d) \to L^2_\alpha(\mathbb{R}^d)$, locally uniformly in t.

Then, if $E \subset \mathbb{R}^d$ is compact with Hausdorff dimension $\dim_{\mathcal{H}}(E) > d - \alpha$, T is stably realizable in E. In other words, there is a nonempty open interval $I \subset \mathbb{R}_+$ such that T is realizable in E with any gap $t \in I$: for all $t \in I$ there exist distinct $x^1, \dots, x^n \in E$ such that $\phi(x^i, x^j) = t$ for (i, j) such that $\mathcal{E}_T(i, j) = 1$.

(We refer to [8; 30] for treatments of the L^2-based Sobolev spaces, L^2_α, $\alpha \in \mathbb{R}$, that we will use.) Corollary 2.1 will be proved in Section 4. As an example, if $\phi(x, y) = |x - y|$ is the Euclidean distance, the corresponding R_t^ϕ are spherical means operators which are smoothing of order $\alpha = (d-1)/2$, and thus the conclusions of Corollary 2.1 hold if the Hausdorff dimension of E is greater than $(d+1)/2$.

More generally, this L^2 Sobolev regularity holds within the conjugate locus for the distance induced by any Riemannian metric, leading immediately to the following.

Corollary 2.2 (realizing trees on Riemannian manifolds). *Let (M, g) be a compact Riemannian manifold without boundary or conjugate points, of dimension $d \geq 2$. Let E be a subset of M of Hausdorff dimension $> (d+1)/2$, and let ρ_M denote the induced Riemannian distance function on M.*

Let T be any tree graph on n vertices, and let $\mathcal{E}_T$ be the corresponding edge map. Then T is realizable in E, in the sense that there exist $x^1, \ldots, x^n \in E$ and a nonempty interval I such that $\rho_M(x^i, x^j) = t$ for $(i, j), t \in I$, such that $\mathcal{E}_T(i, j) = 1$.

2.2. ***Trees of triangles.*** We now illustrate applications of Theorem 1.3 to more complicated configurations, based on the approach briefly described below the statement of Theorem 1.3 for 2-chains of congruent triangles, and this is where we pick up the narrative and provide more details for some specific examples.

Theorem 1.3 yields the existence of trees of k-point configurations for a wide variety of configurations studied in the literature on what are now called Mattila–Sjölin functions. For $d, p \in \mathbb{N}$ and $k \geq 2$, let $\Phi : (\mathbb{R}^d)^k \to \mathbb{R}^p$ be a continuous function which is smooth (except possibly on a lower-dimensional set). For a compact $E \subset \mathbb{R}^d$, the Φ-*configuration set* of E [9] is the compact set

$$\Delta_\Phi(E) := \{\Phi(x^1, \ldots, x^k) : x^1, \ldots, x^k \in E\} \subset \mathbb{R}^p.$$

Then Φ is said to be a *Mattila–Sjölin function* if there is some $s_\Phi < d$ such that $\dim_{\mathcal{H}}(E) > s_\Phi$ ensures that $\Delta_\Phi(E)$ has nonempty interior. See [7; 11; 12; 13; 16; 20; 23; 28; 29] for results of this type.

Our approach in [11; 12; 13] was as follows: Let μ be a Frostman measure supported on E and of finite s-energy. Then the Radon–Nikodym derivative of the *configuration measure* $\nu_\Phi := \Phi_*(\mu \otimes \cdots \otimes \mu)$ on $\mathbb{R}^p$ with respect to Lebesgue measure $d\boldsymbol{t}$ can be represented as a multilinear form $\Lambda_{\boldsymbol{t}}(\mu, \ldots, \mu)$, with multilinear kernel

$$L_{\boldsymbol{t}}(x^1, \ldots, x^k) = \delta(\Phi(x^1, \ldots, x^k) - \boldsymbol{t}),$$

where δ is the Dirac delta at $0 \in \mathbb{R}^p$. The method of partition optimization [11; 12] and its local and microlocal variants [13] allow one to obtain estimates of the form

$$|\Lambda_{\boldsymbol{t}}(f_1, \ldots, f_k)| \leq C \prod_{j=1}^{k} \|f_j\|_{L^2_{r_j}}$$

for negative r_j with a lower bound on $\sum r_j$ depending on Φ. Applying this to $f_1 = \cdots = f_k = \mu$ having finite s-energy (which implies that $\mu \in L^2_{(s-d)/2}$) yields that, for $\boldsymbol{t}$ in a set $S \subset \mathbb{R}^p$ with nonempty interior,

$$0 < \Lambda_{\boldsymbol{t}}(\mu, \mu, \ldots, \mu) := \big\langle \delta(\Phi(x^1, \ldots, x^k) - \boldsymbol{t}), \mu \otimes \cdots \otimes \mu \big\rangle = C < \infty \tag{2-1}$$

if $\dim_{\mathcal{H}}(E) > s_\Phi$, and furthermore $\Lambda_{\boldsymbol{t}}(\mu, \dots, \mu)$ is continuous in $\boldsymbol{t} \in S$. Here, the pairing $\langle \cdot, \cdot \rangle$ is between distributions and Sobolev functions on $\mathbb{R}^{kd}$.

The left side of (2-1) can be rewritten in various equivalent ways by partitioning the variables and integrating out some first. In particular, fixing distinct indices $1 \le i < j \le k$, for points $x, y \in \mathbb{R}^d$, let

$$\hat{\boldsymbol{x}}^{ij} = (x^1, \dots, x^{i-1}, x, x^{i+1}, \dots, x^{j-1}, y, x^{j+1}, \dots x^k).$$

Then (2-1) implies that

$$K(x, y) := \Big\langle \delta(\Phi(\hat{\boldsymbol{x}}^{ij})), \bigotimes_{\substack{l=1 \\ l \neq i,j}}^{k} \mu(x^l) \Big\rangle \tag{2-2}$$

has the property that

$$0 < \iint K(x, y)\mu(x)\mu(y) = C < \infty, \tag{2-3}$$

which is condition (1-3) from Theorem 1.3. The following argument shows that, by replacing μ with its restriction to a chosen subset $F \subset E$, $\mu(F) > 0$, one can preserve (1-3) while also ensuring that (1-4) holds.

To start, from (2-3) it follows that

$$\mu\Big\{y : \int K(x, y)\, d\mu(x) > 3C\Big\} \le \tfrac{1}{3}.$$

Thus, if we define

$$F_1 = \Big\{y \in E : \int K(x, y)\, d\mu(x) \le 3C\Big\},$$

then $\mu(F_1) \ge \frac{2}{3}$. Denoting by μ_1 the restriction of μ to F_1, it follows that

$$\int K(x, y)\, d\mu(x) \le 3 \quad \text{for all } \ y \in F_1, \tag{2-4}$$

and

$$\iint K(x, y)\, d\mu(x)\, d\mu_1(y) \le C.$$

Using the last inequality and the positivity of the integrand, we can change the order of integration and repeat this argument with respect to x. Since

$$\mu\Big\{x : \int K(x, y)\, d\mu_1(y) > 3C\Big\} \le \tfrac{1}{3}, \tag{2-5}$$

the set $F_2 := \big\{x : \int K(x, y)\, d\mu_1(y) \le 3C\big\}$ has $\mu(F_2) \ge \frac{2}{3}$. Denoting $\mu|_{F_2}$ by μ_2, we then have

$$\begin{gathered} \int K(x, y)\, d\mu_1(y) \le 3C \quad \text{for all } \ x \in F_2, \\ \mu(F_1 \cap F_2) \ge 1 - \tfrac{1}{3} - \tfrac{1}{3} = \tfrac{1}{3}, \end{gathered} \tag{2-6}$$

and we denote $F_1 \cap F_2$ by $\tilde{E}$ and $\mu|_{\tilde{E}}$ by $\tilde{\mu}$. Finally, we have

$$\int K(x, y)\, d\tilde{\mu}(x) \le 3C \quad \text{for all } y \in \tilde{E}, \qquad \int K(x, y)\, d\tilde{\mu}(y) \le 3C \quad \text{for all } x \in \tilde{E},$$

so that Young's inequality applies to the integral kernel $\tilde{K} = K|_{\tilde{E}\times\tilde{E}}$, which thus defines a bounded operator $U_{\tilde{K}} : L^2(\tilde{\mu}) \to L^2(\tilde{\mu})$. Erasing all the tildes, we see that (1-4) is satisfied.

Finally, in order to apply Theorem 1.3 to obtain trees of Φ-configurations, we need that $K(x, y)$ be symmetric in x and y, and for this one needs to impose some symmetry conditions on the configuration function Φ. For simplicity, take $i = 1$, $j = 2$ in (2-2); then we demand that for some $A \in \mathrm{GL}(p, \mathbb{R})$ and some permutation $\tilde{\boldsymbol{x}}^{12}$ of the $k-2$ variables in $\hat{\boldsymbol{x}}^{12}$, we have

$$\Phi(y, x, \hat{\boldsymbol{x}}^{12}) = A \circ \Phi(x, y, \tilde{\boldsymbol{x}}^{12}), \tag{2-7}$$

so that $K(y, x) = c \cdot K(x, y)$, with $c = |A|^{-1}$, which preserves (2-3) and is good enough for our purposes.

2.2.1. *Building a tree of congruent triangles.* The mechanism of this paper applies to any configuration for which we can prove that the natural measure associated to the configuration satisfies the assumptions of Theorem 1.3, in the sense described below the statement of that result. We give just one illustrative example, namely the existence of trees of congruent triangles in E, using the following result from [29]; see also [13] for an alternate proof using microlocal analysis.

Theorem 2.3 [29]. *If $E \subset \mathbb{R}^d$, $d \ge 4$, is compact with $\dim_H(E) > \frac{2d+3}{3}$, then the set of congruence classes of triangles with vertices in E,*

$$\big\{(|x-y|, |x-z|, |y-z|) : x, y, z \in E\big\}, \tag{2-8}$$

has nonempty interior in $\mathbb{R}^3$.

Moreover, it is shown that the natural measure supported on

$$\big\{(|x-y|, |x-z|, |y-z|) : x, y, z \in E\big\},$$

namely the configuration measure ν_{triangle} defined by

$$\int f(t^1, t^2, t^3)\, d\nu_{\text{triangle}}(t^1, t^2, t^3) := \iiint f(|x-y|, |x-z|, |y-z|)\, d\mu(x)\, d\mu(y)\, d\mu(z),$$

is continuous away from the degenerate triangles, and from this the conclusion of Theorem 2.3 is ultimately obtained.

Given a side length vector $\vec{t} = (t^1, t^2, t^3)$ in the nonempty interior, say S, of the configuration set guaranteed by Theorem 2.3, we can build a tree of congruent triangles with side lengths $\vec{t}$, with any two triangles joined at exactly one vertex, as follows. At such a $\vec{t}$ the measure ν_{triangle} has a continuous density function, namely

$$\nu_{\text{triangle}} = \lim_{\epsilon \to 0^+} \nu^{\epsilon}_{\text{triangle}},$$

where

$$\nu^{\epsilon}_{\text{triangle}}(\vec{t}) = \iiint \sigma^{\epsilon}_{t^1}(x-y)\sigma^{\epsilon}_{t^2}(x-z)\sigma^{\epsilon}_{t^3}(y-z)\, d\mu(x)\, d\mu(y)\, d\mu(z).$$

We now define the approximate kernel

$$K^{\epsilon}_{\vec{t}}(x, y) = \sigma^{\epsilon}_{t^1}(x-y) \int \sigma^{\epsilon}_{t^2}(x-z)\sigma^{\epsilon}_{t^3}(y-z)\, d\mu(z),$$

which satisfies the equivariance property (2-7) with $A \in \mathrm{GL}(3, \mathbb{R})$ interchanging t^2 and t^3. This is the object to which we apply Theorem 1.3, and the argument is complete. Following the same procedure we can produce an arbitrary tree of triangles, not just two triangles joined at a vertex.

2.2.2. *Trees of equiarea triangles.* This method can be applied to obtain the existence of arbitrary trees for some, but not all, of the k-point configurations for which nonempty interior of configurations sets were established in [12; 13]. One of these concerned areas of triangles in the plane:

Theorem 2.4 [12, Theorem 1.1(i)]. *If $E \subset \mathbb{R}^2$ is compact with $\dim_{\mathcal{H}}(E) > \frac{5}{3}$, then the set of signed areas of triangles determined by triples of points of E,*

$$\left\{\tfrac{1}{2}\det[x-z, y-z] : x, y, z \in E\right\} \subset \mathbb{R}, \tag{2-9}$$

contains an open interval.

The $\mathbb{R}^1$-valued configuration function Φ of three variables in $\mathbb{R}^2$, $\Phi(x, y, z) = \det[x-z, y-z]$ satisfies (2-7) with factor -1, so that the method above applies. Hence, for $\dim_{\mathcal{H}}(E) > \frac{5}{3}$ and for an arbitrary tree T, and areas A in an open interval, there exist copies of T in E and auxiliary points $y^{ij} \in E$ for each (i, j) with $\mathcal{E}_T(i, j) = 1$, such that x^i, x^j and y^{ij} span a triangle of area A.

3. Proof of Theorem 1.3

Our basic scheme is the following. We first prove the result for paths with $k = 2^m$ vertices by utilizing Cauchy–Schwarz and the assumption (1-3). We then induct downwards to fill in the gaps between the dyadic numbers after first pigeonholing to a subset where $U_K 1$ is not too large. Finally, we notice that the flexibility afforded by our arguments allows us to extend the case of a path to a general tree.

3.1. ***The case $k = 2^m$.*** We begin by proving (1-5). Set

$$c_{\text{lower}} := \iint K(x, y)\, d\mu(x)\, d\mu(y),$$

which is > 0 by (1-3), and define

$$C_k(\mu) = \int \cdots \int \prod_{j=1}^{k} K(x^j, x^{j+1}) \prod_{i=1}^{k+1} d\mu(x^i).$$

Suppose that $k = 2^m$ for some m. Then, by repeated application of Cauchy–Schwarz, we have

$$\begin{aligned} C_{2^m}(\mu) &= \int \cdots \int \prod_{j=1}^{k} K(x^j, x^{j+1}) \prod_{i=1}^{k+1} d\mu(x^i) \\ &\geq \left(\iint K(x, y)\, d\mu(x)\, d\mu(y) \right)^{2^m} > c_{\text{lower}}^{2^m} > 0, \end{aligned}$$

where we used (1-3) and the assumption that μ is a probability measure.

3.2. ***Refinement to a subset where $U_K\mathbf{1}$ is not too large.*** In order to deal with general k, we need to do a bit of pigeonholing. Observe that, if $C_{\text{norm}} := \|U_K\|_{L^2 \to L^2}$,

$$\mu\{x : (U_K 1)(x) > \lambda\} \leq \frac{1}{\lambda^2} \int |U_K 1(x)|^2\, d\mu(x) \leq \frac{C_{\text{norm}}^2}{\lambda^2}$$

by (1-4). It follows that if $\lambda = NC_{\text{norm}}$, with $N > 2$ to be determined later, then

$$(U_K 1)(x) \leq NC_{\text{norm}} \quad \text{on a set } E' \text{ with } \mu(E') \leq \frac{1}{N^2}. \tag{3-1}$$

If we replace the constant function 1 in (3-1) by the indicator function of E', the upper bound still holds. Moreover, if we let μ' denote μ restricted to E', we have

$$\begin{aligned} &\iint K(x, y)\, d\mu'(x)\, d\mu'(y) \\ &\qquad = \iint K(x, y)\, d\mu(x)\, d\mu(y) - \iint K(x, y) f(x) f(y)\, d\mu(x)\, d\mu(y) = I - II, \end{aligned}$$

where f is the indicator function of the set where $U_K 1(x) > 2C_{\text{norm}}$. By assumption, $I = c_{\text{lower}} > 0$. Observe that

$$II \leq C_{\text{norm}} \cdot \|f\|_{L^2(\mu)}^2 \leq \frac{C_{\text{norm}}^3}{\lambda^2} \leq \frac{C_{\text{norm}}}{N^2} \leq \frac{c_{\text{lower}}}{2}$$

if we choose

$$N \geq \sqrt{\frac{2C_{\text{norm}}}{c_{\text{lower}}}}.$$

With a slight abuse of notation, we can now rename μ' back to μ, renormalize, and pretend that from the very beginning we had a set E, equipped with the Borel measure μ, such that $(U_K 1)(x)$ is bounded above by some uniform constant C, and both (1-3) and (1-4) hold.

3.3. ***Paths of arbitrary finite length and transition to trees.*** Using the results just obtained in Section 3.2, one sees that for $k \geq 2$,

$$C_k(\mu) \leq C \cdot C_{k-1}(\mu),$$

where C is the upper bound on $(U_K 1)(x)$.

Proceeding by induction we get a lower bound on a path of arbitrary length. In particular, and this notion will come in handy in a moment, having built a path in E with 2^m links, we have also built a path of smaller length.

In order to build an arbitrary tree, we use the simple principle that if T, T' are trees, and T is contained in T', then building T' in E implies that we can build T. Given a tree T, let

$$T(\mu) = \int \cdots \int \prod_{(i,j) \in \mathcal{E}_T} K(x^i, x^j) \prod_{i=1}^{k+1} d\mu(x^i), \tag{3-2}$$

where $\mathcal{E}_T$ is as above.

We shall need the following definition.

Definition 3.1 (a wrist of a tree). Let G be a connected tree graph. We say that w is a wrist of order n if the following conditions hold:

(i) $w \in V$, the vertex set of G.

(ii) $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \{w\}$.

(iii) Vertices from $V_1 \setminus \{w\}$ are not connected by edges to vertices in $V_2 \setminus \{w\}$.

(iv) Let G_1 denote G restricted to V_1. Then G_1 is the union of finitely many chains $C_1, C_2, \ldots, C_n$ such that vertices of C_i only intersect vertices of C_j, $i \neq j$, at w.

Example 3.2. (i) Consider a chain on three vertices, with vertices v_1, v_2, v_3 such that v_1 is connected by an edge to v_2, and v_2 is connected by an edge to v_3, but v_1 and v_3 are not connected (see Figure 2, left). Then v_1 is a wrist because there is a chain with vertices v_1, v_2, v_3 with one endpoint at v_1. The vertex v_3 is a wrist for the same reason. The vertex v_2 is also a wrist because two chains, namely the one with vertices v_2, v_1, and the one with vertices v_2, v_3 have v_2 as an endpoint.

(ii) Consider a complete graph on three vertices (see Figure 2, middle). Then no vertex is a wrist.

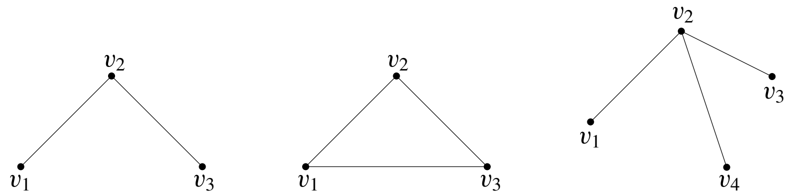

Figure 2. Illustration of Example 3.2.

(iii) Consider a graph on four vertices v_1, v_2, v_3, v_4, where v_1 and v_2 are connected, v_2 and v_3 are connected, v_2 and v_4 are connected, and there are no other edges (see Figure 2, right). Then v_2 is the only wrist.

Our argument is based on the fact that every tree which is not a chain contains a wrist of order > 1.

Lemma 3.3 (any nontrivial tree contains a wrist). *Let T be a finite connected tree. Then either T is a chain, or T contains a wrist of order* > 1.

To prove the lemma, let $v_1, v_2, \ldots, v_m$ denote the (distinct) vertices of degree 1 in T. We move from each v_j until we encounter a vertex of degree ≥ 3. If such a vertex does not exist, then T is clearly a chain. In this way, we assign a vertex w_j, of degree ≥ 3, to each v_j. We claim that there exist i, j, $i \neq j$, such that $w_i = w_j$. Suppose not. Remove all the v_js and the vertices and edges that lead up to, but not including, w_j. The resulting graph T' is still a connected tree. Each vertex w_j in T' has a degree $\geq 3 - 1 = 2$, so the vertices of degree 1 in T' are not any of the v_js or any of the w_js. This means that those vertices were present in the original tree graph T, but this is impossible since we removed them.

Now that we have shown that there exists $i \neq j$ such that $w_i = w_j$, it is not difficult to see that this w_i is a wrist of order > 1, as desired. This completes the proof of the lemma.

Let w_0 denote a wrist point in T, which, as we just proved, is guaranteed to exist. We now rewrite (3-2) in the form

$$\int U_K(w) C_1(w) C_2(w) \ldots C_k(w)\, dw, \tag{3-3}$$

where

$$C_j(w) = \int \cdots \int K(w, x^{j,1}) K(x^{1,1}, x^{j,2}) \ldots K(x^{j,n_j-1}, x^{j,n_j})\, dx^{j,1} \cdots dx^{j,n_j}.$$

Adding vertices and edges, if necessary, we can make all the chains have the same length, $n_{\max}$. One can then estimate (3-2) using Hölder's inequality:

$$\int U_K(w) \Big(\int \cdots \int K(w, x^1) K(x^1, x^2) \cdots K(x^{n_{\max}-1}, x^{n_{\max}})\, dx^1 \cdots dx^{n_{\max}} \Big)^k dw$$
$$= \int \Big(\int \cdots \int K(w, x^1) K(x^1, x^2) \cdots K(x^{n_{\max}-1}, x^{n_{\max}})\, dx^1 \cdots dx^{n_{\max}} \Big)^k U_K(w)\, dw$$

$$\geq \frac{\left(\iint\cdots\int K(w,x^1)K(x^1,x^2)\cdots K(x^{n_{\max}-1},x^{n_{\max}})\,dx^1\cdots dx^{n_{\max}}U_K(w)\,dw\right)^k}{\left(\int U_K(w)\,dw\right)^{k-1}}$$
$$\geq C\left(\iint\cdots\int K(w,x^1)K(x^1,x^2)\cdots K(x^{n_{\max}-1},x^{n_{\max}})\,dx^1\cdots dx^{n_{\max}}U_K(w)\,dw\right)^k$$

since we have an upper bound for $\int U_K(w)\,dw$ by a repeated use of (3-1).

In other words,

$$T(\mu)\geq c\cdot T'(\mu),$$

where T' is the tree obtained from T by removing all but the longest chain emanating from the wrist w. It is clear that T' has fewer vertices (and hence edges) than T. Proceeding in this way shows that given any tree T, there exists a tree T^* containing T and a positive constant c^* such that

$$T^*(\mu)\geq c^*\cdot c_{\text{lower}}>0.$$

This completes the proof of Theorem 1.3.

4. Proof of Corollary 2.1

The proof of Corollary 2.1 follows, in view of Theorem 1.3, from the following results. The first one follows from the proof of the main result in [11] (also see [12]).

Theorem 4.1 (establishing the lower bound (1-3)). *Let ϕ, R_t^ϕ be as in the statement of Corollary 2.1, with $R_t^\phi : L^2(\mathbb{R}^d)\to L^2_\alpha(\mathbb{R}^d)$ for some $\alpha>0$, uniformly for t in a nontrivial interval $I_0\subset\mathbb{R}$. Let E be a compact set of Hausdorff dimension* $\dim_{\mathcal{H}}(E)>d-\alpha$, *and μ a Frostman measure on E of finite s-energy for some $s>d-\alpha$. Then*

$$J(t):=\int R_t^\phi\mu(x)\,d\mu(x)$$

is a continuous function on I_0, and there exists a nonempty open interval $I\subseteq I_0$ and a $c_\delta>0$ such that for all $t\in I$,

$$\int R_t^\phi\mu(x)\,d\mu(x)\geq c_\delta>0. \tag{4-1}$$

Theorem 4.1 establishes that assumption (1-3) in Theorem 1.3 is satisfied (with $K(x,y)$ the Schwartz kernel of R_t^ϕ) uniformly for $t\in I$.

Theorem 4.2 (establishing the upper bound (1-4)). *Let ϕ, R_t^ϕ be as in the statement of Corollary 2.1. Suppose that for some $\alpha>0$ and all $t>0$, $R_t^\phi : L^2(\mathbb{R}^d)\to L^2_\alpha(\mathbb{R}^d)$ is bounded. Let $E\subset\mathbb{R}^d$ be compact, with Hausdorff dimension greater than $d-\alpha$, and μ a Frostman measure on E. Then for any $t>0$,*

$$\|R_t^\phi f\|_{L^2(\mu)}\leq K\,\|f\|_{L^2(\mu)}. \tag{4-2}$$

This establishes the assumption (1-4) in Theorem 1.3.

Remark 4.3. The constant K above only depends (uniformly) on the implicit constants in the assumptions of Corollary 2.1. By standard FIO theory (see, e.g., Section 2 in [10] for similar calculations) K depends only on the ambient dimension d, the Hausdorff dimension of the support of μ, the bounds implicit in the Sobolev estimate for R_t^{ϕ} and the Frostman constant, i.e., the constant such that $\mu(B(x,r)) \le Cr^{\alpha}$ for any $\alpha < d$, where $B(x,r)$ is the ball of radius r (sufficiently small) centered at x in the support of μ.

4.1. ***Proof of Theorem 4.1.*** This is essentially proven in [11], but for the sake of completeness we include the argument. The assumption $\nabla_x\phi(x,y), \nabla_y\phi(x,y) \neq \vec{0}$ on $\{(x,y) : \phi(x,y) = t\}$ allows one to conjugate by an elliptic pseudodifferential operator of any order $r \in \mathbb{R}$, so that $R_t^{\phi} : L_r^2(\mathbb{R}^d) \to L_{r+\alpha}^2(\mathbb{R}^d)$, locally uniformly in t. Since μ has finite s-energy, $\mu \in L^2_{s-d/2}$. Thus $R_t^{\phi}\mu \in L^2_{(s-d)/2+\alpha}$, and this will pair boundedly against μ if $(s-d)/2+\alpha+(s-d)/2 \ge 0$, i.e., if $s > d-\alpha$. Furthermore, by continuity of the integral, this is continuous in t. Since the integral of J in t is positive by the coarea formula, there must be a t_0 at which $J(t_0) > 0$, and hence there is a nonempty open interval on which J is strictly positive.

4.2. ***Proof of Theorem 4.2.*** It is enough to show that

$$\int R_t^{\phi} f\mu(x) g(x)\, d\mu(x) \le C < \infty$$

for g such that $\|g\|_{L^2(\mu)} = 1$. Let $(f\mu)_j$ denote the Littlewood–Paley piece of $f\mu$ on scale $j \ge 0$. Negative scales are straightforward and will be handled separately. We are going to bound

$$\langle R_t^{\phi}(f\mu)_j, (g\mu)_{j'}\rangle = \langle \widehat{R_t^{\phi}(f\mu)_j}, \widehat{(g\mu)_{j'}}\rangle.$$

By a standard orthogonality argument for generalized Radon transforms, the expression above decays rapidly when $|j-j'| \ge 5$. It follows that it suffices to bound

$$\sum_{|j-j'|\ge 5} \langle \widehat{R_t^{\phi}(f\mu)_j}, \widehat{(g\mu)_{j'}}\rangle \tag{4-3}$$
$$\le \sum_{|j-j'|\ge 5} \left(\int |\widehat{R_t^{\phi}(f\mu)_j}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int |\widehat{(g\mu)_{j'}}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$
$$\le C \sum_{|j-j'|\ge 5} 2^{-j\alpha}\left(\int |\widehat{(f\mu)_j}(\xi)|^2 d\xi\right)^{\frac{1}{2}}\left(\int |\widehat{(g\mu)_{j'}}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

We shall need the following basic estimate. See [5; 18] for similar results.

Lemma 4.4. *With the notation above, for any* $\epsilon > 0$,

$$\left(\int |\widehat{(f\mu)_j(\xi)}|^2 \, d\xi\right)^{\frac{1}{2}} \leq C_\epsilon 2^{j(d-s+\epsilon)/2} \|f\|_{L^2(\mu)}.$$

With Lemma 4.4 in tow, the expression in (4-3) is bounded by

$$C 2^{-j\alpha} 2^{j(d-s+\epsilon)} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)},$$

so the sum over j is bounded by $C \|f\|_{L^2(\mu)}$ provided that $s > d - \alpha$, as claimed. This completes the proof of Theorem 4.2, once we establish Lemma 4.4.

To prove Lemma 4.4, we write

$$\int |\widehat{(f\mu)_j(\xi)}|^2 \, d\xi = \int |\widehat{f\mu}(\xi)|^2 \psi^2(2^{-j}\xi) \, d\xi,$$

where ψ is a smooth cut-off function supported in the annulus

$$\left\{\xi \in \mathbb{R}^d : \tfrac{1}{2} \leq |\xi| \leq 4\right\}.$$

This expression is bounded by

$$\int |\widehat{f\mu}(\xi)|^2 \widehat{\rho}(2^{-j}\xi) \, d\xi,$$

where ρ is a suitable cut-off function.

By Fourier inversion and a limiting argument (see [31]), this expression equals

$$2^{dj} \iint \rho(2^j(x-y)) f(x) f(y) \, d\mu(x) \, d\mu(y) = \langle U_j f, f\rangle,$$

where

$$U_j f(x) = \int 2^{dj} \rho(2^j(x-y)) f(y) \, d\mu(y),$$

and $\langle \cdot, \cdot \rangle$ is the $L^2(\mu)$ inner product.

Since

$$\int 2^{dj} \rho(2^j(x-y)) \, d\mu(y) = \int 2^{dj} \rho(2^j(x-y)) \, d\mu(x) \leq C_\epsilon 2^{-j(s-\epsilon)}$$

for any $\epsilon > 0$ since μ is a Frostman measure on E. By Schur's test,

$$U_j : L^2(\mu) \to L^2(\mu) \quad \text{with norm } C_\epsilon 2^{j(d-s-\epsilon)}.$$

By Cauchy–Schwarz,

$$\langle U_j f, f\rangle \leq \|U_j f\|_{L^2(\mu)} \cdot \|f\|_{L^2(\mu)} \leq C_\epsilon 2^{j(d-s-\epsilon)} \|f\|^2_{L^2(\mu)}$$

and the proof is completed by taking square roots.

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