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To the memory of Gary Seitz

Let Γ be a cocompact, oriented Fuchsian group which is not on an explicit finite list of possible exceptions and q a sufficiently large prime power not divisible by the order of any nontrivial torsion element of Γ. Then $|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))| \sim c_{q,n} q^{(1-\chi(\Gamma))n^2}$, where $c_{q,n}$ is periodic in n. Within a fixed congruence class for q and for n, $c_{q,n}$ can be expressed as a Puiseux series in $1/q$. Moreover, this series is essentially the q-expansion of a meromorphic modular form of half-integral weight.

1. Introduction

Let Γ be a cocompact and oriented Fuchsian group (which, in what follows, we shall call simply a Fuchsian group). Concretely, this means that Γ has a presentation

$$\langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_r | z_1^{a_1}, \ldots, z_r^{a_r}, [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_r \rangle,$$

where $a_1, a_2, \ldots, a_r$ is a fixed (possibly empty) nondecreasing sequence of integers $a_i \geq 2$ such that the Euler characteristic

$$\chi(\Gamma) := 2 - 2g - \sum_{i=1}^{r} \Big(1 - \frac{1}{a_i}\Big)$$

is negative. Let $\mathbb{F}_q$ be a finite field. We investigate the asymptotic growth in n of

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the number of homomorphisms from Γ to the group $\mathrm{GL}_n(q)$, which we denote G_n when the value of q is understood.

There are two complementary points of view. On the one hand we can fix n and consider the homomorphism scheme $\mathrm{Hom}(\Gamma, \mathrm{GL}_n) := \mathrm{Hom}(\Gamma, \mathrm{GL}_{n,\mathbb{Z}})$, which is defined over $\mathbb{Z}$. For a fixed characteristic $p > 0$, we can think of the fiber of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ over $\mathrm{Spec}\,\mathbb{F}_p$ as the variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_{n,\mathbb{F}_p})$ of homomorphisms from Γ to GL_n over $\mathbb{F}_p$. (Note that in this paper, a *variety* will be just an affine scheme of finite type over a field; in particular, it need not be either irreducible or reduced.) Applying the Lang–Weil theorem to this variety as q ranges over powers of p, we see that the number of homomorphisms $\rho\colon \Gamma \to \mathrm{GL}_n(q)$ determines its *dimension*, by which we mean the maximum dimension of any of its irreducible components.

On the other hand, for fixed q, we can partition homomorphisms ρ according to the r-tuple of G_n-conjugacy classes

$$(C_1, \dots, C_r) = (\rho(z_1)^{G_n}, \dots, \rho(z_r)^{G_n}).$$

Each C_i must consist of elements of order dividing a_i. For given $(C_1, \dots, C_r)$ satisfying this divisibility condition, the number of homomorphisms $\Gamma \to G_n$ with $\rho(z_i) \in C_i$ for all i is given by a theorem of Hurwitz [Liebeck and Shalev 2004, Proposition 3.2]:

$$|G_n|^{2g-1}|C_1|\cdots|C_r| \sum_{\chi\in\mathrm{Irr}(G_n)} \frac{\chi(C_1)\cdots\chi(C_r)}{\chi(1)^{2g+r-2}}. \tag{1-1}$$

Summing (1-1) over all possible r-tuples $(C_1, \dots, C_r)$ we obtain a formula which can potentially be used for fixed q to understand the asymptotic behavior of $|\mathrm{Hom}(\Gamma, G_n)|$ as $n \to \infty$.

These two ways of counting $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))$ are in some sense complementary. For instance, just as we can use character methods to determine the dimension of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$, we can use the dimension of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ to get an upper bound on $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))$ for all large q. For the large n limit, the first way of counting seems to be the more appropriate, and an analysis of (1-1) has led us to this conjecture:

Conjecture 1. *Let A denote the least common multiple of $a_1, \dots, a_r$, which we take to be 1 if $r = 0$. Let q be a prime power relatively prime to A.*

(a) *There exists a $2A$-periodic sequence $c_{q,1}, c_{q,2}, \dots$ of positive numbers such that*

$$|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))| \sim c_{q,n} q^{(1-\chi(\Gamma))n^2}$$

uniformly in q and Γ, that is,

$$\frac{|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))|}{c_{q,n} q^{(1-\chi(\Gamma))n^2}}$$

approaches 1 in the limit as $n \to \infty$, uniformly in q and Γ.

(b) *There exist a 2-dimensional array* $e_{\Gamma,q,n}$ *of rational numbers and a 2-dimensional array* $f_{\Gamma,q,n}$ *of half-integral weight meromorphic modular forms, periodic in both* q *and* n, *such that*

$$c_{q,n} = (q-1)q^{e_{\Gamma,q,n}} f_{\Gamma,q,n}\left(\frac{i \log q}{2\pi}\right).$$

Moreover, $f_{\Gamma,q,n}$ *is holomorphic on the upper half plane and has integer Fourier coefficients at* $i\infty$.

The problem of estimating the number of representations of a given Fuchsian group over a finite field seems to have been first considered by Liebeck and Shalev [2005b]. There are a number of significant differences in emphasis between that paper and ours; Γ need not be oriented in their paper, and the target of homomorphisms from Γ could be a quasisimple group $G(q)$ instead of $\mathrm{GL}_n(q)$. They were primarily interested in the "geometric" direction, that is, n fixed and $q \to \infty$. A key limitation of their paper is that their method requires $g \geq 2$.

Under this hypothesis, they showed that the contribution in (1-1) from nonlinear characters is negligible, which reduces the problem of estimating $|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))|$ to that of estimating the numbers $j_{q,n}(a_i)$ of elements $x \in \mathrm{GL}_n(q)$ satisfying $x^{a_i} = 1$. They gave an asymptotic formula for $j_{q,n}(a)$ when n is fixed and $q \to \infty$, using work of Lawther [2005]. In the case $r = 0$ (the surface group case) they proved that $|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))|$ is asymptotic to $(q-1)|\mathrm{GL}_n(q)|^{2g-1}$. If $\eta(z)$ is the Dedekind function, then

$$\frac{|\mathrm{GL}_n(q)|}{q^{1/24}\eta\left(\frac{i \log q}{2\pi}\right)} = q^{n^2} \prod_{i=n+1}^{\infty} (1-q^{-i})^{-1} \sim q^{n^2}.$$

Setting $e_{\Gamma,q,n} = \frac{1}{24}(2g-1)$ and $f_{\Gamma,q,n} = \eta^{2g-1}$ for all q, n, we deduce Conjecture 1 for surface groups. In general, their analysis depends crucially on the fact that for $g \geq 2$, the trivial upper bound on $|\chi(C_i)|$ is good enough to allow us to ignore nonlinear characters of G_n. This is certainly not the case when $g = 1$, let alone when $g = 0$. However, the new character bounds developed by Bezrukavnikov, Liebeck, Shalev, and Tiep [Bezrukavnikov et al. 2018] give us hope of making progress even for $g = 0$.

Using these bounds, Liebeck, Shalev, and Tiep proved [Liebeck et al. 2020, Theorem 1.1] that for every Γ satisfying $\chi(\Gamma) < -2$, if $q \equiv 1 \pmod{A}$, then

$$|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))| \leq f(n)q^{(1-\chi(\Gamma))n^2+1}$$

where $f(n)$ does not depend on q. This immediately gives an upper bound for the dimension of the representation variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_{n,K})$ where K is any field in which $A \neq 0$, provided that n is sufficiently large:

$$\dim \mathrm{Hom}(\Gamma, \mathrm{GL}_{n,K}) \leq (1-\chi(\Gamma))n^2 + 1.$$

They also proved a lower bound on dimension:

$$\dim \mathrm{Hom}(\Gamma, \mathrm{GL}_{n,K}) \geq (1-\chi(\Gamma))(n^2-1) - \sum_i a_i.$$

Using ideas from [Bezrukavnikov et al. 2018; Taylor and Tiep 2020], we prove a new exponential character bound Theorem 2.9, which applies to all semisimple elements and which plays an essential role in the proof of the main theorems of this paper. The reason we can prove Conjecture 1 only when q is sufficiently large is that the exponent in our bound only approaches its optimal value as $q \to \infty$.

Proposition 3.2 gives a list, consisting of thirty-one triangle groups and one quadrilateral group, where even for large q, our bounds are not strong enough to prove the conjecture. When Γ is not on this list and q is sufficiently large and relatively prime to the a_i, $|\mathrm{Hom}(\Gamma, G_n)|$ behaves as predicted.

Theorem A. *There exists an absolute constant q_0 such that if Γ is a Fuchsian group which is not on the finite list of groups excluded by Proposition 3.2, then Conjecture 1 holds for Γ for all prime powers $q > q_0$ which are prime to A.*

In particular, the theorem holds for all Fuchsian groups Γ with Euler characteristic less than $-\frac{1}{6}$.

We deduce Theorem A from an analogue of [Liebeck and Shalev 2005b, Theorem 1.2 (i)]. Let $J_{q,n}(a_1, \ldots, a_r)$ denote the cardinality of the set

$$\Big\{(t_1, \ldots, t_r) \in \mathrm{GL}_n(q) \mid t_i^{a_i} = 1 \ \forall i, \ \prod_i \det(t_i) = 1\Big\}. \tag{1-2}$$

Theorem B. *If Γ is not on the excluded list of Proposition 3.2, and $q > q_0$ is prime to A then*

$$|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))| = (1+o(1))(q-1) J_{q,n}(a_1, \ldots, a_r) |\mathrm{GL}_n(q)|^{2g-1},$$

where the term $o(1)$ does not depend on q.

Theorem A allows us to compute the exact dimension of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ when n is sufficiently large. Given Γ and n, we define $\sigma_{\Gamma,n}$ to be either 1 or -1 according to the rule that it is -1 if and only if $a_i \in 2\mathbb{Z}$ implies $n/a_i \in \mathbb{Z}$, and

$$\sum_{\{i \mid a_i \in 2\mathbb{Z}\}} \frac{n}{a_i} \in 1 + 2\mathbb{Z}.$$

Let $\{x\}$ denote the fractional part of x. We have:

Theorem C. *There exists an absolute constant N such that if Γ is not on the excluded list of Proposition 3.2, $n > N$, and K is any field of characteristic $p \geq 0$, such that $p \nmid a_i$ for any i, we have*

$$\dim \mathrm{Hom}(\Gamma, \mathrm{GL}_{n,K}) = \sigma_{\Gamma,n} + (1-\chi(\Gamma))n^2 - \sum_{i=1}^r a_i \Big\{\frac{n}{a_i}\Big\}\Big\{-\frac{n}{a_i}\Big\}.$$

In particular,

$$\dim \operatorname{Hom}(\Gamma, \mathrm{GL}_{n,K}) \geq -\tfrac{1}{2} + (1-\chi(\Gamma))n^2 - \sum_{i=1}^{r} \tfrac{1}{4} a_i.$$

Unfortunately, the letter q has a standard meaning both for finite fields and for modular forms. We use it only in the former sense, but we evaluate modular forms f at $i \log q/(2\pi)$, which amounts to plugging $1/q$ into the q-expansion for f.

2. An asymptotic character bound

The goal of this section is to prove an asymptotic version of the character bounds in [Bezrukavnikov et al. 2018, Theorem 1.1] and [Taylor and Tiep 2020, Theorem 1.9] when G is a finite group of Lie-type A. We will achieve this by combining the approach of [Bezrukavnikov et al. 2018] with the *character level* approach developed in [Guralnick et al. 2020] to bound $|\chi(g)|$.

To this end, let us recall the approach of [Bezrukavnikov et al. 2018]. Throughout this section, q is a prime power and $\mathcal{G} = \mathcal{G}(\overline{\mathbb{F}}_q)$ is the group of $\overline{\mathbb{F}}_q$-points of a connected reductive $\mathbb{F}_q$-group scheme. We assume $G = \mathcal{G}(\mathbb{F}_q) = \mathcal{G}^F$ is the finite group of $\mathbb{F}_q$-points, where $F : \mathcal{G} \to \mathcal{G}$ is the Frobenius endomorphism determined by its structure as a scheme over $\mathbb{F}_q$.

The main case of interest to us will be when the underlying group scheme is GL_n^{ϵ}, where $\epsilon \in \{+,-\}$ and we set $\mathrm{GL}_n^{+} := \mathrm{GL}_n$, the general linear group, and $\mathrm{GL}_n^{-} := \mathrm{GU}_n$, the general unitary group. In this setting,

$$\mathcal{G} = \mathcal{G}_n := \mathrm{GL}_n(\overline{\mathbb{F}}_q)$$

is the general linear group of dimension $n > 0$ and F is either F_q or σF_q, where $F_q : \mathcal{G} \to \mathcal{G}$ is the standard Frobenius endomorphism and $\sigma : \mathcal{G} \to \mathcal{G}$ is the inverse transpose automorphism.

Suppose $L = \mathcal{L}^F$, where $\mathcal{L} < \mathcal{G}$ is a proper F-stable Levi subgroup of $\mathcal{G}$. Assume $g \in \mathcal{G}^F$ is an element such that $\boldsymbol{C}_{\mathcal{G}}(g) \leqslant \mathcal{L}$. By [Taylor and Tiep 2020, Lemma 13.3], for every irreducible character χ of G, we have

$$\chi(g) = {}^*R^{\mathcal{G}}_{\mathcal{L}}(\chi)(g) = \sum_{\eta \in \mathrm{Irr}(L)} \langle \eta, {}^*R^{\mathcal{G}}_{\mathcal{L}}(\chi) \rangle \eta(g), \tag{2-1}$$

where ${}^*R^{\mathcal{G}}_{\mathcal{L}}$ denotes Deligne–Lusztig restriction. We also write $R^{\mathcal{G}}_{\mathcal{L}}$ for Deligne–Lusztig induction.

Following [Bezrukavnikov et al. 2018, Theorem 1.1], we define the constant $\alpha(L)$ to be the maximum over nontrivial unipotent elements $u \in L$ of

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}};$$

if L contains no such elements we take $\alpha(L) = 0$. Here $u^{\mathcal{G}}$ denotes the $\mathcal{G}$-conjugacy class of u, and similarly $u^{\mathcal{L}}$ denotes the $\mathcal{L}$-conjugacy class. From the proof of [Bezrukavnikov et al. 2018, Theorem 1.1], see also [Taylor and Tiep 2020, §2], we get

$$|\eta(g)| \leqslant \eta(1) \leqslant B_1 \left(\frac{q+1}{q-1}\right)^{D/2} \chi(1)^{\alpha(L)}, \tag{2-2}$$

for any $\eta \in \mathrm{Irr}(L)$ with $\langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle \neq 0$, where $B_1 > 0$ is a constant that depends on $\mathcal{G}^F$. Furthermore, $D = \dim v^{\mathcal{G}}$, where $v^{\mathcal{G}} = \mathcal{O}_\chi^*$ is the *wave front set* of χ, defined by work of Kawanaka [1986], Lusztig [1984], and Taylor [2016].

If $f_\eta \in \mathbb{Q}[X]$ is the degree polynomial of η, so that $\eta(1) = f_\eta(q)$, then the constant B_1 is chosen such that $B_1 f_\eta \in \mathbb{Z}[X]$. When the underlying group scheme is GL_n^ϵ we have that $\mathcal{L}^F$ is a direct product of groups $\mathrm{GL}_{n_i}^{\epsilon_i}(q)$. Therefore, in this case, the constant B_1 can be taken to be 1 because the degree polynomial of any irreducible character of $\mathcal{L}^F$ is already contained in $\mathbb{Z}[X]$.

Recall that ${}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)$ is a virtual character but is a true character if $\mathcal{L}$ is split. To bound $|\chi(g)|$ it suffices, by the triangle inequality, (2-1), and (2-2), to bound

$$\sum_{\eta\in\mathrm{Irr}(L)} |\langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle| \leqslant \sum_{\eta\in\mathrm{Irr}(L)} \langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle^2 = \langle {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi), {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle, \tag{2-3}$$

where $\langle -, -\rangle$ is the usual inner product on class functions.

We know from [Bezrukavnikov et al. 2018, Proposition 2.2] and its proof, as well as the arguments in [Taylor and Tiep 2020, §13], that (2-3) is always bounded above by

$$(n!)^2. \tag{2-4}$$

However, we can do significantly better if n is large compared to both q and the *true level*

$$\mathfrak{l}^*(\chi) = j$$

of χ, as defined in [Guralnick et al. 2020, Definition 1(i)]. When $G = \mathrm{GL}_n(q)$ then j is the smallest integer for which χ is a constituent of τ^j, where $\tau(g)$ is the number of fixed points of g acting on the natural module $V = \mathbb{F}_q^n$ of G.

In the next subsections we give upper bounds for (2-3) that incorporate the true level of χ.

2.1. ***Elements with split centralizer in*** $\mathbf{GL}_n(q)$**.** In this subsection we consider the group scheme GL_n so that

$$G = G_n := \mathrm{GL}_n(q).$$

Fix a proper split Levi subgroup $\mathcal{L}$, and let

$$L = \mathcal{L}^F = \mathrm{GL}_{m_1}(q) \times \mathrm{GL}_{m_2}(q) \times \cdots \times \mathrm{GL}_{m_t}(q) \subset G, \tag{2-5}$$

where $m_i \in \mathbb{Z}_{\geq 1}$ and $\sum_{i=1}^{t} m_i = n$. In this case ${}^*R^{\mathcal{G}}_{\mathcal{L}}$ is just Harish-Chandra restriction. With $1 \leq j < \frac{1}{2}n$ fixed, consider a split Levi subgroup $\mathcal{M}$ and set

$$M = \mathcal{M}^F \cong \mathrm{GL}_j(q) \times \mathrm{GL}_{n-j}(q) \subset G.$$

By [Guralnick et al. 2020, Theorem 3.9(i)], $\mathfrak{l}^*(\chi) = j$ implies that χ is an irreducible constituent of the Harish-Chandra induction

$$R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}})$$

for a unique irreducible character α of G_j. Conjugating $\mathcal{M}$ by a suitable element $g \in G$, we may assume that L and M are block-diagonal subgroups in the same basis $(e_1, e_2, \dots, e_n)$ of V.

To bound (2-3), it therefore suffices to bound

$$\langle {}^*R^{\mathcal{G}}_{\mathcal{L}} R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}}), {}^*R^{\mathcal{G}}_{\mathcal{L}} R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}})\rangle.$$

By the Mackey formula for Harish-Chandra restriction and induction [Dipper and Fleischmann 1992, Theorem 1.14],

$$(2\text{-}6) \qquad {}^*R^{\mathcal{G}}_{\mathcal{L}} R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}}) = \sum_{x \in L \backslash \mathcal{S}(\mathcal{L},\mathcal{M})/M} R^{\mathcal{L}}_{\mathcal{L}\cap {}^x\mathcal{M}} {}^*R^{{}^x\mathcal{M}}_{\mathcal{L}\cap {}^x\mathcal{M}}((\alpha \boxtimes 1_{G_{n-j}})^x),$$

where $\mathcal{S}(\mathcal{L}, \mathcal{M})$ is the set of elements $y \in G$ such that $\mathcal{L} \cap {}^y\mathcal{M}$ contains a maximal torus of $\mathcal{G}$, and the summation runs through the (L, M) double cosets of this set.

For our pair of split Levi subgroups $(\mathcal{L}, \mathcal{M})$, there is an explicit description of $L \backslash \mathcal{S}(\mathcal{L}, \mathcal{M})/M$, as described in [Brundan et al. 2001, §2.2c]. Embed the symmetric group S_n in G_n via permutation matrices, and consider the Young subgroups

$$\mathsf{S}_\lambda = \mathsf{S}_{m_1} \times \mathsf{S}_{m_2} \times \cdots \times \mathsf{S}_{m_t}, \qquad \mathsf{S}_\mu = \mathsf{S}_j \times \mathsf{S}_{n-j}$$

of the embedded S_n. Then in (2-6) we can just choose x as representatives of the set $\mathsf{S}_\lambda \backslash \mathsf{S}_n / \mathsf{S}_\mu$, one for each double coset. The set of double cosets $\mathsf{S}_\lambda \backslash \mathsf{S}_n / \mathsf{S}_\mu$ is in bijection with S_λ-orbits on the set of $\mathsf{S}_n / \mathsf{S}_\mu$, which may be identified with the set of j-subsets of $\{1, 2, \dots, n\}$. Hence each such double coset can be labeled uniquely by a t-tuple

$$(2\text{-}7) \qquad \kappa = (k_1, k_2, \dots, k_t), \quad 0 \leq k_i \leq m_i, \ \sum_{i=1}^{t} k_i = j.$$

Correspondingly, we can choose $x = x_\kappa$ to be the element of G that sends the first j basis vectors $e_1, \dots, e_j$ of V to

$$e_1, \dots, e_{k_1}, e_{m_1+1}, e_{m_1+2}, \dots, e_{m_1+k_2}, \dots, e_{m_1+\cdots+m_{t-1}+1}, \dots, e_{m_1+m_2+\cdots+m_{t-1}+k_t}$$

in the increasing order of the subscripts, and sends the last $n - j$ basis vectors $e_{j+1}, \dots, e_n$ to the remaining $n - j$ basis vectors, again in the increasing order of the subscripts. We will say that $x_\kappa(e_i) = e_{x_\kappa(i)}$, $1 \leq i \leq n$.

For the reader's convenience, let us give a justification for this statement in the case $q \geq 3$. Suppose $y \in G$ is such that $\mathcal{L} \cap {}^y\mathcal{M}$ contains a maximal torus $\mathcal{T}$ of $\mathcal{G}$. Then $\mathcal{T}$ is a maximal torus of $(\mathcal{L} \cap {}^y\mathcal{M})^\circ$ which is F-stable and connected. By the Lang–Steinberg theorem, conjugating $\mathcal{T}$ suitably, we may assume that it is F-stable. Then

$$T := \mathcal{T}^F \cong C_{(q^{a_1})-1} \times C_{(q^{a_2})-1} \times \cdots \times C_{(q^{a_s})-1}$$

for some integers $a_1, a_2, \ldots, a_s \geq 1$. Since $q \geq 3$, all cyclic direct factors in this decomposition are nontrivial, and hence $V = \mathbb{F}_q^n$ is a direct sum of s simple $\mathbb{F}_q T$-modules $W_1, \ldots, W_s$ of dimension $a_1, a_2, \ldots, a_s$, which are pairwise non-isomorphic (indeed, they have pairwise distinct kernels). On the other hand, the $\mathbb{F}_q L$-module V decomposes as the sum $\bigoplus_{i=1}^t V_i$ of $\mathbb{F}_q L$-modules, where

$$V_1 := \langle e_1, \ldots, e_{m_1} \rangle_{\mathbb{F}_q},\ V_2 := \langle e_{m_1+1}, \ldots, e_{m_1+m_2} \rangle_{\mathbb{F}_q}, \ldots, V_t := \langle e_{m_1+\cdots+m_{t-1}+1}, \ldots, e_n \rangle_{\mathbb{F}_q}.$$

Since $T \leq L$, each V_i is a direct sum of some of these W_l, $1 \leq l \leq s$. Similarly, since $V = \mathbb{F}_q^j \oplus \mathbb{F}_q^{n-j}$ as an $\mathbb{F}_q {}^yM$-module and $T \leq {}^yM$, each of $\mathbb{F}_q^j$ and $\mathbb{F}_q^{n-j}$ is a direct sum of some of these W_l. Using the left multiplication by L and right multiplication by M if needed, we may assume that $\mathbb{F}_q^j$ is spanned by

$$e_1, \ldots, e_{k_1}, e_{m_1+1}, e_{m_1+2}, \ldots, e_{m_1+k_2}, \ldots, e_{m_1+\cdots+m_{t-1}+1}, \ldots, e_{m_1+m_2+\cdots+m_{t-1}+k_t}$$

(k_i first vectors in the indicated basis of V_i for each $1 \leq i \leq t$), and $\mathbb{F}_q^{n-j}$ is spanned by the remaining $n - j$ basis vectors.

It is well known (and can be proved by an easy induction on $t \geq 1$) that the total number N of t-tuples κ in (2-7) is

$$N = \binom{j+t-1}{j} = t \cdot \frac{t+1}{2} \cdots \frac{t+j-1}{j} \leq t^j \leq n^j \tag{2-8}$$

since $t \leq n$. For each such κ, $x = x_\kappa$ sends e_i to $e_{x(i)}$, and we can write

$$\begin{aligned} {}^xM = xMx^{-1} = \mathrm{GL}(\langle e_{x(1)}, \ldots, e_{x(j)} \rangle_{\mathbb{F}_q}) \times \mathrm{GL}(\langle e_{x(j+1)}, \ldots, e_{x(n)} \rangle_{\mathbb{F}_q}) \\ \cong \mathrm{GL}_j(q) \times \mathrm{GL}_{n-j}(q) \end{aligned}$$

Now, $L \cap {}^xM$ fixes each of the subspaces

$$\langle e_1, \ldots, e_{m_1} \rangle_{\mathbb{F}_q} \cap \langle e_{x(1)}, \ldots, e_{x(j)} \rangle_{\mathbb{F}_q} = \langle e_1, \ldots, e_{k_1} \rangle_{\mathbb{F}_q}$$

and

$$\langle e_1, \ldots, e_{m_1} \rangle_{\mathbb{F}_q} \cap \langle e_{x(j+1)}, \ldots, e_{x(n)} \rangle_{\mathbb{F}_q} = \langle e_{k_1+1}, \ldots, e_{m_1} \rangle_{\mathbb{F}_q}$$

of V_1, and similarly for V_i with $2 \leq i \leq t$. It follows that

$$L \cap {}^xM = \prod_{i=1}^t (K_i \times M_i),$$

where for each $1 \le i \le t$, $K_i \cong \mathrm{GL}_{k_i}(q)$ is contained in the $\mathrm{GL}_j(q)$-factor of ${}^xM$, and $M_i \cong \mathrm{GL}_{m_i-k_i}(q)$ is contained in the $\mathrm{GL}_{n-j}(q)$-factor of ${}^xM$. Moreover, $\prod_{i=1}^t K_i$ is a split Levi subgroup of $\mathrm{GL}_j(q)$, and $\prod_{i=1}^t M_i$ is a split Levi subgroup of $\mathrm{GL}_{n-j}(q)$. Now, applying [Giannelli et al. 2017, Lemma 2.7(i)] twice, we obtain

$$^*R^{^x\mathcal{M}}_{\mathcal{L}\cap {}^x\mathcal{M}}((\alpha \boxtimes 1_{G_{n-j}})^x) = {}^*R^{\mathcal{G}_j}_{K_1\times\cdots\times K_t}(\alpha^x) \boxtimes 1_{M_1\times\cdots\times M_t}.$$

Recall that α^x is an irreducible character of $\mathrm{GL}_j(q)$. So, by (2-4), the total sum of multiplicities of irreducible constituents β in $^*R^{^x\mathcal{M}}_{\mathcal{L}\cap {}^x\mathcal{M}}((\alpha \boxtimes 1_{G_{n-j}})^x)$ is at most

$$(j!)^2. \tag{2-9}$$

Consider any such irreducible constituent

$$\beta = \alpha_1 \boxtimes \alpha_2 \boxtimes \cdots \boxtimes \alpha_t \boxtimes 1_{M_1\times\cdots\times M_t}.$$

By [Guralnick et al. 2020, Lemma 2.5(ii)],

$$R^{\mathcal{L}}_{\mathcal{L}\cap {}^x\mathcal{M}}(\beta) = \boxtimes_{i=1}^{t} R^{\mathcal{G}_{m_i}}_{\mathcal{G}_{k_i}\times\mathcal{G}_{m_i-k_i}}(\alpha_i \boxtimes 1_{G_{m_i-k_i}}).$$

Let $\tau_{q,n}$ denote the permutation character of G_n on $\mathbb{F}_q^n$, see [Guralnick et al. 2020, Equation 3.1]. Then the character

$$\gamma_i := R^{\mathcal{G}_{m_i}}_{\mathcal{G}_{k_i}\times\mathcal{G}_{m_i-k_i}}(\alpha_i \boxtimes 1_{G_{m_i-k_i}})$$

is contained in $(\tau_{m_i,q})^{k_i}$ by [Guralnick et al. 2020, Proposition 3.2]. If $k_i = 0$, then the total number $N(\gamma_i)$ of multiplicities of irreducible constituents of γ_i is 1. If $1 \le k_i \le \frac{1}{2}m_i$, then

$$N(\gamma_i) \le \langle \gamma_i, \gamma_i\rangle \le \langle \tau^{k_i}_{m_i,q}, \tau^{k_i}_{m_i,q}\rangle = \langle \tau^{2k_i}_{m_i,q}, 1_{G_{m_i}}\rangle,$$

which is the number of G_{m_i}-orbits on ordered $2k_i$-tuples of vectors in $\mathbb{F}_q^{m_i}$, and hence is at most $8q^{k_i^2} \le q^{4k_i^2}$ by [Guralnick et al. 2020, Lemma 2.4]. Suppose $\frac{1}{2}m_i < k_i \le m_i$. Then γ_i is a character of degree at most $q^{m_ik_i} < q^{2k_i^2}$, and hence $N(\gamma_i) < q^{2k_i^2}$. Thus in all cases we have

$$N(\gamma_i) \le q^{4k_i^2}.$$

It follows that the total number $N(\beta)$ of multiplicities of irreducible constituents of

$$R^{\mathcal{L}}_{\mathcal{L}\cap {}^x\mathcal{M}}(\beta) = \boxtimes_{i=1}^{t} \beta_i$$

is at most

$$q^{4\sum_{i=1}^t k_i^2} \le q^{4(\sum_{i=1}^t k_i)^2} = q^{4j^2}.$$

Combining this with (2-8) and (2-9), we have proved:

Proposition 2.1. *Let $G = \mathcal{G}^F = \mathrm{GL}_n(q)$ and let χ be any irreducible character G of true level $j \leq \frac{1}{2}n$. If $L = \mathcal{L}^F$ is a proper split Levi subgroup of G, then the total number A of irreducible constituents (counting multiplicities) of the Harish-Chandra restriction ${}^*R^{\mathcal{G}}_{\mathcal{L}}(\chi)$ is at most $n^j(j!)^2q^{4j^2}$.*

Corollary 2.2. *Let $G = \mathrm{GL}_n(q)$ and let $g \in G$ be any element such that $\boldsymbol{C}_G(g)$ is contained in a split Levi subgroup $\mathcal{L}$ of G. Let $\chi \in \mathrm{Irr}(G)$ be of true level $j \leq \frac{1}{2}n$, and let $D = \dim \upsilon^{\mathcal{G}}$, with $\upsilon^{\mathcal{G}} = \mathcal{O}^*_\chi$ being the wave front set of χ. Then*

$$|\chi(g)| \leq n^j(j!)^2q^{4j^2}\left(\frac{q+1}{q-1}\right)^{D/2}\chi(1)^{\alpha(L)}.$$

Proof. As mentioned above, in our case the constant B_1 in (2-2) can be taken to be 1. We now combine (2-1), (2-2), and (2-3) with Proposition 2.1. □

2.2. ***The general case.*** For semisimple elements whose centralizer is a nonsplit Levi subgroup, the bound in Corollary 2.2 can be very poor; for instance, it says nothing at all about character values for elements in anisotropic tori. However, the following result is almost as good for all semisimple elements as Corollary 2.2 is in the split case, and moreover works for both GL_n and GU_n:

Theorem 2.3. *Let $G = \mathrm{GL}^\epsilon_n(q)$ and let $g \in G$ be any element such that $\boldsymbol{C}_{\mathcal{G}}(g)$ is contained in a proper F-stable Levi subgroup $\mathcal{L}_1$. Define $L_1 := \mathcal{L}_1^F$. Let $\chi \in \mathrm{Irr}(G)$ be of true level j, $0 \leq j \leq n$, and let $D = \dim \upsilon^{\mathcal{G}}$, with $\upsilon^{\mathcal{G}} = \mathcal{O}^*_\chi$ being the wave front set of χ. Then*

$$|\chi(g)| \leq n^{3j}\left(\frac{q+1}{q-1}\right)^{D/2}\chi(1)^{\alpha(L_1)}.$$

To prove this result we will use Deligne–Lusztig theory. However, before developing the necessary results about Deligne–Lusztig characters we recall a few facts about cosets. Assume $\mathfrak{G}$ is a group. The set of conjugacy classes of $\mathfrak{G}$ will be denoted by $\mathrm{Cl}(\mathfrak{G})$ and if $x \in \mathfrak{G}$ then $x^{\mathfrak{G}} \in \mathrm{Cl}(\mathfrak{G})$ denotes the conjugacy class containing x. A *subcoset* of $\mathfrak{G}$ is a coset $Hw \subseteq \boldsymbol{N}_{\mathfrak{G}}(H)$ of a subgroup $H \leqslant \mathfrak{G}$. Given any subsets $X, Y \subseteq \mathfrak{G}$ we define

$$\boldsymbol{N}_X(Y) := X \cap \boldsymbol{N}_{\mathfrak{G}}(Y), \quad \boldsymbol{C}_X(Y) := X \cap \boldsymbol{C}_{\mathfrak{G}}(Y),$$

where $\boldsymbol{N}_{\mathfrak{G}}(Y)$ and $\boldsymbol{C}_{\mathfrak{G}}(Y)$ are the usual normalizer and centralizer of Y. As usual

$$XY := \{xy \mid x \in X \text{ and } y \in Y\}.$$

Now assume that $W\gamma \subseteq \mathfrak{G}$ is a finite subcoset. We denote by $\mathrm{cf}(W\gamma)$ the space of W-invariant functions $f : W\gamma \to \mathbb{C}$, which we call class functions. This space has an inner product $\langle -, - \rangle$ and if $Hw \subseteq W\gamma$ is a subcoset then we have induction $\mathrm{Ind}^{W\gamma}_{Hw} : \mathrm{cf}(Hw) \to \mathrm{cf}(W\gamma)$ and restriction maps $\mathrm{Res}^{W\gamma}_{Hw} : \mathrm{cf}(W\gamma) \to \mathrm{cf}(Hw)$ which satisfy Frobenius reciprocity with respect to $\langle -, - \rangle$, see [Bonnafé 2006, § 1.C] or [Taylor and Tiep 2020, § 4].

The function $\pi_w = \pi_w^{W\gamma}$ taking the value $|\boldsymbol{C}_W(w)|$ at any W-conjugate of $w \in W\gamma$ and the value 0 otherwise is clearly contained in $\mathrm{cf}(W\gamma)$. We will need the following elementary calculation.

Lemma 2.4. *For any subcoset $Hw \subseteq W\gamma$ and $x \in W\gamma$ we have*

$$\mathrm{Res}_{Hw}^{W\gamma}(\pi_x^{W\gamma}) = \sum_{\substack{z \in H\backslash W/\boldsymbol{C}_W(x) \\ {}^z x \in Hw}} \frac{|\boldsymbol{C}_W(x)|}{|\boldsymbol{C}_H({}^z x)|} \pi_{{}^z x}^{Hw} = \sum_{\substack{z \in H\backslash W \\ {}^z x \in Hw}} \pi_{{}^z x}^{Hw}$$

Proof. The first equality is easy and the second follows because

$$Hz\boldsymbol{C}_W(x) = \bigsqcup_{c \in \boldsymbol{C}_H({}^z x)\backslash \boldsymbol{C}_W({}^z x)} Hcz. \qquad \square$$

We can also produce class functions in the following way. Consider the subgroup $W\langle\gamma\rangle \leqslant \boldsymbol{N}_{\mathfrak{G}}(W)$ and let $\rho \in \mathrm{Irr}(W)$ be a γ-invariant irreducible character. The representation affording ρ can be extended to a representation of $W\langle\gamma\rangle$ containing γ^n in its kernel, for some $n > 0$. The trace function $\tilde{\rho} : W\langle\gamma\rangle \to \mathbb{C}$ of such a representation is what we call an extension of ρ. Note that the group $W\langle\gamma\rangle$ may be infinite but, by design, $\tilde{\rho}$ factors through a finite quotient.

The restriction $\mathrm{Res}_{W\gamma}^{W\langle\gamma\rangle}(\tilde{\rho})$ of such an extension, which we usually again denote by $\tilde{\rho}$, is called an irreducible character of $W\gamma$. The set of irreducible characters is denoted by $\mathrm{Irr}(W\gamma)$. We say $\mathcal{B} \subseteq \mathrm{Irr}(W\gamma)$ is a basis if it is a basis of $\mathrm{cf}(W\gamma)$. Every basis is orthonormal and is obtained by choosing for each γ-stable $\rho \in \mathrm{Irr}(W)$ exactly one extension to $W\langle\gamma\rangle$, see [Digne and Michel 2020, Proposition 11.6.3]. We need the following analogue of [Taylor and Tiep 2020, Corollary 4.11]:

Lemma 2.5. *Assume $Hw \subseteq W\gamma$ is a subcoset and $\rho_i \in \mathrm{Irr}(W)$, with $i \in \{1, 2\}$, is γ-invariant. If $\tilde{\rho}_i$ is an extension of ρ_i to $W\langle\gamma\rangle$ then*

$$|\langle \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_1), \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_2)\rangle| \leqslant \langle \mathrm{Res}_H^W(\rho_1), \mathrm{Res}_H^W(\rho_2)\rangle$$

Proof. Expanding out in a basis $\mathcal{B} \subseteq \mathrm{Irr}(Hw)$ and using the triangle inequality,

$$\begin{aligned} |\langle \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_1), \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_2)\rangle| &\leqslant \sum_{\tilde{\eta} \in \mathcal{B}} |\langle \tilde{\eta}, \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_1)\rangle \overline{\langle \tilde{\eta}, \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_2)\rangle}| \\ &= \sum_{\tilde{\eta} \in \mathcal{B}} |\langle \mathrm{Ind}_{Hw}^{W\gamma}(\tilde{\eta}), \tilde{\rho}_1\rangle| |\langle \mathrm{Ind}_{Hw}^{W\gamma}(\tilde{\eta}), \tilde{\rho}_2\rangle|. \end{aligned}$$

Therefore, using [Taylor and Tiep 2020, Lemma 4.10] we obtain

$$\begin{aligned}|\langle \mathrm{Res}^{W\gamma}_{Hw}(\tilde{\rho}_1), \mathrm{Res}^{W\gamma}_{Hw}(\tilde{\rho}_2)\rangle| &\leqslant \sum_{\eta\in\mathrm{Irr}(H)} \langle \mathrm{Ind}^W_H(\eta), \rho_1\rangle\langle \mathrm{Ind}^W_H(\eta), \rho_2\rangle \\ &= \sum_{\eta\in\mathrm{Irr}(H)} \langle \eta, \mathrm{Res}^W_H(\rho_1)\rangle\langle \eta, \mathrm{Res}^W_H(\rho_2)\rangle \\ &= \langle \mathrm{Res}^W_H(\rho_1), \mathrm{Res}^W_H(\rho_2)\rangle. \end{aligned}$$ □

Recall that $\mathcal{G} = \mathcal{G}(\overline{\mathbb{F}}_q)$ is the group of $\overline{\mathbb{F}}_q$-points of a connected reductive $\mathbb{F}_q$-group, with Frobenius F. We form the semidirect product $\mathcal{G}\langle F\rangle$ with the infinite cyclic group generated by F, defined so that $FgF^{-1} = F(g)$ for all $g \in \mathcal{G}$. If $\mathcal{H}n \subseteq \mathcal{G}F$ is a subcoset then the centralizer $\boldsymbol{C}_{\mathcal{H}}(n) \leqslant \boldsymbol{C}_{\mathcal{G}}(n)$ is a finite group. Moreover, if $\mathcal{H} \leqslant \mathcal{G}$ is closed and connected, then by the Lang–Steinberg theorem, $\mathcal{H}$ acts transitively by conjugation on $\mathcal{H}n$. If $\mathcal{H}$ is a Levi subgroup of $\mathcal{G}$, resp., maximal torus of $\mathcal{G}$, then we call $\mathcal{H}n$ a *Levi subcoset*, resp., a *toral subcoset*.

We define

$$\mathscr{C}(\mathcal{G}F) := \{(g, n) \in \mathcal{G} \times \mathcal{G}F \mid gn = ng\}$$

to be the set of commuting pairs. The group $\mathcal{G}$ acts by simultaneous conjugation on $\mathscr{C}(\mathcal{G}F)$. We write $[g, n]$ for the orbit of $(g, n) \in \mathscr{C}(\mathcal{G}F)$ and $\mathscr{C}(\mathcal{G}F)/\mathcal{G}$ for the set of orbits.

Lemma 2.6. *The map* $g^{\boldsymbol{C}_{\mathcal{G}}(F)} \mapsto [g, F]$ *is a well-defined bijection*

$$\mathrm{Cl}(\boldsymbol{C}_{\mathcal{G}}(F)) \to \mathscr{C}(\mathcal{G}F)/\mathcal{G}.$$

Proof. Clearly this is injective. If $(g, n) \in \mathscr{C}(\mathcal{G}F)$ then by the Lang–Steinberg theorem $n = F^h$ for some $h \in \mathcal{G}$ so $[g, n] = [{}^h g, F]$. □

Let $\mathrm{cf}(\mathscr{C}(\mathcal{G}F))$ be the set of $\mathcal{G}$-invariant functions $f : \mathscr{C}(\mathcal{G}F) \to \mathbb{C}$. Via Lemma 2.6 we can identify $\mathrm{cf}(\mathscr{C}(\mathcal{G}F))$ with the space $\mathrm{cf}(\boldsymbol{C}_{\mathcal{G}}(F))$ of $\mathbb{C}$-valued class functions on the finite group $\boldsymbol{C}_{\mathcal{G}}(F)$. We define $\mathrm{Irr}(\mathscr{C}(\mathcal{G}F))$ to be those functions corresponding to $\mathrm{Irr}(\boldsymbol{C}_{\mathcal{G}}(F))$. The advantage of working with $\mathscr{C}(\mathcal{G}F)$ is that we can work with the different (inner) forms $\boldsymbol{C}_{\mathcal{G}}(gF)$ of $\mathcal{G}$ simultaneously.

If $\mathcal{L}w \subseteq \mathcal{G}F$ is a Levi subcoset then we can define Deligne–Lusztig induction and restriction maps

$$R^{\mathcal{G}F}_{\mathcal{L}w} : \mathrm{cf}(\mathscr{C}(\mathcal{L}w)) \to \mathrm{cf}(\mathscr{C}(\mathcal{G}F)) \quad\text{and}\quad {}^*R^{\mathcal{G}F}_{\mathcal{L}w} : \mathrm{cf}(\mathscr{C}(\mathcal{G}F)) \to \mathrm{cf}(\mathscr{C}(\mathcal{L}w)).$$

For our purposes this can be done as follows. We start first with the case of a coset $\mathcal{L}F$ where $\mathcal{L} \leqslant \mathcal{G}$ is an F-stable Levi subgroup. Making the identifications $\mathrm{cf}(\boldsymbol{C}_{\mathcal{G}}(F)) \to \mathrm{cf}(\mathscr{C}(\mathcal{G}F))$ and $\mathrm{cf}(\boldsymbol{C}_{\mathcal{L}}(F)) \to \mathrm{cf}(\mathscr{C}(\mathcal{L}F))$ we define

$$R^{\mathcal{G}F}_{\mathcal{L}F} := R^{\mathcal{G}}_{\mathcal{L}}, \quad {}^*R^{\mathcal{G}F}_{\mathcal{L}F} := {}^*R^{\mathcal{G}}_{\mathcal{L}}.$$

Now consider a general Levi subcoset $\mathcal{L}w \subseteq \mathcal{G}F$. We pick an element $g \in \mathcal{G}$ such that $F^g \in \mathcal{L}w$, so that ${}^g(\mathcal{L}w) = \mathcal{L}_1 F$ where $\mathcal{L}_1 := {}^g\mathcal{L}$ is an F-stable Levi subgroup of $\mathcal{G}$. If $\iota_g : \mathcal{G}\langle F\rangle \to \mathcal{G}\langle F\rangle$ is the inner automorphism defined by $\iota_g(x) = {}^g x$ then $\iota_g(\mathcal{L}w) = \mathcal{L}_1 F$ and we define

$$R^{\mathcal{G}F}_{\mathcal{L}w} := R^{\mathcal{G}F}_{\mathcal{L}_1 F} \circ (\iota_g^{-1})^* \quad \text{and} \quad {}^*R^{\mathcal{G}F}_{\mathcal{L}w} := (\iota_g)^* \circ {}^*R^{\mathcal{G}F}_{\mathcal{L}_1 F},$$

where $(\iota_g^{-1})^*$ is the map $f \mapsto f \circ \iota_g^{-1}$ and likewise for $(\iota_g)^*$.

We note that the maps $R^{\mathcal{G}F}_{\mathcal{L}w}$ and ${}^*R^{\mathcal{G}F}_{\mathcal{L}w}$ are defined only up to composition with $(\iota_n)^*$ for some $n \in \boldsymbol{N}_{\mathcal{G}}(\mathcal{L}w) = \boldsymbol{N}_{\boldsymbol{C}_{\mathcal{G}}(w)}(\mathcal{L})\mathcal{L}$. We need the following interpretation of the Mackey formula.

Lemma 2.7. *If $\mathcal{L}w \subseteq \mathcal{G}F$ is a Levi subcoset and $\mathcal{T}x \subseteq \mathcal{G}F$ is a toral subcoset then*

$${}^*R^{\mathcal{G}F}_{\mathcal{L}w} \circ R^{\mathcal{G}F}_{\mathcal{T}x} = \sum_{\substack{z \in \mathcal{L}\backslash\mathcal{G} \\ {}^z(\mathcal{T}x) \subseteq \mathcal{L}w}} R^{\mathcal{L}w}_{{}^z(\mathcal{T}x)} \circ (\iota_z^{-1})^*$$

Proof. Fix elements $h_w, h_x \in \mathcal{G}$ such that $w = F^{h_w}$ and $x = F^{h_x}$ and let $\mathcal{L}_1 := {}^{h_w}\mathcal{L}$ and $\mathcal{T}_1 := {}^{h_x}\mathcal{T}$ be corresponding F-stable subgroups of $\mathcal{G}$. According to the Mackey formula, see [Digne and Michel 2020, Theorem 9.2.6], we have

$${}^*R^{\mathcal{G}}_{\mathcal{L}_1} \circ R^{\mathcal{G}}_{\mathcal{T}_1} = \sum_{\substack{u \in L_1\backslash G/T_1 \\ {}^u\mathcal{T}_1 \leqslant \mathcal{L}_1}} T^{\mathcal{L}_1}_{{}^u\mathcal{T}_1} \circ (\iota_u^{-1})^* = \sum_{\substack{u \in L_1\backslash G \\ {}^u\mathcal{T}_1 \leqslant \mathcal{L}_1}} R^{\mathcal{L}_1}_{{}^u\mathcal{T}_1} \circ (\iota_u^{-1})^*,$$

where $L_1 = \boldsymbol{C}_{\mathcal{L}_1}(F)$, $G = \boldsymbol{C}_{\mathcal{G}}(F)$, and $T_1 = \boldsymbol{C}_{\mathcal{T}_1}(F)$. The second equality follows because if ${}^u\mathcal{T}_1 \leqslant \mathcal{L}_1$ then ${}^uT_1 \leqslant L_1$, so

$$L_1 u T_1 = L_1({}^uT_1)u = L_1 u.$$

Consider the isomorphism of varieties $\psi : \mathcal{G} \to \mathcal{G}$ given by $\psi(v) = h_w v h_x^{-1}$. If $F' : \mathcal{G} \to \mathcal{G}$ is the morphism defined by $F'(v) = wvx^{-1}$ then we have $F\psi = \psi F'$. From this it follows that we have bijections

$$(\mathcal{L}\backslash\mathcal{G})^{F'} \xrightarrow{\psi} \boldsymbol{C}_{\mathcal{L}_1\backslash\mathcal{G}}(F) \longrightarrow L_1\backslash G,$$

where $\boldsymbol{C}_{\mathcal{L}_1\backslash\mathcal{G}}(F) = \{\mathcal{L}_1 u \in \mathcal{L}_1\backslash\mathcal{G} \mid \mathcal{L}_1 u F = \mathcal{L}_1 F u\}$, and $(\mathcal{L}\backslash\mathcal{G})^{F'}$ denotes the cosets fixed by F'. The second bijection is a simple consequence of the Lang–Steinberg theorem. If $z \in \mathcal{G}$ then ${}^{\psi(z)}\mathcal{T}_1 \leqslant \mathcal{L}_1$ if and only if ${}^z\mathcal{T} \leqslant \mathcal{L}$ and $F'(\mathcal{L}z) = \mathcal{L}z$ if and only if ${}^zx \in \mathcal{L}w$. It is clear that the combination of these two conditions is equivalent to the condition ${}^z(\mathcal{T}x) \subseteq \mathcal{L}w$.

Finally, conjugating the expression above we get

$${}^*R^{\mathcal{G}F}_{\mathcal{L}w} \circ R^{\mathcal{G}}_{\mathcal{T}x} = \sum_{\substack{z \in \mathcal{L}\backslash\mathcal{G} \\ {}^z(\mathcal{T}x) \leqslant \mathcal{L}w}} (\iota_{h_w})^* \circ R^{\mathcal{L}_1}_{{}^{\psi(z)}\mathcal{T}_1} \circ (\iota_{\psi(z)}^{-1})^* \circ (\iota_{h_x}^{-1})^*.$$

It suffices to show that $(\iota_{h_w})^* \circ R^{\mathcal{L}_1}_{\psi(z)\mathcal{T}_1} = R^{\mathcal{L}}_{{}^z\mathcal{T}} \circ (\iota_{h_w})^*$ when $F'(z) = z$, where $R^{\mathcal{L}}_{{}^z\mathcal{T}} = R^{\mathcal{L}w}_{({}^z\mathcal{T})w}$ is defined with respect to the Frobenius w on $\mathcal{L}$. However, the arguments to prove this are identical to those used to prove [Digne and Michel 2020, Proposition 11.3.10]; see also the arguments by Bonnafé in [Navarro et al. 2008]. We omit the details. □

From now on we assume the underlying group scheme is GL_n^ϵ so that

$$\mathcal{G} = \mathcal{G}_n = \mathrm{GL}_n(\overline{\mathbb{F}}_q).$$

We will assume $\mathcal{T} \leqslant \mathcal{G}$ is the diagonal maximal torus and we denote by $W := \boldsymbol{N}_{\mathcal{G}}(\mathcal{T})/\mathcal{T} \cong \mathsf{S}_n$ the corresponding Weyl group. The quotient $\boldsymbol{N}_{\mathcal{G}\langle F\rangle}(\mathcal{T})/\mathcal{T} = W\langle F\rangle$ is isomorphic to the semidirect product $W \rtimes \langle F\rangle$, where we identify F with its natural image $\mathcal{T}F$. The coset $WF \subseteq W\langle F\rangle$ is, by definition, the set of toral subcosets $\mathcal{T}n$ where $n \in \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T})$.

We wish to reinterpret Lemma 2.7 in the language of Lusztig's almost characters. Recall that GL_n^ϵ is self-dual. If $(w, s) \in WF \times \mathcal{T}$ is a pair such that $ws = sw$ then we set

$$R^{\mathcal{G}F}_w(s) := R^{\mathcal{G}F}_x(\theta),$$

where $x \in WF$ and $\theta \in \mathrm{Irr}(\mathscr{C}(w))$ correspond to (w, s) under a bijection obtained as in [Digne and Michel 2020, Proposition 11.1.16] from duality.

If $s \in \mathcal{T}$ and $\boldsymbol{C}_{WF}(s) \neq \varnothing$ then, following Lusztig [1984, §8.4], we define

$$\mathcal{R}^{\mathcal{G}F}_s : \mathrm{cf}(\boldsymbol{C}_{WF}(s)) \to \mathrm{cf}(\mathscr{C}(\mathcal{G}F)), \quad f \mapsto \frac{1}{|\boldsymbol{C}_W(s)|} \sum_{w \in \boldsymbol{C}_{WF}(s)} f(w) R^{\mathcal{G}F}_w(s) \in \mathrm{cf}(\mathscr{C}(\mathcal{G}F)),$$

where, as defined above, $\boldsymbol{C}_X(s) = \{w \in X \mid ws = sw\}$ for any subset $X \subseteq W\langle F\rangle$.

Corollary 2.8. *Assume $\mathcal{T} \leqslant \mathcal{L} \leqslant \mathcal{G}$ is a Levi subgroup and $w \in \boldsymbol{N}_{WF}(\mathcal{L})$. Then for any $s \in \mathcal{T}$ with $\boldsymbol{C}_{WF}(s) \neq \varnothing$ we have*

$$^*R^{\mathcal{G}F}_{\mathcal{L}w} \circ \mathcal{R}^{\mathcal{G}F}_s = \sum_{\substack{z \in H\backslash W/\boldsymbol{C}_W(s) \\ \boldsymbol{C}_{Hw}({}^zs) \neq \varnothing}} \mathcal{R}^{\mathcal{L}w}_{{}^zs} \circ \mathrm{Res}^{\boldsymbol{C}_{WF}({}^zs)}_{\boldsymbol{C}_{Hw}({}^zs)} \circ (\iota_z^{-1})^*$$

where $H = \boldsymbol{N}_{\mathcal{L}}(\mathcal{T})/\mathcal{T}$ is the Weyl group of $\mathcal{L}$.

Proof. By linearity it is enough to check both sides agree when evaluated at $\pi_x^{\boldsymbol{C}_{WF}(s)}$ for some $x \in \boldsymbol{C}_{WF}(s)$. But in that case $\mathcal{R}^{\mathcal{G}F}_s(\pi_x^{\boldsymbol{C}_{WF}(s)}) = R^{\mathcal{G}F}_x(s)$. Assume (x, s) corresponds to (y, θ). Evaluating at θ we have by Lemma 2.4 that

$$^*R^{\mathcal{G}F}_{\mathcal{L}w}(R^{\mathcal{G}F}_y(\theta)) = \sum_{\substack{z \in \mathcal{L}\backslash\mathcal{G} \\ {}^zy \subseteq \mathcal{L}w}} R^{\mathcal{L}w}_{{}^zy}({}^z\theta).$$

If ${}^zy \subseteq \mathcal{L}w$ then ${}^z\mathcal{T} \leqslant \mathcal{L}$ so ${}^{lz}\mathcal{T} = \mathcal{T}$ for some $l \in \mathcal{L}$. Therefore, we can take the sum over cosets $\boldsymbol{N}_{\mathcal{L}}(\mathcal{T})\backslash\boldsymbol{N}_{\mathcal{G}}(\mathcal{T})$ or similarly $H\backslash W$. As ${}^zy \subseteq \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T})$ the condition

${}^{z}y \subseteq \mathcal{L}w$ is equivalent to

$$ {}^{z}y \subseteq N_{\mathcal{G}F}(\mathcal{T}) \cap \mathcal{L}w = N_{\mathcal{L}}(\mathcal{T})w $$

which in turn is equivalent to ${}^{z}y \in Hw$. Breaking the sum in Lemma 2.7 along double cosets, as in the proof of Lemma 2.4, gives

$$ {}^{*}R^{\mathcal{G}F}_{\mathcal{L}w}(R^{\mathcal{G}F}_{y}(\theta)) = \sum_{z \in H\backslash W/C_W(s)} \sum_{\substack{c \in C_H({}^{z}s)/C_W({}^{z}s) \\ {}^{cz}y \in Hw}} R^{\mathcal{L}w}_{{}^{cz}y}({}^{cz}\theta). $$

Picking a different double coset representative we can assume that ${}^{z}y \in Hw$. We claim that $C_{Hw}({}^{z}s) = C_H({}^{z}s){}^{z}y$. Certainly ${}^{z}x \in C_{Hw}({}^{z}s)$ by assumption. Now if $d \in C_W({}^{z}s)$ then $d({}^{z}x) \in Hw$ if and only if $d \in C_H({}^{z}s) = H \cap C_W({}^{z}s)$. The statement now follows from Lemma 2.4. □

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. By assumption, χ has true level j. Embed the maximal diagonal torus $\mathcal{T}$ in the natural F-stable Levi subgroup $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_1 \cong \mathcal{G}_j$ and $\mathcal{M}_2 \cong \mathcal{G}_{n-j}$. Note that F stabilizes $\mathcal{M}_1$ and $\mathcal{M}_2$. We will identify $\mathcal{M}\langle F\rangle$ with a subgroup of $\mathcal{M}_1\langle F\rangle \times \mathcal{M}_2\langle F\rangle$, where we again denote by F its restriction to $\mathcal{M}_i$.

By [Guralnick et al. 2020, Theorem 3.9] our assumption on the level of χ implies that χ is a constituent of

$$ R^{\mathcal{G}F}_{\mathcal{M}F}(\alpha \boxtimes 1) $$

for some $\alpha \in \operatorname{Irr}(\mathscr{C}(\mathcal{M}_1 F))$. Note that

$$ C_{\mathcal{M}}(F) = C_{\mathcal{M}_1}(F) C_{\mathcal{M}_2}(F) \cong \mathrm{GL}^{\epsilon}_{j}(q) \times \mathrm{GL}^{\epsilon}_{n-j}(q). $$

If $W_i \leqslant W$ is the subgroup $N_{\mathcal{M}_i}(\mathcal{T})/\mathcal{T}$ then the subgroup $W_1W_2 \leqslant W$ is a direct product with $W_1 \cong \mathsf{S}_j$ and $W_2 \cong \mathsf{S}_{n-j}$.

By [Digne and Michel 2020, Theorem 11.7.3] we have $\alpha = \pm\mathcal{R}^{\mathcal{M}_1 F}_{\tilde{s}}(\tilde{\phi})$ for some $s \in \mathcal{M}_1 \cap \mathcal{T}$, with $C_{W_1 F}(s) \neq \varnothing$, and some irreducible character $\tilde{\phi} \in \operatorname{Irr}(C_{W_1 F}(s))$ afforded by a representation over $\mathbb{Q}$, see [Lusztig 1984, Proposition 3.2]. Note that $C_{W_1 W_2}(s) = C_{W_1}(s) W_2$ is a reflection group and we have

$$ C_{W_1 W_2 F}(s) = C_{W_1 F}(s) W_2. $$

It is known, see [Digne and Michel 2020, Proposition 11.6.6], that

$$ R^{\mathcal{G}F}_{\mathcal{M}F}(\alpha \boxtimes 1_{\mathcal{M}_2 F}) = \pm\mathcal{R}^{\mathcal{G}F}_{s}(\operatorname{Ind}^{C_{WF}(s)}_{C_{W_1 W_2 F}(s)}(\tilde{\phi} \boxtimes 1)) $$

from which it follows that $\chi = \pm\mathcal{R}^{\mathcal{G}F}_{s}(\tilde{\psi})$ for some irreducible constituent $\tilde{\psi} \in \operatorname{Irr}(C_{WF}(s))$ of the induced function $\operatorname{Ind}^{C_{WF}(s)}_{C_{W_1 W_2 F}(s)}(\tilde{\phi} \boxtimes 1)$.

We denote again by $\tilde{\phi}$ and $\tilde{\psi}$ irreducible characters of $\boldsymbol{C}_{W_1}(s)\langle F\rangle$ and $\boldsymbol{C}_W(s)\langle F\rangle$ respectively, yielding $\tilde{\phi}$ and $\tilde{\psi}$ upon restriction to the respective cosets. By [Taylor and Tiep 2020, Lemma 4.10],

$$|\langle \mathrm{Ind}_{\boldsymbol{C}_{W_1W_2F}(s)}^{\boldsymbol{C}_{WF}(s)}(\tilde{\phi}\boxtimes 1), \tilde{\psi}\rangle| \leqslant \langle \mathrm{Ind}_{\boldsymbol{C}_{W_1W_2}(s)}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1), \psi\rangle.$$

In particular, ψ is a constituent of $\mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)$.

Following the proof of Corollary 2.2 it suffices to show that

$$\sum_{\eta\in\mathrm{Irr}(L_1)} |\langle \eta, {}^*R_{\mathcal{L}_1F}^{\mathcal{G}F}(\chi)\rangle| \leqslant \langle {}^*R_{\mathcal{L}_1F}^{\mathcal{G}F}(\chi), {}^*R_{\mathcal{L}_1F}^{\mathcal{G}F}(\chi)\rangle \leqslant n^{3j}. \tag{2-10}$$

A straightforward argument shows that we may find a Levi subgroup $\mathcal{T}\leqslant\mathcal{L}\leqslant\mathcal{G}$, an element $w\in \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T})$, and an element $h\in\mathcal{G}$ such that ${}^h(\mathcal{L}w)=\mathcal{L}_1F$, see [Digne and Michel 2020, Proposition 11.4.1]. With this we need only bound

$$\langle {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\chi), {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\chi)\rangle = \langle {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\mathcal{R}_s^{\mathcal{G}}(\tilde{\psi})), {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\mathcal{R}_s^{\mathcal{G}}(\tilde{\psi}))\rangle.$$

By Corollary 2.8,

$${}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\mathcal{R}_s^{\mathcal{G}}(\tilde{\psi})) = \sum_{\substack{z\in H\backslash W/\boldsymbol{C}_W(s)\\ \boldsymbol{C}_{Hw}({}^zs)\neq\varnothing}} \mathcal{R}_{{}^zs}^{\mathcal{L}w}(\mathrm{Res}_{\boldsymbol{C}_{Hw}({}^zs)}^{\boldsymbol{C}_{WF}({}^zs)}({}^z\tilde{\psi})),$$

where $H=\boldsymbol{N}_{\mathcal{L}}(\mathcal{T})/\mathcal{T}$ is the Weyl group of $\mathcal{L}$. The number of double cosets appearing in this sum is bounded above by

$$|H\backslash W/\boldsymbol{C}_W(s)| \leqslant |W/\boldsymbol{C}_W(s)| \leqslant |W/W_2| = |\mathsf{S}_n/\mathsf{S}_{n-j}| \leqslant n^j$$

because $W_2\leqslant \boldsymbol{C}_W(s)$.

Now, by the disjointness of Deligne–Lusztig characters, see [Digne and Michel 2020, Proposition 11.3.2], the summands are pairwise orthogonal because each ${}^zs$ lies in a distinct $\mathcal{L}$-conjugacy class. So, using Lemma 2.5 it suffices to bound

$$|\langle \mathrm{Res}_{\boldsymbol{C}_{Hw}({}^zs)}^{\boldsymbol{C}_{WF}({}^zs)}(\tilde{\psi}), \mathrm{Res}_{\boldsymbol{C}_{Hw}({}^zs)}^{\boldsymbol{C}_{WF}({}^zs)}(\tilde{\psi})\rangle| \leqslant \langle \mathrm{Res}_{\boldsymbol{C}_H({}^zs)}^{\boldsymbol{C}_W({}^zs)}(\psi), \mathrm{Res}_{\boldsymbol{C}_H({}^zs)}^{\boldsymbol{C}_W({}^zs)}(\psi)\rangle \leqslant \psi(1)^2.$$

But we know ψ is a constituent of $\mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)$ so

$$\psi(1)^2 \leqslant \mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)(1)^2 \leqslant \mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)(1)^2 \leqslant |W/W_2|^2 \leqslant n^{2j}. \qquad \square$$

2.3. ***An asymptotic version of* [Bezrukavnikov et al. 2018, Theorem 1.1] *and* [Taylor and Tiep 2020, Theorem 1.9].** Now we can prove the main result of the section:

Theorem 2.9. *For any $\epsilon>0$, there are some explicit positive constants $N_0=N_0(\epsilon)$ and $q_0=q_0(\epsilon)$ such that the following statement holds for all integers $n\geq N_0$ and all prime powers $q\geq q_0$. Let $\mathcal{G}=\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and let $F:\mathcal{G}\to\mathcal{G}$ be a Frobenius* *endomorphism so that* $\mathcal{G}^F \in \{\mathrm{GL}_n(q), \mathrm{GU}_n(q)\}$. *Suppose we are in one of the following two cases.*

(i) $G := \mathcal{G}^F$ *and* $g \in G$ *is any element such that* $\boldsymbol{C}_{\mathcal{G}}(g)$ *is contained in a proper* F*-stable Levi subgroup* $\mathcal{L}$ *of* $\mathcal{G}$ *and* $L := \mathcal{L}^F$.

(ii) $G := [\mathcal{G}, \mathcal{G}]^F \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q)\}$, $g \in G$, *and either*

(a) $\boldsymbol{C}_{[\mathcal{G},\mathcal{G}]}(g)$ *is contained in a proper split Levi subgroup* $\mathcal{L}$ *of* $[\mathcal{G}, \mathcal{G}]$ *and* $L := \mathcal{L}^F$, *or*

(b) g *is noncentral semisimple with* $L := \boldsymbol{C}_G(g)$.

Then, for all $\chi \in \mathrm{Irr}(G)$,

$$|\chi(g)| \le \chi(1)^{\alpha(L)+\epsilon} \tag{2-11}$$

Proof. Note that (2-11) is obvious if $\alpha := \alpha(L) \ge 1-\epsilon$. So in what follows we will assume

$$\alpha + \epsilon < 1,$$

in particular $0 < \epsilon < 1$.

(A) First we prove (2-11) in the cases of (i) and (ii)(a). Note that the upper bound on $|\chi(g)|$ in Theorem 2.3 is obtained by combining (2-1), (2-2), and (2-10). If $\mathcal{L}$ is split, $\phi := {}^*R^{\mathcal{G}^F}_{\mathcal{L}_1{}^F}(\chi)$ is a true character of $\mathcal{L}_1^F$ (in the notation of the proof of Theorem 2.3), hence this upper bound is actually an upper bound on the degree of ϕ. Arguing as in part (ii) of the proof of [Bezrukavnikov et al. 2018, Theorem 1.1], it therefore suffices to prove (2-11) for

$$G \in \{\mathrm{GL}_n(q), \mathrm{GU}_n(q)\}.$$

Let $j = \mathfrak{l}(\chi)$ be the *level* of χ in the sense of [Guralnick et al. 2020, Definition 1(ii)]. This means that multiplying χ by a suitable linear character of G, we may assume that $\mathfrak{l}^*(\chi) = j$. Applying Theorem 2.3, it suffices to prove

$$n^{3j}\left(\frac{q+1}{q-1}\right)^{D/2} \le \chi(1)^{\epsilon}. \tag{2-12}$$

Note that the degree of any irreducible character of G is a monic polynomial in variable q with integer coefficients, in fact a product of a power of q and cyclotomic polynomials in q. Writing $D = \dim \upsilon^{\mathcal{G}}$, with $\upsilon^{\mathcal{G}} = \mathcal{O}^*_\chi$ being the wave front set of χ, the degree of this polynomial is $\frac{1}{2}D$ [Bezrukavnikov et al. 2018, Equation 2.1]. Since $\Phi_k(q) \ge (q-1)^{\deg \Phi_k}$ for any cyclotomic polynomial Φ_k, we therefore have

$$\chi(1) \ge (q-1)^{D/2}.$$

Choosing $q_0 = q_0(\epsilon) \ge 3$ such that

$$\frac{q_0+1}{q_0-1} \le (q_0-1)^{\epsilon/3},$$

it remains to prove

$$n^{3j} \leq \chi(1)^{2\epsilon/3} \tag{2-13}$$

for $q \geq q_0$ and $n \geq N_0$.

Choosing $N_0 \geq 4$, we have $\frac{1}{4}n^2 - 2 > \frac{1}{16}n^2$ for $n \geq N_0$, and so, when $j > \frac{1}{2}n$, we have

$$\chi(1) > q^{n^2/16} > q^{n^2\epsilon/16}$$

by [Guralnick et al. 2020, Theorem 1.2(ii)]. Next, if $\frac{1}{4}n < j \leq \frac{1}{2}n$, then $j(n-j) > \frac{1}{8}n^2$, and so

$$\chi(1) > q^{n^2/8}$$

by [Guralnick et al. 2020, Theorem 1.2(i)]. If $0 \leq j \leq \frac{1}{4}n$, then $j(n-j) \geq \frac{3}{4}nj$, and so

$$\chi(1) \geq q^{3nj/4} \tag{2-14}$$

again by [Guralnick et al. 2020, Theorem 1.2(i)].

First we work in the setting

$$\tfrac{1}{12}n\epsilon \leq j \leq n.$$

Then (2-14) and the above arguments show that

$$\chi(1) \geq q^{n^2\epsilon/16}.$$

Choose $N_0 \geq 4$ such that

$$n \leq q_0^{n\epsilon^2/72} \tag{2-15}$$

for all $n \geq N_0$. Then for $q \geq q_0$ we now have

$$n^{3j} \leq n^{3n} \leq q_0^{n^2\epsilon^2/24} \leq \chi(1)^{2\epsilon/3},$$

yielding (2-13) in this case.

Assume now that $j \leq \frac{1}{12}n\epsilon \leq \frac{1}{12}n$. Then for $n \geq N_0$ and $q \geq q_0$ we now have by (2-14) and (2-15) that

$$n^{3j} \leq q_0^{n\epsilon^2 j/24} < q^{n\epsilon j/2} \leq \chi(1)^{2\epsilon/3},$$

proving (2-13) in this case as well.

(B) Now we handle the case (ii)(b), embedding G in $\tilde{G} := \mathcal{G}^F$. Letting $\tilde{L} := \boldsymbol{C}_{\tilde{G}}(g)$, note that $\tilde{G} = G\tilde{L}$ and $g \in \tilde{L}$. Letting $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$ lie above χ, by Clifford's theorem we have

$$\tilde{\chi}|_G = \sum_{i=1}^{t} \chi^{x_i},$$

where $x_1, \dots, x_t$ can be chosen from $\tilde{L}$. Since every x_i centralizes g, we have

$$\tilde{\chi}(1) = t\chi(1), \quad \tilde{\chi}(g) = t\chi(g).$$

Let $\mathcal{L} = \boldsymbol{C}_{[\mathcal{G},\mathcal{G}]}(g)$ and $\tilde{\mathcal{L}} = \boldsymbol{C}_{\mathcal{G}}(g)$, so that $L = \mathcal{L}^F$ and $\tilde{L} = \tilde{\mathcal{L}}^F$. We have $\tilde{\mathcal{L}} = \mathcal{L}\boldsymbol{Z}(\mathcal{G})$ because $\mathcal{G} = [\mathcal{G}, \mathcal{G}]\boldsymbol{Z}(\mathcal{G})$, and so every unipotent element of $\tilde{L}$ is contained in L. Moreover, if $u \in \tilde{L}$ is unipotent then $u^{\tilde{\mathcal{L}}} = u^{\mathcal{L}}$ and $u^{\mathcal{G}} = u^{[\mathcal{G},\mathcal{G}]}$, whence $\alpha(\tilde{L}) = \alpha(L)$. By the case (i) proved in (A), $|\tilde{\chi}(g)| \leq \tilde{\chi}(1)^{\alpha+\epsilon}$. As $\alpha + \epsilon < 1$, it follows that $|\chi(g)| \leq \chi(1)^{\alpha+\epsilon}$. □

3. Some numerical estimates

This section is devoted to numerical estimates which allow us to determine a finite list of possible exceptions to Conjecture 1.

Let

$$f_{a,x}(\delta) := \min\Bigl(\Bigl(\frac{1}{a} + \delta\Bigr)x, \frac{1}{2}\Bigl(\frac{1}{a} + \frac{a}{a-1}\delta^2\Bigr)\Bigr).$$

The main goal of this section is to give an explicit finite list of possible exceptions to the rule that if $2 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ are integers such that $\sum_{i=1}^r \frac{1}{a_i} < r - 2$, then

$$\sum_{i=1}^r f_{a_i,x}(\delta_i) < x + \sum_{i=1}^r \frac{a_i\delta_i^2}{a_i - 1}$$

when $x \in \bigl[0, \frac{1}{2}\bigr]$ and $\delta_i \in \bigl[0, \frac{a_i-1}{a_i}\bigr]$ for all i. We will see that when the rule holds, Conjecture 1 holds for the corresponding genus 0 Fuchsian group. There is only one exception with $r \geq 4$, and there are thirty-one with $r = 3$.

Proposition 3.1. *For all $a \geq 2$, $x \in \bigl(0, \frac{1}{2}\bigr]$, and $\delta \in \bigl[0, 1 - \frac{1}{a}\bigr]$, the function*

$$\frac{f_{a,x}(\delta) - \frac{a\delta^2}{a-1}}{x}$$

is bounded above by $\max\bigl(\frac{x}{a}, G_{a,x}, H_{a,x}\bigr)$, *where*

$$G_{a,x} = \begin{cases} \dfrac{(a-1)x+4}{4a} & \text{if } x \leq \dfrac{2\sqrt{3a+1}-4}{3(a-1)}, \\ -\infty & \text{if } x > \dfrac{2\sqrt{3a+1}-4}{3(a-1)}, \end{cases}$$

and

$$H_{a,x} = \begin{cases} \dfrac{\sqrt{(a-1)^2x^2+2(a-1)x-(a-1)}}{a} - \dfrac{(a-1)x}{a} + \dfrac{1-x}{ax} & \text{if } (a-1)x^2+ax \geq 1, \\ -\infty & \text{if } (a-1)x^2+ax < 1. \end{cases}$$

In particular,

$$f_{a,x}(\delta) < \frac{2x}{\sqrt{a}} + \frac{a\delta^2}{a-1}. \tag{3-1}$$

Proof. Let

$$g_{a,x}(\delta) := \Big(\frac{1}{a}+\delta\Big)x - \frac{a\delta^2}{a-1}, \quad h_a(\delta) := \frac{1}{2a} - \frac{a\delta^2}{2(a-1)},$$

so

$$f_{a,x}(\delta) - \frac{a\delta^2}{a-1} = \min(g_{a,x}(\delta), h_a(\delta)).$$

For each fixed integer $a \geq 2$, we wish to determine as a function of $x \in \big(0, \frac{1}{2}\big]$, the (unique) element $\delta_0(x) \in \big[0, \frac{a-1}{a}\big]$ for which $\min(g_{a,x}(\delta), h_a(\delta))$ achieves its maximum as a function of δ.

We note first that as functions of δ, $g_{a,x}(\delta)$ and $h_a(\delta)$ are strictly concave, and $h_a(\delta)$ is decreasing on $[0, \infty)$. Therefore, $\delta_0(x)$ must either be the unique critical point $\frac{(a-1)x}{2a}$ of $g_{a,x}(\delta)$, the minimum solution of $g_{a,x}(\delta) = h_a(\delta)$ in the interval $\big[0, \frac{a-1}{a}\big]$, or one of the endpoints 0 and $\frac{a-1}{a}$ of the interval. For the endpoints, we have

$$\min(g_{a,x}(0), h_a(0)) \leq g_{a,x}(0) = \frac{x}{a},$$

and

$$\min\Big(g_{a,x}\Big(\frac{a-1}{a}\Big), h_a\Big(\frac{a-1}{a}\Big)\Big) \leq h_a\Big(\frac{a-1}{a}\Big) = \frac{1}{a} - \frac{1}{2} \leq 0.$$

If the maximum occurs at $\frac{(a-1)x}{2a}$, it must be

$$g_{a,x}\Big(\frac{(a-1)x}{2a}\Big) = \frac{(a-1)x^2+4x}{4a},$$

and this quantity must be less than or equal to

$$h_a\Big(\frac{(a-1)x}{2a}\Big) = \frac{4-(a-1)x^2}{8a},$$

so

$$x \leq \frac{2\sqrt{3a+1}-4}{3(a-1)}.$$

Thus, for all x,

$$G_{a,x} \leq \frac{1}{a} + \frac{\sqrt{3a+1}-2}{6a} \leq \frac{1}{a} + \frac{1}{\sqrt{2a}} < \frac{2}{\sqrt{a}}. \tag{3-2}$$

The graphs of $g_{a,x}(\delta)$ and $h_a(\delta)$ intersect only if

$$(a-1)x^2 + 2x \geq 1, \tag{3-3}$$

in which case the smaller δ-value satisfying $g_{a,x}(\delta) = h_a(\delta)$ is

$$\delta = \frac{(a-1)x - \sqrt{(a-1)^2x^2 + 2(a-1)x - (a-1)}}{a}.$$

If this is $\delta_0(x)$, we have

$$H_{a,x} = \frac{\sqrt{(a-1)^2x^2+2(a-1)x-(a-1)}\,x}{a} - \frac{(a-1)x^2}{a} + \frac{1-x}{a} \leq \frac{1-x}{a} \leq \frac{1}{a}. \tag{3-4}$$

By (3-3), $x > \frac{1}{2\sqrt{a}}$, so (3-4) implies

$$\frac{g_{a,x}(\delta_0(x))}{x} < \frac{2}{\sqrt{a}}.$$

Together with (3-2), this implies the proposition. □

Proposition 3.2. *Let $r \geq 3$ be an integer, $a_1 \leq a_2 \leq \cdots \leq a_r$ be integers greater or equal to 2, and δ_i be nonnegative numbers with $\frac{1}{a_i} + \delta_i \leq 1$ for all i. We assume that the tuple $a_1 \cdots a_r$ is not in the following list*:

(i) $23c,\ 7 \leq c \leq 24$.

(ii) $24c,\ 5 \leq c \leq 9$.

(iii) $25c,\ 5 \leq c \leq 7$.

(iv) 266.

(v) $33c,\ 4 \leq c \leq 6$.

(vi) 344.

(vii) 2223.

Then for all $x \in \left(0, \frac{1}{2}\right]$,

$$\sum_{i=1}^{r} f_{a_i,x}(\delta_i) < -2rx\epsilon + (r-2)x + \sum_{i=1}^{r} \frac{a_i\delta_i^2}{a_i - 1},$$

where ϵ is a positive constant which does not depend on x, r, the a_i, or the δ_i.

Proof. By (3-1) for $a \geq 100$ and machine computation for $2 \leq a < 100$,

$$\frac{f_{a,x}(\delta) - \dfrac{a\delta^2}{a-1}}{x} \leq \begin{cases} 0.555 & \text{if } a = 2, \\ 0.399 & \text{if } a = 3, \\ 0.318 & \text{if } 4 \leq a < 100, \\ 0.2 & \text{if } 100 \leq a < 10000, \\ 0.02 & \text{if } 10000 \leq a. \end{cases} \tag{3-5}$$

Therefore, if $r \geq 5$,

$$r - 2 - \sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \dfrac{a_i\delta_i^2}{a_i-1}}{x} > r - 2 - 0.56r \geq (2r)0.02.$$

If $r = 4$ and $a_4 \geq 4$, then

$$r - 2 - \sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \dfrac{a_i\delta_i^2}{a_i-1}}{x} > 2 - 0.56 \cdot 3 - 0.318 = 0.002,$$

while if $r = 4$ and $a_3 \geq 3$, then

$$r - 2 - \sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \dfrac{a_i\delta_i^2}{a_i - 1}}{x} > 2 - 0.56 \cdot 2 - 0.4 \cdot 2 = 0.08.$$

The only remaining possibility for $r = 4$ is 2223.

For $r = 3$, we may assume $a_2 \geq 3$, so if $a_3 \geq 10000$,

$$r - 2 - \sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \dfrac{a_i\delta_i^2}{a_i - 1}}{x} > 1 - 0.56 - 0.4 - 0.02 = 0.02.$$

The triples with $a_3 < 10000$ can be handled exhaustively by machine, by partitioning the x-interval $\left[0, \frac{1}{2}\right]$ into subintervals on which $G_{a,x}$ and $H_{a,x}$ are bounded above. □

Lemma 3.3. *Let g be a positive integer and r a nonnegative integer such that if $g = 1$, then $r > 0$. Let $a_1 \leq a_2 \leq \cdots \leq a_r$ be a (possibly empty) sequence of integers at least 2. Then for all $x \in \left[0, \frac{1}{2}\right]$,*

$$\sum_{i=1}^{r} f_{a_i,x}(\delta_i) < -0.22(r+1)x + (2g + r - 2)x + \sum_{i=1}^{r} \frac{a_i\delta_i^2}{a_i - 1}.$$

Proof. If $g = 1$ and $r \geq 1$, (3-5) implies

$$\sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \dfrac{a_i\delta_i^2}{a_i - 1}}{x} < 0.56r \leq -0.44r + r \leq -0.22(r+1) + (2g + r - 2).$$

If $g \geq 2$ and $r \geq 0$,

$$\begin{aligned}\sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \dfrac{a_i\delta_i^2}{a_i - 1}}{x} &\leq 0.56r \\ &< -0.22(r+1) + (2+r) \leq -0.22(r+1) + (2g+r-2).\end{aligned}$$ □

4. Asymptotics of $j_{q,n}(a)$

Let t be an element of $G_n = \mathrm{GL}_n(q)$ of order a. We assume q is prime to a. Let $\zeta = \zeta_a$ be a primitive a-th root of unity in $\overline{\mathbb{F}}_q$, so $z^a - 1 = 0$ has distinct roots $\zeta, \zeta^2, \ldots, \zeta^a = 1$ in $\overline{\mathbb{F}}_q$. Let m_i denote the multiplicity of ζ^i as an eigenvalue of t. We write $i \sim j$ if ζ^i and ζ^j have the same Frobenius orbit. Then:

(1) $m_i \in \mathbb{Z}$ for all i.

(2) $m_1 + \cdots + m_a = n$.

(3) $m_i = m_j$ whenever $i \sim j$.

(4) $m_i \geq 0$ for all i.

The element t is determined up to conjugacy in $\mathrm{GL}_n(q)$ by the vector $(m_1, \ldots, m_a)$. For given n, the vector is determined by $m_1, \ldots, m_{a-1}$, so the number of possibilities is $O(n^{a-1})$.

Let S denote the subset of $\{1, \ldots, a\}$ consisting of the smallest element in each Frobenius orbit, and let l_s be the size of the orbit of s. The centralizer of t in $\mathrm{GL}_n(q)$ can be written $\prod_{s\in S} \mathrm{GL}_{m_s}(q^{l_s})$, so the conjugacy class $C = t^{G_n}$ satisfies

$$(4\text{-}1)\qquad |C| = \frac{q^{n^2}\prod_{j=1}^n (1-q^{-j})}{\prod_{s\in S}\bigl(q^{l_s m_s^2}\prod_{j=1}^{m_s}(1-q^{-l_s j})\bigr)} = \frac{q^{n^2-\sum_{i=1}^a m_i^2} q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} \frac{\prod_{s\in S}\prod_{j=m_s+1}^\infty (1-q^{-l_s j})}{\prod_{j=n+1}^\infty (1-q^{-j})},$$

where $\eta(z)$ is the Dedekind eta-function. The second multiplicand on the right-hand side can be bounded above in terms of a, and it approaches 1 as $\inf_i m_i$ goes to ∞.

Writing

$$(4\text{-}2)\qquad n^2 - \sum_{i=1}^a m_i^2 = n^2\Bigl(1-\frac{1}{a}\Bigr) - \sum_{i=1}^a \Bigl(\frac{n}{a} - m_i\Bigr)^2,$$

we see that if $m_i \le n/2a$, then

$$|C| = O(q^{(1-1/a-1/(4a^2))n^2}),$$

so the sum of $|C|$ over all conjugacy classes with $\inf_i m_i \le n/2a$ is $o(q^{(1-1/a)n^2})$.

We define $j_{q,n,k}(a)$ to be the number of elements $t \in \mathrm{GL}_n(q)$ with $t^a = 1$ and $\det(a) = \zeta^k$. Consider the subset of $\mathbb{Z}^a$ satisfying conditions (1)–(3) and the congruence condition

$$(4\text{-}3)\qquad \sum_{i=1}^r i m_i \equiv k \pmod a,$$

which is equivalent to the condition $\det(t) = \zeta^k$. This is a coset $\lambda_n + \Lambda$, where Λ is a subgroup of $\mathbb{Z}^a$ which does not depend on n and $\lambda_n \in \mathbb{Z}^a$ has coordinate sum n. Moreover, adding 2 to each m_i and $2a$ to n preserves the sets satisfying conditions (1)–(3) and (4-3), so

$$\lambda_{n+2a} = \lambda_n + (2, 2, \ldots, 2).$$

Thus,

$$\lambda'_n := \lambda_n - \Bigl(\frac{n}{a}, \frac{n}{a}, \ldots, \frac{n}{a}\Bigr)$$

is periodic in n with period $2A$ and has coordinate sum 0.

By (4-1) and (4-2),

$$
\begin{aligned}
j_{q,n,k}(a) &= \sum_{(m_1,\dots,m_a)\in(\lambda_n+\Lambda)\cap\mathbb{N}^a} \frac{q^{n^2}\prod_{j=1}^{n}(1-q^{-j})}{\prod_{s\in S}\bigl(q^{l_s m_s^2}\prod_{j=1}^{m_s}(1-q^{-l_s j})\bigr)} \\
&= \sum_{(m_1,\dots,m_a)\in(\lambda_n+\Lambda)\cap\mathbb{N}^a} \frac{q^{n^2-\sum_{i=1}^{a} m_i^2}\, q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} + o(q^{(1-1/a)n^2}) \\
&= \frac{q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} \sum_{(m_1,\dots,m_a)\in\lambda_n+\Lambda} q^{n^2-\sum_{i=1}^{a} m_i^2} + o(q^{(1-1/a)n^2}) \\
&= \frac{q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} q^{(1-1/a)n^2} \sum_{\lambda'\in\lambda'_n+\Lambda} q^{-\lambda'\cdot\lambda'} + o(q^{(1-1/a)n^2}),
\end{aligned}
$$

where the implicit constant on the right-hand side does not depend on q. Defining

$$
\theta_v(z) := \sum_{\lambda\in v+\Lambda} e^{2\pi i(\lambda\cdot\lambda)z} \quad\text{and}\quad f_n(z) := \frac{\eta(z)\theta_{\lambda'_n}(z)}{\prod_{s\in S}\eta(l_s z)},
$$

we have proved the following:

Proposition 4.1. *The periodic sequence $f_1, f_2, f_3, \dots$ of half-integral weight modular forms with integral q-expansions satisfies*

$$
j_{q,n,k}(a) = \left(f_n\left(\frac{i\log q}{2\pi}\right) + o(1)\right) q^{\frac{1}{24}(1-a)} q^{(1-1/a)n^2},
$$

where the $o(1)$ term does not depend on q.

From this, it is easy to deduce:

Corollary 4.2. *Let $a_1, \dots, a_r$ denote positive integers with least common multiple A. Then there exists a $2A$-periodic sequence of meromorphic modular forms $f_1, f_2, f_3, \dots$ with integral Fourier coefficients, holomorphic except possibly at $i\infty$, such that*

$$
J_{q,n}(a_1,\dots,a_r) = \left(f_n\left(\frac{i\log q}{2\pi}\right) + o(1)\right) q^{\frac{1}{24}(r-a_1-\cdots-a_r)} q^{(r-1/a_1-\cdots-1/a_r)n^2}.
$$

Proof. Let

$$
\Sigma(a_1,\dots,a_r) := \Bigl\{(k_1,\dots,k_r)\in(\mathbb{Z}/a_1\mathbb{Z})\times\cdots\times(\mathbb{Z}/a_r\mathbb{Z}) \mid \prod \zeta_{a_i}^{k_i} = 1\Bigr\}.
$$

If $t_i^{a_i}=1$, then $\det(t_i)=\zeta_{a_i}^{k_i}$ for a well-defined $k_i\in\mathbb{Z}/a_i\mathbb{Z}$. Every element $(t_1,\dots,t_r)$ of (1-2) determines $(k_1,\dots,k_r)\in\Sigma(a_1,\dots,a_r)$ such that $\det(t_i)=\zeta_{a_i}^{k_1}$. Therefore,

$$
J_{q,n}(a_1,\dots,a_r) = \sum_{(k_1,\dots,k_r)\in\Sigma(a_1,\dots,a_r)} \prod_{i=1}^{r} j_{q,n,k_i}(a_i),
$$

and the corollary follows immediately from Proposition 4.1. □

5. Counting Fuchsian group representations

We now prove the main results of the paper. We continue with the notation of the previous section.

If $t^a = 1$, and $m_1, \dots, m_a$ are the eigenvalue multiplicities of t, define $\delta := -1/a + \sup_i m_i/n$. Thus, $\delta \geq 0$. Let j be chosen so $m_j = n/a + \delta n$.

Proposition 5.1. *For all $\epsilon > 0$, there exist N and q_0 depending only on ϵ such that if $n > N$, $q > q_0$, $x \in \left(0, \frac{1}{2}\right]$ and χ is an irreducible character of $G_n = \mathrm{GL}_n(q)$ of degree $q^{xn^2} > 1$, then*

$$\frac{\log |t^{G_n}||\chi(t)|}{n^2 \log q} \leq \left(1 - \frac{1}{a} - \frac{a\delta^2}{a-1}\right) + \frac{1}{n} + \epsilon x + f_{a,x}(\delta).$$

Proof. Since

$$\sum_{i \neq j} \left(\frac{n}{a} - m_i\right) \geq \delta n,$$

by the Cauchy–Schwartz inequality,

$$\sum_{i \neq j} \left(\frac{n}{a} - m_i\right)^2 \geq \frac{\delta^2 n^2}{a-1}.$$

By (4-2), the dimension of the centralizer of t in the algebraic group GL_n is

$$\frac{n^2}{a} + \sum_i \left(\frac{n}{a} - m_i\right)^2 \geq \frac{n^2}{a} + \delta^2 n^2 + (a-1)\frac{\delta^2 n^2}{(a-1)^2} = \frac{n^2}{a} + \frac{a\delta^2 n^2}{a-1}. \tag{5-1}$$

The order of the centralizer L of t in G_n is less than or equal to q to the power of the centralizer dimension, so the centralizer bound $|\chi(t)| \leq |L|^{1/2}$ for irreducible characters $\chi \in \mathrm{Irr}(G_n)$ implies

$$\begin{aligned} |t^{G_n}||\chi(t)| \leq \frac{|G_n|}{|L|^{\frac{1}{2}}} &\leq q^{n^2 - \frac{1}{2}\left(\frac{n^2}{a} + \frac{a\delta^2 n^2}{a-1}\right)} (1 - q^{-1})^{-\frac{1}{2}n} \\ &\leq q^{n^2\left(1 - \frac{1}{2a} - \frac{a\delta^2}{2(a-1)}\right) + \frac{1}{2}n}. \end{aligned} \tag{5-2}$$

On the other hand, by [Bezrukavnikov et al. 2018, Theorem 1.10],

$$\alpha(L) \leq \frac{\sup_i m_i}{n} = \frac{1}{a} + \delta.$$

The character bound Theorem 2.9 therefore implies

$$|\chi(t)| \leq \chi(1)^{\frac{1}{a} + \delta + \epsilon} = q^{n^2 x \left(\frac{1}{a} + \delta + \epsilon\right)},$$

so

$$|t^{G_n}||\chi(t)| \leq q^{n^2\left(1 - \frac{1}{a} - \frac{a\delta^2}{a-1}\right) + n} q^{n^2 x\left(\frac{1}{a} + \delta + \epsilon\right)}.$$

Combining this with (5-2), we get

$$\begin{aligned}\frac{\log|t^{G_n}||\chi(t)|}{n^2\log q}&\\ &\le \min\Bigl(\Bigl(1-\frac{1}{a}-\frac{a\delta^2}{a-1}\Bigr)+\frac{1}{n}+x\Bigl(\frac{1}{a}+\delta+\epsilon\Bigr),\Bigl(1-\frac{1}{2a}-\frac{a\delta^2}{2(a-1)}\Bigr)+\frac{1}{2n}\Bigr)\\ &\le \Bigl(1-\frac{1}{a}-\frac{a\delta^2}{a-1}\Bigr)+\frac{1}{n}+\epsilon x+\min\Bigl(x\Bigl(\frac{1}{a}+\delta\Bigr),\frac{1}{2a}+\frac{a\delta^2}{2(a-1)}\Bigr)\\ &= \Bigl(1-\frac{1}{a}-\frac{a\delta^2}{a-1}\Bigr)+\frac{1}{n}+\epsilon x+f_{a,x}(\delta). \qquad\square\end{aligned}$$

Proposition 5.2. *There exist absolute constants $\epsilon > 0$, q_0, and N with the following property. If $2 \le a_1 \le a_2 \le \cdots \le a_r$ is not excluded in Proposition 3.2 above, $q > q_0$ is a prime power relatively prime to all a_i, $n > N$, elements $t_i \in G_n = \mathrm{GL}_n(q)$ satisfy $t_i^{a_i} = 1$, and χ is a nonlinear irreducible character of G_n, then*

$$\frac{\prod_i |t_i^{G_n}|}{|G_n|}\frac{\prod_i |\chi(t_i)|}{\chi(1)^{r-2}} \le 4q^{n^2(-1+\sum_i(1-\frac{1}{a_i}))}\chi(1)^{-\epsilon}.$$

Proof. Let $x := \log_{q^{n^2}}\chi(1)$, which by [Landazuri and Seitz 1974] is at least $1/(2n)$. We fix q_0 as in Proposition 5.1 and ϵ as in Proposition 3.2 and choose $N > 2/\epsilon$. Let $\delta_i := \mu_i - 1/a_i$, where $\mu_i n$ is the highest multiplicity of any eigenvalue of t_i. By Proposition 5.1 and Proposition 3.2,

$$\begin{aligned}\frac{\log\bigl(\prod_i |t_i^{G_n}|\prod_i|\chi(t_i)|\bigr)}{n^2\log q} &\le \sum_{i=1}^r\Bigl(1-\frac{1}{a_i}-\frac{a_i\delta_i^2}{a_i-1}+\frac{1}{n}+\epsilon x+f_{a_i,x}(\delta_i)\Bigr)\\ &= \frac{r}{n}+rx\epsilon+\sum_i\Bigl(1-\frac{1}{a_i}\Bigr)+\sum_i\Bigl(f_{a_i,x}(\delta_i)-\frac{a_i\delta_i^2}{a_i-1}\Bigr)\\ &< \frac{r}{n}-rx\epsilon+\sum_i\Bigl(1-\frac{1}{a_i}\Bigr)+(r-2)x\\ &< -x\epsilon+\sum_i\Bigl(1-\frac{1}{a_i}\Bigr)+(r-2)x.\end{aligned}$$

Since $|G_n| \ge \frac{1}{4}q^{n^2}$, the proposition follows. $\square$

The positive genus variant of this result is as follows:

Lemma 5.3. *There exist absolute constants q_0 and N such that for every $g \ge 1$, every sequence $2 \le a_1 \le \cdots \le a_r$ (assumed nonempty if $g = 1$), every prime power $q > q_0$ relatively prime to a_i for all $i \le r$, every $n > N$, every tuple $(t_1, \ldots, t_r) \in G_n^r$ satisfying $t_i^{a_i} = 1$ for all i, and every nonlinear $\chi \in \mathrm{Irr}(G_n)$, we have*

$$\prod_i |t_i^{G_n}||G_n|^{2g-1}\frac{\prod_i|\chi(t_i)|}{\chi(1)^{2g+r-2}} \le q^{n^2(2g-1+\sum_i(1-\frac{1}{a_i}))}\chi(1)^{-0.22}.$$

Proof. The proof is exactly the same as that of Proposition 5.2 except that we use Lemma 3.3 instead of Proposition 3.2. □

Lemma 5.4. *Let $H_i = \mathrm{GL}_{n_i}(q_i)$, where $\lim_{i\to\infty} n_i = \infty$. Then for all $\epsilon > 0$,*

$$\sum_{\chi\in\mathrm{Irr}(H_i)} \chi(1)^{-\epsilon} = q_i - 1 + o(q_i^{-\frac{1}{3}\epsilon n_i})$$

Proof. By [Liebeck and Shalev 2005a, Theorem 1.2],

$$\sum_{\chi\in\mathrm{Irr}(\mathrm{SL}_{n_i}(q_i))} \chi(1)^{-\frac{1}{2}\epsilon} = 1 + o(1),$$

so if D_i denotes the minimum degree of a nontrivial character of $\mathrm{SL}_{n_i}(q_i)$,

$$\sum_{\chi\in\mathrm{Irr}(\mathrm{SL}_{n_i}(q_i))} \chi(1)^{-\epsilon} = 1 + o(D_i^{-\frac{1}{2}\epsilon}) = 1 + o(q_i^{-\frac{1}{3}\epsilon n_i})$$

by [Landazuri and Seitz 1974]. The relation which assigns to each element of $\mathrm{Irr}(H_i)$ all the elements in $\mathrm{Irr}(\mathrm{SL}_{n_i}(q_i))$ which are constituents of its restriction is at most $q-1$ to 1 and nonincreasing in degree. There are $q_i - 1$ linear characters for H_i, all mapping to the trivial character of $\mathrm{SL}_{n_i}(q_i)$). Therefore,

$$1 - q_i + \sum_{\chi\in\mathrm{Irr}(H_i)} \chi(1)^{-\epsilon} = o((q_i-1)q_i^{-\frac{1}{3}\epsilon n_i}),$$

and the lemma follows. □

We can now prove Theorems A and B.

Proof. We assume first that $g = 0$, so Γ is determined by $2 \le a_1 \le a_2 \le \cdots \le a_r$. A homomorphism $\Gamma \to G_n$ is determined by the images $t_1, \ldots, t_r \in G_n$ of $z_1, \ldots, z_r \in \Gamma$, which satisfy $t_i^{a_i} = 1$ and $t_1 \cdots t_r = 1$. We can partition the set of homomorphisms according to the conjugacy classes $C_1, \ldots, C_r$ to which the t_i belong. By the Frobenius formula, the total number of homomorphisms is

$$\sum_{(C_1,\ldots,C_r)} \frac{|C_1\times\cdots\times C_r|}{|G_n|} \sum_{\chi\in\mathrm{Irr}(G_n)} \frac{\chi(C_1)\cdots\chi(C_r)}{\chi(1)^{r-2}}. \tag{5-3}$$

The determinant condition implies that each linear character in the inner sum contributes 1, and there are a total of $q-1$ such characters. Their total contribution is therefore

$$(q-1)\sum \frac{|C_1\times\cdots\times C_r|}{|G_n|} = (q-1)J_{q,n}(a_1,\ldots,a_r)|G_n|^{-1}. \tag{5-4}$$

By Proposition 5.2 and Lemma 5.4, the contribution of all nonlinear characters χ to (5-3) is $o(q^{-\frac{1}{3}\epsilon n} q^{(1-\chi(\Gamma))n^2})$, where ϵ is the positive absolute constant defined in Proposition 5.2. By Corollary 4.2,

$$|\mathrm{Hom}(\Gamma, G)| = (q-1)q^{\frac{1}{24}(r-a_1-\cdots-a_r)}\left(f_n\left(\frac{i\log q}{2\pi}\right) + o(1)\right)q^{(1-\chi(\Gamma))n^2},$$

which implies Theorem A and Theorem B in the genus 0 case. The proof in the higher genus case is the same except that we use Lemma 5.3 instead of Proposition 5.2. □

Proposition 5.5. *Let a and n be positive integers, $a \geq 2$, and let q be a prime power which is* $1 \pmod a$. *The minimum dimension of the centralizer in* GL_n *of a semisimple element* $t \in \mathrm{GL}_n(q)$ *of order dividing a is* $n^2/a + a\{n/a\}\{-n/a\}$. *If a is odd, t can be chosen to have determinant* 1. *If a is even and* $n/a \notin \mathbb{Z}$, *then t can be chosen to have determinant* 1 *or* -1. *If a is even and* $n/a \in \mathbb{Z}$, *then t must have determinant* $(-1)^{n/a}$; *if this is* -1, *there is no element in* $\mathrm{GL}_n(q)$ *whose centralizer has dimension* $n^2/a + 1$, *but there is an element* $t' \in \mathrm{SL}_n(q)$ *with centralizer dimension* $n^2/a + 2$.

Proof. If the multiplicities $m_1, \dots, m_a$ of the eigenvalues $\zeta_a, \dots, \zeta_a^a$ of a semisimple $t \in \mathrm{GL}_n(q)$ satisfying $t^a = 1$ are written $n/a + \epsilon_i$, then the centralizer of t has dimension

$$\sum_i m_i^2 = \frac{n^2}{a} + \sum_i \epsilon_i^2.$$

As $\sum_i \epsilon_i = 0$, either all are zero (which can only happen in the case that a divides n), or at least one is positive and at least one is negative. In the latter case, if any $\epsilon_i \geq 1$, then by reducing this by 1 and increasing some negative ϵ_j by 1, we decrease $\sum_i \epsilon_i^2$, and likewise if some $\epsilon_i \leq -1$. As all ϵ_i are $\{n/a\} \pmod 1$, each must be $\{n/a\}$ or $\{n/a\} - 1$, and since they sum to zero, there must be $a - a\{n/a\}$ of the former and $a\{n/a\}$ of the latter, implying $\sum_i \epsilon_i^2 = a\{n/a\}\{-n/a\}$.

Next, we claim that as long as $\{n/a\} \neq 0$, there exists some sequence $m_1, \dots, m_a$ consisting of $a\{n/a\}$ copies of $\lceil n/a \rceil$ and $a - a\{n/a\}$ copies of $\lfloor n/a \rfloor$ such that $\prod_i \zeta_a^{im_i}$ is any desired power of ζ_a. To prove this, it suffices to show that if $0 < k < a$, the sums of k-element subsets S of $\{0, 1, \dots, a-1\}$ represent all residue classes $\pmod a$. Indeed, if $S \neq \{a-1, a-2, \dots, a-k\}$, there exists $s \in S$ such that $s + 1 \in \{0, 1, \dots, a-1\} \setminus S$. Thus, the set of sums of k-element subsets S includes all integers from $\binom{k}{2}$ to $\binom{a}{2} - \binom{a-k}{a}$, a total of $k(a-k)+1 \geq a$ consecutive integers, which therefore represent all congruence classes $\pmod a$.

Finally, assume a divides n, so $m_1 = \cdots = m_a = n/a$ gives the minimum value n^2/a of $\sum_i m_i^2$. Any other choice of $(m_1, \dots, m_a)$ must have all ϵ_i integral and at least two nonzero, so $\sum_i m_i^2 \geq n^2/a + 2$. If $m_1 = \cdots = m_a$ and n/a is even or a is odd, then $\sum_i i m_i$ is divisible by a, so $\det(t) = 1$. If a is even and n/a is odd, then $m_1 = \cdots = m_a$ gives $\det(t) = -1$. In this last case, setting $\epsilon_1 = 1$ and $\epsilon_{a/2+1} = -1$ and all other $\epsilon_i = 0$, we get $\sum_i i m_i$ is divisible by a and $\sum_i m_i^2 = n^2/a + 2$. □

Let E_Γ denote the set of i such that a_i is even. As in the statement of Theorem C, for each positive integer n, we define $\sigma_{\Gamma,n} := -1$ if $n/a_i \in \mathbb{Z}$ for all $i \in E_\Gamma$, and $\sum_{i \in E_\Gamma} n/a_i$ is odd; otherwise $\sigma_{\Gamma,n} := 1$.

Proposition 5.6. *Suppose $q \equiv 1 \pmod{a_i}$ for all i. If $(t_1, \dots, t_r)$ is an r-tuple of semisimple elements in $\mathrm{GL}_n(q)$ such that $t_i^{a_i} = 1$ and $\prod_i \det(t_i) = 1$, the minimum possible sum of the dimensions of the centralizers of the t_i in GL_n is*

$$1 - \sigma_{\Gamma,n} + \sum_{i=1}^{r} \Bigl(\frac{n^2}{a_i} + a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\} \Bigr).$$

Proof. If there is at least one a_i which is even and such that $n/a_i \notin \mathbb{Z}$, then we can choose t_i to have either determinant 1 or -1 and centralizer dimension

$$\frac{n^2}{a_i} + a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\}. \tag{5-5}$$

For $j \neq i$, we can choose t_j to have determinant in $\{\pm 1\}$ and centralizer $n^2/a_j + a_j\{n/a_j\}\{-n/a_j\}$. Therefore, we can choose minimal centralizer dimension for all t_i while imposing the condition $\prod_i \det(t_i) = 1$.

If $n/a_i \in \mathbb{Z}$ for all a_i even, and the set of i such that a_i is even and n/a_i is odd has even cardinality, then we may choose t_i whose centralizer has dimension (5-5) for all i and such that $\det(t_i) = 1$ except when a_i is even and n/a_i is odd. In these cases, of which there are an odd number, $\det(t_i) = 1$, so again $\prod_i \det(t_i) = 1$.

What remains is the case $\sigma_{\Gamma,n} = -1$, and here if t_i has centralizer dimension (5-5) for all i, then the product $\prod_i \det(t_i)$ is -1 times a product of elements of odd order, so it cannot be 1. On the other hand, if we choose one t_i with a_i even and n/a_i odd, and choose it to have determinant 1 and centralizer dimension $n^2/a_i + 2$, and all other t_j have minimal centralizer dimension and $\det(t_j) = \pm 1$, then the product $\prod_i \det(t_i)$ equals 1. □

Proof of Theorem C. By Theorem B, there exist q_0 and N such that if $q > q_0$ is relatively prime to A and $n > N$, then

$$\tfrac{1}{2} < \frac{q^{(1-2g)n^2} |\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))|}{q J_{q,n}(a_1, \dots, a_r)} < \tfrac{3}{2}.$$

For any fixed such q and n, let $X_{q,n}$ denote the variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ over the field $\mathbb{F}_q$. Then, for all positive integers m,

$$\tfrac{1}{2} < \frac{q^{(1-2g)mn^2} |X_{q,n}(\mathbb{F}_{q^m})|}{q^m J_{q^m,n}(a_1, \dots, a_r)} < \tfrac{3}{2}.$$

By Proposition 5.6,

$$\begin{aligned}
\dim X_{q,n} &= \limsup_m \log_{q^m} |X_{q,n}(\mathbb{F}_{q^m})| \\
&= 1 + (2g-1)n^2 + \limsup_m \log_{q^m} J_{q^m,n}(a_1, \dots, a_r) \\
&= 1 + (2g-1)n^2 + rn^2 - 1 + \sigma_{\Gamma,n} - \sum_{i=1}^{r} \Bigl(\frac{n^2}{a_i} + a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\} \Bigr) \\
&= \sigma_{\Gamma,n} + (1 - \chi(\Gamma))n^2 - \sum_{i=1}^{r} a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\}.
\end{aligned}$$

The first claim of the theorem follows in the positive characteristic case.

For characteristic zero, we consider the scheme $\mathrm{Hom}(\Gamma, \mathrm{GL}_{n,\mathbb{Z}})$ over $\mathrm{Spec}\,\mathbb{Z}$ whose $\mathbb{F}_p$ fiber is the n-dimensional representation variety of Γ over $\mathbb{F}_p$. By the constructibility of the set of dimensions of irreducible fiber components [EGA IV$_3$ 1966, proposition 9.5.5], the dimension of the generic fiber must be the same as the common dimension of any infinite set of fibers over closed points.

Finally, for the second claim of the theorem, we observe that $\{t\}\{-t\} \le \frac{1}{4}$ for all real t, so for all i,

$$a_i\left\{\frac{n}{a_i}\right\}\left\{-\frac{n}{a_i}\right\} \le \tfrac{1}{4}a_i.$$

We have $\sigma_{\Gamma,n} > -\frac{1}{2}$ unless there is at least one value of i for which a_i is even and n/a_i is integral. For this value of i, $a_i \ge 2$, so

$$\sigma_{\Gamma,n} - \sum_{i=1}^{r} a_i\left\{\frac{n}{a_i}\right\}\left\{-\frac{n}{a_i}\right\} \ge -\tfrac{1}{2} - \sum_{i=1}^{r}\tfrac{1}{4}a_i. \qquad \square$$

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In memoriam
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