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ON DIMENSIONS OF ROCK BLOCKS OF CYCLOTOMIC QUIVER HECKE SUPERALGEBRAS

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To the memory of Gary Seitz

We explicitly compute the dimensions of certain idempotent truncations of RoCK blocks of cyclotomic quiver Hecke superalgebras. Equivalently, this amounts to a computation of the value of the Shapovalov form on certain explicit vectors in the basic representations of twisted affine Kac–Moody Lie algebras of type A.

1. Introduction

Our goal is to obtain a technical (but neat) result computing the dimensions of certain idempotent truncations of RoCK blocks of cyclotomic quiver Hecke superalgebras. Equivalently, this amounts to a computation of the value of the Shapovalov form on certain explicit vectors in the basic representation $V(\Lambda_0)$ of the twisted affine Kac–Moody Lie algebra $\mathfrak{g}$ of type $A_{2\ell}^{(2)}$.

To state the result, let $\mathfrak{g}$ be the Kac–Moody Lie algebra of type $A_{2\ell}^{(2)}$, with a normalized invariant form $(\cdot\,|\,\cdot)$ on the corresponding weight lattice P, and the Weyl group W. Let

$$I = \{0, 1, \dots, \ell\}, \quad J := I \setminus \{\ell\},$$

$\{\alpha_i \mid i \in I\}$ be the simple roots, $\{\Lambda_i \mid i \in I\}$ be the fundamental dominant weights, and $Q_+ \subset P$ be the set of $\mathbb{Z}_{\geq 0}$-linear combinations of the simple roots. For the negative Chevalley generators $\{f_i \mid i \in I\}$ of $\mathfrak{g}$, we have the divided powers $f_i^{(k)} := f_i^k/k! \in U(\mathfrak{g})$ for $k \in \mathbb{Z}_{\geq 0}$. Fix a nonzero highest weight vector v_+ of the irreducible $\mathfrak{g}$-module $V(\Lambda_0)$ with highest weight Λ_0. Let $(\cdot\,,\cdot)$ be the Shapovalov form on $V(\Lambda_0)$ such that $(v_+, v_+) = 1$.

For every $w \in W$ with reduced decomposition $w = r_{i_t} \cdots r_{i_1}$, setting

$$a_k := (r_{i_{k-1}} \cdots r_{i_1} \Lambda_0 \mid \alpha_{i_k}^\vee) \quad (k = 1, \dots, t), \tag{1.1}$$

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it is easy to see that $a_1, \dots, a_t \in \mathbb{Z}_{\geq 0}$ and

$$v_w := f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$$

is a nonzero vector of the weight space $V(\Lambda_0)_{w\Lambda_0}$, which does not depend on the choice of reduced decomposition and satisfies $(v_w, v_w) = 1$.

For every $j \in J$ and $m \in \mathbb{Z}_{\geq 0}$, we define the divided power monomial

$$f(m, j) := f_j^{(m)} \cdots f_1^{(m)} f_0^{(2m)} f_1^{(m)} \cdots f_j^{(m)} f_{j+1}^{(2m)} \cdots f_{\ell-1}^{(2m)} f_\ell^{(m)} \in U(\mathfrak{g}) \tag{1.2}$$

(with $f(0, j)$ interpreted as 1). Recalling the null-root $\delta := 2\alpha_0 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$, note that the monomial $f(m, j)$ has total weight $-m\delta$. More generally, given $d, n \in \mathbb{Z}_{>0}$, a composition $\mu = (\mu_1, \dots, \mu_n)$ of d into n nonnegative parts and a tuple $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$ we define

$$f(\mu, \boldsymbol{j}) := f(\mu_n, j_n) \cdots f(\mu_1, j_1) \in U(\mathfrak{g}). \tag{1.3}$$

For a composition $\omega_d := (1, \dots, 1)$ of d, we also define

$$f(\omega_d) := \sum_{\boldsymbol{j} \in J^d} f(\omega_d, \boldsymbol{j}) = \sum_{j_1, \dots, j_d \in J} f(1, j_d) \cdots f(1, j_1). \tag{1.4}$$

If the weight space $V(\Lambda_0)_{\Lambda_0 - \theta}$ for $\theta \in Q_+$ is nonzero, then we can write $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some w in the Weyl group W and a unique $d \in \mathbb{Z}_{\geq 0}$. We then say that θ is *RoCK* if $(\theta \mid \alpha_0^\vee) \geq 2d$ and $(\theta \mid \alpha_i^\vee) \geq d - 1$ for $i = 1, \dots, \ell$. This is equivalent to the cyclotomic quiver Hecke algebra $R_\theta^{\Lambda_0}$ being a *RoCK block*, as defined in [Kleshchev and Livesey 2022, Section 4.1]. To each composition $\mu = (\mu_1, \dots, \mu_n)$ and tuple $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$, we can define the corresponding divided power idempotents $e(\mu, \boldsymbol{j}) \in R_\theta^{\Lambda_0}$. The idempotents $e(\omega_d, \boldsymbol{j})$ are then distinct and orthogonal to each other for distinct $\boldsymbol{j} \in J^d$, so we also have the idempotent $e(\omega_d) := \sum_{\boldsymbol{j} \in J^d} e(\omega_d, \boldsymbol{j})$.

Main Theorem. *Let weight $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ be RoCK, $n \in \mathbb{Z}_{>0}$, $\mu = (\mu_1, \dots, \mu_n)$ be a composition of d with n parts, and $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$. Set*

$$|\mu, \boldsymbol{j}|_{\ell-1} := \sum_{\substack{1 \leq r \leq n \\ j_r = \ell - 1}} \mu_r. \tag{1.5}$$

Then

$$\begin{aligned} \dim e(\mu, \boldsymbol{j}) R_\theta^{\Lambda_0} e(\omega_d) &= (f(\mu, \boldsymbol{j}) v_w, f(\omega_d) v_w) \\ &= \binom{d}{\mu_1 \cdots \mu_n} 4^{d - |\mu, \boldsymbol{j}|_{\ell-1}} 3^{|\mu, \boldsymbol{j}|_{\ell-1}}. \end{aligned}$$

The first equality in the theorem is a known consequence of the Kang–Kashiwara–Oh categorification, so the main work is to prove of the second equality. This is proved in Theorem 5.6.

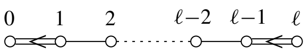
The dimension formula for $e(\mu, \boldsymbol{j})R_\theta^{\Lambda_0}e(\omega_d)$ obtained in the Main Theorem is a shadow of the fact that the Gelfand–Graev idempotent truncation of a RoCK block $R_\theta^{\Lambda_0}$ is isomorphic to a generalized Schur algebra corresponding to a certain Brauer tree algebra. In fact, our Main Theorem is a key step needed for the proof of this isomorphism in [Kleshchev 2024].

2. Shapovalov forms

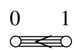
2.1. ***Lie theoretic notation.*** Let $\mathfrak{g}$ be the Kac–Moody Lie algebra of type $A_{2\ell}^{(2)}$ (over $\mathbb{C}$); see [Kac 1990, Chapter 4]. We set

$$p := 2\ell + 1.$$

The Dynkin diagram of $\mathfrak{g}$ has vertices labeled by $I = \{0, 1, \dots, \ell\}$:



if $\ell \geq 2$ and



if $\ell = 1$.

We have the standard Chevalley generators $\{e_i, f_i, h_i \mid i \in I\}$ of $\mathfrak{g}$ and the Chevalley anti-involution

$$\sigma : \mathfrak{g} \to \mathfrak{g}, \quad e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad h_i \mapsto h_i.$$

We have the *weight lattice* P of $\mathfrak{g}$, the subset of all *dominant integral weights* $P_+ \subset P$, and the set $\{\alpha_i \mid i \in I\} \subset P$ of the *simple roots* of $\mathfrak{g}$. We denote by Q the sublattice of P generated by the simple roots and set

$$Q_+ := \left\{ \sum_{i \in I} m_i \alpha_i \;\middle|\; m_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I \right\} \subset Q.$$

For $\theta = \sum_{i \in I} m_i \alpha_i \in Q_+$, its *height* is

$$\operatorname{ht}(\theta) := \sum_{i \in I} m_i.$$

For $\theta \in Q_+$ of height n, we set

$$I^\theta := \{(i_1, \cdots, i_n) \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \theta\}.$$

We have the null-root

$$\delta = \sum_{i=0}^{\ell-1} 2\alpha_i + \alpha_\ell$$

with $\operatorname{ht}(\delta) = 2\ell + 1 = p$.

We denote by $(\,\cdot\,|\,\cdot\,)$ a normalized invariant form on P whose Gram matrix with respect to the linearly independent set $\alpha_0, \alpha_1, \dots, \alpha_\ell$ is

$$\begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 4 & -2 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & 4 & -2 & 0 \\ 0 & 0 & 0 & \dots & -2 & 4 & -4 \\ 0 & 0 & 0 & \dots & 0 & -4 & 8 \end{pmatrix} \quad \text{if } \ell \geq 2 \qquad \text{and} \qquad \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \quad \text{if } \ell = 1.$$

Recall from [Kac 1990, §§3.7, 3.13] the (affine) *Weyl group* W generated by the fundamental reflections $\{r_i \mid i \in I\}$ as a Coxeter group. The form $(\,\cdot\,|\,\cdot\,)$ is W-invariant; see [Kac 1990, Proposition 3.9].

For $\Lambda \in P_+$, we denote by $V(\Lambda)$ the *irreducible integrable* $\mathfrak{g}$*-module* of highest weight Λ. Fix a nonzero highest weight vector $v_+ \in V(\Lambda)_\Lambda$. The *Shapovalov form* is the unique symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ on $V(\Lambda)$ such that $(v_+, v_+) = 1$ and

$$(2.1) \qquad (xv, w) = (v, \sigma(x)w) \quad (x \in \mathfrak{g},\ v, w \in V(\Lambda)).$$

Lemma 2.2. *Let* $\Lambda \in P_+$ *and* $w \in W$ *with a reduced decomposition* $w = r_{i_t} \cdots r_{i_1}$. *Then*

(i) $a_k := (r_{i_{k-1}} \cdots r_{i_1} \Lambda \mid \alpha_{i_k}^\vee) \in \mathbb{Z}_{\geq 0}$ *for all* $k = 1, \dots, t$;

(ii) $v_w := f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$ *is a nonzero vector of the weight space* $V(\Lambda)_{w\Lambda}$, *which does not depend on the choice of a reduced decomposition of* w;

(iii) $(v_w, v_w) = 1$.

Proof. This is well-known. For example, for all the claims, except the independence of a reduced decomposition, one can consult [Kleshchev and Livesey 2022, Lemma 2.4.11]. One way to see the independence of a reduced decomposition is to first note that (iii) determines v_w uniquely up to a sign, and if $f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$ is such vector corresponding to another reduced decomposition, we cannot have $v_w = -f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$, since, by the Kang–Kashiwara–Oh categorification [Kang et al. 2013], the vectors $f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$ and $f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$ correspond to modules over a certain algebra when $V(\Lambda)$ is identified with a Grothendieck group. □

2.2. ***Quantized enveloping algebra.*** Let q be an indeterminate and consider the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials and the field $\mathbb{C}(q)$ of rational functions. For $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we have the following elements of $\mathbb{Z}[q, q^{-1}]$:

$$(2.3) \qquad q_i := q^{(\alpha_i|\alpha_i)/2}, \quad [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i^! := [1]_i [2]_i \cdots [n]_i.$$

Let $U_q(\mathfrak{g})$ be the *quantized enveloping algebra of type* $A_{2\ell}^{(2)}$, i.e., the associative unital $\mathbb{C}(q)$-algebra with generators $\{E_i, F_i, K_i^{\pm 1} \mid i \in I\}$ subject only to the quantum Serre relations

$$\begin{aligned} T_i E_j T_i^{-1} &= q^{(\alpha_i|\alpha_j)} E_j, \\ T_i F_j T_i^{-1} &= q^{-(\alpha_i|\alpha_j)} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{T_i - T_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{i,j}} (-1)^r E_i^{(r)} E_j E_i^{(1-a_{i,j}-r)} &= 0 \quad (i \neq j), \\ \sum_{r=0}^{1-a_{i,j}} (-1)^r F_i^{(r)} F_j F_i^{(1-a_{i,j}-r)} &= 0 \quad (i \neq j), \end{aligned}$$

where we have set $T_i := K_i^{(\alpha_i|\alpha_i)/2}$, $E_i^{(r)} := E_i/[r]_i^!$, $F_i^{(r)} := F_i/[r]_i^!$.

There is a $\mathbb{C}(q)$-linear anti-involution $\sigma_q : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ with

$$\sigma_q : E_i \mapsto q_i F_i T_i^{-1} = q_i^{-1} T_i^{-1} F_i, \quad F_i \mapsto q_i^{-1} T_i E_i = q_i E_i T_i, \quad T_i \mapsto T_i.$$

For $\Lambda \in P_+$, we denote by $V_q(\Lambda)$ the *irreducible integrable module* for $U_q(\mathfrak{g})$ of highest weight Λ. We fix a nonzero highest weight vector $v_{+,q} \in V_q(\Lambda)_\Lambda$, so $E_i v_{+,q} = 0$ and $T_i v_{+,q} = q^{(\alpha_i|\Lambda)} v_{+,q}$ for all $i \in I$. There is a unique symmetric bilinear form $(\cdot\,,\cdot)_q$ on $V_q(\Lambda)$ such that $(v_{+,q}, v_{+,q})_q = 1$ and

$$(xv, w)_q = (v, \sigma_q(x)w)_q \quad (x \in U_q(\mathfrak{g}),\ v, w \in V_q(\Lambda)); \tag{2.4}$$

see [Kashiwara et al. 1996, Appendix D]. We refer to $(\cdot\,,\cdot)_q$ as the (quantum) Shapovalov form.

2.3. ***Putting q to* 1.** Let V_q be a $U_q(\mathfrak{g})$-module which decomposes as a direct sum of finite-dimensional integral weight spaces $V_q = \bigoplus_{\mu \in P} V_{q,\mu}$. Suppose also that there exists a full rank $\mathbb{Z}[q, q^{-1}]$-sublattice $V_{q,\mu,\mathbb{Z}[q,q^{-1}]} \subset V_{q,\mu}$ for every μ such that $V_{q,\mathbb{Z}[q,q^{-1}]} := \bigoplus_{\mu \in P} V_{q,\mu,\mathbb{Z}[q,q^{-1}]}$ is stable under all E_i and F_i. Considering $\mathbb{C}$ as a $\mathbb{Z}[q, q^{-1}]$-module with q acting as 1, we change scalars to get the complex vector space $V_q|_{q=1} := \mathbb{C} \otimes_{\mathbb{Z}[q,q^{-1}]} V_{q,\mathbb{Z}[q,q^{-1}]}$ with linear operators $e_i := 1 \otimes E_i$, $f_i := 1 \otimes F_i$ and $h_i := 1 \otimes ((T_i - T_i^{-1})/(q_i - q_i^{-1}))$ for all i. These linear operators are easily checked to satisfy the Serre relations for $\mathfrak{g}$; see [Jantzen 1996, 5.13; Lusztig 1988, Theorem 4.12 and §4.14]. Thus $V_q|_{q=1}$ becomes a $\mathfrak{g}$-module. (This module depends on the choice of the $\mathbb{Z}[q, q^{-1}]$-sublattice).

Given a symmetric bilinear form $(\cdot\,,\cdot)_q$ on V_q which satisfies (2.4) and is $\mathbb{Z}[q, q^{-1}]$-valued on $V_{q,\mathbb{Z}[q,q^{-1}]}$, we obtain, extending scalars, a symmetric bilinear form $(\cdot\,,\cdot)_q|_{q=1}$ on $V_q|_{q=1}$ satisfying (2.1).

The above constructions can be applied to the irreducible modules $V_q(\Lambda)$ with highest weight $\Lambda \in P_+$ by considering the $\mathbb{Z}[q, q^{-1}]$-sublattice spanned by all vectors of the form $F_{i_1} \cdots F_{i_r} v_+$. Taking into account that the formal characters of $V_q(\Lambda)$ and $V(\Lambda)$ agree by [Lusztig 1988, Theorem 4.12 and §4.14], this shows that there is a unique isomorphism $V(\Lambda) \xrightarrow{\sim} V_q(\Lambda)|_{q=1}$ mapping v_+ onto $1 \otimes v_{+,q}$. Identifying $V(\Lambda)$ and $V_q(\Lambda)|_{q=1}$ under this isomorphism, the quantum Shapovalov form $(\cdot\,,\cdot)_q$ yields the usual Shapovalov form $(\cdot\,,\cdot)$, that is, $(\cdot\,,\cdot)_q|_{q=1} = (\cdot\,,\cdot)$. In particular:

Lemma 2.5. *Let $(\cdot\,,\cdot)_q$ be the Shapovalov form on $V_q(\Lambda)$ and $(\cdot\,,\cdot)$ be the Shapovalov form on $V(\Lambda)$. Then for all $i_1, \dots, i_n, j_1, \dots, j_n \in I$, we have*

$$(F_{i_1} \cdots F_{i_n} v_{+,q}, F_{j_1} \cdots F_{j_n} v_{+,q})_q \in \mathbb{Z}[q, q^{-1}]$$

and

$$(f_{i_1} \cdots f_{i_n} v_+, f_{j_1} \cdots f_{j_n} v_+) = (F_{i_1} \cdots F_{i_n} v_{+,q}, F_{j_1} \cdots F_{j_n} v_{+,q})_q|_{q=1}.$$

Another example of passing from V_q to $V_q|_{q=1}$ will be considered in Section 4.

3. Combinatorics

Recall that we have set $p = 2\ell + 1$.

3.1. ***Partitions, multipartitions, tableaux.*** We denote by $\mathscr{P}$ the set of all partitions and by $\mathscr{P}(n)$ the set of all partitions of $n \in \mathbb{Z}_{\geq 0}$. For $\lambda \in \mathscr{P}(n)$ we write $|\lambda| = n$. Collecting equal parts of $\lambda \in \mathscr{P}$, we can write it in the form

$$\lambda = (l_1^{m_1}, \dots, l_k^{m_k}) \quad \text{with } l_1 > \cdots > l_k > 0 \text{ and } m_1, \dots, m_k \geq 1. \tag{3.1}$$

We then define

$$\|\lambda\|_q := \prod_{r \text{ with } p|l_r} \prod_{s=1}^{m_r} (1 - (-q^2)^s), \tag{3.2}$$

and

$$h(\lambda) := \sum_{r=1}^{k} m_r, \quad h_p(\lambda) := \sum_{r \text{ with } p|l_r} m_r. \tag{3.3}$$

In other words, $h(\lambda)$ is the number of (positive) parts of λ and $h_p(\lambda)$ is the number of (positive) parts of λ divisible by p. If $m_r > 1$ implies $p \mid l_r$ for all $1 \leq r \leq k$ then λ is called *p-strict*. Note that 0-strict also makes sense and means simply *strict*, i.e., all parts are distinct. We denote by $\mathscr{P}_p(n)$ the set of all p-strict partitions of n, and let $\mathscr{P}_p := \bigsqcup_{n \geq 0} \mathscr{P}_p(n)$. We use the similar notation $\mathscr{P}_0(n)$ and $\mathscr{P}_0$ for strict partitions.

For $\lambda \in \mathscr{P}_0$ we define its *parity*:

$$p_\lambda := \begin{cases} 1 & \text{if } \lambda \text{ has odd number of positive even parts,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let λ be a p-strict partition. As usual, we identify λ with its *Young diagram*

$$\lambda = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r\}.$$

We refer to the element $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ as the *node* in row r and column s. We define a preorder '$\leq$' on the nodes via $(r, s) \leq (r', s')$ if and only if $s \leq s'$. For partitions (equivalently Young diagrams) $\alpha \subseteq \lambda$, we define

$$q(\lambda/\alpha) := |\{r \in \mathbb{Z}_{>0} \mid \lambda \backslash \alpha \text{ has a node in column } r \text{ but not in column } r+1\}|. \tag{3.5}$$

We label the nodes with the elements of the set I as follows: the labeling follows the repeating pattern $0, 1, \dots, \ell-1, \ell, \ell-1, \dots, 1, 0$, starting from the first column and going to the right; see Example 3.8 below. If a node $\mathtt{A} \in \lambda$ is labeled with i, we say that $\mathtt{A}$ has *residue* i and write $\operatorname{Res} \mathtt{A} = i$. Recalling α_i's and Q_+ from Section 2.1, define the *residue content* of λ to be

$$\operatorname{cont}(\lambda) := \sum_{\mathtt{A} \in \lambda} \alpha_{\operatorname{Res} \mathtt{A}} \in Q_+.$$

We always write $\operatorname{cont}(\lambda) = \sum_{i \in I} c_i^\lambda \alpha_i$, and

$$c_{\neq 0}^\lambda := c_1^\lambda + \dots + c_\ell^\lambda = |\lambda| - c_0^\lambda. \tag{3.6}$$

Following [Morris 1965; Leclerc and Thibon 1997], we can associate to every $\lambda \in \mathscr{P}_p$ its *$\bar{p}$-core*

$$\operatorname{core}(\lambda) \in \mathscr{P}_p$$

obtained from λ by removing certain nodes. Clearly, from the definition, the number of nodes removed to go from λ to $\operatorname{core}(\lambda)$ is divisible by p, so the *$\bar{p}$-weight* of λ

$$\operatorname{wt}(\lambda) := \frac{|\lambda| - |\operatorname{core}(\lambda)|}{p}$$

is a nonnegative integer. A partition $\rho \in \mathscr{P}_p$ is called a *$\bar{p}$-core* if $\operatorname{core}(\rho) = \rho$. By [Kleshchev and Livesey 2022, Lemma 3.1.39], we have:

Lemma 3.7. *A p-strict partition λ is a $\bar{p}$-core if and only if*

$$\operatorname{cont}(\lambda) = \Lambda_0 - w\Lambda_0 \quad \textit{for some } w \in W.$$

Example 3.8. Let $\ell = 2$, so $p = 5$. The partition $\lambda = (16, 11, 10, 10, 9, 4, 1)$ is 5-strict. The residues of the nodes are

0	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0
0	1	2	1	0	0	1	2	1	0	0					
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1							
0	1	2	1												
0															

The $\bar{5}$-core of λ is (1).

The partition $\lambda \in \mathscr{P}_p$ is determined by its $\bar{p}$-core $\mathrm{core}(\lambda)$ and $\bar{p}$-quotient

$$\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)}), \tag{3.9}$$

which is an *I-multipartition* of d, in other words, $\lambda^{(0)}, \dots, \lambda^{(\ell)}$ are partitions and $|\lambda^{(0)}| + |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = d$. We denote the set of all such multipartitions by $\mathscr{P}^I(d)$, and set

$$\mathscr{P}^I := \bigsqcup_{d \geq 0} \mathscr{P}^I(d).$$

We refer the reader to [Morris and Yaseen 1986, p. 27] and [Kleshchev and Livesey 2022, §2.3b] for details on this. For a $\bar{p}$-core partition ρ, we define

$$\mathscr{P}_p(\rho, d) := \{\lambda \in \mathscr{P}_p \mid \mathrm{core}(\lambda) = \rho \text{ and } \mathrm{wt}(\lambda) = d\}.$$

The map

$$\mathscr{P}_p(\rho, d) \to \mathscr{P}^I(d), \quad \lambda \mapsto \mathrm{cont}(\lambda), \tag{3.10}$$

is a bijection; see [Morris and Yaseen 1986, Theorem 2]. The condition $\lambda \in \mathscr{P}_p(\rho, d)$ is equivalent to $\mathrm{cont}(\lambda) = \mathrm{cont}(\rho) + d\delta$; see [Kleshchev and Livesey 2022, Lemma 2.3.9].

A multipartition $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^I$ is *strict* if its 0-th component $\lambda^{(0)}$ is a strict partition (and $\lambda^{(1)}, \dots, \lambda^{(\ell)}$ are arbitrary partitions). For $d \in \mathbb{Z}_{\geq 0}$, we denote by $\mathscr{P}^I_0(d)$ the set of all strict multipartitions of d. Note that $\mathrm{quot}(\lambda) \in \mathscr{P}^I_0(d)$ if and only if $\lambda \in \mathscr{P}_0(\rho, d)$, so the bijection (3.10) restricts to the bijection

$$\mathscr{P}_0(\rho, d) \to \mathscr{P}^I_0(d), \quad \lambda \mapsto \mathrm{cont}(\lambda). \tag{3.11}$$

We identify a multipartition $\underline{\lambda}$ with its Young diagram

$$\underline{\lambda} = \{(i, r, s) \in I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda^{(i)}_r\}.$$

We refer to the element $(i, r, s) \in I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ as the *node* in row r and column s of component i. For each i, we consider (the Young diagram of) the partition $\lambda^{(i)}$ as a subset $\lambda^{(i)} \subseteq \underline{\lambda}$ consisting of the nodes of $\underline{\lambda}$ in its i-th component.

Let $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$. A *composition* of d with n parts is a tuple $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_1, \dots, \mu_n \in \mathbb{Z}_{\geq 0}$ such that $\mu_1 + \cdots + \mu_n = d$. We denote by $\Lambda(n, d)$ the set of all compositions of d with n parts. We will need a special composition of d with d parts:

$$\omega_d := (1^d) = (1, \dots, 1). \tag{3.12}$$

Recall that J denotes $I \setminus \{\ell\}$. A *colored composition of d with n parts* is a pair $(\mu, \boldsymbol{j})$ where $\mu = (\mu_1, \dots, \mu_n)$ is a composition of d with n parts and $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$. We denote by $\Lambda^{\mathrm{col}}(n, d)$ the set of all colored compositions of d with n parts.

Let $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^I(d)$ and $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. A *colored tableau of shape* $\underline{\lambda}$ *and type* $(\mu, \boldsymbol{j})$ is a function $\mathsf{T} : \underline{\lambda} \to \mathbb{Z}_{>0}$ such that

(1) $\mathsf{T}(i, r, s) \leq \mathsf{T}(i, r, s+1)$ and $\mathsf{T}(i, r, s) \leq \mathsf{T}(i, r+1, s)$ whenever these make sense;

(2) for all $k = 1, \dots, n$, we have $|\mathsf{T}^{-1}(\{k\})| = \mu_k$ and $\mathsf{T}^{-1}(\{k\}) \subseteq \lambda^{(j_k)} \sqcup \lambda^{(j_k+1)}$;

(3) for all $k = 1, \dots, n$, no two nodes of $\mathsf{T}^{-1}(\{k\}) \cap \lambda^{(j_k)}$ are in the same column, and no two nodes of $\mathsf{T}^{-1}(\{k\}) \cap \lambda^{(j_k+1)}$ are in the same row.

Denote by $\mathsf{CT}(\underline{\lambda}; \mu, \boldsymbol{j})$ the set of all colored tableaux of shape $\underline{\lambda}$ and type $(\mu, \boldsymbol{j})$. For $\mathsf{T} \in \mathsf{CT}(\underline{\lambda}; \mu, \boldsymbol{j})$ and $1 \leq k \leq n$ we denote by $q_k(\mathsf{T})$ the number of positive integers r such that $\mathsf{T}^{-1}(k) \cap \lambda^{(0)}$ contains a node in column r but not in column $r+1$. We then set $q(\mathsf{T}) = q_1(\mathsf{T}) + \cdots + q_n(\mathsf{T})$ and define

$$K(\underline{\lambda}; \mu, \boldsymbol{j}) := \sum_{\mathsf{T} \in \mathsf{CT}(\underline{\lambda}; \mu, \boldsymbol{j})} 2^{q(\mathsf{T})}. \tag{3.13}$$

3.2. ***Addable and removable nodes.*** Let λ be a p-strict partition and $i \in I$. A node $\mathtt{A} \in \lambda$ is called *i-removable* (for λ) if one of the following holds:

(R1) $\operatorname{Res} \mathtt{A} = i$ and $\lambda_{\mathtt{A}} := \lambda \setminus \{\mathtt{A}\}$ is again a p-strict partition; such $\mathtt{A}$'s are also called *properly i-removable*.

(R2) The node $\mathtt{B}$ immediately to the right of $\mathtt{A}$ belongs to λ, $\operatorname{Res} \mathtt{A} = \operatorname{Res} \mathtt{B} = i$, and both $\lambda_{\mathtt{B}} = \lambda \setminus \{\mathtt{B}\}$ and $\lambda_{\mathtt{A},\mathtt{B}} := \lambda \setminus \{\mathtt{A}, \mathtt{B}\}$ are p-strict partitions.

A node $\mathtt{B} \notin \lambda$ is called *i-addable* (for λ) if one of the following holds:

(A1) $\operatorname{Res} \mathtt{B} = i$ and $\lambda^{\mathtt{B}} := \lambda \cup \{\mathtt{B}\}$ is again a p-strict partition; such $\mathtt{B}$'s are also called *properly i-addable*.

(A2) The node $\mathtt{A}$ immediately to the left of $\mathtt{B}$ does not belong to λ, $\operatorname{Res} \mathtt{A} = \operatorname{Res} \mathtt{B} = i$, and both $\lambda^{\mathtt{A}} = \lambda \cup \{\mathtt{A}\}$ and $\lambda^{\mathtt{A},\mathtt{B}} := \lambda \cup \{\mathtt{A}, \mathtt{B}\}$ are p-strict partitions.

We note that (R2) and (A2) above are only possible if $i = 0$. For $i \in I$, we denote by $\mathrm{Ad}_i(\lambda)$ (resp. $\mathrm{Re}_i(\lambda)$) the set of all i-removable (resp. i-addable) nodes for λ. We also denote by $\mathrm{PAd}_i(\lambda)$ (resp. $\mathrm{PRe}_i(\lambda)$) the set of all properly i-removable (resp. properly i-addable) nodes for λ.

Let $\lambda \in \mathscr{P}_p$ be written in the form (3.1). Suppose $\mathtt{A} \in \mathrm{PRe}_i(\lambda)$. Then there is $1 \leq r \leq k$ such that

$$\mathtt{A} = (m_1 + \cdots + m_r, l_r).$$

Recalling the preorder '$\leq$' on the nodes defined above, we set

$$\eta_{\mathtt{A}}(\lambda) := \sharp\{\mathtt{C} \in \mathrm{Re}_i(\lambda) \mid \mathtt{C} > \mathtt{A}\} - \sharp\{\mathtt{C} \in \mathrm{Ad}_i(\lambda) \mid \mathtt{C} > \mathtt{A}\},$$

$$\zeta_{\mathtt{A}}(\lambda) := \begin{cases} (1-(-q^2)^{m_r}) & \text{if } p \mid l_r, \\ 1 & \text{otherwise,} \end{cases}$$

$$d_{\mathtt{A}}(\lambda) := q_i^{\eta_{\mathtt{A}}(\lambda)} \zeta_{\mathtt{A}}(\lambda).$$

Note that

$$d_{\mathtt{A}}(\lambda)|_{q=1} = \begin{cases} 0 & \text{if } p \mid l_r \text{ and } m_r \text{ is even,} \\ 2 & \text{if } p \mid l_r \text{ and } m_r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.14}$$

Suppose $\mathtt{B} \in \mathrm{PAd}_i(\lambda)$. Then there is r such that $1 \leq r \leq k+1$ and

$$\mathtt{B} = (m_1 + \cdots + m_{r-1} + 1, l_r + 1),$$

where we interpret l_{k+1} as 0. We define

$$\eta^{\mathtt{B}}(\lambda) := \sharp\{\mathtt{C} \in \mathrm{Ad}_i(\lambda) \mid \mathtt{C} < \mathtt{B}\} - \sharp\{\mathtt{C} \in \mathrm{Re}_i(\lambda) \mid \mathtt{C} < \mathtt{B}\}, \tag{3.15}$$

$$\zeta^{\mathtt{B}}(\lambda) := \begin{cases} (1-(-q^2)^{m_r}) & \text{if } r \leq k \text{ and } p \mid l_r, \\ 1 & \text{otherwise,} \end{cases} \tag{3.16}$$

$$d^{\mathtt{B}}(\lambda) := q_i^{\eta^{\mathtt{B}}(\lambda)} \zeta^{\mathtt{B}}(\lambda). \tag{3.17}$$

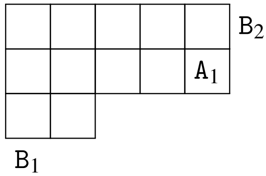
Note that

$$d^{\mathtt{B}}(\lambda)|_{q=1} = \begin{cases} 0 & \text{if } r \leq k,\ p \mid l_r \text{ and } m_r \text{ is even,} \\ 2 & \text{if } r \leq k,\ p \mid l_r \text{ and } m_r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.18}$$

Example 3.19. Let $\ell = 2$ so $p = 5$. The partition $\lambda = (5, 5, 2)$ is 5-strict, and the residues of its boxes are labeled on the diagram

0	1	2	1	0
0	1	2	1	0
0	1			

The only 0-removable node is marked as $\mathtt{A}_1$, and the 0-addable nodes are marked as $\mathtt{B}_1, \mathtt{B}_2$:



We have $d^{\mathtt{B}_1}(\lambda) = 1$ and $d^{\mathtt{B}_2}(\lambda) = (1-q^4)$. On the other hand for the partition $\mu = (5)$ and the node $\mathtt{B} = (1, 6)$, we have $d^{\mathtt{B}}(\mu) = (1+q^2)$.

3.3. ***Symmetric functions.*** We denote by Λ the algebra of symmetric functions in the variables $x_1, x_2, \dots$ over $\mathbb{C}$ with the basis

$$\{\mathsf{s}_\lambda \mid \lambda \in \mathscr{P}\}$$

of *Schur functions* and the inner product $\langle \cdot, \cdot \rangle$ such that $\langle \mathsf{s}_\lambda, \mathsf{s}_\mu \rangle = \delta_{\lambda,\mu}$; see [Macdonald 1995]. We also have the *monomial symmetric functions* m_λ, the *elementary symmetric functions* $\mathsf{e}_r = \mathsf{s}_{(1^r)}$ and the *complete symmetric functions* $\mathsf{h}_r = \mathsf{s}_{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$. *Pieri's rules* [Macdonald 1995, (5.16), (5.17)] say that

$$\mathsf{s}_\lambda \mathsf{h}_r = \sum_\mu \mathsf{s}_\mu \quad \text{and} \quad \mathsf{s}_\lambda \mathsf{e}_r = \sum_\nu \mathsf{s}_\nu, \tag{3.20}$$

where the first sum is over all partitions μ obtained by adding r nodes to λ with no two nodes added in the same column, and the second sum is over all partitions ν obtained by adding r nodes to λ with no two nodes added in the same row.

Suppose that for $s_1, \dots, s_t \in \mathbb{Z}_{\geq 0}$, we have that $\mathsf{f}_{s_u} = \mathsf{e}_{s_u}$ or h_{s_u}. Under the characteristic map [Macdonald 1995, I.7], the symmetric function $\mathsf{f}_{s_1} \cdots \mathsf{f}_{s_t}$ corresponds to an induced representation of the symmetric group $\mathfrak{S}_{s_1+\cdots+s_t}$ of dimension given by the multinomial coefficient $\binom{s_1+\cdots+s_t}{s_1\cdots s_t}$, while the symmetric function $\mathsf{s}_{(1)}^r$ with $r \in \mathbb{Z}_{\geq 0}$ corresponds to the regular representation of the symmetric group $\mathfrak{S}_r$. Hence,

$$(\mathsf{f}_{s_1} \cdots \mathsf{f}_{s_t}, \mathsf{s}_{(1)}^r) = \begin{cases} \binom{r}{s_1\cdots s_t} & \text{if } s_1 + \cdots + s_t = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Denoting by $\mathsf{p}_r \in \Lambda$ the r-th power sum symmetric function, let Ω be the (unital) subalgebra of Λ generated by $\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \dots$. Then Ω has bases

$$\{\mathsf{P}_\lambda \mid \lambda \in \mathscr{P}_0\} \quad \text{and} \quad \{\mathsf{Q}_\lambda \mid \lambda \in \mathscr{P}_0\},$$

where the elements P_λ and Q_λ are *Schur's P- and Q-symmetric functions*; see [Stembridge 1989, §6]. We have $\mathsf{P}_\lambda = 2^{-h(\lambda)}\mathsf{Q}_\lambda$ for all $\lambda \in \mathscr{P}_0$. Let $[\cdot, \cdot]$ be an inner product on Ω such that $[\mathsf{P}_\lambda, \mathsf{Q}_\lambda] = \delta_{\lambda,\mu}$ for all $\lambda, \mu \in \mathscr{P}_0$; see [Stembridge 1989, §§5, 6].

We also have the symmetric functions

$$\mathsf{q}_r = 2\mathsf{P}_{(r)} \in \Omega \quad (r \in \mathbb{Z}_{>0})$$

(and $\mathsf{q}_0 := 1$); see [Stembridge 1989, (5.3), (6.6)]. We have the analogue of the Pieri's rule (which goes back to [Morris 1964] but can be most easily seen from [Stembridge 1989, Theorem 8.3]):

$$\mathsf{P}_\lambda \mathsf{q}_r = \sum_\mu 2^{q(\mu/\lambda)} \mathsf{P}_\mu, \tag{3.22}$$

where the sum is over all strict partitions μ obtained by adding r nodes to λ with no two nodes added in the same column and $q(\mu/\lambda)$ is as in (3.5).

By [Stembridge 1989, Proposition 5.6(b)], the inner product $[\mathsf{q}_{s_1} \dots \mathsf{q}_{s_t}, \mathsf{q}_1^r]$ is the coefficient of $\mathsf{m}_{(s_1,\dots,s_r)}$ in $\mathsf{q}_1^r = (2x_1 + 2x_2 + \dots)^r$ (we may assume that $s_1 \geq \dots \geq s_r$ so $(s_1, \dots, s_r)$ is a partition), whence

$$[\mathsf{q}_{s_1} \dots \mathsf{q}_{s_t}, \mathsf{q}_1^r] = \begin{cases} 2^r \binom{r}{s_1 \cdots s_t} & \text{if } s_1 + \dots + s_t = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

We consider the algebra

$$\mathsf{Sym}^I := \Omega \otimes \Lambda^{(1)} \otimes \dots \otimes \Lambda^{(\ell)},$$

where each algebra $\Lambda^{(i)}$ is just a copy of Λ. This has bases

$$\{\pi_{\underline{\lambda}} := \mathsf{P}_{\lambda^{(0)}} \otimes \mathsf{s}_{\lambda^{(1)}} \otimes \dots \otimes \mathsf{s}_{\lambda^{(\ell)}} \mid \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_0^I\} \tag{3.24}$$

and

$$\{\kappa_{\underline{\lambda}} := \mathsf{Q}_{\lambda^{(0)}} \otimes \mathsf{s}_{\lambda^{(1)}} \otimes \dots \otimes \mathsf{s}_{\lambda^{(\ell)}} \mid \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_0^I\}, \tag{3.25}$$

which are dual with respect to the inner product $(\cdot\,,\cdot)_{\mathsf{Sym}}$ defined as

$$(f_0 \otimes f_1 \otimes \dots \otimes f_\ell, g_0 \otimes g_1 \otimes \dots \otimes g_\ell)_{\mathsf{Sym}} := [f_0, g_0]\langle f_1, g_1\rangle \cdots \langle f_\ell, g_\ell\rangle. \tag{3.26}$$

Let $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. Recalling (3.13), set

$$\Pi_{\mu,\boldsymbol{j}} := \sum_{\underline{\lambda} \in \mathscr{P}_0^I(d)} K(\underline{\lambda}; \mu, \boldsymbol{j}) \pi_{\underline{\lambda}}. \tag{3.27}$$

Example 3.28. Let $n = 1$ so $(\mu, \boldsymbol{j})$ is of the form $((d), j) \in \Lambda^{\mathrm{col}}(1, d)$. Then

$$\Pi_{(d),j} := \begin{cases} 1 \otimes \mathsf{e}_d \otimes 1^{\otimes \ell-1} + 2\sum_{k=1}^d \mathsf{P}_{(k)} \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1} & \text{if } j = 0, \\ \sum_{k=0}^d 1^{\otimes j} \otimes \mathsf{h}_k \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1-j} & \text{if } 1 \leq j < \ell. \end{cases} \tag{3.29}$$

Lemma 3.30. *Let $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. Suppose $n \geq 2$ and set $\nu := (\mu_1, \dots, \mu_{n-1})$, $\boldsymbol{k} := (j_1, \dots, j_{n-1})$, $m := \mu_n$, $j := j_n$. Then*

$$\Pi(\mu, \boldsymbol{j}) = \Pi(\nu, \boldsymbol{k}) \Pi((m), j).$$

Proof. Suppose $j \neq 0$. Using (3.29), we see that $\Pi(\nu, \boldsymbol{k})\Pi((m), j)$ equals

$$\Big(\sum_{\underline{\alpha}\in\mathscr{P}_0^I(d-m)} K(\underline{\alpha}; \nu, \boldsymbol{k})\pi_{\underline{\alpha}}\Big)\Big(\sum_{k=0}^{m} 1^{\otimes j}\otimes \mathsf{h}_k\otimes \mathsf{e}_{m-k}\otimes 1^{\otimes \ell-1-j}\Big)$$
$$= \sum_{\substack{\underline{\alpha}\in\mathscr{P}_0^I(d-m)\\ 0\le k\le m}} K(\underline{\alpha}; \nu, \boldsymbol{k})\mathsf{P}_{\alpha^{(0)}}\otimes \mathsf{s}_{\alpha^{(1)}}\otimes\cdots\otimes \mathsf{s}_{\alpha^{(j)}}\mathsf{h}_k\otimes \mathsf{s}_{\alpha^{(j+1)}}\mathsf{e}_{m-k}\otimes\cdots\otimes \mathsf{s}_{\alpha^{(\ell)}}$$
$$= \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda}; \mu, \boldsymbol{j})\mathsf{P}_{\lambda^{(0)}}\otimes \mathsf{s}_{\lambda^{(1)}}\otimes\cdots\otimes \mathsf{s}_{\lambda^{(j)}}\otimes \mathsf{s}_{\lambda^{(j+1)}}\otimes\cdots\otimes \mathsf{s}_{\lambda^{(\ell)}},$$

where the last equality follows from Pieri's rules (3.20).

Suppose now that $j = 0$. Using (3.29), we see that $\Pi(\nu, \boldsymbol{k})\Pi((m), 0)$ equals

$$\Big(\sum_{\underline{\alpha}\in\mathscr{P}_0^I(d-m)} K(\underline{\alpha}; \nu, \boldsymbol{k})\pi_{\underline{\alpha}}\Big)\Big(\sum_{k=0}^{d} \mathsf{q}_k\otimes \mathsf{e}_{d-k}\otimes 1^{\otimes \ell-1}\Big)$$
$$= \sum_{\substack{\underline{\alpha}\in\mathscr{P}_0^I(d-m)\\ 0\le k\le m}} K(\underline{\alpha}; \nu, \boldsymbol{k})\mathsf{P}_{\alpha^{(0)}}\mathsf{q}_k\otimes \mathsf{s}_{\alpha^{(1)}}\mathsf{e}_{d-k}\otimes \mathsf{s}_{\alpha^{(2)}}\otimes\cdots\otimes \mathsf{s}_{\alpha^{(\ell)}}$$
$$= \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda}; \mu, \boldsymbol{j})\mathsf{P}_{\lambda^{(0)}}\otimes \mathsf{s}_{\lambda^{(1)}}\otimes\cdots\otimes \mathsf{s}_{\lambda^{(\ell)}},$$

where the last equality follows from Pieri's rules (3.22) and (3.20). □

Recalling (3.12), we let

$$\Pi_{\omega_d} := \sum_{\boldsymbol{j}\in J^d} \Pi_{\omega_d, \boldsymbol{j}}. \tag{3.31}$$

Corollary 3.32. *We have*

$$\Pi_{\omega_d} = \sum_{k_0+k_1+\cdots+k_\ell=d} 2^{k_1+\cdots+k_{\ell-1}}\binom{d}{k_0\, k_1 \cdots k_\ell}\mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell}.$$

Proof. We apply induction on d. In the base case $d = 1$, by (3.29) and using $\mathsf{q}_1 = 2\mathsf{P}_{(1)}$ we have

$$\begin{aligned}\Pi_{\omega_1} &= \Pi_{(1),0}+\Pi_{(1),1}+\cdots+\Pi_{(1),\ell-1}\\ &= (1\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes \ell-1}+2\mathsf{P}_{(1)}\otimes 1^{\otimes \ell})+(1\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes \ell-1}+1\otimes 1\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes \ell-2})\\ &\qquad +\cdots+(1^{\otimes \ell-1}\otimes \mathsf{s}_{(1)}\otimes 1+1^{\otimes \ell-1}\otimes 1\otimes \mathsf{s}_{(1)})\\ &= \mathsf{q}_1\otimes 1^{\otimes \ell}+1^{\otimes \ell}\otimes \mathsf{s}_{(1)}+2\sum_{i=1}^{\ell-1} 1^{\otimes i}\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes \ell-i},\end{aligned}$$

as required.

For the inductive step, for $d > 1$, it follows from Lemma 3.30 that $\Pi_{\omega_d} = \Pi_{\omega_{d-1}}\Pi_{\omega_1}$. So, by the inductive assumption, we get

$$\begin{aligned}\Pi_{\omega_d} &= \Pi_{\omega_{d-1}}\Pi_{\omega_1}\\ &= \Bigg(\sum_{m_0+m_1+\cdots+m_\ell=d-1} 2^{m_1+\cdots+m_{\ell-1}}\binom{d-1}{m_0\, m_1 \cdots m_\ell}\mathsf{q}_1^{m_0}\otimes \mathsf{s}_{(1)}^{m_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{m_\ell}\Bigg)\\ &\qquad\qquad \times\Bigg(\sum_{n_0+n_1+\cdots+n_\ell=1} 2^{n_1+\cdots+n_{\ell-1}}\mathsf{q}_1^{n_0}\otimes \mathsf{s}_{(1)}^{n_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{n_\ell}\Bigg)\\ &= \sum_{k_0+k_1+\cdots+k_\ell=d} 2^{k_1+\cdots+k_{\ell-1}}\binom{d}{k_0\, k_1 \cdots k_\ell}\mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell}\end{aligned}$$

thanks to the identity $\binom{d}{k_0\, k_1 \cdots k_\ell} = \sum_{r \text{ with } k_r>0}\binom{d-1}{k_0 \cdots k_{r-1} \cdots k_r-1\, k_{r+1} \cdots k_\ell}$. □

3.4. ***Another description of*** $\boldsymbol{\Pi_{\mu,j}}$**.** Let $M_{n,I}$ denote the set of $n\times I$-matrices with nonnegative integer entries,

$$M_{n,I} = \{(a_{r,i})_{1\le r\le n,\, i\in I} \mid a_{r,i}\in \mathbb{Z}_{\ge 0}\}.$$

For $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$, we define the sets of matrices

$$\begin{aligned} M_{n,I}(\mu) &:= \Big\{(a_{r,i})\in M_{n,I} \;\Big|\; \sum_{i\in I} a_{r,i} = \mu_r \text{ for } r = 1,\dots,n\Big\},\\ M_{n,I}(\boldsymbol{j}) &:= \{(a_{r,i})\in M_{n,I} \mid a_{r,i} = 0 \text{ if } i \ne j_r,\, j_r+1 \text{ for } r=1,\dots,n\},\\ M(\mu,\boldsymbol{j}) &:= M_{n,I}(\mu)\cap M_{n,I}(\boldsymbol{j}).\end{aligned}$$

Let $A = (a_{r,i}) \in M(\mu, \boldsymbol{j})$. For $1 \le r \le n$ and $i \in I\setminus\{0\}$, we define

$$\psi_A(r,i) := \begin{cases}\mathsf{h}_{a_{r,i}} & \text{if } i = j_r,\\ \mathsf{e}_{a_{r,i}} & \text{if } i = j_r+1,\\ 1 & \text{otherwise.}\end{cases}$$

We now set

$$\psi_A^{(0)} := \mathsf{q}_{a_{1,0}}\cdots \mathsf{q}_{a_{n,0}}, \qquad \psi_A^{(i)} := \psi_A(1,i)\cdots\psi_A(n,i) \quad (\text{for } i\in I\setminus\{0\}),$$

and

$$\psi_A := \psi_A^{(0)}\otimes\psi_A^{(1)}\otimes\cdots\otimes\psi_A^{(\ell)} \in \mathsf{Sym}^I.$$

Example 3.33. Suppose $n = 1$ and so $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(1, d)$ is of the form $((d), j)$. The set $M((d), j)$ consists of all matrices of the form

$$\{A(j,k) := (0\ \cdots\ 0\ k\ \ d-k\ \ 0\ \cdots\ 0) \mid 0\le k\le d\}$$

with k in position j. Note that by definition we have

$$\psi_{A_{0,0}} = \mathsf{q}_0 \otimes \mathsf{e}_d \otimes 1^{\otimes \ell-1} = 1 \otimes \mathsf{e}_d \otimes 1^{\otimes \ell-1},$$
$$\psi_{A_{0,k}} = \mathsf{q}_k \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1} = 2\mathsf{P}_{(k)} \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1} \quad (1 \le k \le d),$$
$$\psi_{A_{j,k}} = 1^{\otimes j} \otimes \mathsf{h}_k \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1-j} \quad (1 \le j < \ell).$$

In particular, comparing with (3.29), we deduce that $\Pi_{(d),j} = \sum_{A \in M((d),j)} \psi_A$.

Proposition 3.34. *Let* $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. *Then*

$$\Pi_{\mu,\boldsymbol{j}} = \sum_{A \in M(\mu,\boldsymbol{j})} \psi_A.$$

Proof. We apply induction on n. For the base $n = 1$, see Example 3.33. Suppose $n \ge 2$ and set

$$\nu := (\mu_1, \ldots, \mu_{n-1}), \quad \boldsymbol{k} := (j_1, \ldots, j_{n-1}), \quad m := \mu_n, \quad j := j_n.$$

Then $\Pi(\mu, \boldsymbol{j}) = \Pi(\nu, \boldsymbol{k})\Pi((m), j)$ by Lemma 3.30. By the inductive assumption,

$$\Pi(\nu, \boldsymbol{k}) = \sum_{B \in M(\nu,\boldsymbol{k})} \psi_B \quad \text{and} \quad \Pi((m), j) = \sum_{C \in M((m),j)} \psi_C,$$

so it suffices to observe that

$$\Big(\sum_{B \in M(\nu,\boldsymbol{k})} \psi_B\Big)\Big(\sum_{C \in M((m),j)} \psi_C\Big) = \sum_{A \in M(\mu,\boldsymbol{j})} \psi_A,$$

which comes from the definitions. □

3.5. ***Computing the inner product*** $(\boldsymbol{\Pi_{\mu,j}}, \boldsymbol{\Pi_{\omega_d}})_{\mathsf{Sym}}$**.** Recall the inner product $(\cdot\,,\cdot)_{\mathsf{Sym}}$ from (3.26). Throughout this subsection, we fix $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. For $A = (a_{r,i}) \in M(\mu, \boldsymbol{j})$ and $i \in I$, we define

$$|a_{*,i}| := \sum_{r=1}^{n} a_{r,i}.$$

Then we have compositions

$$a_{*,i} := (a_{1,i}, \ldots, a_{n,i}) \in \Lambda(n, |a_{*,i}|) \quad (i \in I)$$

and multinomial coefficients

$$\binom{|a_{*,i}|}{a_{*,i}} := \binom{|a_{*,i}|}{a_{1,i} \cdots a_{n,i}} = \frac{|a_{*,i}|!}{a_{1,i}! \cdots a_{n,i}!}.$$

Lemma 3.35. *Let* $A \in M(\mu, \boldsymbol{j})$, *and* $k_0, k_1, \ldots, k_\ell \in \mathbb{Z}_{\ge 0}$. *Then*

$$(\psi_A, \mathsf{q}_1^{k_0} \otimes \mathsf{s}_{(1)}^{k_1} \otimes \cdots \otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}} = \begin{cases} 2^{|a_{*,0}|} \prod_{i \in I} \binom{|a_{*,i}|}{a_{*,i}} & \textit{if } |a_{*,i}| = k_i \textit{ for all } i \in I, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We have

$$(\psi_A, \mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}} = (\psi_A^{(0)}\otimes\psi_A^{(1)}\otimes\cdots\otimes\psi_A^{(\ell)}, \mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}}$$
$$= [\psi_A^{(0)}, \mathsf{q}_1^{k_0}]\,(\psi_A^{(1)}, \mathsf{s}_{(1)}^{k_1})\cdots(\psi_A^{(\ell)}, \mathsf{s}_{(1)}^{k_\ell}).$$

Now, by (3.23), we have

$$[\psi_A^{(0)}, \mathsf{q}_1^{k_0}] = [\mathsf{q}_{a_{1,0}}\cdots \mathsf{q}_{a_{n,0}}, \mathsf{q}_1^{k_0}] = \begin{cases} 2^{k_0}\binom{k_0}{a_{1,0}\cdots a_{n,0}} & \text{if } k_0 = a_{1,0}+\cdots+a_{n,0},\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, by (3.21), we have, for $i = 1, \dots, \ell$,

$$(\psi_A^{(i)}, \mathsf{s}_{(1)}^{k_i})) = (\psi_A(1,i)\cdots\psi_A(n,i), \mathsf{s}_{(1)}^{k_i})) \begin{cases} \binom{k_i}{a_{1,i}\cdots a_{n,i}} & \text{if } k_i = a_{1,i}+\cdots+a_{n,i},\\ 0 & \text{otherwise.}\end{cases}$$

This implies the required equality. □

Recall the notation $|\mu, \boldsymbol{j}|_{\ell-1}$ from (1.5).

Theorem 3.36. *Let* $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. *Then*

$$(\Pi_{\mu,\boldsymbol{j}}, \Pi_{\omega_d})_{\mathsf{Sym}} = \binom{d}{\mu_1\cdots\mu_n} 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}.$$

Proof. By Proposition 3.34 and Corollary 3.32, we have that $(\Pi_{\mu,\boldsymbol{j}}, \Pi_{\omega_d})_{\mathsf{Sym}}$ equals

$$\sum_{A\in M(\mu,\boldsymbol{j})}\ \sum_{k_0+k_1+\cdots+k_\ell=d} 2^{k_1+\cdots+k_{\ell-1}}\binom{d}{k_0\, k_1\cdots k_\ell}(\psi_A, \mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}}.$$

By Lemma 3.35, this equals

$$\sum_{A\in M(\mu,\boldsymbol{j})} 2^{|a_{*,1}|+\cdots+|a_{*,\ell-1}|}\binom{d}{|a_{*,0}|\,|a_{*,1}|\cdots|a_{*,\ell}|}2^{|a_{*,0}|}\prod_{i\in I}\binom{|a_{*,i}|}{a_{*,i}}$$
$$= \sum_{A\in M(\mu,\boldsymbol{j})} 2^{|a_{*,0}|+|a_{*,1}|+\cdots+|a_{*,\ell-1}|}\frac{d!}{\prod_{i\in I}\prod_{r=1}^n a_{r,i}!}$$
$$= \sum_{A\in M(\mu,\boldsymbol{j})} 2^{d-|a_{*,\ell}|}\binom{d}{\mu_1\cdots\mu_n}\prod_{r=1}^n\binom{\mu_r}{a_{r,0}\cdots a_{r,\ell}},$$

and it remains to prove that

$$\sum_{A\in M(\mu,\boldsymbol{j})} 2^{d-|a_{*,\ell}|}\prod_{r=1}^n\binom{\mu_r}{a_{r,0}\cdots a_{r,\ell}} = 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}. \tag{3.37}$$

Define

$$d_j := \sum_{\substack{1\le r\le n\\ j_r=j}} \mu_r \qquad (j\in J).$$

In particular, $d_{\ell-1} = |\mu, \boldsymbol{j}|_{\ell-1}$ and $d_0 + d_1 + \cdots + d_{\ell-1} = d$. Note that permuting the parts of $(\mu_1, \ldots, \mu_n)$ and $(j_1, \ldots, j_n)$ by the same permutation in $\mathfrak{S}_n$ does not change the left-hand side of (3.37), so we may assume without loss of generality that $\boldsymbol{j} = (0^{n_0}, 1^{n_1}, \ldots, (\ell-1)^{n_{\ell-1}})$ with $n_0 + n_1 + \cdots + n_{\ell-1} = n$ and

$$\mu = (\lambda_1^{(0)}, \ldots, \lambda_{n_0}^{(0)}, \ldots, \lambda_1^{(\ell-1)}, \ldots, \lambda_{n_{\ell-1}}^{(\ell-1)})$$

with $(\lambda_1^{(j)}, \ldots, \lambda_{n_j}^{(j)}) \in \Lambda(n_j, d_j)$ for all $j \in J$. Then, the matrices $A \in M(\mu, \boldsymbol{j})$ look like

$$A = \begin{pmatrix} B^{(0)} \\ \vdots \\ B^{(\ell-1)} \end{pmatrix},$$

where, for each j, the matrix $B^{(j)} = (b_{r,i}^{(j)})_{1 \le r \le n_j,\, i \in I}$ is an arbitrary matrix with non-negative integer values such that $b_{r,i}^{(j)} = 0$ unless $i \in \{j, j+1\}$ and $b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}$ for all $r = 1, \ldots, n_j$. So, the left-hand side of (3.37) equals XY where

$$X := \prod_{j=0}^{\ell-2} \prod_{r=1}^{n_j} \sum_{b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}} 2^{\lambda_r^{(j)}} \binom{\lambda_r^{(j)}}{b_{r,j}^{(j)}\ b_{r,j+1}^{(j)}}$$

and

$$Y := \prod_{r=1}^{n_{\ell-1}} \sum_{b_{r,\ell-1}^{(\ell-1)} + b_{r,\ell}^{(\ell-1)} = \lambda_r^{(\ell-1)}} 2^{b_{r,\ell-1}^{(\ell-1)}} \binom{\lambda_r^{(\ell-1)}}{b_{r,\ell-1}^{(\ell-1)}\ b_{r,\ell}^{(\ell-1)}}.$$

Now, using the formula $\sum_{a+b=c} \binom{c}{a\ b} = 2^c$, we get

$$X = 2^{d_0 + \cdots + d_{\ell-2}} \prod_{j=0}^{\ell-2} \prod_{r=1}^{n_j} \sum_{b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}} 2^{\lambda_r^{(j)}} = 4^{d_0 + \cdots + d_{\ell-2}} = 4^{d - d_{\ell-1}} = 4^{d - |\mu, \boldsymbol{j}|_{\ell-1}},$$

and, using the formula $\sum_{a+b=c} 2^a \binom{c}{a\ b} = 3^c$, we get

$$Y = \prod_{r=1}^{n_{\ell-1}} 3^{\lambda_r^{(\ell-1)}} = 3^{d_{\ell-1}} = 3^{|\mu, \boldsymbol{j}|_{\ell-1}},$$

completing the proof. □

4. Fock space

4.1. ***Fock spaces $\mathscr{F}_q$ and $\mathscr{F}$.*** The (*q-deformed*) *level*-1 *Fock space* $\mathscr{F}_q$, as defined in [Kashiwara et al. 1996] (see also [Leclerc and Thibon 1997]), is the $\mathbb{Q}(q)$-vector space with basis $\{u_\lambda \mid \lambda \in \mathscr{P}_p\}$ labeled by the p-strict partitions

$$\mathscr{F}_q := \bigoplus_{\lambda\in\mathscr{P}_p} \mathbb{Q}(q)\cdot u_\lambda.$$

There is a structure of a $U_q(\mathfrak{g})$-module on $\mathscr{F}_q$ such that

$$E_i u_\lambda = \sum_{\mathtt{A}\in \mathrm{PRe}_i(\lambda)} d_{\mathtt{A}}(\lambda) u_{\lambda_{\mathtt{A}}}, \tag{4.1}$$

$$F_i u_\lambda = \sum_{\mathtt{B}\in \mathrm{PAd}_i(\lambda)} d^{\mathtt{B}}(\lambda) u_{\lambda^{\mathtt{B}}}, \tag{4.2}$$

$$T_i u_\lambda = q^{(\alpha_i\,|\,\Lambda_0-\mathrm{cont}(\lambda))} u_\lambda. \tag{4.3}$$

Example 4.4. In the set up of Example 3.19 we have

$$F_0 u_{(5,5,2)} = (1-q^4)u_{(6,5,2)} + u_{(5,5,2,1)}.$$

As established in [Kashiwara et al. 1996, Appendix D], there is a bilinear form $(\,\cdot\,,\cdot\,)$ on $\mathscr{F}_q$ which satisfies

$$(u_\lambda, u_\mu)_q = \delta_{\lambda,\mu}\|\lambda\|_q \tag{4.5}$$

and

$$(xv, w)_q = (v, \sigma_q(x)w)_q$$

for all $x \in U_q(\mathfrak{g})$ and $v, w \in \mathscr{F}_q$. The following well-known result allows us to identify $V_q(\Lambda_0)$ with the submodule of $\mathscr{F}_q$ generated by $u_\varnothing$, where $\varnothing$ stands for the partition (0) of 0; cf. [Kleshchev and Livesey 2022, Lemma 2.4.20].

Lemma 4.6. *There exists a unique isomorphism of $U_q(\mathfrak{g})$-modules $V_q(\Lambda_0) \xrightarrow{\sim} U_q(\mathfrak{g})\cdot u_\varnothing$ mapping $v_{+,q}$ onto $u_\varnothing$. Moreover, identifying $V_q(\Lambda_0)$ with the submodule $U_q(\mathfrak{g})\cdot u_\varnothing \subseteq \mathscr{F}_q$ via this isomorphism, the Shapovalov form $(\,\cdot\,,\cdot\,)_q$ on $V_q(\Lambda_0)$ is the restriction of the form $(\,\cdot\,,\cdot\,)_q$ on $\mathscr{F}_q$ to $V_q(\Lambda_0)$.*

We now apply the construction of Section 2.3 to go from the $U_q(\mathfrak{g})$-module $\mathscr{F}_q$ to the $\mathfrak{g}$-module $\mathscr{F}_q|_{q=1} = \mathbb{C}\otimes_{\mathbb{Z}[q,q^{-1}]}\mathscr{F}_{q,\mathbb{Z}[q,q^{-1}]}$ with the lattice $\mathscr{F}_{q,\mathbb{Z}[q,q^{-1}]} := \bigoplus_{\lambda\in\mathscr{P}_p}\mathbb{Z}[q,q^{-1}]\cdot u_\lambda$. We will denote $1\otimes u_\lambda \in \mathbb{C}\otimes_{\mathbb{Z}[q,q^{-1}]}\mathscr{F}_{q,\mathbb{Z}[q,q^{-1}]}$ again by u_λ. So we have a $\mathfrak{g}$-module

$$\mathscr{F}_q|_{q=1} = \bigoplus_{\lambda\in\mathscr{P}_p}\mathbb{C}\cdot u_\lambda$$

with the action

$$e_i u_\lambda = \sum_{\mathtt{A}\in \mathrm{PRe}_i(\lambda)} (d_{\mathtt{A}}(\lambda)|_{q=1}) u_{\lambda_{\mathtt{A}}}, \qquad f_i u_\lambda = \sum_{\mathtt{B}\in \mathrm{PAd}_i(\lambda)} (d^{\mathtt{B}}(\lambda)|_{q=1}) u_{\lambda^{\mathtt{B}}}, \tag{4.7}$$

and the form $(\cdot\,,\cdot) := (\cdot\,,\cdot)_q|_{q=1}$ which satisfies

$$(u_\lambda, u_\mu) = \delta_{\lambda,\mu}\|\lambda\|_q|_{q=1}$$

and $(xv, w) = (v, \sigma(x)w)$ for all $x \in \mathfrak{g}$ and $v, w \in \mathscr{F}_q|_{q=1}$.

Recalling (3.14) and (3.18), it is easy to see that $\mathscr{R} := \mathrm{span}_{\mathbb{C}}(u_\lambda \mid \lambda \in \mathscr{P}_p \setminus \mathscr{P}_0)$ is a $\mathfrak{g}$-submodule of $\mathscr{F}_q|_{q=1}$. Consider the *reduced Fock space*

$$\mathscr{F} := (\mathscr{F}_q|_{q=1})/\mathscr{R}.$$

Denoting $u_\lambda + \mathscr{R} \in (\mathscr{F}_q|_{q=1})/\mathscr{R}$ by u_λ again, we have that

$$\mathscr{F} = \bigoplus_{\lambda\in\mathscr{P}_0} \mathbb{C}\cdot u_\lambda$$

with the action of the Chevalley generators f_i given by

$$f_i u_\lambda = \sum_{\mathtt{B}\in A_i(\lambda)} c(\lambda, \mathtt{B}) u_{\lambda^{\mathtt{B}}}, \tag{4.8}$$

where

$$A_i(\lambda) := \{\mathtt{B} \in \mathrm{PAd}_i(\lambda) \mid \lambda^{\mathtt{B}} \in \mathscr{P}_0\}$$

and, recalling (3.3),

$$c(\lambda, \mathtt{B}) := \begin{cases} 2 & \text{if } h_p(\lambda^{\mathtt{B}}) = h_p(\lambda) - 1, \\ 1 & \text{otherwise.} \end{cases} \tag{4.9}$$

(We are not going to need the action of the Chevalley generators e_i.) Moreover, $\mathscr{F}$ inherits the form $(\cdot\,,\cdot)$ which satisfies

$$(u_\lambda, u_\mu) = \delta_{\lambda,\mu}2^{h_p(\lambda)}, \tag{4.10}$$

and $(xv, w) = (v, \sigma(x)w)$ for all $x \in \mathfrak{g}$ and $v, w \in \mathscr{F}$. We now have from Lemmas 4.6 and 2.5:

Lemma 4.11. *There is a unique isomorphism of $\mathfrak{g}$-modules from $V(\Lambda_0)$ to the submodule $U(\mathfrak{g})\cdot u_\varnothing \subseteq \mathscr{F}$ generated by $u_\varnothing$, mapping v_+ onto $u_\varnothing$. Moreover, identifying $V(\Lambda_0)$ with $U(\mathfrak{g})\cdot u_\varnothing \subseteq \mathscr{F}$ via this isomorphism, the Shapovalov form $(\cdot\,,\cdot)$ on $V(\Lambda_0)$ is the restriction of the form $(\cdot\,,\cdot)$ on $\mathscr{F}$ to $V(\Lambda_0)$.*

Let ρ be a $\bar{p}$-core. By definition, we have $h_p(\rho) = 0$ and $\rho \in \mathscr{P}_0$. Note that by Lemma 3.7, there is $w \in W$ such that $\mathrm{cont}(\rho) = \Lambda_0 - w\Lambda_0$. We have the element $v_w \in V(\Lambda_0)$ defined in Lemma 2.2, and the element $u_\rho \in \mathscr{F}$. The following lemma shows that these agree:

Lemma 4.12. *Let $\iota : V(\Lambda_0) \xrightarrow{\sim} U(\mathfrak{g})\cdot u_\varnothing$, $v_+ \mapsto u_\varnothing$, be the isomorphism of Lemma 4.11. If ρ is a $\bar{p}$-core and $w \in W$ is such that* $\mathrm{cont}(\rho) = \Lambda_0 - w\Lambda_0$ *then $\iota(v_w) = u_\rho$.*

Proof. By Lemma 2.2, for certain $a_1, \dots, a_l$, we have

$$\iota(v_w) = \iota(F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} v_+) = F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} \iota(v_+) = F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing.$$

It follows from Lemma 3.7 that $\mathscr{F}_{w\Lambda_0} = \mathscr{F}_{\Lambda_0 - \mathrm{cont}(\rho)}$ is 1-dimensional and hence spanned by u_ρ. It follows from the formulas (4.8) and (4.9) that $F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing = k u_\rho$ for $k \in \mathbb{Z}_{>0}$. Now, $(u_\rho, u_\rho) = 2^{h_p(\rho)} = 1$. On the other hand,

$$(F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing, F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing) = (\iota(v_w), \iota(v_w)) = (v_w, v_w) = 1$$

by Lemmas 4.11 and 2.2(iii). So $k = 1$. □

4.2. *The elements* χ_λ. In this subsection we introduce a new basis of $\mathscr{F}$. Recalling (3.3), (3.4) and (3.6), we consider the following rescalings of the basis vectors u_λ:

$$\chi_\lambda := 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} u_\lambda \quad (\lambda \in \mathscr{P}_0). \tag{4.13}$$

These elements correspond to the irreducible supercharacters of the double covers of symmetric groups, see [Fayers et al. 2024, §5], and the formula (4.13) was communicated to us by M. Fayers.

Set

$$a(\lambda, \mathtt{B}) := \begin{cases} 2 & \text{if } p_\lambda = 1 \text{ and } p_{\lambda^\mathtt{B}} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.14. *Let $i \in I$ and $\lambda \in \mathscr{P}_0$. Then $f_i \chi_\lambda = \sum_{\mathtt{B} \in A_i(\lambda)} a(\lambda, \mathtt{B}) u_{\lambda^\mathtt{B}}$.*

Proof. We have

$$\begin{aligned} f_i \chi_\lambda &= 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} f_i u_\lambda \\ &= 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} \sum_{\mathtt{B} \in A_i(\lambda)} c(\lambda, \mathtt{B}) u_{\lambda^\mathtt{B}} \\ &= 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} \sum_{\mathtt{B} \in A_i(\lambda)} c(\lambda, \mathtt{B}) 2^{-(p_{\lambda^\mathtt{B}} - h_p(\lambda^\mathtt{B}) - c^{\lambda^\mathtt{B}}_{\neq 0})/2} \chi_{\lambda^\mathtt{B}}, \end{aligned}$$

so we need to prove that

$$a(\lambda, \mathtt{B}) = 2^{(p_\lambda - p_{\lambda^\mathtt{B}} - h_p(\lambda) + h_p(\lambda^\mathtt{B}) - c^\lambda_{\neq 0} + c^{\lambda^\mathtt{B}}_{\neq 0})/2} c(\lambda, \mathtt{B}). \tag{4.15}$$

If $i \neq 0$ then either $p_\lambda = 1$ and $p_{\lambda^\mathtt{B}} = 0$, or $p_\lambda = 0$ and $p_{\lambda^\mathtt{B}} = 1$. Moreover, $c^{\lambda^\mathtt{B}}_{\neq 0} = c^\lambda_{\neq 0} + 1$. Hence

$$\begin{aligned} 2^{(p_\lambda - p_{\lambda^\mathtt{B}} - h_p(\lambda) + h_p(\lambda^\mathtt{B}) - c^\lambda_{\neq 0} + c^{\lambda^\mathtt{B}}_{\neq 0})/2} c(\lambda, \mathtt{B}) &= 2^{(p_\lambda - p_{\lambda^\mathtt{B}} + 1)/2} \\ &= \begin{cases} 2 & \text{if } p_\lambda = 1 \text{ and } p_{\lambda^\mathtt{B}} = 0, \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

which is $a(\lambda, B)$ as required.

On the other hand, if $i = 0$ then $c^{\lambda}_{\neq 0} = c^{\lambda^{B}}_{\neq 0}$. Let λ be written in the form (3.1), and consider the following three cases:

(1) $B = (m_1 + \cdots + m_k + 1, 1)$. In this case, we have $c(\lambda, B) = 1$, $p_\lambda = p_{\lambda^B}$ and $h_p(\lambda) = h_p(\lambda^B)$, which immediately gives (4.15).

(2) $B = (m_1 + \cdots + m_{r-1} + 1, 1)$ for some $1 \le r \le k$ and $p \mid l_r$. In this case, we have $h_p(\lambda^B) = h_p(\lambda) - 1$, $c(\lambda, B) = 2$ and $p_{\lambda^B} \neq p_\lambda$. If $p_{\lambda^B} = 1$ and $p_\lambda = 0$ then both sides of (4.15) equal 1. If $p_{\lambda^B} = 0$ and $p_\lambda = 1$ then both sides of (4.15) equal 2.

(3) $B = (m_1 + \cdots + m_{r-1} + 1, 1)$ for some $1 \le r \le k$ and $p \nmid l_r$. In this case, we have $h_p(\lambda^B) = h_p(\lambda) + 1$, $c(\lambda, B) = 1$ and $p_{\lambda^B} \neq p_\lambda$. If $p_{\lambda^B} = 1$ and $p_\lambda = 0$ then both sides of (4.15) equal 1. If $p_{\lambda^B} = 0$ and $p_\lambda = 1$ then both sides of (4.15) equal 2. □

5. Shapovalov form for RoCK weights

Suppose that $\theta \in Q_+$ satisfies $V(\Lambda_0)_{\Lambda_0-\theta} \neq 0$. Then $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some w in the Weyl group W, and unique $d \in \mathbb{Z}_{\ge 0}$. We say that θ is *RoCK* if $(\theta \mid \alpha_0^\vee) \ge 2d$ and $(\theta \mid \alpha_i^\vee) \ge d - 1$ for $i = 1, \dots, \ell$. This is equivalent to the cyclotomic quiver Hecke superalgebra $R^{\Lambda_0}_\theta$ being a *RoCK block*, as defined in [Kleshchev and Livesey 2022, Section 4.1].

Throughout Section 5, we fix a RoCK weight $\theta \in Q_+$, so that $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some $w \in W$ and $d \in \mathbb{Z}_{\ge 0}$, and $(\theta \mid \alpha_0^\vee) \ge 2d$, $(\theta \mid \alpha_i^\vee) \ge d - 1$ for $i = 1, \dots, \ell$.

We have the element $v_w \in V(\Lambda_0)_{w\Lambda_0}$ defined in Lemma 2.2.

5.1. ***Computation of $f(\mu, j)u_\rho$.*** For each $m \in \mathbb{Z}_{\ge 0}$, $j \in J$, $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$, recall the divided power monomials $f(m, j)$ and $f(\mu, \boldsymbol{j})$ defined in (1.2) and (1.3). We also have a sum $f(\omega_d)$ of monomials defined in (1.4).

Let $\mu \in \mathscr{P}_0(\rho, c)$ with $c \le d$. Recall the notion of a p-quotient $\mathrm{quot}(\mu) = (\mu^{(0)}, \dots, \mu^{(\ell)})$ of μ from (3.9) and the notation (3.5). Recall that $\mathrm{quot}(\mu) \in \mathscr{P}^I_0(c)$ since μ is strict. The following result follows immediately from [Fayers et al. 2024, Proposition 6.6 and (5.1)] and Lemma 4.14.

Lemma 5.1. *Let $j \in J$ and $c, k \in \mathbb{Z}_{\ge 0}$ satisfy $c + k \le d$. For $\alpha \in \mathscr{P}_0(\rho, c)$, in the reduced Fock space $\mathscr{F}$, we have*

$$f(k, j)\chi_\alpha = \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)}) + (k(2\ell-1) + h(\alpha^{(0)}) - h(\lambda^{(0)}) + p_\alpha - p_\lambda)/2} \chi_\lambda,$$

where the sum is over all $\lambda \in \mathscr{P}_0(\rho, c + k)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ *is obtained from* $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ *by adding k nodes to the components $\alpha^{(j)}$ and $\alpha^{(j+1)}$, with no two nodes added in the same column of $\alpha^{(j)}$ or in the same row of $\alpha^{(j+1)}$.*

Corollary 5.2. *Let $j \in J$ and $c, k \in \mathbb{Z}_{\geq 0}$ satisfy $c + k \leq d$. For $\alpha \in \mathscr{P}_0(\rho, c)$, in the reduced Fock space $\mathscr{F}$, we have*

$$f(k, j)u_\alpha = \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})-h(\lambda^{(0)})} u_\lambda,$$

where the sum is over all $\lambda \in \mathscr{P}_0(\rho, c+k)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ *is obtained from* $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ *by adding k nodes to the components $\alpha^{(j)}$ and $\alpha^{(j+1)}$, with no two nodes added in the same column of $\alpha^{(j)}$ or in the same row of $\alpha^{(j+1)}$.*

Proof. Note that $h_p(\alpha) = h(\alpha^{(0)})$. Moreover, for λ's appearing in the sum, we have $c^\lambda_{\neq 0} - c^\alpha_{\neq 0} = k(2\ell - 1)$ and $h_p(\lambda) = h(\lambda^{(0)})$. So we have by (4.13) and Lemma 5.1,

$$\begin{aligned}
f(k, j)u_\alpha &= 2^{(-p_\alpha + h_p(\alpha) + c^\alpha_{\neq 0})/2} f(k, j)\chi_\alpha \\
&= 2^{(-p_\alpha + h_p(\alpha) + c^\alpha_{\neq 0})/2} \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)}) + (k(2\ell-1) + h(\alpha^{(0)}) - h(\lambda^{(0)}) + p_\alpha - p_\lambda)/2} \chi_\lambda \\
&= 2^{q(\lambda^{(0)}/\alpha^{(0)}) + h(\alpha^{(0)}) + (k(2\ell-1) - h(\lambda^{(0)}) - p_\lambda + c^\alpha_{\neq 0})/2} \sum_\lambda 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} u_\lambda \\
&= \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)}) + h(\alpha^{(0)}) - h(\lambda^{(0)})} u_\lambda,
\end{aligned}$$

as required. □

Lemma 5.3. *Let $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. Then*

$$f(\mu, \boldsymbol{j})u_\rho = \sum_{\lambda \in \mathscr{P}_0(\rho, d)} K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) 2^{-h(\lambda^{(0)})} u_\lambda.$$

Proof. We apply induction on n. For the induction base case $n = 1$ we apply Corollary 5.2 to see that

$$f(\mu_1, j_1)u_\rho = \sum_\lambda 2^{q(\lambda^{(0)}/\rho^{(0)}) + h(\rho^{(0)}) - h(\lambda^{(0)})} u_\lambda = \sum_\lambda 2^{q(\lambda^{(0)}/\rho^{(0)})} 2^{-h(\lambda^{(0)})} u_\lambda,$$

where the sums are over all $\lambda \in \mathscr{P}_0(\rho, d)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\mathrm{quot}(\rho) = (\varnothing, \dots, \varnothing)$ by adding k nodes to the components $\rho^{(j)} = \varnothing$ and $\rho^{(j+1)} = \varnothing$, with no two nodes added in the same column of $\rho^{(j)}$ or in the same row of $\rho^{(j+1)}$, and

$$q(\lambda^{(0)}/\rho^{(0)}) = |\{r \in \mathbb{Z}_{>0} \mid \lambda^{(0)} \text{ contains a node in column } r \text{ but not in column } r+1\}|.$$

It follows from the definitions that the λ's appearing in the latter sum are exactly the λ's with $K(\mathrm{quot}(\lambda); (\mu_1), j_1) \neq 0$, and for those λ we have $K(\mathrm{quot}(\lambda); (\mu_1), j_1) = 2^{q(\lambda^{(0)}/\rho^{(0)})}$. This establishes the induction base.

For the inductive step, suppose $n > 1$. Let

$$\nu := (\mu_1, \dots, \mu_{n-1}), \quad \boldsymbol{k} := (j_1, \dots, j_{n-1}), \quad c = \mu_1 + \cdots + \mu_{n-1}.$$

In particular,

$$f(\mu, \boldsymbol{j}) = f(\mu_n, j_n) f(\nu, \boldsymbol{k}).$$

By the inductive assumption, we have

$$\begin{aligned} f(\mu, \boldsymbol{j}) u_\rho &= f(\mu_n, j_n) f(\nu, \boldsymbol{k}) u_\rho \\ &= \sum_{\alpha \in \mathscr{P}_0(\rho, c)} K(\mathrm{quot}(\alpha); \nu, \boldsymbol{k}) 2^{-h(\alpha^{(0)})} f(\mu_n, j_n) u_\alpha \\ &= \sum_{\alpha \in \mathscr{P}_0(\rho, c)} K(\mathrm{quot}(\alpha); \nu, \boldsymbol{k}) 2^{-h(\alpha^{(0)})} \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)}) + h(\alpha^{(0)}) - h(\lambda^{(0)})} u_\lambda, \end{aligned}$$

where the second sum is over all $\lambda \in \mathscr{P}_0(\rho, d)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ by adding μ_n nodes to the components $\alpha^{(j_n)}$ and $\alpha^{(j_n+1)}$, with no two nodes added in the same column of $\alpha^{(j_n)}$ or in the same row of $\alpha^{(j_n+1)}$. It remains to note that λ's appearing in the expression above are exactly those with $K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) \neq 0$, and for such λ we have

$$K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) = \sum_\alpha 2^{q(\lambda^{(0)}/\alpha^{(0)})} K(\mathrm{quot}(\alpha); \nu, \boldsymbol{k}),$$

where the sum is over all $\alpha \in \mathscr{P}_0(\rho, c)$ such that $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ is obtained from $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ by removing μ_n nodes from the components $\lambda^{(j_n)}$ and $\lambda^{(j_n+1)}$, with no two nodes removed in the same column of $\lambda^{(j_n)}$ or in the same row of $\lambda^{(j_n+1)}$. □

Recall the definition of $\Pi_{\mu,\boldsymbol{j}}$ from (3.27).

Corollary 5.4. *Let* $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(m, d)$ *and* $(\nu, \boldsymbol{i}) \in \Lambda^{\mathrm{col}}(n, d)$. *Then*

$$(f(\mu, \boldsymbol{j}) u_\rho, f(\nu, \boldsymbol{i}) u_\rho) = (\Pi_{\mu,\boldsymbol{j}}, \Pi_{\nu,\boldsymbol{i}})_{\mathsf{Sym}}.$$

Proof. For $\lambda \in \mathscr{P}_0(\rho, d)$, we have $h_p(\lambda) = h(\lambda^{(0)})$. So, by Lemma 5.3 and (4.10), taking into account the bijection (3.11), we have

$$\begin{aligned} &(f(\mu, \boldsymbol{j}) u_\rho, f(\nu, \boldsymbol{i}) u_\rho) \\ &\qquad = \sum_{\lambda \in \mathscr{P}_0(\rho, d)} K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) K(\mathrm{quot}(\lambda); \nu, \boldsymbol{i}) 2^{-2h(\lambda^{(0)})} (u_\lambda, u_\lambda) \\ &\qquad = \sum_{\underline{\lambda} \in \mathscr{P}_0^I(d)} K(\underline{\lambda}; \mu, \boldsymbol{j}) K(\underline{\lambda}; \nu, \boldsymbol{i}) 2^{-h(\lambda^{(0)})}. \end{aligned}$$

Since the bases $\{\pi_{\underline{\lambda}} \mid \underline{\lambda} \in \mathscr{P}_0^I\}$ from (3.24) and $\{\kappa_{\underline{\lambda}} \mid \underline{\lambda} \in \mathscr{P}_0^I\}$ from (3.25) are dual to each other with respect to the inner product $(\cdot\,,\cdot)_{\mathsf{Sym}}$, we have

$$\begin{aligned}\sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} & K(\underline{\lambda};\mu,\boldsymbol{j})K(\underline{\lambda};\nu,\boldsymbol{i})2^{-h(\lambda^{(0)})}\\ &= \Bigg(\sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\mu,\boldsymbol{j})2^{-h(\lambda^{(0)})}\kappa_{\underline{\lambda}}, \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\nu,\boldsymbol{i})\pi_{\underline{\lambda}}\Bigg)_{\mathsf{Sym}}\\ &= \Bigg(\sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\mu,\boldsymbol{j})\pi_{\underline{\lambda}}, \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\nu,\boldsymbol{i})\pi_{\underline{\lambda}}\Bigg)_{\mathsf{Sym}}\\ &= (\Pi_{\mu,\boldsymbol{j}},\Pi_{\nu,\boldsymbol{i}})_{\mathsf{Sym}},\end{aligned}$$

as required. □

Theorem 5.5. *Let* $(\mu,\boldsymbol{j}) \in \Lambda^{\mathrm{col}}(m,d)$. *Then*

$$(f(\mu,\boldsymbol{j})u_\rho, f(\omega_d)u_\rho) = \binom{d}{\mu_1\cdots\mu_n} 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}.$$

Proof. By Corollary 5.4, we have $(f(\mu,\boldsymbol{j})u_\rho, f(\omega_d)u_\rho) = (\Pi_{\mu,\boldsymbol{j}},\Pi_{\omega_d})_{\mathsf{Sym}}$, and the theorem follows from Theorem 3.36. □

Recall the vector $v_w \in V(\Lambda_0)_{w\Lambda_0}$ defined in Lemma 2.2.

Theorem 5.6. *Let* $(\mu,\boldsymbol{j}) \in \Lambda^{\mathrm{col}}(m,d)$. *Then*

$$(f(\mu,\boldsymbol{j})v_w, f(\omega_d)v_w) = \binom{d}{\mu_1\cdots\mu_n} 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}.$$

Proof. In view of Lemmas 4.11 and 4.12, this follows from Theorem 5.5. □

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