Pacific Journal of Mathematics

ON p-ADIC L-FUNCTIONS FOR $GSp_4 \times GL_2$

DAVID LOEFFLER AND ÓSCAR RIVERO

Volume 335 No. 2 April 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 335, No. 2, 2025
https://doi.org/10.2140/pjm.2025.335.373

ON p-ADIC L-FUNCTIONS FOR $\mathrm{GSp}_4 \times \mathrm{GL}_2$

DAVID LOEFFLER AND ÓSCAR RIVERO

We use higher Coleman theory to construct a new p-adic L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$. While Loeffler et al. (2021) had considered the p-adic variation of classes in the H^2 of Shimura varieties for GSp_4, here we explore the interpolation of classes in the H^1, which detect critical values for a different range of weights, disjoint from the range covered by this earlier construction. Using the algebraicity result established in our earlier work (Loeffler and Rivero 2024) we further show an interpolation property in terms of complex L-values.

1. Introduction

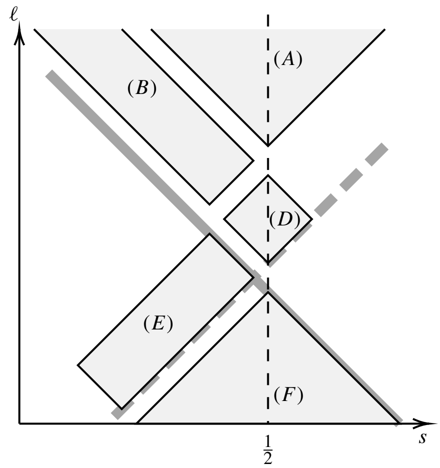
Let π and σ be cuspidal automorphic representations of $\mathrm{GSp}_4/\boldsymbol{Q}$ and $\mathrm{GL}_2/\boldsymbol{Q}$ respectively. Then we have a degree 8 L-function $L(\pi \times \sigma, s)$, associated to the tensor product of the natural degree 4 (spin) and degree 2 (standard) representations of the L-groups of GSp_4 and GL_2. If π and σ are algebraic, then this L-function is expected to correspond to a motive, and we can ask whether it has critical values.

We suppose that π (or, more precisely, its L-packet) corresponds to a holomorphic Siegel modular eigenform of weight (k_1, k_2) for $k_1 \geqslant k_2 \geqslant 2$ integers, and that σ corresponds to a holomorphic elliptic modular form of weight $\ell \geqslant 1$. For $L(\pi \times \sigma, s)$ to be a critical value, we must have $s = -\frac{1}{2}(k_1 + k_2 + \ell - 4) + j$ for $j \in \boldsymbol{Z}$, so that $L(\pi \times \sigma, s) = L(V_p(\pi) \otimes V_p(\sigma), j)$ where $V_p(-)$ are the Galois representations corresponding to π and σ; and the tuple (k_1, k_2, ℓ, j) has to satisfy one of three different sets of (mutually exclusive) inequalities, which we have outlined in more detail in the companion paper [Loeffler and Rivero 2024], corresponding to the cases (A), (D), (F) in Table 1 of the same reference. In this paper, we focus on region (D), which is given by the inequalities

(1) $k_1 - k_2 + 3 \leqslant \ell \leqslant k_1 + k_2 - 3, \quad \max(k_1, \ell) \leqslant j \leqslant \min(k_2 + \ell + 3, k_1 + k_2 - 3).$

Supported by ERC Consolidator grant “Shimura varieties and the BSD conjecture” (Loeffler) and Royal Society Newton International Fellowship NIF\R1\202208 (Rivero). This material is based upon work supported by the National Science Foundation under grant no. DMS-1928930 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2023 semester.
MSC2020: primary 11F46; secondary 11F67, 11R23.
Keywords: p-adic L-functions, higher Coleman theory, Siegel Shimura varieties.

The corresponding values of s and ℓ are illustrated in the diagram below. (The "off-centre" regions (B), (E), and the two grey diagonal lines, will be explained shortly.)



We shall now consider the case when π and σ vary through p-adic families. We consider Coleman families $\underline{\pi}$ for GSp_4 (over some 2-dimensional affinoid space $U \subset \mathcal{W} \times \mathcal{W}$, where $\mathcal{W}$ is the space of characters of $\mathbf{Z}_p^\times$), and similarly $\underline{\sigma}$ for GL_2, over a 1-dimensional affinoid $U' \subset \mathcal{W}$.

Following [Loeffler and Zerbes 2021c] and [Loeffler and Rivero 2024], we may conjecture that there exist three different p-adic L-functions in $\mathcal{O}(U \times U' \times \mathcal{W})$, denoted by $\mathcal{L}^{\diamondsuit}(\underline{\pi} \times \underline{\sigma}, -)$ for $\diamondsuit \in \{(A), (D), (F)\}$, whose values at integer points (k_1, k_2, ℓ, j) satisfying the inequalities (1) interpolate the corresponding complex L-values. (These depend on various auxiliary data, which we suppress for now.)

In [Loeffler and Zerbes 2021b], building on the earlier work [Loeffler et al. 2021], we proved a weakened form of this conjecture for region (F): we constructed a p-adic L-function over a codimension-1 subspace of the parameter space $U \times U' \times \mathcal{W}$, interpolating L-values in region (F) and lying at the "right-hand edge" of the critical strip. Thus, for each (k_1, k_2, ℓ) such that $\ell \leqslant k_1 - k_2 + 1$, our p-adic L-function captures just one among the (possibly) many critical values of the L-function of the weight (k_1, k_2, ℓ) specialisation of $\underline{\pi} \times \underline{\sigma}$. This corresponds to the solid grey diagonal line in the above figure. We also showed that certain (noncritical) values of this p-adic L-function, corresponding to the elongation of the diagonal line to meet region (E), were related to syntomic regulators of Euler system classes constructed in [Hsu et al. 2020]; the region (E) in the above diagram is precisely the range of weights in which the geometric Euler system classes of the same work are defined.

Note 1.1. We would also expect a second Euler system construction for weights in region (B), but this is only conjectural at present.

The goal of this paper is to prove the analogue for region (D) of the first main result proved for region (F) in [Loeffler and Zerbes 2021b]. That is, we define a p-adic L-function interpolating L-values along the "lower right edge" of region (D), i.e., for (k_1, k_2, ℓ, j) satisfying the conditions

$$k_1 - k_2 + 3 \leqslant \ell \leqslant k_1, \quad j = k_2 + \ell - 3,$$

where $s = \frac{1}{2}(\ell - k_1 + k_2 - 2)$. So this p-adic L-function again lives over a codimension-1 subspace of the 4-dimensional parameter space, but a different one from that of [Loeffler and Zerbes 2021b]: it is indicated by the dotted grey line in the figure. We conjecture, but do not prove here, a relation between this new p-adic L-function and syntomic regulators in region (E); we hope to return to this in a subsequent work.

Remark 1.2. Both in the present paper and in [Loeffler and Zerbes 2021b], the reason why we lose one variable in the construction is that we do not know how to work with *nearly holomorphic* modular forms in the framework of higher Coleman theory. More precisely, L-values anywhere in region (F), and in the "lower half" of region (D), can be interpreted algebraically via cup products in coherent cohomology; but the Eisenstein series appearing in these expressions are only holomorphic if s lies at the upper or lower limit of the allowed range — otherwise, they are nearly holomorphic but not holomorphic. We are optimistic that future developments in higher Hida/Coleman theory may circumvent this barrier, allowing the construction of p-adic L-functions over the full 4-dimensional parameter space with interpolating properties in region (F) or region (D).

The main result. It is convenient to reindex the weights by setting $(r_1, r_2) = (k_1 - 3, k_2 - 3)$; for region (D) to be nonempty we need $r_1 \geqslant r_2 \geqslant 0$, and in this case, (r_1, r_2) is the highest weight of the algebraic representation of GSp_4 for which π is cohomological. Let χ_π (resp. χ_σ) the central character of π (resp. σ).

To define the (imprimitive) p-adic L-function, we need to consider the following objects. Here, P denotes a point in U and Q a point in U'.

• A set of local conditions encoded in terms of the local data γ_S, introduced in Sections 5.1 and 5.4, and appearing in the factor $Z_S(\pi_P \times \sigma_Q, \gamma_S)$.

• A degree 8 Euler factor $\mathcal{E}^{(D)}(\pi_P \times \sigma_Q)$, where π_P (resp. σ_Q) stands for the specialization of $\underline{\pi}$ (resp. $\underline{\sigma}$) at the point P (resp. Q). This is consistent with the predictions of [Loeffler and Zerbes 2021c, Section 4.3] and the fact that the Galois representation is 8-dimensional. The precise computation and description of the Euler factor is done in [Loeffler and Rivero 2024, Section 7].

• The completed (complex) L-function $\Lambda(\pi_P \times \sigma_Q, s)$.

• A basis $\underline{\xi} \otimes \underline{\eta}$ of the space $S^1(\underline{\pi}) \otimes S^1(\underline{\sigma})$, as introduced in [Loeffler and Zerbes 2021b, Definition 10.4.1]. The p-adic L-function does depend on that choice.

• The complex (resp. p-adic) period $\Omega_\infty(\pi_P, \sigma_Q)$ (resp. $\Omega_p(\pi_P, \sigma_Q)$), introduced in Definition 5.7 and depending also on the specialization $\xi_P \otimes \eta_Q$ of the canonical differential $\underline{\xi} \otimes \underline{\eta}$ at (P, Q).

• The Gauss sum attached to χ_σ^{-1}, denoted by $G(\chi_\sigma^{-1})$.

Further, we need to introduce the notion of *nice critical point*. We say a point (P, Q) of $U \times U'$ is nice if $P = (r_1, r_2)$ and $Q = (\ell)$ are integer points, with P nice for $\underline{\pi}$ and Q nice for $\underline{\sigma}$, according to the definitions of Section 5. Further, we say (P, Q) is nice critical if we also have $r_1 - r_2 + 3 \le \ell \le r_1 + 3$.

The main theorem we prove in this note, using in a crucial way the algebraicity result of [Loeffler and Rivero 2024], is the following.

Theorem 1.3. *There exists a p-adic L-function $\mathcal{L}_{p,\gamma_S}^{\mathrm{imp}}(\underline{\pi} \times \underline{\sigma})$ satisfying the following interpolation property*: *if (P, Q) is nice critical*, *then*

$$\begin{aligned}\frac{\mathcal{L}_{p,\gamma_S}^{\mathrm{imp}}(\underline{\pi} \times \underline{\sigma})(P, Q)}{\Omega_p(\pi_P, \sigma_Q)} &\\ = Z_S(\pi_P \times \sigma_Q, \gamma_S) \cdot \mathcal{E}^{(D)}(\pi_P \times \sigma_Q) \cdot &\frac{G(\chi_\sigma^{-1})\Lambda\big(\pi_P \times \sigma_Q, \tfrac{1}{2}(\ell - k_1 + k_2 - 2)\big)}{\Omega_\infty(\pi_P, \sigma_Q)},\end{aligned}$$

where (k_1, k_2, ℓ) are such that π_P has weight (k_1, k_2) and σ_Q has weight ℓ.

The approach we follow to establish the theorem is the following:

(1) Use results of Harris and Su (see [Loeffler and Rivero 2024, Section 3]) to express the automorphic period to be computed as a cup product in the coherent cohomology of a Shimura variety associated with $\mathrm{GL}_2 \times \mathrm{GL}_2$.

(2) Use higher Coleman theory to reinterpret the cup product in terms of a pairing in coherent cohomology over certain strata in the adic Shimura varieties.

(3) Use the families of automorphic forms $\underline{\pi}$ and $\underline{\sigma}$ in order to define the p-adic L-function $\mathcal{L}_{p,\gamma_S}^{\mathrm{imp}}(\underline{\pi} \times \underline{\sigma}; \underline{\xi})$.

(4) Derive an interpolation formula at critical points using the compatibility of the cup product with specialisation.

Remark 1.4. For $s = \frac{1}{2}(\ell - k_1 + k_2 - 2)$, we can write $L(\pi_P \times \sigma_Q, s) = L(V, 0)$, where V is the Galois representation $V(\pi_P) \otimes V(\sigma_Q)(k_2 + \ell - 3)$. This Galois representation always has one of its Hodge–Tate weights equal to 0, which gives an intuitive explanation of why it should be "easier" to interpolate L-values along this subspace of the parameter space rather than over the entire 4-dimensional parameter space incorporating arbitrary cyclotomic twists.

If we specialise at a fixed P, giving a one-variable p-adic L-function $L^{\mathrm{imp}}_{p,\gamma_S}(\pi \times \underline{\sigma})$ associated to a fixed π and a GL_2 family $\underline{\sigma}$, and we choose this $\underline{\sigma}$ to be a family of ordinary CM forms (arising from an imaginary quadratic field K in which p is split), then L-values interpolated by $\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})$ can be interpreted as values of the L-function of π twisted by Grössencharacters of K; and the restriction on the value of s implies that the Grössencharacters arising have infinity-types of the form $(n, 0)$. We expect that this L-function should have an interpretation as a "$\mathfrak{p}$-adic L-function", interpolating twists by characters of the ray class group of K modulo $\mathfrak{p}^\infty$, for a specific choice of prime $\mathfrak{p}$ above p; this will be pursued in more detail elsewhere.

Connection with other works. In [Loeffler and Zerbes 2021c], the authors work in the setting of cusp forms for the larger group $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ and conjecture the existence of 6 different p-adic L-functions interpolating Gross–Prasad periods, corresponding to the "sign $+1$" regions (a), (a'), (c), (d), (d') and (f) in the diagrams of the same work. The case of region (f) was covered in [Loeffler et al. 2021] (see also [Loeffler and Zerbes 2021b]) using higher Hida and Coleman theory, and the p-adic L-function for region (c) was announced by Bertolini, Seveso and Venerucci, also using tools from coherent cohomology. Note however that the works cited only cover the case when one of the GL_2-forms is an Eisenstein series.

If one formally replaces one of the two cusp forms by an Eisenstein series, then the Gross–Prasad period becomes Novodvorsky's integral computing the degree 8 L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$; and regions (a), (b), (d), (e), (f) correspond to the regions (A), (B), (D), (E), (F) of the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ figure above (while the arithmetic meaning of the remaining regions (a'), (b'), (d'), (c) is less clear in this case). The methods we develop in the present work for region (D) can be straightforwardly modified to interpolate $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ Gross–Prasad periods along one edge of region (d) (and its mirror-image (d')).

For weights in the "off-centre" regions (B) and (E), the complex L-value $L(\pi \times \sigma, s)$ vanishes to order precisely 1, due to the shape of the archimedean Γ-factors. Beilinson's conjecture predicts the existence of canonical motivic cohomology classes whose complex regulators are related to $L'(\pi \times \sigma, s)$; and we expect the images of these classes in p-adic étale cohomology to form Euler systems. For weights in region (E), an Euler system has been obtained in a recent work of Hsu, Jin and Sakamoto [Hsu et al. 2020]; and Loeffler and Zerbes [2021a] showed that the syntomic regulators of these classes are related to values (outside its domain of interpolation) of the p-adic L-function interpolating critical values in region (F). In the last section of this article, we discuss the kind of reciprocity law one can expect relating the cohomology classes of Hsu et al. [2020] with the p-adic L-function of this article. We hope to come back to this question in a forthcoming work.

Another prior work which treats p-adic interpolation of $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-values is the Ph.D. thesis of M. Agarwal [2007]. Agarwal's construction gives a one-variable p-adic L-function, which appears to correspond to the restriction of our 3-variable function to the line where $k_1 = k_2 = \ell = k$ for a parameter k, although his methods are very different from ours (using an Eisenstein series on the unitary group $U(3, 3)$).

After the release of the preprint version of this article, Z. Liu [2023] gave a construction of a new p-adic L-function in the case of $\mathrm{GSp}_4 \times \mathrm{GL}_2$, using the same integral representation as in [Agarwal 2007]; Liu's construction includes the cyclotomic variable and proceeds in a different way than ours, using the Klingen Eisenstein series on GU(2, 2) with Bessel models of representations of GSp_4. Also, Graham et al. [2023] have developed tools which are likely to allow the construction of a four variable p-adic L-function for regions (D) and (F).

One of the differences among the present paper and most of the works discussed above is that here we interpret Siegel modular forms in the H^1 of the Siegel modular variety, and not in the H^2 as in [Loeffler et al. 2021]; further, the GL_2-form is naturally seen as an element in the H^1 of the modular curve. These choices are a direct reflection of the different weight regions considered in this setting.

2. Setup: groups and Hecke parameters

2.1. *Groups.* We denote by G the group scheme GSp_4 (over $\boldsymbol{Z}$), defined with respect to the antidiagonal matrix

$$J = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix},$$

and we let ν be the multiplier map $G \to \boldsymbol{G}_m$. We define $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, which we embed into G via the embedding

$$\iota : \left[\begin{pmatrix} a & b\\ c & d \end{pmatrix}, \begin{pmatrix} a' & b'\\ c' & d' \end{pmatrix}\right] \mapsto \begin{pmatrix} a &&& b\\ & a' & b' &\\ & c' & d' &\\ c &&& d \end{pmatrix}.$$

We sometimes write h_i for the i-th GL_2 factor of H. We write T for the diagonal torus of G, which is contained in H and is a maximal torus in either H or G.

2.2. *Parabolics.* We write B_G for the upper-triangular Borel subgroup of G, and P_{Si} and P_{Kl} for the standard Siegel and Klingen parabolics containing B, so

$$P_{\mathrm{Si}} = \begin{pmatrix} \star&\star&\star&\star\\ \star&\star&\star&\star\\ &&\star&\star\\ &&\star&\star \end{pmatrix}, \quad P_{\mathrm{Kl}} = \begin{pmatrix} \star&\star&\star&\star\\ &\star&\star&\star\\ &\star&\star&\star\\ &&&\star \end{pmatrix}.$$

We write $B_H = \iota^{-1}(B_G) = \iota^{-1}(P_{\mathrm{Si}})$ for the upper-triangular Borel of H.

We have a Levi decomposition $P_{\mathrm{Si}} = M_{\mathrm{Si}} N_{\mathrm{Si}}$, with $M_{\mathrm{Si}} \cong \mathrm{GL}_2 \times \mathrm{GL}_1$, identified as a subgroup of G via

$$(A, u) \mapsto \begin{pmatrix} A & \\ & uA' \end{pmatrix}, \quad A' := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t A^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

In this paper P_{Si} and M_{Si} will be much more important than P_{Kl} and M_{Kl} (in contrast to [Loeffler et al. 2021]) so we shall often denote them simply by P and M. The intersection $B_M := M \cap B_G$ is the standard Borel of M; its Levi factor is T.

2.3. ***Coefficient sheaves.*** We retain the conventions about algebraic weights and roots as in [Loeffler and Zerbes 2021b]. In particular, we identify characters of T with triples of integers $(r_1, r_2; c)$, with $r_1 + r_2 = c$ modulo 2 corresponding to $\mathrm{diag}(st_1, st_2, st_2^{-1}, st_1^{-1}) \mapsto t_1^{r_1} t_2^{r_2} s^c$. With our present choices of Borel subgroups, a weight $(r_1, r_2; c)$ is dominant for H if $r_1, r_2 \geq 0$, dominant for M_G if $r_1 \geq r_2$, and dominant for G if both of these conditions hold. (We frequently omit the central character c if it is not important in the context.)

For our further use, we briefly recall the conventions of [Loeffler and Zerbes 2021b] about sheaves. The Weyl group acts on the group of characters $X^*(T)$ via $(w \cdot \lambda)(t) = \lambda(w^{-1} t w)$. As discussed in the same work, we can define explicitly $w_G^{\max}$, the longest element of the Weyl group, as well as $\rho = (2, 1; 0)$, which is half the sum of the positive roots for G. There is a functor from representations of P_G to vector bundles on the compactified Siegel Shimura variety; and we let $\mathcal{V}_\kappa$, for $\kappa \in X^\bullet(T)$ that is M_G-dominant, be the image of the irreducible M_G-representation of highest weight κ. Given an integral weight $\nu \in X^\bullet(T)$ such that $\nu + \rho$ is dominant, we define

$$\kappa_i(\nu) = w_i(\nu + \rho) - \rho, \quad 0 \leq i \leq 3,$$

here, as usual, ρ is half the sum of the positive roots and the w_i stand for the Kostant representatives of the Weyl group. These are the weights κ such that representations of infinitesimal character $\nu^\vee + \rho$ contribute to $R\Gamma(S_K^{G,\mathrm{tor}}, \mathcal{V}_\kappa)$, where $\nu^\vee$ is the dual weight of ν and the superscript "tor" stands for the toroidal compactification (for some choice of a projective cone decomposition). If ν is dominant (i.e., $r_1 \geqslant r_2 \geqslant 0$), they are the weights which appear in the *dual BGG complex* computing de Rham cohomology with coefficients in the algebraic G-representation of highest weight ν.

2.4. ***Hecke parameters.*** With the notation of the introduction, let π be a cuspidal automorphic representation of G, and let p be a prime. If π_{f} is unramified at p, we write $\alpha, \beta, \gamma, \delta$ for the Hecke parameters of π_p, and $P_p(X)$ for the polynomial $(1 - \alpha X) \dots (1 - \delta X)$. The Hecke parameters are algebraic integers over a number field E, and are well defined up to the action of the Weyl group. If π is non-CAP, which we shall assume,[1] then the Hecke parameters all have complex absolute value $p^{w/2}$, where $w := r_1 + r_2 + 3$, and they satisfy $\alpha\delta = \beta\gamma = p^w \chi_\pi(p)$, where $\chi_\pi(p)$ is a root of unity.

Let $\mathrm{Iw}_G(p)$ denote the Iwahori subgroup. We shall consider the following operators in the Hecke algebra of level $\mathrm{Iw}_G(p)$, acting on the cohomology of any of the sheaves introduced before:

- The Siegel operator $\mathcal{U}_{\mathrm{Si}} = [\mathrm{diag}(p, p, 1, 1)]$, its dual $\mathcal{U}'_{\mathrm{Si}} = [\mathrm{diag}(1, 1, p, p)]$.
- The Klingen operator $\mathcal{U}_{\mathrm{Kl}} = p^{-r_2} \cdot [\mathrm{diag}(p^2, p, p, 1)]$, as well as its dual $\mathcal{U}'_{\mathrm{Kl}} = p^{-r_2} \cdot [\mathrm{diag}(1, p, p, p^2)]$.
- The Borel operator $\mathcal{U}_B = \mathcal{U}_{\mathrm{Si}} \cdot \mathcal{U}_{\mathrm{Kl}}$, as well as its dual $\mathcal{U}'_B = \mathcal{U}'_{\mathrm{Si}} \cdot \mathcal{U}'_{\mathrm{Kl}}$.

3. Flag varieties and orbits

The key technical input into our interpolation results is a detailed study of certain loci in flag varieties for G and H, making explicit the theory of Boxer and Pilloni [2021] for the groups G and H, and studying how it interacts with restriction from G to H.

We use the usual Roman letters $X, Y, \ldots$ for algebraic varieties or schemes, while calligraphic letters $\mathcal{X}, \mathcal{Y}, \ldots$ or typewriter letters $\mathtt{X}, \mathtt{Y}, \ldots$ denote adic spaces.

3.1. *Kostant representatives.* We write FL_G for the Siegel flag variety $P\backslash G$, with its natural right G-action. There are four orbits for the Borel B_G acting on FL_G, represented by a subset of the Weyl group of G, the *Kostant representatives* (a distinguished set of representatives for the quotient $W_{M_G}\backslash W_G$ where M_G is the Levi of P). We denote these by $w_0, \ldots, w_3$; see [Loeffler and Zerbes 2021b] for explicit matrices.

Definition 3.1. As in [Boxer and Pilloni 2021, Section 3.1], we write $C^G_{w_i}$ for the orbit $P\backslash P w_i B_G$, a locally closed subvariety of FL_G of dimension i.

Remark 3.2. For $g \in G$, we can determine which cell $C^G_{w_i}$ contains the point $Pg \in FL_G$ via a criterion in terms of the span of the rows of the bottom left 2×2 submatrix of g, as in Remark 5.1.2 of [Loeffler and Zerbes 2021b].

Remark 3.3. Note that $C^G_{w_0}$ and $C^G_{w_3}$ are stable under P, while $C^G_{w_1} \sqcup C^G_{w_2}$ forms a single P-orbit.

[1] Equivalently (given our conditions on the weight), we require that π is not a Saito–Kurokawa lift; for lifts of this type, the ratio of two among the Hecke parameters is p, so they cannot all have the same absolute value. Including Saito–Kurokawa lifts would add extra technical complications in our theory; and excluding them is no loss anyway, since if π is such a lift, then $L(\pi \times \sigma, s)$ factors as a product of L-functions of GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$, whose p-adic interpolation is well understood.

Analogously, for the H-flag variety $FL_H = B_H\backslash H$, we have 4 Kostant representatives $w_{00} = \mathrm{id}$, $w_{10} = \left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \mathrm{id}\right)$, similarly w_{01}, w_{11} (with the cell $C^H_{w_{ij}}$ having dimension $i + j$). (This is the whole of the Weyl group of H, since the Levi subgroup of $M_H = T$ is trivial.)

Remark 3.4. Either for G or for H, each cell will determine a subspace of the Iwahori-level Shimura variety (as an adic space), via pullback along the Hodge–Tate period map. This is the locus where the relative position of the Hodge filtration and level structure on the p-divisible group lies in the given Bruhat cell. In particular, the “smallest” cell (w_0 or w_{00}) corresponds to the multiplicative locus, and the “largest” one to the étale locus.

3.2. ***A twisted embedding of flag varieties.*** Consider the elements

$$\tau = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \in M(\boldsymbol{Z}_p), \qquad \tau^\sharp = \iota(w_{01})^{-1} \tau w_2 \in G(\boldsymbol{Z}_p).$$

Note that τ was denoted γ in [Loeffler et al. 2021], but γ was also used for a Satake parameter, so we use a different letter here. The element τ represents the unique open T-orbit for the M-flag variety $B_M\backslash M$.

We will consider the translated embedding $\iota^\sharp : H \to G$ given by $h \mapsto \iota(h)\tau^\sharp$. The map $FL_H \to FL_G$ induced by $\iota^\sharp$ by construction sends $[w_{01}]$ to $[w_2]$. We also have projection maps $\pi_i : FL_H \to FL_{\mathrm{GL}_2} \cong \boldsymbol{P}^1$, and the product $(\iota^\sharp, \pi_1, \pi_2)$ evidently sends w_{01} to $([w_2], [\mathrm{id}], [w])$ (where the unlabelled w is the GL_2 long Weyl element).

If we equate $(x : y) \in \boldsymbol{P}^1$ with the orbit $Bg \in B\backslash G$, where g is any invertible matrix of the form $\left(\begin{smallmatrix} \star & \star \\ x & y \end{smallmatrix}\right)$, then $\iota^\sharp$ sends $((x : y), (X : Y))$ to

$$P_{\mathrm{Si}} \cdot \begin{pmatrix} \star & \cdots & \cdots & \star \\ \star & \cdots & \cdots & \star \\ -X & Y & \star & \star \\ -y & x & \star & \star \end{pmatrix}.$$

Using this and the explicit description of the Bruhat cells in terms of the bottom left corner of the matrix, we see that:

- The preimage of $C^G_{w_0}$ is empty.
- The preimage of $C^G_{w_1}$ is the point $((1 : 0), (0 : 1))$ (the image of $[w_{10}] \in FL_H$).
- The preimage of $C^G_{w_2}$ is a copy of the affine line, corresponding to points of the form

$$B_H\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix}\right) w_{01}.$$

Proof. The condition for the above matrix to lie in $X^G_{w_2}$ is that $\left(\begin{smallmatrix} -X & Y \\ -y & x \end{smallmatrix}\right)$ be singular, that is, $Xx = Yy$; and the condition for it to lie in $C^G_{w_2}$ is that the span of the rows not be $(0:1)$, so $X \neq 0$ and $y \neq 0$. So, without loss of generality we can take $X = 1$ and $y = 1$, leaving the equation $Y = x$; that is, our point was $B_H\left(\left(\begin{smallmatrix} \star & \star \\ x & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \star & \star \\ 1 & x \end{smallmatrix}\right)\right) = B_H\left(\left(\begin{smallmatrix} \star & \star \\ x & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \star & \star \\ -x & 1 \end{smallmatrix}\right)\right) w_{01}$. □

Notation. We write $X^G_w = \bigcup_{w' \leqslant w} C^G_{w'}$ (closed subvariety), and $Y^G_w = \bigcup_{w' \geqslant w} C^G_{w'}$ (open subvariety).

Proposition 3.5. *We have*

$$(\iota^\sharp)^{-1}(X^G_{w_2}) \cap \pi_2^{-1}(Y^{\mathrm{GL}_2}_w) = (\iota^\sharp)^{-1}(C^G_{w_2}) \cap \pi_1^{-1}(C^{\mathrm{GL}_2}_w \cdot w^{-1}) \cap \pi_2^{-1}(C^{\mathrm{GL}_2}_w).$$

(Note that the translate $C^{\mathrm{GL}_2}_w \cdot w^{-1}$ is the "big cell at the origin", $B \backslash B\bar{B}$.)

Proof. Since the single point $(\iota^\sharp)^{-1}(C^G_{w_0} \cup C^G_{w_1}) = [w_{10}]$ does not map to $Y^{\mathrm{GL}_2}_w$ under π_2, we conclude that $(\iota^\sharp)^{-1}(X^G_{w_2}) \cap \pi_2^{-1}(Y^{\mathrm{GL}_2}_w)$ is equal to $(\iota^\sharp)^{-1}(C^G_{w_2})$. We saw above that this subvariety is a copy of the affine line, and its image under the π_i is as stated. □

3.3. *Some tubes.* Let $\mathtt{FL}_G$ denote the analytification of FL_G, as an adic space over $\boldsymbol{Q}_p$, and similarly for H and for $G \times H$, so $\mathtt{FL}_{(G\times H)} = \mathtt{FL}_G \times \boldsymbol{P}^{1,\mathrm{an}} \times \boldsymbol{P}^{1,\mathrm{an}}$.

Now we define loci inside these spaces, using the tubes of various subvarieties of the special fibres. As usual $\mathcal{X}^G_w$ denotes the tube of $X^G_{w,\boldsymbol{F}_p}$ in $\mathtt{FL}_G$, etc. We shall set

$$\mathtt{Z}_0 = \overline{\mathcal{X}^G_{w_2}} \times \boldsymbol{P}^{1,\mathrm{an}} \times \overline{\mathcal{Y}^{\mathrm{GL}_2}_w} \quad \text{and} \quad \mathtt{U}_0 = \mathcal{Y}^G_{w_2} \times \mathcal{X}^{\mathrm{GL}_2}_{\mathrm{id}} \times \boldsymbol{P}^{1,\mathrm{an}}.$$

Then we have that $\mathtt{Z}_0$ is closed, $\mathtt{U}_0$ is open, and both are stable under the action of $\mathrm{Iw}_G \times \mathrm{Iw}_{\mathrm{GL}_2} \times \mathrm{Iw}_{\mathrm{GL}_2}$; and $\mathtt{U}_0 \cap \mathtt{Z}_0$ is a partial closure of the (w_2, id, w) Bruhat cell for $G \times \mathrm{GL}_2 \times \mathrm{GL}_2$.

We need to allow smaller "overconvergence radii", for which we use the action of the element $\eta_G = \mathrm{diag}(p^3, p^2, p, 1)$ and its cousin $\eta = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$.

Definition 3.6. Let us set $\mathtt{Z}_m = \mathtt{Z}^G_m \times \mathtt{Z}^H_m$, where

$$\mathtt{Z}^G_m = \overline{\mathcal{X}^G_{w_2}} \cdot \eta_G^m, \quad \mathtt{Z}^H_m = \boldsymbol{P}^{1,\mathrm{an}} \times \left(\overline{\mathcal{Y}^{\mathrm{GL}_2}_w} \cdot \eta^{-m} \begin{pmatrix} 1 & \boldsymbol{Z}_p \\ 0 & 1 \end{pmatrix}\right).$$

We have $\mathtt{Z}_0 \supseteq \mathtt{Z}_1 \supseteq \mathtt{Z}_2 \ldots$ by [Boxer and Pilloni 2021, Lemma 3.4.1], and $\mathtt{Z}_m$ is stable under $\mathrm{Iw}(p^t)$ for $t \geqslant 3m+1$.

On the other hand, we can define $\mathtt{U}_n = \mathtt{U}^G_n \times \mathtt{U}^H_n$, where

$$\mathtt{U}^G_n = \mathcal{Y}^G_{w_2} \cdot \eta_G^{-m} N_{B_G}(\boldsymbol{Z}_p), \quad \mathtt{U}^H_n = (\mathcal{X}^{\mathrm{GL}_2}_{\mathrm{id}} \cdot \eta^n) \times \boldsymbol{P}^{1,\mathrm{an}}.$$

Again, we have $\mathtt{U}_0 \supseteq \mathtt{U}_1 \supseteq \ldots$, and $\mathtt{U}_n$ is stable under $\mathrm{Iw}(p^t)$ for $t \geqslant n+1$.

3.4. ***Explicit coordinates.*** For $m \in \boldsymbol{Q}$, we define the subsets of the adic projective line given by

$$\mathcal{B}_m = \{|\cdot| : |z| \le p^m\}, \quad \overline{\mathcal{B}}_m = \bigcap_{m'<m} \mathcal{B}_{m'}, \quad \mathcal{B}_m^\circ = \bigcup_{m'>m} \mathcal{B}_{m'}, \quad \overline{\mathcal{B}}_m^\circ = \{|\cdot| : |z| < |p|^m\},$$

that satisfy the inclusions $\mathcal{B}_m^\circ \subset \overline{\mathcal{B}}_m^\circ \subset \mathcal{B}_m \subset \overline{\mathcal{B}}_m$.

We can identify the Zariski-open neighbourhood $U_{w_2} = P\backslash P\,\overline{P}w_2$ of $[w_2] \in FL_G$ with $\boldsymbol{A}^3$, via the map

$$P\backslash P \begin{pmatrix} 1 & & & \\ & 1 & & \\ x & y & 1 & \\ z & x & & 1 \end{pmatrix} w_2.$$

Then one computes that

$$\overline{\mathcal{X}_{w_2}^G} \cap U_{w_2}^{\mathrm{an}} = \{(x, y, z) : x \notin \mathcal{B}_0 \text{ or } y \notin \mathcal{B}_0 \text{ or } z \in \overline{\mathcal{B}}_0^\circ\},$$

and η_G preserves $U_{w_2}^{\mathrm{an}}$ and acts in these coordinates via $(x, y, z) \mapsto (p^{-1}x, p^{-3}y, pz)$. Thus

$$\mathtt{Z}_m^G \cap U_{w_2}^{\mathrm{an}} = \{(x, y, z) : x \notin \mathcal{B}_{-m} \text{ or } y \notin \mathcal{B}_{-3m} \text{ or } z \in \overline{\mathcal{B}}_m^\circ\},$$

and a similar computation identifies $\overline{\mathcal{Y}_w^{\mathrm{GL}_2}}$ with $\overline{\mathcal{B}}_0$, and $\overline{\mathcal{Y}_w^{\mathrm{GL}_2}} \cdot \eta^{-m}\left(\begin{smallmatrix} 1 & \boldsymbol{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ with $\overline{\mathcal{B}}_m + \boldsymbol{Z}_p$.

We can compute $\mathtt{U}_n^G$ in coordinates as

$$\mathtt{U}_n^G \cap U_{w_2}^{\mathrm{an}} = \{(x, y, z) : x \in \mathcal{B}_n + \boldsymbol{Z}_p, y \in \mathcal{B}_{3n} + \boldsymbol{Z}_p\},$$

with no condition on z; and the projection to the first GL_2 coordinate is just $\mathcal{B}_n^\circ$.

Lemma 3.7. *The intersection $\mathtt{Z}_m^G \cap \mathtt{U}_n^G$ is contained in $U_{w_2}^{\mathrm{an}}$ for all $m, n \geqslant 0$.*

Proof. It suffices to check this for $(m, n) = (0, 0)$; see Lemma 3.3.21 of [Boxer and Pilloni 2021]. □

3.5. ***Pullback to H.*** Guided by the zeta-integral computations in [Loeffler and Rivero 2024], we shall consider the map

$$\iota^{\sharp\sharp} : FL_H \to FL_G \times FL_H, \quad h \mapsto \left(\iota^\sharp(h), h_1 \begin{pmatrix} p^t & \\ & 1 \end{pmatrix}, h_2\right) \quad \text{for some } t \geqslant 1.$$

Proposition 3.8. *If $m > 3n \geqslant 0$, then*

$$(\iota^{\sharp\sharp})^{-1}(\mathtt{Z}_m \cap \mathtt{U}_n) = (\iota^{\sharp\sharp})^{-1}(\mathtt{Z}_m),$$

and in particular this preimage is closed in FL_H.

Proof. We know that the pullback of Z_0 is contained in the big cell, so we can compute it in coordinates. We find that the inequalities on (z_1, z_2) for it to land in Z_m are

$$z_1 + z_2 \in \overline{\mathcal{B}_m^\circ}, \quad z_2 \in \overline{\mathcal{B}_m} + \boldsymbol{Z}_p.$$

For $\mathrm{Z}_m \cap \mathrm{U}_n$ we add the extra inequalities

$$z_2 \in \mathcal{B}_{3n} + \boldsymbol{Z}_p, \quad p^t z_1 \in \mathcal{B}_n^\circ.$$

If $m > 3n$, then the latter equations are a consequence of the former. □

3.6. *Period maps and overconvergent cohomology.* We consider the analytifications $\mathcal{S}_{G,K} = (S_K \times \mathrm{Spec}(\boldsymbol{Q}_p))^{\mathrm{an}}$ and $\mathcal{S}_{G,K}^{\mathrm{tor}} = (S_{G,K}^{\mathrm{tor}} \times \mathrm{Spec}(\boldsymbol{Q}_p))^{\mathrm{an}}$, and similarly for H and $G \times H$ (denoted always by calligraphic letters). Write $\mathcal{S}_{G,K^p}^{\mathrm{tor}}$ for the perfectoid space $\varprojlim_{K_p} \mathcal{S}_{G,K^pK_p}^{\mathrm{tor}}$, which allows us to consider the Hodge–Tate period map

$$\pi_{\mathrm{HT},G}^{\mathrm{tor}} : \mathcal{S}_{G,K^p}^{\mathrm{tor}} \to \mathrm{FL}_G,$$

which for every open compact $K_p \subset G(\boldsymbol{Q}_p)$ descends to a map of topological spaces (see [Boxer and Pilloni 2021, Section 4.5])

$$\pi_{\mathrm{HT},G,K_p}^{\mathrm{tor}} : \mathcal{S}_{G,K^pK_p}^{\mathrm{tor}} \to \mathrm{FL}_G / K_p.$$

There are also analogous maps for H and $G \times H$, for which we use the same notation.

The Hodge–Tate period maps for G and H are related by a compatibility property analogous to Theorem 6.2.1 of [Loeffler and Zerbes 2021b]. To formulate this we need to introduce an auxiliary level structure $K_\diamond^H$, defined as follows.

Definition 3.9. Let $K_\triangle^H(p^t) = K_{\mathrm{Iw}}^H(p^t) \cap \tau^\sharp K_{\mathrm{Iw}}^G(p^t)(\tau^\sharp)^{-1}$, concretely given by

$$K_\triangle^H(p^t) = \left\{ h \in H(\boldsymbol{Z}_p) \;\middle|\; h = \left(\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix} \right) \bmod p^t \text{ for some } x, z \right\}.$$

We just put $\mathcal{S}_{H,\triangle}^{\mathrm{tor}}$ for $\mathcal{S}_{G,K}^{\mathrm{tor}}$ to represent the choice of $K_\triangle^H(p^t)$.

Proposition 3.10. *There is a commutative diagram of Hodge–Tate period maps*:

$$\begin{array}{ccc}
\mathcal{S}_{H,\mathrm{Iw}}^{\mathrm{tor}}(p^t) & \xrightarrow{\pi_{\mathrm{Iw}}^H} & \mathrm{FL}_H / K_{H,\mathrm{Iw}}(p^t) \\
\uparrow{\scriptstyle \mathrm{pr}_\triangle} & & \uparrow{\scriptstyle \mathrm{pr}_\triangle} \\
\mathcal{S}_{H,\triangle}^{\mathrm{tor}}(p^t) & \xrightarrow{\pi_\triangle^H} & \mathrm{FL}_H / K_\triangle^H(p^t) \\
\downarrow{\scriptstyle \iota^\sharp} & & \downarrow{\scriptstyle \iota^\sharp} \\
\mathcal{S}_{G,\mathrm{Iw}}^{\mathrm{tor}}(p^t) & \xrightarrow{\pi_{\mathrm{Iw}}^G} & \mathrm{FL}_G / K_{G,\mathrm{Iw}}(p^t)
\end{array}$$

The choices of neighbourhoods we have made are sufficient to get the maps working, including the compatibility with the classical cohomology via compact-support cohomology of $\mathtt{Z}_m$. Let $\mathcal{U}_n^G \subset \mathcal{S}_{G\times H,\mathrm{Iw}}(p^t)$ and $\mathcal{Z}_m^H \subset \mathcal{U}_n^H \subset \mathcal{S}_{H,\triangle}(p^t)$ denote the preimages of the subsets $\mathtt{U}_n^G \subset \mathtt{FL}^G$ and $\mathtt{Z}_m^H \subset \mathtt{U}_n^H \subset \mathtt{FL}^H$ under the Hodge–Tate period maps $\pi^{\mathrm{tor}}_{\mathrm{HT},G\times H,\mathrm{Iw}}$ and $\pi^{\mathrm{tor}}_{\mathrm{HT},H,\triangle}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
R\Gamma(\mathcal{U}_0^G, \mathcal{V}) & \longrightarrow & R\Gamma(\mathcal{S}^{\mathrm{tor}}_{H,\triangle}(p^t), (\iota^\sharp)^*(\mathcal{V})) \\
\uparrow & & \uparrow \\
R\Gamma_{\mathcal{Z}_m}(\mathcal{U}_0^G, \mathcal{V}) & \longrightarrow & R\Gamma_{\mathcal{Z}_m^H}(\mathcal{S}^{\mathrm{tor}}_{H,\triangle}(p^t), (\iota^\sharp)^*(\mathcal{V})) \\
\downarrow & \nearrow & \\
R\Gamma_{\mathcal{Z}_m}(\mathcal{U}_n^G, \mathcal{V}) & &
\end{array}$$

Here, the horizontal maps correspond to $(\iota^\sharp)^*$, while the vertical ones are the usual restriction and corestriction maps.

4. Branching laws and sheaves of distributions

In this section, we introduce the necessary tools to p-adically interpolate the automorphic vector bundles associated to representations of the Levi subgroups M_G and M_H, which are the coefficient systems for the cohomology we study. We keep the notation of [Loeffler and Zerbes 2021b, Section 8] and review some of the more relevant results of the same work, focusing on the changes we need in our setting. Note in particular that the discussions and results of [Loeffler and Zerbes 2021b, Section 6] hold verbatim, with the obvious changes in Proposition 6.4.1.

Along this section, we will frequently consider the projections of the embedding $\iota^{\sharp\sharp}$ on each factor: the first component corresponds to $\iota^\sharp : FL_H \to FL_G$, and the second, referred to as ι_p, is the map

$$\iota_p : FL_H \to FL_H, \quad (h_1, h_2) \mapsto \left(h_1 \begin{pmatrix} p^t & \\ & 1 \end{pmatrix}, h_2 \right).$$

We also write $\upsilon = \upsilon(p^t) = \left(\left(\begin{smallmatrix} p^t & \\ & 1 \end{smallmatrix} \right), 1 \right) \in M(\mathbf{Z}_p)$.

4.1. ***Torsors.*** We begin this section recalling a general procedure for Tate-twisting proétale torsors, referring the reader to [Graham 2024, Section 4.2] for a more extensive discussion on the main properties of this operation. Let $L/\mathbb{Q}_p$ be a finite extension and $\mathcal{X}/L$ a smooth adic space. Let $\mathcal{T}^\times \to \mathcal{X}$ denote the proétale $\mathbb{Z}_p^\times$-torsor parametrising isomorphisms (of proétale sheaves) $\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p(1)$. The action of $\mathbb{Z}_p^\times$

is given as follows: for $\lambda \in \mathbb{Z}_p^\times$ and $\phi : \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p(1)$, we set

$$\phi \cdot \lambda = \phi(\lambda \cdot -).$$

Let M be a smooth adic group over $\operatorname{Spa} L$ and suppose that we have a homomorphism $\mu : \mathbb{Z}_p^\times \to M$ whose image is contained in the centre of M.

Definition 4.1. Let $\mathcal{M} \to \mathcal{X}$ be a (right) proétale M-torsor. We define the twist of $\mathcal{M}$ along μ to be

$${}^{\mu}\mathcal{M} := \mathcal{M} \times^{[\mathbb{Z}_p^\times, \mu]} \mathcal{T}^\times,$$

where the right-hand side is the quotient of $\mathcal{M} \times_{\mathcal{X}} \mathcal{T}^\times$ by the equivalence relation

$$(m \cdot \mu(\lambda), \phi) \sim (m, \phi \cdot \lambda^{-1}) \quad \text{for all } m \in \mathcal{M},\ \phi \in \mathcal{T}^\times,\ \lambda \in \mathbb{Z}_p^\times.$$

This defines a proétale M-torsor ${}^{\mu}\mathcal{M} \to \mathcal{X}$ via the action $(m, \phi) \cdot n = (m \cdot n, \phi)$ for $m \in \mathcal{M},\ \phi \in \mathcal{T}^\times$ and $n \in M$.

The map $x \mapsto x^{-1} : G \to FL_G$ allows us to regard G as a right P_G-torsor over FL_G, and similarly to regard $G/N_G \to FL_G$ as a right M_G-torsor. We consider their analytifications

$$\mathtt{P}^G : \mathcal{G} \to \mathtt{FL}_G \quad \text{and} \quad \mathtt{M}^G : \mathcal{G}/\mathcal{N}_G \to \mathtt{FL}_G,$$

which are torsors over $\mathtt{FL}_G$ under the analytic groups $\mathcal{P}_G$ and $\mathcal{M}_G$ respectively. We similarly define torsors over the flag varieties H, H_1 and H_2.

Definition 4.2. Define $\mathcal{P}^G_{\mathrm{HT}}$ and $\mathcal{M}^G_{\mathrm{HT}}$ to be the pullbacks via π^G_{HT} of the torsors $\mathtt{P}^G$ and $\mathtt{M}^G$; there are right torsors over $\mathcal{S}_{G,\mathrm{Iw}}(p^t)$ for the groups $\mathcal{P}_G$ and $\mathcal{M}_G$. We similarly define $\mathcal{P}^H_{\mathrm{HT}}$ and $\mathcal{M}^H_{\mathrm{HT}}$, $\mathcal{P}^{H_i}_{\mathrm{HT}}$ and $\mathcal{M}^{H_i}_{\mathrm{HT}}$ for $i = 1, 2$.

Using Definition 4.1 we can define ${}^{\mu}\mathcal{P}^G_{\mathrm{HT}}$, ${}^{\mu}\mathcal{M}^G_{\mathrm{HT}}$ and the analogous twisted objects for the torsors corresponding to H and H_i.

Definition 4.3. For $n > 0$, let $\mathcal{M}^1_{G,n}$ be the group of elements which reduce to the identity modulo p^n. Define

$$\mathcal{M}^{\square}_{G,n} = \mathcal{M}^1_{G,n} \cdot B_{M_G}(\mathbf{Z}_p),$$

which is an affinoid analytic subgroup containing $\mathrm{Iw}_{M_G}(p^n)$. A similar definition applies to $M_H = T$; we write the group as $\mathcal{T}^{\square}_n = T(\mathbf{Z}_p)\mathcal{T}^1_n$.

Consider in the same way

$$\mathcal{T}^{\diamond}_n = \{\mathrm{diag}(t_1, t_2, \nu t_2^{-1}, \nu t_1^{-1}) \in \mathcal{T}^{\square}_n : t_1 - t_2 \in \mathcal{B}_n\}.$$

As in [Loeffler and Zerbes 2021b, Proposition 7.2.5], we also consider the étale torsors ${}^{\mu}\mathcal{M}^G_{\mathrm{HT},n}$, ${}^{\mu}\mathcal{M}^G_{\mathrm{HT},n,\mathrm{Iw}}$ and ${}^{\mu}\mathcal{M}^H_{\mathrm{HT},n,\diamond}$ arising as the reduction of structure of the torsors ${}^{\mu}\mathcal{M}^G_{\mathrm{HT}}$ over $\mathcal{U}^G_n$, ${}^{\mu}\mathcal{M}^H_{\mathrm{HT}}$ over $\mathcal{U}^H_{\mathrm{Iw},n}$ and ${}^{\mu}\mathcal{M}^H_{\mathrm{HT}}$ over $\mathcal{U}^H_n$, respectively.

The following proposition is the key statement allowing for p-adic variation, and it mainly follows from the theory developed in [Boxer and Pilloni 2021, Section 4.6].

Proposition 4.4. *We have an equality of $\mathcal{M}^{\square}_{G,n}$-torsors over $\mathcal{U}^{H}_{n,\diamond}$:*

$$(\iota^{\sharp})^{*}({}^{\mu}\mathcal{M}^{G}_{\mathrm{HT},n,\mathrm{Iw}}) = {}^{\mu}\mathcal{M}^{H}_{\mathrm{HT},n,\diamond} \times^{[\mathcal{T}_n^{\diamond},\tau]} \mathcal{M}^{\square}_{G,n},$$

where we regard $\mathcal{T}_n^{\diamond}$ as a subgroup of $\mathrm{Iw}_{M_G}(p^t)\mathcal{M}^{1}_{G,n}$ via conjugation by τ.

Proof. This follows by checking the analogous statement on the flag variety, noting that there is a commutative diagram of adic space:

$$\begin{array}{ccc} K^{H}_{\triangle}(p^t)\mathcal{H}^1_n & \longrightarrow & K^{G}_{\mathrm{Iw}}(p^t)\mathcal{G}^1_n \\ \downarrow & & \downarrow \\ \mathcal{B}^H\backslash\mathcal{B}^H w_{01} K^{H}_{\triangle}(p^t)\mathcal{H}^1_n & \longrightarrow & \mathcal{P}^G\backslash\mathcal{P}^G w_2 K^{G}_{\mathrm{Iw}}(p^t)\mathcal{G}^1_n \end{array}$$

Here, the vertical maps are $h \mapsto \mathcal{B}^H\backslash\mathcal{B}^H h^{-1}$ (on the left) and $g \mapsto \mathcal{P}^G\backslash\mathcal{P}^G w_2 g^{-1}$ (on the right); the lower horizontal map is $\iota^{\sharp}$ is $\mathcal{B}^H h \mapsto \mathcal{P}^G h\tau w_2$, and the map along the top making the diagram commute is $h \mapsto (\tau^{\sharp})^{-1} h \tau^{\sharp}$.

Then we may conclude as in [Loeffler and Zerbes 2021b, Proposition 7.2.7]. □

A straightforward adaptation of these techniques can be applied to the second factor ι_p, yielding to an equality of $\mathcal{M}^{\square}_{H,n}$-torsors over $\mathcal{U}^{H}_{n,\diamond}$,

$$(\iota_p)^{*}({}^{\mu}\mathcal{M}^{H}_{\mathrm{HT},n,\mathrm{Iw}}) = {}^{\mu}\mathcal{M}^{H}_{\mathrm{HT},n,\diamond} \times^{[\mathcal{T}_n^{\diamond},\upsilon]} \mathcal{M}^{\square}_{H,n},$$

where we regard $\mathcal{T}_n^{\diamond}$ as a subgroup of $\mathrm{Iw}_{M_H}(p^t)\mathcal{M}^{1}_{H,n}$ via conjugation by υ. Observe that the conjugation by υ does not introduce denominators in any element of M_H, and hence the previous objects are well defined.

4.2. *Analytic characters and analytic inductions.*

Definition 4.5. Let $n \in \boldsymbol{Q}_{>0}$. We say that a continuous character $\kappa : \boldsymbol{Z}_p^{\times} \to A^{\times}$, for (A, A^{+}) a complete Tate algebra, is n-analytic if it extends to an analytic A-valued function on the affinoid adic space

$$\boldsymbol{Z}_p^{\times} \cdot \mathcal{B}_n \subset \boldsymbol{G}_m^{\mathrm{ad}}.$$

This definition extends to characters $T(\boldsymbol{Z}_p) \to A^{\times}$: the n-analytic characters are exactly those which extend to $\mathcal{T}_n^{\square}$.

Let $n_0 > 0$ and assume that $\kappa_A : T(\boldsymbol{Z}_p) \to A^{\times}$ is an n_0-analytic character. For $? \in \{G, H\}$ and $n \geq n_0$, let $\mathcal{M}^{1}_{?,n}$ be the affinoid subgroup of $\mathcal{M}_?$ defined above, and let B_{M_G} be the Borel of $M_?$.

Definition 4.6. For $n \geq n_0$, define

$$V_{G,\kappa_A}^{n-\mathrm{an}} = \mathrm{anInd}_{(\mathcal{M}_{G,n}^{\square}\cap\mathcal{B}_G)}^{(\mathcal{M}_{G,n}^{\square})}(w_{0,M?}\kappa_A)$$
$$= \{f \in \mathcal{O}(\mathcal{M}_{G,n}^{\square})\hat{\otimes}A : f(mb) = (w_{0,M}\kappa_A)(b^{-1})f(m)\}$$

for all $m \in \mathcal{M}_{G,n}^{\square}$ and for all $b \in \mathcal{M}_{G,n}^{\square} \cap \mathcal{B}_G$.

We define a left action of $\mathcal{M}_{G,n}^{\square}$ on $V_{G,\kappa_A}^{n-\mathrm{an}}$ by $(h \cdot f)(m) = f(h^{-1}m)$.

Write $D_{G,\kappa_A}^{n-\mathrm{an}}$ for the dual space, and $\langle \cdot, \cdot \rangle$ for the pairing between these; we equip $D_{G,\kappa_A}^{n-\mathrm{an}}$ with a left action of the same group $\mathcal{M}_{G,n}^{\square}$ such that $\langle h\mu, hf\rangle = \langle \mu, f\rangle$.

4.3. *Branching laws in families.* Recall that for a Tate algebra A endowed with an n_0-analytic character $\kappa_A : T(\mathbf{Z}_p) \to A^\times$ as above, and additionally with a character $\lambda : (1+\mathcal{B}_n)^\times \to A^\times$, we may define a special vector in $V_{G,\kappa_A}^{n-\mathrm{an}}$ (referred to as the "krakenfish" in [Loeffler and Zerbes 2021b]) by the formula $\mathfrak{K}^\lambda(z) = \lambda(1+z)$.

The following lemma is analogous to [Loeffler and Zerbes 2021b, Lemma 8.3.2], but recall that now the objects involved in the definition of $\mathcal{T}_n^\diamond$ are different.

Lemma 4.7. *The function $\mathfrak{K}^\lambda$ is an eigenvector for $(\tau^\sharp)^{-1}\mathcal{T}_n^\diamond\tau^\sharp$, with eigencharacter $w_{0,M}\,\kappa_A + (\lambda, -\lambda; 0)$.*

Proof. This follows from the same argument that has been done in [Loeffler and Zerbes 2021b, Section 8.3] once we note that the element τ lies in the Siegel parabolic subgroup, and that only the projection to the Levi subgroup matters for the purpose of this computation. □

The following result is a straightforward consequence of the previous lemma.

Proposition 4.8. *Pairing with $\mathfrak{K}^\lambda$ defines a homomorphism of $\mathcal{T}_n^\diamond$-representations*

$$(\iota^\sharp)^*(D_{G,\kappa_A}^{n-\mathrm{an}}) \to D_{H,w_{01,M}\kappa_A+(\lambda,-\lambda;0)}^{n-\mathrm{an}}.$$

4.4. *Labelling of weights.* As above, let (A, A^+) be a Tate algebra over $(\mathbf{Q}_p, \mathbf{Z}_p)$. Given a weight $\nu_A : T(\mathbf{Z}_p) \to A^\times$ for some coefficient ring A, we may define $\kappa_A : T(\mathbf{Z}_p) \to A^\times$ by

$$\kappa_A = -w_{0,M}\,w_2(\nu+\rho) - \rho.$$

If ν_A is $(\nu_1, \nu_2; \omega)$ for some $\nu_i, \omega : \mathbf{Z}_p^\times \to A^\times$, then $\kappa_A = (\nu_1, -2-\nu_2; \omega)$. Its Serre dual is $\kappa'_A = (\kappa_A + 2\rho_{\mathrm{nc}})^\vee$. This can be written as $(\nu_2 - 1, -3-\nu_1; c) = w_2(\nu_A+\rho) - \rho$.

4.5. *Sheaves on G.* Let $1 \leq n < t$ be integers.

Definition 4.9 [Loeffler and Zerbes 2021b, Definition 9.2.1]. The sheaf $\mathcal{V}_{G,\nu_A}^{n-\mathrm{an}}$ over $\mathcal{U}_n^G$ is given by the product

$$\mathcal{V}_{G,\nu_A}^{n-\mathrm{an}} = {}^{\mu}\mathcal{M}_{\mathrm{HT},n,\mathrm{Iw}}^{G} \times^{M_{G,n}^{\square}} V_{G,\kappa_A}^{n-\mathrm{an}}.$$

We define similarly another sheaf $\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A}$ by

$$\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A} = {}^{\mu}\mathcal{M}^{G}_{\mathrm{HT},n,\mathrm{Iw}} \times^{M^{\square}_{G,n}} D^{n-\mathrm{an}}_{G,(\kappa_A+2\rho_{\mathrm{nc}})}.$$

As discussed in [Loeffler and Zerbes 2021b, Definition 9.2.1], the sheaves $\mathcal{V}^{n-\mathrm{an}}_{G,\nu_A}$ and $\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A}$ are sheaves of A-modules compatible with base-change in A. If $A = \boldsymbol{Q}_p$ and $\nu_A = (r_1, r_2; c)$ for integers $r_1 \geq r_2 \geq -1$, we have classical comparison maps

$$\mathcal{V}_{G,\kappa_1} \hookrightarrow \mathcal{V}^{n-\mathrm{an}}_{G,\nu_A}, \quad \mathcal{D}^{n-\mathrm{an}}_{G,\nu_A} \twoheadrightarrow \mathcal{V}_{G,(\kappa_A+2\rho_{\mathrm{nc}})^{\vee}} = \mathcal{V}_{G,\kappa_2}.$$

4.6. *Sheaves on H.* We mimic the same definitions for H, using $w_{01} \in W_H$ in place of w_2. Given an n-analytic character τ_A, we define $\kappa^H_A = -\tau_A - 2\rho_H$, and set

$$\mathcal{V}^{n-\mathrm{an}}_{H,\diamond,\nu_A} = {}^{\mu}\mathcal{M}^{H}_{\mathrm{HT},n,\diamond} \times^{\mathcal{T}^{\diamond}_n} V^{n-\mathrm{an}}_{H,\kappa^H_A} \quad \text{and} \quad \mathcal{D}^{n-\mathrm{an}}_{H,\diamond,\tau_A} = {}^{\mu}\mathcal{M}^{H}_{\mathrm{HT},n,\mathrm{Iw}} \times^{\mathcal{T}^{\diamond}_n} D^{n-\mathrm{an}}_{H,(\kappa^H_A+2\rho_H)}.$$

4.7. *Branching for sheaves.*

Definition 4.10. We say that A-valued, n-analytic characters ν_A and τ_A of $T(\boldsymbol{Z}_p)$ are compatible if $\nu_A = (\nu_1, \nu_2; \nu_1 + \nu_2)$, $\tau_A = (\tau_1, \tau_2; \nu_1 + \nu_2)$ for some characters ν_i, τ_i of $\boldsymbol{Z}_p^{\times}$, and we have the relation

$$\tau_1 - \tau_2 = \nu_1 - \nu_2 - 2.$$

If ν_A, τ_A are compatible, then taking $\lambda = \nu_1 - \tau_1 = \nu_2 - \tau_2 + 2$, we obtain a homomorphism of $\mathcal{T}^{\diamond}_n$-representations

$$D^{n-\mathrm{an}}_{G,(\kappa_A+2\rho_{\mathrm{nc}})} \to D^{n-\mathrm{an}}_{H,-\tau_A}.$$

Proposition 4.11. *Pairing with $\mathfrak{K}^{\lambda}$ induces a morphism of sheaves over $\mathcal{U}^H_n$:*

$$(\iota^{\sharp})^*(\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A}) \to \mathcal{D}^{\mathrm{an}}_{H,\diamond,\tau_A},$$

which is compatible with specialisation in A, and if $A = \boldsymbol{Q}_p$ and $\nu = (r_1, r_2; r_1+r_2)$, $\tau = (t_1, t_2; r_1 + r_2)$ are algebraic weights with $r_1 - r_2 \geq 0$ and $r_i, t_i \geq -1$, then this morphism is compatible with the map of finite dimensional sheaves $(\iota^{\sharp})^*(\mathcal{V}_{\kappa_2}) \to \mathcal{V}^H_{\tau}$, where $\mathcal{V}_{\kappa_2}$ is as in [Loeffler and Rivero 2024, Section 3].

Proof. This follows immediately from the results of Section 4.5. □

4.8. *Locally analytic overconvergent cohomology.* We adopt the same definitions regarding cuspidal, locally analytic, overconvergent cohomology of [Loeffler and Zerbes 2021b, Section 9.5]. In particular,

$$R\Gamma^{G}_{w,\mathrm{an}}(\nu_A, \mathrm{cusp})^{-,\mathrm{fs}} = R\Gamma_{\mathcal{I}^G_{mn}}(\mathcal{U}^G_n, \mathcal{D}^{n,-\mathrm{an}}_{G,\nu_A}(-D_G))^{-,\mathrm{fs}}$$

and similarly for the noncuspidal version. Here, "$-$, fs" is the finite-slope part for the dual Hecke operators $\mathcal{U}'_{\mathrm{Si}}$ and $\mathcal{U}'_{\mathrm{Kl}}$. This complex is independent of m, n and t, and is concentrated in degrees [0, 1, 2].

Given ν_A and τ_A satisfying $\tau_1 - \tau_2 = \nu_1 - \nu_2 - 2$, the previous discussion means that we have a morphism of complexes of A-modules:

$$(\iota^\sharp)^* : R\Gamma^G_{w,\mathrm{an}}(\nu_A, \mathrm{cusp})^{-,\mathrm{fs}} \to R\Gamma_{\mathcal{Z}^H_m}(\mathcal{U}^H_n, \mathcal{D}^{n-\mathrm{an}}_{H,\diamond,\tau_A}(-D_H)). \tag{2}$$

The map ι_p induces in the same way a morphism of sheaves over $\mathcal{U}^H_n$ and an analogous morphism at the level of complexes of A-modules.

4.9. ***Pairings and duality.*** We may define

$$R\Gamma_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{+,\dagger} = \varinjlim R\Gamma(\mathcal{Z}^H_{m,\mathrm{Iw}}(p^t), \mathcal{V}^{\mathrm{an}}_{H,\mathrm{Iw},\tau_A}).$$

The following theorem will be crucially used in the definition of the p-adic L-function. It can be understood as a statement about cup products of overconvergent cohomology on $\mathcal{S}_H$.

Theorem 4.12. *The cup product induces a pairing*

$$H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A, \mathrm{cusp})^{-,\dagger} \times H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{+,\dagger} \to A,$$

whose formation is compatible with base-change in A, and it is also compatible with the Serre duality pairing on classical cohomology when $A = \boldsymbol{Q}_p$ and ν, τ are classical weights.

Proof. The map is defined using the pairing between the cohomology groups $H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A, \mathrm{cusp})^{-,\dagger}$ and $H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{+,\dagger}$. The result in the current form follows from [Boxer and Pilloni 2021, Theorem 6.7.1], from where it is clear that the pairing is compatible with Serre duality for each classical weight. □

4.10. ***A Künneth formula for cohomology with support.*** In order to define the p-adic L-function, we need to p-adically interpolate the cohomological pairing between H^0 and H^1. This may be regarded as a Künneth formula for cohomology with support.

Proposition 4.13. *The cup product induces a pairing*

$$\begin{aligned} H^0_{w_0,\mathrm{an}}(\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}(p^t), \tau_1)^\dagger \times H^1_{w_1,\mathrm{an}}(\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}(p^t), \tau_2, \mathrm{cusp})^\dagger \\ \to H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{-,\dagger}, \end{aligned} \tag{3}$$

where $\tau_A = (\tau_1, \tau_2)$ is a weight for H.

Proof. This follows from [Loeffler and Zerbes 2021b, Theorem 9.6.2] by the general theory as in [Boxer and Pilloni 2021, Theorem 6.7.1]. □

5. The p-adic L-function

In this section we discuss how to use higher Coleman theory to reinterpret the Harris–Su pairing, as discussed in [Loeffler and Rivero 2024, Section 3], in coherent cohomology over certain strata in suitable adic Shimura varieties. In particular, this analysis allows us to perform p-adic interpolation provided that there exist families of cohomology classes interpolating the different elements involved there. We implicitly use some of the results discussed in [Loeffler et al. 2021, Sections 9, 10], as well as Novodvorsky's formula and its interpretation in coherent cohomology discussed in [Loeffler and Rivero 2024].

If not specified otherwise, π and σ are cohomological cuspidal automorphic representations of GSp_4 and of GL_2, defined over some number field E, both globally generic and unramified outside a certain finite set. Let L be some p-adic field with an embedding from E.

5.1. ***Tame test data.*** As in [Loeffler and Zerbes 2021b, Section 10.2], we fix the following data:

- M_0, N_0 are positive integers coprime to p with $M_0^2 \mid N_0$, and χ_0 is a Dirichlet character of conductor M_0 (valued in L).
- M_2, N_2 are positive integers coprime to p with $M_2 \mid N_2$, and χ_2 is a Dirichlet character of conductor M_2 (valued in L).

As usual, we use the hat to denote the adelic counterpart of the characters. We shall consider automorphic representations π of G with conductor N_0 and central character $\hat{\chi}_0$ (up to twists by norm); here "conductor" is the analytic conductor of the associated degree 4 L-function, which always satisfies the divisibility $M_0^2 \mid N_0$. We assume similarly that the representation σ of GL_2 has conductor N_2 and character $\hat{\chi}_2$ (up to twists by norm).

Let S denote the set of primes dividing N_0N_2. By tame test data we mean a pair $\gamma_S = (\gamma_{0,S}, \Phi_S)$ such that:

- $\gamma_{0,S} \in G(\boldsymbol{Q}_S)$, where $\boldsymbol{Q}_S = \prod_{\ell \in S} \boldsymbol{Q}_\ell$.
- $\Phi_S \in C_c^\infty(\boldsymbol{Q}_S^2, L)$, lying in the $(\hat{\chi}_0\hat{\chi}_2)^{-1}$-eigenspace for $\boldsymbol{Z}_S^\times$, where the action is as described in [Loeffler et al. 2022, Section 3].

We let K_S be the quasiparamodular subgroup (in the sense of [Okazaki 2019]) of $G(\boldsymbol{Q}_S)$ of level (N_0, M_0), so that π has one-dimensional invariants under K_S; and we let $\hat{K}_S$ be the open compact subgroup of $G(\boldsymbol{Q}_S)$ defined in [Loeffler and Zerbes 2021b, Section 10.2]. We also use analogous notation for K^p and $\hat{K}^p$, the prime-to-p part of the level and its adelic counterpart, respectively.

5.2. ***p-adic families.*** We use the conventions regarding p-adic families of [Loeffler and Zerbes 2021b, Section 10.4]. In particular, we consider $U \subset \mathcal{W}^2$ an open affinoid disc, and let $\boldsymbol{r}_1, \boldsymbol{r}_2 : \boldsymbol{Z}_p^\times \to \mathcal{O}(U)^\times$ be the universal characters associated to the two factors of $\mathcal{W}^2$. Let ν_U be the character $(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_1 + \boldsymbol{r}_2)$ of $T(\boldsymbol{Z}_p)$.

Definition 5.1. A *family of automorphic representations* $\underline{\pi}$ of tame level N_0 and character χ_0 over an open affinoid disc U is the data of a finite flat covering $\widetilde{U} \to U$ and a homomorphism $\widetilde{U} \to \mathcal{E}$ lifting the inclusion $U \hookrightarrow \mathcal{W}$, such that:

(a) $\widetilde{U}$ is 2-dimensional and smooth.

(b) The restriction of $H^k(\mathcal{M}^{\bullet,-,fs}_{\mathrm{cusp},w_j})$ to $\widetilde{U}$ is zero if $j + k \neq 3$, and the sheaves $S^k(\underline{\pi}) = H^k(\mathcal{M}^{\bullet,-,fs}_{\mathrm{cusp},w_{3-k}})$ are either free over $\mathcal{O}(\widetilde{U})$ of rank 1 for all k (general-type), or free of rank 1 for $k = 1, 2$ and zero for $k = 0, 3$ (Yoshida-type).

(c) The centre of $G(\boldsymbol{A}_\mathrm{f}^p)$ acts on $S^k(\underline{\pi})$ by $|\cdot|^{-(\boldsymbol{r}_1+\boldsymbol{r}_2)}\hat{\chi}_0$.

Assume that the representation π can be interpolated along a finite-slope overconvergent p-adic family of automorphic representations $\underline{\pi}$ over the open affinoid disc U introduced at the beginning of the section. (Given any cohomological π with sufficiently small slope at p, the theory of eigenvarieties guarantees that we can always find a sufficiently small open disc around the weight of π such that this holds.) Let $S^1(\underline{\pi}) = H^1(\mathcal{M}^{\bullet,-,fs}_{\mathrm{cusp},w_2})$ be the sheaf [Loeffler and Zerbes 2021b, Definition 10.4.1], which is free of rank 1 according to the definition we have made. We shall then choose a basis $\underline{\xi}$ of that space. Since the spaces of higher Coleman theory have an action of $G(\boldsymbol{A}_\mathrm{f}^p)$, we can make sense of $\gamma_{0,S} \cdot \underline{\xi}$ as a family of classes at tame level $\hat{K}^p$, which is still an eigenfamily for the Hecke operators away from S.

Definition 5.2. A point $P \in U(L)$ is nice for $\underline{\pi}$ if the weight of P is $(r_1, r_2) \in U \cap \boldsymbol{Z}^2$ with $r_1 \geq r_2 \geq 0$ and the specialisation at P of the system of eigenvalues $\lambda_{\underline{\pi}}^-$ attached to the family $\underline{\pi}$ is the character of a p-stabilised automorphic representation π_P, which is cuspidal, globally generic, and has conductor N_0 and character χ_0.

This implies that the fibre of $S^1(\underline{\pi})$ at P maps isomorphically to the π_P-eigenspace in the classical $H^1(K^p, \kappa_1(\nu), \mathrm{cusp})$; in particular, this eigenspace is 1-dimensional. By the classicality theorems for higher Coleman theory, given a family $\underline{\pi}$, all specialisations of integer weight (r_1, r_2) with $r_1 - r_2$ and r_2 sufficiently large relative to the slope of $\underline{\pi}$ will be nice; and if $\underline{\pi}$ is ordinary, it suffices to assume that $r_1 - r_2 \geq 3$ and $r_2 \geq 0$.

We can consider analogous objects for GL_2. In particular, we may choose a disc $U' \subset \mathcal{W}$ and a finite-slope overconvergent p-adic family of modular eigenforms $\mathcal{G}$ over U' (of weight $\boldsymbol{\ell} + 2$ where $\boldsymbol{\ell}$ is the universal character associated to U'). We also impose that the corresponding spaces $S^0(\mathcal{G})$ and $S^1(\mathcal{G})$ are free of rank 1. Then, we say a point $Q \in U'$ is nice for $\mathcal{G}$ if it lies above an integer $\ell \in U' \cap \boldsymbol{Z}_{\geq 0}$, and the specialisation of $\mathcal{G}$ at Q is a classical form. We further require that the fibre of $S^1(\underline{\sigma})$ at Q maps isomorphically to the σ_Q-eigenspace in the classical H^1 (and in particular, this eigenspace is 1-dimensional). We write σ_ℓ for the corresponding automorphic representation, with the normalisations in [Loeffler and Zerbes 2021b, Definition 10.4.1]. As before, we shall take a basis $\underline{\eta}$ of $S^1(\underline{\sigma})$.

Remark 5.3. The inequalities defining region (D) automatically imply that we are not dealing with noncohomological weights, and hence we do not need to consider an étale covering, as it was the case for region (F).

5.3. ***Construction of the imprimitive p-adic L-function.*** We refer to [Loeffler et al. 2021, Section 7.4] for the construction of the p-adic family of Eisenstein series $\mathcal{E}^{\Phi^{(p)}}(0, \boldsymbol{t}-1)$, which depends on a prime-to-p Schwartz function $\Phi^{(p)}$. According to [Loeffler and Zerbes 2021b, Proposition 10.1.2], it is an overconvergent cusp form of weight $\boldsymbol{t}$ which may be understood as an element in the overconvergent H^0 of $\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}$. As discussed at the end of Section 10.2 of the same reference, $\Phi^{(p)}$ and Φ_S agree up to multiplication by the characteristic function of $\hat{\boldsymbol{Z}}^{S\cup\{p\}}$.

Recall the pairing (3). From now on, let $A = \mathcal{O}(U \times U')$. Next, we can consider

$$\mathcal{E}^{\Phi^{(p)}}(0, \boldsymbol{t}-1) \boxtimes G(\chi_2^{-1})\underline{\eta} \in H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^2), \tau_A)^{+,\dagger},$$

where $\boldsymbol{t} = \boldsymbol{r}_2 - \boldsymbol{r}_1 + \boldsymbol{\ell} - 2$ and the tame level is taken to be $H \cap \hat{K}^p$.

Remark 5.4. The Gauss sum can be normalised away by rescaling $\underline{\eta}$ if the coefficient field L contains a root of unity of order M_2, but we do not assume this here.

Definition 5.5. We let $\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\pi \times \sigma; \underline{\xi}; \underline{\eta})$ denote the element of A defined by

$$\langle (\iota^\sharp)^*(\gamma_{0,S} \cdot \underline{\xi}), \mathcal{E}^{\Phi^{(p)}}(0, \boldsymbol{t}-1) \boxtimes G(\chi_2^{-1})\underline{\eta} \rangle,$$

where we are using the pairing of Theorem 4.12.

This is a three-variable p-adic L-function, where we may vary the weights (r_1, r_2) and we keep the linear condition in terms of (r_1, r_2, ℓ, t), namely $\boldsymbol{t} = \boldsymbol{r}_2 - \boldsymbol{r}_1 + \boldsymbol{\ell} - 2$ (alternatively, $s = \frac{1}{2}(\ell - r_1 + r_2 - 2)$).

Definition 5.6.
- We say a point (P, Q) of $U \times U'$ is nice if $P = (r_1, r_2)$ and $Q = (\ell)$ are integer points, with P nice for $\underline{\pi}$ and Q nice for $\underline{\sigma}$.
- We say that (P, Q) is nice critical if we also have $\ell \le r_1 - r_2 + 1$ (the specialisation t of $\boldsymbol{t}$ at (P, Q) is ≥ -1).
- If instead we have $r_1 - r_2 \le \ell - 2 \le r_1$, we say that P is nice geometric.

5.4. ***The correction term Z_S.*** This section introduces a correction term Z_S which depends on the choice of local data, and which will arise in the interpolation property of the p-adic L-function. Its definition depends on certain Whittaker models properly introduced in [Loeffler and Rivero 2024, Section 6]; since this will have a minor relevance in this work, we just refer the interested reader to our previous paper. Following the notation in [Loeffler and Rivero 2024] we may consider the integral $Z(W, \Phi_1, W^{(\ell)}; s)$.

We set

$$Z_S(\pi \times \sigma, \gamma_S; s) = \frac{Z(\gamma_{0,S} \cdot W_0^{\mathrm{new}}, \Phi_S, W_2^{\mathrm{new}}; s)}{G(\chi_2^{-1}) \prod_{\ell \in S} L(\pi_\ell \times \sigma_\ell, s)},$$

and

$$Z_S(\pi \times \sigma, \gamma_S) = Z_S\Big(\pi \times \sigma, \gamma_S; 1 + \frac{t}{2}\Big),$$

where $t = r_2 - r_1 - 2 + \ell$, as usual, and $G(\chi_2^{-1})$ is the Gauss sum of the character χ_2. Note that for any given π and σ, one can choose γ_S such that $Z_S(\pi \times \sigma, \gamma_S; s) \neq 0$ (this follows from the definition of the L-factor as a GCD of local zeta-integrals).

5.5. *Interpolation property.* We choose a $\overline{\boldsymbol{Q}}$-basis ξ of the new subspace of $H^1(\pi_{\mathrm{f}})$, where $H^1(\pi_{\mathrm{f}})$ is the copy of π_{f} appearing in the degree 1 coherent cohomology of the Siegel Shimura variety. Analogously, we also choose a $\overline{\boldsymbol{Q}}$-basis η of the new subspace of $H^1(\sigma_{\mathrm{f}})$. We write $S^1(\pi, L)$ for the cohomology with L-coefficients, which is an L-vector space.

The element $\xi \otimes \eta$ is an explicit multiple of the standard Whittaker function, and the corresponding multiple defines a complex period $\Omega_\infty(\pi, \sigma) \in \boldsymbol{C}^\times$.

Definition 5.7. Given nonzero $\xi \in S^1(\underline{\pi}, L)$ and $\eta \in S^1(\underline{\sigma}, L)$, we define periods $\Omega_p(\underline{\pi}, \underline{\sigma}) \in L^\times$ and $\Omega_\infty(\underline{\pi}, \underline{\sigma}) \in \boldsymbol{C}^\times$ as in [Loeffler et al. 2021, Section 10.2]. (These periods do depend on the choices of ξ and η up to multiplication by $L^\times$, but we drop that dependence from the notation). We write $\Omega_p(\pi_P, \sigma_Q)$ and $\Omega_\infty(\pi_P, \sigma_Q)$ for the specialization of the periods at (P, Q).

More precisely, the space of Whittaker-E-rational classes is exactly $\Omega_\infty(\pi, \sigma) \cdot H^1(\pi_{\mathrm{f}}) \otimes H^1(\sigma_{\mathrm{f}})$ for a nonzero constant $\Omega_\infty(\pi, \sigma) \in \boldsymbol{C}^\times$ (the Whittaker period).

Below we establish the interpolation property for the p-adic L-function; observe that the algebraicity of the right-hand side was the main result of [Loeffler and Rivero 2024]. Recall the degree-8 Euler factor $\mathcal{E}^{(d)}$ of [Loeffler and Zerbes 2021c], that we call $\mathcal{E}^{(D)}$ in the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ setting of [Loeffler and Rivero 2024].

Theorem 5.8. *The p-adic L-function $\mathcal{L}_{p,\gamma_S}^{\mathrm{imp}}(\underline{\pi} \times \underline{\sigma})$ has the following interpolation property: if (P, Q) is nice critical, with P of weight (r_1, r_2) and Q of weight ℓ, then*

$$\frac{\mathcal{L}_{p,\gamma_S}^{\mathrm{imp}}(\underline{\pi} \times \underline{\sigma})(P, Q)}{\Omega_p(\pi_P, \sigma_Q)} = Z_S(\pi_P \times \sigma_Q, \gamma_S) \cdot \mathcal{E}^{(D)}(\pi_P \times \sigma_Q) \cdot \frac{G(\chi_2^{-1}) \Lambda\big(\pi_P \times \sigma_Q, \frac{1}{2}(\ell - r_1 + r_2 - 2)\big)}{\Omega_\infty(\pi_P, \sigma_Q)},$$

where $\Lambda(\pi_P \times \sigma_Q, s)$ is the completed (complex) L-function and $G(\chi_2^{-1})$ is the Gauss sum of χ_2^{-1}.

Proof. By construction, we have

$$\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})(P, Q) = G(\chi_2^{-1})\langle (\iota^\sharp)^*(\gamma_{0,S} \cdot \xi_P), \mathcal{E}^{\Phi^{(p)}}(0, t-1) \boxtimes \eta_Q \rangle.$$

Along the region given by $\ell - t = r_1 - r_2 + 2$, this expands as the product of $G(\chi_2^{-1})\Lambda\big(\pi_P \times \sigma_Q, \frac{t}{2}\big)$ and a product of normalised local zeta-integrals. The local zeta-integral at p has been evaluated in [Loeffler and Rivero 2024, Section 7] and gives the desired Euler factor. The product of zeta-integrals at the bad primes is by definition $G(\chi_2^{-1}) Z_S(\cdots)$. □

Remark 5.9. Taking into account the discussions of [Loeffler and Rivero 2024, Remark 7.14], it is possible to use this same method to get an improved p-adic L-function where the interpolation property involves a degree seven Euler factor. Further, following the recent work by Graham et al. [2023], it should be possible to extend the previous construction to a p-adic L-function in all four variables.

6. A conjectural reciprocity law

6.1. ***Slope conditions.*** We now recall various notions of *slope* associated to automorphic representations of G and H. Given a cohomological automorphic representation π of G such that π_p has nonzero invariants under Iw_G (with a chosen embedding of its coefficient field into $\overline{\mathbf{Q}}_p$), and a simultaneous eigenspace in $(\pi_p)^{\mathrm{Iw}_G}$ for the operators $\mathcal{U}'_{\mathrm{Si}}$ and $\mathcal{U}'_{\mathrm{Kl}}$, we define the *slope* of this eigenspace to be the pair of rational numbers $\lambda(\mathcal{U}'_{\mathrm{Si}})$, $\lambda(\mathcal{U}'_{\mathrm{Kl}})$ which are the valuations of the eigenvalues for these operators. These slopes play a central role in the classicity criteria as in [Boxer and Pilloni 2021]. (One can use either the usual Hecke operators $\mathcal{U}_{\mathrm{Si}}$ and $\mathcal{U}_{\mathrm{Kl}}$, or the dual operators $\mathcal{U}'_{\mathrm{Si}}$ and $\mathcal{U}'_{\mathrm{Kl}}$, since the same eigenvalues appear for both choices.)

If $\lambda(\mathcal{U}'_{\mathrm{Si}}) = 0$ we say the eigenspace is *Siegel-ordinary*, and similarly *Klingen-ordinary*; and we say that π is Borel-ordinary at p if it is both Siegel- and Klingen-ordinary. The condition of being Siegel ordinary at p may be rephrased by requiring that $v_p(\alpha) = 0$, and being Klingen-ordinary is equivalent to $v_p(\alpha\beta) = r_2 + 1$.

For a cuspidal automorphic representation σ of GL_2, write $\mathfrak{a}$, $\mathfrak{b}$ for the Hecke parameters of σ_p (that is, the parameters corresponding to the action of the dual Hecke operators). We adopt the convention that $v_p(\mathfrak{a}) \le v_p(\mathfrak{b})$ and say that σ is Borel-ordinary at p (with respect to v) if $v_p(\mathfrak{a}) = 0$.

The hypotheses of the classicity theorems in [Boxer and Pilloni 2021] require two (slightly different) notions of "small slope", which we make explicit here. We consider the Hecke operators with the previously discussed normalisations acting on the cohomology of the sheaves $\mathcal{V}_\kappa$. Thus each operator is "minimally integrally normalised" acting on the classical cohomology (slopes are ≥ 0). Write K^p for some fixed choice of open compact away from p. Conjecture 5.9.2 of [Boxer and

Pilloni 2021] predicts lower bounds for the slopes of the Hecke operators acting on the overconvergent cohomology complexes $R\Gamma_w(K^p, \kappa)^{\pm}$ and $R\Gamma(K^p, \kappa, \mathrm{cusp})^{\pm}$, whose precise definitions are given in the same work; there are similar conjectures for the locally analytic cohomology complexes.

For $w \in W_G$, we compute the character $w^{-1} w_G^{\max}(\kappa + \rho) - \rho$ and find out how it pairs with the antidominant cocharacters $\mathrm{diag}(1, 1, x, x,)$ and $\mathrm{diag}(1, x, x, x^2)$ defining the operators $\mathcal{U}'_{\mathrm{Si}}$ and $\mathcal{U}'_{\mathrm{Kl}}$. We take $\kappa = \kappa_2 = (r_2 - 1, -r_1 - 3; r_1 + r_2)$, and subtract r_2 from all entries in the bottom row since this is our normalising constant for $\mathcal{U}'_{\mathrm{Kl}}$. Following the approach in [Boxer and Pilloni 2021, Section 5.11], below we summarise the conjectural slope bounds.

	id	w_1	(w_2)	w_3
$\mathcal{U}'_{\mathrm{Si}}$	$r_1 + 2$	0	(0)	$r_2 + 1$
$\mathcal{U}'_{\mathrm{Kl}}$	$r_1 - r_2 + 1$	$r_1 - r_2 + 1$	(0)	0

We do not know this conjecture in full, but from [Boxer and Pilloni 2021, Theorems 5.9.6, 6.8.3], we do know a weaker statement in which we replace $w^{-1} w_G^{\max}(\kappa_2 + \rho) - \rho$ with $w^{-1} w_G^{\max} \kappa_2$. This gives the following bounds.

	id	w_1	(w_2)	w_3
$\mathcal{U}'_{\mathrm{Si}}$	$r_1 + 2$	-1	(-1)	$r_2 - 2$
$\mathcal{U}'_{\mathrm{Kl}}$	$r_1 - r_2 + 1$	$r_1 - r_2 + 1$	(-3)	-3

The following proposition discusses the conditions of "small slope" and "strictly small slope". The reason for introducing different slope conditions is that the conditions needed to obtain a vanishing theorem are not the same as those needed to obtain classicality theorems; further, there are different kinds of control theorems requiring distinct sets of hypotheses.

Proposition 6.1. *For the weight $\kappa_2 = (r_2 - 1, -r_1 - 3; r_1 + r_2)$ with $r_1 \geq r_2 \geq 0$, we have*:

- *The "small slope" condition* $(-, \mathrm{ss}^M(\kappa_2))$ *is*
$$\lambda(\mathcal{U}'_{\mathrm{Si}}) < r_1 + 2, \quad \lambda(\mathcal{U}'_{\mathrm{Kl}}) < r_1 - r_2 + 1.$$
- *The "strictly small slope" condition* $(-, \mathrm{sss}^M(\kappa_2))$ *is*
$$\lambda(\mathcal{U}'_{\mathrm{Si}}) < r_1 + 2, \quad \lambda(\mathcal{U}'_{\mathrm{Kl}}) < r_1 - r_2 - 2.$$

Proof. This follows from the bounds given previously. □

6.2. ***Ordinary filtrations at p.*** Along the rest of this section we assume that π is both Klingen and Siegel ordinary and that σ is Borel ordinary. This is done just with the purpose of simplifying notation; similar conjectures can be formulated in the more general strictly small-slope setting, but one needs to use the theory of (φ, Γ)-modules over the Robba ring (rather than actual subrepresentations of Galois representations). Further, one expects to be able to formulate integral refinements in the ordinary setting, using Coleman maps instead of the Perrin–Riou map, and making $\mathcal{L}_p$ a p-adic measure instead of just a distribution. We begin by discussing the slope conditions.

Associated with the family $\underline{\pi}$ we have a family of Galois representations $V(\underline{\pi})$, which is a rank 4 $\mathcal{O}(U)$-module with an action of $\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})$, unramified outside pN_0 and with a prescribed trace for Frob_ℓ^{-1}, when $\ell \nmid pN_0$. The Galois representation $V(\underline{\pi})$ has a decreasing filtration by $\mathcal{O}(U)$-submodules stable under $\mathrm{Gal}(\overline{\boldsymbol{Q}}_p/\boldsymbol{Q}_p)$. Borrowing the notation from [Loeffler and Zerbes 2021b, Section 11], we write $\mathcal{F}^i V(\underline{\pi})$ for the codimension i subspace, and similarly for its dual $V(\underline{\pi})^*$. Similarly, there is a 2-step filtration for $V(\underline{\sigma})$. See, e.g., [Loeffler and Zerbes 2021c, Section 9.1] for a precise account of the feature of the different filtrations involved in the picture.

Definition 6.2. We set

$$\mathbb{V}^* = V(\underline{\pi})^* \otimes V(\underline{\sigma})(-1-\boldsymbol{r}_1),$$

and we let

$$\mathcal{F}^{(D)} V(\underline{\pi} \times \underline{\sigma})^* = (\mathcal{F}^1 V(\underline{\pi})^* \otimes \mathcal{F}^1 V(\underline{\sigma})^*) + (\mathcal{F}^3 V(\underline{\pi})^* \otimes V(\underline{\sigma})^*)$$

and

$$\mathcal{F}^{(E)} V(\underline{\pi} \times \underline{\sigma})^* = (\mathcal{F}^1 V(\underline{\pi})^* \otimes \mathcal{F}^1 V(\underline{\sigma})^*) + (\mathcal{F}^2 V(\underline{\pi})^* \otimes V(\underline{\sigma})^*).$$

For a nice weight (P, Q) we write $\mathbb{V}^*_{P,Q}$ for the specialisation of $\mathbb{V}^*$ at (P, Q), so $\mathbb{V}^*_{P,Q} = V(\pi_P)^* \otimes V(\sigma_Q)^*(-1-r_1)$ if $P = (r_1, r_2)$.

In particular, $\mathcal{F}^{(E)}$ has rank 5, $\mathcal{F}^{(D)}$ has rank 4, and the quotient $\mathrm{Gr}^{(e/d)}$ is isomorphic to

$$\mathrm{Gr}^{(E/D)} \cong (\mathrm{Gr}^2\, V(\underline{\pi})^*) \otimes (\mathrm{Gr}^0\, V(\underline{\sigma})^*)(-1-\boldsymbol{r}_1).$$

While Loeffler and Zerbes [2021b] were interested in the quotient $(\mathrm{Gr}^1\, V(\underline{\pi})^*) \otimes (\mathrm{Gr}^1\, V(\underline{\sigma})^*)$, here we are using a different step of the filtration. This is because Loeffler and Zerbes [2021b] were comparing the regions (e) and (f), while here the contrast is between (e) and (d), so the filtrations involved in each factor are different.

6.3. ***p-adic periods and p-adic Eichler–Shimura isomorphisms.*** The representations $\mathrm{Gr}^2\, V(\underline{\pi})(-2-\boldsymbol{r}_1)$ and $\mathrm{Gr}^0\, V(\underline{\sigma})$ are unramified, and hence crystalline as $\mathcal{O}(U)$ (resp. $\mathcal{O}(U')$)-linear representations. Since $\boldsymbol{D}_{\mathrm{cris}}(\boldsymbol{Q}_p(1))$ is canonically $\boldsymbol{Q}_p$, we can therefore define $\boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^{(e/d)}\,\mathbb{V}^*)$ to be an alias for the rank 1 $\mathcal{O}(U \times U')$-module

$$\boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^2\, V(\pi)^*(-2-\boldsymbol{r}_1))\,\hat{\otimes}\,\boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^0\, V(\underline{\sigma})^*).$$

We can then define a Perrin–Riou big logarithm for $\mathrm{Gr}^{(e/d)}\,\mathbb{V}^*$, which is a morphism of $\mathcal{O}(U \times U')$-modules:

$$\mathcal{L}^{\mathrm{PR}} : H^1(\boldsymbol{Q}_p, \mathrm{Gr}^{(e/d)}\,\mathbb{V}^*) \to \boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^{(e/d)}\,\mathbb{V}^*).$$

For nice geometric weights P, this specialises to the Bloch–Kato logarithm map, up to an Euler factor; and for nice critical weights is specialises to the Bloch–Kato dual exponential.

Let P be a nice weight. There is an *Eichler–Shimura* isomorphism

$$\mathrm{ES}^1_{\pi_P} : S^1(\pi_P, L) \cong \boldsymbol{D}_{\mathrm{cris}}\big(\mathrm{Gr}^2(V(\pi_P))\big).$$

Similarly, for GL_2 we have an isomorphism

$$\mathrm{ES}^1_{\sigma_Q} : S^1(\sigma_Q, L) \cong \boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^0\, V(\sigma_Q)).$$

In this case, the existence of a comparison in families is known after Kings et al. [2017], that is, there exists an isomorphism of $\mathcal{O}(U')$-modules:

$$\mathrm{ES}^1_{\underline{\sigma}} : S^1(\underline{\sigma}) \cong \boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^0\, V(\underline{\sigma}))$$

interpolating the isomorphism $\mathrm{ES}^1_{\sigma_Q}$ for varying Q, where $\mathcal{S}^1(\underline{\sigma})$ is the $\mathcal{O}(U')$-module spanned by $\underline{\eta}$.

6.4. ***Euler system classes.*** Suppose that the character $\chi_0\chi_2$ is nontrivial. Then, by the results in [Hsu et al. 2020], associated to the data γ_S, we have a family of cohomology classes

$$z_m(\underline{\pi} \times \underline{\sigma}, \gamma_S) \in H^1(\boldsymbol{Q}(\mu_m), \mathbb{V}^*)$$

for all square-free integers coprime to some finite set T containing both p and the ramified primes. The image of $z_m(\underline{\pi} \times \underline{\sigma}, \gamma_S)$ under localisation at p lands in the image of the injective map from the cohomology of $\mathcal{F}^{(E)}\mathbb{V}^*$ and we can therefore make sense of

$$\mathcal{L}^{\mathrm{PR}}(z_m(\underline{\pi} \times \underline{\sigma}), \gamma_S) \in \boldsymbol{D}_{\mathrm{cris}}(\mathrm{Gr}^{(e/f)}\,\mathbb{V}^*).$$

In this setting, we expect the following result.

Conjecture 6.3. *Under the running assumptions, the equality*

$$\big\langle \mathcal{L}^{\mathrm{PR}}(z_1(\underline{\pi} \times \underline{\sigma}, \gamma_S))(P, Q), \mathrm{ES}^1_{\pi_P}(\xi_P) \otimes \mathrm{ES}^1_{\sigma_Q}(\eta_Q)\big\rangle = \mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})(P, Q)$$

holds for all (P, Q) *in the geometric range.*

The main difficulty for proving the theorem following an analogous strategy to the case of region (F) is the lack of semistable models for the different Shimura varieties involved in this picture (Siegel level). We hope that a better understanding of higher Coleman theory following the new results of Boxer and Pilloni could lead to a proof of the previous conjecture.

Acknowledgements

The authors would like to thank Sarah Zerbes for informative conversations related to this work, and to Andrew Graham for some comments on an earlier version of the paper. They express gratitude to the referee, whose comments contributed to improve the presentation and to correct some inaccuracies.

References

[Agarwal 2007] M. K. Agarwal, *p-adic L-functions for* GSp(4) × GL(2), Ph.D. thesis, University of Michigan, 2007, available at https://www.proquest.com/docview/304842333. MR

[Boxer and Pilloni 2021] G. Boxer and V. Pilloni, "Higher Coleman theory", preprint, 2021. arXiv 2110.10251

[Graham 2024] A. Graham, "On the p-adic interpolation of unitary Friedberg–Jacquet periods", *Algebra Number Theory* **18**:6 (2024), 1117–1188. MR

[Graham et al. 2023] A. Graham, V. Pilloni, and J. R. Jacinto, "p-adic interpolation of Gauss–Manin connections on nearly overconvergent modular forms and p-adic L-functions", preprint, 2023. arXiv 2311.14438

[Hsu et al. 2020] C.-Y. Hsu, Z. Jin, and R. Sakamoto, "Euler systems for $GSp_4 \times GL_2$", preprint, 2020. arXiv 2011.12894

[Kings et al. 2017] G. Kings, D. Loeffler, and S. L. Zerbes, "Rankin–Eisenstein classes and explicit reciprocity laws", *Camb. J. Math.* **5**:1 (2017), 1–122. MR

[Liu 2023] Z. Liu, "p-adic L-functions for GSp(4) × GL(2)", preprint, 2023. arXiv 2308.08533

[Loeffler and Rivero 2024] D. Loeffler and O. Rivero, "Algebraicity of L-values for $GSp_4 \times GL_2$ and $GSp_4 \times GL_2 \times GL_2$", *Q. J. Math.* **75**:2 (2024), 391–412. MR

[Loeffler and Zerbes 2021a] D. Loeffler and S. L. Zerbes, "On p-adic regulators for GSp(4) × GL(2) and GSp(4) × GL(2) × GL(2)", preprint, 2021. arXiv 2011.15098

[Loeffler and Zerbes 2021b] D. Loeffler and S. L. Zerbes, "On the Bloch–Kato conjecture for GSp(4) × GL(2)", preprint, 2021. arXiv 2106.14511

[Loeffler and Zerbes 2021c] D. Loeffler and S. L. Zerbes, "p-adic L-functions and diagonal cycles for GSp(4) × GL(2) × GL(2)", preprint, 2021. arXiv 2011.15064

[Loeffler et al. 2021] D. Loeffler, V. Pilloni, C. Skinner, and S. L. Zerbes, "Higher Hida theory and p-adic L-functions for GSp_4", *Duke Math. J.* **170**:18 (2021), 4033–4121. MR

[Loeffler et al. 2022] D. Loeffler, C. Skinner, and S. L. Zerbes, "Euler systems for GSp(4)", *J. Eur. Math. Soc.* (*JEMS*) **24**:2 (2022), 669–733. MR

[Okazaki 2019] T. Okazaki, "Local Whittaker-newforms for GSp(4) matching to Langlands parameters", preprint, 2019. arXiv 1902.07801

Received April 13, 2024. Revised January 29, 2025.

DAVID LOEFFLER
MATHEMATICS INSTITUTE
UNIVERSITY OF WARWICK
COVENTRY
UNITED KINGDOM
Current address:
UNIDISTANCE SUISSE
BRIG
SWITZERLAND
david.loeffler@unidistance.ch

ÓSCAR RIVERO
SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE
BERKELEY, CA
UNITED STATES
Current address:
DEPARTMENT OF MATHEMATICS
UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
COMPOSTELA
SPAIN
oscar.rivero@usc.es

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 335 No. 2 April 2025
