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In memory of Gary, who influenced us greatly

Let G be a reductive algebraic group over an algebraically closed field \mathbb{k} of prime characteristic not 2, whose Lie algebra is denoted \mathfrak{g} . We call a subvariety \mathfrak{X} of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ *monogamous* if for every $e \in \mathfrak{X}$, the \mathfrak{sl}_2 -triples (e, h, f) with $f \in \mathfrak{X}$ are conjugate under the centraliser $C_G(e)$. Building on work by the first two authors, we show there is a unique maximal closed G -stable monogamous subvariety $\mathcal{V} \subset \mathcal{N}$ and that it is an orbit closure, hence irreducible. We show that \mathcal{V} can also be characterised in terms of Serre's G -complete reducibility.

1. Introduction

Let \mathbb{k} be an algebraically closed field of characteristic $p \neq 2$, and G a simple algebraic \mathbb{k} -group with Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Three elements $e, h, f \in \mathfrak{g}$ form an \mathfrak{sl}_2 -triple if the subalgebra $\langle e, h, f \rangle$ is a homomorphic image of $\mathfrak{sl}_2(\mathbb{k})$. That is, (e, h, f) satisfy the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

When the characteristic is 2 these relations degenerate leading to a qualitatively different theory; see [Stewart and Thomas 2024] for more details. This justifies our underlying assumption of $p \neq 2$. Theorems of Jacobson [1951], Morozov [1942] and Kostant [1959] say that if \mathbb{k} is of characteristic 0, then for any nilpotent $e \in \mathfrak{g}$ there exists an \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{g} which is unique up to conjugacy by the centraliser of e in G .

Over fields of positive odd characteristic, for any nilpotent $e \in \mathfrak{g}$ there exists an \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{g} except in the case G is of type G_2 , $p = 3$, and e is in the $\tilde{A}_{1(3)}$ class [Stewart and Thomas 2018, Theorem 1.7]. We continue the investigation into generalising Kostant's uniqueness theorem to fields of small characteristic. Let \mathfrak{X} be a subset of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. We say that \mathfrak{X} is *monogamous* if the

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following property holds:

Let (e, h, f) and (e, h', f') be \mathfrak{sl}_2 -triples with $e, f, f' \in \mathfrak{X}$. Then (e, h, f) is $C_G(e)$ -conjugate to (e, h', f') .

The main theorem of [Stewart and Thomas 2018] proves that \mathcal{N} is monogamous if and only if $p > h(G)$, where $h(G)$ is the Coxeter number for G . When G is of classical type, Goodwin and Pengelly [2024] showed that there exists a unique maximal G -stable closed subvariety of \mathcal{N} that is monogamous, and gave an explicit description of these. This paper completes the story by treating the exceptional types. Define the following subset of \mathcal{N} :

$$\mathcal{V} := \left\{ x \in \mathcal{N} \mid \begin{array}{l} x^{[p]} = 0, \\ x \text{ is not distinguished in a Levi subalgebra with a factor of type } A_{p-1}, \\ x \text{ is not subregular if } G \text{ is of type } G_2 \text{ and } p = 3. \end{array} \right\}.$$

Theorem 1.1. *Let G be a simple algebraic group over an algebraically closed field \mathbb{k} of characteristic $p > 2$. Then \mathcal{V} is the unique maximal G -stable closed monogamous subvariety of \mathcal{N} . Furthermore, \mathcal{V} is irreducible, being the closure of a single orbit as specified in Tables 1 and 2 below.*

In [Stewart and Thomas 2018], a close relationship was found between uniqueness of \mathfrak{sl}_2 -subalgebras and the existence of so-called non- G -cr \mathfrak{sl}_2 -subalgebras. The notion of G -complete reducibility for subgroups of G is due to Serre [2005], and the natural generalisation to subalgebras of \mathfrak{g} was introduced by McNinch [2007].

G	m	λ
A_{m-1}	$a(p-1) + r$	$((p-1)^a, r)$
$B_{\frac{m-1}{2}}$	$p + a(p-1) + r$ ($r > 0$)	$(p, (p-1)^a, r-1, 1)$ a even $(p, (p-1)^{a-1}, p-2, r+1)$ a odd
	$p + a(p-1)$	$(p, (p-1)^a)$ a even $(p, (p-1)^{a-1}, p-2, 1)$ a odd
	$\leq p$	(m)
$C_{\frac{m}{2}}$	$a(p-1) + r$	$((p-1)^a, r)$
$D_{\frac{m}{2}}$	$p + a(p-1) + r$	$(p, (p-1)^a, r)$ a even $(p, (p-1)^{a-1}, p-2, r, 1)$ a odd
	$\leq p$	$(m-1, 1)$

Table 1. Partition λ corresponding to the orbit O_λ such that $\mathcal{V} = \overline{O_\lambda}$ in the classical types, where $a \geq 0$ and $0 \leq r < p-1$.

G	p	O	G	p	O
G_2	3	$\tilde{A}_1^{(3)}$	E_6	3	A_1^3
	5	$G_2(a_1)$		5	$D_4(a_1)$
	≥ 7	G_2		7	$E_6(a_3)$
F_4	3	$A_1\tilde{A}_1$		11	$E_6(a_1)$
	5	$F_4(a_3)$		≥ 13	E_6
	7	$F_4(a_2)$			
	11	$F_4(a_1)$			
	≥ 13	F_4			
G	p	O	G	p	O
E_7	3	A_1^4	E_8	3	A_1^4
	5	$A_3A_2A_1$		5	A_3^2
	7	$E_7(a_5)$		7	$E_8(a_7)$
	11	$E_7(a_3)$		11	$E_8(a_6)$
	13	$E_7(a_2)$		13	$E_8(a_5)$
	17	$E_7(a_1)$		17	$E_8(a_4)$
	≥ 19	E_7		19	$E_8(a_3)$
				23	$E_8(a_2)$
		29	$E_8(a_1)$		
		≥ 31	E_8		

Table 2. Orbits O such that $\mathcal{V} = \bar{O}$ in the exceptional types.

Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we say that \mathfrak{h} is G -completely reducible (G -cr for short) if for every parabolic subalgebra \mathfrak{p} such that $\mathfrak{h} \subseteq \mathfrak{p}$ there exists some Levi subalgebra \mathfrak{l} of \mathfrak{p} with $\mathfrak{h} \subseteq \mathfrak{l}$.

We say $\mathfrak{X} \subseteq \mathcal{N}$ is A_1 - G -cr if every subalgebra generated by an \mathfrak{sl}_2 -triple (e, h, f) with $e, f \in \mathfrak{X}$ is G -cr.

Theorem 1.2. *Let G be a simple algebraic group over an algebraically closed field \mathbb{k} of characteristic $p > 2$. Then \mathcal{V} is the unique maximal G -stable closed A_1 - G -cr subvariety of \mathcal{N} .*

The proof follows very quickly from Theorem 1.1; see Section 4.

Remark 1.3. It would be interesting to know more about the geometry of the nilpotent variety \mathcal{V} . In type A , Donkin [1990] showed that the closure of each orbit is normal. Orbit closures in the remaining classical types are considered by Xiao and Shu [2015]. For exceptional types G_2, F_4, \dots, E_8 , results of Thomsen [2000] show that our varieties \mathcal{V} are in fact Gorenstein normal varieties with rational singularities as long as $p \geq 5, 11, 7, 11, 13$, respectively.

2. Preliminaries

Throughout, \mathbb{k} is an algebraically closed field of characteristic $p > 2$ and G is a simple \mathbb{k} -group with $\mathfrak{g} = \text{Lie}(G)$. There is an inherited $[p]$ -map on \mathfrak{g} and we use $x^{[p]}$ to denote the image of $x \in \mathfrak{g}$ under this map. The variety of all nilpotent elements in \mathfrak{g} , often called the nilpotent cone, is denoted by \mathcal{N} . The restricted nullcone is the subvariety of \mathcal{N} consisting of elements x such that $x^{[p]} = 0$ and we denote it by \mathcal{N}_p . The distribution of nilpotent elements among \mathfrak{sl}_2 -subalgebras of \mathfrak{g} is insensitive to central isogeny, and so we assume that whenever G is classical, it is one of $\text{SL}(V)$, $\text{Sp}(V)$ or $\text{SO}(V)$ and write $G = \text{Cl}(V)$ for brevity; if G is exceptional, we take it to be simply connected.

Recall that a prime p is bad for G if $p = 2$ and G is of type B , C or D ; if $p \leq 3$ and G is exceptional; or if $p \leq 5$ and G is of type E_8 ; otherwise it is good. In some examples we require a choice of base for the root system associated to \mathfrak{g} ; we use Bourbaki notation [2005]. Finally, we fix a maximal torus T of G .

2.1. Nilpotent orbits and Hasse diagrams. The orbits for the action of G on \mathcal{N} are called nilpotent orbits. There are finitely many such and they are classified. In case G is of exceptional type, we describe an orbit $O = G \cdot x$ by a label indicating a Levi subalgebra in which e is distinguished; for these labels we refer to [Liebeck and Seitz 2012].

When $G = \text{Cl}(V)$, the classification of orbits in terms of the action on V is well-known and can be found in [Jantzen 2004, Section 1], but we recap it here for ease of reference. Set $m = \dim V$. If $G = \text{SL}(V)$, orbits are parametrised by partitions of m according to the Jordan decomposition of their elements' actions on V ; we write $x \sim (\lambda_1, \dots, \lambda_r)$ where $\lambda_1 \geq \dots \geq \lambda_r$ is the partition of m corresponding to x . In types B and C orbits are parametrised by partitions of m with an even number of even parts and an even number of odd parts, respectively. In type D it is slightly more complicated. A partition is called very even if it only has even parts and they all occur with even multiplicity. There is one orbit for each partition of m with an even number of even parts that is not very even; and two orbits for each very even partition of m .

To check that \mathcal{V} is a closed subvariety of \mathcal{N} we require information about the Hasse diagrams for the closure relation on nilpotent orbits. For classical types, apart from type D , the closure order on orbits is precisely the dominance order on partitions. In type D we start with the Hasse diagram for the dominance order on partitions with an even number of even parts. Then we replace each very even partition λ with two nodes λ_1, λ_2 and replace each edge from λ to μ with two edges from λ_i to μ . For exceptional types the picture is actually incomplete in general. But if p is good for G , the existence of Springer morphisms implies that the Hasse diagrams remain the same as those in characteristic 0 [Spaltenstein 1982, Théorème III 5.2]. These can be found in [Spaltenstein 1982, pp. 247–250] and are

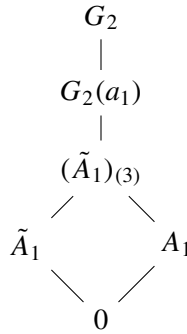
reproduced in [Carter 1993, Section 13.4] with labels closer to those in [Liebeck and Seitz 2012]. However, those in [Carter 1993] are missing edges in the E_6 , E_7 and E_8 diagrams. Specifically, there should be an edge between the following pairs of labels:

$$E_6: (D_4(a_1), A_3).$$

$$E_7: (D_6(a_2), D_5(a_1) + A_1), (D_5(a_1), D_4), (D_4(a_1), 2A_2 + A_1), (D_4(a_1), A_2 + 3A_1).$$

$$E_8: (E_6 + A_1, E_8(b_6)), (E_8(a_7), D_6(a_2)), (A_3 + A_1, A_3).$$

In bad characteristic, there are not even the same number of nilpotent and unipotent orbits; for certain bad primes there are more nilpotent orbits than in characteristic 0. The full Hasse diagram for G_2 when $p = 3$ can be deduced from [Stuhler 1971]:



For the remaining types we will have to work harder to obtain partial information about the closure relations.

We can now prove part of Theorem 1.1.

Lemma 2.1. *The subset $\mathcal{V} \subseteq \mathcal{N}$ is a closed G -stable subvariety; moreover, it is the closure of a single orbit in each case, as specified in Tables 1 and 2.*

Proof. Suppose $G = \text{Cl}(V)$ with $\dim V = m$. An orbit corresponding to a partition λ of m is contained in the restricted nullcone if and only if the largest part of λ is at most p . Let $G = \text{SL}(V)$ or $\text{Sp}(V)$ (resp. $\text{SO}(V)$), and let $x \in \mathcal{N}$ with partition represented by λ . Then x is not distinguished in a Levi subalgebra with a factor of type A_{p-1} precisely when λ contains no parts of size p (resp. at most one part of size p). Now every orbit represented in Table 1 represents a single orbit in \mathcal{V} : for G of type D , each λ given in Table 1 is not very even. Observe that any other orbit in \mathcal{V} is represented by a partition lower than λ in the dominance ordering, and hence is contained in \overline{O}_λ ; and vice-versa, by definition of \mathcal{V} .

Now suppose G is of exceptional type. We use the tables in the corrected arxiv version of [Stewart 2016] to determine the orbits in the restricted nullcone. A nilpotent element x is distinguished in a Levi subalgebra with a factor of type A_{p-1} exactly when the labelling of its orbit contains an A_{p-1} part. Thus in good characteristic, as well as for G of type G_2 , the result then follows by inspecting the Hasse diagrams.

O	A_3^2	$D_4(a_1)A_2$	$A_3A_2A_1$	A_3A_2
λ	$(5, 4^2, 1^3)$	$(5, 3^3, 1^2)$	$(5, 3^2, 2^2, 1)$	$(5, 3^2, 1^5)$
O	$D_4(a_1)A_1$	$D_4(a_1)$	$A_3A_1^2$	$A_2^2A_1^2$
λ	$(5, 3, 2^2, 1^4)$	$(5, 3, 1^8)$	$(5, 2^4, 1^3)$	$(3^5, 1)$
O	A_3A_1	$A_2^2A_1$	A_3	A_2^2
λ	$(5, 2^2, 1^7)$	$(3^4, 2^2)$	$(5, 1^{11})$	$(3^4, 1^4)$
O	$A_2A_1^3$	$A_2A_1^2$	A_2A_1	A_2
λ	$(3^3, 2^2, 1^3)$	$(3^3, 1^7)$	$(3^2, 2^2, 1^6)$	$(3^2, 1^{10})$
O	A_1^4	A_1^3	A_1^2	A_1
λ	$(3, 2^4, 1^5)$	$(2^6, 1^4)$	$(2^4, 1^8)$	$(2^2, 1^{12})$

Table 3. D_8 partitions for nilpotent orbits in \mathcal{V} for E_8 , $p = 5$.

In the remaining cases we use case-by-case analysis. First let G be of type E_8 and $p = 5$. Note that every class is distinguished in $\text{Lie}(L)$ for L some Levi subgroup of G . The Levi subgroups in question are all conjugate to subgroups of M , a maximal subgroup of G of type D_8 . Let V be the 16-dimensional standard module for M . For each nontrivial class in \mathcal{V} we choose a representative e in $\text{Lie}(M)$ and calculate the Jordan block sizes for the action of e on V ; these are in Table 3. Note that for some classes there are many non- M -conjugate choices for e . For example, there are three non- M -conjugate Levi subgroups of M of type A_3^2 ; these correspond to the subsets of simple roots $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 5, 6, 8\}$ and $\{1, 2, 3, 6, 7, 8\}$. A regular nilpotent element of the corresponding Levi subalgebras will act on V with Jordan blocks of sizes (4^4) , (4^4) and $(5, 4^2, 1^3)$, respectively.

Note that the final partition is higher in the dominance order than all other partitions in Table 3. Therefore, the closure of the M -orbit of a representative of the class A_3^2 contains a representative of every class in \mathcal{V} . It remains to prove that there are no more G -classes in the closure of the A_3^2 class. By [Stewart 2016, Table 10], the Jordan block sizes for the adjoint action of nilpotent elements in the A_3^2 -class are $(5^{38}, 4^{12}, 1^{10})$. By embedding G into SL_{248} , it follows that the Jordan block sizes for the adjoint action of every nilpotent element in the closure of the A_3^2 -class will be lower than $(5^{38}, 4^{12}, 1^{10})$ in the dominance order. Using [loc. cit.], we check that every nonrestricted class has a Jordan block of size greater than 5 and all remaining classes (which have labels with an A_4 part) have at least 45 blocks of size 5.

Now let $p = 3$. When G is of type F_4 , the subset \mathcal{V} consists of the zero element and the union of the three classes with labels A_1 , \tilde{A}_1 and $A_1\tilde{A}_1$. All three nontrivial classes have representatives contained in $\text{Lie}(M)$ where M is a subgroup of type B_3 . We may choose these representatives so that the corresponding partitions of 7 are $(2^2, 1^3)$, $(3, 1^4)$ and $(3, 2^2)$, respectively. Therefore, all three classes are contained

in the closure of the $A_1\tilde{A}_1$ -class. By [Liebeck and Seitz 2012, Table 22.1.4], the three classes in \mathcal{V} for G of type E_6 (which are A_1, A_1^2 and A_1^3) are all contained in an F_4 -subalgebra. Therefore the closure of the A_1^3 -class contains all three classes.

When G is of type E_7 , the nonzero elements of \mathcal{V} consist of the union of the five classes with labels $A_1, A_1^2, (A_1^3)^{(1)}, (A_1^3)^{(2)}$ and A_1^4 . All such classes have representatives contained in $\text{Lie}(M)$ where M is a subgroup of type D_6 . We may choose these representatives so that the corresponding partitions of 12 are $(2^2, 1^8), (3, 1^9), (2^6), (3, 2^2, 1^5)$ and $(3, 2^4, 1)$, respectively. Thus, all the classes in \mathcal{V} are contained in the closure of the A_1^4 -class. The discussion in [Liebeck and Seitz 2012, Section 16.1.2] shows that the four nontrivial classes in \mathcal{V} for G of type E_8 (which are A_1, A_1^2, A_1^3 and A_1^4) are contained in an E_7 -subalgebra. Thus the closure of the A_1^4 -class contains all classes in \mathcal{V} .

A final routine use of the tables in [Stewart 2016] allows us to complete the proof. For example, when G is of type E_7 the Jordan block sizes for the adjoint action of a nilpotent element in the A_1^4 -class are $(3^{28}, 2^{14}, 1^{21})$. Every nonrestricted class has a block of size greater than 3 and all other remaining classes have at least 33 blocks of size 3. □

2.2. G -cr subalgebras.

Proposition 2.2. *Suppose $e \in \mathcal{N}_p$. If e is contained in an \mathfrak{sl}_2 -triple then there exists a G -cr subgroup $X \leq G$ of type A_1 such that $\text{Lie}(X)$ contains e .*

Proof. If $G = \text{SL}(V)$ then $e^{[p]} = 0$ implies e has Jordan blocks of size at most p , which means e is regular in a Levi subalgebra of type $A_{r_1} \times \cdots \times A_{r_i}$ where $r_j \leq p - 1$ for each j . The image of $X = \text{SL}_2$ under the completely reducible representation given by $L(r_1) \oplus \cdots \oplus L(r_i)$ satisfies the demands of the theorem, where r_j now represents a (restricted) high weight. So assume G is not of type A . Then if p is good for G , it is very good, and the result follows from [McNinch 2005, Proposition 33, Theorem 52].

So we may assume p is bad, and therefore that G is exceptional. As before, the orbits of \mathcal{N}_p can be worked out from the tables in [Stewart 2016] and there are not very many. By inspection, it follows that the label of every restricted nilpotent class is denoted by sums of A_r for $r < p$ and $D_4(a_1)$ if $G = E_8, p = 5$ or is $G_2(a_1)$ when $G = G_2, p = 3$; the class $(\tilde{A}_1)_{(3)}$ is excluded since it is not contained in an \mathfrak{sl}_2 -triple.

We first deal with the final case. The subsystem subgroup $A_2 < G_2$ contains an A_2 -irreducible subgroup X of type A_1 . By [Stewart 2010, Theorem 1], all simple subgroups of G_2 are G_2 -cr when $p = 3$. The restriction of the nontrivial 7-dimensional G_2 -module to X is $L(2)^2 + L(0)$. It follows that the nilpotent elements contained in $\text{Lie}(X)$ have Jordan blocks of size $(3^2, 1)$ and thus are in the $G_2(a_1)$ class by [Stewart 2016, Table 4].

In the remaining cases, every class is a distinguished element in $\mathfrak{l} = \text{Lie}(L)$ for some Levi subgroup L with simple factors only of type A_r with $r < p$ or D_4 . By [Serre 2005, Proposition 3.2], a subgroup X of L is G -cr if and only if it is L -cr. Furthermore a subgroup X of a central product $L = L_1 L_2$ is L -cr if and only if the projection of X to both L_1 and L_2 is L -cr. Therefore, it suffices to deal with the cases where L is simple and simply connected of type A_r ($r < p$) or D_4 — but these cases have already been tackled. \square

If X is G -cr then so is $\text{Lie}(X)$ by [McNinch 2007, Theorem 1]; so we get the following.

Corollary 2.3. *Suppose $e \in \mathcal{N}_p$. If e is contained in an \mathfrak{sl}_2 -triple then there exists a G -cr subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2$ of \mathfrak{g} containing e .*

The following is used a couple of times, and is [McNinch 2007, Lemma 4].

Lemma 2.4. *Let L be a Levi factor of a parabolic subgroup of G . Suppose that we have a Lie subalgebra $\mathfrak{s} \subset \mathfrak{l} = \text{Lie}(L)$. Then \mathfrak{s} is G -cr if and only if \mathfrak{s} is L -cr.*

Proposition 2.5. *Suppose $e \in \mathcal{N}$ is distinguished in a Levi subalgebra $\mathfrak{l} = \text{Lie}(L)$ with a factor of type A_{p-1} . Then there is an \mathfrak{sl}_2 -triple (e, h, f) such that $\mathfrak{s} := \langle e, h, f \rangle$ is non- G -cr and $f \in \overline{L} \cdot e$.*

Proof. By Lemma 2.4 it suffices to treat the case that $L = \text{SL}(V)$ with $\dim V = p$. In that case, let $\mathfrak{s} = \langle e, h, f \rangle$ be the image of \mathfrak{sl}_2 under the representation given by the p -dimensional baby Verma module $Z_0(0)$; see [Jantzen 1998, Section 5.4]. As $V \downarrow X = Z_0(0)$ is a nontrivial extension of the irreducible module $L(p-2)$ by the trivial module we have that \mathfrak{s} is not L -cr. It is easy to see that one of e or f has a full Jordan block on V and is therefore regular. But the whole of $\mathcal{N}(L)$ is the closure of a regular nilpotent element so we are done. \square

Lemma 2.6. *Let p be a good prime for G and (e, h, f) be an \mathfrak{sl}_2 -triple with $e, f \in \mathcal{N}$. Suppose that e and f are distinguished in Levi subalgebras of \mathfrak{g} with no factors of type A_{p-1} . If $\mathfrak{s} := \langle e, f \rangle$ is G -cr then \mathfrak{s} is a p -subalgebra.*

Proof. Suppose \mathfrak{s} is not a p -subalgebra. Then by [Stewart and Thomas 2018, Lemma 4.3], \mathfrak{s} is L -irreducible in a Levi subalgebra $\mathfrak{l} = \text{Lie}(L)$ with $L = L_1 L_2 \cdots L_r$ and L_1 of type A_{rp-1} , say, for some $r \in \mathbb{N}$. Therefore, the projection $\bar{\mathfrak{s}}$ of \mathfrak{s} to $\mathfrak{l}_1 = \text{Lie}(L_1)$ is also L_1 -irreducible, so that $\bar{\mathfrak{s}}$ acts irreducibly on the rp -dimensional natural L_1 -module. All irreducible representations of \mathfrak{sl}_2 have dimension at most p by [Block 1962, Lemma 5.1], thus $r = 1$. Moreover, the classification of p -dimensional irreducible \mathfrak{sl}_2 -modules in [Jantzen 1998, Section 5.4] shows that the image of e or f in $\bar{\mathfrak{s}}$ is regular in L_1 , a contradiction. \square

3. Monogamy of \mathcal{V}

We start with an observation that \mathcal{V} can be characterised using the following partial order on \mathcal{N} .

Definition 3.1. Let $x, y \in \mathcal{N}$. We say $x \leq y$ (resp. $x < y$) if $\text{rank}(\text{ad}(x)^{p-1}) \leq \text{rank}(\text{ad}(y)^{p-1})$ (resp. $\text{rank}(\text{ad}(x)^{p-1}) < \text{rank}(\text{ad}(y)^{p-1})$).

Note that $\text{rank}(\text{ad}(x)^{p-1})$ can be calculated from the adjoint Jordan blocks of x of size at least p , and if G is exceptional, this can be done by reference to [Stewart 2016, Section 3.1]. The next lemma follows from a simple case-by-case check, using Tables 1 and 2, the Hasse diagrams for nilpotent orbit closures and [Stewart 2016, Section 3.1].

Lemma 3.2. Let $x, y \in \mathcal{N}$ such that $x \in \mathcal{V}$ and $y \notin \mathcal{V}$. Then $x < y$.

Remark 3.3. Comparing ranks of $(p-1)$ -th powers is necessary for the partial order to differentiate nilpotent orbits contained in \mathcal{V} . For example, let G be of type E_6 , $p=5$, and take $x, y \in \mathcal{N}$ to be representatives of the $D_4(a_1)$ and A_4 classes, respectively. Then we have $x \in \mathcal{V}$ and $y \notin \mathcal{V}$. Using [Stewart 2016, Table 16] we see that $\text{rank}(\text{ad}(x)) = \text{rank}(\text{ad}(y)) = 78$, however $\text{rank}(\text{ad}(x)^{p-1}) = 11 < 15 = \text{rank}(\text{ad}(y)^{p-1})$.

Let $\mathfrak{X} \subseteq \mathcal{N}$. We say that \mathfrak{X} is *partially monogamous* if the following holds:

If (e, h, f) and (e, h', f') are two \mathfrak{sl}_2 -triples with $e, f, f' \in \mathfrak{X}$ and $f, f' \leq e$, then f and f' are conjugate under the action of $C_G(e)$.

Lemma 3.4. Let \mathfrak{X} be a subvariety of \mathcal{N}_p . Then \mathfrak{X} is monogamous if and only if it is partially monogamous.

Proof. One direction is trivial. Suppose \mathfrak{X} is partially monogamous but not monogamous. Then there exist \mathfrak{sl}_2 -triples (e, h, f) and (e, h', f') with $e, f, f' \in \mathfrak{X}$ such that (e, h, f) is not $C_G(e)$ -conjugate to (e, h', f') . Since \mathfrak{X} is partially monogamous it follows that either $f \not\leq e$ or $f' \not\leq e$; without loss of generality we assume the former. Thus $\text{rank}(\text{ad}(e)^{p-1}) < \text{rank}(\text{ad}(f)^{p-1})$, and in particular, e and f are not conjugate.

Let $(f, \tilde{h}, \tilde{e})$ be an \mathfrak{sl}_2 -triple with f conjugate to \tilde{e} , which exists by Proposition 2.2. Then the two \mathfrak{sl}_2 -triples $(f, -h, e)$ and $(f, \tilde{h}, \tilde{e})$ satisfy $f, e, \tilde{e} \in \mathfrak{X}$ and $e, \tilde{e} \leq f$. But as \mathfrak{X} is partially monogamous, we have that f is conjugate to \tilde{e} , which is in turn conjugate to e , a contradiction. \square

Theorem 1.1 for classical types follows from Lemma 2.1 and the main theorem of [Goodwin and Pengelly 2024]. For the remainder of this section we suppose G is of exceptional type.

3.1. Bad characteristic. We first treat the case when p is bad for G . Fix $0 \neq e \in \mathcal{V}$ for the remainder of this section. We use the representatives as in [Liebeck and Seitz 2012], presented in [Stewart 2016]. If G is of type G_2 and $p = 3$, then the element e with label $(\tilde{A}_1)_{(3)}$ cannot be extended to an \mathfrak{sl}_2 -triple by [Stewart and Thomas 2018, Theorem 1.7]. So we exclude that case from now on.

Lemma 3.5. *The normaliser $N_G(\langle e \rangle)$ (and centraliser $C_G(e)$) is smooth if and only if the class of e does not occur in the following table:*

G	p	class of e
G_2	3	$G_2(a_1)$
F_4	3	$F_4, \tilde{A}_2 A_1$
E_6	3	$E_6, E_6(a_1), E_6(a_3), A_5, A_2^2 A_1, A_2^2$
E_8	3	$E_8, E_8(a_1), E_8(a_3), E_7, E_6 A_1, E_8(b_6),$ $A_7, E_6, E_6(a_3) A_1, A_5 A_1, A_2^2 A_1^2, A_2^2 A_1$
	5	$E_8, A_4 A_3$

Proof. Every element e has a cocharacter τ for which $\text{im}(\tau)$ is contained in $N_G(\langle e \rangle)$ but not $C_G(e)$. Thus, the dimension of $N_G(\langle e \rangle)$ is precisely $\dim C_G(e) + 1$. Similarly, $\dim \mathfrak{n}_{\mathfrak{g}}(\langle e \rangle) = \dim \mathfrak{c}_{\mathfrak{g}}(\langle e \rangle) + 1$ thanks to the existence of \mathfrak{sl}_2 -triples. Therefore $N_G(\langle e \rangle)$ is smooth precisely when $C_G(e)$ is smooth.

It is straightforward to use Magma to calculate the dimension of $\mathfrak{c}_{\mathfrak{g}}(e)$. Comparing these dimensions with the dimension of $C_G(e)$ presented in [Liebeck and Seitz 2012, Tables 22.1.1–22.1.5] completes the proof. \square

Observe that the set of classes in Lemma 3.5 does not intersect \mathcal{V} , so we may now deduce an important reduction.

Proposition 3.6. *There exists an \mathfrak{sl}_2 -triple (e, \bar{h}, \bar{f}) with \bar{f} conjugate to e and $\bar{h} \in \mathfrak{t} = \text{Lie}(T)$. Moreover, if (e, h, f) is also an \mathfrak{sl}_2 -triple then h is $C_G(e)$ -conjugate to \bar{h} .*

Proof. We know from Proposition 2.2 that there is an \mathfrak{sl}_2 -triple (e, \bar{h}, \bar{f}) with \bar{f} in the same nilpotent class as e . By Lemma 3.5, the group $N_G(\langle e \rangle)$ is smooth. Therefore, all maximal tori in $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$ are $N_G(\langle e \rangle)$ -conjugate. A computation in Magma shows that $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle) \cap \mathfrak{t}$ is a maximal torus of $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$. So we may assume that \bar{h} is contained in \mathfrak{t} (noting that if $(\lambda e, \bar{h}^g, \bar{f}^g)$ is an \mathfrak{sl}_2 -triple then so is $(e, \bar{h}^g, \lambda \bar{f}^g)$).

For the final part, first note that since $[h, e] = 2e$ we have $[h^{[p]^r}, e] = \text{ad}(h)^{[p]^r} e = 2e$ thanks to Fermat's little theorem. Therefore $\mathfrak{h} = \langle h^{[p]^r} \mid r = 0, 1, \dots \rangle$ is an abelian p -closed subalgebra of $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$. It follows from [Strade and Farnsteiner 1988, Chapter 2, Corollary 4.2] that $\mathfrak{h} = \mathfrak{t}' \oplus \mathfrak{n}'$ where \mathfrak{t}' is the set of semisimple elements of \mathfrak{h} . Since \mathfrak{t}' is a torus, the above argument shows that up to $N_G(\langle e \rangle)$ -conjugacy we may assume that \mathfrak{t}' is contained in \mathfrak{t} . In particular, $\bar{h} \in \mathfrak{t}'$.

Because $\mathfrak{c}_{\mathfrak{g}}(\langle e \rangle)$ has codimension 1 in $\mathfrak{n}_{\mathfrak{g}}(\langle e \rangle)$ and $\bar{h} \notin \mathfrak{c}_{\mathfrak{g}}(\langle e \rangle)$ we see that the torus t' decomposes as $t' = \mathfrak{c}_{\mathfrak{g}}(e) \oplus \langle \bar{h} \rangle$. Furthermore, $\mathfrak{n}' \subset \mathfrak{c}_{\mathfrak{g}}(\langle e \rangle)$. It follows that $h = \bar{h} + h'$ for some $h' \in \mathfrak{c}_{\mathfrak{g}}(e) \cap \mathfrak{c}_{\mathfrak{g}}(\bar{h})$.

Since $h = [e, f]$ and $\bar{h} = [e, \bar{f}]$ we also have $h' \in \text{im}(\text{ad}(e))$. Thus

$$h' \in W = \mathfrak{c}_{\mathfrak{g}}(\langle e, h \rangle) \cap \text{im}(\text{ad}(e)).$$

Another Magma check shows that every element in W is p -nilpotent.

In particular, all eigenvalues of h' are 0. Since $h = \bar{h} + h'$ and $[h, f] = -2f$ we must have $[\bar{h}, f] = -2f$. Therefore, $f \in F = \ker(\text{ad}(\bar{h}) + 2I_{\dim \mathfrak{g}})$ and so $h = [e, f] \in \text{im}(\text{ad}(e))(F)$. Note that $\bar{f} \in F$ also, so $\bar{h} \in \text{im}(\text{ad}(e))(F)$ and hence $h' \in \text{im}(\text{ad}(e))(F)$.

Thus $h' \in W \cap \text{im}(\text{ad}(e))(F)$. A final straightforward check in Magma shows that $W \cap \text{im}(\text{ad}(e))(F) = 0$, as required. \square

We now describe an ad-hoc method to prove that if (e, h, f') is an \mathfrak{sl}_2 -triple with $f' \in \mathcal{V}$ and $f' \preceq e$ then f' is uniquely determined up to $C := (C_G(e) \cap C_G(h))$ -conjugacy. In principle, this can be implemented by hand, but for speed and accuracy we have used Magma. Applying Proposition 3.6 and Lemma 3.4 then completes the proof that \mathcal{V} is monogamous.

Setup: By Proposition 3.6, there exists an \mathfrak{sl}_2 -triple (e, h, f) with $h \in \mathfrak{t} = \text{Lie}(T)$ and $f \in \mathcal{V}$ in the same nilpotent class as e . Let (e, h, f') be an \mathfrak{sl}_2 -triple with $f' \in \mathcal{V}$ and $f' \preceq e$. Since

$$(1) \quad [h, f'] = -2f'$$

we have $f' \in F := \ker(\text{ad}(h) + 2I_{\dim(\mathfrak{g})})$. We set up a generic element of the subspace F , namely $\tilde{f} = \sum x_i v_i \in \mathfrak{g}$ where the x_i are variables and $v_1, \dots, v_{\dim(F)}$ is a basis for F . One can view the set of all \tilde{f} as describing a subvariety \mathcal{F} of \mathfrak{g} . In Steps 1 to 3 below, we add in additional equations and thus replace \mathcal{F} with successively smaller sets (still called \mathcal{F} by abuse of notation).

Step 1: The equation

$$(2) \quad [e, \tilde{f}] = h$$

yields a set of linear equations among the x_i . We use these to constrain \tilde{f} and thus reduce the dimension of \mathcal{F} . Now every element of \mathcal{F} forms an \mathfrak{sl}_2 -triple with e .

Example 3.7. We give an example where Step 1 is sufficient. Let G be of type E_7 , $p = 3$ and $e = e_{\alpha_2} + e_{\alpha_5} + e_{\alpha_7}$. Then e is a representative of the $(A_1^3)^{(1)}$ orbit and $e \in \mathcal{V}$ by Lemma 2.1. On this occasion it is obvious that (e, h, f) is an \mathfrak{sl}_2 -triple with $h = h_2 + h_5 + h_7 \in \mathfrak{t}$ and $f = e_{-\alpha_2} + e_{-\alpha_5} + e_{-\alpha_7}$.

Let $F := \ker(\text{ad}(h) + 2I_{\dim(\mathfrak{g})})$. A straightforward calculation shows that the space F is 27-dimensional with a basis of root vectors $v_1 = e_{r_1}, \dots, v_{27} = e_{r_{27}}$ for some set of roots r_1, \dots, r_{27} ; in particular $r_{12} = -\alpha_2$, $r_{13} = -\alpha_5$ and $r_{14} = -\alpha_7$.

We let $\tilde{f} = \sum_i x_i v_i$ as above. We then compute $[e, \tilde{f}] = h$. For $i \neq 12, 13, 14$ we find that the left-hand side has a coordinate of the form λx_i for $\lambda = 1$ or 2 . Thus $x_i = 0$ for $i \neq 12, 13, 14$. On the other hand the coordinate of h_2 is seen to be equal to $x_{14} + 2$. Thus x_{14} is -1 . Similarly, the coordinates of h_5 and h_7 are $x_{13} + 2$ and $x_{12} + 2$, respectively. We have therefore determined all the variables in \tilde{f} and in fact $\tilde{f} = f$, which is sufficient.

Step 2: The adjoint action of C preserves \mathcal{F} . Find a set of variables $\{x_i \mid i \in Z\}$ such that every C -orbit in \mathcal{F} contains a representative with $x_i = 0$ for $i \in Z$. Thus we may assume that these variables are zero in \tilde{f} , further reducing \mathcal{F} .

Example 3.8. We give an example where Steps 1 and 2 are sufficient. Let G be of type G_2 and $p = 3$. Consider $e = e_{10}$ which is a representative of the \tilde{A}_1 orbit, thus contained in \mathcal{V} by Lemma 2.1.

Clearly, if $h = h_{10}$, $f = e_{-10}$, then (e, h, f) is an \mathfrak{sl}_2 -triple with $f \in \mathcal{V}$. Define $F := \ker(\text{ad}(h) + 2I_{\dim(\mathfrak{g})})$. This is 3-dimensional and we build \tilde{f} as above:

$$\tilde{f} = x_1 e_{-11} + x_2 e_{-10} + x_3 e_{21}.$$

After Step 1 we find

$$\tilde{f} = x_1 e_{-11} + e_{-10} + x_3 e_{21}.$$

Now we apply elements of $C = C_G(e) \cap C_G(h)$ to \tilde{f} . First consider $x_{-01}(t) \in C$. We calculate that

$$x_{-01}(t) \cdot \tilde{f} = (t + x_1) e_{-11} + e_{-10} + x_3 e_{21}.$$

Therefore, by setting $t = -x_1$, we see that every C -orbit in \mathcal{F} contains a representative with $x_1 = 0$. We're down to

$$\tilde{f} = e_{-10} + x_3 e_{21}.$$

Finally, conjugation by $x_{31}(t) \in C$ sends \tilde{f} to $e_{-10} + (t + x_3) e_{21}$. Thus we conclude that $\tilde{f} = f$, as required.

Step 3: Finally, we impose the condition that \tilde{f} should represent an element $f' \in \mathcal{V}$ with $f' \preceq e$. Since every element in \mathcal{V} is p -nilpotent, the equation

$$(3) \quad \text{ad}(\tilde{f})^p = 0$$

yields further polynomial equations we want the x_i to satisfy.

Forcing \mathcal{F} to only contain elements f' with $f' \preceq e$ is slightly more subtle since we cannot simply calculate the 'rank' of $M = \text{ad}(\tilde{f})^{p-1}$. Let $R = \text{rank}(\text{ad}(e)^{p-1})$

and ϵ be a map evaluating the remaining variables to choices in \mathbb{k} (so each $f' \in \mathcal{F}$ is simply some $\epsilon(\tilde{f})$). We find a subset r_1, \dots, r_R of rows and subset c_1, \dots, c_R of columns such that, up to the reordering of rows and columns, the corresponding submatrix S of M is upper triangular and all diagonal entries are elements of \mathbb{F}_p^* . Then any element $f' \in \mathcal{F}$ will satisfy $\text{rank}(\text{ad}(f')^{p-1}) \geq R$. We only want those elements $f' \preceq e$ which means $\text{rank}(\text{ad}(f')^{p-1}) \leq R$. Thus, given any row r of M the element $\epsilon(r)$ is in the span of $\epsilon(r_1), \dots, \epsilon(r_R)$. In particular, a row r' of M with zeroes at all columns c_1, \dots, c_R evaluates to zero. This final set of conditions is enough to force all remaining variables to be 0.

Example 3.9. Here we require Step 3. Let G be of type G_2 and $p = 3$. Consider $e = e_{01}$ which is a representative of the A_1 orbit, thus contained in \mathcal{V} by Lemma 2.1.

Take $h = h_{01}$, $f = e_{-01}$. Then (e, h, f) is an \mathfrak{sl}_2 -triple in \mathfrak{g} with $f \in \mathcal{V}$. Define $F := \ker(\text{ad}(h) + 2I_{\dim(\mathfrak{g})})$. This is 5-dimensional and we build \tilde{f} as above:

$$\tilde{f} = x_1e_{-32} + x_2e_{-01} + x_3e_{-10} + x_4e_{11} + x_5e_{32}.$$

After Step 1 we find

$$\tilde{f} = x_1e_{-32} + e_{-01} + x_3e_{-10} + x_4e_{11} + x_5e_{32}.$$

There are no elements of $C = C_G(e) \cap C_G(h)$ which we can use to reduce \tilde{f} , so we move onto Step 3.

The equation $\text{ad}(\tilde{f})^p = 0$ gives many relations amongst the remaining variables but none that allow us to conveniently reduce \tilde{f} . Consider the matrix $M = \text{ad}(\tilde{f})^{p-1}$. The first, eighth, tenth and thirteenth column of M consist only of zeroes, so we remove them, leaving the matrix M' as follows:

$$\begin{pmatrix} x_1x_5 & 0 & 0 & x_5 & 2x_4^2 & 0 & 0 & x_4x_5 & 0 & x_5^2 \\ 0 & 2x_4 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_4 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x_1x_5+x_3x_4 & 0 & 0 & x_3x_5+x_4^2 & 0 & 0 \\ 0 & 2x_1x_4+2x_3^2 & 0 & 0 & 0 & x_1x_5+2x_3x_4 & 0 & 0 & x_3x_5+x_4^2 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 2x_4 & 0 & 0 & x_5 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1x_4+x_3^2 & 0 & 0 & 2x_1x_5+x_3x_4 & 0 & 0 \\ x_1 & 0 & 0 & 1 & x_3 & 0 & 0 & x_4 & 0 & x_5 \\ 0 & 2x_1 & 0 & 0 & 0 & 2x_3 & 0 & 0 & 2x_4 & 0 \\ 0 & 0 & 2x_1 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 \\ x_1^2 & 0 & 0 & x_1 & x_1x_3 & 0 & 0 & 2x_3^2 & 0 & x_1x_5 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_3 & 0 \end{pmatrix}$$

We calculate that

$$R = \text{rank}(\text{ad}(e)^{p-1}) = 1.$$

Therefore, if $\epsilon(\tilde{f}) = f' \preceq e$ for some evaluation map ϵ , the rank of $\epsilon(M')$ is at most one. Observe that $M'_{10,4} = 1$ and so the rank of $\epsilon(M')$ is at least one. It follows that every row of $\epsilon(M')$ is a multiple of the tenth row of $\epsilon(M')$.

Consider the sixth row of M' . This only has nonzero entries in columns 2, 6 and 9, namely $x_3, 2x_4$ and x_5 . Since the tenth row is zero in columns 2, 6 and 9, the sixth row of $\epsilon(M')$ is zero. Hence $x_3 = x_4 = x_5 = 0$.

Similarly, row 11 of $\epsilon(M')$ is zero. Thus $x_1 = 0$, and we conclude that $\tilde{f} = f$.

3.2. Good characteristic. Suppose p is a good prime for G . As in the bad characteristic case, we describe an algorithm to deduce that \mathcal{V} is monogamous. In good characteristic there is a considerable amount of theory at our disposal. In particular, every $e \in \mathcal{N}$ has an associated cocharacter: that is, a homomorphism $\tau : \mathbb{G}_m \rightarrow G$ such that under the adjoint action, we have $\tau(t) \cdot e = t^2 e$ and τ evaluates in the derived subgroup of the Levi subgroup in which e is distinguished.

Lemma 3.10. *Suppose p is good for G , and let (e, h_1, f_1) be an \mathfrak{sl}_2 -triple with $e, f_1 \in \mathcal{V}$. Then there is a cocharacter τ associated to e such that $\text{Lie}(\tau(\mathbb{G}_m)) = \langle h_1 \rangle$. Thus if (e, h_2, f_2) is also an \mathfrak{sl}_2 -triple with $f_2 \in \mathcal{V}$, then h_2 is $C_G(e)$ -conjugate to h_1 . Moreover, if $h_1 = h_2$ and $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ is the grading of \mathfrak{g} with respect to τ we have*

$$f_1 - f_2 \in \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp),$$

where $\mathfrak{g}_e(i) := \mathfrak{c}_{\mathfrak{g}}(e) \cap \mathfrak{g}(i)$.

Proof. We start by proving that h_i is toral. By Lemma 2.6, the subalgebra $\mathfrak{s}_i = \langle e, h_i, f_i \rangle$ is either a p -subalgebra or non- G -cr. In the former case, we are done. In the latter case, the argument in the proof of [Stewart and Thomas 2018, Lemma 6.1] applies, showing h_i is toral.

Now we apply [Stewart and Thomas 2018, Proposition 2.8]. This yields cocharacters τ_i associated to e such that $\text{Lie}(\tau_i(\mathbb{G}_m)) = \langle h_i \rangle$. By [Jantzen 2004, Lemma 5.3], any two cocharacters associated to e are $C_G(e)$ -conjugate. Therefore, h_1 and h_2 are $C_G(e)$ -conjugate and so up to $C_G(e)$ -conjugacy we may assume they are equal. Set $h = h_1 = h_2$.

Since $[e, f_1 - f_2] = h - h = 0$ we know $f_1 - f_2 \in \mathfrak{c}_{\mathfrak{g}}(e)$. Furthermore, $[h, f_1 - f_2] = -2(f_1 - f_2)$ and hence

$$f_1 - f_2 \in \bigoplus_r \mathfrak{g}(-2 + rp).$$

The conclusion follows by noting that $\mathfrak{c}_{\mathfrak{g}}(e)$ is contained in the nonnegative graded part of \mathfrak{g} . □

Fix $0 \neq e \in \mathcal{V}$ for the remainder of this section. Choose a cocharacter τ associated to e such that $h \in \text{Lie}(\tau(\mathbb{G}_m)) \subset \mathfrak{t}$ with $[h, e] = 2e$. In practice, we use the representatives and associated cocharacters given in [Lawther and Testerman 2011]. We know from Pommerening [1977; 1980] and Lemma 3.10 that there exists a unique $\bar{f} \in \mathfrak{g}(-2)$ such that (e, h, \bar{f}) is an \mathfrak{sl}_2 -triple. Furthermore, if (e, h, f) is another \mathfrak{sl}_2 -triple then $f = \bar{f} + f'$ with $f' \in \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp)$. Therefore, we need to prove that if $f \in \mathcal{V}$ then up to $C = C_G(e) \cap C_G(h)$ -conjugacy we have $f = \bar{f}$, i.e., that $f' = 0$.

To do this we use the ad-hoc method from Section 3.1. Indeed, by Lemma 3.4 it suffices to prove that $f = \bar{f}$ when $f \leq e$. We now apply Steps 1–3 starting with the space $F = f + \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp)$.

Example 3.11. We give a final example, this time in good characteristic. Let G be of type E_7 and $p = 7$. Consider

$$e = e_{100000}^0 + e_{010000}^0 + e_{001000}^0 + e_{000100}^0 + e_{000010}^0,$$

which is a representative of the $(A_5)^{(2)}$ orbit; thus $e \in \mathcal{V}$ by Lemma 2.1. By [Lawther and Testerman 2011, p. 109], e has an associated cocharacter with the following τ -weights on simple roots $\tau = \begin{smallmatrix} 2 & 2 \\ -9 \end{smallmatrix} \begin{smallmatrix} 2 & 2 & 2 \\ -5 \end{smallmatrix}$. One uses the inverse of the Cartan matrix to convert this into a sum of coroots, yielding

$$h = 2h_1 + 6h_3 + 5h_4 + 6h_5 + 2h_6 \in \text{Lie}(\tau(\mathbb{G}_m))$$

(this process is how one gets from the diagram of the distinguished cocharacters in Section 11 to the cocharacters given in Table 3 of [ibid.]). The unique $\bar{f} \in \mathfrak{g}(-2)$ such that (e, h, \bar{f}) is an \mathfrak{sl}_2 -triple is then given by

$$\bar{f} = 2e_{-100000}^0 + 6e_{-010000}^0 + 5e_{-001000}^0 + 6e_{-000100}^0 + 2e_{-000010}^0.$$

Let $F = f + \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp)$, which is 6-dimensional. We build a generic element \tilde{f} of F as in Section 3.1 with six variables. Following Step 1 by enforcing the linear equations from $[e, \tilde{f}] = h$ yields

$$\begin{aligned} \tilde{f} = \bar{f} &+ x_1 e_{-123211}^2 + x_2 e_{-001100}^1 + x_2 e_{-011000}^1 + x_3 e_{-000001}^0 + x_4 e_{111111}^0 \\ &- x_5 e_{122110}^1 + x_5 e_{112210}^1 + x_6 e_{234321}^2. \end{aligned}$$

On this occasion $C := C_G(e) \cap C_G(h)$ is finite and we move on to Step 3.

Let $M = \text{ad}(\tilde{f})^{p-1}$. We calculate that

$$R = \text{rank}(\text{ad}(e)^{p-1}) = 13.$$

So if $\epsilon(\tilde{f}) = f' \leq e$ for some evaluation map ϵ , we have that the rank of $\epsilon(M)$ is at most 13.

Ordering the basis of \mathfrak{g} as in Magma, we use the 13×13 submatrix S of M corresponding to the rows r and columns c where

$$r = \{75, 125, 62, 94, 87, 129, 120, 97, 42, 82, 23, 34, 108\},$$

$$c = \{37, 100, 24, 52, 50, 109, 92, 60, 14, 40, 5, 9, 72\}.$$

The submatrix S is upper triangular and all diagonal entries are elements of \mathbb{F}_p^* . The only other nonzero entries in S can be found in row one, which is

$$(1 \ 0 \ 4x_2 \ 0 \ 0 \ 0 \ 5x_5 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0).$$

We find that 42 rows of M have zero entries in every column in c , so each of these rows is zero. An example of such a row is the eighth row of M . In row 8 we find $x_4, 3x_5$ and $-x_6$ in columns 11, 15 and 70, respectively. It follows that $x_4 = x_5 = x_6 = 0$. Similarly the 133-rd row of M then allows us to deduce that $x_1 = x_2 = x_3 = 0$. Thus $\tilde{f} = f$ as required.

4. Proof of Theorems 1.1 and 1.2

Proposition 2.2 shows that for each $e \in \mathcal{V}$ there exists an \mathfrak{sl}_2 -triple (e, h, f) with $\mathfrak{s} = \langle e, h, f \rangle = \text{Lie}(X)$ for a G -cr subgroup $X < G$ of type A_1 . Thus f is G -conjugate to e and hence $f \in \mathcal{V}$. We have demonstrated in Section 3 that any other \mathfrak{sl}_2 -triple (e, h', f') with $f' \in \mathcal{V}$ is $C_G(e)$ -conjugate to (e, h, f) . Therefore $\mathfrak{s}' = \langle e, h', f' \rangle$ is G -conjugate to \mathfrak{s} and hence G -cr.

It remains to prove that \mathcal{V} is the unique maximal closed G -stable subvariety of \mathcal{N} satisfying both the monogamy and A_1 - G -cr conditions.

For G of classical type, it follows from [Goodwin and Pengelly 2024, Theorem 1.1] that \mathcal{V} is maximal with respect to being monogamous and the unique subvariety with this property. For the A_1 - G -cr property, the ingredients are in [ibid.] but let us spell out the details, as these essentially make up the strategy for the groups of exceptional type used below.

Proposition 4.1. *Let G be a simple algebraic group of classical type. Then \mathcal{V} is the unique maximal closed G -stable A_1 - G -cr subvariety of \mathcal{N} .*

Proof. Suppose \mathfrak{X} is a G -stable closed subvariety of \mathcal{N} satisfying the A_1 - G -cr condition and $\mathfrak{X} \not\subseteq \mathcal{V}$. Let $e \in \mathfrak{X} \setminus \mathcal{V}$.

First suppose e is distinguished in a Levi subalgebra $\mathfrak{l} = \text{Lie}(L)$ with L having a factor of type A_{p-1} . Proposition 2.5 shows that e is contained in an \mathfrak{sl}_2 -triple generating a non- G -cr subalgebra, a contradiction (these non- G -cr subalgebras are also exhibited in [Goodwin and Pengelly 2024, Section 2.4]).

By definition of \mathcal{V} , we may now assume that $e^{[p]} \neq 0$. The discussion before Proposition 2.2 in [ibid.] exhibits an \mathfrak{sl}_2 -triple (e, h, f) with $f^{[p]} = 0$ and f in $\overline{G \cdot e}$,

thus $f \in \mathfrak{X}$. The argument in the first paragraph shows that neither e nor f are distinguished in a Levi subalgebra with a factor of type A_{p-1} . By Lemma 2.6, the \mathfrak{sl}_2 -subalgebra $\langle e, f \rangle$ is non- G -cr, a final contradiction. \square

Proposition 4.2. *Let G be a simple algebraic group of exceptional type. The variety \mathcal{V} is the unique maximal closed G -stable subvariety of \mathcal{N} satisfying both the monogamy and A_1 - G -cr conditions.*

Proof. Suppose \mathfrak{X} is a G -stable closed subvariety of \mathcal{N} satisfying either the monogamy or A_1 - G -cr condition and $\mathfrak{X} \not\subseteq \mathcal{V}$.

First suppose there exists $e \in \mathfrak{X}$ which is distinguished in a Levi subalgebra $\mathfrak{l} = \text{Lie}(L)$ with a factor of type A_{p-1} . Then Propositions 2.2 and 2.5 furnish us with two \mathfrak{sl}_2 -triples (e, h, f) and (e, h', f') such that the first generates a G -cr subalgebra and the second generates a non- G -cr subalgebra. Moreover, f is in the same G -class as e and f' is in the closure of the G -class of e . Hence \mathfrak{X} does not satisfy either condition, a contradiction.

Thus, we now assume every element of \mathfrak{X} is distinguished in a Levi subalgebra with no factors of type A_{p-1} . Since $\mathfrak{X} \not\subseteq \mathcal{V}$, there exists a nilpotent class in \mathfrak{X} with representative e distinguished in a Levi subalgebra $\mathfrak{l} = \text{Lie}(L)$ of \mathfrak{g} such that $e^{[p]} \neq 0$.

Suppose p is good for L . From [Premet and Stewart 2019, Section 2.4] we find an \mathfrak{sl}_2 -triple (e, h, f) of \mathfrak{l} with $f^{[p]} = 0$. Since p is good for L , we may simply inspect the Hasse diagrams of each factor of L to deduce that every restricted nilpotent class is contained in the closure of each nonrestricted distinguished class. Thus, $f \in X$. Furthermore, $\mathfrak{s} = \langle e, f \rangle \cong \mathfrak{sl}_2$ is a non- L -cr subalgebra by Lemma 2.6. Hence by Lemma 2.4, \mathfrak{X} does not satisfy the A_1 - G -cr condition. Proposition 2.2 yields an \mathfrak{sl}_2 -triple (f, h', e') which generates a G -cr \mathfrak{sl}_2 -subalgebra, and moreover e' is in the same G -class as f . Therefore, f is contained in two nonconjugate \mathfrak{sl}_2 -triples. Thus \mathfrak{X} does not satisfy the monogamy condition either.

In the remaining cases p is bad for L (and hence for G) so L has an exceptional factor (including the cases $L = G$). For each class, we choose e to be the representative as in [Liebeck and Seitz 2012]. Then [Liebeck and Seitz 2012, Theorem 1(iii)(b)] provides a parabolic subgroup $P = QL$ of G and a 1-dimensional torus $T_1 < Z(L)$ with the following properties: $e \in \mathfrak{q}_{\geq 2} := \text{Lie}(Q_{\geq 2})$ and moreover, the closure of the P -orbit of e is equal to $\mathfrak{q}_{\geq 2}$, where here $Q_{\geq 2}$ is the product of all root groups for which the T_1 -weight is at least 2. Thus, $\mathfrak{q}_{\geq 2} \subseteq \mathfrak{X}$. Unless G is of type G_2 (this case is dealt with momentarily), a straightforward calculation shows that $\mathfrak{q}_{\geq 2}$ contains a representative of the A_{p-1} -class. Thus, so does \mathfrak{X} , which is a contradiction.

Finally, let G be of type G_2 and $p = 3$. The only two classes not contained in \mathcal{V} are the regular and the subregular. Since the closure of the regular class contains the subregular class it suffices to assume \mathfrak{X} contains the subregular class.

A representative for this orbit is $e = e_{\alpha_2} + e_{-3\alpha_1 - \alpha_2}$. This is a regular nilpotent element in $\mathfrak{m} = \text{Lie}(M)$ where M is the standard subsystem subgroup of type A_2 corresponding to the simple roots α_2 and $-3\alpha_1 - 2\alpha_2$.

As in the proof of Proposition 2.5, there exists an \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{m} such that $\mathfrak{s} = \langle e, f \rangle$ is non- M -cr. Furthermore, f is in the orbit labelled A_1 (both as an A_2 -orbit and G_2 -orbit). We claim that \mathfrak{s} is non- G -cr. By Proposition 2.2, the element f is contained in an \mathfrak{sl}_2 -triple generating a G -cr subalgebra and by the claim, the \mathfrak{sl}_2 -triple $(f, -h, e)$ generates a non- G -cr subalgebra. Hence \mathfrak{X} does not satisfy either condition.

For the claim, note that \mathfrak{s} is certainly G -reducible since it is non- M -cr. All G -cr \mathfrak{sl}_2 -subalgebras which are G -reducible are contained in a Levi subalgebra. In this low-rank case, it immediately follows that all such \mathfrak{sl}_2 -subalgebras are G -conjugate to either $\mathfrak{l}_1 = \langle e_{\pm\alpha_1} \rangle$ or $\mathfrak{l}_2 = \langle e_{\pm\alpha_2} \rangle$. Therefore a G -cr \mathfrak{sl}_2 -subalgebra only contains nilpotent elements in the A_1 or A_1 classes. The claim follows since \mathfrak{s} contains e which is in the subregular class. \square

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
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