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This paper lays the foundation for the study of the saturated 2-fusion systems in which the centralizer of some fully centralized involution has a component whose center is nontrivial.

The results in this paper are part of a program to, first, classify a large subclass of the class of simple 2-fusion systems of component type, and then, second, to use the theorem on fusion systems to simplify the proof of the theorem classifying the finite simple groups. See [4; 5] for a description of the program; there is also a bit of discussion of the program below.

Let p be a prime and S a finite p-group. A *fusion system* on S is a category $\mathcal{F}$ whose objects are the subgroups of S and, for subgroups P, Q of S, the set $\hom_{\mathcal{F}}(P, Q)$ of morphisms from P to Q is a set of injective group homomorphisms from P to Q, and that set satisfies two weak axioms. The standard example is the fusion system $\mathcal{F}_S(G)$ for G a finite group and $S \in \mathrm{Syl}_p(G)$, whose morphisms are those induced via conjugation in G. A fusion system is *saturated* if it satisfies two more axioms easily seen to hold in the standard example using Sylow's theorem. See [12] for notation, terminology, and basic definitions and results on fusion systems.

Let $\mathcal{F}$ be a saturated fusion system on a finite 2-group S. Proceeding by analogy with finite groups, one can define the notion of a *normal subsystem* of $\mathcal{F}$, which can then be used to define the notions of *simple* and *quasisimple* systems, *subnormal subsystems* of $\mathcal{F}$, and the set $\mathrm{Comp}(\mathcal{F})$ of *components* of $\mathcal{F}$. For t an involution in S the *centralizer* $C_{\mathcal{F}}(t)$ of t in $\mathcal{F}$ is defined, and if t is *fully centralized* (i.e., $|C_S(t)| \geq |C_S(x)|$ for each conjugate x of t) then $C_{\mathcal{F}}(t)$ is saturated, so we can define $\mathrm{Comp}(C_{\mathcal{F}}(t))$.

Define $\mathfrak{C}(\mathcal{F})$ to be the set of *components of centralizers of involutions* in $\mathcal{F}$; that is, $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ if there exists some involution $t \in S$ and a conjugate $(\bar{t}, \bar{\mathcal{C}})$ of $(t, \mathcal{C})$ such that $\bar{t}$ is fully centralized and $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{F}}(\bar{t}))$; we write $\mathcal{I}(\mathcal{C})$ for the set of such involutions t. We say that $\mathcal{F}$ is of *component type* if $\mathfrak{C}(\mathcal{F})$ is nonempty.

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Our main theorem is a contribution to the case where some $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is *intrinsic*: that is, $Z(\mathcal{C})$ is nontrivial.

Theorem 1. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ has a fully normalized Sylow group. In addition assume the following*:

(a) *$\mathcal{C}$ is realized by some $K \in \mathcal{K}_{\mathrm{qs}}$.*

(b) *$\mathcal{C}$ is maximal in $\mathfrak{C}(\mathcal{F})$.*

(c) *$\mathcal{I}(\mathcal{C}) \cap Z(\mathcal{C}) \neq \varnothing$.*

Then one of the following holds:

(1) *$\mathcal{C}$ is a component of $\mathcal{F}$.*

(2) *$\mathcal{C}$ is nearly standard, so $\widetilde{\mathcal{X}}(\mathcal{C})$ has a unique maximal member Q, and one of the following holds*:

  (i) *$Q = Z(K)$ is of order 2.*

  (ii) *$K \cong \Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $\epsilon \equiv q \bmod 4$, $\mathcal{C}$ is standard, and $Q \cong \mathbb{Z}_4$.*

  (iii) *$K/Z(K) \cong L_3(4)$ and $Q = Z(K) \cong E_4$.*

The definition of a *maximal member* of $\mathfrak{C}(\mathcal{F})$ appears in Notation 6.1.12 in [5]. The definition of a *standard or nearly standard subsystem* appears in Section 9.1 of [5]. The definition of $\widetilde{\mathcal{X}}(\mathcal{C})$ appears in Notation 6.1.2 of [5]. $\mathcal{K}_{\mathrm{qs}}$ is a class of quasisimple groups with centers of even order, defined in Definition 1.3 and listed in Definition 1.3, 1.4, and 1.5; the remaining quasisimple groups with centers of even order have been treated elsewhere.

1. Intrinsic components

Definition 1.1. Let K be a quasisimple group with center of even order, $\overline{K} = K/Z(K)$, and $\bar{x}$ an involution in $\overline{K}$. Following Definition 5.5.1 in [14], define $\bar{x}$ to *split relative to K* if the coset $\bar{x}$ contains an involution. Define $\bar{x}$ to be *stable relative to K* if $\bar{x}$ splits and for some involution y in $\bar{x}$, $C_{\mathrm{Aut}(K)}(\bar{x}) = C_{\mathrm{Aut}(K)}(y)$.

Definition 1.2. We extend a notion in [8] to quasisimple groups: define K to be 2-*small* if for $T \in \mathrm{Syl}_2(K)$ and $\overline{T} \leq S \in \mathrm{Syl}_2(\mathrm{Aut}(K))$, we have $C_S(T) = \overline{Z(T)}$ with $|Z(T) : Z(K)| = 2$.

Definition 1.3. Define $\mathcal{K}_{\mathrm{qs}}$ to be the collection of known quasisimple groups K with $Z(K)$ of even order and $K/Z(K)$ not Goldschmidt, other than

(a) $\hat{A}_n$, $n \geq 5$,

(b) K of Lie type of odd characteristic other than $\Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$,

(c) $\widehat{\mathrm{Sp}_6(2)}$,

(d) $K/Z(K) \cong L_3(4)$ with $\Phi(Z(K)) \neq 1$.

We recall that, from [6], the 2-fusion systems of all the groups $\Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$ appearing in Definition 1.3(b) are isomorphic. Thus in this case we may take K to be $\Omega_6^-(3)$.

1.4. *A quasisimple group K with $K/Z(K)$ sporadic is in $\mathcal{K}_{\mathrm{qs}}$ precisely when*

(1) *$K/Z(K)$ is M_{12}, M_{22}, J_2, Co_1, HS, Suz, Ru, F_{22}, or F_2, and*

(2) *either $|Z(K)| = 2$ or $K/Z(K) \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$.*

Proof. See 6.4.2 in [5]. □

1.5. *A quasisimple group with $K/Z(K) \in \mathrm{Chev}(2)$ is in $\mathcal{K}_{\mathrm{qs}}$ precisely when one of the following holds*:

(1) *$K/Z(K) \cong G_2(4)$ or $F_4(2)$ and $|Z(K)| = 2$.*

(2) *$K/Z(K)$ is $U_6(2)$, $\Omega_8^+(2)$, or ${}^2E_6(2)$ and $Z(K) \cong \mathbb{Z}_2$ or E_4.*

(3) *$K/Z(K) \cong L_3(4)$ and $Z(K) \cong \mathbb{Z}_2$ or E_4.*

Proof. See 6.4.3 in [5]. □

1.6. *Let $K \in \mathcal{K}_{\mathrm{qs}}$, $\overline{K} = K/Z(K)$, and $\bar{z}$ a 2-central involution in $\overline{K}$. Then one of the following holds*:

(1) *$\bar{z}$ is stable relative to K.*

(2) *$\overline{K} \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$.*

(3) *$\overline{K} \cong \Omega_8^+(2)$.*

Proof. This follows from Proposition 6.4.2 in [14]. □

1.7. *Let $K \in \mathcal{K}_{\mathrm{qs}}$ satisfy one of the following*:

(1) *$\overline{K} \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$.*

(2) *$\overline{K} \cong \Omega_8^+(2)$.*

Then for $T \in \mathrm{Syl}_2(K)$, $Z(T) = Z(K)$.

Proof. In each case, $Z(\overline{T}) = \langle \bar{z} \rangle$ is of order 2. Therefore either $Z(T) = Z(K)$ or $\overline{Z(T)} = Z(\overline{T})$. But no involution in $Z(\overline{T})$ is stable from the remark following Proposition 6.4.2 in [14], so the lemma follows. □

1.8. *Let $K \in \mathcal{K}_{\mathrm{qs}}$ and assume*

(1) *neither of the exceptional cases in 1.7 hold, and*

(2) *$K/Z(K)$ is not $G_2(4)$, $F_4(2)$, or $L_3(4)$.*

Then K is 2-small.

Proof. Let $T \in \mathrm{Syl}_2(K)$ and Z be the preimage of $Z(\overline{T})$ in K. By (1) and 1.6, all involutions in $Z(\overline{T})$ are stable, so $Z = Z(T)$. Further $\mathcal{K}_{\mathrm{qs}}$ can be retrieved from Definition 1.3, 1.4, and 1.5, and, for groups K on that list but not in (2) we check using Definition 7.1, 7.2, and 7.3 in [8] that $\overline{K}$ is 2-small. Hence $|Z(\overline{T})| = 2$, so $|Z(T) : Z(K)| = 2$, and for $\overline{T} \leq S \in \mathrm{Syl}_2(\mathrm{Aut}(K))$ we have $C_S(T) = \overline{Z(T)}$, completing the proof. □

2. Terminal components

See Definition 8.1.1 in [5] for the definition of a *terminal component*.

2.1. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ has a fully normalized Sylow group T. In addition assume the following*:

(a) *$\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$.*

(b) *$\mathcal{C}$ is maximal in $\mathfrak{C}(\mathcal{F})$.*

(c) $m(T) > 1$.

Then one of the following holds:

(1) *$\mathcal{C}$ is a component of $\mathcal{F}$.*

(2) *$\mathcal{C}$ is a terminal member of $\mathfrak{C}(\mathcal{F})$.*

Proof. By (a) there is an involution $j \in \mathcal{C}^f$ and $\mathcal{L} \in \mathrm{Comp}(C_{\mathcal{C}}(j))$ such that $j \in Z(\mathcal{L})$ and $\mathcal{L} \in \mathfrak{C}(\mathcal{F})$ with $j \in \mathcal{I}(\mathcal{L})$. See Definition 6.1.17 in [5] for the definition of $\mathcal{C}^{\perp}$, and Definition 6.2.7 in [5] for the definition of $\rho(\mathcal{C})$ and $\rho_0(\mathcal{C})$.

Suppose first that $\mathcal{C}^{\perp} \neq \{\mathcal{C}\}$. Then (1) holds by (c) and Theorem 7.4.14 in [5]. Thus we may assume $\mathcal{C}^{\perp} = \{\mathcal{C}\}$. Therefore if $\rho(\mathcal{C}) = \rho_0(\mathcal{C})$, then (2) holds by Definition 8.1.1 in [5], so we may assume otherwise. Thus by Definition 6.2.7 in [5], there is $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C})$ and an involution $a \in Q_{t_1} - \widetilde{\mathcal{X}}(\mathcal{C}_1)$. Without loss of generality $(t_1, \mathcal{C}_1) = (t, \mathcal{C})$. Let $\alpha \in \mathfrak{A}(a)$ and adopt the bar convention of Notation 6.1.12 in [5]. Now there is a conjugate $(i, \mathcal{E}, \bar{\mathcal{C}})$ of $(j, \mathcal{L}, \mathcal{C})$ under α. As $j \in \mathcal{I}(L) \cap Z(\mathcal{L})$, $i \in \mathcal{I}(\mathcal{E}) \cap Z(\mathcal{E})$, so $\bar{\mathcal{C}}$ pumps up to a component $\mathcal{D}$ of $\mathcal{F}_{\bar{a}}$ by 1.9 in [7]. Thus by (b), $\bar{\mathcal{C}} = \mathcal{D}$, contradicting $a \notin \widetilde{\mathcal{X}}(\mathcal{C}_1)$. □

See Section 9.1 in [5] for the definition of a *standard subsystem* and a *nearly standard subsystem* of $\mathcal{F}$. In particular if $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is nearly standard, then $\widetilde{\mathcal{X}}(\mathcal{C})$ has a unique maximal member Q.

Often we assume the following hypothesis:

Hypothesis 2.2. (1) $\mathcal{F}$ is a saturated fusion system over a finite 2-group S.

(2) $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ has fully normalized Sylow group T and is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$.

(3) $\mathcal{C}$ is terminal in $\mathfrak{C}(\mathcal{F})$.

2.3. *Assume Hypothesis 2.2. Then*:

(1) *$\mathcal{C}$ is nearly standard. Let Q be the unique maximal member of $\widetilde{\mathcal{X}}(\mathcal{C})$ and $Q_0 = C_S(T)$.*

(2) *$Z(K) = Z(\mathcal{C}) \leq Q$.*

(3) *For $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$, $\mathcal{C}\alpha^* \trianglelefteq N_{\mathcal{F}}(X\alpha)$.*

(4) *Set $\Sigma = N_{\mathrm{Aut}_{\mathcal{F}}(Q_0T)}(T)$. Then $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Aut}_{\Sigma}(T)$.*

(5) *If $\mathrm{Aut}_{\mathcal{F}}(T) \leq \mathrm{Aut}(\mathcal{C})$ then $\mathcal{C}$ is standard.*

(6) *If $\sigma \in \Sigma$ with X and $X\sigma$ in $\widetilde{\mathcal{X}}$ then $\sigma_{|T} \in \mathrm{Aut}(\mathcal{C})$.*

(7) *If $Z(T) = Z(K)$ then $\mathcal{C}$ is standard.*

Proof. By Hypothesis 2.2(2), $Z(\mathcal{C}) = Z(K) \neq 1$, so part (1) follows from 9.2.4 in [5] and Hypothesis 2.2(3). Part (2) follows from 6.2.10(3) in [5] and Hypothesis 2.2(3). Part (3) follows from (1) and (S2) in the definition of "nearly standard" in Section 9.1 in [5]. Part (4) follows from 6.1.8(1) in [5]. Part (5) follows from (1) and the definition of "standard". Part (6) is 6.1.8(3) in [5]. Assume the setup of (7). By (2), $X = Z(K) = Z(\mathcal{C}) \leq Q$, so by 8.1.3 in [5], $X \in \widetilde{X}$. By assumption $Z(T) = Z(K) = X$, so Σ acts on X; hence $\Sigma_{|T} \leq \mathrm{Aut}(\mathcal{C})$ by (6). Now (7) follows from (4) and (5). □

In this section Σ is defined as in 2.3(4).

2.4. *Assume Hypothesis 2.2 and $\mathcal{C}$ is not standard. Then*:

(1) *Q_0 is abelian.*

(2) *$Q_0 = QZ(T)$, so $|Q_0 : Q| = |Z(\overline{T})|$.*

(3) *There exists $\sigma \in \Sigma$ with $Q \cap Q\sigma = 1$. Hence $|Q| \leq |Z(\overline{T})|$.*

(4) *If K is 2-small then $Q = Z(K)$ is of order 2.*

(5) *If $K/Z(K)$ is $G_2(4)$, $F_4(2)$, or $L_3(4)$ then either $Q = Z(K)$ is of order 2 or $Q \cong E_4$.*

(6) *Assume the setup of (5). Then for each $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$, $F^*(N_{\mathcal{F}}(X\alpha))$ equals $(Q\alpha) * \mathcal{C}\alpha^*$ and $\alpha : Q_0T \to Q_0T$ induces an isomorphism of $Q * \mathcal{C}$ with $F^*(N_{\mathcal{F}}(X\alpha))$.*

Proof. Part (1) follows from 2.3.5 and 9.2.3(9) in [5].

Choose $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$. By 8.1.4(1) in [5], we may take $T\alpha = T$. Set $\mathcal{B} = \mathcal{C}\alpha^*$; by 2.3(3), $\mathcal{B} \trianglelefteq N_{\mathcal{F}}(X\alpha)$.

As $Q_0 = C_S(T)$, $1 = [Q_0, T]$, and by (1), $Q_0T \leq C_S(X)$, so $(Q_0T)\alpha \leq C_S(X\alpha)$ and hence $1 = [Q_0\alpha, T\alpha] = [Q_0\alpha, T]$. Thus $Q_0\alpha \leq C_S(T) = Q_0$, so $Q_0\alpha = Q_0$.

Define Q_X as in Notation 9.2.2 in [5]; that is,

$$(*) \qquad Q_X\alpha = \theta \cap N_S(X)\alpha, \quad \text{where } \theta = C_{N_S(X\alpha)}(\mathcal{B}).$$

By 9.2.3(6) in [5], $Q_X = N_Q(X)$, so $Q_X = Q$ by (1). As $\mathcal{B} \trianglelefteq N_{\mathcal{F}}(X\alpha)$ and $Q_0 = Q_0\alpha$, we can form $Q_0\mathcal{B}$ in $N_{\mathcal{F}}(X\alpha)$. By Lemma 2.22 in [9], $Q_0\mathcal{B}$ is realized by a group Q_0K_1 with $K_1 \cong K$. As T is Sylow in $\mathcal{B}$, θ centralizes T, and hence is contained in Q_0, so $\theta = C_{Q_0}(\mathcal{B}) = C_{Q_0}(K_1)$. Define $\pi : Q_0K_1 \to \mathrm{Aut}(K_1)$ by $x\pi = c_x$ to be conjugation of K_1 by x. Observe that $\ker(\pi) = \theta$ so $Q_0K_1\pi \cong Q_0\pi\overline{K}_1$. Also $Q_0\pi = \overline{Z(T)}$; this follows from 1.8 unless K is one of the five exceptional groups appearing in 1.8, where the claim can be verified using 2-local facts about the five groups such as in [14]. Hence $Q_0/\theta \cong \overline{Z(T)}$ and $Q_0 = \theta Z(T)$. Now

(∗∗) $$Q\alpha = Q_X\alpha = \theta \cap N_S(X)\alpha.$$

Also $\theta \leq Q_0 = Q_0\alpha \leq N_S(X)\alpha$ so

(∗∗∗) $$Q\alpha = \theta.$$

Thus $Q_0 = \theta Z(T) = Q\alpha Z(T) = Q\alpha Z(T)\alpha$, so (2) holds. Also by (∗) and (∗∗∗), $Q\alpha = C_{N_S(X\alpha)}(\mathcal{B})$, so the first statement in (6) holds. We have seen that α acts on Q_0 and T, so α is an automorphism of the group $Q_0T = QT$, and induces an isomorphism of the fusion systems $Q\mathcal{C}$ and $Q\alpha\mathcal{B}$. This completes the proof of (6). As $\mathcal{C}$ is not standard by hypothesis, 2.3(4)–(5) say there exists $\sigma \in \Sigma$ with $\sigma \notin \mathrm{Aut}(\mathcal{C})$. Then $Q \cap Q\sigma = 1$ by 2.3(6). Then $|Q| \leq |Z(\overline{T})|$ by the second statement in (2), completing the proof of (3). Part (4) follows from (3). Similarly if K is as in (5) then $Z(\overline{T}) \cong E_4$, so (3) implies (5). □

In the next lemma for a saturated 2-fusion system $\mathcal{E}$, $\mathcal{E}^\infty$ is the last term in the Puig series for $\mathcal{E}$, defined in Definition 2.18 of [9].

2.5. *Assume Hypothesis 2.2 with $K/Z(K) \cong F_4(2)$ and $|Q| > 2$. Then $\mathcal{C}$ is standard.*

Proof. Assume $\mathcal{C}$ is not standard. Then $Q \cong E_4$ by 2.4(5). By 2.4(3) there is $\sigma \in \Sigma$ with $Q_0 = Q \times Q\sigma$. Now $Z(\overline{T}) = \langle \bar{z}_l, \bar{z}_s \rangle$ where $\bar{z}_c$ is a root involution. Then there is $y \in Q\sigma$ such that the image of y in $\overline{K}$ is a root involution and $\mathcal{D} = C_{\mathcal{C}}(y)$ is the 2-fusion system of $C_K(y)$. By 8.5 in [3], $\overline{\mathcal{D}}$ is the fusion system of a maximal parabolic of $\overline{K}$ with $\overline{\mathcal{D}}^\infty = \overline{\mathcal{D}}$. As K is quasisimple, the parabolic does not split over $Z(K)$, so $\mathcal{D} = \mathcal{D}^\infty$. Let $\tau = \sigma^{-1} \in \mathrm{Aut}_{\mathcal{F}}(Q_0T)$, so that $Q\sigma\tau = Q$. Now y centralizes $\mathcal{D}$, so $x = y\tau$ centralizes $\mathcal{D}\tau^*$. Choose X and α as in the proof of 2.4 with $X = \langle x \rangle$. Then $x\alpha$ centralizes $\mathcal{E} = \mathcal{D}\tau^*\alpha^*$. As $\mathcal{D} = \mathcal{D}^\infty$, $\mathcal{E} = \mathcal{E}^\infty$ and hence $\mathcal{E} \leq \mathcal{F}_{x\alpha}^\infty = \mathcal{B} = \mathcal{C}\alpha^*$ by 2.4(6). Thus as $Q\alpha$ centralizes $\mathcal{B}$, it centralizes $\mathcal{E}$. By 2.4(6), α induces an isomorphism of the fusion systems $Q\mathcal{C}$ and $Q\alpha\mathcal{B}$. Thus applying α^{-1}, Q centralizes $\mathcal{D}\tau^*$. Then applying σ, $Q\sigma$ centralizes $\mathcal{D}$, contradicting $C_{Q_0}(\mathcal{D}) = Q\langle y \rangle$. □

2.6. *Assume $K/Z(K) \cong L_3(4)$ and $|Z(K)| = 2$. Then:*

(1) $\mathrm{Aut}(T)$ *acts on* $Z(K)$.

(2) *If Hypothesis 2.2 holds for K then $\mathcal{C}$ is standard.*

Proof. Let $Z(K) = \langle k\rangle$; we show k is the unique member of $Z(T)$ that is not a commutator in T, hence establishing (1). One way to see this is to consider a maximal parabolic $\widehat{Y}$ of the covering group $\widehat{K}$ of K with $Z(\widehat{K}) \cong E_4$. Using 1.6, $\widehat{Y}$ is of the form $A \cong E_{64}$ extended by $L_2(4)$, with $A/Z(\widehat{T})$ the natural module for $\widehat{Y}/A$ and $\widehat{Y}$ indecomposable on A as $\widehat{K}$ is quasisimple. Regarding A as an $\mathbb{F}_4$-module for $\widehat{Y}/A$ and taking $\widehat{T} \in \mathrm{Syl}_2(\widehat{Y})$, we claim that $\widehat{Z} = C_A(\widehat{T})$ has five points $Z(\widehat{K})$, $[\widehat{Z}, \widehat{B}]$ for $\mathbb{Z}_3 \cong \widehat{B} \le N_{\widehat{Y}}(\widehat{T})$, and $[A, t_i]$, $1 \le i \le 3$, for $t_i A$ the involutions in $\widehat{T}/A$. If so, the only members of $\widehat{Z}$ that are commutators in $\widehat{T}$ are in $[A, t_i]$ for some i. Now $K = \widehat{K}/\Pi$ for some Π of order 2 in $Z(\widehat{K})$, and thus k is the unique noncommutator in $Z(T)$. Thus to complete the proof of (1) it remains to prove the claim. But if the claim fails, then $[A, \widehat{T}]$ is the point $[\widehat{Z}, \widehat{B}]$, so $[A, \widehat{Y}] = [A, \widehat{T}][A, \widehat{T}_1]$ is of rank 2, where $\widehat{T}_1$ is a second Sylow group of $\widehat{Y}$, contradicting $\widehat{Y}$ indecomposable on A. So (1) is established.

Assume Hypothesis 2.2. By 2.3(2), $X = Z(K) \le Q$, so by (1) and 2.3(6), $\Sigma \le \mathrm{Aut}(\mathcal{C})$. Then (2) follows from 2.3(4)–(5). □

2.7. *Assume $K/Z(K) \cong G_2(4)$ and $|Z(K)| = 2$. Let $M = C_K(Z(T))$. Then*:

(1) $\mathrm{Aut}(M)$ *acts on* $Z(K)$.

(2) *If Hypothesis 2.2 holds for K then $\mathcal{C}$ is standard.*

Proof. Let $Z(K) = \langle k\rangle$ and $Z = Z(T)$. By 18.4 in [10], $\bar{K}$ has two classes of involutions — the long and short root involutions — and $Z(\bar{T})$ is a long root subgroup. By 1.6, $Z(\bar{T}) = \bar{Z}$. Again by [10], $\bar{M} = \bar{L}\bar{R}$, where $\bar{R}$ is the radical of $\bar{M}$ and $\bar{L} \cong L_2(4)$ contains a short root subgroup. From the Atlas, short root involutions lift to elements of order 4 in K, so the preimage L of $\bar{L}$ in K is isomorphic to $\mathrm{SL}_2(5)$. From Example 3.2.4 in [14], R/Z is the sum of two orthogonal modules for $\bar{L}$, so $\bar{R}$ is transitive on complements $\bar{J}$ to $\bar{R}$ in $\bar{M}$, so $Z(J) = \langle k\rangle$ for each such J. This proves (1).

Assume the setup of (2). Then, using 2.4(6), $QC_{\mathcal{C}}(Z) = C_{\mathcal{F}}(Z) \trianglelefteq N_{\mathcal{F}}(Z) = \mathcal{B}$, with $QR \trianglelefteq \mathcal{B}$ and $C_{\mathcal{B}}(QR) \le QR$, so $\mathcal{B}$ is constrained with model B with $C_B(Z) \cong QC_K(Z) = QM$. Also $M = O^2(QM) \trianglelefteq B$. Then $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Aut}_B(T)$ as $Z = Z(T)$ and $B = N_{\mathcal{F}}(Z)$. Then as $M \trianglelefteq \mathrm{Aut}_B(T)$ centralizes k by (1), (2) follows from 2.4(3). □

Theorem 2.8. *Assume Hypothesis 2.2. Then one of the following holds*:

(1) *$Q = Z(K)$ is of order 2.*

(2) *$\mathcal{C}$ is standard.*

(3) *$K/Z(K) \cong L_3(4)$ and $Q = Z(K) \cong E_4$.*

Proof. If $|Q| = 2$ then (1) follows from 2.3(2), so we may assume $|Q| > 2$. By 2.3(7), we may assume $Z(T) \ne Z(K)$, while K is not 2-small by 2.4(4). Hence by 1.7 and 1.8, $K/Z(K)$ appears in case (2) of 1.8. Next $K/Z(K) \cong L_3(4)$ by 2.5 and 2.7, while $Z(K) \cong E_4$ by 1.5(3) and 2.6. Finally (3) holds by 2.4(3). □

3. Tight embedding

Given a saturated fusion system on a 2-group S, a fully normalized abelian subgroup P of S is *tightly embedded* in $\mathcal{F}$ if

(T1) For each $1 \neq X \leq P$ and $\alpha \in \mathfrak{A}(X)$, $P\alpha \trianglelefteq N_{\mathcal{F}}(X\alpha)$, and

(T2) for each $1 \neq X \leq P$, $X^{\mathcal{F}} \cap P = X^{N_{\mathcal{F}}(P)}$.

We can also define tight embedding for groups. Given a finite group G, an abelian 2-subgroup P of G is *tightly embedded* in G if distinct conjugates of P intersect trivially. Equivalently,

(TE1) For each $1 \neq X \leq P$, $P \trianglelefteq N_G(X)$, and

(TE2) for each $1 \neq X \leq P$, $X^G \cap P = X^{N_G(P)}$.

Another equivalent definition is obtained by quantifying over all involutions in P rather than over all nontrivial subgroups of P.

In the remainder of this section we assume the following hypothesis:

Hypothesis 3.1. $\mathcal{F}$ is a saturated fusion system on a finite 2-group S such that $\mathcal{F} = P\mathcal{C}$ where

(1) $\mathcal{C} = O^2(\mathcal{F})$ is tamely realized by $K \in \mathcal{K}_{\text{qs}}$, and

(2) P is an abelian 2-subgroup of S tightly embedded and fully normalized in $\mathcal{F}$.

3.2. (1) *There exists a finite group G with $S \in \mathrm{Syl}_2(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, $K = E(G)$, $F^*(G) = O_2(G)K$, and $G = PK$.*

(2) *$G/K \cong P/(P \cap K)$ is an abelian 2-group.*

Proof. As K tamely realizes $\mathcal{C}$, (1) follows from Lemma 2.22 in [9]. As $G = PK$ and P is an abelian 2-group, (2) holds. □

Set $U = O_2(G)$ and $G^* = G/U$; thus $U = C_G(K)$ and $G^* \leq \mathrm{Aut}(K^*)$ with $K^* \cong K/Z(K)$ simple.

3.3. *Let $1 \neq X \leq P$ and $G_X = \langle P^{N_G(X)} \rangle$. Then*:

(1) *$N_G(X) = PN_K(X)$.*

(2) *$O^2(C_K(X))^* = O^2(C_{K^*}(X^*))$.*

(3) *P is strongly closed in S_X with respect to $N_G(X)$ for $S_X \in \mathrm{Syl}_2(N_G(X))$.*

(4) *$G_X^+ = G_X/O(G_X) = H_0^+ \times H_1^+ \times \cdots \times H_n^+$ where $H_0 \leq P$ and for $1 \leq i \leq n$, H_i^+ is a Goldschmidt group with $P \cap H_i = \Omega_1(S \cap H_i)$.*

(5) *If $N_K(X)/O(N_K(X))$ has no Goldschmidt components then $O(N_K(X))P$ is a normal subgroup of $N_G(X)$.*

Proof. As P is abelian and $G = PK$, (1) holds. Next $k \in C_K(X) = \Delta$ if and only if $[k, X] = 1$ while $k^* \in C_{K^*}(X^*)$ if and only if $1 = [k^*, X^*] = [k, X]^*$ if and only if $[k, X] \leq U$. Thus $O^2(\Delta^*) \leq O^2(C_{K^*}(X^*))$. Let $j^* \in C_{K^*}(X^*)$ be of odd order m; we may choose $|j| = m$. Then $j \in O^2(G) = K$, so $[j, X] \leq U \cap K = Z(K)$ and hence j centralizes $XZ(K)/Z(K)$ and $Z(K)$ so j centralizes X by 24.5 in [2]. Thus $O^2(C_{K^*}(X^*)) \leq O^2(\Delta^*)$, completing the proof of (2).

As P is abelian and tightly embedded in $\mathcal{F}$, from condition (T1) above for $\alpha \in \mathfrak{A}(X)$ we have $P\alpha \trianglelefteq N_{\mathcal{F}}(X\alpha)$. As $P \in \mathcal{F}^f$ there is $\beta \in \mathfrak{A}(P\alpha)$ with $P\alpha\beta = P$; set $\zeta = \alpha\beta$ and observe that $P\zeta = P$ and as $P\alpha \trianglelefteq N_{\mathcal{F}}(X\alpha)$, $\zeta \in \mathfrak{A}(X)$, so replacing α by ζ we may take $P\alpha = P$. Thus $P \trianglelefteq N_{\mathcal{F}}(X\alpha)$, so P is strongly closed in $N_S(X\alpha)$ with respect to $N_{\mathcal{F}}(X\alpha)$. Next by (T2), there is $\gamma \in N_{\mathcal{F}}(P)$ with $(X\alpha)\gamma = X$, so there is $g \in N_G(P)$ with $(X\alpha)^g = X$, so (3) holds with $S_X = N_S(X\alpha)^g$.

Part (4) follows from Goldschmidt's fusion theorem [13], determining groups generated by conjugates of a strongly closed abelian 2-subgroup. As $O^2(G) = K$, $O^2(G_X) \leq N_K(X)$. Then (4) implies (5). □

3.4. *Let $\mathfrak{X}$ be the set of involutions $x \in P$ such that $O(C_{K^*}(x^*)) = 1$ and $C_{K^*}(x^*)$ has no Goldschmidt components. Then*:

(1) *For each $x \in \mathfrak{X}$, $P \trianglelefteq C_G(x)$.*

(2) *If each involution in P is in $\mathfrak{X}$ then P is tightly embedded in G.*

Proof. By 3.3(2), $O(C_G(x))^* = O(C_{K^*}(x^*))$ and if $C_{K^*}(x^*)$ has no Goldschmidt components then, using 3.3(5), neither does $C_G(x)$. Hence (1) follows from 3.3(5). From Hypothesis 3.1(2) and condition (T2) in the definition of tight embedding, $x^{\mathcal{F}} \cap P = x^{N_{\mathcal{F}}(P)}$ so $x^G \cap P = x^{N_G(P)}$, and, together with (1) and the third of the equivalent definitions of tight embedding in groups at the beginning of this section, (2) follows. □

3.5. *Either P is faithful on K or $G = P * K$ is a central product of P with K.*

Proof. Suppose x is an involution in $C_P(K)$. As K is not Goldschmidt and $F^*(G) = UK$ by 3.2(1), it follows from 3.4 that $P \trianglelefteq C_G(x)$. Thus $[P, K] \leq P \cap K \leq Z(K)$ so K centralizes P. The lemma follows as $G = PK$ by 3.2(1). □

3.6. *Assume $|P| = 4$ or P is cyclic. Then*:

(1) *For each involution $x \in P$, $PO(C_K(x)) \trianglelefteq C_G(x)$.*

(2) *If $O(C_{K^*}(x^*)) = 1$ for each involution x in P then P is tightly embedded in G.*

Proof. If $|P| = 4$ then $|P : \langle x \rangle| = 2$, so as P is strongly closed in $C_G(x)$, (1) follows from the Z^*-theorem. Similarly (1) holds when P is cyclic. Then (1), 3.3(2) and 3.4(2) imply (2). □

3.7. *Assume $K/Z(K) \in$* Chev(2). *Then*:

(1) *P is tightly embedded in G.*

(2) *If P is faithful on K, $\Phi(P) = 1$, and $|P| > 2$ then $P \leq UK$.*

Proof. We may assume P is not normal in G, so P is faithful on K by 3.5. We first prove (1), where by 3.4 it suffices to verify the two conditions defining $\mathfrak{X}$. If $x^* \in K^*$ then $F^*(C_{K^*}(x^*)) = O_2(C_{K^*}(x^*))$, so the conditions hold; thus we may assume x^* induces an outer automorphism on K^*, so x^* and its centralizer are listed in [10]. As K appears in 1.5, we find that the conditions are satisfied, unless $K^* \cong L_3(4)$ and x^* induces a graph or graph-field automorphism. In either case we conclude from 3.3(4) that $P^* \cap K^* \neq 1$, so there is an involution $z \in P$ with $z^* \in K^*$. But as we just saw, $F^*(C_{K^*}(z^*)) = O_2(C_{K^*}(z^*))$, so $P \trianglelefteq C_G(z)$ by 3.3(5). Next z^* is stable by 1.6 and P is fully normalized by Hypothesis 3.1(2), so $z \in Z(T)$, where $T = S \cap K$. As $\langle x^T \rangle$ is nonabelian, this contradicts $P \trianglelefteq C_G(z)$ and P abelian. This completes the proof of (1).

Assume the setup of (2). Using (1), the hypotheses of 20.1 in [10] are satisfied; then (2) follows from that lemma. □

3.8. *Assume $K/Z(K)$ is sporadic and either $|P| = 4$ or P is cyclic. Then P is tightly embedded in G.*

Proof. By 3.6(2) it suffices to show $O(C_{K^*}(x^*)) = 1$ for each involution x in P. This follows by inspection of Table 5.3 in [14] for each of the groups listed in 1.4. □

4. Splitting

In this section we assume $\mathcal{C}$ is a quasisimple fusion system on a 2-group T tamely realized by some $K \in \mathcal{K}_{\text{qs}}$.

Recall from Subsection 0.13 in [5] that a *critical split extension* of $\mathcal{C}$ is a pair $(\mathcal{F}, P)$, where $\mathcal{F}$ is a saturated fusion system on a 2-group S, $\mathcal{C} = O^2(\mathcal{F})$, P is a complement in S to a Sylow group T of $\mathcal{C}$, P is isomorphic to E_4, and P is tightly embedded in $\mathcal{F}$. Further $\mathcal{C}$ is said to *split* if there is no nontrivial critical extension $(\mathcal{F}, P)$ of $\mathcal{C}$; that is, for each such pair we have $\mathcal{F} = C_S(\mathcal{C}) * \mathcal{C}$.

Throughout this section we assume $(\mathcal{F}, P)$ is a critical split extension of $\mathcal{C}$.

4.1. (1) *Hypothesis 3.1 is satisfied.*

(2) *There exists a finite group G with $S \in \mathrm{Syl}_2(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, $K = E(G)$, $F^*(G) = O_2(G)K$, and $G = PK$.*

(3) *If $(\mathcal{F}, P)$ is nontrivial then P is faithful on K.*

Proof. Part (1) is immediate; then (1) and 3.2(1) imply (2), and 3.5 implies (3). □

Set $U = O_2(G)$ and $G^* = G/U$.

4.2. *Assume $(\mathcal{F}, P)$ is nontrivial. Then*:

(1) *For some involution* $x \in P$, $x^* \notin K^*$.

(2) $|\mathrm{Out}(K/Z(K))|$ *is even.*

Proof. If (1) fails then $G^* = K^*$, so $G = UK$ and hence $\mathcal{F} = U * \mathcal{C}$, contradicting $(\mathcal{F}, P)$ nontrivial. Thus (1) holds and (1) and 4.1(2) imply (2). □

4.3. *If* $K/Z(K) \in \mathrm{Chev}(2)$ *then* $\mathcal{C}$ *splits.*

Proof. This follows from 3.7(2). □

4.4. *If* $K/Z(K)$ *is sporadic then* $\mathcal{C}$ *splits.*

Proof. Assume $(\mathcal{F}, P)$ is nontrivial. By 3.8, P is tightly embedded in G. Hence, using 3.3(2), for each involution $x \in P$, $O^2(C_{K^*}(x^*))$ acts on $P^* \cong E_4$.

By 4.2 there is an involution x in $P - KU$, and $|\mathrm{Out}(K^*)|$ is even. Inspecting Table 5.3 in [14] in those cases where $|\mathrm{Out}(K^*)|$ is even, we find that K^*, $C_{K^*}(x^*)$ is among the pairs listed in 14.5 in [8]. In each case $O^2(C_K(x))$ is irreducible on $V^* = O_2(C_{K^*}(x^*))$, so $V^* = P^* \cap K^*$ is of order 2, and hence $K^* \cong M_{12}$ and the preimage V of V^* in K is generated by v of order 4. Next by Table 5.3b in [14], there exists an involution $i^* \in C_{K^*}(v^*)$ with $[x^*, i^*] = v^*$. It follows that x inverts v and $P = \langle x, y\rangle$ where $y = uv$ for some $u \in U$ with $u^2 = v^2$. Now $v \in C_K(y)$, so v acts on P, a contradiction as then $[x, v] = v^2 \in P$, whereas P is faithful on K by 3.5. □

We recall that, from [6], the 2-fusion systems of all the groups $\Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$ appearing in Definition 1.3(b) are isomorphic. Thus in this case we may take K to be $\Omega_6^-(3)$.

4.5. *If* K *is* $\Omega_6^{\epsilon}(q)$ *with* $q \equiv \pm 3 \bmod 8$ *and* $q \equiv \epsilon \bmod 4$ *then* $\mathcal{C}$ *splits.*

Proof. Assume $(\mathcal{F}, P)$ is nontrivial. We may take K to be $\Omega_6^-(3)$. Then the classes of involutions in $\mathrm{Aut}(K)$ and their centralizers are listed in Table 2.10 in [1]. Inspecting that list, we find that for x^* an involution in G^*, we have $O(C_{K^*}(x^*)) = 1$ and $C_{K^*}(x^*)$ has no Goldschmidt components. Therefore by 3.4, P is tightly embedded in G and normal in $C_G(x)$ for each involution x in P. As $E_4 \cong P^*$ is $O^2(C_{K^*}(x^*))$-invariant we conclude from Table 2.10 in [1] that x^* is of type $i(4, \delta)$ for $\delta \in \{1, -1\}$. By 4.2(1), we may choose $x^* \notin K^*$, so $x^* \in i(4, -1)$. Thus the remaining two involutions in $P^* = O_2(C_{G^*}(x^*))$ are reflections, whereas we just saw they are in $i(4, \delta)$. □

Theorem 4.6. *Assume* $\mathcal{C}$ *is a quasisimple* 2*-fusion system tamely realized by some* $K \in \mathcal{K}_{\mathrm{qs}}$. *Then* $\mathcal{C}$ *splits.*

Proof. The possibilities for K are listed in Definition 1.3, 1.4, 1.5. Now appeal to 4.3–4.5. □

5. Standard subsystems

In this section we assume the following hypothesis:

Hypothesis 5.1. (1) $\mathcal{F}$ is a saturated fusion system on a finite 2-group S.

(2) $\mathcal{C}$ is a standard subsystem of $\mathcal{F}$ on $T \in \mathcal{F}^f$. Let Q be the unique maximal member of $\widetilde{\mathcal{X}}(\mathcal{C})$ and write $\mathcal{Q}$ for the centralizer in $\mathcal{F}$ of $\mathcal{C}$.

(3) $\mathcal{C}$ is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$.

(4) $\mathcal{C}$ is not a component of $\mathcal{F}$.

5.2. *$\mathcal{F}$ is almost simple.*

Proof. Observe Hypothesis 9.4.1 of [5] is satisfied by (1) and (2) of Hypothesis 5.1, so the result follows from Hypothesis 5.1(4) and 9.4.6 in [5]. □

5.3. *Either $\Phi(Q) = 1$ or Q is cyclic.*

Proof. By Theorem 4.6 and 9.4.10 in [5], either $\Phi(Q) = 1$ or $m(Q) = 1$. Thus we may assume Q is quaternion and it remains to exhibit a contradiction. Let z be the involution in Q.

As $\mathcal{C}$ is standard, $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$, so $\tau = (\mathcal{F}, \Omega)$ is a quaternion fusion packet, where $\Omega = Q^{\mathcal{F}}$. By 5.2, $\mathcal{L} = F^*(\mathcal{F})$ is simple, so by Theorem 1 in [6], and as $\mathcal{C}$ is a component of $\mathcal{F}_z$, we conclude that $\mathcal{L}$ is the 2-fusion system of a group L of Lie type over $\mathbb{F}_q$ for some odd q. Then as $\mathcal{C}$ is a component of $\mathcal{L}_z$, $K \cong \Omega_6^{\epsilon}(q)$. As $\mathcal{Q} \trianglelefteq \mathcal{F}_z$, $\Omega(z) = \{Q\}$, where $\Omega(z) = \{P \in \Omega : z \in P\}$. As $\mathcal{C} = E(C_{\mathcal{F}_z}(\mathcal{Q}))$, $|\Omega| > 1$ from Theorem 1 in [6], and then $\rho = (\mathcal{C}, \Gamma)$ is a quaternion fusion packet, where $\Gamma = \Omega - \{Q\}$. As $K \cong \Omega_6^{\epsilon}(q)$ it follows that $\Gamma = \Omega(t)$ is of order 2, for some $t \in z^{\mathcal{F}}$, contradicting $|\Omega(t)| = |\Omega(z)| = 1$. □

In the remainder of the section we assume:

Hypothesis 5.4. Hypothesis 5.1 holds with $|Q| > 2$.

5.5. (1) *$N_{\mathcal{F}}(Q)$ is tamely realized by a group M with $F^*(M) = Q * K$.*

(2) *Q is tightly embedded in $\mathcal{F}$.*

Proof. By 5.3, Q is abelian, so $Q = O_2(\mathcal{Q})$ and $F^*(N_{\mathcal{F}}(Q)) = Q\mathcal{C}$. Then (1) follows from Lemma 2.22 in [9]. As $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$ and $Q = O_2(\mathcal{Q}) = O^{2'}(\mathcal{Q})$, (2) follows. □

5.6. *It is not the case that Q is weakly closed in $N_S(Q)$ with respect to $\mathcal{F}$.*

Proof. If Q is cyclic the lemma follows from 9.4.7(3) in [5]. If $\Phi(Q) = 1$ it follows from 9.4.11 in [5]. □

Notation 5.7. Set $\Delta = Q^{\mathcal{F}} \cap N_S(Q) - \{Q\}$.

By 5.6, $\Delta \neq \varnothing$. Recall from Definition 3.1.9 in [5] that $\mathcal{P}(Q)$ is the set $\{1 \neq P \leq S : \mathrm{hom}_{\mathcal{F}}(P, Q) \neq \varnothing\}$ and $\mathcal{P}^*(Q)$ is the set of maximal members of $\mathcal{P}(Q)$. Let $\mathcal{P}_Q = \{P \in \mathcal{P}(Q) : P \leq N_S(Q)\}$ and $\mathcal{P}^*_Q$ be the maximal members of $\mathcal{P}_Q$. For example, $\Delta \subseteq \mathcal{P}^*_Q$.

Choose $Q \neq P \in \mathcal{P}^*_Q \cap N_{\mathcal{F}}(Q)^f$, and set $H = PK \leq M$ and $H^* = H/(H \cap Q)$.

5.8. (1) *P is tightly embedded in $N_{\mathcal{F}}(Q)$ and in $P\mathcal{C}$.*

(2) *P is faithful on K.*

Proof. Part (1) is a consequence of 13.1 in [8]. As $Q = C_S(K)$, $C_P(K) \leq Q$; but $P \cap Q = 1$, by, for example, 3.1.12(2) in [5]. Thus (2) holds. □

5.9. (1) *$Z(K)$ is strongly closed in Q with respect to $\mathcal{F}$.*

(2) *Let $P \in \Delta$ and $\phi \in \mathrm{hom}_{\mathcal{F}}(Q, P)$. Then $Z(K)\phi$ is strongly closed in P with respect to $\mathcal{F}$.*

Proof. Let $x \in Z(K)$ and $y \in x^{\mathcal{F}} \cap Q$. As $\mathrm{Aut}_{\mathcal{F}}(Q)$ controls fusion in Q there is $m \in M$ with $x^m = y$. Then as $K \trianglelefteq M$, $y \in Z(K)$, proving (1). Now (1) implies (2). □

6. Sporadic components

We prove:

Theorem 6.1. *Assume Hypothesis 5.1 with $K/Z(K)$ sporadic. Then $|Q| = 2$.*

Assume $\mathcal{F}$ is a counterexample to the theorem. Adopt the notation from Section 5; in particular choose $P \in \Delta$ as in Notation 5.7.

6.2. $\Phi(Q) = 1$.

Proof. Assume otherwise; by 5.3, Q is cyclic, so $P \cong Q$ is also cyclic. Let x be the involution in P. By 5.8(1) and 3.8, P is tightly embedded in H, so $P \trianglelefteq C_H(x)$. Hence by 3.3(2), $O^2(C_{K^*}(x^*))$ centralizes P^*. Inspecting Table 5.3 in [14] for involution centralizers with this property, we conclude $K^* \cong HS$, $C_{K^*}(x^*) \cong S_5/(\mathbb{Z}_4 * Q_8^2)$ and $P^* = Z(O_2(C_{K^*}(x^*)))$.

Let $W = O^2(C_K(x))$ and $R = O_2(W)$. Then $R = \langle r^W \rangle$ for $r^* \in R^* - P^*$ of order 4, so $\Phi(R) = \langle r^2 \rangle$ is of order 2 with $\Phi(R)^* = \langle x^* \rangle$.

Let y be a preimage of x^* in K; by 1.6, x^* is stable, so y is an involution and $x = qy$ for some $q \in Q$ with $q^2 = 1$. Then as Q is cyclic, $q \in Z(K)$ and $x \in K$. Let P_0 be the preimage of P^* in K; then $P_0 = P \times Z(K)$ with $\langle x \rangle = \Phi(P_0)$. Thus $\langle x \rangle = \Phi(R) \leq W$.

Let $\alpha \in \mathfrak{A}(x)$ with $z = x\alpha \in Z(\mathcal{C})$. Let $\mathcal{W} = \mathcal{F}_{S \cap W}(W)$; then $\mathcal{W} = O^2(C_{\mathcal{F}_z}(x))$. Next $\mathcal{U} = \mathcal{W}\alpha^* \leq O^2(\mathcal{F}_z) = \mathcal{C}$. As $W/\langle x, z \rangle \cong A_5/E_{16}$, it follows from Table 5.3m in [14] that $z\alpha \in \{x, xz\}$ and $\mathcal{U} = \mathcal{W}$. This is a contradiction as $Z(K) = \langle x\alpha \rangle = \Phi(R\alpha)$, so $O_2(W/Z(K)) = R\alpha/Z(K)$ is elementary abelian. □

6.3. $|Z(K)| = 2$.

Proof. If not then by 1.4(2), $K/Z(K) \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$, contrary to 6.2. □

Notation 6.4. By 6.3, $Z(K) = \langle t \rangle$ is of order 2. Let $\phi \in \hom_{\mathcal{F}}(Q, P)$ and $x_0 = t\phi$. Let $x \in P - \langle x_0 \rangle$ and $X = \langle x, x_0 \rangle$.

6.5. (1) $x_0 \in Z(C_H(x))$.

(2) x_0^* *centralizes* $O^2(C_{K^*}(x^*))$.

Proof. By 5.9(2), x_0 is strongly closed in P with respect to $\mathcal{F}$. Then as P is strongly closed in S_x with respect to $C_H(x)$ for $S_x \in \mathrm{Syl}_2(C_H(x))$ by 3.3(3), x_0 is strongly closed in S_x. Hence, by the Z^*-theorem, $O(C_H(x))\langle x_0 \rangle \trianglelefteq C_H(x)$. But as we saw during the proof of 3.8, $O(C_H(x)) = 1$, so (1) holds. Then (1) and 3.3(2) imply (2). □

6.6. $X^* = O_2(C_{H^*}(x^*))$, $C_{H^*}(X^*) = X^* \times L^*$, *and one of the following holds*:

(1) $K^* \cong M_{12}$, $X^* \not\leq K^*$ *and* $L^* \cong A_5$.

(2) $X^* \leq K^* \cong J_2$ *and* $L^* \cong A_5$.

(3) $X^* \leq K^* \cong \mathrm{Co}_1$ *and* $L^* \cong G_2(4)$.

(4) $X^* \leq K^* \cong \mathrm{Suz}$ *and* $L^* \cong L_3(4)$.

(5) $X^* \leq K^* \cong \mathrm{Ru}$ *and* $L^* \cong \mathrm{Sz}(8)$.

Proof. We appeal to 6.5 and inspect the tables in Section 5.3 of [14] for involutions x^* and a 4-group X^* centralizing $O^2(C_{K^*}(x^*))$. □

We are now in a position to derive a contradiction that will establish Theorem 6.1. From 6.6 there is an involution $y \in X$ with $y^* \in K^*$. Thus $y = ck$ for some $c \in Q$ and $k \in K$ with $k^* = y^*$. But from Section 5.3 in [14], $|k| = 4$. Then as $\Phi(Q) = 1$ by 6.2, $y = ck$ is also of order 4, contradicting $\Phi(P) = 1$. This completes the proof of Theorem 6.1.

7. $\Omega_6^-(3)$

We prove:

Theorem 7.1. *Assume Hypothesis 5.1 with* $K \cong \Omega_6^{\epsilon}(q)$ *and* $|Q| > 2$. *Then we may take* $q = 3$ *and we have*:

(1) $Q \cong \mathbb{Z}_4$.

(2) *Let* x *be the involution in* P; *then* $x \in K$ *is in* $i(2, -1)$ *in* $K = \Omega_6^-(3)$, *so* $O^2(C_{K^*}(x^*)) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$.

Assume $\mathcal{F}$ is a counterexample to the theorem. Adopt the notation from Section 5; in particular choose $P \in \Delta$ as in Notation 5.7. Note that by Definition 1.3(b), $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$; indeed the 2-fusion systems of any two groups satisfying these congruences are isomorphic, so, for example, we may take $(q, \epsilon) = (3, -1)$.

7.2. *Q is cyclic.*

Proof. Assume otherwise; by 5.3, $\Phi(Q) = 1$. Adopt Notation 6.4; applying 5.9(2) as in the proof of 6.5, x_0^* centralizes $O^2(C_{K^*}(x^*))$. We may take $K \cong \Omega_6^-(3)$. Then inspecting the list of involution centralizers in Table 2.10 of [1] for involutions x^* and a 4-group X^* centralizing $O^2(C_{K^*}(x^*))$ we conclude that $x^* \in i(4, \delta)$ for some $\delta \in \{1, -1\}$. But then for $y^* \in X^* - \langle x^* \rangle$, y^* is a reflection, a contradiction as $(xx_0)^*$ also serves in the role of x^*. □

Let x be the involution in P. Inspecting the list of centralizers in Table 2.10 of [1] we find that $O(C_{K^*}(i^*)) = 1$ for each involution $i^* \in H^*$, so by 3.6(2):

7.3. *P is tightly embedded in H, so $P \trianglelefteq C_H(x)$ and P^* centralizes $O^2(C_{K^*}(x^*))$.*

7.4. $|P^*| = 4$ *and either*

(1) $x^* \in i(4, +)$ *and* $O^2(C_{K^*}(x^*)) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$, *or*

(2) *x^* is a projective involution and $C_{K^*}(x^*) \cong U_3(3)$.*

Proof. This time we inspect Table 2.10 in [1] for centralizers in which some cyclic group of order at least 4 centralizes $O^2(C_{K^*}(x^*))$. □

7.5. *Case* (1) *of 7.4 holds.*

Proof. Assume instead that 7.4(2) holds. Let $F = \mathbb{F}_9$, V be a 4-dimensional unitary space over F, and $G = \mathrm{GU}(V)$. Let $B = \{v_1, \dots, v_4\}$ be an orthonormal basis for V, $\lambda \in F$ of order 4, $w_i \in C_G(v_i^\perp)$ with $v_i w_i = \lambda v_i$, $Q_i = \langle w_i \rangle$, and $t_i = w_i^2$. Then $C_G(t_i) = Q_i \times L_i$ where $L_i = C_G(v_i) \cong \mathrm{GU}_3(3)$. Therefore Q_i is tightly embedded in G.

Next, setting $Z = \langle -\mathrm{id}_V \rangle \leq Z(G)$ and $G^+ = G/Z$, we have $\mathrm{SU}(V)^+ \cong K$ and $\mathrm{SU}(V)^+ Q_1^+$ is a split extension with w_1 inducing an automorphism on $\mathrm{SU}(V)^+$ quasiequivalent to that of a generator of P on K; so $H = PK \cong \mathrm{SU}(V)^+ Q_1^+$.

Let $\Sigma = \{Q_i : 1 \leq i \leq 4\}$ and $W_0 = \langle \Sigma \rangle$. Then $\Sigma = Q_1^G \cap C_G(W_0)$ as the weight spaces of W_0 are the Fv_i. Hence as the weak closure of Q_1 in a Sylow 2-subgroup S_G of G is abelian (since $Q_1 \cong \mathbb{Z}_4$ is tightly embedded), W_0 is that weak closure and $\Sigma = Q_1^G \cap S_G$ is of order 4. Moreover $\mathrm{Aut}_G(W_0)$ acts as $\mathrm{Sym}(\Sigma)$ on Σ and $U_0 = \langle t_1^{N_G(W_0)} \rangle$ is of rank 4 with $t_1^{+N_G(W_0)} = U_0^+ - U_1^+$ for a unique hyperplane U_1 of U_0, and a generator t^+ for $Z(\mathrm{SU}(V)^+)$ is not contained in U_0^+ as $|t| = 4$. In particular $\prod_i t_i^+ = 1$.

As $H \cong \mathrm{SU}(V)^+ Q_1^+$ it follows that $P^H \cap N_S(Q) = \theta$ is of order 4 and $\Theta = Q^{\mathcal{F}} = \theta \cup \{Q\}$ is of order 5. Moreover $W = \langle \Theta \rangle$ is abelian with $\mathrm{Aut}_{\mathcal{F}}(W)$ transitive on Θ, so as $\mathrm{Aut}_{N_{\mathcal{F}}(Q)}(W_0)$ induces $\mathrm{Sym}(\theta)$ on θ, $\mathrm{Aut}_{\mathcal{F}}(W)$ induces $\mathrm{Sym}(\Theta)$ on Θ. Let x_i be the involution in $P_i \in \theta$ and y the involution in Q. As $\prod_i t_i^+ = 1$ we have $\prod_i x_i = 1$. Then as $\prod_i x_i = 1$ and $\mathrm{Aut}_{\mathcal{F}}(\Theta)$ induces $\mathrm{Sym}(\Theta)$ on Θ, we conclude that $y = x_1x_2x_3 \in U = \langle x_i : 1 \leq i \leq 4\rangle$, contradicting $t^+ \notin U_0^+$. □

We are now in a position to complete the proof of Theorem 7.1. By 7.4 and 7.5, 7.4(1) holds. Thus $x^* \in i(4,+)$ induces an inner automorphism on K, so $x = ck$ for some $c \in Q$ and $k \in K$ with $k^* = x^*$. Then k is of type $i(4,+)$ or $i(2,-)$ in $K \cong \Omega_6^-(3)$. Further $1 = x^2 = c^2k^2 = c^2$, so as $Q \cong \mathbb{Z}_4$ by 7.4, $c \in Z(K)$ so $x \in K$ is in $i(4,+)$ or $i(2,-)$. Let t be the involution in Q, $X = \langle t, x\rangle$, and $\alpha \in \mathfrak{A}(x)$ with $x\alpha = t$. As P centralizes $O^2(C_K(x)) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$ and $x \in P$, we may take $X\alpha = X$. Then as $Z(O^2(C_K(x)))$ is generated by an element in $i(4,+)$ and $x\alpha = t \notin O^2(C_K(x))$, we conclude that x is in $i(2,-)$. This completes the proof of Theorem 7.1.

8. Chev(2)

In this section we assume the following hypothesis:

Hypothesis 8.1. Hypothesis 5.1 is satisfied with $K/Z(K) \in \mathrm{Chev}(2)$.

We continue to adopt the notation from Section 5, and choose $P \in \Delta$ as in Notation 5.7.

8.2. *P is tightly embedded in H.*

Proof. This is 3.7(1). □

In the remainder of the section we assume:

Hypothesis 8.3. Hypothesis 8.1 holds with $\Phi(Q) = 1$ and $|Q| > 2$.

8.4. *$P \leq QK$.*

Proof. This is 3.7(2). □

8.5. *Assume $m(Q) > 2$. Then for each $h \in H$, $N_{P^h}(P) \leq C_H(P)$.*

Proof. See the proof of 15.18 and 21.2 in [8]. □

8.6. *Assume $K/Z(K) \cong L_3(4)$. Then:*

(1) $Q = Z(K) \cong E_4$.

(2) $P^* = Z(T^*)$.

Proof. By choice of P in Notation 5.7, $P \in N_{\mathcal{F}}(Q)^f$ and by 8.4, $P^* \leq T^*$, so as K^* has one class of involutions there exists an involution $z \in P$ with $z^* \in Z(T^*)$, and hence $z \in Z(T)Q$. Let $J = TQ$ and suppose $m(Q) > 2$. Then $z \in Z(J)$ so for $h \in H$, $P^h \cap J \leq C_{P^h}(z) \leq N_{P^h}(P)$ by 8.2, so $P^h \cap J \leq C_H(P)$ by 8.5. But

$J \le \langle z^H \cap J\rangle Q$, so $P \le Z(J) = Z(T)Q$. Therefore as $|Z(J):Q| = 4 < |P|$, $1 \ne P \cap Q$, a contradiction.

We've shown $m(Q) \le 2$, so as $|Q| > 2$ by Hypothesis 8.3, we have $Q \cong E_4$. As $J \le C_H(z) \le N_H(P)$, we have $[P, T] \le P \cap T$. Indeed if $P^* \ne Z(T^*)$ then $[P^*, T^*] = Z(T^*)$, so $Z(T)Q \le PQ$, contradicting $|P| = |Q| = 4$. Hence $PQ = Z(J)$, so the lemma holds if $Q = Z(K)$. Thus we may assume $|Z(K)| = 2$ and it remains to produce a contradiction.

Next $J = J(N_S(Q))$, so $\mathcal{N} = N_{\mathcal{F}}(J)$ controls fusion of conjugates of members of Q in $Z(J)$. Therefore a generator k of $Z(K)$ is fused to a member of P in $\mathcal{N}$. This is a contradiction as k is the only member of $[J, J]$ which is not a commutator in J, which we saw during the proof of 2.6. In any event this is a contradiction. □

8.7. *$K/Z(K)$ is not $U_6(2)$ or $\Omega_8^+(2)$.*

Proof. Assume otherwise. By 8.2, P is tightly embedded in H, and by 8.5, either

(a) $|P| = 4$, or

(b) for each $h \in H$, $N_{P^h}(P) \le C_H(P)$.

Therefore Hypothesis 22.1 in [10] is satisfied, so as $Z(K) \ne 1$, 22.2 in [10] supplies a contradiction. Note that the case $K/Z(K) \cong L_3(4)$ should have been excluded in Hypothesis 21.1, and it was not treated in 21.3 in [10]. □

8.8. *$K/Z(K)$ is not $G_2(4)$, $F_4(2)$, or ${}^2E_6(2)$.*

Proof. We can repeat the proof of 8.3 in [11]. Note the appeal to 3.9 in paragraph four of that proof should be an appeal to 2.7. □

Theorem 8.9. *Assume Hypothesis 8.1 with $\Phi(Q) = 1$. Then either $|Q| = 2$ or $K/Z(K) \cong L_3(4)$ and $Q = Z(K) \cong E_4$.*

Proof. Assume otherwise; then Hypothesis 8.3 is satisfied. Then $K \in \mathcal{K}_{qs}$, with $K/Z(K) \in \mathrm{Chev}(2)$. In particular K appears in 1.5. But we have eliminated the cases in 1.5(1)–(2) in 8.7 and 8.8, and 8.6 completes the proof when 1.5(3) holds. □

9. Mopping up

In this section we assume the following hypothesis:

Hypothesis 9.1. Hypothesis 8.1 is satisfied with Q cyclic and $|Q| > 2$.

We continue to adopt the notation from Section 5, and choose $P \in \Delta$ as in Notation 5.7. Let x be the involution in P and $\langle u\rangle = U = \Omega_2(P)$. Let $S_P = PT$, $\mathcal{H} = P\mathcal{C}$, and $W = \langle U^{\mathcal{H}}\rangle$.

9.2. *$\Phi(P) \le QK$, so $x^* \in K^*$.*

Proof. From [14] a Sylow 2-subgroup of $\mathrm{Out}(K)$ is of exponent at most 2. □

9.3. *$K/Z(K)$ is not $G_2(4)$.*

Proof. Assume otherwise; from 9.2, $x^* \in K^*$, so x^* is in one of the two classes of involutions of K^* described in 18.2 of [10], and $C_{K^*}(x^*)$ is described in 18.4 of [10]. In particular $C_{H^*}(O^2(C_{K^*}(x^*))) = X^*$ is the root subgroup of the root involution x^*, whereas from 3.3(2) and 8.2, $P^* \leq X^*$ and P is cyclic by Hypothesis 9.1. □

9.4. *$K/Z(K)$ is not $L_3(4)$.*

Proof. Assume otherwise; then K^* has one class of involutions, so as $P \in N_{\mathcal{F}}(Q)^f$, $x^* \in Z(T^*)$ and then by 1.6, $x \in Z(QT)$. Hence by 8.2, $[QT, U] \leq U$, whereas T^* normalizes no $\mathbb{Z}_4$-subgroup of H^*. □

Given a finite group G, a *near transposition* in G is an involution t such that whenever $s \in t^G$ with $\langle s, t\rangle$ a 2-group, we have $[s, t] = 1$.

9.5. *Let $R \in \mathcal{H}^{\mathrm{frc}}$ with $R \trianglelefteq S_P$, let Y be a model for $N_{\mathcal{H}}(R)$, and set $Y^+ = Y/R$. Then*:

(1) $\Phi(W) \leq R$.

(2) *If $u \notin R$ then u^+ is a near transposition in Y^+.*

Proof. This follows from 18.4 in [8]. □

9.6. *$K/Z(K)$ is not $U_6(2)$.*

Proof. Assume otherwise; regard K^* as the image of $\widehat{K} = \mathrm{SU}(V)$ for a 6-dimensional unitary space V over $\mathbb{F}_4$. Let V_3 be a T-invariant 3-dimensional totally singular subspace of V, $Y = N_H(V_3)$, and $R = O_2(Y)$. Then $R^* \cong E_{2^9}$ and $Y^+ \cong L_3(4)$ is irreducible on R^*. By 9.5, $x \in \Phi(W) \leq R$ and as $\Phi(R^*) = 1$, $u \notin R$. Thus u^+ is a near transposition in Y^+ by 9.5(2). This contradicts 17.2 in [8], which says that $L_3(4)$ has no near transpositions. □

9.7. *$K/Z(K)$ is not $\Omega_8^+(2)$.*

Proof. Assume otherwise; then K^* has three maximal parabolics Y_i^*, $1 \leq i \leq 3$, such that $R_i^* = O_2(Y_i^*)$ is the orthogonal module for $Y_i^*/R_i^* \cong \Omega_6^+(2)$. As Q is cyclic, $|Z(K)| = 2$, so there is a unique i with $\Phi(R_i) = 1$, say $i = 1$. By 9.5(1), $x \in R_1$, and as $\Phi(R_1^*) = 1$, $u \notin R_1$. Hence by 9.5(2), u^+ is a near transposition in Y^+, where $Y = N_H(R_1^*)$. It follows that $Y^+ \cong O_6^+(2)$ and u^+ is a transvection on the orthogonal space R_1^*. Then $[R_1, u] = \langle x\rangle$ with x^* a nonsingular point in R_1^*, so $X^* = O^2(C_{K^*}(x^*))$ is $[R_1^*, X^*]$ extended by $\mathrm{Sp}_6(2)$. This is a contradiction as such an X^* centralizes no element of Y^* of order 4. □

9.8. *$K/Z(K)$ is not ${}^2E_6(2)$.*

Proof. Assume otherwise; we repeat the argument in 18.15, 18.6, and 18.19 in [8].

Let z^* be the long root involution in $Z(T^*)$ and $Y^* = C_{K^*}(z^*)$. Then $R^* = O_2(Y^*) \cong 2^{1+20}$ is extraspecial with $Y^+ \cong U_6(2)$ irreducible on $\widetilde{R} = R^*/\langle z^*\rangle$. In particular Y^* centralizes no element of order 4, so $x^* \neq z^*$. Then as $\Phi(R^*) = \langle z^*\rangle$, $u^* \notin R^*$, so u^+ is a near transposition in Y^+. Hence by 17.7(1) in [8], u^+ is a transvection in Y^+. As $\Phi(R) \leq Z(K)\langle z\rangle$, $|R : C_R(x)| \leq 4$, so as $[C_R(x), u] \leq \langle x\rangle$ we have $m([\widetilde{R}, u]) \leq 3$. Then by 17.7(2) in [8], $m(\widetilde{R}) \leq 18$, a contradiction. □

9.9. *$K/Z(K)$ is not $F_4(2)$.*

Proof. Assume otherwise and for $c \in \{l, s\}$ let z_c^* be the c-root involution in $Z(T^*)$. By 1.6, $z_c \in Z(T)$.

Suppose $u^* \notin K^*$; then $z_l^{*u} = z_s^*$ so $[z_l, u] = z_l z_s = x' \in Z(T)$, and hence $x = x'$. But then $T = C_T(u)\langle z_l\rangle$, a contradiction.

Hence $u^* \in K^*$, so $u \in Y = C_{PK}(z_l)$. Now from Section 8 in [3], $R^* = O_2(Y^*) = R_1^* Z(R^*)$ where $R_1^* \cong 2^{1+8}$ and $Z(R^*) \cong E_{2^7}$, and $Y^*/R^* \cong \mathrm{Sp}_6(2)$ with $\langle z_l^*\rangle = Z(Y^*)$. Hence $x^* \neq z_l^*$ and as $\Phi(R^*) = \langle z_l^*\rangle$, we conclude that $u^* \notin R^*$. Therefore u^+ is a near transposition in $Y^+ \cong \mathrm{Sp}_6(2)$, so u^+ is a transvection. Arguing as in the proof of the previous lemma, $m([\widetilde{R}, u]) \leq 3$. This contradicts $m([R/Z(R), u]) = 4$. □

Theorem 9.10. *Assume Hypothesis 8.1 holds with Q cyclic. Then $|Q| = 2$.*

Proof. By Hypothesis 8.1, $K \in \mathcal{K}_{\mathrm{qs}}$ with $K/Z(K) \in \mathrm{Chev}(2)$. Thus K is described in 1.5. But now the various cases arising in 1.5 are treated in this section, establishing the theorem. □

Theorem 2. *Assume Hypothesis 5.1. Then one of the following holds*:

(1) $|Q| = 2$.

(2) $K \cong \Omega_6^\epsilon(q)$ *with* $q \equiv \pm 3 \bmod 8$ *and* $\epsilon \equiv q \bmod 4$, *and* $Q \cong \mathbb{Z}_4$.

(3) $K/Z(K) \cong L_3(4)$ *and* $Q = Z(K) \cong E_4$.

Proof. By Hypothesis 5.1(3), $\mathcal{C}$ is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$. The class $\mathcal{K}_{\mathrm{qs}}$ is defined in Definition 1.3; in particular one of the following holds:

(i) $K/Z(K)$ is sporadic.

(ii) $K/Z(K) \in \mathrm{Chev}(2)$.

(iii) $K \cong \Omega_6^\epsilon(q)$ with $q \equiv \pm 3 \bmod 8$ and $\epsilon \equiv q \bmod 4$.

This follows as $K/Z(K)$ is a known finite simple group that is not Goldschmidt, $Z(K)$ is of even order, coverings of alternating groups are excluded in Definition 1.3(a), and coverings of groups of Lie type and odd characteristic, with exception of the orthogonal group in (iii), are excluded in Definition 1.3(b).

In case (i) conclusion (1) of Theorem 2 holds by Theorem 6.1. In case (iii) conclusion (1) or (2) of Theorem 2 holds by Theorem 7.1. This leaves case (ii), where Hypothesis 8.1 is satisfied.

By 5.3 either $\Phi(Q) = 1$ or Q is cyclic. In the first case conclusion (1) or (3) of Theorem 2 holds by Theorem 8.9. Thus we may assume Hypothesis 9.1 is satisfied. Here conclusion (1) of Theorem 2 holds by Theorem 9.10, completing the proof. □

Finally we supply a proof of Theorem 1. So assume the hypothesis of that theorem. First we check that the hypotheses of 2.1 are satisfied. Condition 2.1(a) holds by hypothesis (c) of Theorem 1. Condition 2.1(b) holds by hypothesis (b) of Theorem 1. And condition 2.1(c) holds by hypothesis (a) of Theorem 1.

By 2.1, either conclusion (1) of Theorem 1 is satisfied, or $\mathcal{C}$ is terminal, and we may assume the latter. Thus Hypothesis 2.2 is satisfied. Hence by 2.3(1), $\mathcal{C}$ is nearly standard, so Q exists. We may assume conclusion (2)(i) of Theorem 1 does not hold, so $|Q| > 2$. Hence by Theorem 2.8 either conclusion (2)(iii) holds or $\mathcal{C}$ is standard, and we may assume the latter. Therefore Hypothesis 5.1 is satisfied. But now Theorem 2 completes the proof.

References

[1] M. Aschbacher, "A characterization of some finite groups of characteristic 3", *J. Algebra* **76**:2 (1982), 400–441. MR Zbl

[2] M. Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics **10**, Cambridge Univ. Press, 1986. MR Zbl

[3] M. Aschbacher, "Overgroup lattices in finite groups of Lie type containing a parabolic", *J. Algebra* **382** (2013), 71–99. MR Zbl

[4] M. Aschbacher, "Finite simple groups and fusion systems", pp. 1–15 in *Groups St Andrews* 2017 *in Birmingham*, edited by C. M. Campbell et al., London Math. Soc. Lecture Note Ser. **455**, Cambridge Univ. Press, 2019. MR Zbl

[5] M. Aschbacher, *On fusion systems of component type*, Mem. Amer. Math. Soc. **1236**, 2019. MR Zbl

[6] M. Aschbacher, *Quaternion fusion packets*, Contemporary Mathematics **765**, Amer. Math. Soc., Providence, RI, 2021. MR Zbl

[7] M. Aschbacher, "Walter's theorem for fusion systems", *Proc. Lond. Math. Soc.* (3) **122**:4 (2021), 569–615. MR Zbl

[8] M. Aschbacher, "Fusion systems with 2-small components", *Trans. Amer. Math. Soc.* **376**:9 (2023), 6063–6121. MR Zbl

[9] M. Aschbacher and B. Oliver, "Fusion systems", *Bull. Amer. Math. Soc.* (*N.S.*) **53**:4 (2016), 555–615. MR Zbl

[10] M. Aschbacher and G. M. Seitz, "Involutions in Chevalley groups over fields of even order", *Nagoya Math. J.* **63** (1976), 1–91. MR Zbl

[11] M. Aschbacher and G. M. Seitz, "On groups with a standard component of known type", *Osaka Math. J.* **13**:3 (1976), 439–482. MR Zbl

[12] M. Aschbacher, R. Kessar, and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series **391**, Cambridge Univ. Press, 2011. MR Zbl

[13] D. M. Goldschmidt, "2-fusion in finite groups", *Ann. of Math.* (2) **99** (1974), 70–117. MR Zbl

[14] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups, III: Almost simple K-groups*, Mathematical Surveys and Monographs **40.3**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

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In memoriam
Gary Seitz