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OF DISTINGUISHED UNIPOTENT ELEMENTS  
IN REDUCTIVE GROUPS**

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# ON GOOD $A_1$ SUBGROUPS, SPRINGER MAPS, AND OVERGROUPS OF DISTINGUISHED UNIPOTENT ELEMENTS IN REDUCTIVE GROUPS

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*Dedicated to the fond memory of Gary Seitz*

Suppose  $G$  is a simple algebraic group defined over an algebraically closed field of good characteristic  $p$ . In 2018 Korhonen showed that if  $H$  is a connected reductive subgroup of  $G$  which contains a distinguished unipotent element  $u$  of  $G$  of order  $p$ , then  $H$  is  $G$ -irreducible in the sense of Serre. We present a short and uniform proof of this result under an extra hypothesis using so-called *good*  $A_1$  subgroups of  $G$ , introduced by Seitz. In the process we prove some new results about good  $A_1$  subgroups of  $G$  and their properties. We also formulate a counterpart of Korhonen's theorem for overgroups of  $u$  which are finite groups of Lie type. Moreover, we generalize both results above by removing the restriction on the order of  $u$  under a mild condition on  $p$  depending on the rank of  $G$ , and we present an analogue of Korhonen's theorem for Lie algebras.

## 1. Introduction and main results

Throughout,  $G$  is a connected reductive linear algebraic group defined over an algebraically closed field  $k$  of characteristic  $p$  and  $H$  is a closed subgroup of  $G$ .

Following Serre [35], we say that  $H$  is  $G$ -completely reducible ( $G$ -cr for short) provided that whenever  $H$  is contained in a parabolic subgroup  $P$  of  $G$ , it is contained in a Levi subgroup of  $P$ , and that  $H$  is  $G$ -irreducible ( $G$ -ir for short) provided  $H$  is not contained in any proper parabolic subgroup of  $G$  at all. Clearly, if  $H$  is  $G$ -irreducible, it is trivially  $G$ -completely reducible, and an overgroup of a  $G$ -irreducible subgroup is again  $G$ -irreducible; for an overview of this concept see [4], [34] and [35]. Note that in case  $G = \mathrm{GL}(V)$  a subgroup  $H$  is  $G$ -completely reducible exactly when  $V$  is a semisimple  $H$ -module and it is  $G$ -irreducible precisely

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when  $V$  is an irreducible  $H$ -module. Recall that if  $H$  is  $G$ -completely reducible, then the identity component  $H^\circ$  of  $H$  is reductive [35, Proposition 4.1].

A unipotent element  $u$  of  $G$  is *distinguished* provided any torus in the centralizer  $C_G(u)$  of  $u$  in  $G$  is central in  $G$ . Likewise, a nilpotent element  $X$  of the Lie algebra  $\mathfrak{g}$  of  $G$  is *distinguished* provided any torus in the centralizer  $C_G(X)$  of  $X$  in  $G$  is central in  $G$ , see [10, Section 5.9] and [16, Section 4.1]. For instance, regular unipotent elements in  $G$  are distinguished, and so are regular nilpotent elements in  $\mathfrak{g}$  [39, III, 1.14] (or [10, Proposition 5.1.5]). The converse is true in type  $A$ , since a distinguished unipotent or nilpotent element must clearly consist of a single Jordan block. Overgroups of regular unipotent elements have attracted much attention in the literature, e.g., see [22; 32; 42; 44] and [6].

The following remarkable result was proved by Korhonen.

**Theorem 1.1** [17, Theorem 6.5]. *Suppose  $G$  is simple and  $p$  is good for  $G$ . Let  $H$  be a reductive subgroup of  $G$ . Suppose  $H^\circ$  contains a distinguished unipotent element of  $G$  of order  $p$ . Then  $H$  is  $G$ -irreducible.*

One can easily extend this theorem to arbitrary connected reductive  $G$  by reducing to the simple case, see Remark 6.3.

Korhonen's proof of Theorem 1.1 depends on checks for the various possible Dynkin types for simple  $G$ . E.g., for  $G$  simple of exceptional type, Korhonen's argument relies on long exhaustive case-by-case investigations from [20], where all connected reductive non- $G$ -cr subgroups are classified in the exceptional type groups in good characteristic. For classical  $G$ , Korhonen requires an intricate classification of all  $\mathrm{SL}_2$ -representations on which a nontrivial unipotent element of  $\mathrm{SL}_2$  acts with at most one Jordan block of size  $p$ . Our main aim is to give a short uniform proof of Theorem 1.1 in Section 6 without resorting to further case-by-case checks, but imposing an extra hypothesis which allows us to use a landmark result by Seitz (see Section 5.1).

**Theorem 1.2.** *Suppose  $p$  is good for  $G$ . Let  $H$  be a connected reductive subgroup of  $G$ . Suppose  $H$  contains a distinguished unipotent element of  $G$  of order  $p$ . Suppose also that:*

(†) *There exists a Springer map  $\phi$  for  $H$  such that  $\phi(u)$  is a distinguished element of  $\mathfrak{g}$ .*

*Then  $H$  is  $G$ -irreducible.*

For a discussion of Springer maps, see Section 4.1.

**Remark 1.3.** Suppose as in Theorem 1.1, that  $G$  is simple classical with natural module  $V$ , and  $p \geq \dim V > 2$ . Then, thanks to [15, Proposition 3.2],  $V$  is semisimple as an  $H^\circ$ -module, and by [35, (3.2.2(b))], this is equivalent to  $H^\circ$  being

$G$ -cr. Then  $H$  is  $G$ -ir, by [Lemma 3.1](#). This gives a short uniform proof of the conclusion of [Theorem 1.1](#) in this case, as the bound  $p \geq \dim V > 2$  ensures that every distinguished unipotent element (including the regular ones) is of order  $p$ . The conclusion can fail if the bound is not satisfied, see [Theorem 1.5](#).

We say that a subgroup of  $G$  is of *type  $A_1$*  if it is isomorphic to  $SL_2$  or  $PGL_2$ . Our proof of [Theorem 1.2](#) involves the notion of a *good  $A_1$  subgroup*, which was introduced by Seitz in [33]. We consider the interaction of good  $A_1$  subgroups with associated cocharacters and Springer maps; we identify a useful class of Springer maps ([Definition 5.16](#)), which we call *logarithmic Springer maps*, and we prove some results that are of interest in their own right (see [Corollary 5.20](#) and [Lemma 5.30](#)). Our main result on good  $A_1$  subgroups is the following (see [Section 5.2](#) for definitions).

**Theorem 1.4.** *Suppose  $p$  is good for  $G$  and let  $A$  be an  $A_1$  subgroup of  $G$ . The following are equivalent.*

- (i)  $A$  is subprincipal.
- (ii)  $A$  is optimal.
- (iii)  $A$  is good.

[Theorem 1.1](#) covers the situation when  $p$  is good for  $G$ . There are only a few cases when  $G$  is simple,  $p$  is bad for  $G$ , and  $G$  admits a distinguished unipotent element of order  $p$ , by work of Proud, Saxl, and Testerman [31, Lemmas 4.1, 4.2] (see [Lemmas 2.5](#) and [2.7](#)). In this case the conclusion of [Theorem 1.1](#) fails precisely in one instance, as observed in [17, Proposition 1.2] ([Example 2.6](#)), else it is valid ([Example 2.8](#)). Combining the cases when  $p$  is bad for  $G$  with [Theorem 1.2](#), we recover Korhonen's main theorem [17, Theorem 1.3] (assuming that  $(\dagger)$  from [Theorem 1.2](#) holds).

**Theorem 1.5.** *Suppose  $G$  is simple and let  $H$  be a reductive subgroup of  $G$ . Suppose  $H^\circ$  contains a distinguished unipotent element of  $G$  of order  $p$ , and suppose that  $(\dagger)$  holds. Then  $H$  is  $G$ -irreducible, unless  $p = 2$ ,  $G$  is of type  $C_2$ , and  $H$  is a type  $A_1$  subgroup of  $G$ .*

Our next goal is an extension of [Theorem 1.2](#) to finite groups of Lie type in  $G$ . Let  $\sigma : G \rightarrow G$  be a Steinberg endomorphism of  $G$ , so that the finite fixed point subgroup  $G_\sigma = G(q)$  is a finite group of Lie type over the field  $\mathbb{F}_q$  of  $q$  elements. For a Steinberg endomorphism  $\sigma$  of  $G$  and a connected reductive  $\sigma$ -stable subgroup  $H$  of  $G$ ,  $\sigma$  is also a Steinberg endomorphism for  $H$  with finite fixed point subgroup  $H_\sigma = H \cap G_\sigma$  [40, 7.1(b)]. Obviously, one cannot directly appeal to [Theorem 1.2](#) to deduce anything about  $H_\sigma$ , because  $(H_\sigma)^\circ$  is trivial. For the notion of a  $q$ -Frobenius endomorphism, see [Section 2.3](#).

**Theorem 1.6.** *Let  $H$  be a connected reductive subgroup of  $G$  and suppose  $p$  is good for  $G$ . Let  $\sigma : G \rightarrow G$  be a Steinberg endomorphism stabilizing  $H$  such that  $\sigma|_H$  is a  $q$ -Frobenius endomorphism of  $H$ . If  $G$  admits components of exceptional type, then assume  $q > 7$ . Suppose  $H_\sigma$  contains a distinguished unipotent element of  $G$  of order  $p$ , and suppose that  $(\dagger)$  holds. Then  $H_\sigma$  is  $G$ -irreducible.*

Combining [Theorem 1.6](#) with the aforementioned results from [\[31\]](#), we are able to deduce the following analogue of [Theorem 1.5](#) for finite subgroups of Lie type in  $G$ .

**Theorem 1.7.** *Let  $H$  be a connected reductive subgroup of  $G$ . Let  $\sigma : G \rightarrow G$  be a Steinberg endomorphism stabilizing  $H$  such that  $\sigma|_H$  is a  $q$ -Frobenius endomorphism of  $H$ . If  $G$  is of exceptional type, then assume  $q > 7$ . Suppose  $H_\sigma$  contains a distinguished unipotent element of  $G$  of order  $p$ , and suppose that  $(\dagger)$  holds. Then  $H_\sigma$  is  $G$ -irreducible, unless  $p = 2$ ,  $G$  is of type  $C_2$ , and  $H$  is a type  $A_1$  subgroup of  $G$ .*

In the special instance in [Theorems 1.6](#) and [1.7](#) when  $H_\sigma$  contains a regular unipotent element  $u$  from  $G$ , the conclusion of both theorems holds without any restriction on the order of  $u$  and without any restriction on  $q$  (and without any exceptions of the type seen in [Theorem 1.7](#)), see [\[6, Theorem 1.3\]](#).

In our final main result we show that we can remove condition  $(\dagger)$  and the condition that  $u$  has order  $p$  from [Theorem 1.2](#), at the cost of increasing our bound on  $p$ . We also obtain an analogue under the hypothesis that  $\text{Lie}(H)$  contains a distinguished nilpotent element of  $\mathfrak{g}$ . For a unipotent element  $u \in G$  to be distinguished is a mere condition on the structure of the centralizer  $C_G(u)$  of  $u$  in  $G$ . The extra condition for  $u$  to have order  $p$  is thus somewhat artificial. This restriction in [Theorems 1.1](#) and [1.2](#) is due to the methods used in [\[17\]](#) and in our proofs in [Section 6](#), which require the unipotent element to lie in a subgroup of type  $A_1$ ; such an element must obviously have order  $p$ .

To state our theorem, we need to introduce an invariant  $a(G)$  of  $G$  from [\[35, Section 5.2\]](#): for  $G$  simple, set  $a(G) = \text{rk}(G) + 1$ , where  $\text{rk}(G)$  is the rank of  $G$ . For reductive  $G$ , let  $a(G) = \max\{1, a(G_1), \dots, a(G_r)\}$ , where  $G_1, \dots, G_r$  are the simple components of  $G$ .

**Theorem 1.8.** *Suppose  $p \geq a(G)$ . Let  $H$  be a reductive subgroup of  $G$ . Suppose  $H^\circ$  contains a distinguished unipotent element of  $G$  or  $\text{Lie}(H)$  contains a distinguished nilpotent element of  $\mathfrak{g}$ . Then  $H$  is  $G$ -irreducible.*

[Section 2](#) contains background material. In [Section 3](#) we prove [Theorem 1.8](#), along with some analogues for finite subgroups of Lie type. In [Section 4](#) we discuss Springer maps and associated cocharacters. We recall Seitz's notion of good  $A_1$  subgroups in [Section 5](#) and we prove [Theorem 1.4](#) in [Section 5.2](#) (see [Theorem 5.24](#)). [Theorems 1.2](#) and [1.5–1.7](#) are proved in [Section 6](#).

## 2. Preliminaries

**2.1. Notation.** Throughout, we work over an algebraically closed field  $k$  of characteristic  $p$ . For convenience we assume that  $p > 0$  unless otherwise stated; most of our results hold for  $p = 0$  with obvious modifications and in many cases the proof is much easier (see [Remarks 3.3\(vi\)](#), for example). All affine varieties are considered over  $k$  and are identified with their sets of  $k$ -points. A linear algebraic group  $H$  over  $k$  has identity component  $H^\circ$ ; if  $H = H^\circ$ , then we say that  $H$  is *connected*. We denote by  $R_u(H)$  the *unipotent radical* of  $H$ ; if  $R_u(H)$  is trivial, then we say  $H$  is *reductive*.

Throughout,  $G$  denotes a connected reductive linear algebraic group over  $k$ . All subgroups of  $G$  considered are closed. By  $\mathcal{D}G$  we denote the derived subgroup of  $G$ , and likewise for subgroups of  $G$ . We denote the Lie algebra of  $G$  by  $\text{Lie}(G)$  or by  $\mathfrak{g}$ . If  $p > 0$  then we denote the  $p$ -power map on  $\mathfrak{g}$  by  $X \mapsto X^{[p]}$ . By a Levi subgroup of  $G$  we mean a Levi subgroup of some parabolic subgroup of  $G$ . Recall that a homomorphism  $f : G_1 \rightarrow G_2$  of connected algebraic groups is a *central isogeny* if  $f$  is surjective,  $\ker(f)$  is finite and the kernel of the derivative  $df$  is central in  $\text{Lie}(G_1)$ .

Let  $Y(G) = \text{Hom}(\mathbb{G}_m, G)$  denote the set of cocharacters of  $G$ . For  $\mu \in Y(G)$  and  $g \in G$  we define the *conjugate cocharacter*  $g \cdot \mu \in Y(G)$  by  $(g \cdot \mu)(t) = g\mu(t)g^{-1}$  for  $t \in \mathbb{G}_m$ ; this gives a left action of  $G$  on  $Y(G)$ . For  $H$  a subgroup of  $G$ , let  $Y(H) := Y(H^\circ) = \text{Hom}(\mathbb{G}_m, H)$  denote the set of cocharacters of  $H$ . There is an obvious inclusion  $Y(H) \subseteq Y(G)$ .

Fix a Borel subgroup  $B$  of  $G$  containing a maximal torus  $T$ . Let  $\Phi = \Phi(G, T)$  be the root system of  $G$  with respect to  $T$ , let  $\Phi^+ = \Phi(B, T)$  be the set of positive roots of  $G$ , and let  $\Sigma = \Sigma(G, T)$  be the set of simple roots of  $\Phi^+$ . For each  $\alpha \in \Phi$  we have a root subgroup  $U_\alpha$  of  $G$ . For  $\alpha$  in  $\Phi$ , let  $x_\alpha : \mathbb{G}_a \rightarrow U_\alpha$  be a parametrization of the root subgroup  $U_\alpha$  of  $G$ .

We denote the unipotent variety of  $G$  by  $\mathcal{U}_G$  and the nilpotent cone of  $\mathfrak{g}$  by  $\mathcal{N}_G$ . We define

$$\mathcal{U}_G^{(1)} = \{u \in \mathcal{U}_G \mid u^p = 1\}$$

and

$$\mathcal{N}_G^{(1)} = \{X \in \mathcal{N}_G \mid X^{[p]} = 0\}.$$

If  $u \in \mathcal{U}_G$  then we have a unique decomposition  $u = u_1 \cdots u_r$ , where  $u_i \in G_i$  and the  $G_i$  are the simple factors of  $\mathcal{D}G$ ; we call  $u_i$  the *projection of  $u$  onto  $G_i$* . Clearly  $u$  is distinguished in  $G$  if and only if  $u_i$  is distinguished in  $G_i$  for each  $i$ .

**2.2. Good primes.** A prime  $p$  is said to be *good* for  $G$  if it does not divide any coefficient of any positive root when expressed as a linear combination of simple ones. Else  $p$  is called *bad* for  $G$  [[39](#), Section 4]. Explicitly, if  $G$  is simple,  $p$  is

good for  $G$  provided  $p > 2$  in case  $G$  is of Dynkin type  $B_n$ ,  $C_n$ , or  $D_n$ ;  $p > 3$  in case  $G$  is of Dynkin type  $E_6$ ,  $E_7$ ,  $F_4$  or  $G_2$  and  $p > 5$  in case  $G$  is of type  $E_8$ . If  $G$  is semisimple then we say that  $p$  is *separably good* for  $G$  if  $p$  is good for  $G$  and the canonical map from  $G_{\text{sc}}$  to  $G$  is separable, where  $G_{\text{sc}}$  is the simply connected cover of  $G$ . For arbitrary connected reductive  $G$  we say that  $p$  is *separably good* for  $G$  if it is separably good for  $[G, G]$ . We observe that if  $L$  is a Levi subgroup of  $G$  and  $p$  is good for  $G$ , then it is also good for  $L$ .

**2.3. Steinberg endomorphisms of  $G$ .** Recall that there is a basic dichotomy for endomorphisms of a simple algebraic group: either the endomorphism is an automorphism, or its set of fixed points is finite [40, Theorem 10.13]. Given a reductive group  $G$ , a *Steinberg endomorphism of  $G$*  is a surjective homomorphism  $\sigma : G \rightarrow G$  such that the corresponding fixed point subgroup  $G_\sigma := \{g \in G \mid \sigma(g) = g\}$  of  $G$  is finite. If  $\mathcal{S}$  is a  $\sigma$ -stable set of closed subgroups of  $G$ , then  $\mathcal{S}_\sigma$  denotes the subset consisting of all  $\sigma$ -stable members of  $\mathcal{S}$ .

If  $G$  is a reductive group defined over the finite field  $\mathbb{F}_q$  (i.e., with some  $\mathbb{F}_q$ -structure), then the corresponding standard Frobenius endomorphism  $\sigma_q : G \rightarrow G$  is an example of a Steinberg endomorphism, and in this case we also write  $G_\sigma = G(q)$ . In this situation, there exist a  $\sigma_q$ -stable maximal torus  $T$  and Borel subgroup  $B \supseteq T$ , and with respect to a chosen parametrization of the root groups as above, we have  $\sigma_q(x_\alpha(t)) = x_\alpha(t^q)$  for each  $\alpha \in \Phi$  and  $t \in \mathbb{G}_a$ , see [13, Theorem 1.15.4(a)].

Recall that a *generalized Frobenius endomorphism* of a reductive group  $G$  is an endomorphism of  $G$  for which some power is a standard Frobenius endomorphism  $\sigma_q$ . If  $G$  is simple, then every Steinberg endomorphism of  $G$  is actually a generalized Frobenius morphism [13, Theorem 2.1.11]. Further, when  $G$  is simple and  $p$  is good for  $G$ , every such endomorphism has the form  $\sigma = \tau\sigma_q$ , where  $\tau$  is an algebraic automorphism of  $G$  of finite order,  $\sigma_q$  is a standard  $q$ -power Frobenius endomorphism of  $G$ , and  $\sigma_q$  and  $\tau$  commute, see [40, Section 11]. Conversely, it is clear (for arbitrary reductive  $G$ ) that any endomorphism  $\sigma$  which factorizes in this way is a generalized Frobenius endomorphism. However, if  $G$  is not simple and  $p$  is bad for  $G$ , then a generalized Frobenius map may fail to factor into a field and algebraic automorphism of  $G$ , e.g., see [14, Example 1.3].

Following [31], we call a generalized Frobenius endomorphism  $\sigma$  a  *$q$ -Frobenius endomorphism* provided  $\sigma = \tau\sigma_q$ , where  $\tau$  is an algebraic automorphism of  $G$  of finite order,  $\sigma_q$  is a standard  $q$ -power Frobenius endomorphism of  $G$ , and  $\sigma_q$  and  $\tau$  commute.

**2.4. Bala–Carter theory.** We recall some relevant results and concepts from Bala–Carter theory. Suppose  $p$  is good for  $G$ . A parabolic subgroup  $P$  of  $G$  admits a dense open orbit on its unipotent radical  $R_u(P)$ , the so-called *Richardson orbit*, see [10, Theorem 5.2.1]. A parabolic subgroup  $P$  of  $G$  is called *distinguished* provided

$\dim(\mathcal{D}P/R_u(P)) = \dim(R_u(P)/\mathcal{D}R_u(P))$ , see [30, Section 2.1]. For  $G$  simple, the distinguished parabolic subgroups of  $G$  (up to  $G$ -conjugacy) were worked out in [2] and [3], see [10, pp. 174–177]. The notion of a distinguished parabolic subgroup of  $G$  also makes sense in case  $p$  is bad for  $G$ , see [16, Section 4.10].

The following is the celebrated Bala–Carter theorem [10, Theorems 5.9.5, 5.9.6], which is valid in good characteristic, thanks to work of Pommerening [28; 29]. For the Lie algebra versions see also [16, Proposition 4.7, Theorem 4.13].

**Theorem 2.1.** *Suppose  $p$  is good for  $G$ .*

(i) *There is a bijective map between the  $G$ -conjugacy classes of distinguished unipotent elements of  $G$  and conjugacy classes of distinguished parabolic subgroups of  $G$ . The unipotent class corresponding to a given parabolic subgroup  $P$  contains the dense  $P$ -orbit on  $R_u(P)$ .*

(ii) *There is a bijective map between the  $G$ -conjugacy classes of unipotent elements of  $G$  and conjugacy classes of pairs  $(L, P)$ , where  $L$  is a Levi subgroup of  $G$  and  $P$  is a distinguished parabolic subgroup of  $\mathcal{D}L$ . The unipotent class corresponding to the pair  $(L, P)$  contains the dense  $P$ -orbit on  $R_u(P)$ .*

**Remark 2.2.** (i) Let  $1 \neq u \in \mathcal{U}_G$ . Let  $S$  be a maximal torus of  $C_G(u)$ . Then  $u$  is distinguished in the Levi subgroup  $C_G(S)$  of  $G$ , since  $S$  is the unique maximal torus of  $C_{C_G(S)}(u)$ . Conversely, if  $L$  is a Levi subgroup of  $G$  with  $u$  distinguished in  $L$ , then the connected center of  $L$  is a maximal torus of  $C_G(u)^\circ$ , see [16, Remark 4.7].

(ii) Let  $\sigma : G \rightarrow G$  be a Steinberg endomorphism of  $G$  and let  $1 \neq u \in G_\sigma$  be unipotent. Then  $C_G(u)^\circ$  is  $\sigma$ -stable. The set of all maximal tori of  $C_G(u)^\circ$  is  $\sigma$ -stable and  $C_G(u)^\circ$  is transitive on that set [38, Theorem 6.4.1]. Thus the Lang–Steinberg theorem, see [39, I, 2.7], provides a  $\sigma$ -stable maximal torus, say  $S$ , of  $C_G(u)^\circ$ . Then, by part (i),  $L = C_G(S)$  is a  $\sigma$ -stable Levi subgroup of  $G$  and  $u$  is distinguished in  $L$ .

**2.5. Cocharacters and parabolic subgroups of  $G$ .** Let  $\lambda \in Y(G)$ . Recall that  $\lambda$  affords a  $\mathbb{Z}$ -grading on  $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j, \lambda)$ , where

$$\mathfrak{g}(j, \lambda) := \{X \in \mathfrak{g} \mid \text{Ad}(\lambda(t))X = t^j X \text{ for every } t \in \mathbb{G}_m\}$$

is the  $j$ -weight space of  $\text{Ad}(\lambda(\mathbb{G}_m))$  on  $\mathfrak{g}$ , see [10, Section 5.5] or [16, Section 5.1]. Let  $\mathfrak{p}_\lambda := \bigoplus_{j \geq 0} \mathfrak{g}(j, \lambda)$ . Then there is a unique parabolic subgroup  $P_\lambda$  with  $\text{Lie}(P_\lambda) = \mathfrak{p}_\lambda$  and  $C_G(\lambda) := C_G(\lambda(\mathbb{G}_m))$  is a Levi subgroup of  $P_\lambda$ . Since all maximal tori in  $G$  are conjugate, it suffices to describe these subgroups and subalgebras when  $\lambda \in Y(T)$  for our fixed maximal torus  $T$ . In this case, letting  $X(T) = \text{Hom}(T, \mathbb{G}_m)$  denote the character group of  $T$ , we have  $U_\alpha \subseteq P_\lambda$  if and only if  $\langle \lambda, \alpha \rangle \geq 0$ , where  $\langle \cdot, \cdot \rangle : Y(T) \times X(T) \rightarrow \mathbb{Z}$  is the usual pairing between cocharacters and characters.

We have  $U_\alpha \subseteq C_G(\lambda)$  if and only if  $\langle \lambda, \alpha \rangle = 0$ , and  $R_u(P_\lambda)$  is generated by the  $U_\alpha$  with  $\langle \lambda, \alpha \rangle > 0$ , see the proof of [38, Proposition 8.4.5].

Set  $J := \{\alpha \in \Sigma \mid \langle \alpha, \lambda \rangle = 0\}$ . Then  $P_\lambda = P_J = \langle T, U_\alpha \mid \langle \alpha, \lambda \rangle \geq 0 \rangle$  is the *standard parabolic subgroup* of  $G$  associated with  $J \subseteq \Sigma$ .

Let  $\rho = \sum_{\alpha \in \Sigma} c_{\alpha\rho} \alpha$  be the highest root in  $\Phi^+$ . Define  $\text{ht}_J(\rho) := \sum_{\alpha \in \Sigma \setminus J} c_{\alpha\rho}$ . In view of Theorem 2.1, the following gives the order of a distinguished unipotent element in good characteristic.

**Lemma 2.3** [43, order formula (0.4)]. *Suppose  $p$  is good for  $G$ . Let  $P = P_J$  be a distinguished parabolic subgroup of  $G$  and let  $u$  be in the Richardson orbit of  $P$  on  $R_u(P)$ . Then the order of  $u$  is  $\min\{p^a \mid p^a > \text{ht}_J(\rho)\}$ .*

**2.6. Overgroups of type  $A_1$ .** It has been understood for some time now that if  $p$  is good for  $G$  then one can study unipotent elements of  $G$  having order  $p$  by embedding them in  $A_1$  subgroups of  $G$ . The existence of  $A_1$  overgroups for unipotent elements of order  $p$  is guaranteed by the following fundamental results of Testerman [43, Theorem 0.1] if  $p$  is good for  $G$  and else by Proud, Saxl, and Testerman [31]; these results were originally proved for semisimple  $G$  but the extension to arbitrary connected reductive  $G$  is immediate.

**Theorem 2.4** [43, Theorems 0.1, 0.2]. *Suppose  $p$  is good for  $G$ . Let  $\sigma$  be  $\text{id}_G$  or a Steinberg endomorphism of  $G$ . Let  $u \in G_\sigma$  be unipotent of order  $p$ . Then there exists a  $\sigma$ -stable subgroup of  $G$  of type  $A_1$  containing  $u$ .*

The proof of Theorem 2.4 is based on case-by-case checks and depends in part on computer calculations involving explicit unipotent class representatives. For a uniform proof of the theorem, we refer the reader to McNinch [23]. Conditions to ensure  $G$ -complete reducibility of such a subgroup were given in [26].

We now consider  $A_1$  overgroups of distinguished unipotent elements in arbitrary characteristic. There are only a few instances when  $G$  is simple,  $p$  is bad for  $G$ , and  $G$  admits a distinguished unipotent element of order  $p$ . We recall the relevant results concerning the existence of  $A_1$  overgroups of such elements.

**Lemma 2.5** [31, Lemma 4.1]. *Let  $G$  be simple classical of type  $B_l, C_l$ , or  $D_l$  and suppose  $p = 2$ . Then  $G$  admits a distinguished involution  $u$  if and only if  $G$  is of type  $C_2$  and  $u$  belongs to the subregular class  $\mathcal{C}$  of  $G$ . If  $\sigma$  is  $\text{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$  and  $u \in \mathcal{C} \cap G_\sigma$ , then there exists a  $\sigma$ -stable subgroup  $A$  of  $G$  of type  $A_1$  containing  $u$ .*

**Example 2.6.** Let  $G$  be simple of type  $C_2$  and let  $p = 2$ . Let  $\sigma$  be  $\text{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$ , and suppose  $u \in G_\sigma$  is a distinguished unipotent element of order 2. Then Lemma 2.5 provides a  $\sigma$ -stable subgroup  $A$  of type  $A_1$  containing  $u$ . Thanks to [17, Proposition 1.2], there are such subgroups  $A$  which are not  $G$ -ir. In fact, according to the same reference, there are two  $G$ -conjugacy

classes of such  $A_1$  subgroups in  $G$ , see [Example 5.15](#) below. Since  $A$  is contained in a proper parabolic subgroup of  $G$ , so is  $A_\sigma$ . So the latter is also not  $G$ -ir. By [Lemma 3.1](#) below,  $A$  and  $A_\sigma$  are not  $G$ -cr, either.

**Lemma 2.7** [[31](#), Lemmas 3.3, 4.2]. *Let  $G$  be simple of exceptional type and suppose  $p$  is bad for  $G$ . Then  $G$  admits a distinguished unipotent element  $u$  of order  $p$  if and only if  $G$  is of type  $G_2$ ,  $p = 3$ , and  $u$  belongs either to the subregular class  $G_2(a_1)^1$  or to the class  $A_1^{(3)}$  of  $G$ . Moreover, if  $\sigma$  is  $\text{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$  and  $u \in G_2(a_1) \cap G_\sigma$ , then there exists a  $\sigma$ -stable subgroup  $A$  of  $G$  of type  $A_1$  containing  $u$ . In case  $u \in A_1^{(3)}$ , there is no overgroup of  $u$  in  $G$  of type  $A_1$ .*

**Example 2.8.** Let  $G$  be simple of type  $G_2$  and  $p = 3$ . Let  $H$  be a reductive subgroup of  $G$  containing a distinguished unipotent element  $u$  from  $G$ . Then, as  $p = 3 = a(G_2)$ , it follows from [Theorem 1.8](#) that  $H^\circ$  is  $G$ -ir, and so is  $H$ . This applies in particular to the subgroup  $A$  of  $G$  of type  $A_1$  containing  $u$  when  $u \in G_2(a_1)$ . Since 3 is not a good prime for  $G$ , [Theorem 1.1](#) does not apply in this case. See also [[41](#), Corollary 2].

In case of the presence of a  $q$ -Frobenius endomorphism of  $G$  stabilizing  $H$ , we show in our proof of [Theorem 1.7](#) that  $H_\sigma$  is also  $G$ -ir.

**Theorem 2.9** [[31](#), Theorem 5.1]. *Let  $G$  be semisimple and suppose  $p$  is bad for  $G$ . Let  $\sigma$  be  $\text{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$ . Let  $u \in G_\sigma$  be unipotent of order  $p$ . If  $p = 3$ , and  $G$  has a simple component of type  $G_2$ , assume that the projection of  $u$  into this component does not lie in the class  $A_1^{(3)}$ . Then there exists a  $\sigma$ -stable subgroup of  $G$  of type  $A_1$  containing  $u$ .*

**Corollary 2.10.** *Let  $G$  be simple of type  $G_2$ ,  $p = 3$  and let  $\sigma$  be  $\text{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$ . Let  $u \in A_1^{(3)} \cap G_\sigma$ . Then there is no proper semisimple subgroup  $H$  of  $G$  containing  $u$ . In particular, any such  $u$  is **semiregular**, that is,  $C_G(u)$  does not contain a noncentral semisimple element of  $G$ .*

*Proof.* By way of contradiction, suppose  $H$  is a proper semisimple subgroup of  $G$  containing  $u$ . Since  $p = 3$  is good for  $H$  (e.g., see [[41](#), Corollary 3]), there is a  $\sigma$ -stable  $A_1$  subgroup  $A$  in  $H$  containing  $u$ , by [Theorem 2.4](#). It follows from [Lemma 2.7](#) that  $u \in G_2(a_1)$  which contradicts the hypothesis that  $u \in A_1^{(3)}$ .  $\square$

The following result is needed in the proof of [Theorem 1.2](#) below.

**Lemma 2.11.** *Suppose  $G$  is semisimple and  $p = 3$  is good for  $G$ . Let  $H$  be a connected reductive subgroup of  $G$ . Let  $u \in H$  be a unipotent element of order 3 which is distinguished in  $G$ . Then  $H$  does not admit a simple component of type  $G_2$ .*

<sup>1</sup>Throughout, we use the Bala–Carter notation for distinguished classes in the exceptional groups, see [[10](#), Section 5.9].

*Proof* (see [17, p. 387]). Since  $p$  is good for  $G$ , every simple component of  $G$  is of classical type. Let  $V'$  be the natural module of the simple component  $G'$  of  $G$ , and let  $H'$  be the projection of  $H$  into  $G'$ . Since the projection  $u'$  of  $u$  into  $G'$  has order 3, the largest Jordan block size of  $u'$  on  $V'$  is at most 3. Since  $u'$  is distinguished in  $G'$ , the Jordan block sizes of  $u'$  are distinct and of the same parity. Hence  $\dim V' \leq 4$ . Since a nontrivial representation of a simple algebraic group of type  $G_2$  has dimension at least 5,  $H'$  does not have a simple component of type  $G_2$ . Hence  $H$  has no simple component of type  $G_2$ .  $\square$

In summary, we see that if  $1 \neq u \in \mathcal{U}_G^{(1)}$  then  $u$  is contained in an  $A_1$  subgroup of  $G$  unless  $p = 3$  and  $G$  has a simple  $G_2$  factor such that the projection of  $u$  onto this factor lies in the class  $A_1^{(3)}$ .

### 3. Variations on Theorems 1.2 and 1.6

In this section we prove [Theorem 1.8](#). We also state and prove some related results for finite subgroups of Lie type. We need the following analogue of [6, Corollary 4.6], which shows that in order to derive the  $G$ -irreducibility of  $H$  in [Theorem 1.8](#), it suffices to show that  $H$  is  $G$ -cr, see also [17, Lemma 6.1]. This also applies to [Theorems 1.2 and 1.6](#).

**Lemma 3.1.** *Let  $H$  be a  $G$ -completely reducible subgroup of  $G$ . Suppose that  $H$  contains a distinguished unipotent element  $u$  of  $G$  or  $\text{Lie}(H)$  contains a distinguished nilpotent element  $X$  of  $\mathfrak{g}$ . Then  $H$  is  $G$ -irreducible.*

*Proof.* Suppose  $H$  is contained in a parabolic subgroup  $P$  of  $G$ . Then, by hypothesis,  $H$  is contained in a Levi subgroup  $L$  of  $P$ . As the latter is the centralizer of a torus  $S$  in  $G$ ,  $S$  centralizes  $u$  (resp.,  $X$ ) and so  $S$  is central in  $G$ . Hence  $L = G$ , which implies  $P = G$ .  $\square$

Along with [Lemma 3.1](#), the following theorem of Serre immediately yields [Theorem 1.8](#).

**Theorem 3.2** [35, Theorem 4.4]. *Suppose  $p \geq a(G)$  and  $(H : H^\circ)$  is prime to  $p$ . Then  $H^\circ$  is reductive if and only if  $H$  is  $G$ -completely reducible.*

*Proof of [Theorem 1.8](#).* Since  $p \geq a(G)$ , [Theorem 3.2](#) applied to  $H^\circ$  shows the latter is  $G$ -cr. Thus  $H^\circ$  is  $G$ -ir by [Lemma 3.1](#), and so is  $H$ .  $\square$

**Remarks 3.3.** (i) The characteristic restriction in [Theorem 1.8](#) (and [Theorem 3.2](#)) is needed, see [Theorem 1.5](#).

(ii) The condition in [Theorem 1.8](#) that the distinguished unipotent element of  $G$  belongs to  $H^\circ$  (as opposed to  $H$ ) is also necessary, as for instance the finite unipotent subgroup of  $G$  generated by a given distinguished unipotent element of  $G$  is not  $G$ -cr [[35](#), Proposition 4.1].

- (iii) Under the given hypotheses, [Theorem 1.8](#) applies to an arbitrary distinguished unipotent element of  $G$ , irrespective of its order. For [Theorem 1.1](#) to achieve the same uniform result,  $p$  has to be sufficiently large to guarantee that the chosen element has order  $p$ . For  $G$  simple classical with natural module  $V$ , this requires the bound  $p \geq \dim V$ , see [Remark 1.3](#). For  $G$  simple of exceptional type, this requires the following bounds:  $p > 11$  for  $E_6$ ,  $p > 17$  for  $E_7$ ,  $p > 29$  for  $E_8$ ,  $p > 11$  for  $F_4$ , and  $p > 5$  for  $G_2$ , see [[43](#), Proposition 2.2]. So in many cases the bound  $p \geq a(G)$  from [Theorem 1.8](#) is better.
- (iv) For an instance when  $p$  is bad for  $G$  so that [Theorem 1.1](#) does not apply, but [Theorem 1.8](#) does, see [Example 2.8](#).
- (v) [Theorem 1.8](#) generalizes [[6](#), Theorem 3.2] which consists of the analogue in the special instance when the distinguished element is regular in  $G$  (or  $\mathfrak{g}$ ). Note that in this case no restriction on  $p$  is needed, see [[6](#), Theorem 3.2; [22](#), Theorem 1; [44](#), Theorem 1.2].
- (vi) In characteristic 0, a subgroup  $H$  of  $G$  is  $G$ -cr if and only if it is reductive [[35](#), Proposition 4.1]. So in that case the conclusion of [Theorem 1.8](#) follows directly from [Lemma 3.1](#).

Once again, in the presence of a Steinberg endomorphism  $\sigma$  of  $G$ , one cannot appeal to [Theorem 1.8](#) directly to deduce anything about  $H_\sigma$ , because  $(H_\sigma)^\circ$  is trivial. In [Corollary 3.5](#) we present an analogue of [Theorem 1.8](#) for the finite groups of Lie type  $H_\sigma$  under an additional condition stemming from [[7](#)].

Note that for  $S$  a torus in  $G$ , we have  $C_G(S) = C_G(s)$  for some  $s \in S$ , see [[8](#), III, Proposition 8.18].

**Proposition 3.4** [[7](#), Proposition 3.2]. *Let  $H \subseteq G$  be connected reductive groups. Let  $\sigma : G \rightarrow G$  be a Steinberg endomorphism that stabilizes  $H$  and a maximal torus  $T$  of  $H$ . Suppose:*

- (i)  $C_G(T) = C_G(t)$  for some  $t \in T_\sigma$ .
- (ii)  $H_\sigma$  meets every  $T$ -root subgroup of  $H$  nontrivially.

*Then  $H_\sigma$  and  $H$  belong to the same parabolic and the same Levi subgroups of  $G$ . In particular,  $H$  is  $G$ -completely reducible if and only if  $H_\sigma$  is  $G$ -completely reducible; similarly,  $H$  is  $G$ -irreducible if and only if  $H_\sigma$  is  $G$ -irreducible.*

Without condition (i), the proposition is false in general, see [[7](#), Example 3.2]. The following is an immediate consequence of [Theorem 1.8](#) and [Proposition 3.4](#).

**Corollary 3.5.** *Suppose  $G$ ,  $H$  and  $\sigma$  satisfy the hypotheses of [Proposition 3.4](#). Suppose in addition that  $p \geq a(G)$ . If  $H_\sigma$  contains a distinguished unipotent element of  $G$ , then  $H_\sigma$  is  $G$ -irreducible.*

**Corollary 3.5** generalizes [6, Theorem 1.3] which consists of the analogue in the special instance when the distinguished element is regular in  $G$ . Note that in this case no restriction on  $p$  is needed.

**Example 3.6** below shows that the conditions in **Corollary 3.5** hold generically.

**Example 3.6.** Let  $\sigma_q : \mathrm{GL}(V) \rightarrow \mathrm{GL}(V)$  be a standard Frobenius endomorphism which stabilizes a connected reductive subgroup  $H$  of  $\mathrm{GL}(V)$  and a maximal torus  $T$  of  $H$ . Pick  $l \in \mathbb{N}$  such that firstly all the different  $T$ -weights of  $V$  are still distinct when restricted to  $T_{\sigma_q^l}$  and secondly there is a  $t \in T_{\sigma_q^l}$ , such that  $C_{\mathrm{GL}(V)}(T) = C_{\mathrm{GL}(V)}(t)$ . Then for every  $n \geq l$ , both conditions in **Corollary 3.5** are satisfied for  $\sigma = \sigma_q^n$ . Thus there are only finitely many powers of  $\sigma_q$  for which the conditions in **Corollary 3.5** can fail. The argument here readily generalizes to a Steinberg endomorphism of a connected reductive  $G$  which induces a generalized Frobenius morphism on  $H$ .

## 4. Springer maps and associated cocharacters

**4.1. Springer maps.** The notion of a Springer isomorphism was introduced in [37]. A *Springer isomorphism* is a  $G$ -equivariant isomorphism of varieties  $\phi : \mathcal{U}_G \rightarrow \mathcal{N}_G$ . It follows from work of Springer [37, Theorem 3.1] that a Springer isomorphism  $\phi$  exists if  $p$  is good and  $G$  is simple and simply connected. We follow Springer and consider  $G$ -equivariant maps from  $\mathcal{U}_G$  to  $\mathcal{N}_G$ , but note that several other authors consider  $G$ -equivariant maps from  $\mathcal{N}_G$  to  $\mathcal{U}_G$  instead (see, e.g., [36]).

We wish to consider versions of Springer maps for arbitrary connected reductive  $G$ . To prove existence, we need to weaken the definition slightly.

**Definition 4.1.** A *Springer map (for  $G$ )* is a  $G$ -equivariant homeomorphism of varieties  $\phi : \mathcal{U}_G \rightarrow \mathcal{N}_G$ .

**Remark 4.2.** It follows from  $G$ -equivariance that if  $\phi$  is a Springer map then  $\phi(1) = 0$  and for any  $u \in \mathcal{U}_G$ ,  $u$  is distinguished if and only if  $\phi(u)$  is distinguished.

**Remark 4.3.** If  $p$  is good for  $G$  then there exists a Springer map  $\phi$  for  $G$ , see [25, Proposition 5]. Below we sketch the argument briefly, following [25, Proposition 5] and [36, Section 1.2]. Note first that a Springer map is uniquely determined by its value on a single regular unipotent element  $u$  of  $G$ : this follows from  $G$ -equivariance, and because the orbit  $G \cdot u$  is dense in  $\mathcal{U}_G$ . If  $G$  is simple and  $p$  is separably good for  $G$  then we can prove existence of a Springer isomorphism by reversing this argument. Fix a regular unipotent element  $u \in G$ , and choose  $X \in \mathcal{N}_G$  such that  $C_G(u) = C_G(X)$ . We have an obvious isomorphism from  $G \cdot u$  to  $G \cdot X$ . Because  $\mathcal{U}_G$  and  $\mathcal{N}_G$  are normal (for references, see [34, Lecture 2]), one can show that this map extends to a unique  $G$ -equivariant isomorphism from  $\mathcal{U}_G$  to  $\mathcal{N}_G$ . Let us say that  $G$  is *of separable type* if it is of the form  $G = G_1 \times \cdots \times G_r$ , where each  $G_i$

is simple and  $p$  is separably good for  $G$ . A similar argument to the above works for  $G$  of separable type: for  $\mathcal{U}_G = \mathcal{U}_{G_1} \times \cdots \times \mathcal{U}_{G_r}$  is normal since each  $\mathcal{U}_{G_i}$  is, and likewise  $\mathcal{N}_G$  is normal.

Now let  $G$  be an arbitrary connected reductive group and assume  $p$  is good for  $G$ . Since  $\mathcal{U}_G \subseteq \mathcal{D}G$  and  $\mathcal{N}_G \subseteq \text{Lie}(\mathcal{D}G)$ , there is no harm in assuming that  $G$  is semisimple. Choose a central isogeny  $\pi$  from  $\tilde{G}$  to  $G$ , where  $\tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_r$  with each  $\tilde{G}_i$  simple and  $p$  separably good for  $\tilde{G}$ . Then  $\pi$  (resp.,  $d\pi$ ) gives a homeomorphism from  $\mathcal{U}_{\tilde{G}}$  to  $\mathcal{U}_G$  (resp., from  $\mathcal{N}_{\tilde{G}}$  to  $\mathcal{N}_G$ ) [23, Lemma 27]. If  $\tilde{\phi}$  is a Springer map for  $\tilde{G}$  then the composition

$$\mathcal{U}_G \rightarrow \mathcal{U}_{\tilde{G}} \xrightarrow{\tilde{\phi}} \mathcal{N}_{\tilde{G}} \rightarrow \mathcal{N}_G$$

is a Springer map for  $G$ . This gives a bijection between the set of Springer maps for  $\tilde{G}$  and the set of Springer maps for  $G$ . Since  $\tilde{G}$  admits a Springer isomorphism, it follows that  $G$  admits a Springer map.

Note that if  $G$  is of separable type then any Springer map  $\phi$  for  $G$  is an isomorphism. For fix a regular unipotent element  $u \in G$  and let  $X = \phi(u)$ . By the above discussion, there is a unique Springer isomorphism  $\phi'$  for  $G$  such that  $\phi'(u) = X$ ; the uniqueness implies that  $\phi' = \phi$ . It also follows from the construction in the previous paragraph that if  $G$  is an arbitrary connected reductive group and  $p$  is good for  $G$  then the restriction of  $\phi$  to any maximal unipotent subgroup  $U$  of  $G$  gives an isomorphism of varieties from  $U$  to  $\text{Lie}(U)$ .

**Remark 4.4.** Let  $G_1, G_2$  be connected reductive groups and let  $\phi_i$  be a Springer map for  $G_i$  for  $i = 1, 2$ . We claim that the map  $\phi_1 \times \phi_2 : \mathcal{U}_{G_1 \times G_2} \rightarrow \mathcal{N}_{G_1 \times G_2}$  given by  $(\phi_1 \times \phi_2)((u_1, u_2)) = (\phi_1(u_1), \phi_2(u_2))$  is a Springer map for  $G_1 \times G_2$ . It is clear that  $\phi_1 \times \phi_2$  is a  $(G_1 \times G_2)$ -equivariant bijection. The Zariski topology on the product of varieties is not the product topology, so it is not immediately clear that  $\phi_1 \times \phi_2$  is a homeomorphism. To see this, we can pass to the case when  $G_1$  and  $G_2$  are of separable type, by Remark 4.3. Then  $\phi_1$  and  $\phi_2$  are isomorphisms, so  $\phi_1 \times \phi_2$  is an isomorphism, and the claim follows. We show in Lemma 4.14 that every Springer map for  $G_1 \times G_2$  arises in this way.

**Remark 4.5.** It follows from  $G$ -equivariance that a Springer map  $\phi$  gives rise to a bijective map from the set of unipotent conjugacy classes of  $G$  to the set of nilpotent conjugacy classes of  $\mathfrak{g}$ . Serre [24, Section 10, Corollary] shows that this map does not depend on the choice of Springer map (the proof given in the same work is for simple  $G$ , but the extension to arbitrary  $G$  follows easily from Remarks 4.3 and 4.4). In particular, the condition in  $(\dagger)$  does not depend on the choice of Springer map for  $H$ .

**Remark 4.6.** Springer maps need not exist in bad characteristic. For instance, a simple group  $G$  of type  $F_4$  with  $p = 2$  does not admit a Springer map, because the

numbers of unipotent classes in  $G$  and nilpotent  $G$ -orbits in  $\text{Lie}(G)$  are different (see [10, Section 5.11]).

**Lemma 4.7** [36, Section 1.2, Remark 1]. *Let  $\phi$  be a Springer map for  $G$ . Then  $\phi(u^p) = \phi(u)^{[p]}$  for any  $u \in \mathcal{U}_G$ .*

**Remark 4.8.** It follows from Lemma 4.7 that any Springer map for  $G$  induces a homeomorphism from  $\mathcal{U}_G^{(1)}$  to  $\mathcal{N}_G^{(1)}$ .

In Section 4.2 we define the notion of an associated cocharacter for an element  $u \in \mathcal{U}_G$ , using a fixed Springer map to give a correspondence between  $\mathcal{U}_G$  and  $\mathcal{N}_G$ . In many contexts one can fix a single Springer map once and for all. We need, however, to consider the interaction of Springer maps with subgroups of  $G$ . This motivates the following definition.

**Definition 4.9.** Let  $M$  be a connected subgroup of  $G$ . We say that a Springer map  $\phi$  for  $G$  is  $M$ -compatible if  $\phi(\mathcal{U}_M) \subseteq \mathcal{N}_M$ , and we say that  $M$  is Springer-compatible if there exists an  $M$ -compatible Springer map for  $G$ .

If  $\phi$  is  $M$ -compatible then in fact  $\phi(\mathcal{U}_M) = \mathcal{N}_M$ , since  $\dim(\mathcal{U}_M) = \dim(\mathcal{N}_M)$ ; note that dimension can be defined in a purely topological way (via Krull dimension), so it is preserved by homeomorphisms. Note also that when  $M$  is reductive and  $\phi$  is an  $M$ -compatible Springer map, the restriction of  $\phi$  to  $\mathcal{U}_M$  gives a Springer map for  $M$ , which we denote by  $\phi_M$ .

**Example 4.10** [27, (3.3.1)(a)]. Let  $M$  be a connected reductive subgroup of the form  $C_G(S)^\circ$ , where  $S \subseteq G$ . It follows from  $G$ -equivariance that any Springer map for  $G$  is  $M$ -compatible, so  $M$  is Springer-compatible.

**Example 4.11.** The arguments in Remark 4.3 show that if  $G_i$  is a simple factor of  $G$  then any Springer map for  $G$  is  $G_i$ -compatible, so  $G_i$  is Springer-compatible.

**Example 4.12.** Assume  $p > h(G)$ , where  $h(G)$  denotes the Coxeter number of  $G$ . The map  $\log : \mathcal{U}_G \rightarrow \mathcal{N}_G$  from [34, Theorem 3] is a Springer map. Let  $H$  be a connected reductive subgroup of  $G$ . We see that  $\log$  is  $H$ -compatible if and only if  $H$  is saturated in the sense of [34, Lecture 3]. For some properties of saturated subgroups, see [7] and [34].

**Example 4.13.** Let  $G = \text{SL}_2 \times \text{SL}_2$ . For  $q$  a positive power of  $p$ , let  $H_q$  be  $\text{SL}_2$  diagonally embedded in  $G$  with a  $q$ -Frobenius twist in one of the factors: say, the second factor. Note that  $\text{Lie}(H_q) = \text{Lie}(\text{SL}_2) \oplus 0$ , so  $\text{Lie}(H_q)$  contains no nilpotent elements that are distinguished in  $\mathfrak{g}$ . It follows from Remark 4.2 that no Springer map for  $G$  is  $H_q$ -compatible, so  $H_q$  is not Springer-compatible.

We can find a similar example for  $G$  simple. Let  $G$  be a simple group of type  $G_2$  and assume  $p > 2$ . Define  $H_q$  to be  $\text{SL}_2$  diagonally embedded in the  $A_1 \tilde{A}_1$  regular subgroup of  $G$  with a  $q$ -Frobenius twist in one of the factors, and let  $1 \neq u \in H_q$

be unipotent. Then  $u$  is a distinguished unipotent element of  $G$  by [19, Table 10, Section 4.1], but  $\text{Lie}(H_q)$  contains no nilpotent elements that are distinguished in  $\mathfrak{g}$ , so  $H_q$  is not Springer-compatible. We are grateful to Adam Thomas for this example.

**Lemma 4.14.** *Let  $G_1, G_2$  be connected reductive groups and let  $\phi$  be a Springer map for  $G_1 \times G_2$ . Then  $\phi$  is  $G_1$ -compatible and  $G_2$ -compatible. Also,  $\phi = \phi_1 \times \phi_2$ , where  $\phi_i$  is the restriction of  $\phi$  to  $G_i$ .*

*Proof.* By Remark 4.4, we can reduce to the case when  $G_1 \times G_2$  is of separable type. The  $G_i$ -compatibility of  $\phi$  follows easily from the  $(G_1 \times G_2)$ -equivariance. Now fix regular  $u_1 \in \mathcal{U}_{G_1}$  and  $u_2 \in \mathcal{U}_{G_2}$ , and set  $X = (X_1, X_2)$ , where  $X_i = \phi_i(u_i)$  for  $1 \leq i \leq 2$ . Then  $X_i$  is a regular element of  $\text{Lie}(G_i)$  for  $1 \leq i \leq 2$ ,  $u = (u_1, u_2)$  is a regular element of  $G$  and  $X$  is a regular element of  $\text{Lie}(G)$ . Clearly, we have  $C_{G_i}(u_i) = C_{G_i}(X_i)$  for  $1 \leq i \leq 2$ .

Let  $\phi'_i$  be the unique Springer isomorphism for  $G_i$  such that  $\phi'_i(u_i) = X_i$ . We have

$$(\phi'_1 \times \phi'_2)((u_1, u_2)) = (\phi'_1(u_1), \phi'_2(u_2)) = (X_1, X_2) = \phi((u_1, u_2)),$$

so  $\phi = \phi'_1 \times \phi'_2$ . Moreover,

$$(\phi_1(u_1), 0) = \phi((u_1, 0)) = (\phi'_1 \times \phi'_2)((u_1, 0)) = (\phi'_1(u_1), 0),$$

so  $\phi_1(u_1) = \phi'_1(u_1)$ , so  $\phi_1 = \phi'_1$ . Likewise  $\phi_2 = \phi'_2$ , and the result follows.  $\square$

**4.2. Cocharacters associated to nilpotent and unipotent elements.** The Jacobson–Morozov theorem allows one to associate an  $\mathfrak{sl}(2)$ -triple to any given nonzero element of  $\mathcal{N}_G$  in characteristic zero or large positive characteristic. This is an indispensable tool in the Dynkin–Kostant classification of the nilpotent orbits in characteristic zero as well as in the Bala–Carter classification of unipotent conjugacy classes of  $G$  in large prime characteristic, see [10, Section 5.9]. In good characteristic there is a replacement for  $\mathfrak{sl}(2)$ -triples, so-called *associated cocharacters*, see Definition 4.15 below. These cocharacters are important tools in the classification theory of unipotent classes and nilpotent orbits of reductive algebraic groups in good characteristic, see, for instance, [16, Section 5] and [30]. We recall the relevant concept of cocharacters associated to a nilpotent element following [16, Section 5.3].

**Definition 4.15.** Let  $X \in \mathcal{N}_G$ . A cocharacter  $\lambda \in Y(G)$  of  $G$  is *associated* to  $X$  (in  $G$ ) provided  $X \in \mathfrak{g}(2, \lambda)$  and there exists a Levi subgroup  $L$  of  $G$  such that  $X$  is distinguished nilpotent in  $\text{Lie}(L)$  and  $\lambda(\mathbb{G}_m) \leq \mathcal{D}L$ . Following [12, Definition 2.13], we write

$$\Omega_G^a(X) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } X\}$$

for the set of cocharacters of  $G$  associated to  $X$ . Likewise, for  $M$  a connected reductive subgroup of  $G$  such that  $X \in \text{Lie}(M)$ , we write  $\Omega_M^a(X)$  for the set of cocharacters of  $M$  that are associated to  $X$ . This notation stems from the fact that associated cocharacters are destabilizing cocharacters of  $G$  for  $X$  in the sense of Kempf–Rousseau theory, see [24] and [30].

Let  $u \in \mathcal{U}_G$ . A cocharacter  $\lambda \in Y(G)$  of  $G$  is *associated to  $u$*  (in  $G$ ) provided it is associated to  $\phi(u)$ , where  $\phi : \mathcal{U}_G \rightarrow \mathcal{N}_G$  is a fixed Springer map as in Section 4.1, see [25, Section 3]. We write

$$\Omega_{G,\phi}^a(u) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } u\}$$

for the set of cocharacters of  $G$  associated to  $u$ . Likewise, for  $M$  a connected reductive subgroup of  $G$  containing  $u$  and  $\phi'$  a Springer map for  $M$ , we write  $\Omega_{M,\phi'}^a(u)$  for the set of cocharacters of  $M$  that are associated to  $u$ . If  $\phi$  is understood then we sometimes write  $\Omega_G^a(u)$  instead of  $\Omega_{G,\phi}^a(u)$ .

**Remark 4.16.** Let  $u \in \mathcal{U}_G$ ,  $\lambda \in \Omega_G^a(u)$ , and  $g \in C_G(u)$ . Then  $g \cdot \lambda$  is also associated to  $u$ , see [16, Section 5.3]. Proposition 4.19(ii) gives a converse to this property.

**Remark 4.17.** Suppose that  $G_1, \dots, G_r$  are connected reductive groups and set  $G = G_1 \times \dots \times G_r$ . Let  $u_i \in \mathcal{U}_{G_i}$  for each  $1 \leq i \leq r$  and let  $L$  be a Levi subgroup of  $G$ . Then  $L = L_1 \times \dots \times L_r$  for some Levi subgroups  $L_i$  of  $G_i$ . Set  $u = (u_1, \dots, u_r) \in L$ . It is clear that  $u$  is distinguished in  $L$  if and only if  $u_i$  is distinguished in  $L_i$  for each  $i$ . Likewise, if  $X = (X_1, \dots, X_r) \in \mathcal{N}_L$  then  $X$  is distinguished in  $\text{Lie}(L)$  if and only if  $X_i$  is distinguished in  $\text{Lie}(L_i)$  for each  $i$ .

Fix a Springer map for  $G$ . Let  $\lambda \in Y(G)$ . We can write  $\lambda = \lambda_1 \times \dots \times \lambda_r$  for some  $\lambda_i \in Y(G_i)$ . It follows from the previous paragraph that  $\lambda$  is associated to  $X$  in  $\text{Lie}(G)$  if and only if  $\lambda_i$  is associated to  $X_i$  in  $\text{Lie}(G_i)$  for each  $i$  [16, Section 5.6]. We deduce the analogous statement for  $u$  from Remark 4.4: if  $\phi = \phi_1 \times \dots \times \phi_r$  is a Springer map for  $G$  then  $\lambda$  is associated to  $u$  in  $G$  if and only if  $\lambda_i$  is associated to  $u_i$  in  $G_i$  for each  $i$ .

Let  $\psi : \tilde{G} \rightarrow G$  be an epimorphism of connected reductive groups such that  $\ker(d\psi)$  is central in  $\text{Lie}(\tilde{G})$ . Let  $\tilde{u} \in \mathcal{U}_{\tilde{G}}$ , let  $\tilde{X} \in \mathcal{N}_{\tilde{G}}$ , let  $\tilde{L}$  be a Levi subgroup of  $\tilde{G}$  and let  $\tilde{\lambda} \in Y(\tilde{G})$ . Set  $u = \psi(\tilde{u})$ ,  $X = d\psi(\tilde{X})$ ,  $L = \psi(\tilde{L})$  and  $\lambda = \psi \circ \tilde{\lambda}$ . Let  $\tilde{\phi}$  be a Springer map for  $\tilde{G}$  and let  $\phi$  be the corresponding Springer map for  $G$  as described in Remark 4.3. Using [16, Section 4.3] and Remark 4.4 we get analogues of the above statements:  $u$  is distinguished in  $L$  if and only if  $\tilde{u}$  is distinguished in  $\tilde{L}$ ,  $X$  is distinguished in  $\text{Lie}(L)$  if and only if  $\tilde{X}$  is distinguished in  $\text{Lie}(\tilde{L})$  and  $\lambda$  is associated to  $X$  (resp., to  $u$ ) if and only if  $\tilde{\lambda}$  is associated to  $\tilde{X}$  (resp., to  $\tilde{u}$ ).

**Remark 4.18.** The notion of an associated cocharacter for an element  $u \in \mathcal{U}_G$  depends on the choice of the Springer map for  $G$ ; see [24, Remark 23]. We do,

however, have the following. Let  $\phi_1$  and  $\phi_2$  be Springer maps for  $G$ . Let  $1 \neq u_1 \in \mathcal{U}_G$  and let  $\lambda \in \Omega_{G, \phi_1}^a(u_1)$ . Then  $\lambda \in \Omega_{G, \phi_2}^a(u_2)$ , where  $u_2 = \phi_2^{-1}(\phi_1(u_1))$ . Note that  $u_2$  is conjugate to  $u_1$  by [Remark 4.5](#).

We require some basic facts about cocharacters associated to unipotent elements. The following results are [[16](#), Lemma 5.3, Proposition 5.9] for nilpotent elements (see also [[30](#), Theorem 2.3, Proposition 2.5]); the versions for unipotent elements follow immediately.

**Proposition 4.19.** *Suppose  $p$  is good for  $G$ . Let  $1 \neq u \in \mathcal{U}_G$ .*

- (i)  $\Omega_G^a(u) \neq \emptyset$ , i.e., cocharacters of  $G$  associated to  $u$  exist.
- (ii)  $C_G(u)^\circ$  acts transitively on  $\Omega_G^a(u)$ .
- (iii) Let  $\lambda \in \Omega_G^a(u)$  and let  $P_\lambda$  be the parabolic subgroup of  $G$  defined by  $\lambda$  as in [Section 2.5](#). Then  $P_\lambda$  depends only on  $u$  and not on the choice of  $\lambda$ .
- (iv) Let  $\lambda \in \Omega_G^a(u)$  and let  $P(u) := P_\lambda$  be as in (iii). Then  $C_G(u) \subseteq P(u)$ .

If  $u$  is distinguished in  $G$ , then the parabolic subgroup  $P(u)$  of  $G$  from [Proposition 4.19\(iii\)](#) is a distinguished parabolic subgroup of  $G$  and  $u$  belongs to the Richardson orbit of  $P(u)$  on its unipotent radical, see [Theorem 2.1\(i\)](#); see also [[24](#), Proposition 22].

**Remark 4.20.** Let  $p > 0$  and suppose  $1 \neq u \in \mathcal{U}_G^{(1)}$  is contained in a subgroup  $A$  of  $G$  of type  $A_1$ . Such a subgroup  $A$  always exists when  $p$  is good, and when  $p$  is bad there is essentially only one exception, due to Testerman [[43](#)] and Proud, Saxl, and Testerman [[31](#)], see [Theorems 2.4](#) and [2.9](#). Then, since  $p$  is good for  $A$ , by [Proposition 4.19\(i\)](#) there exists a cocharacter  $\lambda \in \Omega_A^a(u)$ . Note that  $\lambda(\mathbb{G}_m)$  is a maximal torus in  $A$ .

It follows from the work of Pommerening [[28](#); [29](#)] that the description of the unipotent classes in characteristic 0 is identical to the one for  $G$  when  $p$  is good for  $G$ . In both instances these are described by so-called *weighted Dynkin diagrams*. As a result, a cocharacter associated to a unipotent element in good characteristic acts with the same weights on the Lie algebra of  $G$  as its counterpart does in characteristic 0. This fact is used in the proof of the following result by Lawther [[18](#), Theorem 1]; see also the proof of [[33](#), Proposition 4.2] and [[24](#), Remark 31]. The result is stated in [[18](#), Theorem 1] for  $G$  simple, but the extension to arbitrary connected reductive  $G$  is immediate, using arguments like those in [Remark 4.17](#); note that if  $\psi : \tilde{G} \rightarrow G$  is an epimorphism of connected reductive groups such that  $\ker(d\psi)$  is central in  $\text{Lie}(\tilde{G})$  then  $d\psi$  gives an isomorphism from  $\text{Lie}(\tilde{U})$  onto  $\text{Lie}(\psi(\tilde{U}))$ , where  $\tilde{U}$  is any maximal unipotent subgroup of  $\tilde{G}$ , so the weights of  $\tilde{\lambda} \in Y(\tilde{G})$  on  $\text{Lie}(\tilde{G})$  are the same as the weights of  $\psi \circ \tilde{\lambda}$  on  $\text{Lie}(G)$ .

**Lemma 4.21.** *Let  $u \in \mathcal{U}_G$ . Suppose  $p$  is good for  $G$ . Let  $\lambda \in \Omega_G^a(u)$ . Denote by  $\omega_G$  the highest weight of  $\lambda(\mathbb{G}_m)$  on  $\mathfrak{g}$ . Then  $u$  has order  $p$  if and only if  $\omega_G \leq 2p - 2$ .*

The concept of associated cocharacters is not only a convenient replacement for  $\mathfrak{sl}(2)$ -triples from the Jacobson–Morozov theory, it is a very powerful tool in the classification theory of unipotent conjugacy classes and nilpotent orbits. Specifically, in [30] Premet showcases a conceptual and uniform proof of Pommerening’s extension of the Bala–Carter [Theorem 2.1](#) to good characteristic. His proof uses the fact that associated cocharacters are *optimal* in the geometric invariant theory sense of Kempf–Rousseau–Hesselink.

**4.3. Cocharacters associated to distinguished elements.** The linchpin of our proofs of [Theorems 1.2](#) and [1.6](#) is the following collection of facts.

**Lemma 4.22** [[12](#), Lemma 3.1]. *Suppose  $p$  is good for  $G$ . Let  $M$  be a connected reductive subgroup of  $G$ . Let  $X \in \text{Lie}(M)$  be a distinguished nilpotent element of  $\mathfrak{g}$ . Then  $\Omega_M^a(X) = \Omega_G^a(X) \cap Y(M)$ .*

The assertion of the lemma fails in general if  $X$  is not distinguished in  $\mathfrak{g}$ , even when  $p$  is good for both  $M$  and  $G$ , e.g., see [[16](#), Remark 5.12]. However, we do have the following result for all nilpotent elements in good characteristic.

**Lemma 4.23** [[12](#), Corollary 3.22]. *Suppose  $p$  is good for  $G$ . Let  $L \subseteq G$  be a Levi subgroup of  $G$ . Let  $X \in \mathcal{N}_L$ . Then  $\Omega_L^a(X) = \Omega_G^a(X) \cap Y(L)$ .*

We need group-theoretic analogues of [Lemmas 4.22](#) and [4.23](#). For the former we need an extra Springer compatibility assumption, otherwise the result can fail (see [Remark 6.1](#)).

**Lemma 4.24.** *Suppose  $p$  is good for  $G$ . Let  $M$  be a connected reductive subgroup of  $G$ . Suppose  $M$  is Springer-compatible and let  $\phi$  be an  $M$ -compatible Springer map. Let  $u \in M$  be a distinguished unipotent element of  $G$ . Then*

$$\Omega_{M, \phi_M}^a(u) = \Omega_{G, \phi}^a(u) \cap Y(M).$$

*Proof.* Let  $X = \phi(u) = \phi_M(u)$ . Then

$$\Omega_{M, \phi_M}^a(u) = \Omega_M^a(X) = \Omega_G^a(X) \cap Y(M) = \Omega_{G, \phi}^a(u) \cap Y(M),$$

where the middle equality is from [Lemma 4.22](#). □

**Lemma 4.25.** *Suppose  $p$  is good for  $G$ . Let  $L \subseteq G$  be a Levi subgroup of  $G$  and let  $\phi$  be a Springer map for  $G$ . Let  $u \in \mathcal{U}_L$ . Then  $\Omega_{L, \phi_L}^a(u) = \Omega_{G, \phi}^a(u) \cap Y(L)$ .*

*Proof.* Since  $L = C_G(S)$  for some torus  $S$ ,  $\phi$  is  $L$ -compatible by [Example 4.10](#). The result now follows by the same argument as in [Lemma 4.24](#). □

## 5. Good $A_1$ subgroups

**5.1. Good  $A_1$  overgroups.** In his seminal work [33], Seitz defines an important class of  $A_1$  overgroups of an element  $1 \neq u \in \mathcal{U}_G^{(1)}$  for  $G$  simple (see [33, Section 1]). He establishes the existence and fundamental properties of these overgroups provided  $p$  is good for  $G$ . We recall some of these results and generalize them to arbitrary connected reductive  $G$ .

**Definition 5.1.** Following [23, Section 1], we say that a homomorphism  $\beta : \mathrm{SL}_2 \rightarrow G$  is *good* if each weight of the corresponding representation of  $\mathrm{SL}_2$  on  $\mathfrak{g}$  is at most  $2p - 2$ . We say that a subgroup  $A$  of  $G$  of type  $A_1$  is a *good  $A_1$  subgroup* of  $G$ , or is *good for  $G$* , if it is the image of a good homomorphism. Else we call  $A$  a *bad  $A_1$  subgroup* of  $G$ . This is of course independent of the choice of a maximal torus of  $A$ . For  $1 \neq u \in \mathcal{U}_G^{(1)}$ , we define

$$\mathcal{A}(u) := \mathcal{A}_G(u) := \{A \subseteq G \mid A \text{ is a good } A_1 \text{ subgroup of } G \text{ containing } u\}$$

and analogously, for a connected reductive subgroup  $M$  of  $G$  we write  $\mathcal{A}_M(u)$  for the set of all good  $A_1$  subgroups of  $M$  containing  $u$ .

Clearly any conjugate of a good  $A_1$  homomorphism (resp., subgroup) is good. If  $A \subseteq H \subseteq G$  are connected reductive groups such that  $A$  is a good  $A_1$  subgroup of  $G$ , then  $A$  is obviously also a good  $A_1$  subgroup of  $H$ . We see in Lemma 5.30 that the converse holds under some extra hypotheses. The converse is false in general, however, e.g., just take  $A = H$  to be a bad  $A_1$  subgroup of  $G$ .

**Example 5.2.** Let  $V$  be an  $\mathrm{SL}_2$ -module such that weights of a maximal torus  $T$  of  $\mathrm{SL}_2$  on  $V$  are less than  $p$ . Then the weights of  $T$  in the induced action on  $\mathrm{Lie}(\mathrm{GL}(V)) \cong V \otimes V^*$  are at most  $2p - 2$ . Thus the induced subgroup  $A$  in  $\mathrm{GL}(V)$  is a good  $A_1$ . In this situation the highest weights of  $T$  on each composition factor of  $V$  are restricted, so  $V$  is a semisimple  $\mathrm{SL}_2$ -module; see [1, Corollary 3.9]. Hence  $A$  is  $\mathrm{GL}(V)$ -cr; this is a special case of Theorem 5.4(iii) below.

We record parts of the main theorems from [33] for our purposes, using the notation above. These were formulated and proved in [33] for simple  $G$ , but we need extensions to arbitrary connected reductive  $G$ . To obtain this, we need the following lemma.

**Lemma 5.3.** *Let  $G$  be a connected reductive group. Let  $\beta_1, \beta_2 : \mathrm{SL}_2 \rightarrow G$  be good homomorphisms with the same image  $A$ . Then  $\beta_1$  and  $\beta_2$  are conjugate by an element of  $A$ .*

*Proof.* Assume first that  $A \cong \mathrm{SL}_2$ . Let  $1 \leq i \leq 2$ . Then we can regard  $\beta_i$  as an element of  $\mathrm{End}(\mathrm{SL}_2)$ , so it is an inner endomorphism followed by a Frobenius  $q$ -th power map for  $q = p^r$  for some  $r \geq 0$ . Let  $T$  be a maximal torus of  $\mathrm{SL}_2$ . If  $r \geq 1$  then the highest weight of  $T$  is at least  $2q$ , since  $\mathrm{SL}_2$  acts on  $\mathrm{Lie}(A)$  with highest

weight 2, which contradicts the goodness assumption. Therefore,  $\beta_i \in \text{Aut}(\text{SL}_2)$ . But all automorphisms of  $\text{SL}_2$  are inner. The result follows.

For  $A \cong \text{PGL}_2$ , we can factor  $\beta_i$  as

$$\text{SL}_2 \rightarrow \text{PGL}_2 \xrightarrow{\beta'_i} \text{PGL}_2,$$

where the first map is the canonical projection. One can now apply an argument like the above one to the maps  $\beta'_i : \text{PGL}_2 \rightarrow A$ .  $\square$

**Theorem 5.4.** *Let  $G$  be connected reductive. Suppose  $p$  is good for  $G$  and let  $1 \neq u \in \mathcal{U}_G^{(1)}$ . Then the following hold:*

- (i)  $\mathcal{A}(u) \neq \emptyset$ .
- (ii)  $R_u(C_G(u))$  acts transitively on  $\mathcal{A}(u)$ .
- (iii) Let  $A \in \mathcal{A}(u)$ . Then  $A$  is  $G$ -completely reducible.
- (iv) There is a unique 1-dimensional unipotent subgroup  $U$  of  $G$  such that  $u \in U$  and  $U$  is contained in a good  $A_1$  subgroup of  $G$ .

**Notation 5.5.** We denote the subgroup  $U$  from [Theorem 5.4\(iv\)](#) by  $\mathcal{U}(u)$ .

*Proof of [Theorem 5.4](#).* For  $G$  simple see [\[33, Theorems 1.1–1.3\]](#). Now let  $G$  be connected reductive. Since  $\text{SL}_2$  and  $\text{PGL}_2$  are perfect, any  $A_1$  subgroup of  $G$  is contained in  $\mathcal{D}G$ . Hence without loss we can assume that  $G$  is semisimple; note for (iii) that a subgroup of  $\mathcal{D}G$  is  $\mathcal{D}G$ -cr if and only if it is  $G$ -cr [\[5, Proposition 2.8\]](#). Moreover, let  $\psi : \tilde{G} \rightarrow G$  be a central isogeny of connected reductive groups. If  $\tilde{A}$  is an  $A_1$  subgroup of  $\tilde{G}$  then  $\tilde{A}$  is good for  $\tilde{G}$  if and only if  $\psi(\tilde{A})$  is good for  $G$ : see the argument of the paragraph preceding [Lemma 4.21](#). Note also for (iii) that if  $\tilde{H} \subseteq \tilde{G}$  then  $\tilde{H}$  is  $\tilde{G}$ -cr if and only if  $\psi(\tilde{H})$  is  $G$ -cr [\[4, Lemma 2.12\]](#). Hence we can assume without loss that  $G = G_1 \times \cdots \times G_r$ , where each  $G_i$  is simple.

We need a description of good  $A_1$  subgroups of  $G$  in terms of good  $A_1$  subgroups of the  $G_i$ . Let  $T$  be a maximal torus of  $\text{SL}_2$ . Denote by  $\pi_i$  the projection from  $G$  to  $G_i$ . Let  $\beta : \text{SL}_2 \rightarrow G$  be a homomorphism and define  $\beta_i := \pi_i \circ \beta$ . For notational convenience, we assume that each  $\beta_i$  is nontrivial. The weights of  $T$  on  $\text{Lie}(G_i)$  form a subset of the set of weights of  $T$  on  $\text{Lie}(G)$ , since  $\text{Lie}(G) = \bigoplus \text{Lie}(G_i)$ . Therefore, if  $\beta$  is a good homomorphism for  $G$ , then  $\beta_i$  is a good homomorphism for  $G_i$  or trivial. Conversely, if  $\beta_i : \text{SL}_2 \rightarrow G_i$  is a nontrivial homomorphism for each  $i$ , define  $\beta := \beta_1 \times \cdots \times \beta_r$  to be the diagonal embedding into  $G$ . Then the maximal weight  $\omega_G$  of  $T$  on  $\text{Lie}(G)$  is given by  $\max\{\omega_{G_i}\}$ , where  $\omega_{G_i}$  is the maximal weight of  $T$  on  $\text{Lie}(G_i)$ . Thus,  $\beta$  is good if and only if the  $\beta_i$  are good. Now (i) and (iii) are immediate from the above observations, [\[4, Lemma 2.12\]](#) and the results for  $G$  simple.

For (ii), let  $A^1$  and  $A^2$  be good  $A_1$  subgroups of  $G = G_1 \times \cdots \times G_r$  containing  $u = (u_1, \dots, u_r)$  with  $u_i \neq 1$  for each  $i$ . Choose two good homomorphisms

$\beta^1, \beta^2 : \mathrm{SL}_2 \rightarrow G$  such that  $\mathrm{Im}(\beta^i) = A^i$ . By the observations above, there are good homomorphisms  $\beta_i^1, \beta_i^2 : \mathrm{SL}_2 \rightarrow G_i$  with images  $A_i^1, A_i^2$  containing  $u_i$ . Now [33, Theorem 1.1(ii)] implies that  $A_i^2 = g_i A_i^1 g_i^{-1}$  for some  $g_i \in R_u(C_{G_i}(u_i))$ . Lemma 5.3 (applied to  $G_i$ ) implies that  $h_i g_i \cdot \beta_i^1 = \beta_i^2$  for some  $h_i \in A_i^2$ . Hence  $\beta^2 = hg \cdot \beta^1$ , where  $g = (g_1, \dots, g_r) \in R_u(C_G(u))$  and  $h = (h_1, \dots, h_r) \in A^2$ . It follows that  $A^2 = g \cdot A^1$ .

For (iv), let  $G = G_1 \times \dots \times G_r$  and let  $u = (u_1, \dots, u_r) \in \mathcal{U}_G^{(1)}$  with  $u_i \neq 1$  for each  $i$ . Choose an  $A \in \mathcal{A}_G(u)$  which is the image of the good homomorphism  $\beta$ . We get good homomorphisms  $\beta_i$  with images  $A_i \in \mathcal{A}_{G_i}(u_i)$ , and  $\beta = \beta_1 \times \dots \times \beta_r$ , as before. Without loss we can assume the  $\beta_i$  are nontrivial. Fix a 1-dimensional unipotent subgroup  $V$  of  $\mathrm{SL}_2$ . After conjugating  $\beta$  by an element of  $A$ , we can assume that  $\mathcal{U}(u_i) = \beta_i(V)$  for each  $i$ . Define  $\mathcal{U}(u) = (\beta_1 \times \dots \times \beta_r)(V)$ . This is a 1-dimensional unipotent subgroup of  $G$  containing  $u$  and is contained in the good  $A_1$  subgroup  $A$ . This proves the existence. For the uniqueness, let  $U'$  be another 1-dimensional unipotent subgroup of  $G$  such that  $u \in U' \subseteq A'$  for some  $A' \in \mathcal{A}_G(u)$ . By (ii),  $A = gA'g^{-1}$  for some  $g \in C_G(u)$ , and so  $gU'g^{-1} = \mathcal{U}(u)$ . Write  $g = (g_1, \dots, g_r)$  with  $g_i \in C_{G_i}(u_i)$ . By [33, Theorem 1.2(i)]  $g_i$  centralizes  $\mathcal{U}(u_i)$ , and hence  $g$  centralizes  $\mathcal{U}(u)$ . Thus,  $U' = \mathcal{U}(u)$ .  $\square$

**Example 5.6.** Let  $H_q$  be the bad  $A_1$  subgroup of  $G = \mathrm{SL}_2 \times \mathrm{SL}_2$  from Example 4.13. Here  $\beta_1 = \mathrm{id}_{\mathrm{SL}_2}$ , while  $\beta_2 : \mathrm{SL}_2 \rightarrow \mathrm{SL}_2$  is the  $q$ -th power map, which is not a good homomorphism. On the other hand, the projection of  $H_q$  onto each factor is just  $\mathrm{SL}_2$ , which is a good  $A_1$  subgroup of  $\mathrm{SL}_2$ , so we cannot detect the badness of  $H_q$  just by looking at its images in the simple factors of  $G$ .

**Remark 5.7.** Let  $1 \neq u \in \mathcal{U}_G^{(1)}$ . We claim that

$$(5.8) \quad C_G(\mathcal{U}(u)) = C_G(u) = C_G(\mathrm{Lie}(\mathcal{U}(u))).$$

To see this, suppose first that  $\tilde{G}$  is of the form  $G_1 \times \dots \times G_r$ , where each  $G_i$  is simple, and let  $\pi_i : \tilde{G} \rightarrow G_i$  be the canonical projection. Let  $\tilde{u} = (u_1, \dots, u_r) \in \mathcal{U}_{\tilde{G}}^{(1)}$  with  $u_i \neq 1$  for each  $i$ . Choose a good homomorphism  $\tilde{\beta} : \mathrm{SL}_2 \rightarrow \tilde{G}$  such that  $\mathcal{U}(\tilde{u}) \subseteq \tilde{A} := \mathrm{Im}(\mathrm{SL}_2)$ , and set  $\beta_i = \pi_i \circ \tilde{\beta}$  and  $A_i = \beta_i(\mathrm{SL}_2)$ . It follows from [33, Theorem 1.2(i)] that

$$C_{G_i}(\mathcal{U}(u_i)) = C_{G_i}(u_i) = C_{G_i}(\mathrm{Lie}(\mathcal{U}(u_i)))$$

for each  $i$ . We deduce from the arguments in the proof of Theorem 5.4 that  $C_{\tilde{G}}(\mathcal{U}(\tilde{u})) = C_{\tilde{G}}(\tilde{u}) = C_{\tilde{G}}(\mathrm{Lie}(\mathcal{U}(\tilde{u})))$ ; note that  $d\beta_i : \mathrm{Lie}(\mathrm{SL}_2) \rightarrow \mathrm{Lie}(A_i)$  is surjective for each  $i$  because  $\beta_i$  does not involve a Frobenius twist.

If  $\psi : \tilde{G} \rightarrow G$  is a central isogeny and  $1 \neq \tilde{u} \in \mathcal{U}_{\tilde{G}}^{(1)}$ , then it is clear that  $\mathcal{U}(u) = \psi(\mathcal{U}(\tilde{u}))$ , where  $u = \psi(\tilde{u})$ , and we deduce that

$$(5.9) \quad C_G(\mathcal{U}(u)) = C_G(u) = C_G(\mathrm{Lie}(\mathcal{U}(u))).$$

Now let  $G$  be an arbitrary connected reductive group and let  $1 \neq u \in \mathcal{U}_G^{(1)}$ . Then  $\mathcal{U}(u) \subseteq \mathcal{D}G$ . Now (5.8) follows easily from (5.9) applied to the semisimple group  $\mathcal{D}G$ .

We deduce from (5.8) and Theorem 5.4(ii) that  $\mathcal{U}(u)$  is contained in every good  $A_1$  overgroup of  $u$ .

**Lemma 5.10.** *Suppose  $p$  is good for  $G$ . Let  $1 \neq u \in \mathcal{U}_G^{(1)}$  and let  $A$  be an  $A_1$  subgroup of  $G$  containing  $\mathcal{U}(u)$ . Then  $A$  is good in  $G$ .*

*Proof.* Let  $A'$  be a good  $A_1$  subgroup containing  $\mathcal{U}(u)$ . Then  $A$  and  $A'$  have a common maximal unipotent subgroup  $\mathcal{U}(u)$ . By [21, Theorem 1.1],  $A$  and  $A'$  are  $G$ -conjugate. Hence  $A$  is good, because  $A'$  is.  $\square$

**Lemma 5.11.** *Suppose  $p$  is good for  $G$ . Let  $A$  be an  $A_1$  subgroup of  $G$  and let  $\lambda \in Y(A)$ . Suppose that*

- (i)  $\lambda \in \Omega_G^a(X)$  for some  $0 \neq X \in \mathcal{N}_G^{(1)}$ , or
- (ii)  $\lambda \in \Omega_{G,\phi}^a(u)$  for some  $1 \neq u \in \mathcal{U}_G^{(1)}$  and some Springer map  $\phi$  for  $G$ .

*Then  $A$  is a good  $A_1$  subgroup of  $G$ .*

*Proof.* Let  $\phi$  be a Springer map for  $G$ , and let  $\lambda \in \Omega_{G,\phi}^a(u)$  for some  $1 \neq u \in \mathcal{U}_G^{(1)}$ . It follows from Lemma 4.21 that the weights of  $\lambda$  on  $\mathfrak{g}$  are at most  $2p - 2$ . Define  $\beta : \mathrm{SL}_2 \rightarrow A$  to be an isomorphism if  $A \cong \mathrm{SL}_2$ , and the usual central isogeny  $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$  followed by an isomorphism from  $\mathrm{PGL}_2$  onto  $A$  if  $A \cong \mathrm{PGL}_2$ . Then there exists  $\mu : \mathbb{G}_m \rightarrow \mathrm{SL}_2$  such that  $\mu$  is an isomorphism onto a maximal torus of  $\mathrm{SL}_2$  and  $\lambda = \beta \circ \mu$ . The weights of  $\mu$  on  $\mathfrak{g}$  are at most  $2p - 2$  by construction, so  $A$  is good. Hence  $A$  is good if (ii) holds.

If (i) holds then  $\lambda \in \Omega_{G,\phi}^a(u)$ , where  $u := \phi^{-1}(X)$ . But  $u \in \mathcal{U}_G^{(1)}$ , by Lemma 4.7, so (ii) holds, so  $A$  is good by the argument above.  $\square$

In the next theorem we recall parts of the analogue of Theorem 5.4 for finite overgroups of type  $A_1$ .

**Theorem 5.12.** *Let  $G$  be connected reductive. Suppose  $p$  is good for  $G$ . Let  $\sigma : G \rightarrow G$  be a Steinberg endomorphism of  $G$ . Suppose  $u \in G_\sigma$  is unipotent of order  $p$ .*

- (i)  $\mathcal{A}(u)_\sigma \neq \emptyset$ .
- (ii)  $R_u(C_G(u))_\sigma$  acts transitively on  $\mathcal{A}(u)_\sigma$ .
- (iii) Let  $A \in \mathcal{A}(u)_\sigma$ . Suppose that  $q > 7$  if  $G$  is of exceptional type. Then  $A_\sigma$  is  $\sigma$ -completely reducible.
- (iv) There is a unique  $\sigma$ -stable 1-dimensional unipotent subgroup  $U$  of  $G$  such that  $u \in U$  and  $U$  is contained in a good  $A_1$  subgroup of  $G$ .

*Proof.* (i)–(iii) The simple case is proved by Seitz in [33, Theorem 1.4]. For connected reductive groups we use an argument similar to the one in the proof of Theorem 5.4.

(iv) By (i) we can choose some  $A \in \mathcal{A}(u)_\sigma$ . Now  $\mathcal{U}(u) \subseteq A$  by Remark 5.7. Clearly  $\mathcal{U}(u)$  is the unique 1-dimensional unipotent subgroup of  $A$  that contains  $u$ , so  $\mathcal{U}(u)$  must be  $\sigma$ -stable. Hence  $U := \mathcal{U}(u)$  has the desired properties.  $\square$

**Remark 5.13.** Parts (i) and (ii) of Theorem 5.12 follow from parts (i) and (ii) of Theorem 5.4 and the Lang–Steinberg theorem, see [33, Proposition 9.1].

**Remark 5.14.** (i) Concerning the terminology in Theorem 5.12(iii), following [14], a subgroup  $H$  of  $G$  is said to be  $\sigma$ -completely reducible, provided that whenever  $H$  lies in a  $\sigma$ -stable parabolic subgroup  $P$  of  $G$ , it lies in a  $\sigma$ -stable Levi subgroup of  $P$ . This notion is motivated by certain rationality questions concerning  $G$ -complete reducibility, see [14] for details. For a  $\sigma$ -stable subgroup  $H$  of  $G$ , this property is equivalent to  $H$  being  $G$ -cr, thanks to [14, Theorem 1.4].

(ii) Apart from the special conjugacy class of good  $A_1$  subgroups in  $G$  asserted in Theorem 5.4, there might be a plethora of conjugacy classes of bad  $A_1$  subgroups in  $G$  even when  $p$  is good for  $G$ . Just take a nonsemisimple representation

$$\beta : \mathrm{SL}_2 \rightarrow \mathrm{SL}(V) = G$$

in characteristic  $p > 0$ . Then the  $A_1$  subgroup  $\beta(\mathrm{SL}_2)$  is bad in  $G$ , while  $p$  is good for  $G$ . For a concrete example, see [16, Remark 5.12]. This can only happen if  $p$  is sufficiently small compared to the rank of  $G$ , thanks to Theorem 3.2.

The subgroups  $H_q$  of  $\mathrm{SL}_2 \times \mathrm{SL}_2$  in Example 4.13 are also bad  $A_1$  subgroups, see Remark 6.1.

(iii) The proofs of Theorems 5.4 and 5.12 for  $G$  simple by Seitz in [33] depend on separate considerations for each Dynkin type and involve in part intricate arguments for the component groups of centralizers of unipotent elements. In [24], McNinch presents uniform proofs of Seitz’s theorems for  $G$  strongly standard reductive, which are almost entirely free of any case-by-case checks, utilizing methods from geometric invariant theory. However, McNinch’s argument (see [24, Theorem 44]) of the conjugacy result in Theorem 5.4(ii) depends on the fact that for a good  $A_1$  subgroup  $A$  of  $G$ , the  $A$ -module  $\mathfrak{g}$  is a tilting module. The latter is established by Seitz in [33, Theorem 1.1].

In [33, Section 9], Seitz exhibits instances when there is no good  $A_1$  overgroup of an element of order  $p$  when  $p$  is bad for  $G$ . As we explain next, Example 2.6 gives a counterexample to Theorem 5.4(iii) in case  $p$  is bad for  $G$ : that is, it gives a good  $A_1$  subgroup  $A$  such that  $A$  is not  $G$ -cr. Specifically, we show that some

of the  $A_1$  subgroups in that example are good  $A_1$  subgroups of  $G$ , but thanks to [Example 2.6](#), they are not  $G$ -cr.

**Example 5.15** ([Example 2.6](#) continued). Let  $G$  be simple of type  $C_2$  and  $p = 2$ . Let  $\sigma$  be  $\text{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$ . Let  $\mathcal{C}$  denote the subregular unipotent class of  $G$ . Suppose  $u \in \mathcal{C} \cap G_\sigma$ . Then by [Example 2.6](#) there are  $\sigma$ -stable subgroups  $A$  of  $G$  of type  $A_1$  containing  $u$  that are not  $G$ -cr. Specifically, let  $E$  be the natural module for  $\text{SL}_2$ . Consider the two conjugacy classes of embeddings of  $\text{SL}_2$  into  $G = \text{Sp}(V)$ , where we take either  $V \cong E \perp E$  or  $V \cong E \otimes E$ , as an  $\text{SL}_2$ -module. The images of both embeddings meet the class  $\mathcal{C}$  nontrivially. One checks that the highest weight of a maximal torus of  $\text{SL}_2$  on  $\mathfrak{g}$  is 4 in the second instance. So in this case the image of  $\text{SL}_2$  in  $G$  is not a good  $A_1$ . In contrast, in the first instance the highest weight of a maximal torus of  $\text{SL}_2$  on  $\mathfrak{g}$  is  $2 = 2p - 2$ , by [Example 5.2](#). So the image of  $\text{SL}_2$  in  $G$  is a good  $A_1$  in  $\text{SL}(V)$ , and so it is a good  $A_1$  in  $G$  as well.

**5.2. Characterizations of good  $A_1$  subgroups.** In this section we investigate some other types of  $A_1$  subgroup which were introduced by McNinch. We prove that these other notions are all equivalent to goodness ([Theorem 5.24](#)). The key ingredient we need is work of Sobaje, who proved the existence of a Springer map for  $G$  with especially nice properties. We assume throughout the section that  $p$  is good for  $G$ .

We recall a construction from [[33](#), Proposition 5.2] (see also [[36](#)]). Let  $P$  be a parabolic subgroup of  $G$ , and set  $U = R_u(P)$ . It can be shown that any Springer map for  $G$  maps  $U$  to  $\text{Lie}(U)$ . Suppose  $U$  has nilpotency class less than  $p$ ; in this case we say that  $P$  is *restricted*. In particular, any distinguished parabolic subgroup of  $G$  corresponding to a distinguished unipotent element of order  $p$  is restricted [[24](#), Proposition 24]. We endow  $\text{Lie}(U)$  with the structure of an algebraic group using the Baker–Campbell–Hausdorff formula. There is a unique  $P$ -equivariant isomorphism of algebraic groups  $\exp_p : \text{Lie}(U) \rightarrow U$  such that the derivative of  $\exp_p$  is the identity on  $\text{Lie}(U)$  (this is established in [[33](#), Proposition 5.2] for semisimple  $G$ , but the extension to connected reductive  $G$  is immediate). We denote the inverse of  $\exp_p$  by  $\log_p : \text{Lie}(U) \rightarrow U$ .

**Definition 5.16.** We say that a Springer map  $\phi$  for  $G$  is *logarithmic* if the following holds: for any  $1 \neq u \in \mathcal{U}_G^{(1)}$ , the restriction of  $\phi$  gives an isomorphism  $\phi_u$  of algebraic groups from  $\mathcal{U}(u)$  to  $\text{Lie}(\mathcal{U}(u))$ , and  $d\phi_u$  is the identity on  $\text{Lie}(\mathcal{U}(u))$ .

**Proposition 5.17.** (i) *There exists a logarithmic Springer map for  $G$ .*

(ii) *Let  $\phi$  be a logarithmic Springer map for  $G$ . Then for every restricted parabolic subgroup  $P$ , the restriction of  $\phi$  to  $R_u(P)$  is  $\log_p$ .*

(iii) *Any two logarithmic Springer maps induce the same map from  $\mathcal{U}_G^{(1)}$  to  $\mathcal{N}_G^{(1)}$ .*

*Proof.* First assume that  $G$  is simple and  $p$  is separably good for  $G$ . Part (ii) follows from [36, Proposition 2.1]. For part (i), let  $\varphi : \mathcal{N}_G \rightarrow \mathcal{U}_G$  be a  $G$ -equivariant isomorphism of varieties as in [36, Theorem 4.1]. Fix a maximal unipotent subgroup  $U$  of  $G$ . By [36, Theorem 1.1],  $d\varphi : \text{Lie}(U) \rightarrow \text{Lie}(U)$  is a scalar multiple of the identity. Condition (1) of [36, Theorem 4.1] implies that this scalar is 1, so  $d\varphi$  is the identity map. Let  $1 \neq u \in \mathcal{U}_G^{(1)}$  and set  $X = \varphi^{-1}(u)$ . Then  $X \in \mathcal{N}_G^{(1)}$  by Remark 4.8, so by [36, Corollary 4.3(1)],  $\varphi$  gives an isomorphism from  $kX$  onto a 1-dimensional unipotent subgroup  $U'$  of  $G$  which is contained in a good  $A_1$  subgroup of  $G$ . By construction,  $U' = \mathcal{U}(u)$ . Since  $d\varphi$  is the identity map,  $X$  belongs to  $\text{Lie}(\mathcal{U}(u))$ , so  $\varphi$  gives an isomorphism of algebraic groups from  $\text{Lie}(\mathcal{U}(u))$  to  $\mathcal{U}(u)$ . It follows that  $\varphi^{-1}$  is a logarithmic Springer map for  $G$ , so (i) is proved.

Now let  $1 \neq u \in \mathcal{U}_G^{(1)}$ . Choose a good  $A_1$  overgroup  $A$  of  $u$  in  $G$ . Choose a maximal torus  $T$  of  $A$  such that  $T$  normalizes  $\mathcal{U}(u)$ . Definition 5.16 and the  $T$ -equivariance of  $\varphi^{-1}$  imply that the map from  $\mathcal{U}(u)$  to  $\text{Lie}(\mathcal{U}(u))$  induced by  $\varphi^{-1}$  does not depend on the choice of  $\varphi^{-1}$ . This proves part (iii).

The result now follows for arbitrary connected reductive  $G$  using Remark 4.3.  $\square$

**Remark 5.18.** If  $p > h(G)$  then the map  $\log$  from Example 4.12 is a logarithmic Springer map (see [34, Theorem 3] and [24, Remark 27]). In this case any Borel subgroup of  $G$  is a restricted parabolic, so the restriction of any logarithmic Springer map for  $G$  to  $R_u(B)$  is  $\log_B$  by Proposition 5.17(ii). Hence  $\log$  is the unique logarithmic Springer map for  $G$ .

**Remark 5.19.** We saw above that the condition on  $\phi$  in Definition 5.16 implies part (ii) of Proposition 5.17. Sobaje observes at the beginning of [36, Section 2] that the converse also holds. The reason is that every  $1 \neq u \in \mathcal{U}_G^{(1)}$  belongs to  $R_u(P)$  for some restricted parabolic subgroup  $P$  of  $G$ : this follows from [9, Theorem 2.4]. We also deduce that the restriction of  $\log_P$  to  $\mathcal{U}(u)$  is  $\phi_u$  for every restricted parabolic subgroup  $P$  of  $G$  and every  $u \in R_u(P)$  such that  $u$  has order  $p$ .

**Corollary 5.20.** *Let  $\phi$  be a logarithmic Springer map for  $G$ . Then for any  $A_1$  subgroup  $A$  of  $G$ ,  $A$  is good for  $G$  if and only if  $\phi$  is  $A$ -compatible.*

*Proof.* Suppose  $A$  is good. Let  $1 \neq u \in \mathcal{U}_A$ . Then  $\mathcal{U}(u) \subseteq A$  and

$$\phi(\mathcal{U}(u)) = \text{Lie}(\mathcal{U}(u)) \subseteq \text{Lie}(A),$$

so  $\phi$  is  $A$ -compatible. Conversely, suppose  $\phi$  is  $A$ -compatible. Let  $1 \neq u \in \mathcal{U}_A$  and set  $X = \phi(u) \in \text{Lie}(\mathcal{U}(u))$ . Now  $X \in \text{Lie}(A)$  by the  $A$ -compatibility, so  $kX \subseteq \text{Lie}(A)$ . Hence  $\mathcal{U}(u) = \phi^{-1}(kX) \subseteq A$  by the  $A$ -compatibility. We deduce from Lemma 5.10 that  $A$  is good for  $G$ .  $\square$

We now recall the other types of  $A_1$  subgroup that we need, namely optimal and subprincipal  $A_1$  subgroups. These were introduced by McNinch in [23] and [24].

**Definition 5.21.** We call a homomorphism  $\beta : \mathrm{SL}_2 \rightarrow G$  *optimal* if there is a maximal torus  $T$  of  $\mathrm{SL}_2$  such that the restriction  $\lambda$  of  $\beta$  to  $T \cong \mathbb{G}_m$  is a cocharacter associated in  $G$  to some nilpotent  $0 \neq X \in \mathrm{Im}(d\beta)$ . We call an  $A_1$  subgroup of  $G$  *optimal* if it is the image of an optimal homomorphism.

**Remark 5.22.** This is equivalent to the definition in [24, Section 1]: for it is clear that if  $T$  is the standard maximal torus of  $\mathrm{SL}_2$  and  $\lambda$  is associated to some nilpotent  $0 \neq X \in \mathrm{Lie}(\mathrm{SL}_2)$  then  $X$  is a scalar multiple of  $d\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ .

**Definition 5.23.** Fix a Springer map  $\phi$  for  $G$ . We call a homomorphism  $\beta : \mathrm{SL}_2 \rightarrow G$  *subprincipal* if there is a maximal torus  $T$  of  $\mathrm{SL}_2$  such that the restriction  $\lambda$  of  $\beta$  to  $T \cong \mathbb{G}_m$  is a cocharacter associated in  $G$  to some nilpotent  $0 \neq X \in \mathrm{Im}(d\beta)$  and  $\phi^{-1}(X)$  is  $G$ -conjugate to an element of  $\mathrm{Im}(\beta)$ . Note that the latter condition does not depend on the choice of  $\phi$ , by Remark 4.5. We call an  $A_1$  subgroup of  $G$  *subprincipal* if it is the image of a subprincipal homomorphism.

The next result implies Theorem 1.4.

**Theorem 5.24.** *Let  $A$  be an  $A_1$  subgroup of  $G$ . Let  $\phi$  be a logarithmic Springer map for  $G$ . The following conditions are equivalent.*

- (i)  $A$  is subprincipal.
- (ii)  $A$  is optimal.
- (iii) There exist  $u \in \mathcal{W}_G^{(1)}$  and  $\lambda \in Y(A)$  such that  $\lambda \in \Omega_{G,\phi}^a(u)$ .
- (iv)  $A$  is good.

*Proof.* The implication (i)  $\implies$  (ii) is immediate from the definitions, and (iii)  $\implies$  (iv) follows from Lemma 5.11. If  $A$  is optimal then there exist  $0 \neq X \in \mathcal{N}_A$  and  $\lambda \in Y(A)$  such that  $\lambda \in \Omega_G^a(X)$ . Then  $\lambda \in \Omega_{G,\phi}^a(u')$ , where  $u' = \phi^{-1}(X)$ , and  $u'$  has order  $p$  by Lemma 4.7. Hence (ii)  $\implies$  (iii).

By [23, Remark 21], there exists at least one subprincipal  $A_1$  subgroup  $A$  of  $G$  such that  $u \in A$ , and  $A$  is good by the arguments above. It is clear from the definition that any  $C_G(u)$ -conjugate of  $A$  is subprincipal. Since  $C_G(u)$  acts transitively on  $\mathcal{A}(u)$  (Theorem 5.4(ii)), it follows that any good  $A_1$  subgroup of  $G$  that contains  $u$  is subprincipal. This shows that (iv)  $\implies$  (i). Hence (i)–(iv) are all equivalent.  $\square$

**Remark 5.25.** If the equivalent conditions from Theorem 5.24 hold then there exist  $\lambda \in Y(A)$  and  $0 \neq X \in \mathcal{N}_A$  such that  $\lambda \in \Omega_G^a(X)$ . Then  $\lambda \in \Omega_G^a(u)$ , where  $u = \phi^{-1}(X)$ , which belongs to  $A$  by Corollary 5.20. Hence we can take the element  $u$  from Theorem 5.24(iii) to belong to  $A$  if we wish.

**Remark 5.26.** It is implicit in the discussion in [23, Section 1] that a subprincipal  $A_1$  subgroup of  $G$  is good. McNinch also proved that goodness and optimality are

equivalent for  $A_1$  subgroups under the extra assumption that  $G$  is strongly standard (see [24, Proposition 53]).

**Proposition 5.27.** *Let  $L$  be a Levi subgroup of  $G$  and let  $A$  be a good  $A_1$  subgroup of  $L$ . Then  $A$  is a good  $A_1$  subgroup of  $G$ .*

*Proof.* Since  $p$  is good for  $G$ ,  $p$  is good for  $L$ . By Theorem 5.24,  $A$  is optimal in  $L$ , so there exist  $0 \neq X \in \mathcal{N}_A$  and  $\lambda \in Y(A)$  such that  $\lambda \in \Omega_L^a(X)$ . Lemma 4.23 implies that  $\lambda \in \Omega_G^a(X)$ , so  $A$  is optimal in  $G$ . Hence  $A$  is a good  $A_1$  subgroup of  $G$  by Theorem 5.24.  $\square$

**Corollary 5.28.** *Let  $A$  be a good  $A_1$  subgroup of  $G$  and let  $1 \neq u \in \mathcal{U}_A$ . Then there is a Levi subgroup  $L$  of  $G$  such that  $A \subseteq L$  and  $u$  is a distinguished unipotent element of  $L$ .*

*Proof.* Pick a Levi subgroup  $L'$  of  $G$  such that  $u$  is a distinguished unipotent element of  $L'$ . By Theorem 5.4(i) we can choose a good  $A_1$  subgroup  $A'$  of  $L'$  such that  $u \in A'$ . Now  $A'$  is a good  $A_1$  subgroup of  $G$  by Proposition 5.27, so there exists  $g \in C_G(u)$  such that  $gA'g^{-1} = A$  (Theorem 5.4(ii)). Then  $A \subseteq L$ , where  $L := gL'g^{-1}$ . Clearly,  $L$  is a Levi subgroup of  $G$  and  $u$  is a distinguished unipotent element of  $L$ .  $\square$

**Corollary 5.29.** *Let  $\phi$  be a logarithmic Springer map for  $G$  and let  $L$  be a Levi subgroup of  $G$ . Then  $\phi_L$  is a logarithmic Springer map for  $L$ .*

*Proof.* Let  $1 \neq u \in \mathcal{U}_L^{(1)}$ . Choose a good  $A_1$  overgroup  $A$  of  $u$  in  $L$ . Then  $A$  is good for  $G$  by Proposition 5.27, so  $\mathcal{U}(u) \subseteq A$ . We see that  $\mathcal{U}(u)$  is both the unique 1-dimensional overgroup of  $u$  that is contained in a good  $A_1$  subgroup of  $L$ , and the unique 1-dimensional overgroup of  $u$  that is contained in a good  $A_1$  subgroup of  $G$ . The result now follows from the definition of a logarithmic Springer map.  $\square$

**Lemma 5.30.** *Let  $H$  be a connected reductive subgroup of  $G$ , and assume  $p$  is good for  $H$ . Let  $u \in \mathcal{U}_H^{(1)}$  such that  $u$  is distinguished in  $G$ . Let  $A \in \mathcal{A}_H(u)$ . Suppose there is a Springer map  $\phi$  for  $H$  such that  $\phi(u)$  is a distinguished element of  $\mathfrak{g}$ . Then  $A$  is good for  $G$ .*

*Proof.* By Theorem 5.24,  $A$  is a subprincipal  $A_1$  subgroup of  $H$ , so there exist  $\lambda \in Y(A)$  and  $0 \neq X \in \mathcal{N}_A$  such that  $\lambda$  is associated to  $X$  in  $H$  and  $\phi(u)$  is  $H$ -conjugate to  $X$ . Since by hypothesis  $\phi(u)$  is a distinguished element of  $\mathfrak{g}$ ,  $X$  is also a distinguished element of  $\mathfrak{g}$ . Lemma 4.22 implies that  $\lambda$  is associated to  $X$  in  $G$ . Hence  $A$  is an optimal  $A_1$  subgroup of  $G$ , so  $A$  is a good  $A_1$  subgroup of  $G$  by Theorem 5.24.  $\square$

**Remark 5.31.** Let  $H$  be a connected reductive subgroup of  $G$  and assume  $p$  is good for  $H$ . Suppose  $H$  is Springer-compatible. Let  $A$  be an  $A_1$  subgroup of  $H$  containing a distinguished unipotent element  $u$  of  $G$ . Let  $\phi$  be the restriction to  $H$

of any  $H$ -compatible Springer map for  $G$ . Then  $\phi(u)$  is a distinguished element of  $\mathfrak{g}$  by Remark 4.2, so the hypotheses of Lemma 5.30 hold. Hence if  $A$  is good for  $H$  then  $A$  is good for  $G$ .

The following relates the set of cocharacters of  $G$  that are associated to some  $1 \neq u \in \mathcal{U}_G^{(1)}$  to those stemming from good  $A_1$  overgroups of  $u$  in  $G$ .

**Corollary 5.32.** *Let  $1 \neq u \in \mathcal{U}_G^{(1)}$ . Let  $\phi$  be a logarithmic Springer map for  $G$ . We have a disjoint union*

$$\Omega_{G,\phi}^a(u) = \dot{\bigcup}_{A \in \mathcal{A}(u)} \Omega_{A,\phi_A}^a(u),$$

where  $\phi_A$  denotes the restriction of  $\phi$  to  $A$ .

*Proof.* Note that it makes sense to speak of the restriction of  $\phi$  to a good  $A_1$  subgroup  $A$  of  $G$ , by Corollary 5.20. We first prove that the union above is disjoint. Let  $A, \tilde{A} \in \mathcal{A}(u)$  and suppose there exists some

$$\lambda \in \Omega_{A,\phi_A}^a(u) \cap \Omega_{\tilde{A},\phi_{\tilde{A}}}^a(u).$$

Then  $A$  and  $\tilde{A}$  share the common Borel subgroup  $\lambda(\mathbb{G}_m)\mathcal{U}(u)$ . It follows from [21, Lemma 2.4] that  $A = \tilde{A}$ .

Let  $A \in \mathcal{A}(u)$  and let  $\lambda \in \Omega_{A,\phi_A}^a(u)$ . By Corollary 5.28 there is a Levi subgroup  $L$  of  $G$  such that  $A \subseteq L$  and  $u$  is a distinguished unipotent element of  $L$ . It follows from Lemma 4.24 (applied to the inclusion  $A \subseteq L$ ) and Lemma 4.25 that  $\lambda \in \Omega_{G,\phi}^a(u)$ . Hence

$$\Omega_{G,\phi}^a(u) \supseteq \dot{\bigcup}_{A \in \mathcal{A}(u)} \Omega_{A,\phi_A}^a(u).$$

Since we have  $\mathcal{A}(u) \neq \emptyset$  (Theorem 5.4(i)),  $C_G(u)$  acts transitively on both  $\mathcal{A}(u)$  (Theorem 5.4(ii)) and  $\Omega_{G,\phi}^a(u)$  (Proposition 4.19(ii)), and we see that the reverse inclusion follows.  $\square$

### 6. Proofs of Theorems 1.2 and 1.5–1.7

Armed with the results from above, we prove Theorems 1.2 and 1.6 simultaneously.

*Proof of Theorems 1.2 and 1.6.* We may assume that  $G$  is semisimple, since any unipotent element of  $G$  is contained in the derived subgroup  $\mathcal{D}G^\circ$ . Likewise, we may also assume that  $H$  is connected and semisimple, as any unipotent element of  $H^\circ$  is contained in the derived subgroup  $\mathcal{D}H^\circ$ , and  $H$  is  $G$ -ir if  $\mathcal{D}H^\circ$  is. Let  $u \in \mathcal{U}_H^{(1)}$  be distinguished in  $G$ .

First suppose  $p$  is bad for  $H$ . If  $p > 2$  then  $H$  admits a simple component  $H'$  of exceptional type. If  $u \in H$  is a distinguished unipotent element of  $G$  then the projection  $u'$  of  $u$  onto  $H'$  is a distinguished unipotent element of  $H'$ , so  $p = 3$  and  $H'$  is of type  $G_2$ , by Lemma 2.7. But this is impossible by Lemma 2.11 since  $p$  is good for  $G$ . Hence  $p = 2$ . It follows that each simple component of  $G$  is

of type  $A$ . Now distinguished unipotent elements are regular in type  $A$ , so  $u$  is a regular element of  $G$ . It follows from [6, Theorem 1.1] (resp., [6, Theorem 1.3]) that  $H$  (resp.,  $H_\sigma$ ) is  $G$ -ir.

Therefore we can assume that  $p$  is good for  $H$ . By Theorem 5.4(i) (resp., Theorem 5.12(i)) there is a good  $A_1$  subgroup (resp., good  $\sigma$ -stable  $A_1$  subgroup)  $A$  of  $H$  such that  $u \in A$ . By Lemma 5.30 and hypothesis  $(\dagger)$ ,  $A$  is a good  $A_1$  subgroup of  $G$ . Hence  $A$  (resp.,  $A_\sigma$ ) is  $G$ -cr by Theorem 5.4(iii) (resp., Theorem 5.12(iii)), so  $A$  (resp.,  $A_\sigma$ ) is  $G$ -ir by Lemma 3.1. We conclude that  $H$  (resp.,  $H_\sigma$ ) is  $G$ -ir.  $\square$

**Remark 6.1.** If we remove hypothesis  $(\dagger)$  from Lemma 5.30, Theorem 1.2, etc., then our arguments break down. For instance, let  $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ ,  $q$  and  $H_q$  be as in Example 4.13. Let  $u$  be any unipotent element of  $H_q$  such that the projection of  $u$  onto each  $\mathrm{SL}_2$ -factor of  $H_q$  is nontrivial; then  $u$  is distinguished (in fact, regular) in  $G$ . It is easy to see that  $H_q$  is not a good  $A_1$  subgroup of  $G$  and there does not exist  $\lambda \in Y(H_q)$  such that  $\lambda$  is associated to  $u$  in  $G$ ; in particular, the conclusion of Lemma 5.30 does not hold for  $H_q$ . Of course Theorem 1.1 still applies, alternately so does Theorem 1.8, so  $H_q$  is  $G$ -ir.

As a consequence of Theorems 1.2 and 1.6 we obtain the following.

**Corollary 6.2.** *Let  $G$  be a connected reductive group. Suppose  $p$  is good for  $G$ . Let  $\sigma$  be  $\mathrm{id}_G$  or a  $q$ -Frobenius endomorphism of  $G$ . Let  $u \in G_\sigma$  be unipotent of order  $p$ . Suppose  $u$  is distinguished in the  $\sigma$ -stable Levi subgroup  $L$  of  $G$  (see Remark 2.2(ii)). Let  $H$  be a  $\sigma$ -stable connected reductive subgroup of  $L$  containing  $u$ , and suppose there is a Springer map  $\phi$  for  $L$  such that  $\phi(u)$  is a distinguished element of  $\mathrm{Lie}(L)$ . Then  $H_\sigma$  is  $G$ -completely reducible.*

*Proof.* As  $p$  is also good for  $L$  (see Section 2.2), it follows from Theorem 1.2 (resp., Theorem 1.6) applied to  $L$  that  $H_\sigma$  is  $L$ -ir and so is  $L$ -cr. Thus,  $H_\sigma$  is  $G$ -cr, by [35, Proposition 3.2].  $\square$

**Remark 6.3.** In the setting of Theorem 1.1 the following argument allows us to reduce the case when  $G$  is connected reductive to the simple case. As in the proof of Theorem 1.2 above, we can assume that  $G$  is semisimple. Let  $G_1, \dots, G_r$  be the simple factors of  $G$ . Multiplication gives an isogeny from  $G_1 \times \dots \times G_r$  to  $G$ . Thus, by [4, Lemma 2.12(ii)(b)] and [16, Section 4.3], we can replace  $G$  with  $G_1 \times \dots \times G_r$ , so we can assume  $G$  is the product of its simple factors. Finally, thanks to [4, Lemma 2.12(i)] and [16, Section 4.3], we can reduce to the case when  $G$  is simple.

Finally, we address Theorems 1.5 and 1.7.

*Proof of Theorems 1.5 and 1.7.* By Theorems 1.2 and 1.6, the only cases we need to consider are when  $p$  is bad for  $G$ . If  $G$  is classical, then we are in the situation of Lemma 2.5 and Example 2.6, so we are done.

We are left to consider the case when  $G$  is of exceptional type. Then owing to [Lemma 2.7](#),  $G$  is of type  $G_2$  and  $p = 3$ . There is no harm in assuming that  $H$  is semisimple. It follows from [Example 2.8](#) that  $H$  is  $G$ -ir. Thus [Theorem 1.5](#) follows. So consider the setting of [Theorem 1.7](#) when  $\sigma|_H$  is a  $q$ -Frobenius endomorphism of  $H$  in this case. By [Corollary 2.10](#),  $u$  belongs to the subregular class of  $G_2$ . It follows from the proof of [Lemma 2.7](#) in [\[31\]](#) that  $u$  is contained in a  $\sigma$ -stable maximal rank subgroup of  $G$  of type  $A_1\tilde{A}_1$  and this type is unique. Since  $H$  is proper and semisimple,  $H \subseteq M$ , where  $M$  is a  $\sigma$ -stable maximal rank subgroup of  $G$  of type  $A_1\tilde{A}_1$ . Since  $p$  is good for  $H$ , there is a  $\sigma$ -stable subgroup  $A$  of  $H$  of type  $A_1$  containing  $u$ , by [Theorem 2.4](#). Thus  $A \subseteq H \subseteq M$ . Since  $u$  is also distinguished in  $M$  and  $p = 3$  is good for  $M$ , [Theorem 1.6](#) shows that  $A_\sigma$  is  $M$ -ir. Note that  $M$  is the centralizer of a semisimple element of  $G$  of order 2 (by Derizotis' criterion, see [\[11, 2.3\]](#)). Since  $A_\sigma$  is  $M$ -cr, it is  $G$ -cr, owing to [\[4, Corollary 3.21\]](#). Once again, by [Lemma 3.1](#),  $A_\sigma$  is  $G$ -ir and so is  $H_\sigma$ . [Theorem 1.7](#) follows.  $\square$

**Remark 6.4.** In [\[17, Section 7\]](#), Korhonen gives counterexamples to [Theorem 1.1](#) when the order of the distinguished unipotent element of  $G$  is greater than  $p$  (even when  $p$  is good for  $G$  [\[17, Proposition 7.1\]](#)). [Theorem 1.8](#) implies that this can only happen when  $p < a(G)$ . For instances of overgroups of distinguished unipotent elements of  $G$  of order greater than  $p$  for  $p \geq a(G)$  (and  $p$  good for  $G$ ), so that [Theorem 1.8](#) applies, see [Examples 6.6](#) and [6.7](#).

**Remark 6.5.** In view of [Remark 6.4](#), it is natural to ask for instances of  $G$ ,  $u$  and  $H$  when the conclusion of [Theorem 1.8](#) holds even when  $p < a(G)$  but  $p$  is still good for  $G$ . If  $p$  is good for  $G$  and  $G$  is simple classical, nonregular distinguished unipotent elements always belong to a maximal rank semisimple subgroup  $H$  of  $G$ , by [\[43, Propositions 3.1, 3.2\]](#). For  $G$  simple of exceptional type this is also the case in almost all instances of nonregular distinguished unipotent elements, see [\[43, Lemma 2.1\]](#). Each such  $H$  is obviously  $G$ -irreducible. This is independent of  $p$  of course and thus applies in particular when  $p < a(G)$ . For instance, let  $G$  be of type  $E_7$ ,  $p = 5$ , and suppose  $u$  belongs to the distinguished class  $E_7(a_3)$  (resp.,  $E_7(a_4)$ ,  $E_7(a_5)$ ). Then  $\text{ht}_J(\rho) = 9$  (resp.,  $7, 5$ ), so  $u$  has order  $5^2$ , by [Lemma 2.3](#) in each case. Since  $u$  does not have order 5, [Theorem 1.1](#) does not apply, and since  $5 < 8 = a(G)$  neither does [Theorem 1.8](#). Nevertheless, in each case  $u$  is contained in a maximal rank subgroup  $H$  of type  $A_1D_6$ , see [\[43, p. 52\]](#), and each such  $H$  is  $G$ -ir.

We close the section with several additional higher order examples in good characteristic when [Theorem 1.1](#) does not apply but [Theorem 1.8](#) does.

**Example 6.6.** Let  $G$  be of type  $E_6$ . Suppose  $p$  is good for  $G$ . In [\[43, Lemma 2.7\]](#), Testerman exhibits the existence of a simple subgroup  $H$  of  $G$  of type  $C_4$  whose regular unipotent class belongs to the subregular class  $E_6(a_1)$  of  $G$ . Let  $u$  be regular

unipotent in  $H$ . For  $p = 7$ , the order of  $u$  is  $7^2$ , by [Lemma 2.3](#), so [Theorem 1.1](#) can't be invoked to say anything about  $H$ . However, for  $p = 7 = a(G)$ , we infer from [Theorem 1.8](#) that  $H$  is  $G$ -ir.

**Example 6.7.** Let  $G$  be of type  $E_8$ . Suppose  $p = 11$ . Let  $u$  be in the distinguished class  $E_8(a_3)$  (resp.,  $E_8(a_4)$ ,  $E_8(b_4)$ ,  $E_8(a_5)$ , or  $E_8(b_5)$ ). From the corresponding weighted Dynkin diagram corresponding to  $u$  we get  $\text{ht}_J(\rho) = 17$  (resp., 14, 13, 11, or 11), see [\[10, p. 177\]](#). It follows from [Lemma 2.3](#) that in each of these instances  $u$  has order  $11^2$ . So we can't appeal to [Theorem 1.1](#) to deduce anything about reductive overgroups of  $u$ . But as  $11 = p \geq a(G) = 9$ , [Theorem 1.8](#) applies and allows us to conclude that each such overgroup is  $G$ -ir. For example, in each instance above,  $u$  is contained in a maximal rank subgroup  $H$  of  $G$  of type  $A_1E_7$  or  $D_8$ , see [\[43, p. 52\]](#).

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