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ON DIMENSIONS OF ROCK BLOCKS OF CYCLOTOMIC QUIVER HECKE SUPERALGEBRAS

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To the memory of Gary Seitz

We explicitly compute the dimensions of certain idempotent truncations of RoCK blocks of cyclotomic quiver Hecke superalgebras. Equivalently, this amounts to a computation of the value of the Shapovalov form on certain explicit vectors in the basic representations of twisted affine Kac–Moody Lie algebras of type A .

1. Introduction

Our goal is to obtain a technical (but neat) result computing the dimensions of certain idempotent truncations of RoCK blocks of cyclotomic quiver Hecke superalgebras. Equivalently, this amounts to a computation of the value of the Shapovalov form on certain explicit vectors in the basic representation $V(\Lambda_0)$ of the twisted affine Kac–Moody Lie algebra \mathfrak{g} of type $A_{2\ell}^{(2)}$.

To state the result, let \mathfrak{g} be the Kac–Moody Lie algebra of type $A_{2\ell}^{(2)}$, with a normalized invariant form $(\cdot | \cdot)$ on the corresponding weight lattice P , and the Weyl group W . Let

$$I = \{0, 1, \dots, \ell\}, \quad J := I \setminus \{\ell\},$$

$\{\alpha_i \mid i \in I\}$ be the simple roots, $\{\Lambda_i \mid i \in I\}$ be the fundamental dominant weights, and $Q_+ \subset P$ be the set of $\mathbb{Z}_{\geq 0}$ -linear combinations of the simple roots. For the negative Chevalley generators $\{f_i \mid i \in I\}$ of \mathfrak{g} , we have the divided powers $f_i^{(k)} := f_i^k/k! \in U(\mathfrak{g})$ for $k \in \mathbb{Z}_{\geq 0}$. Fix a nonzero highest weight vector v_+ of the irreducible \mathfrak{g} -module $V(\Lambda_0)$ with highest weight Λ_0 . Let (\cdot, \cdot) be the Shapovalov form on $V(\Lambda_0)$ such that $(v_+, v_+) = 1$.

For every $w \in W$ with reduced decomposition $w = r_{i_t} \cdots r_{i_1}$, setting

$$(1.1) \quad a_k := (r_{i_{k-1}} \cdots r_{i_1} \Lambda_0 \mid \alpha_{i_k}^\vee) \quad (k = 1, \dots, t),$$

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it is easy to see that $a_1, \dots, a_\ell \in \mathbb{Z}_{\geq 0}$ and

$$v_w := f_{i_\ell}^{(a_\ell)} \cdots f_{i_1}^{(a_1)} v_+$$

is a nonzero vector of the weight space $V(\Lambda_0)_{w\Lambda_0}$, which does not depend on the choice of reduced decomposition and satisfies $(v_w, v_w) = 1$.

For every $j \in J$ and $m \in \mathbb{Z}_{\geq 0}$, we define the divided power monomial

$$(1.2) \quad f(m, j) := f_j^{(m)} \cdots f_1^{(m)} f_0^{(2m)} f_1^{(m)} \cdots f_j^{(m)} f_{j+1}^{(2m)} \cdots f_{\ell-1}^{(2m)} f_\ell^{(m)} \in U(\mathfrak{g})$$

(with $f(0, j)$ interpreted as 1). Recalling the null-root $\delta := 2\alpha_0 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$, note that the monomial $f(m, j)$ has total weight $-m\delta$. More generally, given $d, n \in \mathbb{Z}_{>0}$, a composition $\mu = (\mu_1, \dots, \mu_n)$ of d into n nonnegative parts and a tuple $\mathbf{j} = (j_1, \dots, j_n) \in J^n$ we define

$$(1.3) \quad f(\mu, \mathbf{j}) := f(\mu_n, j_n) \cdots f(\mu_1, j_1) \in U(\mathfrak{g}).$$

For a composition $\omega_d := (1, \dots, 1)$ of d , we also define

$$(1.4) \quad f(\omega_d) := \sum_{\mathbf{j} \in J^d} f(\omega_d, \mathbf{j}) = \sum_{j_1, \dots, j_d \in J} f(1, j_d) \cdots f(1, j_1).$$

If the weight space $V(\Lambda_0)_{\Lambda_0 - \theta}$ for $\theta \in Q_+$ is nonzero, then we can write $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some w in the Weyl group W and a unique $d \in \mathbb{Z}_{\geq 0}$. We then say that θ is *RoCK* if $(\theta | \alpha_0^\vee) \geq 2d$ and $(\theta | \alpha_i^\vee) \geq d - 1$ for $i = 1, \dots, \ell$. This is equivalent to the cyclotomic quiver Hecke algebra $R_\theta^{\Lambda_0}$ being a *RoCK block*, as defined in [Kleshchev and Livesey 2022, Section 4.1]. To each composition $\mu = (\mu_1, \dots, \mu_n)$ and tuple $\mathbf{j} = (j_1, \dots, j_n) \in J^n$, we can define the corresponding divided power idempotents $e(\mu, \mathbf{j}) \in R_\theta^{\Lambda_0}$. The idempotents $e(\omega_d, \mathbf{j})$ are then distinct and orthogonal to each other for distinct $\mathbf{j} \in J^d$, so we also have the idempotent $e(\omega_d) := \sum_{\mathbf{j} \in J^d} e(\omega_d, \mathbf{j})$.

Main Theorem. *Let weight $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ be RoCK, $n \in \mathbb{Z}_{>0}$, $\mu = (\mu_1, \dots, \mu_n)$ be a composition of d with n parts, and $\mathbf{j} = (j_1, \dots, j_n) \in J^n$. Set*

$$(1.5) \quad |\mu, \mathbf{j}|_{\ell-1} := \sum_{\substack{1 \leq r \leq n \\ j_r = \ell-1}} \mu_r.$$

Then

$$\begin{aligned} \dim e(\mu, \mathbf{j}) R_\theta^{\Lambda_0} e(\omega_d) &= (f(\mu, \mathbf{j}) v_w, f(\omega_d) v_w) \\ &= \binom{d}{\mu_1 \cdots \mu_n} 4^{d - |\mu, \mathbf{j}|_{\ell-1}} 3^{|\mu, \mathbf{j}|_{\ell-1}}. \end{aligned}$$

The first equality in the theorem is a known consequence of the Kang–Kashiwara–Oh categorification, so the main work is to prove of the second equality. This is proved in Theorem 5.6.

The dimension formula for $e(\mu, j)R_\theta^{\Lambda_0}e(\omega_d)$ obtained in the Main Theorem is a shadow of the fact that the Gelfand–Graev idempotent truncation of a RoCK block $R_\theta^{\Lambda_0}$ is isomorphic to a generalized Schur algebra corresponding to a certain Brauer tree algebra. In fact, our Main Theorem is a key step needed for the proof of this isomorphism in [Kleshchev 2024].

2. Shapovalov forms

2.1. Lie theoretic notation. Let \mathfrak{g} be the Kac–Moody Lie algebra of type $A_{2\ell}^{(2)}$ (over \mathbb{C}); see [Kac 1990, Chapter 4]. We set

$$p := 2\ell + 1.$$

The Dynkin diagram of \mathfrak{g} has vertices labeled by $I = \{0, 1, \dots, \ell\}$:

$$\begin{array}{c} 0 \quad 1 \quad 2 \quad \dots \quad \ell-2 \quad \ell-1 \quad \ell \\ \circ \leftarrow \circ \text{---} \circ \dots \text{---} \circ \text{---} \circ \leftarrow \circ \end{array} \quad \text{if } \ell \geq 2 \quad \text{and} \quad \begin{array}{c} 0 \quad 1 \\ \circ \leftarrow \circ \end{array} \quad \text{if } \ell = 1.$$

We have the standard Chevalley generators $\{e_i, f_i, h_i \mid i \in I\}$ of \mathfrak{g} and the Chevalley anti-involution

$$\sigma : \mathfrak{g} \rightarrow \mathfrak{g}, \quad e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad h_i \mapsto h_i.$$

We have the *weight lattice* P of \mathfrak{g} , the subset of all *dominant integral weights* $P_+ \subset P$, and the set $\{\alpha_i \mid i \in I\} \subset P$ of the *simple roots* of \mathfrak{g} . We denote by Q the sublattice of P generated by the simple roots and set

$$Q_+ := \left\{ \sum_{i \in I} m_i \alpha_i \mid m_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I \right\} \subset Q.$$

For $\theta = \sum_{i \in I} m_i \alpha_i \in Q_+$, its *height* is

$$\text{ht}(\theta) := \sum_{i \in I} m_i.$$

For $\theta \in Q_+$ of height n , we set

$$I^\theta := \{(i_1, \dots, i_n) \in I^n \mid \alpha_{i_1} + \dots + \alpha_{i_n} = \theta\}.$$

We have the null-root

$$\delta = \sum_{i=0}^{\ell-1} 2\alpha_i + \alpha_\ell$$

with $\text{ht}(\delta) = 2\ell + 1 = p$.

We denote by $(\cdot | \cdot)$ a normalized invariant form on P whose Gram matrix with respect to the linearly independent set $\alpha_0, \alpha_1, \dots, \alpha_\ell$ is

$$\begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 4 & -2 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 4 & -2 & 0 \\ 0 & 0 & 0 & \cdots & -2 & 4 & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & 8 \end{pmatrix} \quad \text{if } \ell \geq 2 \quad \text{and} \quad \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \quad \text{if } \ell = 1.$$

Recall from [Kac 1990, §§3.7, 3.13] the (affine) *Weyl group* W generated by the fundamental reflections $\{r_i \mid i \in I\}$ as a Coxeter group. The form $(\cdot | \cdot)$ is W -invariant; see [Kac 1990, Proposition 3.9].

For $\Lambda \in P_+$, we denote by $V(\Lambda)$ the *irreducible integrable \mathfrak{g} -module* of highest weight Λ . Fix a nonzero highest weight vector $v_+ \in V(\Lambda)_\Lambda$. The *Shapovalov form* is the unique symmetric bilinear form (\cdot, \cdot) on $V(\Lambda)$ such that $(v_+, v_+) = 1$ and

$$(2.1) \quad (xv, w) = (v, \sigma(x)w) \quad (x \in \mathfrak{g}, v, w \in V(\Lambda)).$$

Lemma 2.2. *Let $\Lambda \in P_+$ and $w \in W$ with a reduced decomposition $w = r_{i_t} \cdots r_{i_1}$. Then*

- (i) $a_k := (r_{i_{k-1}} \cdots r_{i_1} \Lambda \mid \alpha_{i_k}^\vee) \in \mathbb{Z}_{\geq 0}$ for all $k = 1, \dots, t$;
- (ii) $v_w := f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$ is a nonzero vector of the weight space $V(\Lambda)_{w\Lambda}$, which does not depend on the choice of a reduced decomposition of w ;
- (iii) $(v_w, v_w) = 1$.

Proof. This is well-known. For example, for all the claims, except the independence of a reduced decomposition, one can consult [Kleshchev and Livesey 2022, Lemma 2.4.11]. One way to see the independence of a reduced decomposition is to first note that (iii) determines v_w uniquely up to a sign, and if $f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$ is such vector corresponding to another reduced decomposition, we cannot have $v_w = -f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$, since, by the Kang–Kashiwara–Oh categorification [Kang et al. 2013], the vectors $f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$ and $f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$ correspond to modules over a certain algebra when $V(\Lambda)$ is identified with a Grothendieck group. \square

2.2. Quantized enveloping algebra. Let q be an indeterminate and consider the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials and the field $\mathbb{C}(q)$ of rational functions. For $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we have the following elements of $\mathbb{Z}[q, q^{-1}]$:

$$(2.3) \quad q_i := q^{(\alpha_i \mid \alpha_i)/2}, \quad [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := [1]_i [2]_i \cdots [n]_i.$$

Let $U_q(\mathfrak{g})$ be the *quantized enveloping algebra of type $A_{2\ell}^{(2)}$* , i.e., the associative unital $\mathbb{C}(q)$ -algebra with generators $\{E_i, F_i, K_i^{\pm 1} \mid i \in I\}$ subject only to the quantum Serre relations

$$\begin{aligned} T_i E_j T_i^{-1} &= q^{(\alpha_i|\alpha_j)} E_j, \\ T_i F_j T_i^{-1} &= q^{-(\alpha_i|\alpha_j)} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{T_i - T_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{i,j}} (-1)^r E_i^{(r)} E_j E_i^{(1-a_{i,j}-r)} &= 0 \quad (i \neq j), \\ \sum_{r=0}^{1-a_{i,j}} (-1)^r F_i^{(r)} F_j F_i^{(1-a_{i,j}-r)} &= 0 \quad (i \neq j), \end{aligned}$$

where we have set $T_i := K_i^{(\alpha_i|\alpha_i)/2}$, $E_i^{(r)} := E_i/[r]_i!$, $F_i^{(r)} := F_i/[r]_i!$.

There is a $\mathbb{C}(q)$ -linear anti-involution $\sigma_q : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ with

$$\sigma_q : E_i \mapsto q_i F_i T_i^{-1} = q_i^{-1} T_i^{-1} F_i, \quad F_i \mapsto q_i^{-1} T_i E_i = q_i E_i T_i, \quad T_i \mapsto T_i.$$

For $\Lambda \in P_+$, we denote by $V_q(\Lambda)$ the *irreducible integrable module* for $U_q(\mathfrak{g})$ of highest weight Λ . We fix a nonzero highest weight vector $v_{+,q} \in V_q(\Lambda)_\Lambda$, so $E_i v_{+,q} = 0$ and $T_i v_{+,q} = q^{(\alpha_i|\Lambda)} v_{+,q}$ for all $i \in I$. There is a unique symmetric bilinear form $(\cdot, \cdot)_q$ on $V_q(\Lambda)$ such that $(v_{+,q}, v_{+,q})_q = 1$ and

$$(2.4) \quad (xv, w)_q = (v, \sigma_q(x)w)_q \quad (x \in U_q(\mathfrak{g}), v, w \in V_q(\Lambda));$$

see [Kashiwara et al. 1996, Appendix D]. We refer to $(\cdot, \cdot)_q$ as the (quantum) Shapovalov form.

2.3. Putting q to 1. Let V_q be a $U_q(\mathfrak{g})$ -module which decomposes as a direct sum of finite-dimensional integral weight spaces $V_q = \bigoplus_{\mu \in P} V_{q,\mu}$. Suppose also that there exists a full rank $\mathbb{Z}[q, q^{-1}]$ -sublattice $V_{q,\mu, \mathbb{Z}[q, q^{-1}]} \subset V_{q,\mu}$ for every μ such that $V_{q, \mathbb{Z}[q, q^{-1}]} := \bigoplus_{\mu \in P} V_{q,\mu, \mathbb{Z}[q, q^{-1}]}$ is stable under all E_i and F_i . Considering \mathbb{C} as a $\mathbb{Z}[q, q^{-1}]$ -module with q acting as 1, we change scalars to get the complex vector space $V_q|_{q=1} := \mathbb{C} \otimes_{\mathbb{Z}[q, q^{-1}]} V_{q, \mathbb{Z}[q, q^{-1}]}$ with linear operators $e_i := 1 \otimes E_i$, $f_i := 1 \otimes F_i$ and $h_i := 1 \otimes ((T_i - T_i^{-1})/(q_i - q_i^{-1}))$ for all i . These linear operators are easily checked to satisfy the Serre relations for \mathfrak{g} ; see [Jantzen 1996, 5.13; Lusztig 1988, Theorem 4.12 and §4.14]. Thus $V_q|_{q=1}$ becomes a \mathfrak{g} -module. (This module depends on the choice of the $\mathbb{Z}[q, q^{-1}]$ -sublattice).

Given a symmetric bilinear form $(\cdot, \cdot)_q$ on V_q which satisfies (2.4) and is $\mathbb{Z}[q, q^{-1}]$ -valued on $V_{q, \mathbb{Z}[q, q^{-1}]}$, we obtain, extending scalars, a symmetric bilinear form $(\cdot, \cdot)_{q=1}$ on $V_q|_{q=1}$ satisfying (2.1).

The above constructions can be applied to the irreducible modules $V_q(\Lambda)$ with highest weight $\Lambda \in P_+$ by considering the $\mathbb{Z}[q, q^{-1}]$ -sublattice spanned by all vectors of the form $F_{i_1} \cdots F_{i_r} v_+$. Taking into account that the formal characters of $V_q(\Lambda)$ and $V(\Lambda)$ agree by [Lusztig 1988, Theorem 4.12 and §4.14], this shows that there is a unique isomorphism $V(\Lambda) \xrightarrow{\sim} V_q(\Lambda)|_{q=1}$ mapping v_+ onto $1 \otimes v_{+,q}$. Identifying $V(\Lambda)$ and $V_q(\Lambda)|_{q=1}$ under this isomorphism, the quantum Shapovalov form $(\cdot, \cdot)_q$ yields the usual Shapovalov form (\cdot, \cdot) , that is, $(\cdot, \cdot)_q|_{q=1} = (\cdot, \cdot)$. In particular:

Lemma 2.5. *Let $(\cdot, \cdot)_q$ be the Shapovalov form on $V_q(\Lambda)$ and (\cdot, \cdot) be the Shapovalov form on $V(\Lambda)$. Then for all $i_1, \dots, i_n, j_1, \dots, j_n \in I$, we have*

$$(F_{i_1} \cdots F_{i_n} v_{+,q}, F_{j_1} \cdots F_{j_n} v_{+,q})_q \in \mathbb{Z}[q, q^{-1}]$$

and

$$(f_{i_1} \cdots f_{i_n} v_+, f_{j_1} \cdots f_{j_n} v_+) = (F_{i_1} \cdots F_{i_n} v_{+,q}, F_{j_1} \cdots F_{j_n} v_{+,q})_q|_{q=1}.$$

Another example of passing from V_q to $V_q|_{q=1}$ will be considered in Section 4.

3. Combinatorics

Recall that we have set $p = 2\ell + 1$.

3.1. Partitions, multipartitions, tableaux. We denote by \mathcal{P} the set of all partitions and by $\mathcal{P}(n)$ the set of all partitions of $n \in \mathbb{Z}_{\geq 0}$. For $\lambda \in \mathcal{P}(n)$ we write $|\lambda| = n$. Collecting equal parts of $\lambda \in \mathcal{P}$, we can write it in the form

$$(3.1) \quad \lambda = (l_1^{m_1}, \dots, l_k^{m_k}) \quad \text{with } l_1 > \cdots > l_k > 0 \text{ and } m_1, \dots, m_k \geq 1.$$

We then define

$$(3.2) \quad \|\lambda\|_q := \prod_{r \text{ with } p|l_r} \prod_{s=1}^{m_r} (1 - (-q^2)^s),$$

and

$$(3.3) \quad h(\lambda) := \sum_{r=1}^k m_r, \quad h_p(\lambda) := \sum_{r \text{ with } p|l_r} m_r.$$

In other words, $h(\lambda)$ is the number of (positive) parts of λ and $h_p(\lambda)$ is the number of (positive) parts of λ divisible by p . If $m_r > 1$ implies $p \mid l_r$ for all $1 \leq r \leq k$ then λ is called p -strict. Note that 0-strict also makes sense and means simply *strict*, i.e., all parts are distinct. We denote by $\mathcal{P}_p(n)$ the set of all p -strict partitions of n , and let $\mathcal{P}_p := \bigsqcup_{n \geq 0} \mathcal{P}_p(n)$. We use the similar notation $\mathcal{P}_0(n)$ and \mathcal{P}_0 for strict partitions.

For $\lambda \in \mathcal{P}_0$ we define its *parity*:

$$(3.4) \quad p_\lambda := \begin{cases} 1 & \text{if } \lambda \text{ has odd number of positive even parts,} \\ 0 & \text{otherwise.} \end{cases}$$

Let λ be a p -strict partition. As usual, we identify λ with its *Young diagram*

$$\lambda = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r\}.$$

We refer to the element $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ as the *node* in row r and column s . We define a preorder ‘ \leq ’ on the nodes via $(r, s) \leq (r', s')$ if and only if $s \leq s'$. For partitions (equivalently Young diagrams) $\alpha \subseteq \lambda$, we define

$$(3.5) \quad q(\lambda/\alpha) := |\{r \in \mathbb{Z}_{>0} \mid \lambda \setminus \alpha \text{ has a node in column } r \text{ but not in column } r+1\}|.$$

We label the nodes with the elements of the set I as follows: the labeling follows the repeating pattern $0, 1, \dots, \ell-1, \ell, \ell-1, \dots, 1, 0$, starting from the first column and going to the right; see Example 3.8 below. If a node $A \in \lambda$ is labeled with i , we say that A has *residue* i and write $\text{Res } A = i$. Recalling α_i ’s and Q_+ from Section 2.1, define the *residue content* of λ to be

$$\text{cont}(\lambda) := \sum_{A \in \lambda} \alpha_{\text{Res } A} \in Q_+.$$

We always write $\text{cont}(\lambda) = \sum_{i \in I} c_i^\lambda \alpha_i$, and

$$(3.6) \quad c_{\neq 0}^\lambda := c_1^\lambda + \dots + c_\ell^\lambda = |\lambda| - c_0^\lambda.$$

Following [Morris 1965; Leclerc and Thibon 1997], we can associate to every $\lambda \in \mathcal{P}_p$ its \bar{p} -*core*

$$\text{core}(\lambda) \in \mathcal{P}_p$$

obtained from λ by removing certain nodes. Clearly, from the definition, the number of nodes removed to go from λ to $\text{core}(\lambda)$ is divisible by p , so the \bar{p} -*weight* of λ

$$\text{wt}(\lambda) := \frac{|\lambda| - |\text{core}(\lambda)|}{p}$$

is a nonnegative integer. A partition $\rho \in \mathcal{P}_p$ is called a \bar{p} -*core* if $\text{core}(\rho) = \rho$. By [Kleshchev and Livesey 2022, Lemma 3.1.39], we have:

Lemma 3.7. *A p -strict partition λ is a \bar{p} -core if and only if*

$$\text{cont}(\lambda) = \Lambda_0 - w \Lambda_0 \quad \text{for some } w \in W.$$

Example 3.8. Let $\ell = 2$, so $p = 5$. The partition $\lambda = (16, 11, 10, 10, 9, 4, 1)$ is 5-strict. The residues of the nodes are

0	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0
0	1	2	1	0	0	1	2	1	0	0					
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1							
0	1	2	1												
0															

The $\bar{5}$ -core of λ is (1).

The partition $\lambda \in \mathcal{P}_p$ is determined by its \bar{p} -core $\text{core}(\lambda)$ and \bar{p} -quotient

$$(3.9) \quad \text{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)}),$$

which is an I -multipartition of d , in other words, $\lambda^{(0)}, \dots, \lambda^{(\ell)}$ are partitions and $|\lambda^{(0)}| + |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = d$. We denote the set of all such multipartitions by $\mathcal{P}^I(d)$, and set

$$\mathcal{P}^I := \bigsqcup_{d \geq 0} \mathcal{P}^I(d).$$

We refer the reader to [Morris and Yaseen 1986, p. 27] and [Kleshchev and Livesey 2022, §2.3b] for details on this. For a \bar{p} -core partition ρ , we define

$$\mathcal{P}_p(\rho, d) := \{\lambda \in \mathcal{P}_p \mid \text{core}(\lambda) = \rho \text{ and } \text{wt}(\lambda) = d\}.$$

The map

$$(3.10) \quad \mathcal{P}_p(\rho, d) \rightarrow \mathcal{P}^I(d), \quad \lambda \mapsto \text{cont}(\lambda),$$

is a bijection; see [Morris and Yaseen 1986, Theorem 2]. The condition $\lambda \in \mathcal{P}_p(\rho, d)$ is equivalent to $\text{cont}(\lambda) = \text{cont}(\rho) + d\delta$; see [Kleshchev and Livesey 2022, Lemma 2.3.9].

A multipartition $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathcal{P}^I$ is *strict* if its 0-th component $\lambda^{(0)}$ is a strict partition (and $\lambda^{(1)}, \dots, \lambda^{(\ell)}$ are arbitrary partitions). For $d \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{P}_0^I(d)$ the set of all strict multipartitions of d . Note that $\text{quot}(\lambda) \in \mathcal{P}_0^I(d)$ if and only if $\lambda \in \mathcal{P}_0(\rho, d)$, so the bijection (3.10) restricts to the bijection

$$(3.11) \quad \mathcal{P}_0(\rho, d) \rightarrow \mathcal{P}_0^I(d), \quad \lambda \mapsto \text{cont}(\lambda).$$

We identify a multipartition $\underline{\lambda}$ with its Young diagram

$$\underline{\lambda} = \{(i, r, s) \in I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r^{(i)}\}.$$

We refer to the element $(i, r, s) \in I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ as the *node* in row r and column s of component i . For each i , we consider (the Young diagram of) the partition $\lambda^{(i)}$ as a subset $\lambda^{(i)} \subseteq \underline{\lambda}$ consisting of the nodes of $\underline{\lambda}$ in its i -th component.

Let $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$. A *composition* of d with n parts is a tuple $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_1, \dots, \mu_n \in \mathbb{Z}_{\geq 0}$ such that $\mu_1 + \dots + \mu_n = d$. We denote by $\Lambda(n, d)$ the set of all compositions of d with n parts. We will need a special composition of d with d parts:

$$(3.12) \quad \omega_d := (1^d) = (1, \dots, 1).$$

Recall that J denotes $I \setminus \{\ell\}$. A *colored composition* of d with n parts is a pair (μ, \mathbf{j}) where $\mu = (\mu_1, \dots, \mu_n)$ is a composition of d with n parts and $\mathbf{j} = (j_1, \dots, j_n) \in J^n$. We denote by $\Lambda^{\text{col}}(n, d)$ the set of all colored compositions of d with n parts.

Let $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathcal{P}^I(d)$ and $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$. A *colored tableau of shape $\underline{\lambda}$ and type (μ, \mathbf{j})* is a function $T : \underline{\lambda} \rightarrow \mathbb{Z}_{>0}$ such that

- (1) $T(i, r, s) \leq T(i, r, s + 1)$ and $T(i, r, s) \leq T(i, r + 1, s)$ whenever these make sense;
- (2) for all $k = 1, \dots, n$, we have $|T^{-1}(\{k\})| = \mu_k$ and $T^{-1}(\{k\}) \subseteq \lambda^{(j_k)} \sqcup \lambda^{(j_{k+1})}$;
- (3) for all $k = 1, \dots, n$, no two nodes of $T^{-1}(\{k\}) \cap \lambda^{(j_k)}$ are in the same column, and no two nodes of $T^{-1}(\{k\}) \cap \lambda^{(j_{k+1})}$ are in the same row.

Denote by $\text{CT}(\underline{\lambda}; \mu, \mathbf{j})$ the set of all colored tableaux of shape $\underline{\lambda}$ and type (μ, \mathbf{j}) . For $T \in \text{CT}(\underline{\lambda}; \mu, \mathbf{j})$ and $1 \leq k \leq n$ we denote by $q_k(T)$ the number of positive integers r such that $T^{-1}(k) \cap \lambda^{(0)}$ contains a node in column r but not in column $r + 1$. We then set $q(T) = q_1(T) + \dots + q_n(T)$ and define

$$(3.13) \quad K(\underline{\lambda}; \mu, \mathbf{j}) := \sum_{T \in \text{CT}(\underline{\lambda}; \mu, \mathbf{j})} 2^{q(T)}.$$

3.2. Addable and removable nodes. Let λ be a p -strict partition and $i \in I$. A node $A \in \lambda$ is called *i -removable* (for λ) if one of the following holds:

- (R1) $\text{Res } A = i$ and $\lambda_A := \lambda \setminus \{A\}$ is again a p -strict partition; such A 's are also called *properly i -removable*.
- (R2) The node B immediately to the right of A belongs to λ , $\text{Res } A = \text{Res } B = i$, and both $\lambda_B = \lambda \setminus \{B\}$ and $\lambda_{A,B} := \lambda \setminus \{A, B\}$ are p -strict partitions.

A node $B \notin \lambda$ is called *i -addable* (for λ) if one of the following holds:

- (A1) $\text{Res } B = i$ and $\lambda^B := \lambda \cup \{B\}$ is again a p -strict partition; such B 's are also called *properly i -addable*.
- (A2) The node A immediately to the left of B does not belong to λ , $\text{Res } A = \text{Res } B = i$, and both $\lambda^A = \lambda \cup \{A\}$ and $\lambda^{A,B} := \lambda \cup \{A, B\}$ are p -strict partitions.

We note that (R2) and (A2) above are only possible if $i = 0$. For $i \in I$, we denote by $\text{Ad}_i(\lambda)$ (resp. $\text{Re}_i(\lambda)$) the set of all i -removable (resp. i -addable) nodes for λ . We also denote by $\text{PAD}_i(\lambda)$ (resp. $\text{PRE}_i(\lambda)$) the set of all properly i -removable (resp. properly i -addable) nodes for λ .

Let $\lambda \in \mathcal{P}_p$ be written in the form (3.1). Suppose $A \in \text{PRE}_i(\lambda)$. Then there is $1 \leq r \leq k$ such that

$$A = (m_1 + \cdots + m_r, l_r).$$

Recalling the preorder ‘ \leq ’ on the nodes defined above, we set

$$\eta_A(\lambda) := \#\{\mathbf{C} \in \text{Re}_i(\lambda) \mid \mathbf{C} > A\} - \#\{\mathbf{C} \in \text{Ad}_i(\lambda) \mid \mathbf{C} > A\},$$

$$\zeta_A(\lambda) := \begin{cases} (1 - (-q^2)^{m_r}) & \text{if } p \mid l_r, \\ 1 & \text{otherwise,} \end{cases}$$

$$d_A(\lambda) := q_i^{\eta_A(\lambda)} \zeta_A(\lambda).$$

Note that

$$(3.14) \quad d_A(\lambda)|_{q=1} = \begin{cases} 0 & \text{if } p \mid l_r \text{ and } m_r \text{ is even,} \\ 2 & \text{if } p \mid l_r \text{ and } m_r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Suppose $B \in \text{PAD}_i(\lambda)$. Then there is r such that $1 \leq r \leq k + 1$ and

$$B = (m_1 + \cdots + m_{r-1} + 1, l_r + 1),$$

where we interpret l_{k+1} as 0. We define

$$(3.15) \quad \eta^B(\lambda) := \#\{\mathbf{C} \in \text{Ad}_i(\lambda) \mid \mathbf{C} < B\} - \#\{\mathbf{C} \in \text{Re}_i(\lambda) \mid \mathbf{C} < B\},$$

$$(3.16) \quad \zeta^B(\lambda) := \begin{cases} (1 - (-q^2)^{m_r}) & \text{if } r \leq k \text{ and } p \mid l_r, \\ 1 & \text{otherwise,} \end{cases}$$

$$(3.17) \quad d^B(\lambda) := q_i^{\eta^B(\lambda)} \zeta^B(\lambda).$$

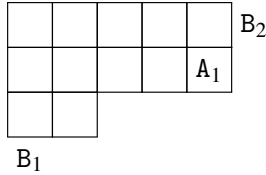
Note that

$$(3.18) \quad d^B(\lambda)|_{q=1} = \begin{cases} 0 & \text{if } r \leq k, p \mid l_r \text{ and } m_r \text{ is even,} \\ 2 & \text{if } r \leq k, p \mid l_r \text{ and } m_r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Example 3.19. Let $\ell = 2$ so $p = 5$. The partition $\lambda = (5, 5, 2)$ is 5-strict, and the residues of its boxes are labeled on the diagram

0	1	2	1	0
0	1	2	1	0
0	1			

The only 0-removable node is marked as A_1 , and the 0-addable nodes are marked as B_1, B_2 :



We have $d^{B_1}(\lambda) = 1$ and $d^{B_2}(\lambda) = (1 - q^4)$. On the other hand for the partition $\mu = (5)$ and the node $B = (1, 6)$, we have $d^B(\mu) = (1 + q^2)$.

3.3. Symmetric functions. We denote by Λ the algebra of symmetric functions in the variables x_1, x_2, \dots over \mathbb{C} with the basis

$$\{s_\lambda \mid \lambda \in \mathcal{P}\}$$

of *Schur functions* and the inner product $\langle \cdot, \cdot \rangle$ such that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$; see [Macdonald 1995]. We also have the *monomial symmetric functions* m_λ , the *elementary symmetric functions* $e_r = s_{(1^r)}$ and the *complete symmetric functions* $h_r = s_{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$. *Pieri's rules* [Macdonald 1995, (5.16), (5.17)] say that

$$(3.20) \quad s_\lambda h_r = \sum_{\mu} s_\mu \quad \text{and} \quad s_\lambda e_r = \sum_{\nu} s_\nu,$$

where the first sum is over all partitions μ obtained by adding r nodes to λ with no two nodes added in the same column, and the second sum is over all partitions ν obtained by adding r nodes to λ with no two nodes added in the same row.

Suppose that for $s_1, \dots, s_t \in \mathbb{Z}_{\geq 0}$, we have that $f_{s_u} = e_{s_u}$ or h_{s_u} . Under the characteristic map [Macdonald 1995, I.7], the symmetric function $f_{s_1} \cdots f_{s_t}$ corresponds to an induced representation of the symmetric group $\mathfrak{S}_{s_1 + \dots + s_t}$ of dimension given by the multinomial coefficient $\binom{s_1 + \dots + s_t}{s_1 \cdots s_t}$, while the symmetric function $s_{(1)^r}^r$ with $r \in \mathbb{Z}_{\geq 0}$ corresponds to the regular representation of the symmetric group \mathfrak{S}_r . Hence,

$$(3.21) \quad (f_{s_1} \cdots f_{s_t}, s_{(1)^r}^r) = \begin{cases} \binom{r}{s_1 \cdots s_t} & \text{if } s_1 + \dots + s_t = r, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting by $p_r \in \Lambda$ the r -th power sum symmetric function, let Ω be the (unital) subalgebra of Λ generated by p_1, p_3, p_5, \dots . Then Ω has bases

$$\{P_\lambda \mid \lambda \in \mathcal{P}_0\} \quad \text{and} \quad \{Q_\lambda \mid \lambda \in \mathcal{P}_0\},$$

where the elements P_λ and Q_λ are *Schur's P- and Q-symmetric functions*; see [Stembridge 1989, §6]. We have $P_\lambda = 2^{-h(\lambda)} Q_\lambda$ for all $\lambda \in \mathcal{P}_0$. Let $[\cdot, \cdot]$ be an inner product on Ω such that $[P_\lambda, Q_\lambda] = \delta_{\lambda, \mu}$ for all $\lambda, \mu \in \mathcal{P}_0$; see [Stembridge 1989, §§5, 6].

We also have the symmetric functions

$$q_r = 2P_{(r)} \in \Omega \quad (r \in \mathbb{Z}_{>0})$$

(and $q_0 := 1$); see [Stembridge 1989, (5.3), (6.6)]. We have the analogue of the Pieri's rule (which goes back to [Morris 1964] but can be most easily seen from [Stembridge 1989, Theorem 8.3]):

$$(3.22) \quad P_\lambda q_r = \sum_{\mu} 2^{q(\mu/\lambda)} P_\mu,$$

where the sum is over all strict partitions μ obtained by adding r nodes to λ with no two nodes added in the same column and $q(\mu/\lambda)$ is as in (3.5).

By [Stembridge 1989, Proposition 5.6(b)], the inner product $[q_{s_1} \dots q_{s_r}, q_1^r]$ is the coefficient of $m_{(s_1, \dots, s_r)}$ in $q_1^r = (2x_1 + 2x_2 + \dots)^r$ (we may assume that $s_1 \geq \dots \geq s_r$ so (s_1, \dots, s_r) is a partition), whence

$$(3.23) \quad [q_{s_1} \dots q_{s_r}, q_1^r] = \begin{cases} 2^r \binom{r}{s_1, \dots, s_r} & \text{if } s_1 + \dots + s_r = r, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the algebra

$$\text{Sym}^I := \Omega \otimes \Lambda^{(1)} \otimes \dots \otimes \Lambda^{(\ell)},$$

where each algebra $\Lambda^{(i)}$ is just a copy of Λ . This has bases

$$(3.24) \quad \{\pi_{\underline{\lambda}} := P_{\lambda^{(0)}} \otimes s_{\lambda^{(1)}} \otimes \dots \otimes s_{\lambda^{(\ell)}} \mid \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \mathcal{P}_0^I\}$$

and

$$(3.25) \quad \{\kappa_{\underline{\lambda}} := Q_{\lambda^{(0)}} \otimes s_{\lambda^{(1)}} \otimes \dots \otimes s_{\lambda^{(\ell)}} \mid \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \mathcal{P}_0^I\},$$

which are dual with respect to the inner product $(\cdot, \cdot)_{\text{Sym}}$ defined as

$$(3.26) \quad (f_0 \otimes f_1 \otimes \dots \otimes f_\ell, g_0 \otimes g_1 \otimes \dots \otimes g_\ell)_{\text{Sym}} := [f_0, g_0] \langle f_1, g_1 \rangle \dots \langle f_\ell, g_\ell \rangle.$$

Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$. Recalling (3.13), set

$$(3.27) \quad \Pi_{\mu, \mathbf{j}} := \sum_{\underline{\lambda} \in \mathcal{P}_0^I(d)} K(\underline{\lambda}; \mu, \mathbf{j}) \pi_{\underline{\lambda}}.$$

Example 3.28. Let $n = 1$ so (μ, \mathbf{j}) is of the form $((d), j) \in \Lambda^{\text{col}}(1, d)$. Then

$$(3.29) \quad \Pi_{(d), j} := \begin{cases} 1 \otimes e_d \otimes 1^{\otimes \ell-1} + 2 \sum_{k=1}^d P_{(k)} \otimes e_{d-k} \otimes 1^{\otimes \ell-1} & \text{if } j = 0, \\ \sum_{k=0}^d 1^{\otimes j} \otimes h_k \otimes e_{d-k} \otimes 1^{\otimes \ell-1-j} & \text{if } 1 \leq j < \ell. \end{cases}$$

Lemma 3.30. Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$. Suppose $n \geq 2$ and set $v := (\mu_1, \dots, \mu_{n-1})$, $\mathbf{k} := (j_1, \dots, j_{n-1})$, $m := \mu_n$, $j := j_n$. Then

$$\Pi(\mu, \mathbf{j}) = \Pi(v, \mathbf{k}) \Pi((m), j).$$

Proof. Suppose $j \neq 0$. Using (3.29), we see that $\Pi(v, \mathbf{k})\Pi((m), j)$ equals

$$\begin{aligned} & \left(\sum_{\underline{\alpha} \in \mathcal{P}_0^1(d-m)} K(\underline{\alpha}; v, \mathbf{k}) \pi_{\underline{\alpha}} \right) \left(\sum_{k=0}^m 1^{\otimes j} \otimes \mathbf{h}_k \otimes \mathbf{e}_{m-k} \otimes 1^{\otimes \ell-1-j} \right) \\ &= \sum_{\substack{\underline{\alpha} \in \mathcal{P}_0^1(d-m) \\ 0 \leq k \leq m}} K(\underline{\alpha}; v, \mathbf{k}) P_{\alpha^{(0)}} \otimes s_{\alpha^{(1)}} \otimes \cdots \otimes s_{\alpha^{(j)}} \mathbf{h}_k \otimes s_{\alpha^{(j+1)}} \mathbf{e}_{m-k} \otimes \cdots \otimes s_{\alpha^{(\ell)}} \\ &= \sum_{\underline{\lambda} \in \mathcal{P}_0^1(d)} K(\underline{\lambda}; \mu, \mathbf{j}) P_{\lambda^{(0)}} \otimes s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(j)}} \otimes s_{\lambda^{(j+1)}} \otimes \cdots \otimes s_{\lambda^{(\ell)}}, \end{aligned}$$

where the last equality follows from Pieri’s rules (3.20).

Suppose now that $j = 0$. Using (3.29), we see that $\Pi(v, \mathbf{k})\Pi((m), 0)$ equals

$$\begin{aligned} & \left(\sum_{\underline{\alpha} \in \mathcal{P}_0^1(d-m)} K(\underline{\alpha}; v, \mathbf{k}) \pi_{\underline{\alpha}} \right) \left(\sum_{k=0}^d \mathbf{q}_k \otimes \mathbf{e}_{d-k} \otimes 1^{\otimes \ell-1} \right) \\ &= \sum_{\substack{\underline{\alpha} \in \mathcal{P}_0^1(d-m) \\ 0 \leq k \leq m}} K(\underline{\alpha}; v, \mathbf{k}) P_{\alpha^{(0)}} \mathbf{q}_k \otimes s_{\alpha^{(1)}} \mathbf{e}_{d-k} \otimes s_{\alpha^{(2)}} \otimes \cdots \otimes s_{\alpha^{(\ell)}} \\ &= \sum_{\underline{\lambda} \in \mathcal{P}_0^1(d)} K(\underline{\lambda}; \mu, \mathbf{j}) P_{\lambda^{(0)}} \otimes s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(\ell)}}, \end{aligned}$$

where the last equality follows from Pieri’s rules (3.22) and (3.20). □

Recalling (3.12), we let

$$(3.31) \quad \Pi_{\omega_d} := \sum_{j \in J^d} \Pi_{\omega_d, j}.$$

Corollary 3.32. *We have*

$$\Pi_{\omega_d} = \sum_{k_0+k_1+\cdots+k_{\ell}=d} 2^{k_1+\cdots+k_{\ell-1}} \binom{d}{k_0 k_1 \cdots k_{\ell}} \mathbf{q}_1^{k_0} \otimes s_{(1)}^{k_1} \otimes \cdots \otimes s_{(1)}^{k_{\ell}}.$$

Proof. We apply induction on d . In the base case $d = 1$, by (3.29) and using $\mathbf{q}_1 = 2P_{(1)}$ we have

$$\begin{aligned} \Pi_{\omega_1} &= \Pi_{(1),0} + \Pi_{(1),1} + \cdots + \Pi_{(1),\ell-1} \\ &= (1 \otimes s_{(1)} \otimes 1^{\otimes \ell-1} + 2P_{(1)} \otimes 1^{\otimes \ell}) + (1 \otimes s_{(1)} \otimes 1^{\otimes \ell-1} + 1 \otimes 1 \otimes s_{(1)} \otimes 1^{\otimes \ell-2}) \\ &\quad + \cdots + (1^{\otimes \ell-1} \otimes s_{(1)} \otimes 1 + 1^{\otimes \ell-1} \otimes 1 \otimes s_{(1)}) \\ &= \mathbf{q}_1 \otimes 1^{\otimes \ell} + 1^{\otimes \ell} \otimes s_{(1)} + 2 \sum_{i=1}^{\ell-1} 1^{\otimes i} \otimes s_{(1)} \otimes 1^{\otimes \ell-i}, \end{aligned}$$

as required.

For the inductive step, for $d > 1$, it follows from Lemma 3.30 that $\Pi_{\omega_d} = \Pi_{\omega_{d-1}} \Pi_{\omega_1}$. So, by the inductive assumption, we get

$$\begin{aligned} \Pi_{\omega_d} &= \Pi_{\omega_{d-1}} \Pi_{\omega_1} \\ &= \left(\sum_{m_0+m_1+\dots+m_{\ell}=d-1} 2^{m_1+\dots+m_{\ell-1}} \binom{d-1}{m_0 \ m_1 \ \dots \ m_{\ell}} \mathfrak{q}_1^{m_0} \otimes s_{(1)}^{m_1} \otimes \dots \otimes s_{(1)}^{m_{\ell}} \right) \\ &\quad \times \left(\sum_{n_0+n_1+\dots+n_{\ell}=1} 2^{n_1+\dots+n_{\ell-1}} \mathfrak{q}_1^{n_0} \otimes s_{(1)}^{n_1} \otimes \dots \otimes s_{(1)}^{n_{\ell}} \right) \\ &= \sum_{k_0+k_1+\dots+k_{\ell}=d} 2^{k_1+\dots+k_{\ell-1}} \binom{d}{k_0 \ k_1 \ \dots \ k_{\ell}} \mathfrak{q}_1^{k_0} \otimes s_{(1)}^{k_1} \otimes \dots \otimes s_{(1)}^{k_{\ell}} \end{aligned}$$

thanks to the identity $\binom{d}{k_0 \ k_1 \ \dots \ k_{\ell}} = \sum_{r \text{ with } k_r > 0} \binom{d-1}{k_0 \ \dots \ k_{r-1} \ \dots \ k_{r-1} \ k_{r+1} \ \dots \ k_{\ell}}$. \square

3.4. Another description of $\Pi_{\mu, j}$. Let $M_{n, I}$ denote the set of $n \times I$ -matrices with nonnegative integer entries,

$$M_{n, I} = \{(a_{r, i})_{1 \leq r \leq n, i \in I} \mid a_{r, i} \in \mathbb{Z}_{\geq 0}\}.$$

For $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$, we define the sets of matrices

$$\begin{aligned} M_{n, I}(\mu) &:= \left\{ (a_{r, i}) \in M_{n, I} \mid \sum_{i \in I} a_{r, i} = \mu_r \text{ for } r = 1, \dots, n \right\}, \\ M_{n, I}(\mathbf{j}) &:= \{(a_{r, i}) \in M_{n, I} \mid a_{r, i} = 0 \text{ if } i \neq j_r, j_r + 1 \text{ for } r = 1, \dots, n\}, \\ M(\mu, \mathbf{j}) &:= M_{n, I}(\mu) \cap M_{n, I}(\mathbf{j}). \end{aligned}$$

Let $A = (a_{r, i}) \in M(\mu, \mathbf{j})$. For $1 \leq r \leq n$ and $i \in I \setminus \{0\}$, we define

$$\psi_A(r, i) := \begin{cases} h_{a_{r, i}} & \text{if } i = j_r, \\ e_{a_{r, i}} & \text{if } i = j_r + 1, \\ 1 & \text{otherwise.} \end{cases}$$

We now set

$$\psi_A^{(0)} := \mathfrak{q}_{a_{1,0}} \cdots \mathfrak{q}_{a_{n,0}}, \quad \psi_A^{(i)} := \psi_A(1, i) \cdots \psi_A(n, i) \quad (\text{for } i \in I \setminus \{0\}),$$

and

$$\psi_A := \psi_A^{(0)} \otimes \psi_A^{(1)} \otimes \dots \otimes \psi_A^{(\ell)} \in \text{Sym}^I.$$

Example 3.33. Suppose $n = 1$ and so $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(1, d)$ is of the form $((d), j)$. The set $M((d), j)$ consists of all matrices of the form

$$\{A(j, k) := (0 \ \dots \ 0 \ k \ d - k \ 0 \ \dots \ 0) \mid 0 \leq k \leq d\}$$

with k in position j . Note that by definition we have

$$\begin{aligned} \psi_{A_{0,0}} &= \mathfrak{q}_0 \otimes \mathfrak{e}_d \otimes 1^{\otimes \ell-1} = 1 \otimes \mathfrak{e}_d \otimes 1^{\otimes \ell-1}, \\ \psi_{A_{0,k}} &= \mathfrak{q}_k \otimes \mathfrak{e}_{d-k} \otimes 1^{\otimes \ell-1} = 2P_{(k)} \otimes \mathfrak{e}_{d-k} \otimes 1^{\otimes \ell-1} \quad (1 \leq k \leq d), \\ \psi_{A_{j,k}} &= 1^{\otimes j} \otimes \mathfrak{h}_k \otimes \mathfrak{e}_{d-k} \otimes 1^{\otimes \ell-1-j} \quad (1 \leq j < \ell). \end{aligned}$$

In particular, comparing with (3.29), we deduce that $\Pi_{(d),j} = \sum_{A \in M((d),j)} \psi_A$.

Proposition 3.34. *Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$. Then*

$$\Pi_{\mu, \mathbf{j}} = \sum_{A \in M(\mu, \mathbf{j})} \psi_A.$$

Proof. We apply induction on n . For the base $n = 1$, see Example 3.33. Suppose $n \geq 2$ and set

$$\mathbf{v} := (\mu_1, \dots, \mu_{n-1}), \quad \mathbf{k} := (j_1, \dots, j_{n-1}), \quad m := \mu_n, \quad j := j_n.$$

Then $\Pi(\mu, \mathbf{j}) = \Pi(\mathbf{v}, \mathbf{k})\Pi((m), j)$ by Lemma 3.30. By the inductive assumption,

$$\Pi(\mathbf{v}, \mathbf{k}) = \sum_{B \in M(\mathbf{v}, \mathbf{k})} \psi_B \quad \text{and} \quad \Pi((m), j) = \sum_{C \in M((m), j)} \psi_C,$$

so it suffices to observe that

$$\left(\sum_{B \in M(\mathbf{v}, \mathbf{k})} \psi_B \right) \left(\sum_{C \in M((m), j)} \psi_C \right) = \sum_{A \in M(\mu, \mathbf{j})} \psi_A,$$

which comes from the definitions. □

3.5. Computing the inner product $(\Pi_{\mu, \mathbf{j}}, \Pi_{\omega_d})_{\text{Sym}}$. Recall the inner product $(\cdot, \cdot)_{\text{Sym}}$ from (3.26). Throughout this subsection, we fix $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$. For $A = (a_{r,i}) \in M(\mu, \mathbf{j})$ and $i \in I$, we define

$$|a_{*,i}| := \sum_{r=1}^n a_{r,i}.$$

Then we have compositions

$$a_{*,i} := (a_{1,i}, \dots, a_{n,i}) \in \Lambda(n, |a_{*,i}|) \quad (i \in I)$$

and multinomial coefficients

$$\binom{|a_{*,i}|}{a_{*,i}} := \binom{|a_{*,i}|}{a_{1,i} \cdots a_{n,i}} = \frac{|a_{*,i}|!}{a_{1,i}! \cdots a_{n,i}!}.$$

Lemma 3.35. *Let $A \in M(\mu, \mathbf{j})$, and $k_0, k_1, \dots, k_\ell \in \mathbb{Z}_{\geq 0}$. Then*

$$(\psi_A, \mathfrak{q}_1^{k_0} \otimes \mathfrak{s}_{(1)}^{k_1} \otimes \cdots \otimes \mathfrak{s}_{(1)}^{k_\ell})_{\text{Sym}} = \begin{cases} 2^{|a_{*,0}|} \prod_{i \in I} \binom{|a_{*,i}|}{a_{*,i}} & \text{if } |a_{*,i}| = k_i \text{ for all } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} (\psi_A, \mathfrak{q}_1^{k_0} \otimes \mathfrak{s}_{(1)}^{k_1} \otimes \cdots \otimes \mathfrak{s}_{(1)}^{k_\ell})_{\text{Sym}} &= (\psi_A^{(0)} \otimes \psi_A^{(1)} \otimes \cdots \otimes \psi_A^{(\ell)}, \mathfrak{q}_1^{k_0} \otimes \mathfrak{s}_{(1)}^{k_1} \otimes \cdots \otimes \mathfrak{s}_{(1)}^{k_\ell})_{\text{Sym}} \\ &= [\psi_A^{(0)}, \mathfrak{q}_1^{k_0}] (\psi_A^{(1)}, \mathfrak{s}_{(1)}^{k_1}) \cdots (\psi_A^{(\ell)}, \mathfrak{s}_{(1)}^{k_\ell}). \end{aligned}$$

Now, by (3.23), we have

$$[\psi_A^{(0)}, \mathfrak{q}_1^{k_0}] = [\mathfrak{q}_{a_{1,0}} \cdots \mathfrak{q}_{a_{n,0}}, \mathfrak{q}_1^{k_0}] = \begin{cases} 2^{k_0} \binom{k_0}{a_{1,0} \cdots a_{n,0}} & \text{if } k_0 = a_{1,0} + \cdots + a_{n,0}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by (3.21), we have, for $i = 1, \dots, \ell$,

$$(\psi_A^{(i)}, \mathfrak{s}_{(1)}^{k_i}) = (\psi_A(1, i) \cdots \psi_A(n, i), \mathfrak{s}_{(1)}^{k_i}) \begin{cases} \binom{k_i}{a_{1,i} \cdots a_{n,i}} & \text{if } k_i = a_{1,i} + \cdots + a_{n,i}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the required equality. □

Recall the notation $|\mu, \mathbf{j}|_{\ell-1}$ from (1.5).

Theorem 3.36. *Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$. Then*

$$(\Pi_{\mu, \mathbf{j}}, \Pi_{\omega_d})_{\text{Sym}} = \binom{d}{\mu_1 \cdots \mu_n} 4^{d-|\mu, \mathbf{j}|_{\ell-1}} \mathfrak{z}^{|\mu, \mathbf{j}|_{\ell-1}}.$$

Proof. By Proposition 3.34 and Corollary 3.32, we have that $(\Pi_{\mu, \mathbf{j}}, \Pi_{\omega_d})_{\text{Sym}}$ equals

$$\sum_{A \in M(\mu, \mathbf{j})} \sum_{k_0 + k_1 + \cdots + k_\ell = d} 2^{k_1 + \cdots + k_{\ell-1}} \binom{d}{k_0 k_1 \cdots k_\ell} (\psi_A, \mathfrak{q}_1^{k_0} \otimes \mathfrak{s}_{(1)}^{k_1} \otimes \cdots \otimes \mathfrak{s}_{(1)}^{k_\ell})_{\text{Sym}}.$$

By Lemma 3.35, this equals

$$\begin{aligned} \sum_{A \in M(\mu, \mathbf{j})} 2^{|a_{*,1}| + \cdots + |a_{*,\ell-1}|} \binom{d}{|a_{*,0}| |a_{*,1}| \cdots |a_{*,\ell}|} 2^{|a_{*,0}|} \prod_{i \in I} \binom{|a_{*,i}|}{a_{*,i}} \\ = \sum_{A \in M(\mu, \mathbf{j})} 2^{|a_{*,0}| + |a_{*,1}| + \cdots + |a_{*,\ell-1}|} \frac{d!}{\prod_{i \in I} \prod_{r=1}^n a_{r,i}!} \\ = \sum_{A \in M(\mu, \mathbf{j})} 2^{d-|a_{*,\ell}|} \binom{d}{\mu_1 \cdots \mu_n} \prod_{r=1}^n \binom{\mu_r}{a_{r,0} \cdots a_{r,\ell}}, \end{aligned}$$

and it remains to prove that

$$(3.37) \quad \sum_{A \in M(\mu, \mathbf{j})} 2^{d-|a_{*,\ell}|} \prod_{r=1}^n \binom{\mu_r}{a_{r,0} \cdots a_{r,\ell}} = 4^{d-|\mu, \mathbf{j}|_{\ell-1}} \mathfrak{z}^{|\mu, \mathbf{j}|_{\ell-1}}.$$

Define

$$d_j := \sum_{\substack{1 \leq r \leq n \\ j_r = j}} \mu_r \quad (j \in J).$$

In particular, $d_{\ell-1} = |\mu, \mathbf{j}|_{\ell-1}$ and $d_0 + d_1 + \dots + d_{\ell-1} = d$. Note that permuting the parts of (μ_1, \dots, μ_n) and (j_1, \dots, j_n) by the same permutation in \mathfrak{S}_n does not change the left-hand side of (3.37), so we may assume without loss of generality that $\mathbf{j} = (0^{n_0}, 1^{n_1}, \dots, (\ell-1)^{n_{\ell-1}})$ with $n_0 + n_1 + \dots + n_{\ell-1} = n$ and

$$\mu = (\lambda_1^{(0)}, \dots, \lambda_{n_0}^{(0)}, \dots, \lambda_1^{(\ell-1)}, \dots, \lambda_{n_{\ell-1}}^{(\ell-1)})$$

with $(\lambda_1^{(j)}, \dots, \lambda_{n_j}^{(j)}) \in \Lambda(n_j, d_j)$ for all $j \in J$. Then, the matrices $A \in M(\mu, \mathbf{j})$ look like

$$A = \begin{pmatrix} B^{(0)} \\ \vdots \\ B^{(\ell-1)} \end{pmatrix},$$

where, for each j , the matrix $B^{(j)} = (b_{r,i}^{(j)})_{1 \leq r \leq n_j, i \in I}$ is an arbitrary matrix with non-negative integer values such that $b_{r,i}^{(j)} = 0$ unless $i \in \{j, j+1\}$ and $b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}$ for all $r = 1, \dots, n_j$. So, the left-hand side of (3.37) equals XY where

$$X := \prod_{j=0}^{\ell-2} \prod_{r=1}^{n_j} \sum_{b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}} 2^{\lambda_r^{(j)}} \begin{pmatrix} \lambda_r^{(j)} \\ b_{r,j}^{(j)} \ b_{r,j+1}^{(j)} \end{pmatrix}$$

and

$$Y := \prod_{r=1}^{n_{\ell-1}} \sum_{b_{r,\ell-1}^{(\ell-1)} + b_{r,\ell}^{(\ell-1)} = \lambda_r^{(\ell-1)}} 2^{b_{r,\ell-1}^{(\ell-1)}} \begin{pmatrix} \lambda_r^{(\ell-1)} \\ b_{r,\ell-1}^{(\ell-1)} \ b_{r,\ell}^{(\ell-1)} \end{pmatrix}.$$

Now, using the formula $\sum_{a+b=c} \binom{c}{a,b} = 2^c$, we get

$$X = 2^{d_0 + \dots + d_{\ell-2}} \prod_{j=0}^{\ell-2} \prod_{r=1}^{n_j} \sum_{b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}} 2^{\lambda_r^{(j)}} = 4^{d_0 + \dots + d_{\ell-2}} = 4^{d-d_{\ell-1}} = 4^{d-|\mu, \mathbf{j}|_{\ell-1}},$$

and, using the formula $\sum_{a+b=c} 2^a \binom{c}{a,b} = 3^c$, we get

$$Y = \prod_{r=1}^{n_{\ell-1}} 3^{\lambda_r^{(\ell-1)}} = 3^{d_{\ell-1}} = 3^{|\mu, \mathbf{j}|_{\ell-1}},$$

completing the proof. □

4. Fock space

4.1. Fock spaces \mathcal{F}_q and \mathcal{F} . The (q -deformed) level-1 Fock space \mathcal{F}_q , as defined in [Kashiwara et al. 1996] (see also [Leclerc and Thibon 1997]), is the $\mathbb{Q}(q)$ -vector

space with basis $\{u_\lambda \mid \lambda \in \mathcal{P}_p\}$ labeled by the p -strict partitions

$$\mathcal{F}_q := \bigoplus_{\lambda \in \mathcal{P}_p} \mathbb{Q}(q) \cdot u_\lambda.$$

There is a structure of a $U_q(\mathfrak{g})$ -module on \mathcal{F}_q such that

$$(4.1) \quad E_i u_\lambda = \sum_{A \in \text{PRE}_i(\lambda)} d_A(\lambda) u_{\lambda_A},$$

$$(4.2) \quad F_i u_\lambda = \sum_{B \in \text{PAd}_i(\lambda)} d^B(\lambda) u_{\lambda^B},$$

$$(4.3) \quad T_i u_\lambda = q^{(\alpha_i | \Lambda_0 - \text{cont}(\lambda))} u_\lambda.$$

Example 4.4. In the set up of Example 3.19 we have

$$F_0 u_{(5,5,2)} = (1 - q^4) u_{(6,5,2)} + u_{(5,5,2,1)}.$$

As established in [Kashiwara et al. 1996, Appendix D], there is a bilinear form (\cdot, \cdot) on \mathcal{F}_q which satisfies

$$(4.5) \quad (u_\lambda, u_\mu)_q = \delta_{\lambda, \mu} \|\lambda\|_q$$

and

$$(xv, w)_q = (v, \sigma_q(x)w)_q$$

for all $x \in U_q(\mathfrak{g})$ and $v, w \in \mathcal{F}_q$. The following well-known result allows us to identify $V_q(\Lambda_0)$ with the submodule of \mathcal{F}_q generated by u_\emptyset , where \emptyset stands for the partition (0) of 0 ; cf. [Kleshchev and Livesey 2022, Lemma 2.4.20].

Lemma 4.6. *There exists a unique isomorphism of $U_q(\mathfrak{g})$ -modules $V_q(\Lambda_0) \xrightarrow{\sim} U_q(\mathfrak{g}) \cdot u_\emptyset$ mapping $v_{+,q}$ onto u_\emptyset . Moreover, identifying $V_q(\Lambda_0)$ with the submodule $U_q(\mathfrak{g}) \cdot u_\emptyset \subseteq \mathcal{F}_q$ via this isomorphism, the Shapovalov form $(\cdot, \cdot)_q$ on $V_q(\Lambda_0)$ is the restriction of the form $(\cdot, \cdot)_q$ on \mathcal{F}_q to $V_q(\Lambda_0)$.*

We now apply the construction of Section 2.3 to go from the $U_q(\mathfrak{g})$ -module \mathcal{F}_q to the \mathfrak{g} -module $\mathcal{F}_q|_{q=1} = \mathbb{C} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathcal{F}_{q, \mathbb{Z}[q, q^{-1}]}$ with the lattice $\mathcal{F}_{q, \mathbb{Z}[q, q^{-1}]} := \bigoplus_{\lambda \in \mathcal{P}_p} \mathbb{Z}[q, q^{-1}] \cdot u_\lambda$. We will denote $1 \otimes u_\lambda \in \mathbb{C} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathcal{F}_{q, \mathbb{Z}[q, q^{-1}]}$ again by u_λ . So we have a \mathfrak{g} -module

$$\mathcal{F}_q|_{q=1} = \bigoplus_{\lambda \in \mathcal{P}_p} \mathbb{C} \cdot u_\lambda$$

with the action

$$(4.7) \quad e_i u_\lambda = \sum_{A \in \text{PRE}_i(\lambda)} (d_A(\lambda)|_{q=1}) u_{\lambda_A}, \quad f_i u_\lambda = \sum_{B \in \text{PAd}_i(\lambda)} (d^B(\lambda)|_{q=1}) u_{\lambda^B},$$

and the form $(\cdot, \cdot) := (\cdot, \cdot)_q|_{q=1}$ which satisfies

$$(u_\lambda, u_\mu) = \delta_{\lambda, \mu} \|\lambda\|_q|_{q=1}$$

and $(xv, w) = (v, \sigma(x)w)$ for all $x \in \mathfrak{g}$ and $v, w \in \mathcal{F}_q|_{q=1}$.

Recalling (3.14) and (3.18), it is easy to see that $\mathcal{R} := \text{span}_{\mathbb{C}}(u_\lambda \mid \lambda \in \mathcal{P}_p \setminus \mathcal{P}_0)$ is a \mathfrak{g} -submodule of $\mathcal{F}_q|_{q=1}$. Consider the *reduced Fock space*

$$\mathcal{F} := (\mathcal{F}_q|_{q=1})/\mathcal{R}.$$

Denoting $u_\lambda + \mathcal{R} \in (\mathcal{F}_q|_{q=1})/\mathcal{R}$ by u_λ again, we have that

$$\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}_0} \mathbb{C} \cdot u_\lambda$$

with the action of the Chevalley generators f_i given by

$$(4.8) \quad f_i u_\lambda = \sum_{\mathbb{B} \in A_i(\lambda)} c(\lambda, \mathbb{B}) u_{\lambda^{\mathbb{B}}},$$

where

$$A_i(\lambda) := \{\mathbb{B} \in \text{PAd}_i(\lambda) \mid \lambda^{\mathbb{B}} \in \mathcal{P}_0\}$$

and, recalling (3.3),

$$(4.9) \quad c(\lambda, \mathbb{B}) := \begin{cases} 2 & \text{if } h_p(\lambda^{\mathbb{B}}) = h_p(\lambda) - 1, \\ 1 & \text{otherwise.} \end{cases}$$

(We are not going to need the action of the Chevalley generators e_i .) Moreover, \mathcal{F} inherits the form (\cdot, \cdot) which satisfies

$$(4.10) \quad (u_\lambda, u_\mu) = \delta_{\lambda, \mu} 2^{h_p(\lambda)},$$

and $(xv, w) = (v, \sigma(x)w)$ for all $x \in \mathfrak{g}$ and $v, w \in \mathcal{F}$. We now have from Lemmas 4.6 and 2.5:

Lemma 4.11. *There is a unique isomorphism of \mathfrak{g} -modules from $V(\Lambda_0)$ to the submodule $U(\mathfrak{g}) \cdot u_\emptyset \subseteq \mathcal{F}$ generated by u_\emptyset , mapping v_+ onto u_\emptyset . Moreover, identifying $V(\Lambda_0)$ with $U(\mathfrak{g}) \cdot u_\emptyset \subseteq \mathcal{F}$ via this isomorphism, the Shapovalov form (\cdot, \cdot) on $V(\Lambda_0)$ is the restriction of the form (\cdot, \cdot) on \mathcal{F} to $V(\Lambda_0)$.*

Let ρ be a \bar{p} -core. By definition, we have $h_p(\rho) = 0$ and $\rho \in \mathcal{P}_0$. Note that by Lemma 3.7, there is $w \in W$ such that $\text{cont}(\rho) = \Lambda_0 - w\Lambda_0$. We have the element $v_w \in V(\Lambda_0)$ defined in Lemma 2.2, and the element $u_\rho \in \mathcal{F}$. The following lemma shows that these agree:

Lemma 4.12. *Let $\iota : V(\Lambda_0) \xrightarrow{\sim} U(\mathfrak{g}) \cdot u_\emptyset$, $v_+ \mapsto u_\emptyset$, be the isomorphism of Lemma 4.11. If ρ is a \bar{p} -core and $w \in W$ is such that $\text{cont}(\rho) = \Lambda_0 - w\Lambda_0$ then $\iota(v_w) = u_\rho$.*

Proof. By Lemma 2.2, for certain a_1, \dots, a_l , we have

$$\iota(v_w) = \iota(F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} v_+) = F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} \iota(v_+) = F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\emptyset.$$

It follows from Lemma 3.7 that $\mathcal{F}_{w\Lambda_0} = \mathcal{F}_{\Lambda_0\text{-cont}(\rho)}$ is 1-dimensional and hence spanned by u_ρ . It follows from the formulas (4.8) and (4.9) that $F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\emptyset = ku_\rho$ for $k \in \mathbb{Z}_{>0}$. Now, $(u_\rho, u_\rho) = 2^{h_p(\rho)} = 1$. On the other hand,

$$(F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\emptyset, F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\emptyset) = (\iota(v_w), \iota(v_w)) = (v_w, v_w) = 1$$

by Lemmas 4.11 and 2.2(iii). So $k = 1$. \square

4.2. The elements χ_λ . In this subsection we introduce a new basis of \mathcal{F} . Recalling (3.3), (3.4) and (3.6), we consider the following rescalings of the basis vectors u_λ :

$$(4.13) \quad \chi_\lambda := 2^{(p_\lambda - h_p(\lambda) - c_{\neq 0}^\lambda)/2} u_\lambda \quad (\lambda \in \mathcal{P}_0).$$

These elements correspond to the irreducible supercharacters of the double covers of symmetric groups, see [Fayers et al. 2024, §5], and the formula (4.13) was communicated to us by M. Fayers.

Set

$$a(\lambda, B) := \begin{cases} 2 & \text{if } p_\lambda = 1 \text{ and } p_{\lambda^B} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.14. *Let $i \in I$ and $\lambda \in \mathcal{P}_0$. Then $f_i \chi_\lambda = \sum_{B \in A_i(\lambda)} a(\lambda, B) u_{\lambda^B}$.*

Proof. We have

$$\begin{aligned} f_i \chi_\lambda &= 2^{(p_\lambda - h_p(\lambda) - c_{\neq 0}^\lambda)/2} f_i u_\lambda \\ &= 2^{(p_\lambda - h_p(\lambda) - c_{\neq 0}^\lambda)/2} \sum_{B \in A_i(\lambda)} c(\lambda, B) u_{\lambda^B} \\ &= 2^{(p_\lambda - h_p(\lambda) - c_{\neq 0}^\lambda)/2} \sum_{B \in A_i(\lambda)} c(\lambda, B) 2^{-(p_{\lambda^B} - h_p(\lambda^B) - c_{\neq 0}^{\lambda^B})/2} \chi_{\lambda^B}, \end{aligned}$$

so we need to prove that

$$(4.15) \quad a(\lambda, B) = 2^{(p_\lambda - p_{\lambda^B} - h_p(\lambda) + h_p(\lambda^B) - c_{\neq 0}^\lambda + c_{\neq 0}^{\lambda^B})/2} c(\lambda, B).$$

If $i \neq 0$ then either $p_\lambda = 1$ and $p_{\lambda^B} = 0$, or $p_\lambda = 0$ and $p_{\lambda^B} = 1$. Moreover, $c_{\neq 0}^{\lambda^B} = c_{\neq 0}^\lambda + 1$. Hence

$$\begin{aligned} 2^{(p_\lambda - p_{\lambda^B} - h_p(\lambda) + h_p(\lambda^B) - c_{\neq 0}^\lambda + c_{\neq 0}^{\lambda^B})/2} c(\lambda, B) &= 2^{(p_\lambda - p_{\lambda^B} + 1)/2} \\ &= \begin{cases} 2 & \text{if } p_\lambda = 1 \text{ and } p_{\lambda^B} = 0, \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

which is $a(\lambda, B)$ as required.

On the other hand, if $i = 0$ then $c_{\neq 0}^\lambda = c_{\neq 0}^{\lambda^B}$. Let λ be written in the form (3.1), and consider the following three cases:

- (1) $B = (m_1 + \dots + m_k + 1, 1)$. In this case, we have $c(\lambda, B) = 1$, $p_\lambda = p_{\lambda^B}$ and $h_p(\lambda) = h_p(\lambda^B)$, which immediately gives (4.15).
- (2) $B = (m_1 + \dots + m_{r-1} + 1, 1)$ for some $1 \leq r \leq k$ and $p \mid l_r$. In this case, we have $h_p(\lambda^B) = h_p(\lambda) - 1$, $c(\lambda, B) = 2$ and $p_{\lambda^B} \neq p_\lambda$. If $p_{\lambda^B} = 1$ and $p_\lambda = 0$ then both sides of (4.15) equal 1. If $p_{\lambda^B} = 0$ and $p_\lambda = 1$ then both sides of (4.15) equal 2.
- (3) $B = (m_1 + \dots + m_{r-1} + 1, 1)$ for some $1 \leq r \leq k$ and $p \nmid l_r$. In this case, we have $h_p(\lambda^B) = h_p(\lambda) + 1$, $c(\lambda, B) = 1$ and $p_{\lambda^B} \neq p_\lambda$. If $p_{\lambda^B} = 1$ and $p_\lambda = 0$ then both sides of (4.15) equal 1. If $p_{\lambda^B} = 0$ and $p_\lambda = 1$ then both sides of (4.15) equal 2. □

5. Shapovalov form for RoCK weights

Suppose that $\theta \in Q_+$ satisfies $V(\Lambda_0)_{\Lambda_0 - \theta} \neq 0$. Then $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some w in the Weyl group W , and unique $d \in \mathbb{Z}_{\geq 0}$. We say that θ is *RoCK* if $(\theta \mid \alpha_0^\vee) \geq 2d$ and $(\theta \mid \alpha_i^\vee) \geq d - 1$ for $i = 1, \dots, \ell$. This is equivalent to the cyclotomic quiver Hecke superalgebra $R_\theta^{\Lambda_0}$ being a *RoCK block*, as defined in [Kleshchev and Livesey 2022, Section 4.1].

Throughout Section 5, we fix a RoCK weight $\theta \in Q_+$, so that $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some $w \in W$ and $d \in \mathbb{Z}_{\geq 0}$, and $(\theta \mid \alpha_0^\vee) \geq 2d$, $(\theta \mid \alpha_i^\vee) \geq d - 1$ for $i = 1, \dots, \ell$.

We have the element $v_w \in V(\Lambda_0)_{w\Lambda_0}$ defined in Lemma 2.2.

5.1. Computation of $f(\mu, \mathbf{j})u_\rho$. For each $m \in \mathbb{Z}_{\geq 0}$, $j \in J$, $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(n, d)$, recall the divided power monomials $f(m, j)$ and $f(\mu, \mathbf{j})$ defined in (1.2) and (1.3). We also have a sum $f(\omega_d)$ of monomials defined in (1.4).

Let $\mu \in \mathcal{P}_0(\rho, c)$ with $c \leq d$. Recall the notion of a p -quotient $\text{quot}(\mu) = (\mu^{(0)}, \dots, \mu^{(\ell)})$ of μ from (3.9) and the notation (3.5). Recall that $\text{quot}(\mu) \in \mathcal{P}_0^I(c)$ since μ is strict. The following result follows immediately from [Fayers et al. 2024, Proposition 6.6 and (5.1)] and Lemma 4.14.

Lemma 5.1. *Let $j \in J$ and $c, k \in \mathbb{Z}_{\geq 0}$ satisfy $c + k \leq d$. For $\alpha \in \mathcal{P}_0(\rho, c)$, in the reduced Fock space \mathcal{F} , we have*

$$f(k, j)\chi_\alpha = \sum_{\lambda} 2^{q(\lambda^{(0)}/\alpha^{(0)}) + (k(2\ell - 1) + h(\alpha^{(0)}) - h(\lambda^{(0)}) + p_\alpha - p_\lambda)/2} \chi_\lambda,$$

where the sum is over all $\lambda \in \mathcal{P}_0(\rho, c + k)$ such that $\text{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\text{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ by adding k nodes to the components $\alpha^{(j)}$ and $\alpha^{(j+1)}$, with no two nodes added in the same column of $\alpha^{(j)}$ or in the same row of $\alpha^{(j+1)}$.

Corollary 5.2. *Let $j \in J$ and $c, k \in \mathbb{Z}_{\geq 0}$ satisfy $c + k \leq d$. For $\alpha \in \mathcal{P}_0(\rho, c)$, in the reduced Fock space \mathcal{F} , we have*

$$f(k, j)u_\alpha = \sum_{\lambda} 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})-h(\lambda^{(0)})} u_\lambda,$$

where the sum is over all $\lambda \in \mathcal{P}_0(\rho, c+k)$ such that $\text{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\text{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ by adding k nodes to the components $\alpha^{(j)}$ and $\alpha^{(j+1)}$, with no two nodes added in the same column of $\alpha^{(j)}$ or in the same row of $\alpha^{(j+1)}$.

Proof. Note that $h_p(\alpha) = h(\alpha^{(0)})$. Moreover, for λ 's appearing in the sum, we have $c_{\neq 0}^\lambda - c_{\neq 0}^\alpha = k(2\ell - 1)$ and $h_p(\lambda) = h(\lambda^{(0)})$. So we have by (4.13) and Lemma 5.1,

$$\begin{aligned} f(k, j)u_\alpha &= 2^{(-p_\alpha+h_p(\alpha)+c_{\neq 0}^\alpha)/2} f(k, j)\chi_\alpha \\ &= 2^{(-p_\alpha+h_p(\alpha)+c_{\neq 0}^\alpha)/2} \sum_{\lambda} 2^{q(\lambda^{(0)}/\alpha^{(0)})+(k(2\ell-1)+h(\alpha^{(0)})-h(\lambda^{(0)})+p_\alpha-p_\lambda)/2} \chi_\lambda \\ &= 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})+(k(2\ell-1)-h(\lambda^{(0)})-p_\lambda+c_{\neq 0}^\alpha)/2} \sum_{\lambda} 2^{(p_\lambda-h_p(\lambda)-c_{\neq 0}^\lambda)/2} u_\lambda \\ &= \sum_{\lambda} 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})-h(\lambda^{(0)})} u_\lambda, \end{aligned}$$

as required. \square

Lemma 5.3. *Let $(\mu, j) \in \Lambda^{\text{col}}(n, d)$. Then*

$$f(\mu, j)u_\rho = \sum_{\lambda \in \mathcal{P}_0(\rho, d)} K(\text{quot}(\lambda); \mu, j) 2^{-h(\lambda^{(0)})} u_\lambda.$$

Proof. We apply induction on n . For the induction base case $n = 1$ we apply Corollary 5.2 to see that

$$f(\mu_1, j_1)u_\rho = \sum_{\lambda} 2^{q(\lambda^{(0)}/\rho^{(0)})+h(\rho^{(0)})-h(\lambda^{(0)})} u_\lambda = \sum_{\lambda} 2^{q(\lambda^{(0)}/\rho^{(0)})} 2^{-h(\lambda^{(0)})} u_\lambda,$$

where the sums are over all $\lambda \in \mathcal{P}_0(\rho, d)$ such that $\text{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\text{quot}(\rho) = (\emptyset, \dots, \emptyset)$ by adding k nodes to the components $\rho^{(j)} = \emptyset$ and $\rho^{(j+1)} = \emptyset$, with no two nodes added in the same column of $\rho^{(j)}$ or in the same row of $\rho^{(j+1)}$, and

$$q(\lambda^{(0)}/\rho^{(0)}) = |\{r \in \mathbb{Z}_{>0} \mid \lambda^{(0)} \text{ contains a node in column } r \text{ but not in column } r+1\}|.$$

It follows from the definitions that the λ 's appearing in the latter sum are exactly the λ 's with $K(\text{quot}(\lambda); (\mu_1), j_1) \neq 0$, and for those λ we have $K(\text{quot}(\lambda); (\mu_1), j_1) = 2^{q(\lambda^{(0)}/\rho^{(0)})}$. This establishes the induction base.

For the inductive step, suppose $n > 1$. Let

$$v := (\mu_1, \dots, \mu_{n-1}), \quad \mathbf{k} := (j_1, \dots, j_{n-1}), \quad c = \mu_1 + \dots + \mu_{n-1}.$$

In particular,

$$f(\mu, \mathbf{j}) = f(\mu_n, j_n) f(v, \mathbf{k}).$$

By the inductive assumption, we have

$$\begin{aligned} f(\mu, \mathbf{j})u_\rho &= f(\mu_n, j_n) f(v, \mathbf{k})u_\rho \\ &= \sum_{\alpha \in \mathcal{P}_0(\rho, c)} K(\text{quot}(\alpha); v, \mathbf{k}) 2^{-h(\alpha^{(0)})} f(\mu_n, j_n)u_\alpha \\ &= \sum_{\alpha \in \mathcal{P}_0(\rho, c)} K(\text{quot}(\alpha); v, \mathbf{k}) 2^{-h(\alpha^{(0)})} \sum_{\lambda} 2^{q(\lambda^{(0)}/\alpha^{(0)}) + h(\alpha^{(0)}) - h(\lambda^{(0)})} u_\lambda, \end{aligned}$$

where the second sum is over all $\lambda \in \mathcal{P}_0(\rho, d)$ such that $\text{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\text{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ by adding μ_n nodes to the components $\alpha^{(j_n)}$ and $\alpha^{(j_n+1)}$, with no two nodes added in the same column of $\alpha^{(j_n)}$ or in the same row of $\alpha^{(j_n+1)}$. It remains to note that λ 's appearing in the expression above are exactly those with $K(\text{quot}(\lambda); \mu, \mathbf{j}) \neq 0$, and for such λ we have

$$K(\text{quot}(\lambda); \mu, \mathbf{j}) = \sum_{\alpha} 2^{q(\lambda^{(0)}/\alpha^{(0)})} K(\text{quot}(\alpha); v, \mathbf{k}),$$

where the sum is over all $\alpha \in \mathcal{P}_0(\rho, c)$ such that $\text{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ is obtained from $\text{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ by removing μ_n nodes from the components $\lambda^{(j_n)}$ and $\lambda^{(j_n+1)}$, with no two nodes removed in the same column of $\lambda^{(j_n)}$ or in the same row of $\lambda^{(j_n+1)}$. \square

Recall the definition of $\Pi_{\mu, \mathbf{j}}$ from (3.27).

Corollary 5.4. *Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(m, d)$ and $(v, \mathbf{i}) \in \Lambda^{\text{col}}(n, d)$. Then*

$$(f(\mu, \mathbf{j})u_\rho, f(v, \mathbf{i})u_\rho) = (\Pi_{\mu, \mathbf{j}}, \Pi_{v, \mathbf{i}})_{\text{Sym}}.$$

Proof. For $\lambda \in \mathcal{P}_0(\rho, d)$, we have $h_p(\lambda) = h(\lambda^{(0)})$. So, by Lemma 5.3 and (4.10), taking into account the bijection (3.11), we have

$$\begin{aligned} (f(\mu, \mathbf{j})u_\rho, f(v, \mathbf{i})u_\rho) &= \sum_{\lambda \in \mathcal{P}_0(\rho, d)} K(\text{quot}(\lambda); \mu, \mathbf{j}) K(\text{quot}(\lambda); v, \mathbf{i}) 2^{-2h(\lambda^{(0)})} (u_\lambda, u_\lambda) \\ &= \sum_{\underline{\lambda} \in \mathcal{P}_0^t(d)} K(\underline{\lambda}; \mu, \mathbf{j}) K(\underline{\lambda}; v, \mathbf{i}) 2^{-h(\lambda^{(0)})}. \end{aligned}$$

Since the bases $\{\pi_{\underline{\lambda}} \mid \underline{\lambda} \in \mathcal{P}_0^I\}$ from (3.24) and $\{\kappa_{\underline{\lambda}} \mid \underline{\lambda} \in \mathcal{P}_0^I\}$ from (3.25) are dual to each other with respect to the inner product $(\cdot, \cdot)_{\text{Sym}}$, we have

$$\begin{aligned} \sum_{\underline{\lambda} \in \mathcal{P}_0^I(d)} K(\underline{\lambda}; \mu, \mathbf{j}) K(\underline{\lambda}; \nu, \mathbf{i}) 2^{-h(\lambda^{(0)})} \\ &= \left(\sum_{\underline{\lambda} \in \mathcal{P}_0^I(d)} K(\underline{\lambda}; \mu, \mathbf{j}) 2^{-h(\lambda^{(0)})} \kappa_{\underline{\lambda}}, \sum_{\underline{\lambda} \in \mathcal{P}_0^I(d)} K(\underline{\lambda}; \nu, \mathbf{i}) \pi_{\underline{\lambda}} \right)_{\text{Sym}} \\ &= \left(\sum_{\underline{\lambda} \in \mathcal{P}_0^I(d)} K(\underline{\lambda}; \mu, \mathbf{j}) \pi_{\underline{\lambda}}, \sum_{\underline{\lambda} \in \mathcal{P}_0^I(d)} K(\underline{\lambda}; \nu, \mathbf{i}) \pi_{\underline{\lambda}} \right)_{\text{Sym}} \\ &= (\Pi_{\mu, \mathbf{j}}, \Pi_{\nu, \mathbf{i}})_{\text{Sym}}, \end{aligned}$$

as required. □

Theorem 5.5. *Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(m, d)$. Then*

$$(f(\mu, \mathbf{j})u_{\rho}, f(\omega_d)u_{\rho}) = \binom{d}{\mu_1 \cdots \mu_n} 4^{d-|\mu, \mathbf{j}|_{\ell-1}} 3^{|\mu, \mathbf{j}|_{\ell-1}}.$$

Proof. By Corollary 5.4, we have $(f(\mu, \mathbf{j})u_{\rho}, f(\omega_d)u_{\rho}) = (\Pi_{\mu, \mathbf{j}}, \Pi_{\omega_d})_{\text{Sym}}$, and the theorem follows from Theorem 3.36. □

Recall the vector $v_w \in V(\Lambda_0)_{w\Lambda_0}$ defined in Lemma 2.2.

Theorem 5.6. *Let $(\mu, \mathbf{j}) \in \Lambda^{\text{col}}(m, d)$. Then*

$$(f(\mu, \mathbf{j})v_w, f(\omega_d)v_w) = \binom{d}{\mu_1 \cdots \mu_n} 4^{d-|\mu, \mathbf{j}|_{\ell-1}} 3^{|\mu, \mathbf{j}|_{\ell-1}}.$$

Proof. In view of Lemmas 4.11 and 4.12, this follows from Theorem 5.5. □

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References

- [Fayers et al. 2024] M. Fayers, A. Kleshchev, and L. Morotti, “Decomposition numbers for abelian defect RoCK blocks of double covers of symmetric groups”, *J. Lond. Math. Soc.* (2) **109**:2 (2024), art. id. e12852. MR Zbl
- [Jantzen 1996] J. C. Jantzen, *Lectures on quantum groups*, Grad. Stud. Math. **6**, Amer. Math. Soc., Providence, RI, 1996. MR Zbl
- [Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge Univ. Press, 1990. MR Zbl
- [Kang et al. 2013] S.-J. Kang, M. Kashiwara, and S.-j. Oh, “Supercategorification of quantum Kac–Moody algebras”, *Adv. Math.* **242** (2013), 116–162. MR Zbl

- [Kashiwara et al. 1996] M. Kashiwara, T. Miwa, J.-U. H. Petersen, and C. M. Yung, “Perfect crystals and q -deformed Fock spaces”, *Selecta Math. (N.S.)* **2**:3 (1996), 415–499. MR Zbl
- [Kleshchev 2024] A. Kleshchev, “RoCK blocks of double covers of symmetric groups and generalized Schur algebras”, preprint, 2024. arXiv 2411.03653
- [Kleshchev and Livesey 2022] A. Kleshchev and M. Livesey, “RoCK blocks for double covers of symmetric groups and quiver Hecke superalgebras”, 2022. To appear in Mem. Amer. Math. Soc. arXiv 2201.06870
- [Leclerc and Thibon 1997] B. Leclerc and J.-Y. Thibon, “ q -deformed Fock spaces and modular representations of spin symmetric groups”, *J. Phys. A* **30**:17 (1997), 6163–6176. MR Zbl
- [Lusztig 1988] G. Lusztig, “Quantum deformations of certain simple modules over enveloping algebras”, *Adv. Math.* **70**:2 (1988), 237–249. MR Zbl
- [Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, 1995. MR Zbl
- [Morris 1964] A. O. Morris, “A note on the multiplication of Hall functions”, *J. Lond. Math. Soc.* **39** (1964), 481–488. MR Zbl
- [Morris 1965] A. O. Morris, “The spin representation of the symmetric group”, *Canadian J. Math.* **17** (1965), 543–549. MR Zbl
- [Morris and Yaseen 1986] A. O. Morris and A. K. Yaseen, “Some combinatorial results involving shifted Young diagrams”, *Math. Proc. Cambridge Philos. Soc.* **99**:1 (1986), 23–31. MR Zbl
- [Stembridge 1989] J. R. Stembridge, “Shifted tableaux and the projective representations of symmetric groups”, *Adv. Math.* **74**:1 (1989), 87–134. MR Zbl

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
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