Pacific Journal of Mathematics

D_4-TYPE SUBGROUPS OF $F_4(q)$

R. LAWTHER

Volume 336 No. 1-2 May 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 336, No. 1-2, 2025
https://doi.org/10.2140/pjm.2025.336.249

D_4-TYPE SUBGROUPS OF $F_4(q)$

R. LAWTHER

We treat the action of the simple group $F_4(q)$ on the cosets of subgroups $D_4(q)$, ${}^2D_4(q)$ and ${}^3D_4(q)$ and their extensions by graph automorphisms. We obtain the ranks and decompose the corresponding permutation characters; we show that, even allowing for the application of field automorphisms, the only two primitive multiplicity-free actions arising are those of $F_4(2)$ on cosets of $D_4(2).S_3$ and ${}^3D_4(2).3$. For these two actions, we calculate the subdegrees; we find that all suborbits are self-paired, but that the action gives rise to no distance-transitive graph.

1. Introduction

This paper represents a further contribution to the programs described in [15], namely the classification of finite primitive permutation groups with the property that the action is on the vertices of a distance-transitive graph, or (more generally) all suborbits are self-paired, or (more generally still) the permutation character is multiplicity-free. Results of [1; 23] essentially reduce these classifications to the consideration of almost simple groups; many cases have already been studied and resolved. In [15] the case of the action of $F_4(q)$ on cosets of $B_4(q)$ was treated; in this paper, which in many ways may be seen as a continuation of [15], we consider two further actions of $F_4(q)$, on cosets of $D_4(q).S_3$ and of ${}^3D_4(q).3$. In conjunction with [17] (about which we shall say more later), we shall show that each of these actions is only multiplicity-free if $q = 2$ (and, moreover, that if $q > 2$ the application of field automorphisms never makes the action multiplicity-free); in these two cases we shall calculate the subdegrees, and show that all suborbits are self-paired, but that the action gives rise to no distance-transitive graph.

In fact, we shall consider other subgroups beside the two just mentioned. We shall say that a subgroup of $F_4(q)$ is a D_4-*type subgroup* if it is of the form ${}^mD_4(q).\Gamma$, where $m \in \{1, 2, 3\}$ (interpreting ${}^1D_4(q)$ to mean simply $D_4(q)$), and Γ is a group of graph automorphisms of ${}^mD_4(q)$. For completeness, we shall decompose the permutation characters of the actions of $F_4(q)$ on cosets of all D_4-type subgroups.

MSC2020: 20G05.
Keywords: permutation characters, coset actions, primitive groups.

We let G be a simple algebraic group of type F_4 over k, the algebraic closure of $\mathbb{F}_p$, where p is a prime. We let T_0 be a maximal torus of G, and Φ be the set of roots of G relative to T_0; we choose a Borel subgroup B of G containing T_0, and let Φ^+ and Σ be the sets of positive and simple roots determined by B. As in [24; 25], we let

$$\Sigma = \left\{\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4, \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\right\},$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ form an orthonormal basis of a 4-dimensional Euclidean space; then

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j : 1 \le i < j \le 4\} \cup \{\epsilon_i : 1 \le i \le 4\} \cup \left\{\tfrac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\},$$

and $\Phi = \Phi^+ \cup -\Phi^+$. For convenience, where there is no danger of confusion, we shall write $\pm i \pm j$ for $\pm\epsilon_i \pm \epsilon_j$, $\pm i$ for $\pm\epsilon_i$, and $+{-}{-}{-}$ for $\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$, etc. in what follows. We write $N = N_G(T_0)$, and $W = N/T_0$ for the Weyl group of G.

Given a root $\alpha \in \Phi$, we let U_α be the corresponding root subgroup of G; we write $U = \prod_{\alpha\in\Phi^+} U_\alpha$. For each α there is an isomorphism $x_\alpha : k \to U_\alpha$ such that for all $\lambda \in k$ and $t \in T_0$ we have ${}^t x_\alpha(\lambda) = x_\alpha(\alpha(t)\lambda)$ (where as usual a group element as superscript denotes conjugation on the appropriate side). We assume that the isomorphisms x_α are chosen as in [4, Section 1.9] such that if for all $\lambda \in k^*$ we set

$$n_\alpha(\lambda) = x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda), \quad h_\alpha(\lambda) = n_\alpha(\lambda)n_\alpha(-1),$$

then as in [3, Lemma 6.4.4] for all $\lambda \in k^*$ we have $n_\alpha(\lambda) \in N$ and $h_\alpha(\lambda) \in T_0$; the maps $h_\alpha : k^* \to T_0$ are the coroots. For all $\lambda, \mu \in k$ we also have the Chevalley commutator relations

$$[x_\alpha(\lambda), x_\beta(\mu)] = \begin{cases} 1 & \text{if } \alpha+\beta \notin \Phi, \\ x_{\alpha+\beta}(N_{\alpha,\beta}\lambda\mu) & \text{if } \alpha+\beta \in \Phi,\ 2\alpha+\beta, \alpha+2\beta \notin \Phi, \\ x_{\alpha+\beta}(N_{\alpha,\beta}\lambda\mu)x_{2\alpha+\beta}\left(-\tfrac{1}{2}N_{\alpha,\beta}N_{\alpha,\alpha+\beta}\lambda^2\mu\right) & \text{if } \alpha+\beta, 2\alpha+\beta \in \Phi, \\ x_{\alpha+\beta}(N_{\alpha,\beta}\lambda\mu)x_{\alpha+2\beta}\left(\tfrac{1}{2}N_{\alpha,\beta}N_{\alpha+\beta,\beta}\lambda\mu^2\right) & \text{if } \alpha+\beta, \alpha+2\beta \in \Phi, \end{cases}$$

where we assume that the structure constants $N_{\alpha,\beta}$ are as given in [25]. For each α we write $n_\alpha = n_\alpha(1)$, and $w_\alpha = n_\alpha T_0 \in W$.

Much as in [24], we shall write elements of T_0 in the form $(\mu_1, \mu_2, \mu_3, \mu_4; \nu)$ with $\mu_1, \mu_2, \mu_3, \mu_4, \nu \in k^*$ and $\nu^2 = \mu_1\mu_2\mu_3\mu_4$, where for $\lambda \in k^*$ we set

$$\begin{aligned} h_{2-3}(\lambda) &= (1, \lambda, \lambda^{-1}, 1; 1), \\ h_{3-4}(\lambda) &= (1, 1, \lambda, \lambda^{-1}; 1), \\ h_4(\lambda) &= (1, 1, 1, \lambda^2; \lambda), \\ h_{+---}(\lambda) &= (\lambda, \lambda^{-1}, \lambda^{-1}, \lambda^{-1}; \lambda^{-1}). \end{aligned}$$

The action of W on $\boldsymbol{T}_0$ is then determined by

$$
\begin{aligned}
{}^{w_{2-3}}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\mu_1, \mu_3, \mu_2, \mu_4; \nu),\\
{}^{w_{3-4}}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\mu_1, \mu_2, \mu_4, \mu_3; \nu),\\
{}^{w_4}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\mu_1, \mu_2, \mu_3, {\mu_4}^{-1}; \nu{\mu_4}^{-1}),\\
{}^{w_{+---}}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\nu, \nu{\mu_3}^{-1}{\mu_4}^{-1}, \nu{\mu_2}^{-1}{\mu_4}^{-1}, \nu{\mu_2}^{-1}{\mu_3}^{-1}; \mu_1).
\end{aligned}
$$

We let

$$
\begin{aligned}
\boldsymbol{H} &= \langle \boldsymbol{U}_\alpha : \alpha \in \Phi(\boldsymbol{H})\rangle,\\
\boldsymbol{C} &= \langle \boldsymbol{U}_\alpha : \alpha \in \Phi(\boldsymbol{C})\rangle,\\
\boldsymbol{A} &= \langle \boldsymbol{U}_\alpha : \alpha \in \Phi(\boldsymbol{A})\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi(\boldsymbol{H}) &= \{\pm\epsilon_i \pm \epsilon_j : 1 \le i < j \le 4\},\\
\Phi(\boldsymbol{C}) &= \left\{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_3 \pm \epsilon_4), \pm\epsilon_3, \pm\epsilon_4, \pm\tfrac{1}{2}(\epsilon_1 - \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\},\\
\Phi(\boldsymbol{A}) &= \left\{\pm\epsilon_4, \pm\tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 \pm \epsilon_4)\right\};
\end{aligned}
$$

then $\boldsymbol{H}$, $\boldsymbol{C}$ and $\boldsymbol{A}$ are simple algebraic groups over k of types D_4, C_3 and $\tilde{A}_2$, respectively, where as usual a tilde denotes a root system comprising short roots, and each is simply connected, as it is easy to see that the $\mathbb{Z}$-linear span of the relevant coroots equals the group of cocharacters of the relevant torus. We let

$$
\begin{aligned}
W_{\boldsymbol{H}} &= \langle w_{1-2}, w_{2-3}, w_{3-4}, w_{3+4}\rangle,\\
W_{\boldsymbol{C}} &= \langle w_{3-4}, w_4, w_{+---}\rangle,\\
W_{\boldsymbol{A}} &= \langle w_4, w_{+---}\rangle,
\end{aligned}
$$

so that $W_{\boldsymbol{H}}$, $W_{\boldsymbol{C}}$ and $W_{\boldsymbol{A}}$ are the Weyl groups of $\boldsymbol{H}$, $\boldsymbol{C}$ and $\boldsymbol{A}$, respectively.

We write

$$
\tau_1 = 1, \qquad \tau_2 = h_{+---}(-1)n_4, \qquad \tau_3 = n_4 n_{+---}.
$$

For $m \in \{1, 2, 3\}$ the element τ_m is of order m and acts on $\boldsymbol{H}$ as a graph automorphism; moreover $\tau_m \in \boldsymbol{A}$, and $C_{\boldsymbol{A}}(\tau_m)$ is connected (if $p \neq m$ then τ_m is semisimple, so the connectedness of $C_{\boldsymbol{A}}(\tau_m)$ follows from the simple connectedness of $\boldsymbol{A}$ and [4, Theorem 3.5.6]; if instead $p = m$ then τ_m is unipotent, and an easy calculation shows the connectedness of $C_{\boldsymbol{A}}(\tau_m)$). If $p \neq 2$, as in [13] we may set $y_2 = x_4(1)n_4x_4\left(\frac{1}{2}\right)$; then ${\tau_2}^{y_2} = h_{+---}(-1) = (-1, -1, -1, -1; -1)$. If instead $p = 2$ we may set $y_2 = x_{-4}(1)$; then ${\tau_2}^{y_2} = x_4(1)$. Likewise, if $p \neq 3$ we may take $\omega \in k^*$ with $\omega^3 = 1 \neq \omega$, and set $y_3 = x_{+--+}(\omega^2)x_{+---}(-\omega)x_4(1)n_4n_{+---}n_4x_4\left(\frac{1}{3}\right)x_{+---}\left(\frac{1-\omega}{3}\right)x_{+--+}\left(\frac{1+2\omega}{3}\right)$; then ${\tau_3}^{y_3} = h_4(\omega) = (1, 1, 1, \omega^2; \omega)$. If instead $p = 3$ we may set $y_3 = x_{-++-}(1)$ $x_{-4}(1)x_{-+++}(-1)$; then ${\tau_3}^{y_3} = x_4(1)x_{+---}(1)$.

We let q be a power of p, and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be the Frobenius map determined by $x_\alpha(\lambda)^F = x_\alpha(\lambda^q)$ for all $\lambda \in k$ and $\alpha \in \Phi$. As in [15; 25] we set $x = (4, q-1)$, $y = (3, q-1)$ and $z = (3, q+1)$. We take $d \in \{0, 1\}$ and $e \in \{0, \pm 1\}$ such that $q \equiv d \pmod 2$ and $q \equiv e \pmod 3$. For $m \in \{1, 2, 3\}$ we set

$$f = \begin{cases} 3 & \text{if } m = 1, \\ 1 & \text{if } m = 2, \\ 0 & \text{if } m = 3; \end{cases}$$

the element τ_m commutes with F, and as $\tau_m \in C_{\boldsymbol{A}}(\tau_m)$, by the Lang–Steinberg theorem we may take $g_m \in C_{\boldsymbol{A}}(\tau_m)$ with ${g_m}^F.{g_m}^{-1} = \tau_m$ (choosing $g_1 = 1$), and let $\boldsymbol{H}_m = \boldsymbol{H}^{g_m}$.

For any F-stable subset $\boldsymbol{X}$ of $\boldsymbol{G}$ we write $\boldsymbol{X}^F$ for the set of points of $\boldsymbol{X}$ fixed by F. We set $G = \boldsymbol{G}^F$, $T_0 = {\boldsymbol{T}_0}^F$, $B = \boldsymbol{B}^F$, $U_\alpha = {\boldsymbol{U}_\alpha}^F$ for $\alpha \in \Phi$, $U = \boldsymbol{U}^F$, $N = \boldsymbol{N}^F$, $A = \boldsymbol{A}^F$, $C = \boldsymbol{C}^F$ and $H_m = (\boldsymbol{H}_m)^F$ for $m \in \{1, 2, 3\}$; then $G = F_4(q)$ and $H_m = {}^mD_4(q)$ with $H_m < G$, and we have

$$\begin{aligned} |G| &= q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1), \\ |H_1| &= q^{12}(q^2-1)(q^6-1)(q^4-1)^2, \\ |H_2| &= q^{12}(q^2-1)(q^6-1)(q^8-1), \\ |H_3| &= q^{12}(q^2-1)(q^6-1)(q^8+q^4+1). \end{aligned}$$

For $m \in \{1, 2, 3\}$ we define the map $F_m = F\tau_m : \boldsymbol{G} \to \boldsymbol{G}$ by $g^{F_m} = (g^F)^{\tau_m}$ for $g \in \boldsymbol{G}$, and set $\hat{H}_m = {}^{g_m}H_m = \boldsymbol{H}^{F_m}$ (where we extend the superscript notation to mean the set of points fixed by F_m).

Now as $g_1 = \tau_1 = 1$, and for $m \in \{2, 3\}$ we have chosen g_m to commute with τ_m, given $m, r \in \{1, 2, 3\}$ with $\{m, r\} \neq \{2, 3\}$ we have ${\tau_r}^{g_m} = \tau_r$. It follows that, up to conjugacy, the D_4-type subgroups of G are H_1, H_2 and H_3; $H_1.2$ and $H_2.2$, where $H_m.2 = H_m\langle\tau_2\rangle$; $H_1.3$ and $H_3.3$, where $H_m.3 = H_m\langle\tau_3\rangle$; and $H_1.S_3 = H_1\langle\tau_2, \tau_3\rangle$. Of these, only $H_1.S_3$ and $H_3.3$ are maximal subgroups of G, since $H_2.2$ lies inside $B_4(q) = \langle U_{\pm(1-2)}, U_{\pm(2-3)}, U_{\pm(3-4)}, U_{\pm 4}\rangle$.

The structure of the rest of the paper is as follows. In Section 2 we develop a method for decomposing the permutation characters which occur here. In Section 3 we apply this to the cases where the D_4-type subgroup is H_m for $m \in \{1, 2, 3\}$; in Section 4 we do the same for the other D_4-type subgroups. Finally in Section 5 we consider the contributions of the actions treated here to the classification programs mentioned above.

2. Decomposing permutation characters

In this section we begin by giving information about conjugacy classes, and then explain a method for calculating scalar products of characters which uses it; this will allow us to decompose the permutation characters of interest here. After some preliminary results we prove the claim on which the method is based.

In what follows we shall make several statements to the effect that a certain expression is a 'polynomial in q'. By this we shall mean that, for a stated set of zero or more parameters which may take only finitely many possible values, if we fix their values there is a polynomial with rational coefficients such that, for any appropriate q, the expression equals the evaluation of the polynomial at q.

2.1. *Conjugacy classes in D_4-type subgroups.* We begin by providing information concerning conjugacy classes in D_4-type subgroups of G, which will be needed in much of the rest of the paper. We shall see that, for $m, r \in \{1, 2, 3\}$ with $\{m, r\} \neq \{2, 3\}$, it suffices to consider the H_m-classes in the coset $H_m\tau_r$; equivalently we may consider the $\hat{H}_m$-classes in the coset $\hat{H}_m\tau_r$ and then conjugate by g_m (note that $\hat{H}_1 = H_1$ as $g_1 = 1$, and that $\hat{H}_m\tau_r = {}^{g_m}(H_m\tau_r)$ as g_m commutes with τ_r).

Of fundamental importance is Jordan decomposition, but the way in which this is applied will depend on whether or not r equals p. We define

$$\bar{r} = \frac{r}{(r, p)} = \begin{cases} r & \text{if } r \neq p, \\ 1 & \text{if } r = p; \end{cases}$$

given an element of the coset $\hat{H}_m\tau_r$, its semisimple part lies in the coset $\hat{H}_m\tau_{\bar{r}}$, and indeed all semisimple elements of $\boldsymbol{H}\langle\tau_r\rangle$ lie in $\boldsymbol{H}\langle\tau_{\bar{r}}\rangle$. If $\bar{r} = r$ we begin with semisimple classes in $\hat{H}_m\tau_r$ and then take unipotent classes in $\hat{H}_m$ lying in centralizers of semisimple elements; if however $\bar{r} \neq r$ we instead begin with unipotent classes in $\hat{H}_m\tau_r$ and then take semisimple classes in $\hat{H}_m$ lying in centralizers of unipotent elements. In the former case, for each unipotent class in the $\hat{H}_m$-centralizer we shall need information concerning which unipotent class in the $\boldsymbol{G}$-centralizer contains it (sometimes partial information is sufficient for our purposes); frequently inspection allows us to determine the class precisely, but if necessary we can always calculate Jordan structure on an appropriate module (the natural module for groups of classical type, the 26-dimensional module for F_4 itself) and use results from either [18] or [14].

First take $r = 1$, so that our concern is with the $\hat{H}_m$-classes in $\hat{H}_m$ itself. We start with semisimple $\hat{H}_m$-classes in $\hat{H}_m$. Recall that any semisimple element of $\hat{H}_m = \boldsymbol{H}^{F_m}$ lies in an F_m-stable maximal torus of $\boldsymbol{H}$, and that there is a bijection between the $\boldsymbol{H}^{F_m}$-classes of F_m-stable maximal tori of $\boldsymbol{H}$ and the F_m-conjugacy classes of $W_{\boldsymbol{H}}$; as $F_m = F\tau_m$, and F acts trivially on W while τ_m corresponds to the Weyl group element 1, w_4 or w_4w_{+---} according as $m = 1$, 2 or 3, we see that the F_m-conjugacy classes of $W_{\boldsymbol{H}}$ correspond to the $W_{\boldsymbol{H}}$-classes in the coset $W_{\boldsymbol{H}}$, $W_{\boldsymbol{H}}w_4$ or $W_{\boldsymbol{H}}w_{+---}w_4$, respectively. In fact the semisimple classes of $B_4(q)$ are listed in the Appendix of [15], from which it is straightforward to deduce the semisimple

$\hat{H}_m$-classes in $\hat{H}_m$ for $m = 1$ and $m = 2$, while those for $m = 3$ are given in [5, Table 2.1]. The $\boldsymbol{G}$-centralizer of a semisimple element is determined by the roots with respect to a maximal torus containing it which take the value 1 at it, and so may easily be determined by inspection (in fact for $m = 1$ and $m = 2$ this information is given in the Appendix of [15]).

Next consider $\hat{H}_m$-classes in $\hat{H}_m$ which are not semisimple. Given a semisimple element of $\hat{H}_m$, the unipotent classes lying in its centralizer in either $\hat{H}_m$ or $\boldsymbol{G}$ may usually be obtained from the various results and tables in [18] (see Theorems 3.1, 7.1 and Tables 8.1a, 8.2a, 8.4a, 8.5a, 22.2.4); the exceptions are that no table is given for $B_4(q)$ (which here appears as the centralizer of a semisimple element only in odd characteristic) but it may be constructed using the results given in Chapters 3–7 therein (alternatively a list is given in [25], which of course also gives the unipotent classes of $F_4(q)$), and that ${}^3D_4(q)$ is not treated but results are given in [5]. In this way we obtain a complete set of representatives of the $\hat{H}_m$-classes in $\hat{H}_m$; conjugation by g_m then gives a complete set of representatives of the H_m-classes in H_m.

Now take $r \in \{2, 3\}$, so that $m \in \{1, r\}$. We note that for given r the $\hat{H}_m$-classes in $\hat{H}_m \tau_r$ for the two values of m are closely related by Shintani descent (see [6, I.7.2; 22, Proposition 5]): there is a bijection between the sets of $\hat{H}_1$-classes in $\hat{H}_1 \tau_r$ and $\hat{H}_r$-classes in $\hat{H}_r \tau_r$ which preserves centralizer orders. Indeed, we may proceed as follows: given an H_1-class in $H_1 \tau_r$ (i.e., an $\hat{H}_1$-class in $\hat{H}_1 \tau_r$), we may take a class representative $h\tau_r$ and use the Lang–Steinberg theorem to write h in the form $\bar{x}^{F_r}.\bar{x}^{-1}$ for some $\bar{x} \in \boldsymbol{H}$; Shintani descent gives the corresponding $\hat{H}_r$-class in $\hat{H}_r \tau_r$ as that containing $h^\dagger \tau_r$ where $h^\dagger = \bar{x}^{-1}.\bar{x}^{F_1}$; conjugating by g_r then gives the corresponding H_r-class in $H_r \tau_r$ as that containing $h' \tau_r$, where $h' = (h^\dagger)^{g_r}$.

First assume $r \neq p$; then $\tau_r = \tau_{\bar{r}}$ is a semisimple element of G. Here we begin by observing that $\tau_{\bar{r}}$ is a quasisemisimple automorphism of $\boldsymbol{H}$ (recall from [28, Section 9] that this means that it stabilizes both a Borel subgroup and a maximal torus therein). We make use of [20, 1.14]: we write $\boldsymbol{T}_{\bar{r}} = \boldsymbol{T}_0 \cap C_{\boldsymbol{H}}(\tau_{\bar{r}})$, so that $\boldsymbol{T}_{\bar{r}}$ is a maximal torus of $C_{\boldsymbol{H}}(\tau_{\bar{r}})$; we let $\boldsymbol{N}_{\bar{r}} = \{n \in \boldsymbol{H} : {}^n(\boldsymbol{T}_{\bar{r}} \tau_{\bar{r}}) = \boldsymbol{T}_{\bar{r}} \tau_{\bar{r}}\}$, and then its identity component $\boldsymbol{N}_{\bar{r}}{}^\circ$ is equal to $\boldsymbol{T}_{\bar{r}}$, so that $\boldsymbol{N}_{\bar{r}}/\boldsymbol{T}_{\bar{r}}$ is finite; any quasisemisimple element lying in $\boldsymbol{H}\tau_{\bar{r}}$ is $\boldsymbol{H}$-conjugate to an element of $\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}$; any two elements of $\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}$ which are $\boldsymbol{H}$-conjugate are in fact conjugate by an element of $\boldsymbol{N}_{\bar{r}}$, and thus lie in the same orbit under the action of the finite group $\boldsymbol{N}_{\bar{r}}/\boldsymbol{T}_{\bar{r}}$. Using this we may prove the following.

Lemma 2.1. *Any semisimple element of $\hat{H}_m \tau_{\bar{r}}$ is $\hat{H}_m$-conjugate to an element $(s\tau_{\bar{r}})^{x'}$ where $s \in \boldsymbol{T}_{\bar{r}}$ and $x' \in \boldsymbol{H}$ with ${x'}^{F_m}.{x'}^{-1} \in \boldsymbol{N}_{\bar{r}}$.*

Proof. Suppose $\check{s} \in \hat{H}_m$ with $\check{s}\tau_{\bar{r}}$ semisimple; then $\check{s}$ is quasisemisimple, so there exist $h \in \boldsymbol{H}$ and $s \in \boldsymbol{T}_{\bar{r}}$ such that $\check{s}\tau_{\bar{r}} = (s\tau_{\bar{r}})^h$. As both $\check{s}$ and $\tau_{\bar{r}}$ are F_m-stable we have $(s\tau_{\bar{r}})^h = (s^{F_m}.\tau_{\bar{r}})^{h^{F_m}}$, so $s\tau_{\bar{r}} = (s^{F_m}.\tau_{\bar{r}})^{h^{F_m}.h^{-1}}$; as $\boldsymbol{T}_{\bar{r}}$ is also F_m-stable, both

$s\tau_{\bar r}$ and $s^{F_m}.\tau_{\bar r}$ lie in $\boldsymbol{T}_{\bar r}\tau_{\bar r}$, so there exists $n \in \boldsymbol{N}_{\bar r}$ such that $s\tau_{\bar r} = (s^{F_m}.\tau_{\bar r})^n$. Using the Lang–Steinberg theorem we may write $n = x'^{F_m}.x'^{-1}$ for some $x' \in \boldsymbol{H}$; then

$$((s\tau_{\bar r})^{x'})^{F_m} = (s^{F_m}.\tau_{\bar r})^{x'^{F_m}} = (s^{F_m}.\tau_{\bar r})^{nx'} = (s\tau_{\bar r})^{x'},$$

so that $(s\tau_{\bar r})^{x'}$ is F_m-stable, and

$$(\check{s}\tau_{\bar r})^{h^{-1}n^{-1}} = (s\tau_{\bar r})^{n^{-1}} = s^{F_m}.\tau_{\bar r} = (\check{s}\tau_{\bar r})^{(h^{F_m})^{-1}},$$

so that $h^{-1}n^{-1}h^{F_m} \in C_{\boldsymbol{H}}(\check{s}\tau_{\bar r})$. As $\boldsymbol{H}$ is simply connected and $\check{s}\tau_{\bar r}$ is a quasisemisimple automorphism of $\boldsymbol{H}$, by [28, Theorem 8.2] $C_{\boldsymbol{H}}(\check{s}\tau_{\bar r})$ is connected, so we may again use the Lang–Steinberg theorem to write $h^{-1}n^{-1}h^{F_m} = c^{-1}c^{F_m}$ for some $c \in C_{\boldsymbol{H}}(\check{s}\tau_{\bar r})$; then as $h^{-1}x'(x'^{F_m})^{-1}h^{F_m} = c^{-1}c^{F_m}$ we see that $ch^{-1}x' \in \boldsymbol{H}$ is F_m-stable and thus lies in $\hat{H}_m$, and $(\check{s}\tau_{\bar r})^{ch^{-1}x'} = (\check{s}\tau_{\bar r})^{h^{-1}x'} = (s\tau_{\bar r})^{x'}$, so that $\check{s}\tau_{\bar r}$ is $\hat{H}_m$-conjugate to $(s\tau_{\bar r})^{x'}$ as required. □

Thus $\boldsymbol{T}_{\bar r}$ and $\boldsymbol{N}_{\bar r}/\boldsymbol{T}_{\bar r}$ play the roles of 'maximal torus' and 'Weyl group' for the coset $\hat{H}_m\tau_{\bar r}$. We find that they are as follows: for $\bar r = 2$ we have

$$\boldsymbol{T}_2 = \left\{\left(\lambda, \mu, \frac{\nu^2}{\lambda\mu}, 1; \nu\right) : \lambda, \mu, \nu \in k^*\right\},$$
$$\boldsymbol{N}_2/\boldsymbol{T}_2 = \langle n_{1-2}\boldsymbol{T}_2, n_{2-3}\boldsymbol{T}_2, n_{3-4}n_{3+4}\boldsymbol{T}_2, (-1,-1,-1,-1;-1)\boldsymbol{T}_2\rangle;$$

for $\bar r = 3$ we have

$$\boldsymbol{T}_3 = \left\{\left(\lambda, \mu, \frac{\lambda}{\mu}, 1; \lambda\right) : \lambda, \mu \in k^*\right\},$$
$$\boldsymbol{N}_3/\boldsymbol{T}_3 = \langle n_{2-3}\boldsymbol{T}_3, n_{1-2}n_{3-4}n_{3+4}\boldsymbol{T}_3, (\omega^2, \omega, \omega, \omega^2; 1)\boldsymbol{T}_3\rangle.$$

In each case we see that $|\boldsymbol{N}_{\bar r}/\boldsymbol{T}_{\bar r}| = |C_W(\tau_{\bar r}\boldsymbol{T}_0)|$. For $\bar r = 2$ we in fact have $\boldsymbol{N}_2/\boldsymbol{T}_2 \cong C_W(\tau_2\boldsymbol{T}_0) = C_W(w_4) = \langle w_{1-2}, w_{2-3}, w_{3-4}w_{3+4}\rangle \times \langle w_4\rangle = W(B_3) \times 2$; however for $\bar r = 3$ we have $\boldsymbol{N}_3/\boldsymbol{T}_3 \cong S_3 \times S_3$ but $C_W(\tau_3\boldsymbol{T}_0) = C_W(w_4w_{+---}) = \langle w_{2-3}, w_{1-2}w_{3-4}w_{3+4}\rangle \times \langle w_4w_{+---}\rangle = W(G_2) \times 3$. Write

$$n_0 = n_{1-2}n_{1+2}n_{3-4}n_{3+4}, \qquad h_0 = \begin{cases} (-1,-1,-1,-1;-1) & \text{if } r = 2,\\ (\omega^2, \omega, \omega, \omega^2; 1) & \text{if } r = 3.\end{cases}$$

In $\boldsymbol{N}_{\bar r}/\boldsymbol{T}_{\bar r}$ we then have conjugacy class representatives $n\boldsymbol{T}_{\bar r}$ as follows: if $\bar r = 2$ we may take $n = n'n''$, where

$$n' \in \{1, n_0, h_0, h_0n_0\}, \quad n'' \in \{1, n_{3-4}n_{3+4}, n_{1-2}, n_{1-2}n_{2-4}n_{2+4}, n_{1-2}n_{2-3}\};$$

if $\bar r = 3$ we may take $n = n'n''$, where

$$n' \in \{1, n_0, h_0\}, \quad n'' \in \{1, n_{2-3}, n_{1+3}n_{2-3}\}.$$

For each such element n, using the Lang–Steinberg theorem we may write $n = x'^{F_m}.x'^{-1}$ for some $x' \in \boldsymbol{H}$; by Lemma 2.1 the various F_m-stable elements $(s\tau_{\bar r})^{x'}$

with $s \in \boldsymbol{T}_{\bar{r}}$ between them represent all semisimple $\hat{H}_m$-classes in $\hat{H}_m\tau_{\bar{r}}$. Note that the condition for F_m-stability of $(s\tau_{\bar{r}})^{x'}$ is $s\tau_{\bar{r}} = (s^{F_m}.\tau_{\bar{r}})^n$; as $s \in \boldsymbol{T}_{\bar{r}}$ we have $s^{F_{\bar{r}}} = s^{F_1}$, so that there is a natural correspondence between the semisimple $\hat{H}_1$-classes in $\hat{H}_1\tau_{\bar{r}}$ and the semisimple $\hat{H}_{\bar{r}}$-classes in $\hat{H}_{\bar{r}}\tau_{\bar{r}}$, as we expect from Shintani descent.

Observe that $\boldsymbol{T}_{\bar{r}}$ commutes with the element $y_{\bar{r}}$ defined above, and we have seen that ${\tau_{\bar{r}}}^{y_{\bar{r}}} = h_{+---}(-1)$ or $h_4(\omega)$ according as $\bar{r} = 2$ or 3, so for all $s \in \boldsymbol{T}_{\bar{r}}$ we have $(s\tau_{\bar{r}})^{y_{\bar{r}}} = s{\tau_{\bar{r}}}^{y_{\bar{r}}} \in \boldsymbol{T}_0$; moreover a straightforward calculation shows that $\boldsymbol{T}_0 \cap \boldsymbol{H}^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}$, so that $\boldsymbol{T}_0 \cap (\boldsymbol{H}\langle\tau_{\bar{r}}\rangle)^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}\langle{\tau_{\bar{r}}}^{y_{\bar{r}}}\rangle$. Write

$$\Upsilon_2 = \langle\epsilon_4\rangle, \quad \Upsilon_3 = \left\langle\epsilon_4, \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\right\rangle;$$

then $\Upsilon_{\bar{r}}$ is a root subsystem of Φ of type $\tilde{A}_{\bar{r}-1}$ such that $\boldsymbol{T}_{\bar{r}} = \bigcap_{\beta\in\Upsilon_{\bar{r}}} \ker\beta$. Indeed, given a root subsystem Υ of Φ of type $\tilde{A}_{\bar{r}-1}$, we define

$$\ker_{\bar{r}}\Upsilon = \bigcap_{\beta\in\Upsilon} \ker\beta \cup \bigcap_{\beta\in\Upsilon} (\ker\bar{r}\beta \setminus \ker\beta)$$

(if $\bar{r} = 2$ and $\beta \in \Upsilon$ then $\ker_{\bar{r}}\Upsilon = \ker 2\beta$; if however $\bar{r} = 3$ and $\{\beta_1, \beta_2\}$ is a simple system of Υ then $\ker_{\bar{r}}\Upsilon = \{s' \in \boldsymbol{T}_0 : \beta_1(s') = \beta_2(s') \in \{1, \omega^{\pm1}\}\}$ because if $s' \in \boldsymbol{T}_0$ and $\beta_1(s'), \beta_2(s') \in \{1, \omega^{\pm1}\}$ with $\beta_1(s') \neq \beta_2(s')$ then exactly one of β_1, β_2 and $\beta_1 + \beta_2$ takes the value 1 at s'); from the expression above for ${\tau_{\bar{r}}}^{y_{\bar{r}}}$ we see that $\boldsymbol{T}_{\bar{r}}\langle{\tau_{\bar{r}}}^{y_{\bar{r}}}\rangle = \ker_{\bar{r}}\Upsilon_{\bar{r}}$.

Note that if we set $\boldsymbol{T}_1 = \boldsymbol{T}_0$, $\boldsymbol{N}_1 = \boldsymbol{N} \cap \boldsymbol{H}$, $y_1 = 1$ and $\Upsilon_1 = \varnothing$, then the above (apart from the comment about Shintani descent) also holds for $\bar{r} = 1$.

For each semisimple $\hat{H}_m$-class representative $(s\tau_{\bar{r}})^{x'}$ we may then take a set of representatives of the unipotent conjugacy classes in the $\hat{H}_m$-centralizer, and form products in the usual way, to obtain a complete set of representatives of the $\hat{H}_m$-classes in $\hat{H}_m\tau_{\bar{r}}$; finally, conjugation by g_m gives a complete set of representatives of the H_m-classes in $H_m\tau_{\bar{r}} = H_m\tau_r$. Indeed, as a check we find that the sum of the sizes of the classes so obtained is $|H_m|$.

Now assume $r = p$; then τ_r is a unipotent element of G. Here we refer to [21, Tables X and VIII], which list representatives and centralizer orders for the unipotent H_1-classes lying in $H_1\tau_r$ for $r = 2$ and $r = 3$; the elements are given as $u_i\tau_r$ for $1 \le i \le 10$ and $1 \le i \le 7$, respectively. Using Shintani descent as described above it is straightforward to obtain representatives ${u_i}^{\dagger}\tau_r$ for the unipotent $\hat{H}_r$-classes lying in $\hat{H}_r\tau_r$. Using the expression for ${\tau_r}^{y_r}$ given above, we may also identify the unipotent $\boldsymbol{G}$-classes containing the elements $u_i\tau_r$ and ${u_i}^{\dagger}\tau_r$, either by inspection or by computing Jordan blocks on the 26-dimensional module for $\boldsymbol{G}$ and referring to [14] (whose notation we use), as described above; we find that for each i the elements $u_i\tau_r$ and ${u_i}^{\dagger}\tau_r$ lie in the same $\boldsymbol{G}$-class, which we give in Table 1.

i	$(u_i\tau_2)^G = (u_i{}^\dagger\tau_2)^G$
1	$\tilde{A}_1$
2	$A_1\tilde{A}_1$
3, 4	B_2
5	$C_3(a_1)$
6	$A_2\tilde{A}_1$
7	$C_3(a_1)^{(2)}$
8, 9	B_3
10	$F_4(a_1)$

i	$(u_i\tau_3)^G = (u_i{}^\dagger\tau_3)^G$
1	$\tilde{A}_2$
2	$\tilde{A}_2A_1$
3	C_3
4, 5, 6	$F_4(a_2)$
7	F_4

Table 1. Unipotent G-classes meeting $\hat{H}_m\tau_r$ for $r = 2$ and $r = 3$.

For each unipotent H_1-class representative $u_i\tau_r$, we may then identify its centralizer $C_{H_1}(u_i\tau_r)$; we take representatives s of the semisimple classes therein and form products $su_i\tau_r$. Combining the results for the various values of i gives a complete set of representatives of the H_1-classes in $H_1\tau_r$; again, as a check we find that the sum of the sizes of the classes so obtained is $|H_1|$. In fact in almost all cases the elements s commute with both $u_i{}^\dagger$ and τ_r, so as they are stable under F_1 they are also stable under $F_r = F_1\tau_r$ and hence lie in $C_{\hat{H}_r}(u_i{}^\dagger\tau_r)$; in the exceptional cases we must replace the elements s by conjugates of them. In this way we also obtain a complete set of representatives of the $\hat{H}_r$-classes in $\hat{H}_r\tau_r$; conjugation by g_r then gives a complete set of representatives of the H_r-classes in $H_r\tau_r$.

Before concluding this section we note that the semisimple G-classes in G are listed in [25] for $p \neq 2$ and in [24] for $p = 2$, each of which groups together classes containing elements with equal centralizers. These groupings, which we shall call *types*, may be described combinatorially, in terms of Weyl group elements by which maximal tori are twisted and root systems of centralizers; the finitely many possibilities for the type are the same for all q (and p), although for a given q not all need occur, either because p is a bad prime for G or because q is small. For each type, both [25] and [24] use a single notation for varying q (although the notations used by [25] and [24] are different).

Here we shall similarly say that a *type* of H_m-class in $H_m\tau_r$ is a collection of such H_m-classes all of which contain elements with equal G-centralizers whose semisimple parts also have equal G-centralizers. The finitely many possibilities for the type of such a semisimple H_m-class are again the same for all q (although once more not all occur for all q); for each type we may likewise use a single notation for varying q. Classes which are not semisimple are however more complicated, because the behavior of unipotent classes in bad characteristic differs from that in good characteristic; nevertheless we may do the following.

Note that the triple (d, e, x) must be one of

$$(0, 1, 1),\ (0, -1, 1),\ (1, 0, 2),\ (1, 0, 4), (1, 1, 2),\ (1, 1, 4),\ (1, -1, 2),\ (1, -1, 4)$$

(the first two cover the cases where $p = 2$, the second two those where $p = 3$, and the last four those where $p > 3$). Fix one such triple (d, e, x) and restrict attention to the prime powers q associated to it, of which there are infinitely many. We then find that there are finitely many possibilities for the type of H_m-class in $H_m\tau_r$, that they are the same for all such q, and that for a given type of H_m-class in $H_m\tau_r$ the order of the H_m-centralizer is a polynomial in q: if $p = r$ then for each unipotent class appearing in Table 1 we have obtained a collection of semisimple elements lying in the centralizer, and after forming products to obtain class representatives the statements follow by inspection; if instead $p \neq r$ then for a given type of semisimple H_m-class in $H_m\tau_r$ inspection of tables of unipotent classes in the various simple factors appearing in the centralizer (see, for example, [18] as mentioned above) shows that the parametrization of such unipotent classes is the same for all q, and that for a given unipotent class the centralizer order is a polynomial in q, from which the statements follow.

Indeed from the above we see that more is true: for a fixed triple (d, e, x) and type of H_m-class in $H_m\tau_r$, if we take one such H_m-class and let s be the semisimple part of an element thereof, then the number of H_m-classes of the type concerned which contain elements with semisimple part s is the same for all q.

Thus for a fixed triple (d, e, x) we may extend the notion of 'type of H_m-class in $H_m\tau_r$' to cover all prime powers q associated to it, and H_m-centralizer orders are polynomials in q. This will be crucial to the approach to calculating character scalar products which we now describe.

2.2. ***Character scalar products.*** Recall that, for any subgroup H of G, when G acts on the cosets of H the value at $g \in G$ of the permutation character ${1_H}^G$ is the number of cosets $g'H$ for $g' \in G$ with $gg'H = g'H$. For any generalized character χ of G we have by Frobenius reciprocity

$$({1_H}^G, \chi)_G = (1_H, \chi|_H)_H = \frac{1}{|H|}\sum_{g\in H}\chi(g) = \sum_{[g]\subset H}\frac{\chi(g)}{|C_H(g)|},$$

where the final sum is over all conjugacy classes $[g]$ in H.

We shall be interested in the cases where $H = H_m\langle\tau_r\rangle$, for $m, r \in \{1, 2, 3\}$ with $\{m, r\} \neq \{2, 3\}$; by taking first $r = 1$, and then $r \in \{2, 3\}$ (and considering the inverses of classes if $r = 3$), in the final sum it will suffice to treat the $H_m\langle\tau_r\rangle$-classes which lie in $H_m\tau_r$, which are of course simply the H_m-classes lying in $H_m\tau_r$. We may then see the scalar product as a sum of contributions from the different types of H_m-class in $H_m\tau_r$, with each contribution being the fraction with numerator given by the sum of the character values concerned and denominator equal to the common order of the $H_m\langle\tau_r\rangle$-centralizer (which is r times the order of the H_m-centralizer).

Now take χ to be a generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$, where $\boldsymbol{T}$ is an F-stable maximal torus of $\boldsymbol{G}$ and θ is a linear character of $\boldsymbol{T}^F$, and consider the contribution to the scalar product $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})_G$ from a given type of H_m-class in $H_m\tau_r$. At the end of Section 2.1 we observed that, for a fixed triple (d, e, x), the order of the H_m-centralizer appearing in the denominator is a polynomial in q, and we shall see that the same is true of the number of H_m-classes of the type concerned. We wish to show that the same is also true of the numerator; in order to make such a statement, we first need to say a little more about the characters $R_{\boldsymbol{T},\theta}$.

To begin with, an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$ may be written as ${}^g\boldsymbol{T}_0$ for some $g \in \boldsymbol{G}$, and then we have $g^{-1}g^F \in \boldsymbol{N}$; if we write $w = g^{-1}g^F\boldsymbol{T}_0 \in W$, we say that $\boldsymbol{T}$ is obtained from $\boldsymbol{T}_0$ by twisting with w — although different choices for g may give different elements w of W, the F-conjugacy class $[w]$ of w in W is uniquely determined by $\boldsymbol{T}$, and indeed there is a bijection between F-conjugacy classes in W and G-classes of F-stable maximal tori of $\boldsymbol{G}$. (All of this is well known; see, for example, [4, Section 3.3].) In the present case F acts trivially on W, so that F-conjugacy in W is simply conjugacy.

Next, if $\boldsymbol{T} = {}^g\boldsymbol{T}_0$ as above the finite group $\boldsymbol{T}^F$ is equal to ${}^g({\boldsymbol{T}_0}^{(Fw^{-1})})$, where we write ${\boldsymbol{T}_0}^{(Fw^{-1})} = \{s_0 \in \boldsymbol{T}_0 : s_0 = {}^w({s_0}^F)\}$. Thus given a linear character θ of $\boldsymbol{T}^F$, we may write $\theta = {}^g\theta_0$, where θ_0 is the linear character of ${\boldsymbol{T}_0}^{(Fw^{-1})}$ defined by $\theta_0(s_0) = {}^g\theta_0({}^gs_0)$. We may then define $\Phi_{\theta_0} = \{\alpha \in \Phi : \ker\alpha \geq \ker\theta_0\}$ to be the root subsystem of Φ comprising those roots whose kernel contains $\ker\theta_0$; for fixed $\boldsymbol{T}$ and θ, different choices for g may give different root subsystems Φ_{θ_0}, but it is straightforward to see that the set of possible Φ_{θ_0} forms a single orbit under the action of W. Moreover, in the present case it is easy to see that two root subsystems of Φ lie in the same W-orbit if and only if they are isomorphic (where we require an isomorphism to preserve root lengths).

We may therefore associate to each generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$ a pair $([w], [\Phi'])$ consisting of an F-conjugacy class $[w]$ in W and an isomorphism class $[\Phi']$ of root subsystems of Φ. There are finitely many such pairs, and they are the same for all q, although for a given q not all pairs may be associated to a character $R_{\boldsymbol{T},\theta}$. We are now in a position to state our claim concerning the numerator in the contribution to the scalar product $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})_G$ from a given type of H_m-class in $H_m\tau_r$.

Claim 1. *For a fixed triple (d, e, x), pair $([w], [\Phi'])$ and type of H_m-class in $H_m\tau_r$, the sum over the corresponding H_m-classes $[su]$, where s and u are commuting semisimple and unipotent elements, respectively, of the character values $R_{\boldsymbol{T},\theta}(su)$ is a polynomial in q. Moreover the degree of this polynomial is at most $d_1 + d_2$, where $d_1 = \dim\big(Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle\big)$ is the degree of the polynomial giving the number of H_m-classes of the type concerned, and $d_2 = \frac{1}{2}(\dim C_{\boldsymbol{G}}(su) - \dim \boldsymbol{T})$.*

We shall prove Claim 1 in Section 2.4, following some preliminary results in Section 2.3. Once this has been done it will follow that, for a fixed triple (d, e, x), pair $([w], [\Phi'])$ and type of H_m-class, the contribution to the scalar product $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})_G$ from the type of H_m-class is a ratio of two polynomials in q. Since this scalar product is an integer, it will suffice to determine the *nonnegligible part* of the contribution, where given two polynomials p_1 and p_2 with $p_2 \neq 0$ we let p_3 be the unique polynomial such that $p_1 - p_2p_3$ is of strictly smaller degree than p_2, and say that the nonnegligible part of $p_1(q)/p_2(q)$ is $p_3(q)$. The scalar product will then be the sum of the nonnegligible parts of the contributions from the finitely many different types of H_m-class, and will therefore be a polynomial in q; by taking linear combinations of Deligne–Lusztig characters it will in fact follow that the same is true of all multiplicities of irreducible characters in the permutation character ${1_{H_m\langle\tau_r\rangle}}^G$.

Note that the degree bound in Claim 1 depends only on the type of H_m-class; we call the type of H_m-class *relevant* if this bound is greater than or equal to the degree in q of the order of the H_m-centralizer, and *irrelevant* otherwise. Nonnegligible parts of contributions from irrelevant types of H_m-class are then zero for all $R_{\boldsymbol{T},\theta}$; it therefore suffices to consider relevant types of H_m-class, and our calculations in subsequent sections will begin by identifying these. We shall see that this significantly reduces the number of classes requiring consideration.

We conclude this section by observing that the decompositions we shall obtain will of necessity be more complicated than was the case in the action of G on cosets of $B_4(q)$ treated in [15]. On one level this is simply because the subgroups H_m are considerably smaller than $B_4(q)$ (the ratio of orders is approximately q^8); however, there is a more fundamental difference between the actions here and that of [15]. In [16] a criterion was established for certain subgroups $\boldsymbol{H}$ of algebraic groups $\boldsymbol{G}$ to be spherical (i.e., to have finitely many orbits on the flag variety $\boldsymbol{G}/\boldsymbol{B}$); the proof proceeded by taking fixed points under Frobenius morphisms F and considering scalar products of permutation characters ${1_{\boldsymbol{H}^F}}^{\boldsymbol{G}^F}$ and principal series characters ${1_{\boldsymbol{B}^F}}^{\boldsymbol{G}^F}$. It was shown that if $\boldsymbol{H}$ is of the form $\langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha : \alpha \in \Psi\rangle$ for some subsystem Ψ of Φ, then $\boldsymbol{H}$ is spherical if Ψ satisfies the following condition: there do not exist $\alpha, \beta \in \Phi \setminus \Psi$ with $\alpha + \beta \in \Phi \setminus \Psi$. (Although it is not stated in [16] that this condition is also necessary for sphericality, it is nevertheless clear from the proof that if it is not met then the scalar product $({1_{\boldsymbol{H}^F}}^{\boldsymbol{G}^F}, {1_{\boldsymbol{B}^F}}^{\boldsymbol{G}^F})$ grows with q.) In the case of B_4, the condition is satisfied; correspondingly, the multiplicities in ${1_{B_4(q)}}^{F_4(q)}$ of unipotent characters lying in the principal series are forced to be constants rather than polynomials in q of positive degree (and indeed they are all 0 or 1). In the case of D_4, however, the condition is not satisfied (as we may take α and β to be the two short simple roots of G); thus some multiplicities in ${1_{H_m\langle\tau_r\rangle}}^G$ must turn out to be polynomials in q of positive degree.

$\Phi(\boldsymbol{T})_s$	p	$Z(C_{\boldsymbol{G}}(s))/Z(C_{\boldsymbol{G}}(s))^\circ$
$A_3\tilde{A}_1$	$\neq 2$	$\mathbb{Z}_4$
$A_2\tilde{A}_2$	$\neq 3$	$\mathbb{Z}_3$
B_4, A_1C_3, A_3, A_1B_2, ${A_1}^2\tilde{A}_1$, ${A_1}^2$	$\neq 2$	$\mathbb{Z}_2$
F_4, B_3, C_3, $A_2\tilde{A}_1$, $A_1\tilde{A}_2$, B_2, A_2, $\tilde{A}_2$, $A_1\tilde{A}_1$, A_1, $\tilde{A}_1$, $\varnothing$	any	1

Table 2. Root subsystems $\Phi(\boldsymbol{T})_s$ of $\Phi(\boldsymbol{T})$.

2.3. ***Preliminary results.*** We begin with some results concerning root subsystems. Let $\boldsymbol{T}$ be any maximal torus of $\boldsymbol{G}$. Write $\Phi(\boldsymbol{T}) \subset \mathrm{Hom}(\boldsymbol{T}, k^*)$ for the root system of $\boldsymbol{G}$ with respect to the maximal torus $\boldsymbol{T}$; for a root subsystem Ψ of $\Phi(\boldsymbol{T})$ define

$$\ker\Psi = \bigcap_{\alpha\in\Psi} \ker\alpha.$$

For $s \in \boldsymbol{T}$ define $\Phi(\boldsymbol{T})_s$ to be the root subsystem $\{\alpha \in \Phi(\boldsymbol{T}) : \alpha(s) = 1\}$ of $\Phi(\boldsymbol{T})$; by [4, Theorems 3.5.6, 3.5.3] $C_{\boldsymbol{G}}(s)$ is generated by $\boldsymbol{T}$ and the root groups relative to $\boldsymbol{T}$ corresponding to the roots in $\Phi(\boldsymbol{T})_s$, so $Z(C_{\boldsymbol{G}}(s)) = \ker\Phi(\boldsymbol{T})_s$.

Lemma 2.2. *Up to the action of the Weyl group of $\boldsymbol{T}$, the possibilities for the root subsystem $\Phi(\boldsymbol{T})_s$ and the component group $Z(C_{\boldsymbol{G}}(s))/Z(C_{\boldsymbol{G}}(s))^\circ$ are listed in Table 2.*

Proof. The root subsystems $\Phi(\boldsymbol{T})_s$ may be obtained using [9, Construction 4.1], which also gives the restrictions on the characteristic p: for example, the subsystem ${A_1}^2\tilde{A}_1$ is formed from the extended Dynkin diagram by removing the nodes corresponding to the first and third simple roots, whose coefficients in the highest root are 2 and 4, respectively, so the order of any element s of $\boldsymbol{T}$ having $\Phi(\boldsymbol{T})_s = {A_1}^2\tilde{A}_1$ must be a positive integer linear combination of 2 and 4 and thus must be even, whence p cannot be 2. A simple calculation in each case then gives the component group $Z(C_{\boldsymbol{G}}(s))/Z(C_{\boldsymbol{G}}(s))^\circ$. □

Given a root subsystem Ψ of $\Phi(\boldsymbol{T})$, we define

$$\Psi^{(p)} = \{\beta \in \Phi(\boldsymbol{T}) : p^i\beta \in \mathbb{Z}\Psi \text{ for some } i \in \mathbb{N}\};$$

clearly $\Psi^{(p)}$ is a root subsystem of $\Phi(\boldsymbol{T})$ containing Ψ, and we call it the p-closure of Ψ. We say that Ψ is p-closed if $\Psi^{(p)} = \Psi$; certainly $(\Psi^{(p)})^{(p)} = \Psi^{(p)}$, i.e., $\Psi^{(p)}$ is p-closed. Observe that the eight instances of pairs (Ψ, p) missing from Table 2 are not p-closed; indeed a simple calculation in each case shows that these p-closures are as given in Table 3.

Lemma 2.3. *Given a root subsystem Ψ of $\Phi(\boldsymbol{T})$, the following are true*:

(i) $\ker\Psi = \ker\Psi^{(p)}$;

Ψ	p	$\Psi^{(p)}$
$A_2\tilde{A}_2$	3	F_4
B_4, $A_3\tilde{A}_1$, A_1C_3	2	F_4
A_3, ${A_1}^2\tilde{A}_1$	2	B_3
A_1B_2	2	C_3
${A_1}^2$	2	B_2

Table 3. p-closures of root subsystems Ψ of $\Phi(\boldsymbol{T})$.

(ii) *there exists* $s \in \ker \Psi$ *such that* $\Psi^{(p)} = \Phi(\boldsymbol{T})_s$;

(iii) $\Psi^{(p)} = \bigcap \{\Phi(\boldsymbol{T})_s : s \in \boldsymbol{T},\ \Phi(\boldsymbol{T})_s \supseteq \Psi\}$; *and*

(iv) Ψ *is not p-closed if and only if* Ψ *is listed in Table 3.*

Proof. As the only p-th root of unity in k is 1, for $i \in \mathbb{N}$ we have $\ker p^i\beta = \ker\beta$. Thus if $\beta \in \Psi^{(p)}$ so that $p^i\beta \in \mathbb{Z}\Psi$, we have $\ker\Psi \subseteq \ker p^i\beta = \ker\beta$, proving (i). Since the only root subsystems of $\Phi(\boldsymbol{T})$ which are not of the form $\Phi(\boldsymbol{T})_s$ for some $s \in \boldsymbol{T}$ are the eight listed in Table 3, which are not p-closed, they cannot be $\Psi^{(p)}$; thus $\Psi^{(p)} = \Phi(\boldsymbol{T})_s$ for some $s \in \boldsymbol{T}$, and then for all $\alpha \in \Psi$ we have $\alpha(s) = 1$, proving (ii). Given $s \in \boldsymbol{T}$ with $\Phi(\boldsymbol{T})_s \supseteq \Psi$, for all $\alpha \in \Phi(\boldsymbol{T})_s$ we have $\alpha(s) = 1$; if $\beta \in \Psi^{(p)}$ then for some $i \in \mathbb{N}$ we have $p^i\beta \in \mathbb{Z}\Psi$, so $s \in \ker p^i\beta = \ker\beta$, whence $\Psi^{(p)} \subseteq \Phi(\boldsymbol{T})_s$. Thus $\Psi^{(p)}$ is contained in all the root subsystems of $\Phi(\boldsymbol{T})$ of the form $\Phi(\boldsymbol{T})_s$ which contain Ψ, and by (ii) it is one such root subsystem, proving (iii). Finally (iii) implies that every root subsystem of $\Phi(\boldsymbol{T})$ of the form $\Phi(\boldsymbol{T})_s$ is p-closed, proving (iv). □

Write $W(\boldsymbol{T}) = N_{\boldsymbol{G}}(\boldsymbol{T})/\boldsymbol{T}$ for the Weyl group of $\boldsymbol{T}$; given a root subsystem Ψ of $\Phi(\boldsymbol{T})$, let $\Psi^\perp$ be the root subsystem of $\Phi(\boldsymbol{T})$ consisting of the roots orthogonal to those in Ψ, and write $W(\Psi)$ and $W(\Psi^\perp)$ for the subgroups of $W(\boldsymbol{T})$ generated by reflections in the roots lying in Ψ and $\Psi^\perp$, respectively.

Lemma 2.4. *Given a root subsystem* Ψ *of* $\Phi(\boldsymbol{T})$, *the index of* $W(\Psi)W(\Psi^\perp)$ *in* $N_{W(\boldsymbol{T})}(W(\Psi))$ *is* 1 *or* 2 *according as the group of graph automorphisms of* Ψ *afforded by* $W(\boldsymbol{T})$ *is trivial or not*; *if it is nontrivial, then the coset* $N_{W(\boldsymbol{T})}(W(\Psi)) \setminus W(\Psi)W(\Psi^\perp)$ *contains the long word of* $W(\boldsymbol{T})$, *unless* $\Psi = {A_1}^2$ *or* ${A_1}^2\tilde{A}_1$, *in which case it contains the reflection in a short root in* $\Phi(\boldsymbol{T})$ *which is half the sum or difference of two long roots in* Ψ.

Proof. This is an easy calculation; note that Ψ has a nontrivial graph automorphism if and only if $\Psi = A_3\tilde{A}_1$, A_3, $A_2\tilde{A}_2$, $A_2\tilde{A}_1$, $A_1\tilde{A}_2$, A_2, $\tilde{A}_2$, ${A_1}^2\tilde{A}_1$ or ${A_1}^2$, and all graph automorphisms are afforded by elements of $W(\boldsymbol{T})$, unless $\Psi = A_2\tilde{A}_2$ in which case the only such graph automorphism acts nontrivially on each simple factor. □

For each root subsystem Ψ of $\Phi(\boldsymbol{T})$ set

$$\boldsymbol{Z}_\Psi = \{s \in \boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle : \Phi(\boldsymbol{T})_s \supseteq \Psi\}, \quad \tilde{\boldsymbol{Z}}_\Psi = \{s \in \boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle : \Phi(\boldsymbol{T})_s = \Psi\}.$$

Then $\boldsymbol{Z}_\Psi = \ker \Psi \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$, so for all $s \in \boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$ we have

$$\boldsymbol{Z}_{\Phi(\boldsymbol{T})_s} = Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle.$$

For a subset $\boldsymbol{X}$ of $\boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$, let $\mathbb{1}_{\boldsymbol{X}}$ denote the indicator function of $\boldsymbol{X}$.

Lemma 2.5. *There exist integers $n_{\Psi,\Psi'}$ for root subsystems Ψ, Ψ' of $\Phi(\boldsymbol{T})$, with $n_{\Psi,\Psi} = 1$, and $n_{\Psi,\Psi'} = 0$ if $\Psi' \not\supseteq \Psi$, such that $\mathbb{1}_{\tilde{\boldsymbol{Z}}_\Psi} = \sum_{\Psi'} n_{\Psi,\Psi'} \mathbb{1}_{\boldsymbol{Z}_{\Psi'}}$.*

Proof. For each root subsystem Ψ of $\Phi(\boldsymbol{T})$, the set $\boldsymbol{Z}_\Psi$ is the disjoint union of the sets $\tilde{\boldsymbol{Z}}_{\Psi'}$ as Ψ' runs over the root subsystems of $\Phi(\boldsymbol{T})$ which contain Ψ, so $\mathbb{1}_{\boldsymbol{Z}_\Psi} = \sum_{\Psi' \supseteq \Psi} \mathbb{1}_{\tilde{\boldsymbol{Z}}_{\Psi'}}$. We may now use downward induction on subsystem size: for $\Psi = \Phi(\boldsymbol{T})$ we have $\mathbb{1}_{\tilde{\boldsymbol{Z}}_\Psi} = \mathbb{1}_{\boldsymbol{Z}_\Psi}$, while for a proper subsystem Ψ of $\Phi(\boldsymbol{T})$ we have $\mathbb{1}_{\tilde{\boldsymbol{Z}}_\Psi} = \mathbb{1}_{\boldsymbol{Z}_\Psi} - \sum_{\Psi' \supset \Psi} \mathbb{1}_{\tilde{\boldsymbol{Z}}_{\Psi'}}$, and by induction we may assume each term $\mathbb{1}_{\tilde{\boldsymbol{Z}}_{\Psi'}}$ with $\Psi' \supset \Psi$ is an integer linear combination of terms $\mathbb{1}_{\boldsymbol{Z}_{\Psi''}}$ with $\Psi'' \supseteq \Psi' \supset \Psi$. The result follows. □

The next result concerns commuting elements of $\boldsymbol{N}$. For convenience we confine our attention to elements of the group $\langle n_\alpha : \alpha \in \Phi\rangle$, which we call N'; note that elements of N' are F-stable, and N' has the normal subgroup $\{s \in \boldsymbol{T}_0 : s^2 = 1\} = \langle h_\alpha(-1) : \alpha \in \Phi\rangle \cong {\mathbb{Z}_{d+1}}^4$, with the quotient being naturally isomorphic to W. As in Section 2.1, write $n_0 = n_{1-2}n_{1+2}n_{3-4}n_{3+4} \in N'$, so that $n_0\boldsymbol{T}_0$ is the long word w_0 of W, which is central in W.

Lemma 2.6. *For each $w \in W$ there exists $n \in N'$ with $n\boldsymbol{T}_0 = w$ such that for all $w' \in C_W(w)$ there exists $n' \in C_{N'}(n)$ with $n'\boldsymbol{T}_0 = w'$.*

Proof. It suffices to treat representatives w of the 25 conjugacy classes in W; in Table 4 we list 25 corresponding elements n lying in N', and for each we give elements n' of N' such that the elements $w' = n'\boldsymbol{T}_0$ of W generate $C_W(w)$. It is in each case a straightforward calculation to check that n commutes with each element n' listed. □

Now suppose $\boldsymbol{T}$ is F-stable, and θ is a linear character of $\boldsymbol{T}^F$. We recall that the generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$ takes values as follows: given $s, u \in G$ commuting semisimple and unipotent elements, respectively, by [4, Theorem 7.2.8] we have

$$R_{\boldsymbol{T},\theta}(su) = \frac{1}{|C_G(s)|} \sum_{\substack{x' \in G \\ s^{x'} \in \boldsymbol{T}}} \theta(s^{x'}) Q^{C_{\boldsymbol{G}}(s)}_{{}^{x'}\boldsymbol{T}}(u)$$

(as already observed, because $\boldsymbol{G}$ is simply connected the centralizer $C_{\boldsymbol{G}}(s)$ is connected). Here $Q^{C_{\boldsymbol{G}}(s)}_{{}^{x'}\boldsymbol{T}}$ is the appropriate Green function; it is defined to be the

n	$n' \in C_{N'}(n)$
1	$n_{2-3}, n_{3-4}, n_4, n_{+---}$
n_{3-4}	$n_{3-4}, n_2, n_{+---}, n_{3+4}$
n_4	$n_4, n_{1-2}, n_{2-3}, n_3 h_{3-4}(-1)$
$n_{3-4}n_{3+4}$	$n_{3-4}, n_4, n_{1-2}, n_2$
$n_{1-2}n_4$	$n_{1-2}, n_4, n_{1+2}, n_3 h_{3-4}(-1)$
$n_{2-3}n_{3-4}$	$n_{2-3}n_{3-4}, n_1, n_{+---}, n_0$
$n_{+---}n_4$	$n_{+---}n_4, n_{2-3}, n_{1+2}, n_0$
$n_{3-4}n_4$	$n_{3-4}n_4, n_{1-2}, n_2 h_{3-4}(-1)$
$n_{3-4}n_{1-2}n_{1+2}$	$n_{3+4}, n_2, n_{+--+}, n_{3-4}$
$n_{3-4}n_{3+4}n_2 h_{1-2}(-1)$	$n_1, n_{2-3}, n_{3-4}, n_4 h_{1-2}(-1)$
$n_{1-2}n_{2-3}n_{3-4}$	$n_{1-2}n_{2-3}n_{3-4}, n_{++++}, n_0$
$n_{1-2}n_{3-4}n_4$	$n_{3-4}n_4, n_{1-2}, n_{1+2}$
$n_{+--+}n_{3-4}n_4$	$n_{+--+}n_{3-4}n_4, n_{1+2}$
$n_{2-3}n_{3-4}n_4$	$n_{2-3}n_{3-4}n_4, n_1 h_{+--+}(-1)$
$n_{++--}n_4 n_{1-2}$	$n_{++--}n_4 n_{1-2}, n_0$
$n_{2-3}n_{3-4}n_1$	$n_{2-3}n_{3-4}n_1, n_0$
n_0	$n_{2-3}, n_{3-4}, n_4, n_{+---}$
$n_{++++}n_1 n_{2-3}n_{3-4}$	$n_{++++}n_1, n_{2-3}n_{3-4}, n_{1-4}n_{3+4}, n_0$
$n_{1-2}n_{1+2}n_{3-4}n_4$	$n_{3-4}n_4, n_{1-2}, n_2 h_{3-4}(-1)$
$n_{+-+-}n_1 n_2 n_4$	$n_{+-+-}n_3 h_{2-3}(-1), n_{2-4}, n_{1+2}, n_0$
$n_{1-2}n_{2-3}n_3 n_4 h_{+---}(-1)$	$n_{1-2}n_{2-3}n_3, n_4, n_{+-+-}$
$n_{1-2}n_2 n_{3-4}n_4 h_{3-4}(-1)$	$n_{1-2}n_2 h_{3-4}(-1), n_{3-4}n_4, n_{1-3}n_{2-4}, n_{2+3}n_{+---}$
$n_{1-2}n_{2-3}n_{3-4}n_4$	$n_{1-2}n_{2-3}n_{3-4}n_4$
$n_{+---}n_{3-4}n_{2-3}n_2$	$n_{+---}n_{3-4}n_{2-3}n_2$
$n_{+---}n_{3+4}n_{2-3}n_3 h_{1-2}(-1)$	$n_{++++}n_1, n_{2-3}n_{3-4}, n_{1-4}n_{3+4}, n_0$

Table 4. Commuting elements of N'.

restriction of the generalized Deligne–Lusztig character $R_{x'\boldsymbol{T},1}$ for the group $C_G(s)$ to the set of unipotent elements therein.

Some of the values taken by Green functions are given in [4]: if the unipotent element is the identity then by [4, Theorem 7.5.1] the value is, up to sign, the p-part of the index of the maximal torus concerned (and in particular, if the torus is maximally split the sign is '+', as may be seen by comparing the statement of [4, Theorem 7.5.1] with the definitions on [4, pp. 197, 199]); at the other extreme, if the unipotent element is regular then by [4, Proposition 8.4.1] the value is 1. Using [4, Proposition 3.3.5] for the first of these we see that both are polynomials in q (for all q, not just those for a fixed triple (d, e, x)).

For the purposes of this paper we are interested in Green functions for groups which occur as centralizers of semisimple elements of $\boldsymbol{G}$; an easy application of [4, Property 7.1.9] reduces to the consideration of simple groups appearing as factors in these centralizers. For these groups Green functions have been known for some time: for groups of type A, see Green's original paper [8]; for other groups of classical type, see [11] for $p > 2$ and [27] for $p = 2$; for the group F_4, see [26] for $p > 3$, [7] for $p = 3$ and [22] for $p = 2$. It follows that for a fixed triple (d, e, x) all Green function values are polynomials in q.

(It is in fact known that Green function values in general are polynomials in q, provided one restricts to those q lying in a given residue class modulo an appropriate modulus; the term 'polynomial on residue classes' has been used to describe this phenomenon. Frank Lübeck has in fact recently completed the determination of all (ordinary) Green functions for simple groups by computing those for $E_8(q)$ in bad characteristic in [19], the introduction to which contains a useful summary of the position; the author, who is far from being an expert on these matters, is grateful to him for his helpful comments.)

We end this discussion of Green functions with two lemmas. The first uses the orthogonality of Green functions to give a bound on degrees.

Lemma 2.7. *Given an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$, $s \in \boldsymbol{T}^F$ and $u \in C_G(s)$ unipotent, for a fixed triple (d, e, x) the degree as a polynomial in q of the Green function value $Q_{\boldsymbol{T}}^{C_G(s)}(u)$ is at most $\frac{1}{2}(\dim C_{\boldsymbol{G}}(su) - \dim \boldsymbol{T})$.*

Proof. By [4, Proposition 7.6.2] we have

$$\sum_{u' \in C_G(s) \text{ unipotent}} Q_{\boldsymbol{T}}^{C_G(s)}(u')^2 = \frac{|C_G(s)| \cdot |N_G(\boldsymbol{T})|}{|\boldsymbol{T}^F|^2};$$

restricting to the class $u^{C_G(s)}$ we have

$$|u^{C_G(s)}| \cdot Q_{\boldsymbol{T}}^{C_G(s)}(u)^2 \leq \frac{|C_G(s)| \cdot |N_G(\boldsymbol{T})|}{|\boldsymbol{T}^F|^2},$$

whence

$$\begin{aligned} Q_{\boldsymbol{T}}^{C_G(s)}(u) &\leq \frac{|C_{C_G(s)}(u)|^{1/2} \cdot |N_G(\boldsymbol{T})|^{1/2}}{|\boldsymbol{T}^F|} \\ &= \left(\frac{|C_G(su)|}{|\boldsymbol{T}^F|}\right)^{1/2} \cdot |N_G(\boldsymbol{T}) : \boldsymbol{T}^F|^{1/2}. \end{aligned}$$

As this is true for all q concerned, the result follows. □

The second concerns the effect on a Green function $Q_{\boldsymbol{T}}^{C_G(s)}$ of conjugating by an element of G which normalizes $C_{\boldsymbol{G}}(s)$.

Lemma 2.8. *Given an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$, $s \in \boldsymbol{T}^F$ and $u \in C_G(s)$ unipotent, if $x' \in G$ normalizes $C_{\boldsymbol{G}}(s)$ then $Q_{\boldsymbol{T}}^{C_G(s)}(u) = Q_{{}^{x'}\boldsymbol{T}}^{C_G(s)}({}^{x'}u)$.*

Proof. As in [4, Section 7.2] we have Lang's map $L : C_{\boldsymbol{G}}(s) \to C_{\boldsymbol{G}}(s)$ defined by $L(g) = g^{-1}g^F$. Take a Borel subgroup $\boldsymbol{B}'$ of $C_{\boldsymbol{G}}(s)$ containing $\boldsymbol{T}$, let $\boldsymbol{U}'$ be its unipotent radical and write $\tilde{\boldsymbol{X}} = L^{-1}(\boldsymbol{U}')$. It is then shown in the proof of

[4, Theorem 7.2.8] that $Q_{\boldsymbol{T}}^{C_G(s)}(u) = (1/|\boldsymbol{T}^F|)\mathscr{L}(u, \tilde{\boldsymbol{X}})$, where $\mathscr{L}(u, \tilde{\boldsymbol{X}})$ is the Lefschetz number of u on $\tilde{\boldsymbol{X}}$. Conjugating everything by x', and noting that ${}^{x'}\tilde{\boldsymbol{X}} = L^{-1}({}^{x'}\boldsymbol{U}')$, we have

$$Q_{{}^{x'}\boldsymbol{T}}^{C_{\boldsymbol{G}}(s)}({}^{x'}u) = \frac{1}{|({}^{x'}\boldsymbol{T})^F|}\mathscr{L}({}^{x'}u, {}^{x'}\tilde{\boldsymbol{X}}).$$

As x' is F-stable we have $({}^{x'}\boldsymbol{T})^F = {}^{x'}\boldsymbol{T}^F$; and if we take the map $f : \tilde{\boldsymbol{X}} \to {}^{x'}\tilde{\boldsymbol{X}}$ given by conjugation by x' and apply [4, Property 7.1.5] we obtain $\mathscr{L}(u, \tilde{\boldsymbol{X}}) = \mathscr{L}({}^{x'}u, {}^{x'}\tilde{\boldsymbol{X}})$. Thus $Q_{\boldsymbol{T}}^{C_G(s)}(u) = Q_{{}^{x'}\boldsymbol{T}}^{C_G(s)}({}^{x'}u)$ as required. □

2.4. ***Proof of Claim 1.*** Take a triple (d, e, x), a pair $([w], [\Phi'])$ and a type of H_m-class in $H_m\tau_r$; take a representative w of the (F-)conjugacy class $[w]$ in W, and take $n \in N$ with $n\boldsymbol{T}_0 = w$ as in Lemma 2.6. All of these will be fixed throughout this section; the statement that an expression is 'a polynomial in q' or that a set or number is 'independent of q' will always make this assumption.

Take a prime power q associated to (d, e, x), and let F be the corresponding Frobenius map. Take $g \in \boldsymbol{G}$ satisfying $g^{-1}g^F = n$; write $\boldsymbol{T} = {}^g\boldsymbol{T}_0$, so that $\boldsymbol{T}$ is an F-stable maximal torus of $\boldsymbol{G}$ twisted by w, and $T = \boldsymbol{T}^F$. Take a linear character θ of T such that, if as in Section 2.2 we write $\theta = {}^g\theta_0$ and $\Phi_{\theta_0} = \{\alpha \in \Phi : \ker\alpha \geq \ker\theta_0\}$, the root subsystem Φ_{θ_0} of Φ lies in the isomorphism class $[\Phi']$. Take an H_m-class $[su]$ in $H_m\tau_r$ of the type concerned, where s and u are commuting semisimple and unipotent elements, respectively, such that $s \in T \cap H_m\tau_{\bar{r}}$ and $u \in H_m\tau_{r/\bar{r}}$; write $s = {}^g s_0$ with $s_0 \in \boldsymbol{T}_0$, and $\Phi_{s_0} = \Phi(\boldsymbol{T}_0)_{s_0}$.

Our first result links w and s_0.

Lemma 2.9. *We have ${s_0}^F = {s_0}^w$, and the element w normalizes $C_W(s_0)$.*

Proof. As ${}^g s_0 = ({}^g s_0)^F = {}^{g^F}{s_0}^F$ we have ${s_0}^F = {s_0}^{g^{-1}g^F} = {s_0}^w$. If $c \in C_W(s_0)$, as F acts trivially on W we have ${s_0}^{wcw^{-1}} = ({s_0}^F)^{cw^{-1}} = ({s_0}^F)^{(c^F)w^{-1}} = (({s_0}^c)^F)^{w^{-1}} = ({s_0}^F)^{w^{-1}} = s_0$, whence $wcw^{-1} \in C_W(s_0)$ as required. □

Note that Lemma 2.9 implies that w acts by conjugation on the set of right cosets of $C_W(s_0)$; let the number of fixed points in this action be l, and choose $w_{(1)} = 1, \ldots, w_{(l)} \in W$ such that the fixed points are $C_W(s_0)w_{(i)}$ for $i \leq l$.

Lemma 2.10. *The distinct $\boldsymbol{G}$-conjugates of s lying in T are ${}^g({s_0}^{w_{(i)}})$ for $i \leq l$. Moreover, for each $i \leq l$, if we take $n_{(i)} \in \boldsymbol{N}$ with $n_{(i)}\boldsymbol{T}_0 = w_{(i)}$, there exists $y_{(i)} \in C_{\boldsymbol{G}}(s)$ such that if we set $x_i = y_{(i)}.{}^g n_{(i)}$ then $x_i \in G$ and $s^{x_i} = {}^g({s_0}^{w_{(i)}})$.*

Proof. Any $\boldsymbol{G}$-conjugate of $s = {}^g s_0$ lying in $\boldsymbol{T} = {}^g\boldsymbol{T}_0$ is of the form $s^{{}^g n'} = {}^g({s_0}^{n'})$ for some $n' \in \boldsymbol{N}$; if it is also F-stable and we write $w' = n'\boldsymbol{T}_0$ then ${}^g({s_0}^{n'}) = {}^{g^F}(({s_0}^F)^{(n')^F}) = {}^{g^F}({s_0}^{n(n')^F})$, so $n(n')^F(g^F)^{-1}gn'^{-1} \in C_{\boldsymbol{G}}(s_0)$, whence $ww'w^{-1}w'^{-1} \in C_W(s_0)$, i.e., $ww'w^{-1} \in C_W(s_0)w'$. Thus for some $i \leq l$ the conjugate is ${}^g(s_0{}^{w_{(i)}})$; as distinct values i give distinct conjugates, the first statement follows. Given $n_{(i)} \in \boldsymbol{N}$ with $n_{(i)}\boldsymbol{T}_0 = w_{(i)}$, we have $s^{{}^g n_{(i)}} = (s^F)^{({}^g n_{(i)})^F} = s^{({}^g n_{(i)})^F}$, so ${}^g n_{(i)}(({}^g n_{(i)})^F)^{-1} \in C_{\boldsymbol{G}}(s)$; by [4, Theorem 3.5.6] $C_{\boldsymbol{G}}(s)$ is connected, so by the Lang–Steinberg theorem there exists $y_{(i)} \in C_{\boldsymbol{G}}(s)$ with $y_{(i)}{}^{-1}y_{(i)}{}^F = {}^g n_{(i)}(({}^g n_{(i)})^F)^{-1}$. Set $x_i = y_{(i)}.{}^g n_{(i)}$; then $x_i \in G$ and $s^{x_i} = {}^g(s_0{}^{w_{(i)}})$ as required. □

In Lemma 2.10 we shall assume that $n_{(1)} = y_{(1)} = x_1 = 1$. Now set

$$\boldsymbol{Z} = \boldsymbol{Z}_{\Phi(\boldsymbol{T})_s} = Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle, \quad \tilde{\boldsymbol{Z}} = \tilde{\boldsymbol{Z}}_{\Phi(\boldsymbol{T})_s} = \{\tilde{s} \in \boldsymbol{Z} : C_{\boldsymbol{G}}(\tilde{s}) = C_{\boldsymbol{G}}(s)\}.$$

Note that as $s \in \boldsymbol{T}$ we have $\boldsymbol{T} \le C_{\boldsymbol{G}}(s)$, so $Z(C_{\boldsymbol{G}}(s)) \le C_{\boldsymbol{G}}(\boldsymbol{T}) = \boldsymbol{T}$ and hence $\boldsymbol{Z} \le \boldsymbol{T}$. Write $Z = \boldsymbol{Z}^F$, so that $Z \le T \cap H_m\langle\tau_{\bar{r}}\rangle$, and $\tilde{Z} = \tilde{\boldsymbol{Z}}^F$; for each root subsystem Ψ of $\Phi(\boldsymbol{T})$ write $Z_\Psi = \boldsymbol{Z}_\Psi{}^F$ and $\tilde{Z}_\Psi = \tilde{\boldsymbol{Z}}_\Psi{}^F$.

Our next result indicates that the elements of $\tilde{Z}$ all behave similarly when it comes to taking conjugates in T.

Lemma 2.11. *If $\tilde{s} \in \tilde{Z}$, the distinct $\boldsymbol{G}$-conjugates of $\tilde{s}$ lying in T are $\tilde{s}^{x_i}$ for $i \le l$.*

Proof. Take $\tilde{s} \in \tilde{Z}$. For $i \le l$ we have $\tilde{s}^{x_i} \in G$, and as $y_{(i)} \in C_{\boldsymbol{G}}(s) = C_{\boldsymbol{G}}(\tilde{s})$ and ${}^g n_{(i)} \in {}^g\boldsymbol{N} = N_{\boldsymbol{G}}(\boldsymbol{T})$ we have $\tilde{s}^{x_i} = \tilde{s}^{y_{(i)}.{}^g n_{(i)}} = \tilde{s}^{{}^g n_{(i)}} \in \boldsymbol{T}$, so $\tilde{s}^{x_i} \in T$. For $i, i' \le l$ with $i \ne i'$, the elements $\tilde{s}^{x_i}$ and $\tilde{s}^{x_{i'}}$ are distinct since x_i and $x_{i'}$ lie in distinct right cosets of $C_G(s) = C_G(\tilde{s})$. Interchanging the roles of s and $\tilde{s}$ we see that the distinct $\boldsymbol{G}$-conjugates of $\tilde{s}$ lying in T are $\tilde{s}^{x_i}$ for $i \le l$ as required. □

It is useful to characterize the set $\boldsymbol{Z}$ in terms of root subsystems of Φ.

Lemma 2.12. *The following are true*:

(i) *there is a root subsystem Υ of Φ of type $\tilde{A}_{\bar{r}-1}$ with $s_0 \in \ker_{\bar{r}} \Upsilon \setminus \ker \Upsilon$*;

(ii) *if Υ' is another such root subsystem then $\langle\Phi_{s_0}, \Upsilon\rangle = \langle\Phi_{s_0}, \Upsilon'\rangle$*;

(iii) *for any such root subsystem Υ we have $\boldsymbol{Z} = {}^g(\ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon)$.*

Proof. As $s \in \boldsymbol{H}_m\tau_{\bar{r}}$ we have ${}^{g_m}s \in \boldsymbol{H}\tau_{\bar{r}}$; because ${}^{g_m}s$ is a quasisemisimple element of $\boldsymbol{H}\tau_{\bar{r}}$, as in Section 2.1 there exists $h \in \boldsymbol{H}$ such that ${}^{hg_m}s \in \boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}$. Set $g' = y_{\bar{r}}{}^{-1}hg_m$; then ${}^{g'}s \in (\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}})^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}{}^{y_{\bar{r}}} \subset \boldsymbol{T}_0$, so $Z(C_{\boldsymbol{G}}({}^{g'}s)) \le \boldsymbol{T}_0$. As ${}^{g'g}s_0 = {}^{g'}s$ is a conjugate of s_0 lying in $\boldsymbol{T}_0$, there exists $n' \in \boldsymbol{N}$ such that ${}^{g'g}s_0 = s_0{}^{n'}$, and then if we write $c = n'g'g$ we have $c \in C_{\boldsymbol{G}}(s_0)$; set $w' = n'\boldsymbol{T}_0 \in W$ and $\Upsilon = {}^{w'}\Upsilon_{\bar{r}}$, then we have $s_0{}^{w'} = {}^{g'g}s_0 \in \ker_{\bar{r}} \Upsilon_{\bar{r}} \setminus \ker \Upsilon_{\bar{r}}$ so that $s_0 \in \ker_{\bar{r}} \Upsilon \setminus \ker \Upsilon$, proving (i).

To prove (ii), suppose Υ' is also a root subsystem of Φ of type $\tilde{A}_{\bar{r}-1}$ with $s_0 \in \ker_{\bar{r}} \Upsilon' \setminus \ker \Upsilon'$. The result is immediate if $\bar{r} = 1$; we shall treat separately the cases $\bar{r} = 2$ and $\bar{r} = 3$.

First suppose $\bar{r} = 2$; write $\Upsilon = \langle\beta\rangle$ and $\Upsilon' = \langle\beta'\rangle$, so that $\beta(s_0) = \beta'(s_0) = -1$. There are 12 possibilities for a root subsystem of Φ of type $\tilde{A}_1$. If $\Upsilon' = \Upsilon$ the result is immediate. If not, then according as β' is or is not orthogonal to β, either both $\beta + \beta'$ and $\beta - \beta'$ are long roots, or exactly one of $\beta + \beta'$ and $\beta - \beta'$ is a short root;

thus we may take a root $\gamma \in \Phi \cap \{\beta \pm \beta'\}$, and then as $\beta(s_0) = \beta'(s_0) = -1$ we have $\gamma(s_0) = 1$, so $\gamma \in \Phi_{s_0}$, and hence $\beta' \in \langle \Phi_{s_0}, \Upsilon \rangle$ and $\beta \in \langle \Phi_{s_0}, \Upsilon' \rangle$, proving (ii) in the case where $\bar{r} = 2$.

Now suppose instead $\bar{r} = 3$; write $\Upsilon = \langle \beta_1, \beta_2 \rangle$ and $\Upsilon' = \langle \beta_1', \beta_2' \rangle$, so that $\beta_1(s_0), \beta_2(s_0), \beta_1'(s_0), \beta_2'(s_0) \in \{\omega^{\pm 1}\}$. There are 16 possibilities for a root subsystem of Φ of type $\tilde{A}_2$ (the 12 positive short roots fall into three sets of 4 mutually orthogonal roots, namely $\{1, 2, 3, 4\}$, $\{+{-}{-}{-}, +{-}{+}{+}, +{+}{-}{+}, +{+}{+}{-}\}$ and $\{+{-}{-}{+}, +{-}{+}{-}, +{+}{-}{-}, +{+}{+}{+}\}$, and such a subsystem must contain exactly one root from each set, with the choices in any two sets determining that in the third). If $\Upsilon' = \Upsilon$ the result is immediate. If not, then in at least two of the three sets of 4 mutually orthogonal short roots the root β' which lies in Υ' is different from the root β which lies in Υ, and then both $\beta + \beta'$ and $\beta - \beta'$ are long roots. Since in each case $\beta(s_0), \beta'(s_0) \in \{\omega^{\pm 1}\}$, for one of the two long roots (say γ) we have $\gamma(s_0) = 1$, so $\gamma \in \Phi_{s_0}$, and hence $\beta' \in \langle \Phi_{s_0}, \Upsilon \rangle$ and $\beta \in \langle \Phi_{s_0}, \Upsilon' \rangle$, proving (ii) in the case where $\bar{r} = 3$.

Finally, given $\check{s} \in Z(C_{\boldsymbol{G}}(s))$, write $\check{s} = {}^g\check{s}_0$ with $\check{s}_0 \in Z(C_{\boldsymbol{G}}(s_0))$; then as $c \in C_{\boldsymbol{G}}(s_0)$ we have ${}^c\check{s}_0 = \check{s}_0$, so

$$ {}^{g'}\check{s} = {}^{g'g}\check{s}_0 = {}^{n'^{-1}c}\check{s}_0 = \check{s}_0{}^{n'} = \check{s}_0{}^{w'}. $$

Since ${}^{g'}\check{s} \in Z(C_{\boldsymbol{G}}({}^{g'}s)) \le \boldsymbol{T}_0$, we have

$$ \begin{aligned} \check{s} \in \boldsymbol{H}_m\langle \tau_{\bar{r}} \rangle &\iff {}^{g_m}\check{s} \in \boldsymbol{H}\langle \tau_{\bar{r}} \rangle \\ &\iff {}^{hg_m}\check{s} \in \boldsymbol{H}\langle \tau_{\bar{r}} \rangle \\ &\iff {}^{g'}\check{s} \in (\boldsymbol{H}\langle \tau_{\bar{r}} \rangle)^{y_{\bar{r}}} \\ &\iff \check{s}_0{}^{w'} = {}^{g'}\check{s} \in \boldsymbol{T}_0 \cap (\boldsymbol{H}\langle \tau_{\bar{r}} \rangle)^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}\langle \tau_{\bar{r}}{}^{y_{\bar{r}}} \rangle = \ker_{\bar{r}} \Upsilon_{\bar{r}} \\ &\iff \check{s}^g = \check{s}_0 \in \ker_{\bar{r}} {}^{w'}\Upsilon_{\bar{r}} = \ker_{\bar{r}} \Upsilon. \end{aligned} $$

Thus

$$ \boldsymbol{Z} = Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle \tau_{\bar{r}} \rangle = {}^g\big(Z(C_{\boldsymbol{G}}(s_0)) \cap \ker_{\bar{r}} \Upsilon\big) = {}^g(\ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon), $$

proving (iii). □

Define

$$ I = \{i \le l : s^{x_i} \in \boldsymbol{Z}\}; $$

this set will be of some importance. Using Lemma 2.12 we may characterize I in terms of the Weyl group elements $w_{(i)}$.

Lemma 2.13. *We have*

$$ I = \{i \le l : w_{(i)} \text{ preserves both } \Phi_{s_0} \text{ and } \langle \Phi_{s_0}, \Upsilon \rangle\}, $$

where the root subsystem Υ is as in Lemma 2.12.

Proof. Take $i \le l$. As $s^{x_i} = s^{{}^g n_{(i)}} = {}^g(s_0{}^{n_{(i)}}) = {}^g(s_0{}^{w_{(i)}})$, and $|\Phi_{s_0{}^{w_{(i)}}}| = |\Phi_{s_0}|$, using the characterization of $\boldsymbol{Z}$ given in Lemma 2.12(iii) we have

$$\begin{aligned} s^{x_i} \in \boldsymbol{Z} &\iff s_0{}^{w_{(i)}} \in \ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon \\ &\iff s_0{}^{w_{(i)}} \in \ker \Phi_{s_0} \cap (\ker_{\bar{r}} \Upsilon \setminus \ker \Upsilon) \\ &\iff s_0 \in \ker {}^{w_{(i)}}\Phi_{s_0} \cap (\ker_{\bar{r}} {}^{w_{(i)}}\Upsilon \setminus \ker {}^{w_{(i)}}\Upsilon) \\ &\iff {}^{w_{(i)}}\Phi_{s_0} = \Phi_{s_0} \text{ and } {}^{w_{(i)}}\langle \Phi_{s_0}, \Upsilon\rangle = \langle \Phi_{s_0}, {}^{w_{(i)}}\Upsilon\rangle = \langle \Phi_{s_0}, \Upsilon\rangle \end{aligned}$$

by Lemma 2.12(ii). □

Thus the set I is independent of q. Our next result shows that membership of I has consequences.

Lemma 2.14. *If $i \in I$, conjugation by x_i preserves the algebraic groups $C_{\boldsymbol{G}}(s)$ and $\boldsymbol{Z}$, the finite groups $C_G(s)$ and Z, and the set $\tilde{Z}$; in particular $s^{x_i} \in \tilde{Z}$.*

Proof. Take $i \in I$, so $s^{x_i} \in \boldsymbol{Z}$. As $\dim C_{\boldsymbol{G}}(s) = \dim C_{\boldsymbol{G}}(s^{x_i})$, both centralizers are connected, and $\boldsymbol{G}$-centralizers of elements of $\boldsymbol{Z}$ contain $C_{\boldsymbol{G}}(s)$, we have $C_{\boldsymbol{G}}(s) = C_{\boldsymbol{G}}(s^{x_i}) = C_{\boldsymbol{G}}(s)^{x_i}$. In addition, by Lemma 2.13 $w_{(i)}$ preserves Φ_{s_0} and $\langle \Phi_{s_0}, \Upsilon\rangle$, so $n_{(i)}$ normalizes $\ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon$; thus by Lemma 2.12(iii) ${}^g n_{(i)}$ normalizes $\boldsymbol{Z}$, and as $y_{(i)} \in C_{\boldsymbol{G}}(s)$ commutes with $\boldsymbol{Z}$ we see that x_i normalizes $\boldsymbol{Z}$. As x_i lies in G it also normalizes $C_G(s)$ and Z; thus given $\tilde{s} \in \tilde{Z}$ we have $\tilde{s}^{x_i} \in Z$, and $C_{\boldsymbol{G}}(\tilde{s}^{x_i}) = C_{\boldsymbol{G}}(\tilde{s})^{x_i} = C_{\boldsymbol{G}}(s)^{x_i} = C_{\boldsymbol{G}}(s)$, so $\tilde{s}^{x_i} \in \tilde{Z}$ as required. □

We now show that the elements of I give rise to permutations.

Lemma 2.15. *Given $i \in I$, there exists a permutation π_i of $\{1, \dots, l\}$ such that for all $i' \le l$ and $\tilde{s} \in \tilde{Z}$ we have $x_{\pi_i(i')} \in C_G(s)x_i x_{i'}$ and $\tilde{s}^{x_{\pi_i(i')}} = (\tilde{s}^{x_i})^{x_{i'}}$.*

Proof. Take $i \in I$. By Lemma 2.14 we have $s^{x_i} \in \tilde{Z}$, so by Lemma 2.11

$$\{(s^{x_i})^{x_1}, \dots, (s^{x_i})^{x_l}\} = \{s^{x_1}, \dots, s^{x_l}\};$$

thus there exists a permutation π_i of $\{1, \dots, l\}$ such that for all $i' \le l$ we have $s^{x_{\pi_i(i')}} = (s^{x_i})^{x_{i'}}$, whence $x_{\pi_i(i')} \in C_G(s)x_i x_{i'}$. For all $\tilde{s} \in \tilde{Z}$ we have $C_G(\tilde{s}) = C_G(s)$, so $\tilde{s}^{x_{\pi_i(i')}} = (\tilde{s}^{x_i})^{x_{i'}}$ as required. □

Define $\Pi = \{\pi_i : i \in I\}$; as may be expected, this is a group.

Lemma 2.16. *The set Π is a group which preserves I and acts without fixed points on $\{1, \dots, l\}$.*

Proof. Take $i, i' \in I$. Lemmas 2.14 and 2.15 give $s^{x_{\pi_i(i')}} = (s^{x_i})^{x_{i'}} \in \boldsymbol{Z}$, so $\pi_i(i') \in I$; thus π_i preserves I. For all $i'' \le l$ we have $s^{x_{\pi_i(\pi_{i'}(i''))}} = (s^{x_i})^{x_{\pi_{i'}(i'')}} = ((s^{x_i})^{x_{i'}})^{x_{i''}} = (s^{x_{\pi_i(i')}})^{x_{i''}} = s^{x_{\pi_{\pi_i(i')}(i'')}}$, so $\pi_i(\pi_{i'}(i'')) = \pi_{\pi_i(i')}(i'')$; therefore $\pi_i \circ \pi_{i'} = \pi_{\pi_i(i')}$. Since $\pi_1 = 1$ because $x_1 = 1$, and $\pi_i \circ \pi_{\pi_i{}^{-1}(1)} = \pi_1$, we see that Π is a group, with $\pi_i{}^{-1} = \pi_{\pi_i{}^{-1}(1)}$. Also if $i \ne i'$ then for all $i'' \le l$ we have $s^{x_{\pi_i(i'')}} = (s^{x_i})^{x_{i''}} \ne (s^{x_{i'}})^{x_{i''}} = s^{x_{\pi_{i'}(i'')}}$, so $\pi_i(i'') \ne \pi_{i'}(i'')$ as required. □

Our next result concerns root subsystems Ψ of $\Phi(\boldsymbol{T})$ which contain $\Phi(\boldsymbol{T})_s$.

Lemma 2.17. *Given a root subsystem Ψ of $\Phi(\boldsymbol{T})$ containing $\Phi(\boldsymbol{T})_s$, the order $|Z_\Psi|$ is a polynomial in q.*

Proof. Write $\Psi = {}^g\Psi_0$, so that Ψ_0 is a root subsystem of Φ containing Φ_{s_0}. As $\boldsymbol{Z}_\Psi = \ker\Psi \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$ and $\boldsymbol{Z} = \boldsymbol{Z}_{\Phi(\boldsymbol{T})_s} = \ker\Phi(\boldsymbol{T})_s \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$, if we take a root subsystem Υ as in Lemma 2.12 we have

$$\boldsymbol{Z}_\Psi = \boldsymbol{Z} \cap \ker\Psi = {}^g(\ker\Phi_{s_0} \cap \ker_{\bar{r}}\Upsilon) \cap {}^g\ker\Psi_0 = {}^g(\ker\Psi_0 \cap \ker_{\bar{r}}\Upsilon)$$

by Lemma 2.12(iii). Now

$$\begin{aligned} |{}^g(\ker\Psi_0 \cap \ker_{\bar{r}}\Upsilon) : {}^g\ker\langle\Psi_0,\Upsilon\rangle| &= |(\ker\Psi_0 \cap \ker_{\bar{r}}\Upsilon) : \ker\langle\Psi_0,\Upsilon\rangle| \\ &= \begin{cases} 1 & \text{if } \Psi_0 \supseteq \langle\Phi_{s_0},\Upsilon\rangle, \\ \bar{r} & \text{if } \Psi_0 \not\supseteq \langle\Phi_{s_0},\Upsilon\rangle. \end{cases} \end{aligned}$$

Set $\overline{\Psi}_0 = \langle\Psi_0,\Upsilon\rangle$ and $\overline{\Psi} = {}^g\overline{\Psi}_0$; then we have ${}^g\ker\overline{\Psi}_0 = \ker\overline{\Psi} = \ker\overline{\Psi}^{(p)}$ by Lemma 2.3(i), and it suffices to prove that the order $|Z_{\overline{\Psi}}|$ is a polynomial in q.

By Lemma 2.3(ii) there exists $\check{s} \in \ker\overline{\Psi}$ such that $\overline{\Psi}^{(p)} = \Phi(\boldsymbol{T})_{\check{s}}$, so $\boldsymbol{Z}_{\overline{\Psi}} = Z(C_{\boldsymbol{G}}(\check{s}))$. The connected reductive group $C_{\boldsymbol{G}}(\check{s})$ has derived group $C_{\boldsymbol{G}}(\check{s})'$ and F-stable maximal torus $\boldsymbol{T}$; thus as in [4, Section 3.3] $\boldsymbol{T} \cap C_{\boldsymbol{G}}(\check{s})'$ is an F-stable maximal torus of $C_{\boldsymbol{G}}(\check{s})'$. Applying [4, Proposition 3.3.5] to $\boldsymbol{G}$ and $C_{\boldsymbol{G}}(\check{s})$ shows that the orders $|\boldsymbol{T}^F|$ and $\left|(\boldsymbol{T} \cap C_{\boldsymbol{G}}(\check{s})')^F\right|$ are both polynomials in q; by [4, Proposition 3.3.7] the order $\left|\left(Z(C_{\boldsymbol{G}}(\check{s}))^\circ\right)^F\right|$ is their ratio, so as it is an integer for all q it must itself be a polynomial in q. The possibilities for the component group $Z(C_{\boldsymbol{G}}(\check{s}))/Z(C_{\boldsymbol{G}}(\check{s}))^\circ$ are given in Table 2: in all but the first two cases F must act trivially on $Z(C_{\boldsymbol{G}}(\check{s}))/Z(C_{\boldsymbol{G}}(\check{s}))^\circ$; consideration of these two cases shows that the action of F on $Z(C_{\boldsymbol{G}}(\check{s}))/Z(C_{\boldsymbol{G}}(\check{s}))^\circ$ is independent of q (as the triple (d,e,x) is fixed), so the order $|Z_{\overline{\Psi}}| = \left|Z(C_{\boldsymbol{G}}(\check{s}))^F\right|$ is also a polynomial in q as required. □

As a consequence, certain sums of values taken by θ are polynomials in q.

Lemma 2.18. *Given $i' \leq l$ and $s' \in Z$, the sums*

$$\sum_{\check{s} \in Z_{\Phi(\boldsymbol{T})_{s'}}} \theta(\check{s}^{x_{i'}}) \quad \text{and} \quad \sum_{\check{s} \in \tilde{Z}_{\Phi(\boldsymbol{T})_{s'}}} \theta(\check{s}^{x_{i'}})$$

are polynomials in q.

Proof. Given $s' \in Z$, as $\Phi(\boldsymbol{T})_{s'} \supseteq \Phi(\boldsymbol{T})_s$ Lemma 2.17 shows that the order $|Z_{\Phi(\boldsymbol{T})_{s'}}|$ is a polynomial in q. The sum over a finite group of the values taken by a linear character is either 0 or the group order (as can be seen by considering the scalar product of the character with the trivial character). Applying this for $i' \leq l$ to the group $Z_{\Phi(\boldsymbol{T})_{s'}}$ and its linear character whose value at the element $\check{s} = {}^g\check{s}_0$ is $\theta(\check{s}^{x_{i'}}) = {}^g\theta_0(({}^g\check{s}_0)^{{}^g n_{(i')}}) = \theta_0(\check{s}_0{}^{w_{(i')}})$, where $\theta = {}^g\theta_0$, shows that the first sum is a polynomial in q; Lemmas 2.5 and 2.17 then show that the same is true of the second sum. □

So far we have considered only semisimple elements. Recall that we have the H_m-class $[su]$; write $u_1 = u$ and let $[su_1], \ldots, [su_a]$ be the distinct H_m-classes in $H_m\tau_r$ of the type concerned containing elements with semisimple part s, where for $j \le a$ the element u_j is unipotent, commutes with s and lies in $H_m\tau_{r/\bar{r}}$. For any $\tilde{s} \in \tilde{Z}$, likewise $[\tilde{s}u_1], \ldots, [\tilde{s}u_a]$ are then the distinct H_m-classes in $H_m\tau_r$ of the type concerned containing elements with semisimple part $\tilde{s}$. As explained at the end of Section 2.1, the number a of these classes is independent of q.

Take $j \le a$. If $i \in I$, then as $u_j \in C_{\boldsymbol{G}}(s) = C_{\boldsymbol{G}}(s^{x_i})$ by Lemma 2.14, the element $s^{x_i}u_j$ has semisimple part s^{x_i} and unipotent part u_j. For $s^{x_i}u_j$ to lie in the G-class $(su_j)^G$ there must exist $g' \in G$ with $s^{g'} = s^{x_i}$ and $u_j{}^{g'} = u_j$; the first condition gives $g' \in C_G(s)x_i$, and then the second forces x_i to preserve the $C_G(s)$-class $u_j{}^{C_G(s)}$. We therefore set

$$I_j = \{i \in I : \ x_i \text{ preserves } u_j{}^{C_G(s)}\}.$$

Lemma 2.19. *For $j \le a$ the set I_j is independent of q.*

Proof. Take $i \in I$; then x_i normalizes $C_G(s)$ by Lemma 2.14. We seek to show that the action of x_i on the unipotent classes in $C_G(s)$ is the same for all q associated to the triple (d, e, x); the result will then follow.

We have $s = {}^g s_0$. Since we know by [4, Theorem 3.5.6] that $C_{\boldsymbol{G}}(s_0)$ is connected, by [4, Theorem 3.5.3] we have $C_{\boldsymbol{G}}(s_0) = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha : \alpha(s_0) = 1\rangle = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha : \alpha \in \Phi_{s_0}\rangle$; thus unipotent classes in $C_G(s)$ lie in $\langle {}^g\boldsymbol{U}_\alpha : \alpha \in \Phi_{s_0}\rangle$, and $C_W(s_0) = \langle w_\alpha : \alpha \in \Phi_{s_0}\rangle = W(\Phi_{s_0})$. By Lemma 2.9 we know that w normalizes $C_W(s_0)$; by Lemma 2.13 we know that $w_{(i)}$ preserves Φ_{s_0} and so normalizes $W(\Phi_{s_0})$. Thus both w and $w_{(i)}$ lie in $N_W(C_W(s_0)) = N_W(W(\Phi_{s_0}))$.

By Lemma 2.4 we see that the index of $W(\Phi_{s_0})W(\Phi_{s_0}{}^\perp)$ in $N_W(W(\Phi_{s_0}))$ is 1 or 2. We shall say that we are in case (i) if the index is 1, case (ii) if the index is 2 and $\Phi_{s_0} \notin \{A_1{}^2, A_1{}^2\tilde{A}_1\}$, and case (iii) if the index is 2 and $\Phi_{s_0} \in \{A_1{}^2, A_1{}^2\tilde{A}_1\}$. In case (ii) we then have $N_W(W(\Phi_{s_0})) = W(\Phi_{s_0})W(\Phi_{s_0}{}^\perp)\langle w_0\rangle$, where $w_0 = w_{1-2}w_{1+2}w_{3-4}w_{3+4}$ is the long word of W; in case (iii) we may assume the long roots in Φ_{s_0} are $\pm\epsilon_3 \pm \epsilon_4$ (and if there are short roots in Φ_{s_0} they are $\pm\epsilon_2$), and then we have $N_W(W(\Phi_{s_0})) = W(\Phi_{s_0})W(\Phi_{s_0}{}^\perp)\langle w_4\rangle$. Set

$$W^* = \begin{cases} W(\Phi_{s_0}) & \text{in case (i)},\\ W(\Phi_{s_0}) & \text{in case (ii)},\\ W(\Phi_{s_0})\langle w_4\rangle & \text{in case (iii)}, \end{cases} \qquad W^\dagger = \begin{cases} W(\Phi_{s_0}{}^\perp) & \text{in case (i)},\\ W(\Phi_{s_0}{}^\perp)\langle w_0\rangle & \text{in case (ii)},\\ W(\Phi_{s_0}{}^\perp) & \text{in case (iii)}; \end{cases}$$

then $N_W(W(\Phi_{s_0})) = W^* \times W^\dagger$. As $w, w_{(i)} \in N_W(W(\Phi_{s_0}))$ we may write $w = w^*w^\dagger$ and $w_{(i)} = w_{(i)}{}^*w_{(i)}{}^\dagger$ with $w^*, w_{(i)}{}^* \in W^*$ and $w^\dagger, w_{(i)}{}^\dagger \in W^\dagger$. Moreover the right coset $W(\Phi_{s_0})w_{(i)}$ is fixed under conjugation by w, so we have ${}^w w_{(i)} \in W(\Phi_{s_0})w_{(i)}$ and therefore $[w, w_{(i)}] \in W(\Phi_{s_0}) \le W^*$; as $[w, w_{(i)}] = [w^*, w_{(i)}{}^*].[w^\dagger, w_{(i)}{}^\dagger]$ we must have $[w^\dagger, w_{(i)}{}^\dagger] = 1$, whence $[w, w_{(i)}] = [w^*, w_{(i)}{}^*]$.

Note that we may choose $w_{(i)}$ to be any representative of the appropriate right coset of $W(\Phi_{s_0})$. In cases (i) and (ii) we may choose ${w_{(i)}}^* = 1$, and then $[w^*, {w_{(i)}}^*] = 1$. In case (iii) the fact that $W(\Phi_{s_0})$ is a proper subgroup of W^* makes the situation rather more complicated. Write $w^* = w^{**}{w_4}^b$ and ${w_{(i)}}^* = {w_{(i)}}^{**}{w_4}^c$ with $w^{**}, {w_{(i)}}^{**} \in W(\Phi_{s_0})$ and $b, c \in \{0, 1\}$. If $c = 0$ we may choose ${w_{(i)}}^{**} = 1$; if $c = b = 1$ we may choose ${w_{(i)}}^{**} = w^{**}$; if $c = 1$, $b = 0$ and w^{**} involves an even number of reflections in long roots we may choose ${w_{(i)}}^{**} = 1$ — in each of these instances we then have $[w^*, {w_{(i)}}^*] = 1$. If however $c = 1$, $b = 0$ and w^{**} involves an odd number of reflections in long roots then for any choice of ${w_{(i)}}^{**}$ we have $[w^*, {w_{(i)}}^*] = w_{3-4}w_{3+4}$. Thus overall the commutator $[w, w_{(i)}]$ is 1, unless we are in the particular situation in case (iii) where it is $w_{3-4}w_{3+4}$.

Suppose $[w, w_{(i)}] = 1$; then w commutes with $w_{(i)}$, and as the element n was chosen as in Lemma 2.6, we may assume that in Lemma 2.10 the element $n_{(i)}$ with $n_{(i)}\boldsymbol{T}_0 = w_{(i)}$ was chosen to lie in $C_{N'}(n)$. It follows that $({}^g n_{(i)})^F = {}^{g.g^{-1}g^F}{n_{(i)}}^F = {}^g({}^n n_{(i)}) = {}^g n_{(i)}$, so that ${}^g n_{(i)} \in G$; we may therefore assume that in Lemma 2.10 the element $y_{(i)}$ was chosen to be 1, whence $x_i = {}^g n_{(i)}$. Thus x_i is an F-stable element of $N_{\boldsymbol{G}}(\boldsymbol{T})$ corresponding to a fixed element of $W(\boldsymbol{T}) = N_{\boldsymbol{G}}(\boldsymbol{T})/\boldsymbol{T}$; any two such elements act identically on the set of unipotent classes in $C_G(s)$, because they differ by an F-stable element of $\boldsymbol{T}$, which thus lies in $C_G(s)$. As the element of $W(\boldsymbol{T})$ is fixed, we see that the action of x_i on the set of unipotent classes in $C_G(s)$ is the same for all q associated to the triple (d, e, x).

Now suppose instead that we are in the particular situation in case (iii) where $[w, w_{(i)}] = w_{3-4}w_{3+4}$. Observe that, if $w_{(i)}$ and $w_{(i')}$ represent two different right cosets of $W(\Phi_{s_0})$ and both involve w_4, the action of x_i on unipotent classes in $C_G(s)$ determines that of $x_{i'}$, since the quotient corresponds to a right coset representative $w_{(i'')}$ which commutes with w and therefore by the above its action is known; we may thus choose the right coset representative $w_{(i)}$ to be w_4. By conjugating w by w_4 if necessary we may assume that $w = w_{3-4}w^\dagger$ for some $w^\dagger \in W^\dagger = W({\Phi_{s_0}}^\perp) \le \langle w_{1-2}, w_2\rangle$. Again we have the element n. Write $n = n_{3-4}n^\dagger$, where inspection of Table 4 shows that we may take $n^\dagger \in \langle n_{1-2}, n_2\rangle$; by taking $n_{(i)} = n_4h_{1-2}(-1)$ we may ensure that $n_{(i)}$ commutes with $n^\dagger$. We then have

$$\begin{aligned}
n_{(i)}({}^n n_{(i)})^{-1} &= n_{(i)}n_{3-4}n^\dagger {n_{(i)}}^{-1}(n^\dagger)^{-1}{n_{3-4}}^{-1}\\
&= n_{(i)}n_{3-4}{n_{(i)}}^{-1}{n_{3-4}}^{-1}\\
&= n_4h_{1-2}(-1)n_{3-4}h_{1-2}(-1){n_4}^{-1}{n_{3-4}}^{-1}\\
&= n_4n_{3-4}{n_4}^{-1}{n_{3-4}}^{-1}\\
&= n_{3+4}{n_{3-4}}^{-1}.
\end{aligned}$$

Take $\lambda, \mu \in k^*$ satisfying $\lambda^{q^2-1} = -1$ and $\mu^q - \mu = \lambda^{q+1}$, and set

$$y_{(i)} = {}^g(x_{3+4}(\mu)h_{3+4}(\lambda)n_{3+4}x_{3+4}(-\lambda^{1-q})x_{3-4}(\lambda^{1-q})h_{3-4}(\lambda)n_{3-4}x_{3-4}(-\mu));$$

then $y_{(i)} \in C_{\boldsymbol{G}}(s)$ and calculation shows that ${y_{(i)}}^{-1}{y_{(i)}}^F = {}^g(n_{3+4}{n_{3-4}}^{-1}) = {}^g(n_{(i)}({}^n n_{(i)})^{-1}) = {}^g n_{(i)}.{}^{gn}{n_{(i)}}^{-1} = {}^g n_{(i)}(({}^g n_{(i)})^F)^{-1}$ as required in Lemma 2.10, while the square of the F-stable element $x_i = y_{(i)}.{}^g n_{(i)}$ fixes ${}^g\langle \boldsymbol{U}_{\pm 3 \pm 4}\rangle$ pointwise.

Assume $\Phi_{s_0} = {A_1}^2$. If $p > 2$ and we write the nontrivial unipotent classes of $({}^g\langle \boldsymbol{U}_{\pm(3-4)}\rangle)^F$ as $\mathcal{C}_1$ and $\mathcal{C}_2$, then those of $({}^g\langle \boldsymbol{U}_{\pm(3+4)}\rangle)^F$ are ${\mathcal{C}_1}^{x_i}$ and ${\mathcal{C}_2}^{x_i}$; the element x_i then acts on the unipotent classes in $C_G(s)$ by fixing $\{1\}$, $\mathcal{C}_1{\mathcal{C}_1}^{x_i}$ and $\mathcal{C}_2{\mathcal{C}_2}^{x_i}$, and interchanging $\mathcal{C}_1$ with ${\mathcal{C}_1}^{x_i}$, $\mathcal{C}_2$ with ${\mathcal{C}_2}^{x_i}$, and $\mathcal{C}_1{\mathcal{C}_2}^{x_i}$ with $\mathcal{C}_2{\mathcal{C}_1}^{x_i}$. If instead $p = 2$ and we write the nontrivial unipotent class of $({}^g\langle \boldsymbol{U}_{\pm(3-4)}\rangle)^F$ as $\mathcal{C}$, then that of $({}^g\langle \boldsymbol{U}_{\pm(3+4)}\rangle)^F$ is $\mathcal{C}^{x_i}$; the element x_i then acts on the unipotent classes in $C_G(s)$ by fixing $\{1\}$ and $\mathcal{C}\mathcal{C}^{x_i}$, and interchanging $\mathcal{C}$ with $\mathcal{C}^{x_i}$. Thus in all characteristics the action of x_i on the unipotent classes in $C_G(s)$ is the same for all q associated to the triple (d, e, x).

Now assume instead $\Phi_{s_0} = {A_1}^2\tilde{A}_1$. The action of x_i on the unipotent classes in $({}^g\langle \boldsymbol{U}_{\pm 2}\rangle)^F$ is determined as in the cases above where w commutes with $w_{(i)}$; combined with the previous paragraph this now determines the action of x_i on the unipotent classes in $C_G(s)$, which again therefore is the same for all q associated to the triple (d, e, x). □

Take $j \le a$. Define $\Pi_j = \{\pi_i : i \in I_j\}$; clearly Π_j is a subgroup of Π which preserves I_j. Let ${I_j}'$ be a set of orbit representatives for the action of Π_j on $\{1, \dots, l\}$; by Lemma 2.19 we may choose ${I_j}'$ to be independent of q, and as by Lemma 2.16 each orbit has size $|I_j|$ we have $|I_j| \cdot |{I_j}'| = l$. Our final lemma in this section gives, for $\tilde{s} \in \tilde{Z}$, the value taken by the generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$ at $\tilde{s}u_j$.

Lemma 2.20. *Given $j \le a$ and $\tilde{s} \in \tilde{Z}$, we have*

$$R_{\boldsymbol{T},\theta}(\tilde{s}u_j) = \sum_{i' \in {I_j}'} Q^{C_{\boldsymbol{G}}(s)}_{x_{i'}\boldsymbol{T}}(u_j) \sum_{i \in I_j} \theta((\tilde{s}^{x_i})^{x_{i'}}).$$

Proof. Take $j \le a$ and $\tilde{s} \in \tilde{Z}$. Since $C_{\boldsymbol{G}}(\tilde{s}) = C_{\boldsymbol{G}}(s)$, by Lemma 2.11 the set $\{x' \in G : \tilde{s}^{x'} \in \boldsymbol{T}\}$ is the disjoint union of the right cosets $C_G(s)x_1, \dots, C_G(s)x_l$; each element of this set is thus of the form $cx_{\pi_i(i')}$ for unique $c \in C_G(s)$, $i \in I_j$ and $i' \in {I_j}'$. By Lemma 2.15 we have $x_{\pi_i(i')} = c'x_i x_{i'}$ for some $c' \in C_G(s)$, and $\tilde{s}^{cx_{\pi_i(i')}} = \tilde{s}^{x_{\pi_i(i')}} = (\tilde{s}^{x_i})^{x_{i'}}$; then

$$\begin{aligned} Q^{C_{\boldsymbol{G}}(s)}_{cx_{\pi_i(i')}\boldsymbol{T}}(u_j) &= Q^{C_{\boldsymbol{G}}(s)}_{cc'x_i x_{i'}\boldsymbol{T}}({}^{cc'}u_j) \\ &= Q^{C_{\boldsymbol{G}}(s)}_{x_{i'}\boldsymbol{T}}({u_j}^{x_i}) \\ &= Q^{C_{\boldsymbol{G}}(s)}_{x_{i'}\boldsymbol{T}}(u_j), \end{aligned}$$

where we obtain the first equality because $Q^{C_{\boldsymbol{G}}(s)}_{cc'x_i x_{i'}\boldsymbol{T}}$ is a $C_G(s)$-class function, the second by Lemma 2.8 and the third because x_i preserves ${u_j}^{C_G(s)}$.

The formula of [4, Theorem 7.2.8] quoted before Lemma 2.7 now gives

$$R_{\boldsymbol{T},\theta}(\tilde{s}u_j) = \frac{1}{|C_G(s)|} \sum_{\substack{c \in C_G(s) \\ i \in I_j \\ i' \in I_j{}'}} \theta(\tilde{s}^{cx_{\pi_i(i')}}) Q^{C_{\boldsymbol{G}}(s)}_{{}^{cx_{\pi_i(i')}}\boldsymbol{T}}(u_j)$$

$$= \sum_{\substack{i \in I_j \\ i' \in I_j{}'}} \theta((\tilde{s}^{x_i})^{x_{i'}}) Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) = \sum_{i' \in I_j{}'} Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) \sum_{i \in I_j} \theta((\tilde{s}^{x_i})^{x_{i'}})$$

as required. □

We may now prove our claim.

Proposition 2.21. *Claim 1 is true.*

Proof. Take $j \leq a$. By Lemma 2.14, for all $i \in I_j$ and $\tilde{s} \in \tilde{Z}$ the element $\tilde{s}^{x_i} u_j$ has semisimple part $\tilde{s}^{x_i} \in \tilde{Z}$ and unipotent part u_j, and lies in the G-class containing $\tilde{s}u_j$; thus $(\tilde{s}u_j)^G \cap \tilde{Z}u_j = \{\tilde{s}^{x_i} u_j : i \in I_j\}$. Choose a subset $\tilde{Z}_j$ of $\tilde{Z}$ such that each element of $\tilde{Z}$ lies in the set $\{\tilde{s}^{x_i} : i \in I_j\}$ for precisely one element $\tilde{s}$ of $\tilde{Z}_j$; then for distinct elements $\tilde{s} \in \tilde{Z}_j$ the classes $(\tilde{s}u_j)^G$ are distinct, and by Lemma 2.20 we have

$$\sum_{\tilde{s} \in \tilde{Z}_j} R_{\boldsymbol{T},\theta}(\tilde{s}u_j) = \sum_{i' \in I_j{}'} Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) \sum_{\tilde{s} \in \tilde{Z}_j} \sum_{i \in I_j} \theta((\tilde{s}^{x_i})^{x_{i'}})$$

$$= \sum_{i' \in I_j{}'} Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) \sum_{\tilde{s} \in \tilde{Z}} \theta(\tilde{s}^{x_{i'}}).$$

For each $i' \in I_j{}'$, by Lemma 2.18 the sum $\sum_{\tilde{s} \in \tilde{Z}} \theta(\tilde{s}^{x_{i'}})$ is a polynomial in q; since $I_j{}'$ is independent of q, and as stated in Section 2.3 we know that the Green functions here are also polynomials in q, we see that $\sum_{\tilde{s} \in \tilde{Z}_j} R_{\boldsymbol{T},\theta}(\tilde{s}u_j)$ is likewise a polynomial in q. Summing over j (and recalling that a is independent of q) we see that the sum of the values taken by $R_{\boldsymbol{T},\theta}$ over the H_m-classes of the type concerned is a polynomial in q as required.

Finally we consider degrees. Fix $j \leq a$ and $i' \in I_j{}'$. The degree of the polynomial $\sum_{\tilde{s} \in \tilde{Z}} \theta(\tilde{s}^{x_{i'}})$ is at most $\dim \boldsymbol{Z} = \dim\big(Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle\big) = d_1$; moreover taking $\theta = 1$ and observing that $|\tilde{Z}| = |I_j| \cdot |\tilde{Z}_j|$ we see that d_1 equals the degree of the polynomial giving the number of H_m-classes of the given type. Since by Lemma 2.7 the degree of $Q^{C_G(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j)$ is at most d_2, it follows that the degree of each polynomial $\sum_{\tilde{s} \in \tilde{Z}_j} R_{\boldsymbol{T},\theta}(\tilde{s}u_j)$ is at most $d_1 + d_2$, whence the same is true of the polynomial at the end of the previous paragraph. □

To conclude this section we observe that if $s \in H_m$ is a regular semisimple element of $\boldsymbol{G}$, so that $\Phi(\boldsymbol{T})_s = \varnothing$ and $\boldsymbol{Z} = C_{\boldsymbol{G}}(s) = \boldsymbol{T}$, and we take $j \leq a$ and $i' \in I_j{}'$, then if $\theta \neq 1$ the degree of the polynomial $\sum_{\tilde{s} \in \tilde{Z}} \theta(\tilde{s}^{x_{i'}})$ is strictly less than $\dim \boldsymbol{T}$ (whereas if $\theta = 1$ it equals $\dim \boldsymbol{T}$): this follows from Lemmas 2.5

and 2.18, because if $\theta \neq 1$ then $\sum_{\check{s}\in Z} \theta(\check{s}^{x_{i'}}) = 0$, while for each nonempty root subsystem Ψ' the sum $\sum_{\check{s}\in Z_{\Psi'}} \theta(\check{s}^{x_{i'}})$ is a polynomial in q of degree less than $\dim \boldsymbol{T}$. It follows that if $s \in H_m$ is regular, the bound $d_1 + d_2$ (which then equals $\dim \boldsymbol{T}$) is not achieved if $\theta \neq 1$.

3. The permutation characters ${1_{H_m}}^G$

In this section we employ the method of Section 2 to decompose the permutation characters ${1_{H_m}}^G$. In Section 3.1 we identify the relevant types of H_m-class in H_m, and give information about them and the notation for them which we will use throughout. In Section 3.2 we treat the geometric conjugacy class of unipotent characters; in Section 3.3 we deal with the other geometric conjugacy classes; finally in Section 3.4 we combine the results to obtain the full decomposition of ${1_{H_m}}^G$.

3.1. ***Relevant types of H_m-class in H_m.*** As explained in Section 2.2, in calculating scalar products of ${1_{H_m}}^G$ with generalized Deligne–Lusztig characters it suffices to consider contributions from types of H_m-class in H_m which are relevant, meaning that the degree bound given in Claim 1 is greater than or equal to the degree in q of the order of the H_m-centralizer. We find that most types of H_m-class in H_m are irrelevant.

We take as an example the type of semisimple H_m-class $[s]$ for $m \in \{1, 2\}$ denoted by t_{15} in [15]; according as $p \neq 2$ or $p = 2$ such H_m-classes lie in ones called h_{37} in [25] or h_6 in [24], and they contain elements $\left(1, 1, \lambda\mu, \frac{\lambda}{\mu}; \lambda\right)$ for $\lambda, \mu \in k^*$ satisfying $\lambda^{q-1} = \mu^{q-1} = 1$ and $\lambda^2, \mu^2, \lambda\mu^{\pm 1} \neq 1$. The number of such H_m-classes is $\frac{1}{8}(q-3)(q-5)$ if $p \neq 2$ and $\frac{1}{8}(q-2)(q-4)$ if $p = 2$, so that $\dim Z(C_{\boldsymbol{G}}(s)) = 2$; the H_m-centralizer is $\langle T_0, U_{\pm(1-2)}, U_{\pm(1+2)}\rangle$, of type ${A_1}^2$, while the $\boldsymbol{G}$-centralizer is $\langle \boldsymbol{T}_0, \boldsymbol{U}_{\pm(1-2)}, \boldsymbol{U}_{\pm 2}\rangle$ of type C_2. Write $\epsilon = 1$ or -1 according as $m = 1$ or 2. There are four types of H_m-class $[su]$, depending on whether the projection of u on each A_1 factor of $C_{H_m}(s)$ is trivial or not. If both are trivial then $u = 1$, and so $\dim C_{\boldsymbol{G}}(su) = 12$ while $|C_{H_m}(su)| = q^2(q^2-1)^2(q-1)(q-\epsilon)$. If exactly one is trivial then it is clear that u lies in the C_2-class labeled $W(1) + V(2)$ in [18, Table 8.1a] (this is the class with Bala–Carter label A_1), and so $\dim C_{\boldsymbol{G}}(su) = 8$ while $|C_{H_m}(su)| = q^2(q^2-1)(q-1)(q-\epsilon)$. Finally if neither is trivial then consideration of Jordan blocks on the natural C_2-module shows that u lies in the C_2-class labeled $W(2)$ (if $p \neq 2$) or $V(2)^2$ (if $p = 2$) in [18, Table 8.1a], and so $\dim C_{\boldsymbol{G}}(su) = 6$ while $|C_{H_m}(su)| = q^2(q-1)(q-\epsilon)$. Thus the bound $d_1 + d_2 = \dim\big(Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle\big) + \frac{1}{2}(\dim C_{\boldsymbol{G}}(su) - \dim \boldsymbol{T})$ given in Claim 1 is 6, 4 or 3, respectively, while the degree of the order of the H_m-centralizer is 8, 6 or 4, respectively.

Treating all types of H_m-class in this manner, we find that for a type of H_m-class to be relevant the root system of the $\boldsymbol{G}$-centralizer of the semisimple part of an element in one of the classes concerned must be C_3, $A_1\tilde{A}_2$, $\tilde{A}_2$, $\tilde{A}_1$ or $\varnothing$; moreover, if it is $\tilde{A}_2$ the nonnegligible part of the contribution can be of degree 1 in q, but in the remaining cases it can only be a constant. We shall deal with each of these five possibilities in turn.

In order to do so we require some further notation. Firstly, there are 25 conjugacy classes in W; in [15, Section 4] the following representatives $w_{(1)}, \ldots, w_{(25)}$ are listed.

$w_{(1)} = 1$	$w_{(10)} = w_2 w_3 w_4$	$w_{(18)} = w_1 w_{3-4} w_{2-3} w_{++++}$
$w_{(2)} = w_{3-4}$	$w_{(11)} = w_{3-4} w_{2-3} w_{1-2}$	$w_{(19)} = w_1 w_2 w_4 w_{3-4}$
$w_{(3)} = w_4$	$w_{(12)} = w_{1-2} w_4 w_{3-4}$	$w_{(20)} = w_1 w_2 w_4 w_{+-+-}$
$w_{(4)} = w_3 w_4$	$w_{(13)} = w_4 w_{3-4} w_{+--+}$	$w_{(21)} = w_3 w_{2-3} w_{1-2} w_4$
$w_{(5)} = w_{1-2} w_4$	$w_{(14)} = w_4 w_{3-4} w_{2-3}$	$w_{(22)} = w_2 w_{1-2} w_4 w_{3-4}$
$w_{(6)} = w_{3-4} w_{2-3}$	$w_{(15)} = w_{1-2} w_4 w_{++--}$	$w_{(23)} = w_4 w_{3-4} w_{2-3} w_{1-2}$
$w_{(7)} = w_4 w_{+---}$	$w_{(16)} = w_1 w_{3-4} w_{2-3}$	$w_{(24)} = w_2 w_{2-3} w_{3-4} w_{+---}$
$w_{(8)} = w_4 w_{3-4}$	$w_{(17)} = w_1 w_2 w_3 w_4$	$w_{(25)} = w_3 w_{2-3} w_{3+4} w_{+---}$
$w_{(9)} = w_1 w_2 w_{3-4}$		

Next, for $w \in W$ we write $T_w = \{s \in \boldsymbol{T}_0 : (s^F)^w = s\}$; moreover we write $T_{(n)}$ for $T_{w_{(n)}}$. The tori $T_{(1)}, \ldots, T_{(25)}$ are also given in [15, Section 4]; for reasons of space we shall not reproduce them here. For each $n \in \{1, \ldots, 25\}$ we may choose $g \in \boldsymbol{G}$ satisfying $(g^F)^{-1}g \in \boldsymbol{N}$ and $(g^F)^{-1}g\boldsymbol{T}_0 = w_{(n)}$, and define $\boldsymbol{T}_{(n)} = {}^g\boldsymbol{T}_0$; then $\boldsymbol{T}_{(n)}$ is an F-stable maximal torus of $\boldsymbol{G}$, and we have ${\boldsymbol{T}_{(n)}}^F = {}^gT_{(n)}$. The $\boldsymbol{T}_{(n)}$ are representatives of the G-classes of F-stable maximal tori of $\boldsymbol{G}$, with $\boldsymbol{T}_{(n)}$ being obtained from $\boldsymbol{T}_0$ by twisting with $w_{(n)}$. (In fact as things stand the twisting element is ${w_{(n)}}^{-1}$, but this is only defined up to F-conjugacy in W, which is simply conjugacy because the map F fixes each element of W, and all conjugacy classes in W are self-inverse, as is evident from the fact that the character table of W given in [10] has only real entries.)

Now for each of the five possibilities mentioned above, we shall concentrate on the semisimple classes. We shall give the notation used in [25] (for $p \neq 2$) and [24] (for $p = 2$) for the semisimple G-classes (and shall hereafter employ the former, which is of the form 'h_ℓ' for some ℓ); we shall give the form of elements within them (where instead of an actual G-class representative we will provide an element of $\boldsymbol{T}_0$ lying in the appropriate $\boldsymbol{G}$-class, as is customary); we shall state how such a G-class meets H_m, giving the notation used in [15] for the corresponding class in $B_4(q)$ in the cases of H_1 and H_2, and that used in [5] in the case of H_3; we shall specify the centralizers in G and H_m; we shall give as much information as we shall need on the number of H_m-classes (as noted in Section 2.4, this number is a polynomial in q, and we shall give the leading one or two terms as appropriate, with

[25]	[24]	[15]	[5]	ϵ	# H_m-classes	n									
h_7	h_2	t_{10}	s_3	1	$\frac{1}{2}q+\cdots$	1	2	3	4	5	7	8	9	12	13
h_8	h_{14}	t_{37}	s_7	-1	$\frac{1}{2}q+\cdots$	17	9	10	4	5	20	19	2	12	15

Table 5. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system C_3.

'$+\cdots$' denoting the presence of lower-degree terms which may be ignored); and we shall indicate the tori $T_{(n)}$ which meet the classes (again, up to $\boldsymbol{G}$-conjugacy). Finally, for each such type of semisimple H_m-class we shall specify (if this is not obvious) the corresponding types of H_m-class which are relevant.

In what follows we often write $\epsilon = \pm 1$; we set $A_2^1(q) = A_2(q)$ and $A_2^{-1}(q) = {}^2A_2(q)$, and we write T_m^ϵ for a maximal torus of $A_2^\epsilon(q)$ of order $(q-\epsilon)^2$, q^2-1 or $q^2+\epsilon q+1$ according as $m = 1$, 2 or 3.

There are two types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system C_3; they contain elements $\left(1, \lambda, \frac{1}{\lambda}, 1; 1\right)$ for appropriate λ. Each such class meets H_m in a single class. The centralizers in G and H_m are $C_3(q).T_1$ and $A_1(q^m).A_1(q)^{3-m}.T_1$, respectively, where T_1 is a torus of order $q-\epsilon$. The notation used for the type of class in [25], [24], [15] and [5], the value of ϵ, the leading term in the number of H_m-classes, and the ten values of n such that the class meets the torus $T_{(n)}$ are given in Table 5. For each such h_ℓ, there are two types of relevant H_m-class: the H_m-classes concerned are the semisimple classes themselves, and the classes of regular elements of H_m whose semisimple parts are of type h_ℓ, in which the unipotent part lies in the class in $C_3(q)$ labeled $W(2)+V(2)$ in [18, Table 8.2a].

There are two types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $A_1\tilde{A}_2$; they contain elements $\left(\lambda, \frac{1}{\lambda}, \lambda^2, 1; \lambda\right)$ for appropriate λ. Each such class meets H_m in a single class. The centralizers in G and H_m are $A_1(q).A_2^\epsilon(q).T_1$ and $A_1(q).T_m^\epsilon.T_1$, respectively, where T_1 is a torus of order $q-\epsilon$. The notation used for the type of class in [25], [24], [15] and [5], the value of ϵ, the leading term in the number of H_m-classes, and the six values of n such that the class meets the torus $T_{(n)}$ are given in Table 6. For each such h_ℓ, there is only one type of relevant H_m-class: the H_m-classes concerned contain regular elements of H_m with semisimple parts of type h_ℓ, in which the unipotent part projects nontrivially on the $A_1(q)$ factor and trivially on the $A_2^\epsilon(q)$ factor.

[25]	[24]	[15]	[5]	ϵ	# H_m-classes	n					
h_9	h_5	t_{23}	s_5	1	$\frac{1}{2}q+\cdots$	1	2	3	5	7	15
h_{10}	h_{17}	t_{50}	s_{10}	-1	$\frac{1}{2}q+\cdots$	17	9	10	5	20	13

Table 6. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $A_1\tilde{A}_2$.

[25]	[24]	[15]	[5]	ϵ	$\|T\|$	# H_m-classes	n
h_{31}	h_9	t_{27}	s_6	1	$(q-1)^2$	$\frac{1}{12}q^2-\frac{2}{3}q+\cdots$	1 3 7
h_{32}	h_{21}	t_{54}	s_{15}	-1	$(q+1)^2$	$\frac{1}{12}q^2-\frac{1}{3}q+\cdots$	17 10 20
h_{33}	h_{28}	t_{86}	s_8	1	q^2-1	$\frac{1}{4}q^2-\frac{1}{2}q+\cdots$	2 5 15
h_{34}	h_{42}	t_{92}	s_{11}	-1	q^2-1	$\frac{1}{4}q^2-\frac{1}{2}q+\cdots$	9 5 13
h_{35}	h_{45}	t_{108}	s_{12}	1	q^2+q+1	$\frac{1}{6}q^2+\frac{1}{6}q+\cdots$	6 16 18
h_{36}	h_{48}	t_{112}	s_{13}	-1	q^2-q+1	$\frac{1}{6}q^2-\frac{1}{6}q+\cdots$	21 14 25

Table 7. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $\tilde{A}_2$.

For each of the remaining three possibilities, the semisimple elements concerned are regular in H_m; thus the centralizer in H_m is a maximal torus, and for each h_ℓ the only H_m-classes containing elements with semisimple parts of type h_ℓ are the semisimple H_m-classes themselves.

There are six types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $\tilde{A}_2$; they contain elements $\left(\lambda, \mu, \frac{\lambda}{\mu}, 1; \lambda\right)$ for appropriate λ and μ. Each such class meets H_m in a single class. The centralizers in G and H_m are $A_2^\epsilon(q).T$ and $T_m^\epsilon.T$, respectively, where T is a torus. The notation used for the type of class in [25], [24], [15] and [5], the values of ϵ and $|T|$, the leading terms in the number of H_m-classes, and the three values of n such that the class meets the torus $T_{(n)}$ are given in Table 7.

There are ten types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $\tilde{A}_1$; they contain elements $\left(\lambda, \mu, \frac{\nu^2}{\lambda\mu}, 1; \nu\right)$ for appropriate λ, μ and ν. Each such class fails to meet H_3, but meets $B_4(q)$ in two classes, one of which is regular in $B_4(q)$; this regular class in $B_4(q)$ fails to meet H_2, but meets H_1 in two classes, while the other class in $B_4(q)$ meets both H_1 and H_2 in a single class. The number of classes of a given type in H_m is thus f times the number of such classes in H_2. The centralizers in G and H_m are $A_1(q).T$ and $T_1.T$, respectively, where T is a torus and T_1 is a torus of order $q-\epsilon$. The notation used for the type of class in [25], [24] and [15], the values of ϵ and $|T|$, the leading term in the number of H_m-classes, and the two values of n such that the class meets the torus $T_{(n)}$ are given in Table 8; in each case the first t_j listed in the third column meets both H_1 and H_2, while the second meets just H_1.

Finally, there are 25 types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $\varnothing$; they contain elements $\left(\lambda, \mu, \frac{\nu^2}{\lambda\mu\pi}, \pi; \nu\right)$ for appropriate λ, μ, ν and π. Each such class meets H_m for exactly one value of m, and does so in six H_m-classes; this is because $W_{\boldsymbol{H}}$ is normal in W, with quotient $\langle W_{\boldsymbol{H}} w_4, W_{\boldsymbol{H}} w_{+---}\rangle \cong S_3$, so that each class in W meets exactly one of the cosets $W_{\boldsymbol{H}}$, $W_{\boldsymbol{H}} w_4$ and $W_{\boldsymbol{H}} w_4 w_{+---}$. The centralizer in both G and H of such an element is a torus T. The notation used for the type of class in [25], [24] and either [15] or [5], the value of m such that the class

[25]	[24]	[15]	ϵ	$\lvert T \rvert$	# H_m-classes	n
h_{66}	h_{10}	t_{28}, t_{29}	1	$(q-1)^3$	$\frac{1}{48} f q^3 + \cdots$	1 3
h_{67}	h_{22}	t_{55}, t_{56}	−1	$(q+1)^3$	$\frac{1}{48} f q^3 + \cdots$	17 10
h_{68}	h_{33}	t_{67}, t_{122}	1	$(q^2-1)(q-1)$	$\frac{1}{16} f q^3 + \cdots$	4 3
h_{69}	h_{37}	t_{74}, t_{123}	−1	$(q^2-1)(q+1)$	$\frac{1}{16} f q^3 + \cdots$	4 10
h_{70}	h_{29}	t_{87}, t_{88}	1	$(q^2-1)(q-1)$	$\frac{1}{8} f q^3 + \cdots$	2 5
h_{71}	h_{43}	t_{93}, t_{94}	−1	$(q^2-1)(q+1)$	$\frac{1}{8} f q^3 + \cdots$	9 5
h_{72}	h_{54}	t_{99}, t_{129}	1	$(q^2+1)(q-1)$	$\frac{1}{8} f q^3 + \cdots$	11 8
h_{73}	h_{59}	t_{102}, t_{130}	−1	$(q^2+1)(q+1)$	$\frac{1}{8} f q^3 + \cdots$	11 19
h_{74}	h_{46}	t_{109}, t_{110}	1	q^3-1	$\frac{1}{6} f q^3 + \cdots$	6 16
h_{75}	h_{49}	t_{113}, t_{114}	−1	q^3+1	$\frac{1}{6} f q^3 + \cdots$	21 14

Table 8. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $\tilde{A}_1$.

meets H_m, the value of $|T|$, the leading term in the number of H_m-classes, and the single value of n such that the class meets the torus $T_{(n)}$ are given in Table 9. (Note that the W-classes containing $w_{(4)}$ and $w_{(11)}$ each split into two $W(B_4)$-classes, so that there are two types of semisimple class given in [15]; in each case the second of the two $W(B_4)$-classes splits further into two $W_{\boldsymbol{H}}$-classes, so that there are in fact three types of semisimple class in H_1 in these cases.)

As stated above, from now on we shall refer to the types of semisimple class described above by the notation of [25]. Thus we have h_7 and h_8; h_9 and h_{10}; $h_{31}, \ldots, h_{36}$; $h_{66}, \ldots, h_{75}$; and $h_{76}, \ldots, h_{100}$. Of these five collections, for the first three each semisimple G-class meets each H_m in a single class; in the fourth, each G-class meets H_m in f classes; in the fifth, each G-class meets just one H_m and does so in 6 classes. In all cases, in the calculations to follow we shall need to consider the classes of regular elements of H_m whose semisimple parts are of the types concerned; for h_7 and h_8 we must also treat the semisimple classes themselves.

3.2. ***Unipotent characters.*** We first consider the geometric conjugacy class of unipotent characters of G; the $R_{\boldsymbol{T},\theta}$ lying in this class are those with $\theta = 1$. We write $R_{(n)}$ for $R_{\boldsymbol{T}_{(n)},1}$.

We begin with types h_7 and h_8; we must treat the semisimple classes and those containing regular elements of H_m. We shall take as an example the contribution from elements with semisimple part s of type h_ℓ to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ in the case $(\ell, n) = (7, 1)$. The number of classes is $\frac{1}{2}q + \cdots$; there are $\frac{1152}{48} = 24$ distinct conjugates of s lying in $T_{(1)}$. For the semisimple classes, the Green function value is

$$\frac{(q^2-1)(q^4-1)(q^6-1)}{(q-1)^3} = (q^4+q^2+1)(q^2+1)(q+1)^3,$$

[25]	[24]	[15] or [5]	m	$\|T\|$	# H_m-classes	n
h_{76}	h_{12}	t_{30}	1	$(q-1)^4$	$\frac{1}{192}q^4+\cdots$	1
h_{77}	h_{24}	t_{57}	1	$(q+1)^4$	$\frac{1}{192}q^4+\cdots$	17
h_{78}	h_{30}	t_{89}	1	$(q^2-1)(q-1)^2$	$\frac{1}{16}q^4+\cdots$	2
h_{79}	h_{36}	t_{68}	2	$(q^2-1)(q-1)^2$	$\frac{1}{48}q^4+\cdots$	3
h_{80}	h_{39}	t_{81}, t_{124}	1	$(q^2-1)^2$	$\frac{3}{32}q^4+\cdots$	4
h_{81}	h_{44}	t_{105}	2	$(q^2-1)^2$	$\frac{1}{8}q^4+\cdots$	5
h_{82}	h_{47}	t_{111}	1	$(q^3-1)(q-1)$	$\frac{1}{6}q^4+\cdots$	6
h_{83}	h_{68}	s_6	3	$(q^3-1)(q-1)$	$\frac{1}{12}q^4+\cdots$	7
h_{84}	h_{56}	t_{100}	2	$(q^2+1)(q-1)^2$	$\frac{1}{16}q^4+\cdots$	8
h_{85}	h_{57}	t_{95}	1	$(q^2-1)(q+1)^2$	$\frac{1}{16}q^4+\cdots$	9
h_{86}	h_{58}	t_{75}	2	$(q^2-1)(q+1)^2$	$\frac{1}{48}q^4+\cdots$	10
h_{87}	h_{60}	t_{107}, t_{131}	1	q^4-1	$\frac{3}{8}q^4+\cdots$	11
h_{88}	h_{62}	t_{125}	2	q^4-1	$\frac{1}{8}q^4+\cdots$	12
h_{89}	h_{72}	s_{11}	3	$(q^3+1)(q-1)$	$\frac{1}{4}q^4+\cdots$	13
h_{90}	h_{51}	t_{117}	2	$(q^3+1)(q-1)$	$\frac{1}{6}q^4+\cdots$	14
h_{91}	h_{73}	s_8	3	$(q^3-1)(q+1)$	$\frac{1}{4}q^4+\cdots$	15
h_{92}	h_{52}	t_{116}	2	$(q^3-1)(q+1)$	$\frac{1}{6}q^4+\cdots$	16
h_{93}	h_{74}	s_{12}	3	$(q^2+q+1)^2$	$\frac{1}{24}q^4+\cdots$	18
h_{94}	h_{64}	t_{103}	2	$(q^2+1)(q+1)^2$	$\frac{1}{16}q^4+\cdots$	19
h_{95}	h_{71}	s_{15}	3	$(q^3+1)(q+1)$	$\frac{1}{12}q^4+\cdots$	20
h_{96}	h_{50}	t_{115}	1	$(q^3+1)(q+1)$	$\frac{1}{6}q^4+\cdots$	21
h_{97}	h_{63}	t_{128}	1	$(q^2+1)^2$	$\frac{1}{16}q^4+\cdots$	22
h_{98}	h_{65}	t_{132}	2	q^4+1	$\frac{1}{4}q^4+\cdots$	23
h_{99}	h_{76}	s_{14}	3	q^4-q^2+1	$\frac{1}{4}q^4+\cdots$	24
h_{100}	h_{75}	s_{13}	3	$(q^2-q+1)^2$	$\frac{1}{24}q^4+\cdots$	25

Table 9. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $\varnothing$.

while the order of the centralizer in H_m is a polynomial in q with leading term q^{10}; thus the contribution here is

$$\frac{24\cdot(q^4+q^2+1)(q^2+1)(q+1)^3\cdot\left(\frac{1}{2}q+\cdots\right)}{q^{10}+\cdots},$$

whose nonnegligible part is 12. For the classes of regular elements of H_m, the Green function value is a polynomial in q with leading term $3q^3$ (as may be seen from [11; 22]), while the order of the centralizer in H_m is $q^4+\cdots$; thus the contribution here is

$$\frac{24 \cdot (3q^3 + \cdots) \cdot \left(\frac{1}{2}q + \cdots\right)}{q^4 + \cdots},$$

with nonnegligible part 36. Treating the other cases (ℓ, n) similarly gives the following, in which the two values listed in each case correspond to the semisimple and regular classes of H_m, respectively.

h_7			h_8		
$R_{(1)}$	12	36	$R_{(17)}$	−12	−36
$R_{(2)}$	−3	3	$R_{(9)}$	3	−3
$R_{(3)}$	−6	−6	$R_{(10)}$	6	6
$R_{(4)}$	2	−2	$R_{(4)}$	−2	2
$R_{(5)}$	1	1	$R_{(5)}$	−1	−1
$R_{(7)}$	3	0	$R_{(20)}$	−3	0
$R_{(8)}$	2	−2	$R_{(19)}$	−2	2
$R_{(9)}$	−1	−3	$R_{(2)}$	1	3
$R_{(12)}$	−1	1	$R_{(12)}$	1	−1
$R_{(13)}$	−1	0	$R_{(15)}$	1	0

Next we consider types h_9 and h_{10}; here we treat only the classes of regular elements of H_m. As an example, take the contribution from such elements su with s of type h_ℓ to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ in the case $(\ell, n) = (9, 1)$. The number of classes is $\frac{1}{2}q + \cdots$; there are $\frac{1152}{12} = 96$ distinct conjugates of s lying in $T_{(1)}$. Since the projections of u in the $A_1(q)$ and $A_2(q)$ factors of $C_G(s)$ are regular and trivial, respectively, the Green function value at u is

$$\frac{1 \cdot (q^2 - 1)(q^3 - 1)}{(q - 1)^2} = (q^2 + q + 1)(q + 1);$$

and $|C_{H_m}(su)|$ is a polynomial in q with leading term q^4. Thus the contribution is

$$\frac{96 \cdot (q^2 + q + 1)(q + 1) \cdot \left(\frac{1}{2}q + \cdots\right)}{q^4 + \cdots},$$

whose nonnegligible part is 48. Dealing similarly with the other pairs (ℓ, n) gives the following.

h_9 :

$R_{(1)}$	$R_{(2)}$	$R_{(3)}$	$R_{(5)}$	$R_{(7)}$	$R_{(15)}$
48	4	−12	−2	3	1

h_{10} :

$R_{(17)}$	$R_{(9)}$	$R_{(10)}$	$R_{(5)}$	$R_{(20)}$	$R_{(13)}$
−48	−4	12	2	−3	−1

For types $h_{31}, \dots, h_{36}$ the calculations are a little more delicate; consideration of leading terms alone is insufficient, since here nonnegligible parts of contributions are linear polynomials in q rather than merely constants. As an example, take the contribution from semisimple elements s of type h_ℓ to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ in the case $(\ell, n) = (31, 1)$. The number of classes is $\frac{1}{12}q^2 - \frac{2}{3}q + \cdots$; there are $\frac{1152}{6} = 192$ distinct conjugates of s lying in $T_{(1)}$. The Green function value is

$$\frac{(q^2-1)(q^3-1)}{(q-1)^2} = (q^2+q+1)(q+1);$$

and

$$|C_{H_m}(s)| = (q-1)^2|T_m^1| = q^4 - (f+1)q^3 + \cdots .$$

Thus the contribution is

$$\frac{192 \cdot (q^3+2q^2+\cdots) \cdot \frac{1}{12}(q^2-8q+\cdots)}{q^4-(f+1)q^3+\cdots} = 16(q+(f-5)+\cdots).$$

Dealing similarly with the other pairs (ℓ, n) gives the following.

h_{31}	$R_{(1)}$	$16q+16f-80$	$R_{(3)}$	$-4q-4f+28$	$R_{(7)}$	$q+f-8$
h_{32}	$R_{(17)}$	$-16q+16f+112$	$R_{(10)}$	$4q-4f-20$	$R_{(20)}$	$-q+f+4$
h_{33}	$R_{(2)}$	$4q+4f-4$	$R_{(5)}$	$-2q-2f+6$	$R_{(15)}$	$q+f-4$
h_{34}	$R_{(9)}$	$-4q+4f+12$	$R_{(5)}$	$2q-2f-2$	$R_{(13)}$	$-q+f$
h_{35}	$R_{(6)}$	$q+f+1$	$R_{(16)}$	$-q-f+1$	$R_{(18)}$	$4q+4f-8$
h_{36}	$R_{(21)}$	$-q+f+1$	$R_{(14)}$	$q-f+1$	$R_{(25)}$	$-4q+4f-8$

The remaining types h_ℓ are more easily handled. For types $h_{66}, \dots, h_{75}$ the Green function value concerned is always $\pm q + 1$. Proceeding as above we obtain the following.

h_{66}	$R_{(1)}$	$12f$	$R_{(3)}$	$-f$
h_{67}	$R_{(17)}$	$-12f$	$R_{(10)}$	f
h_{68}	$R_{(4)}$	$-2f$	$R_{(3)}$	$3f$
h_{69}	$R_{(4)}$	$2f$	$R_{(10)}$	$-3f$
h_{70}	$R_{(2)}$	$6f$	$R_{(5)}$	$-f$

h_{71}	$R_{(9)}$	$-6f$	$R_{(5)}$	f
h_{72}	$R_{(11)}$	$-f$	$R_{(8)}$	$2f$
h_{73}	$R_{(11)}$	f	$R_{(19)}$	$-2f$
h_{74}	$R_{(6)}$	$3f$	$R_{(16)}$	$-f$
h_{75}	$R_{(21)}$	$-3f$	$R_{(14)}$	f

Finally, the types $h_{76}, \dots, h_{100}$ contain regular semisimple elements of G, so that the Green function value is just 1. For each such type h_ℓ, the classes meet $T_{(n)}$ for just one value of n, and H_m for just one value of m. It follows that for this choice of n and m the nonnegligible part of the contribution from classes of type h_ℓ

	H_1	H_2	H_3
$R_{(1)}$	$16q+106$	$16q+44$	$16q+16$
$R_{(2)}$	$4q+40$	$4q+14$	$4q+4$
$R_{(3)}$	$-4q-2$	$-4q+4$	$-4q+4$
$R_{(4)}$	6	0	0
$R_{(5)}$	-8	2	4
$R_{(6)}$	$q+19$	$q+5$	$q+1$
$R_{(7)}$	$q+1$	$q-1$	$q+1$
$R_{(8)}$	6	4	0
$R_{(9)}$	$-4q+4$	$-4q+2$	$-4q+4$
$R_{(10)}$	$4q-14$	$4q$	$4q+4$
$R_{(11)}$	6	0	0
$R_{(12)}$	0	2	0
$R_{(13)}$	$-q+1$	$-q-1$	$-q+1$
$R_{(14)}$	$q+1$	$q+3$	$q+1$
$R_{(15)}$	$q+1$	$q-1$	$q+1$
$R_{(16)}$	$-q-5$	$-q+1$	$-q+1$
$R_{(17)}$	$-16q+34$	$-16q+20$	$-16q+16$
$R_{(18)}$	$4q+4$	$4q-4$	$4q-5$
$R_{(19)}$	-6	0	0
$R_{(20)}$	$-q+1$	$-q-1$	$-q+1$
$R_{(21)}$	$-q+1$	$-q-1$	$-q+1$
$R_{(22)}$	6	0	0
$R_{(23)}$	0	2	0
$R_{(24)}$	0	0	3
$R_{(25)}$	$-4q+4$	$-4q-4$	$-4q-5$

Table 10. Scalar products $({1_{H_m}}^G, R_{(n)})_G$.

to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ is simply $|C_W(w_{(n)})|$ times the coefficient of q^4 in the number of such classes, as given above; this value is always 6, 2 or 3 according as $m = 1$, 2 or 3.

Summing all of the above nonnegligible parts gives the values in Table 10 for the scalar products $({1_{H_m}}^G, R_{(n)})_G$.

There are now two steps in forming the irreducible unipotent characters of G from the Deligne–Lusztig generalized characters $R_{(n)}$. Firstly, for each irreducible character ϕ of W we form

$$R_\phi = \sum_{n=1}^{25} \frac{\phi(w_{(n)})R_{(n)}}{|C_W(w_{(n)})|}.$$

The class functions obtained in this way are among the set of almost characters. The second step is to transform the almost characters by nonabelian Fourier transform matrices to obtain the irreducible characters.

For the first of these steps, the character table of W is given in [10], and we have

$$(1_{H_m}{}^G, R_\phi)_G = \sum_{n=1}^{25} \frac{\phi(w_{(n)})}{|C_W(w_{(n)})|}(1_{H_m}{}^G, R_{(n)})_G;$$

it is therefore straightforward to calculate the scalar products of $1_{H_m}{}^G$ with those almost characters of the form R_ϕ. We find that the only such scalar products which are nonzero for some m are as follows.

ϕ	$(1_{H_m}{}^G, R_\phi)_G$
$\phi_{1,0}, \phi_{9,2}, \phi_{12,4}$	1
$\phi''_{8,3}$	$q+f$
$\phi''_{8,9}$	q
$\phi_{4,1}, \phi''_{2,4}, \phi''_{9,6}$	f
$\phi''_{6,6}$	$f-1$
$\phi''_{1,12}$	$3, -1, 0$ according as $m = 1, 2, 3$

The second step is complicated by the fact that not all almost characters are of the form R_ϕ; however, it turns out that it is still possible to deduce the scalar products of $1_{H_m}{}^G$ with all irreducible unipotent characters. The almost characters R_ϕ for $\phi = \phi_{1,0}, \phi_{9,2}, \phi''_{8,3}$ and $\phi''_{8,9}$ all lie in families of size one, and are thus themselves irreducible characters χ_ϕ; they therefore appear in $1_{H_m}{}^G$ with multiplicities 1, 1, $q+f$ and q, respectively. The almost characters R_ϕ for $\phi = \phi_{4,1}$ and $\phi''_{2,4}$ lie in a family of size four, whose other members are $R_{\phi'_{2,4}}$ and a class function Y_0, say; the corresponding unipotent characters are $\chi_{\phi_{4,1}}, \chi_{\phi''_{2,4}}, \chi_{\phi'_{2,4}}$ and $\chi_{B_2,1}$ (in the notation of [4, 13.9]), and we have

$$\chi_{\phi_{4,1}} = \tfrac{1}{2}(R_{\phi_{4,1}} + R_{\phi''_{2,4}} + R_{\phi'_{2,4}} + Y_0),$$
$$\chi_{\phi''_{2,4}} = \tfrac{1}{2}(R_{\phi_{4,1}} + R_{\phi''_{2,4}} - R_{\phi'_{2,4}} - Y_0),$$
$$\chi_{\phi'_{2,4}} = \tfrac{1}{2}(R_{\phi_{4,1}} - R_{\phi''_{2,4}} + R_{\phi'_{2,4}} - Y_0),$$
$$\chi_{B_2,1} = \tfrac{1}{2}(R_{\phi_{4,1}} - R_{\phi''_{2,4}} - R_{\phi'_{2,4}} + Y_0).$$

If we let $(1_{H_m}{}^G, Y_0)_G = y_0$, then the values above imply that the scalar products of $1_{H_m}{}^G$ with the four irreducible characters are

$$f + \tfrac{1}{2}y_0, \quad f - \tfrac{1}{2}y_0, \quad -\tfrac{1}{2}y_0, \quad \tfrac{1}{2}y_0;$$

as these must all be nonnegative, it follows that $y_0 = 0$ and the scalar products are $f, f, 0, 0$.

The remaining almost characters R_ϕ listed above, those for which $\phi = \phi_{12,4}$, $\phi''_{9,6}$, $\phi''_{6,6}$ and $\phi''_{1,12}$, all lie in a single family of size 21; of the other 17 almost characters in this family, only 7 are also of the form R_ϕ. The corresponding nonabelian Fourier transform matrix is given in [4, p. 456] and repeated in the Appendix of [12]. We shall use an analysis similar to that employed in [12] to determine the scalar products with irreducible characters. We number the rows and columns of the 21×21 matrix in the order in which they appear in [4, p. 456]; for $1 \le i, j \le 21$ let m_i and n_j be the scalar products of ${1_{H_m}}^G$ with the i-th almost character and j-th irreducible character in the family, respectively. Each n_j is therefore the linear combination of the m_i with coefficients given by the j-th column of the matrix. The values calculated above imply that $m_i = 0$ for $i = 2, 3, 6, 9, 13, 15, 18$. In addition, there are two pairs of irreducible characters, and correspondingly two pairs of almost characters, which are complex conjugates of each other; since the values taken by ${1_{H_m}}^G$ are all real, it follows that the scalar products concerned must be equal, so that $m_{16} = m_{17}$, $m_{20} = m_{21}$, $n_{16} = n_{17}$ and $n_{20} = n_{21}$. Note that the scalar product with $R_{\phi''_{1,12}}$ means that the cases with $m = 1$, 2 and 3 must be handled separately.

First assume that $m = 1$; we thus have $m_1 = 1$, $m_5 = 2$, and $m_{10} = m_{12} = 3$. By adding together the appropriate columns of the matrix we see that

$$\begin{aligned} n_{15} + n_{16} + n_{17} &= -1 + m_4, \\ n_{18} + n_{19} + n_{20} + n_{21} &= 1 - m_4; \end{aligned}$$

thus $m_4 = 1$, and $n_{15} = \cdots = n_{21} = 0$. Since we then have $n_{20} = -\frac{1}{4}m_{11}$ and $n_{16} = \frac{1}{3}m_{16}$, it follows that $m_{11} = m_{16} = 0$. Next,

$$n_6 = -\tfrac{1}{2}m_7 \quad \text{and} \quad n_7 = \tfrac{1}{2}m_7,$$

so that $m_7 = n_6 = n_7 = 0$. Similarly,

$$n_8 = \tfrac{1}{2}m_8 \quad \text{and} \quad n_9 = -\tfrac{1}{2}m_8,$$

so that $m_8 = n_8 = n_9 = 0$;

$$n_{18} = -\tfrac{1}{2}m_{19} \quad \text{and} \quad n_{19} = \tfrac{1}{2}m_{19},$$

so that $m_{19} = n_{18} = n_{19} = 0$;

$$n_{14} = \tfrac{1}{2}m_{14} \quad \text{and} \quad n_2 + n_3 = -\tfrac{1}{2}m_{14},$$

so that $m_{14} = n_{14} = n_2 = n_3 = 0$; and

$$n_{11} = -\tfrac{1}{2}m_{20} \quad \text{and} \quad n_{13} = \tfrac{1}{2}m_{20},$$

so that $m_{20} = n_{11} = n_{13} = 0$. All the m_i having now been determined, the remaining n_j may be found; we obtain $n_1 = n_4 = 1$, $n_5 = 2$, $n_{10} = n_{12} = 3$. It follows that the irreducible constituents of ${1_{H_1}}^G$ lying in this family are $\chi_{\phi_{12,4}}$, $\chi_{\phi''_{9,6}}$, $\chi_{\phi''_{1,12}}$, $\chi_{\phi''_{6,6}}$, $\chi_{F_4^{\mathrm{II}}[1]}$, with multiplicities 1, 3, 3, 2, 1, respectively.

The analyses for $m = 2$ and $m = 3$ are very similar. In both cases we find that the irreducible constituents of ${1_{H_m}}^G$ lying in the family all have multiplicity 1; for $m = 2$ they are $\chi_{\phi''_{4,7}}$, $\chi_{\phi_{16,5}}$, $\chi_{B_2,\epsilon'}$ and $\chi_{F_4[-1]}$, while for $m = 3$ they are $\chi_{\phi'_{6,6}}$, $\chi_{F_4[\theta]}$ and $\chi_{F_4[\theta^2]}$. This completes the treatment of the geometric conjugacy class of unipotent characters of G.

3.3. ***Other geometric conjugacy classes.*** We now turn to the other geometric conjugacy classes, which contain the $R_{\boldsymbol{T},\theta}$ with $\theta \neq 1$. In most such instances, the contribution to the scalar product $({1_{H_m}}^G, R_{\boldsymbol{T},\theta})_G$ from a relevant type of H_m-class will have zero nonnegligible part; this is because adding together the roots of unity $\theta(s)$, as s runs through the elements of $\boldsymbol{T}$ lying in H_m-classes of the type concerned, usually results in a cancellation of terms. In fact, this is exactly what happens for all $\theta \neq 1$ in the case of types of H_m-class containing regular semisimple elements of G, as noted at the end of Section 2.4; we shall therefore not need to consider types $h_{76}, \dots, h_{100}$ any further here.

We shall use notation akin to that of [15] for the geometric conjugacy classes of characters of G. Recall that there is a bijective correspondence between geometric conjugacy classes of G and semisimple classes of the dual group, which in this case is isomorphic to G itself; in [15] we have provided for each n an explicit correspondence, involving certain roots of unity ξ_i in k^* and ζ_i in $\mathbb{C}^*$, between the linear characters of the torus $T_{(n)}$ and the elements of the dual torus. We shall say that a geometric conjugacy class is of type κ_c if the corresponding semisimple class is termed h_c in [25]. Writing $\mathbb{Z}_n$ for the integers modulo n for appropriate $n \in \mathbb{N}$, we shall define a set S_c, and an equivalence relation $\sim$ on it, such that the set $\bar{S}_c$ of equivalence classes $[i]$ for $i \in S_c$ parametrizes the semisimple classes of type h_c; accordingly an individual geometric conjugacy class of type κ_c will be denoted by $\kappa_{c,[i]}$. (This notation differs slightly from that employed in [15], where such geometric conjugacy classes were indexed by the element i of S_c rather than the equivalence class $[i]$. The only such equivalence relation required there was that defined by $i \sim -i$, so that the list of all such geometric conjugacy classes could be obtained as $\kappa_{c,i}$ as i ran through the first half of the set S_c; here by contrast more complicated equivalence relations will be needed, so that it will be clearer to index geometric conjugacy classes explicitly by equivalence classes.) The irreducible characters lying in a given geometric conjugacy class are parametrized by the unipotent characters of the centralizer of an element of the corresponding semisimple class; an irreducible character lying in the geometric conjugacy class $\kappa_{c,[i]}$ will be written in the form $\chi^*_{\kappa_{c,[i]}}$, where the superscript indicates the corresponding unipotent character.

We shall refrain from giving full details, because there are several types requiring consideration, and the calculation for each is fairly involved (as may be surmised from the treatment of the unipotent geometric conjugacy class above). Instead we shall deal at some length with two types of geometric conjugacy class, κ_{31} and κ_7, and indicate briefly how the behavior for other classes is related to one or other of these. Note that it was only in handling semisimple classes of types $h_{76}, \ldots, h_{100}$ that H_1, H_2 and H_3 had to be treated separately in the calculations above; since these classes play no further role, at all stages it will be possible to work with all three permutation characters ${1_{H_m}}^G$ simultaneously.

For convenience we repeat from [15] the notation used for roots of unity, as this will be needed in much of what follows. We let $\xi \in k^*$ be a primitive $(q^4+1)(q^{12}-1)$-st root of unity, and for $s = 1, \ldots, 15$ we set $\xi_s = \xi^{r_s}$, where r_s and $(q^4+1)(q^{12}-1)/r_s = o(\xi_s)$ (the order of ξ_s in the multiplicative group k^*) are

$$
\begin{aligned}
r_1 &= (q+1)(q^2+1)(q^4+1)(q^8+q^4+1), & o(\xi_1) &= q-1,\\
r_2 &= (q-1)(q^2+1)(q^4+1)(q^8+q^4+1), & o(\xi_2) &= q+1,\\
r_3 &= (q^2+1)(q^4+1)(q^8+q^4+1), & o(\xi_3) &= q^2-1,\\
r_4 &= (q^2-1)(q^4+1)(q^8+q^4+1), & o(\xi_4) &= q^2+1,\\
r_5 &= (q-1)(q^3+1)(q^4+1)(q^6+1), & o(\xi_5) &= q^2+q+1,\\
r_6 &= (q+1)(q^3-1)(q^4+1)(q^6+1), & o(\xi_6) &= q^2-q+1,\\
r_7 &= (q^3+1)(q^4+1)(q^6+1), & o(\xi_7) &= q^3-1,\\
r_8 &= (q^3-1)(q^4+1)(q^6+1), & o(\xi_8) &= q^3+1,\\
r_9 &= (q+1)(q^4+1)(q^8+q^4+1), & o(\xi_9) &= (q^2+1)(q-1),\\
r_{10} &= (q-1)(q^4+1)(q^8+q^4+1), & o(\xi_{10}) &= (q^2+1)(q+1),\\
r_{11} &= (q^4+1)(q^8+q^4+1), & o(\xi_{11}) &= q^4-1,\\
r_{12} &= q^{12}-1, & o(\xi_{12}) &= q^4+1,\\
r_{13} &= (q^2-q+1)(q^4+1)(q^6+1), & o(\xi_{13}) &= (q^3-1)(q+1),\\
r_{14} &= (q^2+q+1)(q^4+1)(q^6+1), & o(\xi_{14}) &= (q^3+1)(q-1),\\
r_{15} &= (q^4+q^2+1)(q^8-1), & o(\xi_{15}) &= q^4-q^2+1.
\end{aligned}
$$

Likewise we write

$$\zeta = e^{2\pi i/(q^4+1)(q^{12}-1)} \in \mathbb{C}^*,$$

and for $s = 1, \ldots, 15$ we set $\zeta_s = \zeta^{r_s}$.

3.3.1. *Geometric conjugacy classes of type κ_{31}.* The geometric conjugacy classes of type κ_{31} correspond to semisimple classes in G containing elements

$$({\xi_1}^{i+j}, {\xi_1}^{i}, {\xi_1}^{j}, 1; {\xi_1}^{i+j}) \quad \text{with } {\xi_1}^{i}, {\xi_1}^{j}, {\xi_1}^{i\pm j}, {\xi_1}^{2i+j}, {\xi_1}^{i+2j} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 7; the number of classes is $\frac{1}{12}(q^2 - 8q + 10 + 3d + 2y)$. To parametrize these geometric conjugacy classes, we set

$$S_{31} = \{(i, j) \in {\mathbb{Z}_{q-1}}^2 : i, j, i \pm j, 2i + j, i + 2j \neq 0\},$$

so that $(i, j) \in S_{31}$ corresponds to the element given; we define an equivalence relation on S_{31} by

$$(i, j) \sim (j, i) \sim (-i, -j) \sim (i + j, -j),$$

and let $\bar{S}_{31}$ be the set of equivalence classes $[(i, j)]$. The geometric conjugacy classes of type κ_{31} are in bijective correspondence with $\bar{S}_{31}$; we shall write $\kappa_{31,[(i,j)]}$ for the class corresponding to $[(i, j)] \in \bar{S}_{31}$.

There are three distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{31,[(i,j)]}$; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as

$$(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00j}), \quad (\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0j}), \quad (\boldsymbol{T}_{(6)}, \theta^{(6)}_{-(q^2+q+1)i,2i+j}).$$

For convenience of reference, we repeat the definition of the characters θ here:

$$\theta^{(1)}_{i00j}({\xi_1}^a, {\xi_1}^b, {\xi_1}^c, {\xi_1}^{2d-a-b-c}; {\xi_1}^d) = {\zeta_1}^{ia+jd},$$
$$\theta^{(2)}_{i0j}({\xi_1}^a, {\xi_1}^{2c-a-b}, {\xi_3}^b, {\xi_3}^{qb}; {\xi_1}^c) = {\zeta_1}^{ia+jc},$$
$$\theta^{(6)}_{-(q^2+q+1)i,2i+j}({\xi_1}^{2b-a}, {\xi_7}^a, {\xi_7}^{qa}, {\xi_7}^{q^2a}; {\xi_1}^b) = {\zeta_1}^{-ia+(2i+j)b}.$$

We shall take each character $R_{\boldsymbol{T},\theta}$ in turn, and find the scalar product with ${1_{H_m}}^G$; to do this we shall take each type of class handled above containing elements with semisimple parts lying in the torus concerned, and calculate the nonnegligible part of its contribution. We begin with the pair $(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00j})$.

The semisimple classes of type h_7 contain elements $(1, {\xi_1}^a, {\xi_1}^{-a}, 1; 1)$ with ${\xi_1}^{2a} \neq 1$; each such element has 24 distinct conjugates in $T_{(1)}$. Of these, 6 are of the form $(1, *, *, *; 1)$, and are thus sent to 1 by $\theta^{(1)}_{i00j}$; the remaining 18 are sent to various other powers of ζ_1. On summing over the $\frac{1}{2}(q - 2 - d)$ classes of such elements, the values obtained from the conjugates of the form $(1, *, *, *; 1)$ combine to give a sum of $3(q - 2 - d)$; on the other hand the values obtained from the conjugates not of the form $(1, *, *, *; 1)$ cancel each other out to give a sum of $9(d - 1)$, which is too small to affect the rest of the calculation. We may now proceed as in the unipotent case already treated, multiplying by the Green function value and dividing by the order of the centralizer in H_m; since the leading term is obtained from only 6 of the 24 values taken by $\theta^{(1)}_{i00j}$ on each class, each nonnegligible part is one quarter of the corresponding value in the unipotent case.

Thus the nonnegligible part of the contribution from the semisimple classes is 3, while that from the regular classes is 9.

The classes with semisimple part of type h_9 behave in a very similar fashion. The semisimple elements here are $({\xi_1}^a, {\xi_1}^{-a}, {\xi_1}^{2a}, 1; {\xi_1}^a)$ with ${\xi_1}^{2a}, {\xi_1}^{3a} \neq 1$; each such element has 96 distinct conjugates in $T_{(1)}$. Again, 6 of these are of the form $(1, *, *, *; 1)$, and are thus sent to 1 by $\theta^{(1)}_{i00j}$, while the remainder are sent to various other powers of ζ_1. Only the former need be considered, because summing over the $\frac{1}{2}(q-1-d-y)$ classes of such elements produces cancellation among the latter. Since the proportion of roots of unity which are 1 is $\frac{6}{96}$, multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a rational polynomial whose leading term is one sixteenth of that obtained in the unipotent case; thus the nonnegligible part of the contribution from the classes of regular elements of H_m with semisimple part of type h_9 is 3.

The semisimple classes of type h_{31} require somewhat more care. The elements are $({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$ with ${\xi_1}^a, {\xi_1}^b, {\xi_1}^{a\pm b}, {\xi_1}^{2a+b}, {\xi_1}^{a+2b} \neq 1$ (as above); each such element has 192 distinct conjugates in $T_{(1)}$. However, it is not sufficient here simply to count how many are sent to 1 by $\theta^{(1)}_{i00j}$; because the calculation will involve not just the leading term, but the next term as well, we must consider the sum of roots of unity more carefully than was necessary in the previous two paragraphs.

Let s be the element given. The 192 distinct conjugates of s in $T_{(1)}$ are obtained as $s^{w'w''w'''}$, where $w' \in \langle w_1, w_2, w_3\rangle$, $w'' \in \{1, w_{1-2}, w_{1-3}, w_{1-4}\}$ and $w''' \in \langle w_{2-3}, w_{3-4}\rangle$. Since the effect of w''' is simply to permute the second, third and fourth coefficients, we have

$$\theta^{(1)}_{i00j}(s^{w'w''w'''}) = \theta^{(1)}_{i00j}(s^{w'w''});$$

moreover, conjugation by $w_2w_3w_4$ inverts s. Thus the sum of the values taken by $\theta^{(1)}_{i00j}$ on the conjugates of s is

$$\sum_{w',w''} 6(\theta^{(1)}_{i00j}(s^{w'w''}) + \theta^{(1)}_{i00j}(s^{w'w''})^{-1}),$$

where w' is restricted to run over $\{1, w_1, w_2, w_3\}$. For each of the 16 possibilities for the pair (w', w''), we consider the sum of the values over the different classes of type h_{31}. There are $\frac{1}{12}(q^2-8q+10+3d+2y)$ such classes; the conditions which a and b must satisfy mean that we must sum over the square $0 \leq a, b \leq q-2$, subtract the sums over the six lines $a=0$, $b=0$, $a-b=0$, $a+b=0$, $2a+b=0$ and $a+2b=0$ (where all equalities are of course taken modulo $q-1$), and then divide by 12 to allow for the fact that the points (a, b), (b, a), $(-a, -b)$ and $(a, a-b)$ all give the same class (for points (a, b) lying on more than one such line a further compensation is really required, but this is too small to affect the nonnegligible part of the contribution).

If $(w', w'') = (w_1, w_{1-4})$ then $\theta^{(1)}_{i00j}(s^{w'w''}) = 1$; thus the sum over the different classes here is $q^2 - 8q + \cdots$ (note that we require only the terms of positive degree in q here, since any remaining terms are small enough to be ignored). If $(w', w'') = (w_1, w_{1-2})$ we have $\theta^{(1)}_{i00j}(s^{w'w''}) = {\zeta_1}^{ia}$; the sum of these values over the square and five of the lines is zero, but the sum over the line $a = 0$ is $q - 1$, so that the sum over the classes is $-q + \cdots$. If $(w', w'') = (w_3, w_{1-4})$ or (w_3, w_{1-2}) we have $\theta^{(1)}_{i00j}(s^{w'w''}) = {\zeta_1}^{ja}$ or ${\zeta_1}^{(i+j)a}$, respectively, which behave entirely similarly. Likewise if $(w', w'') = (w_1, w_{1-3})$, (w_2, w_{1-4}) or (w_2, w_{1-3}) we have $\theta^{(1)}_{i00j}(s^{w'w''}) = {\zeta_1}^{ib}$, ${\zeta_1}^{jb}$ or ${\zeta_1}^{(i+j)b}$, respectively, while if $(w', w'') = (w_1, 1)$, $(1, w_{1-4})$ or $(1, 1)$ we have $\theta^{(1)}_{i00j}(s^{w'w''}) = {\zeta_1}^{-i(a+b)}$, ${\zeta_1}^{j(a+b)}$ or ${\zeta_1}^{(i+j)(a+b)}$, respectively; thus there are nine pairs (w', w'') giving a sum $-q + \cdots$. If however $(w', w'') = (w_2, w_{1-2})$ then $\theta^{(1)}_{i00j}(s^{w'w''}) = {\zeta_1}^{-ia+jb}$; the sum of these values over the square and all six of the lines is zero. The remaining five pairs behave similarly. We therefore obtain a total sum of $q^2 - 17q + \cdots$. Multiplying by the Green function value and dividing by the order of the centralizer in H_m gives $q + f - 14$ as the nonnegligible part of the contribution from the semisimple classes of type h_{31}.

Finally, we treat the semisimple classes of type h_{66}, which contain elements $({\xi_1}^{a+b}, {\xi_1}^{a+c}, {\xi_1}^{b+c}, 1; {\xi_1}^{a+b+c})$ with both ${\xi_1}^a, {\xi_1}^b, {\xi_1}^c, {\xi_1}^{a\pm b}, {\xi_1}^{a\pm c}, {\xi_1}^{b\pm c} \neq 1$ and ${\xi_1}^{a+b+c} \neq 1, {\xi_1}^{-a}, {\xi_1}^{-b}, {\xi_1}^{-c}$. Each such element has 576 distinct conjugates in $T_{(1)}$, but none of these is of the form $(1, *, *, *; 1)$. In this case, therefore, summing over all classes of such elements produces cancellation in all instances; accordingly, the nonnegligible part of the contribution from these classes is 0. This completes the consideration of the pair $(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00j})$.

We turn to the pair $(\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0j})$. The semisimple classes of type h_7 are as above; the elements lying in $T_{(2)}$ are $(1, 1, {\xi_1}^a, {\xi_1}^a; {\xi_1}^a)$, $({\xi_1}^a, {\xi_1}^a, 1, 1; {\xi_1}^a)$ and $({\xi_1}^a, {\xi_1}^{-a}, 1, 1; 1)$, together with their inverses. None is of the form $(1, *, *, *; 1)$, so that the values taken by $\theta^{(2)}_{i0j}$ are all roots of unity other than 1. Summing over the classes therefore produces cancellation as above, and so the nonnegligible part of the contribution from both the semisimple classes and the regular classes is 0.

The semisimple classes of type h_8 contain elements $(1, {\xi_2}^a, {\xi_2}^{-a}, 1; 1)$ with ${\xi_2}^{2a} \neq 1$; the elements lying in $T_{(2)}$ are $(1, 1, {\xi_2}^a, {\xi_2}^{-a}; 1)$ and its inverse. Since each of these is sent to 1 by $\theta^{(2)}_{i0j}$, the contributions here are the same as in the unipotent case: the nonnegligible parts are 1 from the semisimple classes and 3 from the regular classes.

The semisimple classes of type h_9 are again as above. The element given has 8 conjugates lying in $T_{(2)}$, of which only $(1, {\xi_1}^{-2a}, {\xi_1}^a, {\xi_1}^a; 1)$ and its inverse are of the form $(1, *, *, *; 1)$; thus the nonnegligible part of the contribution from the classes of regular elements of H_m with semisimple part of type h_9 is one quarter of that in the unipotent case, and therefore is 1.

The semisimple classes of type h_{33} contain elements $({\xi_1}^a, {\xi_3}^a, {\xi_3}^{qa}, 1; {\xi_1}^a)$ with ${\xi_1}^a, {\xi_2}^a \neq 1$; there are $\frac{1}{4}(q^2 - 2q + d)$ such classes. As with type h_{31} above, simply counting the number of roots of unity equal to 1 is insufficient here. If we let s be the element given, then to cover all classes we must sum from $a = 0$ to $q^2 - 2$, subtract the sums of the terms in which a is a multiple of $q + 1$ or $q - 1$, and divide by 4 (again, an adjustment should really be made for terms with $a = 0$ or $a = \frac{1}{2}(q^2 - 1)$, but this is too small to affect the final nonnegligible parts). There are 16 conjugates of s in $T_{(2)}$. Of these, 4 are of the form $(1, *, *, *; 1)$, so that the value taken by $\theta^{(2)}_{i0j}$ is 1; summing over the classes here gives $q^2 - 2q + d$. For the remaining 12 conjugates, the value taken by $\theta^{(2)}_{i0j}$ is $\zeta_1{}^{\pm ia}$, $\zeta_1{}^{\pm ja}$ or $\zeta_1{}^{\pm(i+j)a}$; summing from $a = 0$ to $q^2 - 2$ gives zero, as does the sum of the terms with a a multiple of $q+1$, but the terms with a a multiple of $q-1$ sum to $q+1$. Thus the total sum is $q^2 - 5q + \cdots$; multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a contribution with nonnegligible part $q + f - 4$.

Finally, we treat the semisimple classes of type h_{70}, which contain elements $(\xi_1{}^{2b-a}, \xi_3{}^a, \xi_3{}^{qa}, 1; \xi_1{}^b)$ with $\xi_3{}^{(q\pm1)a}, \xi_1{}^b, \xi_1{}^{a-b}, \xi_1{}^{a-2b} \neq 1$. No conjugate in $T_{(2)}$ is of the form $(1, *, *, *; 1)$, so that summing over all classes of such elements produces cancellation in all instances; accordingly, the nonnegligible part of the contribution from these classes is 0. This completes consideration of the pair $(\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0j})$.

The third and final pair is $(\boldsymbol{T}_{(6)}, \theta^{(6)}_{-(q^2+q+1)i,2i+j})$; there are only two types of class requiring consideration. For the semisimple classes of type h_{35}, the elements are $(\xi_5{}^a, \xi_5{}^{-qa}, \xi_5{}^{(q+1)a}, 1; \xi_5{}^a)$ with $\xi_5{}^{3a} \neq 1$; all 6 conjugates lying in $T_{(6)}$ are sent to 1 by $\theta^{(6)}_{-(q^2+q+1)i,2i+j}$, so the contribution is the same as in the unipotent case, with nonnegligible part $q + f + 1$. For the semisimple classes of type h_{74}, the elements are $(\xi_7{}^{(q+1)a}, \xi_7{}^{(q^2+q)a}, \xi_7{}^{(q^2+1)a}, 1; \xi_7{}^{(q^2+q+1)a})$ with $\xi_5{}^a, \xi_1{}^a \neq 1$; here none of the conjugates is sent to 1 by $\theta^{(6)}_{-(q^2+q+1)i,2i+j}$, so the contribution has nonnegligible part 0.

Table 11 lists these nonnegligible parts of contributions to scalar products with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{31,[(i,j)]}$, where for types h_7 and h_8 the first row relates to the semisimple classes and the second row to the classes of regular elements of H_m. It follows that the scalar product of ${1_{H_m}}^G$ with each of the three characters $R_{\boldsymbol{T},\theta}$ treated here is $q + f + 1$. Taking linear combinations given by the character table of the Weyl group of type A_2, we see that the only constituent of ${1_{H_m}}^G$ lying in the geometric conjugacy class $\kappa_{31,[(i,j)]}$ is the semisimple character, whose multiplicity is $q + f + 1$; we shall call this character $\chi^1_{\kappa_{31,[(i,j)]}}$.

3.3.2. *Geometric conjugacy classes of types κ_{32}, κ_{33}, κ_{34}, κ_{35} and κ_{36}.* The geometric conjugacy classes of types κ_{32}, κ_{33}, κ_{34}, κ_{35} and κ_{36} correspond to semisimple classes in G containing elements as follows:

$$\kappa_{32} : (\xi_2{}^{i+j}, \xi_2{}^i, \xi_2{}^j, 1; \xi_2{}^{i+j}) \qquad \text{with } \xi_2{}^i, \xi_2{}^j, \xi_2{}^{i\pm j}, \xi_2{}^{2i+j}, \xi_2{}^{i+2j} \neq 1;$$

$$\kappa_{33} : (\xi_1{}^i, \xi_3{}^i, \xi_3{}^{qi}, 1; \xi_1{}^i) \qquad \text{with } \xi_1{}^i, \xi_2{}^i \neq 1;$$

	$\boldsymbol{T}_{(1)}$	$\boldsymbol{T}_{(2)}$	$\boldsymbol{T}_{(6)}$
h_7	3 9	0 0	– –
h_8	– –	1 3	– –
h_9	3	1	–
h_{31}	$q+f-14$	–	–
h_{33}	–	$q+f-4$	–
h_{35}	–	–	$q+f+1$
h_{66}	0	–	–
h_{70}	–	0	–
h_{74}	–	–	0
	$q+f+1$	$q+f+1$	$q+f+1$

Table 11. Contributions to scalar products $(1_{H_m}{}^G, R_{\boldsymbol{T},\theta})_G$ for $R_{\boldsymbol{T},\theta}$ in $\kappa_{31,[(i,j)]}$.

$\kappa_{34} : ({\xi_2}^i, {\xi_3}^{-i}, {\xi_3}^{qi}, 1; {\xi_2}^i)$ with ${\xi_1}^i, {\xi_2}^i \neq 1$;
$\kappa_{35} : ({\xi_5}^i, {\xi_5}^{-qi}, {\xi_5}^{(q+1)i}, 1; {\xi_5}^i)$ with ${\xi_5}^{3i} \neq 1$;
$\kappa_{36} : ({\xi_6}^i, {\xi_6}^{qi}, {\xi_6}^{-(q-1)i}, 1; {\xi_6}^i)$ with ${\xi_6}^{3i} \neq 1$.

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 7; the number of classes is $\frac{1}{12}(q^2-4q-2+3d+2z)$, $\frac{1}{4}(q^2-2q+d)$, $\frac{1}{4}(q^2-2q+d)$, $\frac{1}{6}(q^2+q+1-y)$ or $\frac{1}{6}(q^2-q+1-z)$, respectively. To parametrize these geometric conjugacy classes, we set

$$S_{32} = \{(i, j) \in {\mathbb{Z}_{q+1}}^2 : i, j, i \pm j, 2i + j, i + 2j \neq 0\},$$
$$S_{33} = S_{34} = \{i \in \mathbb{Z}_{q^2-1} : (q \pm 1)i \neq 0\},$$
$$S_{35} = \{i \in \mathbb{Z}_{q^2+q+1} : 3i \neq 0\},$$
$$S_{36} = \{i \in \mathbb{Z}_{q^2-q+1} : 3i \neq 0\},$$

so that $(i, j) \in S_{32}$ or $i \in S_c$ for $c = 33, \ldots, 36$ corresponds to the element given; define an equivalence relation on the set S_{32} by

$$(i, j) \sim (j, i) \sim (-i, -j) \sim (i + j, -j),$$

and similarly define equivalence relations on S_{33}, S_{34}, S_{35} and S_{36} by

$$i \sim -i \sim qi.$$

Let $\bar{S}_{32}$ denote the set of equivalence classes $[(i, j)]$ of S_{32}, and $\bar{S}_c$ denote the set of equivalence classes $[i]$ of S_c for $c = 33, \ldots, 36$. The geometric conjugacy classes of type κ_c are then in bijective correspondence with the set $\bar{S}_c$ for $c = 32, \ldots, 36$; we shall write $\kappa_{32,[(i,j)]}$ for the class corresponding to $[(i, j)] \in \bar{S}_{32}$, and $\kappa_{c,[i]}$ for the class corresponding to $[i] \in \bar{S}_c$ for $c = 33, \ldots, 36$.

We shall give less detail for these types, as the behavior in each case is very similar to κ_{31}. For each κ_c there are three characters $R_{\boldsymbol{T},\theta}$ in each geometric conjugacy class; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as follows:

$$\begin{aligned}
\kappa_{32,[(i,j)]} &: (\boldsymbol{T}_{(17)}, \theta^{(17)}_{i00j}),\ (\boldsymbol{T}_{(9)}, \theta^{(9)}_{-ij0}),\ (\boldsymbol{T}_{(21)}, \theta^{(21)}_{-(q^2-q+1)i,2i+j});\\
\kappa_{33,[i]} &: (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00i}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0i}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q^2+q+1)i});\\
\kappa_{34,[i]} &: (\boldsymbol{T}_{(10)}, \theta^{(10)}_{00i}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{-ii}),\ (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q^2-q+1)i});\\
\kappa_{35,[i]} &: (\boldsymbol{T}_{(7)}, \theta^{(7)}_{(q-1)i0}),\ (\boldsymbol{T}_{(15)}, \theta^{(15)}_{(q^2-1)i}),\ (\boldsymbol{T}_{(18)}, \theta^{(18)}_{i0});\\
\kappa_{36,[i]} &: (\boldsymbol{T}_{(20)}, \theta^{(20)}_{(q+1)i0}),\ (\boldsymbol{T}_{(13)}, \theta^{(13)}_{(q^2-1)i}),\ (\boldsymbol{T}_{(25)}, \theta^{(25)}_{i0}).
\end{aligned}$$

In each case, we find as with κ_{31} that the scalar product with ${1_{H_m}}^G$ is the same for all three; thus the only constituent of ${1_{H_m}}^G$ lying in each such geometric conjugacy class is the semisimple character, which we call $\chi^1_{\kappa_{32,[(i,j)]}}$ or $\chi^1_{\kappa_{c,[i]}}$ for $c = 33, \ldots, 36$. The multiplicities of these constituents are $q - f - 1$ for $\kappa_{32,[(i,j)]}$; $q + f - 1$ for $\kappa_{33,[i]}$; $q - f + 1$ for $\kappa_{34,[i]}$; $q + f - 1$ for $\kappa_{35,[i]}$; and $q - f + 2$ for $\kappa_{36,[i]}$.

3.3.3. *Geometric conjugacy classes of types κ_9 and κ_{10}.* The geometric conjugacy classes of types κ_9 and κ_{10} correspond to semisimple classes in G containing elements as follows:

$$\begin{aligned}
\kappa_9 &: ({\xi_1}^i, {\xi_1}^{-i}, {\xi_1}^{2i}, 1; {\xi_1}^i) \quad \text{with } {\xi_1}^{2i}, {\xi_1}^{3i} \neq 1,\\
\kappa_{10} &: ({\xi_2}^i, {\xi_2}^{-i}, {\xi_2}^{2i}, 1; {\xi_2}^i) \quad \text{with } {\xi_2}^{2i}, {\xi_2}^{3i} \neq 1.
\end{aligned}$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 6; the number of classes is $\frac{1}{2}(q - 1 - d - y)$ or $\frac{1}{2}(q + 1 - d - z)$, respectively. To parametrize these geometric conjugacy classes, we write $\epsilon = 1$ for κ_9 and $\epsilon = -1$ for κ_{10}, and for $c = 9, 10$ we set

$$S_c = \{i \in \mathbb{Z}_{q-\epsilon} : 2i, 3i \neq 0\},$$

so that $i \in S_c$ corresponds to the element given; we define an equivalence relation on S_c by

$$i \sim -i,$$

and let $\bar{S}_c$ be the set of equivalence classes $[i]$. The geometric conjugacy classes of type κ_c are in bijective correspondence with $\bar{S}_c$; we shall write $\kappa_{c,[i]}$ for the class corresponding to $[i] \in \bar{S}_c$. There are six distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{c,[i]}$; by a temporary abuse of notation we may say that $\kappa_{9,[i]}$ is the union of $\kappa_{31,[(i,i)]}$ and $\kappa_{33,[(q+1)i]}$, while $\kappa_{10,[i]}$ is the union of $\kappa_{32,[(i,i)]}$ and $\kappa_{34,[(q-1)i]}$ (by which we mean that the characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{9,[i]}$ are obtained from those lying in $\kappa_{31,[(i,j)]}$ and $\kappa_{33,[i]}$ by setting $j=i$ in the former and replacing i by $(q+1)i$ in the latter, and similarly for $\kappa_{10,[i]}$).

We find that the calculations to find nonnegligible parts of contributions proceed almost exactly as for types κ_{31} and κ_{33}, or κ_{32} and κ_{34}; the only differences occur with semisimple classes of types h_9 and h_{31} for κ_9, or h_{10} and h_{32} for κ_{10}. For type h_9, it was found in the treatment of κ_{31} above that only 6 of the 96 conjugates of $({\xi_1}^a, {\xi_1}^{-a}, {\xi_1}^{2a}, 1; {\xi_1}^a)$ were sent to 1 by $\theta^{(1)}_{i00j}$; here we find that setting $j=i$ means that an extra 12 conjugates of the form $({\xi_1}^{-a}, *, *, *; {\xi_1}^a)^{\pm 1}$ are sent to 1, and thus the nonnegligible part of the contribution from these classes increases by 6 from 3 to 9. For type h_{31}, on the other hand, the treatment above divided into a consideration of 16 cases, in 6 of which the sum of the roots of unity concerned was too small to affect matters; here we find that these 6 behave in the same manner as the other 9 where the root of unity is not simply 1, and accordingly the nonnegligible part of the contribution from these classes decreases by 6 from $q+f-14$ to $q+f-20$. Since these two changes cancel each other out (and the same is true for κ_{10} with types h_{10} and h_{32}), the scalar products with ${1_{H_m}}^G$ of the $R_{\boldsymbol{T},\theta}$ are the same as above: for $\kappa_{9,[i]}$ we have

$$\begin{aligned} q+f+1 \quad &\text{for } (\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00i}),\ (\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0i}),\ (\boldsymbol{T}_{(6)}, \theta^{(6)}_{-(q^2+q+1)i,3i}),\\ -q-f+1 \quad &\text{for } (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00(q+1)i}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0(q+1)i}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q^2+q+1)(q+1)i}), \end{aligned}$$

while for $\kappa_{10,[i]}$ we have

$$\begin{aligned} -q+f+1 \quad &\text{for } (\boldsymbol{T}_{(17)}, \theta^{(17)}_{i00i}), (\boldsymbol{T}_{(9)}, \theta^{(9)}_{-ii0}), (\boldsymbol{T}_{(21)}, \theta^{(21)}_{-(q^2-q+1)i,3i}),\\ q-f+1 \quad &\text{for } (\boldsymbol{T}_{(10)}, \theta^{(10)}_{00(q-1)i}), (\boldsymbol{T}_{(5)}, \theta^{(5)}_{-(q-1)i,(q-1)i}), (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q^2-q+1)(q-1)i}). \end{aligned}$$

Taking linear combinations given by the character table of the Weyl group of type A_2A_1, we see that there are two constituents of ${1_{H_m}}^G$ lying in each such geometric conjugacy class. The first constituent, which has multiplicity 1, is the semisimple character. The second constituent, which has multiplicity $q+\epsilon f$, corresponds to the unipotent character of the centralizer $A_1(q).A_2^\epsilon(q).T_1$ whose restrictions to the $A_1(q)$ and $A_2^\epsilon(q)$ factors are the Steinberg and the trivial characters, respectively. We shall call these characters $\chi^{1,1}_{\kappa_{c,[i]}}$ and $\chi^{\mathrm{St},1}_{\kappa_{c,[i]}}$.

3.3.4. *Geometric conjugacy classes of types κ_3 and κ_4.* The geometric conjugacy classes of types κ_3 and κ_4 correspond to semisimple classes in G containing elements as follows:

$$\kappa_3 : ({\xi_1}^i, {\xi_1}^{-i}, {\xi_1}^{-i}, 1; {\xi_1}^i) \quad \text{with } {\xi_1}^i \neq 1 = {\xi_1}^{3i},$$
$$\kappa_4 : ({\xi_2}^i, {\xi_2}^{-i}, {\xi_2}^{-i}, 1; {\xi_2}^i) \quad \text{with } {\xi_2}^i \neq 1 = {\xi_2}^{3i}.$$

If we write $\epsilon = 1$ for κ_3 and $\epsilon = -1$ for κ_4, the centralizer in G is a product of two groups $A_2^\epsilon(q)$ (one factor involving long root groups and the other short), while the number of such classes is $\frac{1}{2}(y-1)$ for κ_3 and $\frac{1}{2}(z-1)$ for κ_4. Since there is at most one such class, we shall simply call it κ_c for $c = 3, 4$, with the understanding that the class does not exist unless q is congruent to ϵ modulo 3. There are nine distinct characters $R_{\boldsymbol{T},\theta}$ lying in κ_c; by a temporary abuse of notation similar to that above we may say that κ_3 is the union of $\kappa_{9,[(q-1)/3]}$ and $\kappa_{35,[(q^2+q+1)/3]}$, while κ_4 is the union of $\kappa_{10,[(q+1)/3]}$ and $\kappa_{36,[(q^2-q+1)/3]}$.

We find that the calculations to find nonnegligible parts of contributions proceed exactly as for types κ_9, κ_{10}, κ_{35} and κ_{36} above. Thus the scalar products with ${1_{H_m}}^G$ of the $R_{\boldsymbol{T},\theta}$ are the same as above: for κ_3, writing q_i^+ for $(q^i-1)/3$ we have

$$\begin{aligned} q+f+1 &\quad \text{for } (\boldsymbol{T}_{(1)}, \theta^{(1)}_{q_1^+00q_1^+}),\ (\boldsymbol{T}_{(2)}, \theta^{(2)}_{q_1^+0q_1^+}),\ (\boldsymbol{T}_{(6)}, \theta^{(6)}_{q_3^+0}),\\ -q-f+1 &\quad \text{for } (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00q_2^+}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0q_2^+}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q+1)q_3^+}),\\ q+f-2 &\quad \text{for } (\boldsymbol{T}_{(7)}, \theta^{(7)}_{q_3^+0}),\ (\boldsymbol{T}_{(15)}, \theta^{(15)}_{(q+1)q_3^+}),\ (\boldsymbol{T}_{(18)}, \theta^{(18)}_{(q^2+q+1)/3,0}), \end{aligned}$$

while for κ_4, writing q_i^- for $(q^i-(-1)^i)/3$ we have

$$\begin{aligned} -q+f+1 &\quad \text{for } (\boldsymbol{T}_{(17)}, \theta^{(17)}_{q_1^-00q_1^-}),\ (\boldsymbol{T}_{(9)}, \theta^{(9)}_{-q_1^-q_1^-0}),\ (\boldsymbol{T}_{(21)}, \theta^{(21)}_{q_3^-0}),\\ q-f+1 &\quad \text{for } (\boldsymbol{T}_{(10)}, \theta^{(10)}_{00q_2^-}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{-q_2^-q_2^-)}),\ (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q-1)q_3^-}),\\ -q+f-2 &\quad \text{for } (\boldsymbol{T}_{(20)}, \theta^{(20)}_{q_3^-0}),\ (\boldsymbol{T}_{(13)}, \theta^{(13)}_{(q-1)q_3^-}),\ (\boldsymbol{T}_{(25)}, \theta^{(25)}_{(q^2-q+1)/3,0}). \end{aligned}$$

Taking linear combinations given by the character table of the Weyl group of type A_2A_2, we see that there are again two constituents of ${1_{H_m}}^G$ lying in each such geometric conjugacy class. To describe them, we shall say that the $A_2^\epsilon(q)$ factors of the centralizer involving long and short root groups are the long and short factors, respectively. The first constituent, which has multiplicity 1, corresponds to the unipotent character of the centralizer whose restrictions to the long and short factors are ρ and the trivial character, respectively (where ρ is the third unipotent character of the long factor after the trivial and Steinberg characters). The second constituent, which has multiplicity $q+\epsilon(f-1)$, corresponds to the unipotent character of the centralizer whose restrictions to the long and short factors are the Steinberg and the trivial characters, respectively. We shall call these characters $\chi_{\kappa_c}^{\rho,1}$ and $\chi_{\kappa_c}^{\mathrm{St},1}$.

3.3.5. *Geometric conjugacy classes of type κ_7.* The geometric conjugacy classes of type κ_7 correspond to semisimple classes in G containing elements

$$(1, {\xi_1}^i, {\xi_1}^{-i}, 1; 1) \quad \text{with } {\xi_1}^{2i} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 5; the number of classes is $\frac{1}{2}(q-2-d)$. To parametrize these geometric conjugacy classes, we set

$$S_7 = \{i \in \mathbb{Z}_{q-1} : 2i \neq 0\},$$

so that $i \in S_7$ corresponds to the element given; we define an equivalence relation on S_7 by

$$i \sim -i,$$

and let $\bar{S}_7$ be the set of equivalence classes $[i]$. The geometric conjugacy classes of type κ_7 are in bijective correspondence with $\bar{S}_7$; we shall write $\kappa_{7,[i]}$ for the class corresponding to $[i] \in \bar{S}_7$.

There are ten distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as

$$(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i000}), \quad (\boldsymbol{T}_{(2)}, \theta^{(2)}_{i00}), \quad (\boldsymbol{T}_{(3)}, \theta^{(3)}_{i00}), \quad (\boldsymbol{T}_{(4)}, \theta^{(4)}_{i00}),$$
$$(\boldsymbol{T}_{(5)}, \theta^{(5)}_{-(q+1)i,(q+1)i}), \quad (\boldsymbol{T}_{(6)}, \theta^{(6)}_{0i}), \quad (\boldsymbol{T}_{(8)}, \theta^{(8)}_{i0}),$$
$$(\boldsymbol{T}_{(10)}, \theta^{(10)}_{00(q+1)i}), \quad (\boldsymbol{T}_{(11)}, \theta^{(11)}_{q(q^2+1)(q+1)i/2,i}), \quad (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q^3+1)i}).$$

We shall give rather less detail here than for κ_{31}, both to save space and because the calculations are similar to those that have already been seen; we shall therefore not bother to repeat from [15] the definitions of the characters θ here.

To begin with, in all instances involving semisimple classes of types other than $h_{31}, \ldots, h_{36}$ the nonnegligible part of the contribution is a constant; as was seen in the treatment of κ_{31}, the value concerned is then determined by the proportion of conjugates sent to 1. As an example, take the pair $(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i000})$ and semisimple classes of type h_7; as has been stated, the elements are $(1, {\xi_1}^a, {\xi_1}^{-a}, 1; 1)$ with ${\xi_1}^{2a} \neq 1$. Of the 24 conjugates of such an element in $T_{(1)}$, there are 12 of the form $(1, *, *, *; *)$, which are therefore sent to 1 by $\theta^{(1)}_{i000}$. Thus the nonnegligible part of the contribution from classes with semisimple part of type h_7 is one half of the value in the unipotent case, and is therefore 6 for the semisimple classes and 18 for the regular classes. All other such cases are similar; we thus need say no more about these contributions.

We must therefore consider the semisimple classes of types $h_{31}, \ldots, h_{36}$. For some of the instances where the classes of such a type h_ℓ meet one of the $T_{(n)}$ involved here, the character θ is such that all conjugates are taken to 1; as a result the contribution is the same as in the unipotent case. This occurs when the pair

(ℓ, n) is (32, 10), (34, 5), (35, 6) or (36, 14); the pairs left to be considered are (31, 1), (31, 3), (33, 2) and (33, 5).

We begin with $(\ell, n) = (31, 1)$. When dealing with this instance in the treatment of κ_{31}, we divided into 16 cases. Of these, we find that 4 are such that the conjugates are sent to 1 by $\theta^{(1)}_{i000}$, giving a sum over the different classes of $4q^2 - 32q + \cdots$; the remaining 12 each give a sum $-q + \cdots$. The total sum is therefore $4q^2 - 44q + \cdots$; multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a contribution with nonnegligible part $4q + 4f - 32$.

The instance where $(\ell, n) = (31, 3)$ is a little different. As before, we let $s = ({\xi_1}^{a+b}, {\xi_1}^{a}, {\xi_1}^{b}, 1; {\xi_1}^{a+b})$; there are then 48 conjugates of s in $T_{(3)}$. None of these is of the form $(1, *, *, *; *)$ to be sent to 1 by $\theta^{(3)}_{i00}$; instead, the sum over the conjugates of the values taken by $\theta^{(3)}_{i00}$ is

$$8({\zeta_1}^{ia} + {\zeta_1}^{-ia} + {\zeta_1}^{ib} + {\zeta_1}^{-ib} + {\zeta_1}^{i(a+b)} + {\zeta_1}^{-i(a+b)}).$$

As before, we sum over the square $0 \le a, b \le q - 2$, subtract the sums over the six lines $a = 0$, $b = 0$, $a - b = 0$, $a + b = 0$, $2a + b = 0$ and $a + 2b = 0$ (where all equalities are once more taken modulo $q - 1$), and then divide by 12. For each of the six powers of ζ_1, the sums over the square and five of the lines are zero, but the sum over the other line is $q - 1$. Thus the total sum is $-4q + \cdots$; proceeding as usual then gives a contribution with nonnegligible part 4.

Next we take $(\ell, n) = (33, 2)$. Of the 16 conjugates lying in $T_{(2)}$ mentioned in the treatment of κ_{31}, we find that 8 are sent to 1 by $\theta^{(2)}_{i00}$, giving a sum of $2q^2 - 4q + 2d$; the remaining 8 give a sum $-2q - 2$. The total sum is therefore $2q^2 - 6q + \cdots$; multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a contribution with nonnegligible part $2q + 2f - 4$.

Lastly we turn to $(\ell, n) = (33, 5)$. As before, we let $s = ({\xi_1}^{a}, {\xi_3}^{a}, {\xi_3}^{qa}, 1; {\xi_1}^{a})$; there are 8 conjugates of s in $T_{(5)}$. None of these is sent to 1 by $\theta^{(5)}_{-(q+1)i,(q+1)i}$; instead, the sum over the conjugates of the values taken by $\theta^{(5)}_{-(q+1)i,(q+1)i}$ is

$$4({\zeta_1}^{ia} + {\zeta_1}^{-ia}).$$

As before, we then sum from $a = 0$ to $q^2 - 2$, subtract the sums of the terms in which a is a multiple of $q + 1$ or $q - 1$, and then divide by 4; this gives a total sum of $-2q + \cdots$, and proceeding as usual then gives a contribution with nonnegligible part 2.

Table 12 lists all nonnegligible parts of contributions to scalar products with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$; again, for types h_7 and h_8 the first row relates to the semisimple classes and the second to classes of regular elements of H_m. We have therefore found the scalar products of ${1_{H_m}}^G$ with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$. On taking linear combinations given by the character table of the Weyl group C_3, we obtain nonzero scalar products with four of the resulting characters. Two of these are

	$\boldsymbol{T}_{(1)}$	$\boldsymbol{T}_{(2)}$	$\boldsymbol{T}_{(3)}$	$\boldsymbol{T}_{(4)}$	$\boldsymbol{T}_{(5)}$	$\boldsymbol{T}_{(6)}$	$\boldsymbol{T}_{(8)}$	$\boldsymbol{T}_{(10)}$	$\boldsymbol{T}_{(11)}$	$\boldsymbol{T}_{(14)}$
h_7	6 18	-1 1	-2 -2	0 0	1 1	– –	– –	– –	– –	– –
h_8	– –	1 3	– –	-2 2	-1 -1	– –	– –	6 6	– –	– –
h_9	12	2	0	–	0	–	–	–	–	–
h_{10}	–	–	–	–	2	–	–	12	–	–
h_{31}	$4q+4f-32$	–	4	–	–	–	–	–	–	–
h_{32}	–	–	–	–	–	–	–	$4q-4f-20$	–	–
h_{33}	–	$2q+2f-4$	–	–	2	–	–	–	–	–
h_{34}	–	–	–	–	$2q-2f-2$	–	–	–	–	–
h_{35}	–	–	–	–	–	$q+f+1$	–	–	–	–
h_{36}	–	–	–	–	–	–	–	–	–	$q-f+1$
h_{66}	f	–	0	–	–	–	–	–	–	–
h_{67}	–	–	–	–	–	–	–	f	–	–
h_{68}	–	–	f	0	–	–	–	–	–	–
h_{69}	–	–	–	f	–	–	–	0	–	–
h_{70}	–	f	–	–	0	–	–	–	–	–
h_{71}	–	–	–	–	f	–	–	–	–	–
h_{72}	–	–	–	–	–	–	f	–	0	–
h_{73}	–	–	–	–	–	–	–	–	f	–
h_{74}	–	–	–	–	–	f	–	–	–	–
h_{75}	–	–	–	–	–	–	–	–	–	f
	$4q+5f+4$	$2q+3f+2$	f	f	$2q-f+2$	$q+2f+1$	f	$4q-3f+4$	f	$q+1$

Table 12. Contributions to scalar products $({1_{H_m}}^G, R_{\boldsymbol{T},\theta})_G$ for $R_{\boldsymbol{T},\theta}$ in $\kappa_{7,[i]}$.

irreducible characters: they correspond to the trivial character and the unipotent character $\chi_{1,2}$, and the multiplicities are $q+f+1$ and $q+1$, respectively. (Here for a pair of partitions (α, β) with $|\alpha|+|\beta|=3$ we write $\chi_{\alpha,\beta}$ for the corresponding unipotent character of $C_3(q)$ lying in the principal series, as in [4, Section 13.8].) The remaining two lie in a family of size four, and the scalar products are both f; an analysis using the nonabelian Fourier transform matrix which is entirely similar to that employed with the unipotent characters $\chi_{\phi_{4,1}}$ and $\chi_{\phi''_{2,4}}$ above shows that there are two other constituents, each with multiplicity f, corresponding to the unipotent characters $\chi_{2,1}$ and $\chi_{-,3}$. We shall call these four characters $\chi_{\kappa_{7,[i]}}^{1}$, $\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}$, $\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}$ and $\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}$.

3.3.6. *Geometric conjugacy classes of type κ_8.* The geometric conjugacy classes of type κ_8 correspond to semisimple classes in G containing elements

$$(1, {\xi_2}^i, {\xi_2}^{-i}, 1; 1) \quad \text{with } {\xi_2}^{2i} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 5; the number of classes is $\frac{1}{2}(q-d)$. To parametrize these geometric conjugacy classes, we set

$$S_8 = \{i \in \mathbb{Z}_{q+1} : 2i \neq 0\},$$

so that $i \in S_8$ corresponds to the element given; we define an equivalence relation on S_8 by

$$i \sim -i,$$

and let $\bar{S}_8$ be the set of equivalence classes $[i]$. The geometric conjugacy classes of type κ_8 are in bijective correspondence with $\bar{S}_8$; we shall write $\kappa_{8,[i]}$ for the class corresponding to $[i] \in \bar{S}_8$.

There are ten distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{8,i}$; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as

$$(\boldsymbol{T}_{(17)}, \theta^{(17)}_{i000}),\ (\boldsymbol{T}_{(9)}, \theta^{(9)}_{i00}),\ (\boldsymbol{T}_{(10)}, \theta^{(10)}_{i00}),\ (\boldsymbol{T}_{(4)}, \theta^{(4)}_{0i0}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0(q-1)i}),\ (\boldsymbol{T}_{(21)}, \theta^{(21)}_{0i}),$$
$$(\boldsymbol{T}_{(19)}, \theta^{(19)}_{i0}),\ (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00(q-1)i}),\ (\boldsymbol{T}_{(11)}, \theta^{(11)}_{q(q^2+1)(q-1)i/2,i}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q^3-1)i}).$$

The working is very similar to that of κ_7; we again find that there are four constituents of ${1_{H_m}}^G$ in each such geometric conjugacy class. They correspond to the trivial character and the unipotent characters $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{-,3}$ of the centralizer, and the multiplicities are $q+f-1$, $q-1$, f and f, respectively. We shall call these four characters $\chi^{1}_{\kappa_{8,[i]}}$, $\chi^{\chi_{1,2}}_{\kappa_{8,[i]}}$, $\chi^{\chi_{2,1}}_{\kappa_{8,[i]}}$ and $\chi^{\chi_{-,3}}_{\kappa_{8,[i]}}$.

3.3.7. *The geometric conjugacy class of type κ_1.* The geometric conjugacy class of type κ_1 occurs only in odd characteristic, when it corresponds to the semisimple class in G containing the involution

$$(1, -1, -1, 1; 1).$$

The centralizer in G is the product of groups $C_3(q)$ and $A_1(q)$. Since there is at most one such class, we shall simply call it κ_1, with the understanding that the class does not exist unless q is odd. There are twenty distinct characters $R_{\boldsymbol{T},\theta}$ lying in κ_1; by a temporary abuse of notation similar to those employed previously we may say that κ_1 is the union of $\kappa_{7,[(q-1)/2]}$ and $\kappa_{8,[(q+1)/2]}$.

As before, the calculations to find nonnegligible parts of contributions proceed exactly as for types κ_7 and κ_8. We find that there are six constituents of ${1_{H_m}}^G$. In two cases, the restriction to the $A_1(q)$ factor of the corresponding unipotent character of the centralizer is the trivial character; the restrictions to the $C_3(q)$ factor are the trivial character and the unipotent character $\chi_{1,2}$, and both multiplicities are 1. In the other four cases, the restriction to the $A_1(q)$ factor is the Steinberg character; the restrictions to the $C_3(q)$ factor are the trivial character and the unipotent characters $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{-,3}$, and the multiplicities are $q+f$, q, f and f, respectively. We shall call these six characters $\chi_{\kappa_1}^{1,1}$, $\chi_{\kappa_1}^{\chi_{1,2},1}$, $\chi_{\kappa_1}^{1,\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}}$ and $\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}}$.

3.4. ***The complete decomposition of*** $\mathbf{1_{H_m}}^{\mathbf{G}}$**.** If we now add together the degrees of the constituents of $1_{H_m}{}^G$ found so far, taken with multiplicity, we obtain

$$\begin{array}{ll} q^{12}(q^8+q^4+1)(q^4+1) & \text{if } m=1,\\ q^{12}(q^{12}-1) & \text{if } m=2,\\ q^{12}(q^8-1)(q^4-1) & \text{if } m=3, \end{array}$$

which in each case is equal to $|G:H_m|$. We have therefore proved the following.

Proposition 3.1. *If $G=F_4(q)$ and $H_m={}^mD_4(q)$, the decomposition of $1_{H_m}{}^G$ into irreducible characters is*

$$\begin{aligned} &\chi_{\phi_{1,0}}+\chi_{\phi_{9,2}}+(q+f)\chi_{\phi''_{8,3}}+q\chi_{\phi''_{8,9}}+f\chi_{\phi_{4,1}}+f\chi_{\phi''_{2,4}}\\ &\quad+\left\{\begin{array}{ll} \chi_{\phi_{12,4}}+3\chi_{\phi''_{9,6}}+3\chi_{\phi''_{1,12}}+2\chi_{\phi''_{6,6}}+\chi_{F_4{}^{\mathrm{II}}[1]} & \textit{if } m=1\\ \chi_{\phi''_{4,7}}+\chi_{\phi_{16,5}}+\chi_{B_2,\epsilon'}+\chi_{F_4[-1]} & \textit{if } m=2\\ \chi_{\phi'_{6,6}}+\chi_{F_4[\theta]}+\chi_{F_4[\theta^2]} & \textit{if } m=3 \end{array}\right\}\\ &\qquad+\chi_{\kappa_1}^{1,1}+\chi_{\kappa_1}^{\chi_{1,2},1}+(q+f)\chi_{\kappa_1}^{1,\mathrm{St}}+q\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}}+f\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}}+f\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}}\\ &\qquad+(q+f-1)\chi_{\kappa_3}^{\mathrm{St},1}+\chi_{\kappa_3}^{\rho,1}+(q-f+1)\chi_{\kappa_4}^{\mathrm{St},1}+\chi_{\kappa_4}^{\rho,1}\\ &\qquad\quad+\sum_{[i]\in\bar{S}_7}((q+f+1)\chi_{\kappa_{7,[i]}}^{1}+(q+1)\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}+f\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}+f\chi_{\kappa_{7,[i]}}^{\chi_{-,3}})\\ &\qquad\quad+\sum_{[i]\in\bar{S}_8}((q+f-1)\chi_{\kappa_{8,[i]}}^{1}+(q-1)\chi_{\kappa_{8,[i]}}^{\chi_{1,2}}+f\chi_{\kappa_{8,[i]}}^{\chi_{2,1}}+f\chi_{\kappa_{8,[i]}}^{\chi_{-,3}})\\ &\qquad\quad+\sum_{[i]\in\bar{S}_9}(\chi_{\kappa_{9,[i]}}^{1,1}+(q+f)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1})+\sum_{[i]\in\bar{S}_{10}}(\chi_{\kappa_{10,[i]}}^{1,1}+(q-f)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1})\\ &\qquad\quad+\sum_{[(i,j)]\in\bar{S}_{31}}(q+f+1)\chi_{\kappa_{31,[(i,j)]}}^{1}+\sum_{[(i,j)]\in\bar{S}_{32}}(q-f-1)\chi_{\kappa_{32,[(i,j)]}}^{1}\\ &\qquad\quad+\sum_{[i]\in\bar{S}_{33}}(q+f-1)\chi_{\kappa_{33,[i]}}^{1}+\sum_{[i]\in\bar{S}_{34}}(q-f+1)\chi_{\kappa_{34,[i]}}^{1}\\ &\qquad\quad+\sum_{[i]\in\bar{S}_{35}}(q+f-2)\chi_{\kappa_{35,[i]}}^{1}+\sum_{[i]\in\bar{S}_{36}}(q-f+2)\chi_{\kappa_{36,[i]}}^{1}. \end{aligned}$$

It is now straightforward to calculate the ranks of the three actions as the sums of the squares of the multiplicities of the constituents; we obtain the following.

Corollary 3.2. *The rank of the action of $G = F_4(q)$ on cosets of $H_m = {}^mD_4(q)$ is*

$$\begin{array}{ll} q^4+q^3+9q^2+17q+24 & \text{if } m=1,\\ q^4+q^3+q^2+q+4 & \text{if } m=2,\\ q^4+q^3-q+3 & \text{if } m=3. \end{array}$$

Indeed, this is confirmed by a separate calculation of $(1_{H_m}{}^G, 1_{H_m}{}^G)_G$, which does not require Deligne–Lusztig theory. If H is any subgroup of G, then, for $g \in G$,

$$1_H{}^G(g) = \sum_{[h]\subseteq[g]} \frac{|C_G(h)|}{|C_H(h)|},$$

where the sum is over the H-classes $[h]$ lying in the G-class $[g]$; thus

$$(1_H{}^G, 1_H{}^G)_G = \sum_{[g]\subset G} \frac{1_H{}^G(g)^2}{|C_G(g)|} = \sum_{[g]\subset G} |C_G(g)| \Bigl(\sum_{[h]\subseteq[g]} \frac{1}{|C_H(h)|} \Bigr)^2.$$

Knowledge of the fusion of classes from H into G enables this to be calculated; applying this to each H_m gives the values above.

Before concluding this section we note that the three permutation characters $1_{H_1}{}^G$, $1_{H_2}{}^G$ and $1_{H_3}{}^G$ are given by a formula which, outside a single family of unipotent characters, is linear in the parameter f. As the values taken by f are 3, 1 and 0, respectively, another way of saying this is that if we define the generalized character

$$\psi(G; H) = 1_{H_1}{}^G - 3.1_{H_2}{}^G + 2.1_{H_3}{}^G,$$

then the coefficient in $\psi(G; H)$ of any irreducible character lying outside this family is zero.

The reason for this may be traced back to the contributions to scalar products $(1_{H_m}{}^G, R_{\boldsymbol{T},\theta})$ from regular semisimple classes of types $h_{76}, \dots, h_{100}$ (these were the only contributions whose nonnegligible parts could not be expressed as linear polynomials in f): for each such type h_ℓ, the nonnegligible part was nonzero only when $\theta = 1$, so that $R_{\boldsymbol{T},\theta} = R_{(n)}$ for some n, and only for one value of m, when its value was 6, 2 or 3 according as $m = 1$, 2 or 3. From this we see that for all n we have $(\psi(G; H), R_{(n)})_G = 6(-1)^{m-1}$ where the regular semisimple elements in the torus $T_{(n)}$ lie in H_m; it follows that $\psi(G; H) = 6R_{\phi''_{1,12}}$, where $R_{\phi''_{1,12}}$ is an almost character of degree q^{12}, after which applying the appropriate nonabelian Fourier transform matrix produces the observed linear combination of irreducible unipotent characters.

If we successively remove long simple roots to reduce from $\boldsymbol{G}$ to $\boldsymbol{C}$ and then to $\boldsymbol{A}$, in each case replacing $\boldsymbol{H}$ by its intersection with the reduced group (first a group $A_1{}^3$, then a 2-dimensional torus), the behavior is very similar. The groups $(H \cap C)_m$ are $A_1(q)^3$, $A_1(q^2)A_1(q)$ and $A_1(q^3)$, respectively, while the groups $(H \cap A)_m$ are tori of order $(q-1)^2$, q^2-1 and q^2+q+1, respectively. Using the notation of [4, Section 13.8], we find that $\psi(C;\, H\cap C)=6R_{\phi_{111,-}}$, where $R_{\phi_{111,-}}$ is an almost character of degree q^6; the family here is of size 4, and applying the appropriate nonabelian Fourier transform matrix gives

$$6R_{\phi_{111,-}} = 3\chi_{\phi_{111,-}} + 3\chi_{\phi_{1,11}} - 3\chi_{\phi_{-,21}} - 3\chi_{B_2,\epsilon}.$$

Likewise $\psi(A;\, H\cap A)=6R_{\phi_{111}}$, where $R_{\phi_{111}}$ is an almost character of degree q^3; this time the family is of size 1, so $6R_{\phi_{111}}=6\chi_{\phi_{111}}$. (In fact in this last case the unipotent character $\chi_{\phi_{111}}$ is the Steinberg character St; indeed it follows from [4, Proposition 7.5.4, Corollary 7.6.5] that $\psi(A;\, H\cap A)=6\,\mathrm{St}$.)

4. Extensions of H_m by graph automorphisms

In this section we shall decompose the permutation characters ${1_{H_m.\Gamma}}^G$, where Γ is a nontrivial group of graph automorphisms of H_m. Recall that the cases to be considered are as follows: $H_m.2=H_m\langle\tau_2\rangle$ for $m=1,2$; $H_m.3=H_m\langle\tau_3\rangle$ for $m=1,3$; and $H_1.S_3=H_1\langle\tau_2,\tau_3\rangle$. Note that each constituent of ${1_{H_m.\Gamma}}^G$ is also one of ${1_{H_m}}^G$, so we need only consider the types of geometric conjugacy class treated in Section 3.

For convenience, writing $r=|\Gamma|$ we shall define an integer q_r which is close to $\frac{q}{r}$, and express multiplicities in ${1_{H_m.\Gamma}}^G$ in terms of q_r. Although the details of the calculations to follow will depend upon whether or not the characteristic p divides r, the use of the notation q_r means that in each case it will still be possible to give a single expression for the decomposition of ${1_{H_m.\Gamma}}^G$ which is valid for all characteristics.

In contrast to the results obtained in Section 3, we shall see that the multiplicity of a constituent may depend upon the particular geometric conjugacy class in which it lies, rather than simply being determined by the type of the geometric conjugacy class. We shall therefore require some further notation: we set

$$\epsilon^2_{i,j} = \begin{cases} 1 & \text{if } 2\mid(q-1) \text{ and either } 2\nmid i \text{ or } 2\nmid j \text{ (or both)},\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^2_{i} = \epsilon^2_{i,i} = \begin{cases} 1 & \text{if } 2\mid(q-1) \text{ and } 2\nmid i,\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^{3,+}_{i} = \begin{cases} 1 & \text{if } 3\mid(q-1) \text{ and } 3\nmid i,\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^{3,-}_{i} = \begin{cases} 1 & \text{if } 3\mid(q+1) \text{ and } 3\nmid i,\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^{4} = \begin{cases} 1 & \text{if } 4\mid(q-1),\\ 0 & \text{otherwise.}\end{cases}$$

We shall again use the method described in Section 2 to determine scalar products $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})$ as sums of contributions from types of H_m-class. For those classes lying in H_m the value is that obtained in Section 3, multiplied by a factor of $|H_m|/|H_m\langle\tau_r\rangle| = 1/r$; we therefore consider types of H_m-class lying in $H_m\tau_r$. Much as in Section 3, we begin in Section 4.1 by determining the relevant types of H_m-class in $H_m\tau_r$; we then decompose the permutation characters ${1_{H_m.\Gamma}}^G$, firstly in Section 4.2 where $\Gamma = \langle\tau_2\rangle$ and $m \in \{1, 2\}$, and secondly in Section 4.3 where $\Gamma = \langle\tau_3\rangle$ and $m \in \{1, 3\}$; finally in Section 4.4 we combine these results to decompose the permutation character ${1_{H_m.\Gamma}}^G$ where $\Gamma = \langle\tau_2, \tau_3\rangle$ and $m = 1$.

4.1. ***Relevant types of H_m-class in $H_m\tau_r$ for $r \in \{2, 3\}$.*** Using the information on H_m-classes in $H_m\tau_r$ described in Section 2.1, as in Section 3.1 we treat all types of H_m-class in $H_m\tau_r$ to determine those which are relevant. We find that the number of such types is small; we describe them here.

In each case the elements concerned are simply of the form $s\tau_r$, and the nonnegligible parts of contributions are simply constants; as we shall explain, the H_m-classes are related to those of types $h_{31}, \ldots, h_{36}$ or $h_{66}, \ldots, h_{75}$, and we shall refer to the information given in Tables 7 and 8 of Section 3.1 concerning numbers of H_m-classes and the tori $T_{(n)}$ which meet them. In particular we note the following concerning what is stated there. For all these types we may take s of the form $\big(\lambda, \mu, \frac{\nu^2}{\lambda\mu}, 1; \nu\big)$. For types h_ℓ with $\ell \in \{31, \ldots, 36\}$ (which here occur for both $r = 2$ and $r = 3$), we have $\nu = \lambda$ so that the $\boldsymbol{G}$-centralizer has root system $\tilde{A}_2$, with Weyl group $\langle w_4, w_{+---}\rangle$; there exists $w_{[\ell]} \in \langle w_{2-3}, w_{1-2}w_{3-4}w_{3+4}\rangle$ such that $s^{F_m} = {}^{w_{[\ell]}}s$, so that s lies in the three tori $T_{w_{[\ell]}}$, $T_{w_{[\ell]}w_4}$ and $T_{w_{[\ell]}w_4w_{+---}}$, and up to conjugacy they are listed in that order in Table 7. For types h_ℓ with $\ell \in \{66, \ldots, 75\}$ (which here occur only for $r = 2$), we have $\nu \neq \lambda$ so that the $\boldsymbol{G}$-centralizer has root system $\tilde{A}_1$, with Weyl group $\langle w_4\rangle$; there exists $w_{[\ell]} \in \langle w_{1-2}, w_{2-3}, w_{3-4}w_{3+4}\rangle$ such that $s^{F_m} = {}^{w_{[\ell]}}s$, so that s lies in the two tori $T_{w_{[\ell]}}$ and $T_{w_{[\ell]}w_4}$, and up to conjugacy they are listed in that order in Table 8.

If $p = r$, the expression for ${\tau_r}^{y_r}$ given in Section 1 shows that τ_r lies in the unipotent class $\tilde{A}_1$ or $\tilde{A}_2$ according as $r = 2$ or 3. If the type of the H_m-class containing s is h_ℓ, we shall denote the type of the H_m-class containing $s\tau_r$ by $h_\ell\tau_r$; in each case the number of H_m-classes and the tori containing s are as given in Tables 7 and 8 (if we replace f by 1 in the latter table).

If $p \neq r$, the classes all consist of semisimple elements whose $\boldsymbol{H}$-centralizer is a torus, of dimension $5 - r$. This time the expression for ${\tau_r}^{y_r}$ given in Section 1 shows that $s{\tau_r}^{y_r} = s(-1, -1, -1, -1; -1)$ or $s(1, 1, 1, \omega^2; \omega)$ according as $r = 2$ or 3. The situation here is however somewhat more complicated than in the above paragraph.

First suppose $r = 2$, and s lies in an H_m-class of type h_ℓ for some $\ell \in \{31, \ldots, 36\}$. Here things are simple: we shall denote the type of the H_m-class containing $s\tau_2$

by $h_\ell\tau_2$; the corresponding element $s{\tau_2}^{y_2}$ is of the form $\left(-\lambda, -\mu, -\frac{\lambda}{\mu}, -1; -\lambda\right)$, its $\boldsymbol{G}$-centralizer has root system $\tilde{A}_1$, up to conjugacy it lies in the first and second tori given in Table 7, and the number of H_m-classes is as given there.

Next suppose $r = 2$, and s lies in an H_m-class of type h_ℓ for some $\ell \in \{66, \ldots, 75\}$. We have $(({\tau_2}^{y_2})^{F_m})^{w_{[\ell]}} = {\tau_2}^{y_2}$; thus $((s{\tau_2}^{y_2})^{F_m})^{w_{[\ell]}} = s{\tau_2}^{y_2}$ so that $s{\tau_2}^{y_2} \in T_{w_{[\ell]}}$, and we obtain a type of H_m-class in $H_m\tau_2$ which we may call $h_\ell\tau_2$; however, the number of H_m-classes is one half of that given in Table 8 (replacing f by 1), because conjugation by w_4 multiplies $s{\tau_2}^{y_2}$ by $(1, 1, 1, 1; -1)$ which lies in the torus $T_{w_{[\ell]}}$. To compensate for this we may take $s^* \in \boldsymbol{T}_2$ satisfying

$$((s^*)^{F_m})^{w_{[\ell]}} = s^*(1, 1, 1, 1; -1)$$

(so that s^* depends only on ℓ and not on s), and then $((s^*s{\tau_2}^{y_2})^{F_m})^{w_{[\ell]}w_4} = s^*s{\tau_2}^{y_2}$, so that $s^*s{\tau_2}^{y_2} \in T_{w_{[\ell]}w_4}$, and we obtain another type of H_m-class in $H_m\tau_2$ which we may call $h_\ell{}'\tau_2$; again the number of H_m-classes is one half of that given in Table 8 (replacing f by 1). The elements $s{\tau_2}^{y_2}$ and $s^*s{\tau_2}^{y_2}$ are of the form $\left(-\lambda, -\mu, -\frac{\nu^2}{\lambda\mu}, -1; -\nu\right)$, their $\boldsymbol{G}$-centralizer has root system $\varnothing$, and up to conjugacy they lie in the first and second tori, respectively, given in Table 8.

Finally suppose $r = 3$, and s lies in an H_m-class of type h_ℓ for some $\ell \in \{31, \ldots, 36\}$. Recall that we take $e \in \{0, \pm 1\}$ such that $q \equiv e \pmod 3$ (so here $e = \pm 1$ since $p \neq 3$); temporarily take $\ell' \in \{0, 1\}$ such that $\ell \equiv \ell' \pmod 2$. The details from now on depend on the pair (e, ℓ'). The element ${\tau_3}^{y_3}$ is fixed or inverted by F_m according as $e = 1$ or -1, and is fixed or inverted by $w_{[\ell]}$ according as $\ell' = 1$ or 0; and it is inverted by w_4. First assume $(e, \ell') = (1, 0)$ or $(-1, 1)$. Then $(({\tau_3}^{y_3})^{F_m})^{w_{[\ell]}} = ({\tau_3}^{y_3})^{-1}$, so $(({\tau_3}^{y_3})^{F_m})^{w_{[\ell]}w_4} = {\tau_3}^{y_3}$; thus $((s{\tau_3}^{y_3})^{F_m})^{w_{[\ell]}w_4} = s{\tau_3}^{y_3}$ so that $s{\tau_3}^{y_3} \in T_{w_{[\ell]}w_4}$, and we obtain a type of H_m-class in $H_m\tau_3$ which we may call $h_\ell\tau_3$, whose elements lie in the second torus given in Table 7, with the number of H_m-classes being as given there. Now assume $(e, \ell') = (1, 1)$ or $(-1, 0)$. Then $(({\tau_3}^{y_3})^{F_m})^{w_{[\ell]}} = {\tau_3}^{y_3}$; thus $((s{\tau_3}^{y_3})^{F_m})^{w_{[\ell]}} = s{\tau_3}^{y_3}$ so that $s{\tau_3}^{y_3} \in T_{w_{[\ell]}}$, and we obtain a type of H_m-class in $H_m\tau_3$ which we may call $h_\ell\tau_3$; however, the number of H_m-classes is one third of that given in Table 7, because conjugation by w_4w_{+---} multiplies $s{\tau_3}^{y_3}$ by $(\omega^2, \omega, \omega, 1; \omega^2)$ which lies in the torus $T_{w_{[\ell]}}$. To compensate for this we may take $s^* \in \boldsymbol{T}_3$ satisfying

$$((s^*)^{F_m})^{w_{[\ell]}} = s^*(\omega, \omega^2, \omega^2, 1; \omega)$$

(so that s^* depends only on ℓ and not on s), and then

$$((s^*s{\tau_3}^{y_3})^{F_m})^{w_{[\ell]}w_4w_{+---}} = s^*s{\tau_3}^{y_3},$$

so that $s^*s{\tau_3}^{y_3} \in T_{w_{[\ell]}w_4w_{+---}}$, and we obtain another type of H_m-class in $H_m\tau_3$ which we may call $h_\ell{}'\tau_3$; this time the number of H_m-classes is two thirds of that given in Table 7, because replacing w_4w_{+---} by its inverse $w_{+---}w_4$ gives another

type	# H_m-classes	n
$h_{31}\tau_2$	$\frac{1}{12}q^2+\cdots$	1 3 7
$h_{32}\tau_2$	$\frac{1}{12}q^2+\cdots$	17 10 20
$h_{33}\tau_2$	$\frac{1}{4}q^2+\cdots$	2 5 15
$h_{34}\tau_2$	$\frac{1}{4}q^2+\cdots$	9 5 13
$h_{35}\tau_2$	$\frac{1}{6}q^2+\cdots$	6 16 18
$h_{36}\tau_2$	$\frac{1}{6}q^2+\cdots$	21 14 25

Table 13. H_m-classes in $H_m\tau_2$ related to those of types $h_{31}, \ldots, h_{36}$.

collection of H_m-classes and the maximal tori $T_{w_{[\ell]}w_4w_{+---}}$ and $T_{w_{[\ell]}w_{+---}w_4}$ are conjugate. The elements $s\tau_3{}^{y_3}$ and $s^*s\tau_3{}^{y_3}$ are of the form $\left(\lambda, \mu, \frac{\lambda}{\mu}, \omega^2; \omega\lambda\right)$, their $\boldsymbol{G}$-centralizer has root system $\varnothing$, and up to conjugacy they lie in the first and third tori, respectively, given in Table 7.

We conclude this section by summarizing the notation and the information about numbers of H_m-classes and the tori $T_{(n)}$ containing the semisimple parts of elements therein. Recall that we take $d \in \{0, 1\}$ with $q \equiv d \bmod 2$, as well as $e \in \{0, \pm 1\}$ with $q \equiv e \bmod 3$ as in the previous paragraph. We observe that, for fixed ℓ and r, the number of H_m-classes of type $h_\ell\tau_r$, combined with those of type $h_\ell{}'\tau_r$ if they exist, is given by the same polynomial in q for both values of d (if $r = 2$) or for all three values of e (if $r = 3$). If $r = 2$ we have H_m-classes as given in Table 13, where the last entries in the final column are to be ignored if $d = 1$, and Table 14; if $r = 3$ we have H_m-classes as given in Table 15.

4.2. ***The characters $\mathbf{1}_{H_m.2}{}^G$ for $m = 1, 2$.*** Recall that we take $d \in \{0, 1\}$ with $q \equiv d \bmod 2$. We define

$$q_2 = \tfrac{1}{2}(q+d), \qquad f_2 = \tfrac{1}{2}(f+1) = \begin{cases} 2 & \text{if } m = 1, \\ 1 & \text{if } m = 2. \end{cases}$$

We proceed as in Section 3, determining nonnegligible parts of contributions to the scalar product $(1_{H_m.2}{}^G, R_{\boldsymbol{T},\theta})$ from the various types of class. Contributions from classes lying in H_m are of course exactly as already calculated, except for a factor of $\frac{1}{2}$; it remains to consider the classes in $H_m\tau_2$.

4.2.1. *Unipotent characters.* We begin with the classes of type $h_\ell\tau_2$ with $\ell \in \{31, \ldots, 36\}$; for example, we take $\ell = 31$, in which case the elements concerned have semisimple parts lying in tori $T_{(n)}$ for $n = 1$, 3 and 7 (the last of which is absent if $d = 1$). If $d = 0$, we see from the Appendix of [8] that the Green function value is $2q+1$, 1 or $-q+1$ according as $n = 1$, 3 or 7; thus the contributions to

	$d=0$			$d=1$	
type	# H_m-classes	n		# H_m-classes	n
$h_{66}\tau_2$	$\frac{1}{48}q^3+\cdots$	1	3	$\frac{1}{96}q^3+\cdots$	1
$h_{66}{}'\tau_2$				$\frac{1}{96}q^3+\cdots$	3
$h_{67}\tau_2$	$\frac{1}{48}q^3+\cdots$	17	10	$\frac{1}{96}q^3+\cdots$	17
$h_{67}{}'\tau_2$				$\frac{1}{96}q^3+\cdots$	10
$h_{68}\tau_2$	$\frac{1}{16}q^3+\cdots$	4	3	$\frac{1}{32}q^3+\cdots$	4
$h_{68}{}'\tau_2$				$\frac{1}{32}q^3+\cdots$	3
$h_{69}\tau_2$	$\frac{1}{16}q^3+\cdots$	4	10	$\frac{1}{32}q^3+\cdots$	4
$h_{69}{}'\tau_2$				$\frac{1}{32}q^3+\cdots$	10
$h_{70}\tau_2$	$\frac{1}{8}q^3+\cdots$	2	5	$\frac{1}{16}q^3+\cdots$	2
$h_{70}{}'\tau_2$				$\frac{1}{16}q^3+\cdots$	5
$h_{71}\tau_2$	$\frac{1}{8}q^3+\cdots$	9	5	$\frac{1}{16}q^3+\cdots$	9
$h_{71}{}'\tau_2$				$\frac{1}{16}q^3+\cdots$	5
$h_{72}\tau_2$	$\frac{1}{8}q^3+\cdots$	11	8	$\frac{1}{16}q^3+\cdots$	11
$h_{72}{}'\tau_2$				$\frac{1}{16}q^3+\cdots$	8
$h_{73}\tau_2$	$\frac{1}{8}q^3+\cdots$	11	19	$\frac{1}{16}q^3+\cdots$	11
$h_{73}{}'\tau_2$				$\frac{1}{16}q^3+\cdots$	19
$h_{74}\tau_2$	$\frac{1}{6}q^3+\cdots$	6	16	$\frac{1}{12}q^3+\cdots$	6
$h_{74}{}'\tau_2$				$\frac{1}{12}q^3+\cdots$	16
$h_{75}\tau_2$	$\frac{1}{6}q^3+\cdots$	21	14	$\frac{1}{12}q^3+\cdots$	21
$h_{75}{}'\tau_2$				$\frac{1}{12}q^3+\cdots$	14

Table 14. H_m-classes in $H_m\tau_2$ related to those of types $h_{66},\ldots,h_{75}$.

the scalar product with $R_{(n)}$ are

$$\frac{192\cdot(2q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},\quad \frac{48\cdot 1\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},\quad \frac{12\cdot(-q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},$$

having nonnegligible parts 16, 0 and $-\frac{1}{2}$, respectively. If however $d=1$, the Green function value is $q+1$ or $-q+1$ according as $n=1$ or 3; thus the contributions to the scalar product with $R_{(n)}$ are

$$\frac{576\cdot(q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},\quad \frac{48\cdot(-q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},$$

having nonnegligible parts 24 and -2, respectively. The other instances may be dealt with similarly.

	$e=0$		$e=1$		$e=-1$	
type	# H_m-classes	n	# H_m-classes	n	# H_m-classes	n
$h_{31}\tau_3$	$\frac{1}{12}q^2+\cdots$	1 3 7	$\frac{1}{36}q^2+\cdots$	1	$\frac{1}{12}q^2+\cdots$	3
$h_{31}'\tau_3$			$\frac{1}{18}q^2+\cdots$	7		
$h_{32}\tau_3$	$\frac{1}{12}q^2+\cdots$	17 10 20	$\frac{1}{12}q^2+\cdots$	10	$\frac{1}{36}q^2+\cdots$	17
$h_{32}'\tau_3$					$\frac{1}{18}q^2+\cdots$	20
$h_{33}\tau_3$	$\frac{1}{4}q^2+\cdots$	2 5 15	$\frac{1}{12}q^2+\cdots$	2	$\frac{1}{4}q^2+\cdots$	5
$h_{33}'\tau_3$			$\frac{1}{6}q^2+\cdots$	15		
$h_{34}\tau_3$	$\frac{1}{4}q^2+\cdots$	9 5 13	$\frac{1}{4}q^2+\cdots$	5	$\frac{1}{12}q^2+\cdots$	9
$h_{34}'\tau_3$					$\frac{1}{6}q^2+\cdots$	13
$h_{35}\tau_3$	$\frac{1}{6}q^2+\cdots$	6 16 18	$\frac{1}{18}q^2+\cdots$	6	$\frac{1}{6}q^2+\cdots$	16
$h_{35}'\tau_3$			$\frac{1}{9}q^2+\cdots$	18		
$h_{36}\tau_3$	$\frac{1}{6}q^2+\cdots$	21 14 25	$\frac{1}{6}q^2+\cdots$	14	$\frac{1}{18}q^2+\cdots$	21
$h_{36}'\tau_3$					$\frac{1}{9}q^2+\cdots$	25

Table 15. H_m-classes in $H_m\tau_3$ related to those of types $h_{31},\dots,h_{36}$.

Combining the two cases, we obtain the following table of nonnegligible parts.

$h_{31}\tau_2$	$R_{(1)}$	$8(d+2)$	$R_{(3)}$	$-2d$	$R_{(7)}$	$\frac{1}{2}(d-1)$
$h_{32}\tau_2$	$R_{(17)}$	$-8(d+2)$	$R_{(10)}$	$2d$	$R_{(20)}$	$-\frac{1}{2}(d-1)$
$h_{33}\tau_2$	$R_{(2)}$	$2(d+2)$	$R_{(5)}$	$-d$	$R_{(15)}$	$\frac{1}{2}(d-1)$
$h_{34}\tau_2$	$R_{(9)}$	$-2(d+2)$	$R_{(5)}$	d	$R_{(13)}$	$-\frac{1}{2}(d-1)$
$h_{35}\tau_2$	$R_{(6)}$	$\frac{1}{2}(d+2)$	$R_{(16)}$	$-\frac{1}{2}d$	$R_{(18)}$	$2(d-1)$
$h_{36}\tau_2$	$R_{(21)}$	$-\frac{1}{2}(d+2)$	$R_{(14)}$	$\frac{1}{2}d$	$R_{(25)}$	$-2(d-1)$

Note that the coefficients of d above are precisely the same, apart from the factor of $\frac{1}{2}$ already mentioned, as those of q obtained from consideration of types $h_{31},\dots,h_{36}$ in Section 3.2. This means that when the sets of contributions are added and q is replaced by $2q_2-d$, the terms in d cancel to leave linear polynomials in q_2.

We now consider the classes of type $h_\ell\tau_2$ or $h_\ell{}'\tau_2$ with $\ell\in\{66,\dots,75\}$; for example, we take $\ell=66$, in which case the elements concerned have semisimple parts lying in tori $T_{(n)}$ for $n=1$ and 3. In all cases the element is regular in H_m, so the Green function value is 1. If $d=0$ the contributions to the scalar product with $R_{(n)}$ are

$$\frac{576\cdot 1\cdot\left(\frac{1}{48}q^3+\cdots\right)}{2(q^3+\cdots)},\qquad \frac{48\cdot 1\cdot\left(\frac{1}{48}q^3+\cdots\right)}{2(q^3+\cdots)},$$

having nonnegligible parts 6 and $\frac{1}{2}$, respectively. If however $d=1$, the contributions to the scalar product with $R_{(n)}$ are

$$\frac{1152\cdot 1\cdot\left(\frac{1}{96}q^3+\cdots\right)}{2(q^3+\cdots)},\qquad \frac{96\cdot 1\cdot\left(\frac{1}{96}q^3+\cdots\right)}{2(q^3+\cdots)}$$

(the former from $h_{66}\tau_2$ and the latter from $h_{66}{}'\tau_2$), having nonnegligible parts 6 and $\frac{1}{2}$, respectively. Once more, the other instances may be dealt with similarly, to produce the following table of nonnegligible parts.

$h_{66}\tau_2$ or $h_{66}{}'\tau_2$	$R_{(1)}$	6	$R_{(3)}$	$\frac{1}{2}$	$h_{71}\tau_2$ or $h_{71}{}'\tau_2$	$R_{(9)}$	3	$R_{(5)}$	$\frac{1}{2}$
$h_{67}\tau_2$ or $h_{67}{}'\tau_2$	$R_{(17)}$	6	$R_{(10)}$	$\frac{1}{2}$	$h_{72}\tau_2$ or $h_{72}{}'\tau_2$	$R_{(11)}$	$\frac{1}{2}$	$R_{(8)}$	1
$h_{68}\tau_2$ or $h_{68}{}'\tau_2$	$R_{(4)}$	1	$R_{(3)}$	$\frac{3}{2}$	$h_{73}\tau_2$ or $h_{73}{}'\tau_2$	$R_{(11)}$	$\frac{1}{2}$	$R_{(19)}$	1
$h_{69}\tau_2$ or $h_{69}{}'\tau_2$	$R_{(4)}$	1	$R_{(10)}$	$\frac{3}{2}$	$h_{74}\tau_2$ or $h_{74}{}'\tau_2$	$R_{(6)}$	$\frac{3}{2}$	$R_{(16)}$	$\frac{1}{2}$
$h_{70}\tau_2$ or $h_{70}{}'\tau_2$	$R_{(2)}$	3	$R_{(5)}$	$\frac{1}{2}$	$h_{75}\tau_2$ or $h_{75}{}'\tau_2$	$R_{(21)}$	$\frac{3}{2}$	$R_{(14)}$	$\frac{1}{2}$

Summing the nonnegligible parts gives the values in Table 16 for the scalar products $(1_{H_m.2}{}^G, R_{(n)})_G$.

	H_1	H_2
$R_{(1)}$	$16q_2+75$	$16q_2+44$
$R_{(2)}$	$4q_2+27$	$4q_2+14$
$R_{(3)}$	$-4q_2+1$	$-4q_2+4$
$R_{(4)}$	5	2
$R_{(5)}$	-3	2
$R_{(6)}$	q_2+12	q_2+5
$R_{(7)}$	q_2	q_2-1
$R_{(8)}$	4	3
$R_{(9)}$	$-4q_2+1$	$-4q_2$
$R_{(10)}$	$4q_2-5$	$4q_2+2$
$R_{(11)}$	4	1
$R_{(12)}$	0	1
$R_{(13)}$	$-q_2+1$	$-q_2$

	H_1	H_2
$R_{(14)}$	q_2+1	q_2+2
$R_{(15)}$	q_2	q_2-1
$R_{(16)}$	$-q_2-2$	$-q_2+1$
$R_{(17)}$	$-16q_2+7$	$-16q_2$
$R_{(18)}$	$4q_2$	$4q_2-4$
$R_{(19)}$	-2	1
$R_{(20)}$	$-q_2+1$	$-q_2$
$R_{(21)}$	$-q_2+1$	$-q_2$
$R_{(22)}$	3	0
$R_{(23)}$	0	1
$R_{(24)}$	0	0
$R_{(25)}$	$-4q_2+4$	$-4q_2$

Table 16. Scalar products $(1_{H_m.2}{}^G, R_{(n)})_G$.

We may now proceed as before to find the scalar products of ${1_{H_m.2}}^G$ with irreducible unipotent characters. On taking linear combinations given by the character table of W, we find that the only scalar products of ${1_{H_m.2}}^G$ with almost characters R_ϕ which are nonzero for some m are as follows.

ϕ	$({1_{H_m.2}}^G, R_\phi)_G$
$\phi_{1,0}, \phi_{9,2}$	1
$\phi''_{8,3}$	$q_2 + f_2$
$\phi''_{8,9}$	q_2
$\phi_{4,1}, \phi''_{2,4}$	f_2
$\phi_{12,4}, \phi''_{4,7}, \phi_{16,5}$	$\frac{1}{2}$
$\phi''_{6,6}$	$f_2 - 1$
$\phi''_{9,6}$	$f_2 - \frac{1}{2}$
$\phi''_{1,12}$	$2f_2 - \frac{5}{2}$

For $\phi = \phi_{1,0}, \phi_{9,2}, \phi''_{8,3}$ and $\phi''_{8,9}$ we again have irreducible characters χ_ϕ, appearing in ${1_{H_m.2}}^G$ with multiplicities 1, 1, $q_2 + f_2$ and q_2, respectively. For $\phi = \phi_{4,1}$ and $\phi''_{2,4}$, in the family of size four, as in Section 3.2 we obtain two irreducible characters $\chi_{\phi_{4,1}}$ and $\chi_{\phi''_{2,4}}$, each appearing with multiplicity f_2. Separate analyses of the family of size 21 for the two values of m lead to the following: if $m = 1$ we have constituents $\chi_{\phi_{12,4}}, \chi_{\phi''_{9,6}}, \chi_{\phi''_{1,12}}$ and $\chi_{\phi''_{6,6}}$, with multiplicities 1, 2, 1 and 1, respectively; if $m = 2$ we have constituents $\chi_{\phi''_{4,7}}$ and $\chi_{\phi_{16,5}}$, each with multiplicity 1. This completes the treatment of unipotent characters.

4.2.2. *Other geometric conjugacy classes.* We begin with the geometric conjugacy classes of type κ_{31}; here the only types of H_m-class we need consider are $h_\ell \tau_2$ for $\ell \in \{31, 33, 35\}$. We briefly deal with type $h_{31}\tau_2$, containing elements $s\tau_2$ with $s = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$; the other types are very similar. The calculations are in some ways simpler than those in Section 3.3.1, because the nonnegligible parts are constants and it therefore suffices to consider leading terms of polynomials; on the other hand, some of the details depend on the precise geometric conjugacy class rather than just its type. Recall that there are three distinct characters $R_{\boldsymbol{T},\theta}$ lying in the geometric conjugacy class $\kappa_{31,[(i,j)]}$, and the one with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ (in which s lies) has $\theta = \theta^{(1)}_{i00j}$.

If $d = 0$, we saw in Section 3.3.1 that of the 192 conjugates of s lying in $T_{(1)}$, only 12 were of the form $(1, *, *, *; 1)$ and thus sent to 1 by $\theta^{(1)}_{i00j}$, with the values taken at other elements producing cancellation. Since the Green function value at τ_2 is $2q + 1$, the contribution to the scalar product from these elements is

$$\frac{12\cdot(2q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)}=1+\cdots.$$

If instead $d=1$, we have $(s\tau_2)^{y_2}=(-{\xi_1}^{a+b},-{\xi_1}^{a},-{\xi_1}^{b},-1;-{\xi_1}^{a+b})$; of the conjugates of this element in $T_{(1)}$, we need only consider those of the form $(\pm 1,*,*,*;\pm 1)$, since the values obtained from the others will again produce cancellation. There are 12 conjugates of the form $(-1,*,*,*;1)$, where the value taken by $\theta^{(1)}_{i00j}$ is $(-1)^i$; likewise there are 12 each of the forms $(1,*,*,*;-1)$ and $(-1,*,*,*;-1)$, where the value is $(-1)^j$ or $(-1)^{i+j}$, respectively. Since $(-1)^i+(-1)^j+(-1)^{i+j}=3-4\epsilon^2_{i,j}$, and the Green function value here is $q+1$, we obtain a contribution to the scalar product of

$$\frac{12(3-4\epsilon^2_{i,j})\cdot(q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)}=\tfrac{3}{2}-2\epsilon^2_{i,j}+\cdots.$$

Thus for both values of d the extra nonnegligible part is $1+\frac{1}{2}d-2\epsilon^2_{i,j}$. We saw in Section 3.3.1 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(1)}$ is $q+f+1$; halving and adding the nonnegligible part just found gives $q_2+f_2+1-2\epsilon^2_{i,j}$.

We find that the additional contributions from H_m-classes of type $h_{33}\tau_2$ and $h_{35}\tau_2$ to the scalar product with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(2)}$ and $\boldsymbol{T}_{(6)}$, respectively, are the same as those just calculated; thus the scalar product of ${1_{H_m.2}}^G$ with each of the three characters $R_{\boldsymbol{T},\theta}$ treated here is $q_2+f_2+1-2\epsilon^2_{i,j}$, which is therefore the multiplicity of the semisimple character $\chi^1_{\kappa_{31,[(i,j)]}}$ in ${1_{H_m.2}}^G$.

The geometric conjugacy classes of types $\kappa_{32},\dots,\kappa_{36}$ behave very similarly. For κ_{33} and κ_{35} the types of class requiring attention are as for κ_{31}; for κ_{32}, κ_{34} and κ_{36} they are $h_\ell\tau_2$ for $\ell\in\{32,34,36\}$. For κ_{35} and κ_{36}, the additional contribution is in fact 0 if $d=1$, because the elements $(s\tau_2)^{y_2}$ do not lie in the tori concerned; if however $d=0$, the extra nonnegligible part is $\pm\frac{1}{2}$. On the other hand, for κ_{33} and κ_{34} the nonnegligible part of the additional contribution is 0 if $d=0$, because the Green function in each case is merely 1 and is thus too small to affect matters; however if $d=1$ we have an extra term $\pm\frac{1}{2}(-1)^i$. Upon combining these terms with those already found, and taking linear combinations as before, we obtain the following multiplicities in ${1_{H_m.2}}^G$:

$$\begin{aligned}
\chi^1_{\kappa_{32,[(i,j)]}}&:q_2-f_2+1-2\epsilon^2_{i,j},\\
\chi^1_{\kappa_{33,[i]}}&:q_2+f_2-1-\epsilon^2_{i},\\
\chi^1_{\kappa_{34,[i]}}&:q_2-f_2+1-\epsilon^2_{i},\\
\chi^1_{\kappa_{35,[i]}}&:q_2+f_2-2,\\
\chi^1_{\kappa_{36,[i]}}&:q_2-f_2+1.
\end{aligned}$$

Next we turn to the geometric conjugacy classes of types κ_9, κ_{10}, κ_3 and κ_4. As described in Sections 3.3.3 and 3.3.4, these may loosely be regarded as the unions of certain of those of types $\kappa_{31}, \dots, \kappa_{36}$ just considered, for appropriate values of the parameters. The details of the calculations of additional nonnegligible parts are just as above; note that if $d = 1$ then $\epsilon^2_{(q\pm1)i}$ and $\epsilon^2_{(q\pm1)/3}$ are both zero because the subscripts are even. Upon taking linear combinations to obtain the irreducible characters, we obtain the following multiplicities in ${1_{H_m.2}}^G$:

$$\begin{aligned}
\chi^{\mathrm{St},1}_{\kappa_{9,[i]}} &: q_2 + f_2 - \epsilon_i^2, & \chi^{1,1}_{\kappa_{9,[i]}} &: 1 - \epsilon_i^2,\\
\chi^{\mathrm{St},1}_{\kappa_{10,[i]}} &: q_2 - f_2 + 1 - \epsilon_i^2, & \chi^{1,1}_{\kappa_{10,[i]}} &: \epsilon_i^2,\\
\chi^{\mathrm{St},1}_{\kappa_3} &: q_2 + f_2 - 1, & \chi^{\rho,1}_{\kappa_3} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_4} &: q_2 - f_2 + 1, & \chi^{\rho,1}_{\kappa_4} &: 0.
\end{aligned}$$

We now treat the geometric conjugacy classes of type κ_7, where there are ten distinct characters $R_{\boldsymbol{T},\theta}$ lying in the geometric conjugacy class $\kappa_{7,[i]}$; the one with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ has $\theta = \theta^{(1)}_{i000}$, and we must consider H_m-classes of types $h_{31}\tau_2$ and $h_{66}\tau_2$. For the former we may again take $s = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$ and argue much as above. If $d = 0$ there are 48 conjugates of s of the form $(1, *, *, *; *)$, so the contribution to the scalar product is

$$\frac{48 \cdot (2q+1) \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{2(q^3 + \cdots)} = 4 + \cdots .$$

If instead $d = 1$, there are 48 conjugates of $(s\tau_2)^{y_2}$ of the form $(1, *, *, *; *)$ and 96 of the form $(-1, *, *, *; *)$, so the contribution is

$$\frac{(48 + 96(-1)^i) \cdot (q+1) \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{2(q^3 + \cdots)} = 2 + 4(-1)^i + \cdots .$$

For H_m-classes of type $h_{66}\tau_2$ we may take $s = ({\xi_1}^{a+b}, {\xi_1}^{a+c}, {\xi_1}^{b+c}, 1; {\xi_1}^{a+b+c})$; as the element is regular the Green function value is 1. If $d = 0$, there are again 48 conjugates of s of the form $(1, *, *, *; *)$, so the contribution is

$$\frac{48 \cdot 1 \cdot \left(\frac{1}{48}q^3 + \cdots\right)}{2(q^3 + \cdots)} = \frac{1}{2} + \cdots .$$

If instead $d = 1$, there are 96 conjugates of $(s\tau_2)^{y_2}$ of the form $(-1, *, *, *; *)$ (and none of the form $(1, *, *, *; *)$), so the contribution is

$$\frac{96(-1)^i \cdot 1 \cdot \left(\frac{1}{96}q^3 + \cdots\right)}{2(q^3 + \cdots)} = \frac{1}{2}(-1)^i + \cdots .$$

Thus for both values of d the extra nonnegligible part is $2d+\frac{9}{2}-9\epsilon_i^2$. We saw in Section 3.3.5 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(1)}$ is $4q+5f+4$; halving and adding the nonnegligible part just found gives $4q_2+5f_2+4-9\epsilon_i^2$.

For each of the nine other characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$, there is precisely one type $h_\ell\tau_2$ or $h_\ell{}'\tau_2$ with $\ell\in\{66,\dots,75\}$ giving a contribution with nonnegligible part $\frac{1}{2}$ or $\frac{1}{2}(-1)^i$ according as $d=0$ or 1. In addition, for the character with $\boldsymbol{T}=\boldsymbol{T}_{(2)}$ we obtain a contribution from type $h_{33}\tau_2$ with nonnegligible part 2 or $1+2(-1)^i$ according as $d=0$ or 1; likewise for the character with $\boldsymbol{T}=\boldsymbol{T}_{(6)}$ we obtain a contribution from type $h_{35}\tau_2$ with nonnegligible part 1 or $\frac{1}{2}+(-1)^i$ according as $d=0$ or 1. (There are also various other types where the contribution has zero nonnegligible part.) Combining with the values found in Section 3.3.5 we obtain the following scalar products of ${1_{H_m.2}}^G$ with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$:

$$\begin{aligned}
\boldsymbol{T}=\boldsymbol{T}_{(1)}: &\quad 4q_2+5f_2+4-9\epsilon_i^2,\\
\boldsymbol{T}=\boldsymbol{T}_{(10)}: &\quad 4q_2-3f_2+4-\epsilon_i^2,\\
\boldsymbol{T}=\boldsymbol{T}_{(2)}: &\quad 2q_2+3f_2+2-5\epsilon_i^2,\\
\boldsymbol{T}=\boldsymbol{T}_{(5)}: &\quad 2q_2-f_2+2-\epsilon_i^2,\\
\boldsymbol{T}=\boldsymbol{T}_{(6)}: &\quad q_2+2f_2+2-3\epsilon_i^2,\\
\boldsymbol{T}=\boldsymbol{T}_{(14)}: &\quad q_2+1-\epsilon_i^2,\\
\boldsymbol{T}=\boldsymbol{T}_{(3)},\boldsymbol{T}_{(4)},\boldsymbol{T}_{(8)},\boldsymbol{T}_{(11)}: &\quad f_2-\epsilon_i^2.
\end{aligned}$$

Using the character table of the Weyl group C_3 and the appropriate nonabelian Fourier transform matrix as in Section 3.3.5 shows that the irreducible characters $\chi_{\kappa_{7,[i]}}^{1}$ and $\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}$ have multiplicities in ${1_{H_m.2}}^G$ equal to $q_2+f_2+1-2\epsilon_i^2$ and $q_2+1-\epsilon_i^2$, respectively, while both $\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}$ and $\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}$ have multiplicity $f_2-\epsilon_i^2$.

The geometric conjugacy classes of type κ_8 behave entirely similarly; here we find that the irreducible characters $\chi_{\kappa_{8,[i]}}^{1}$ and $\chi_{\kappa_{8,[i]}}^{\chi_{1,2}}$ have multiplicities in ${1_{H_m.2}}^G$ equal to q_2+f_2-1 and $q_2-\epsilon_i^2$, respectively, while both $\chi_{\kappa_{8,[i]}}^{\chi_{2,1}}$ and $\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}$ have multiplicity $f_2-1+\epsilon_i^2$.

The geometric conjugacy class κ_1 may again be treated in similar fashion; we find that the irreducible characters $\chi_{\kappa_1}^{1,1}$ and $\chi_{\kappa_1}^{\chi_{1,2},1}$ both have multiplicity ϵ^4 in ${1_{H_m.2}}^G$, while $\chi_{\kappa_1}^{1,\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}}$ and $\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}}$ have multiplicities $q_2+f_2-1+\epsilon^4$, q_2, $f_2-1+\epsilon^4$ and $f_2-1+\epsilon^4$, respectively.

4.2.3. *The complete decomposition of* ${1_{H_m.2}}^G$ *for* $m=1,2$. Combining the multiplicities obtained above gives the complete decomposition of ${1_{H_m.2}}^G$ for $m=1,2$ as follows.

Proposition 4.1. *If $G = F_4(q)$ and $H_m = {}^mD_4(q)$ for $m = 1, 2$, the decomposition of ${1_{H_m.2}}^G$ into irreducible characters is*

$$\begin{aligned}
&\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + (q_2+f_2)\chi_{\phi''_{8,3}} + q_2\chi_{\phi''_{8,9}} + f_2\chi_{\phi_{4,1}} + f_2\chi_{\phi''_{2,4}} \\
&\quad + \begin{Bmatrix} \chi_{\phi_{12,4}} + 2\chi_{\phi''_{9,6}} + \chi_{\phi''_{1,12}} + \chi_{\phi''_{6,6}} & \text{if } m=1 \\ \chi_{\phi''_{4,7}} + \chi_{\phi_{16,5}} & \text{if } m=2 \end{Bmatrix} \\
&\quad + \epsilon^4\chi_{\kappa_1}^{1,1} + \epsilon^4\chi_{\kappa_1}^{\chi_{1,2},1} + (q_2+f_2-1+\epsilon^4)\chi_{\kappa_1}^{1,\mathrm{St}} + q_2\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}} \\
&\qquad + (f_2-1+\epsilon^4)\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}} + (f_2-1+\epsilon^4)\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}} \\
&\quad + (q_2+f_2-1)\chi_{\kappa_3}^{\mathrm{St},1} + \chi_{\kappa_3}^{\rho,1} + (q_2-f_2+1)\chi_{\kappa_4}^{\mathrm{St},1} \\
&\quad + \sum_{[i]\in\bar{S}_7}\Big((q_2+f_2+1-2\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{1} + (q_2+1-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{1,2}} \\
&\qquad\qquad + (f_2-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{2,1}} + (f_2-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}\Big) \\
&\quad + \sum_{[i]\in\bar{S}_8}\Big((q_2+f_2-1)\chi_{\kappa_{8,[i]}}^{1} + (q_2-\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{1,2}} \\
&\qquad\qquad + (f_2-1+\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{2,1}} + (f_2-1+\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}\Big) \\
&\quad + \sum_{[i]\in\bar{S}_9}((1-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{1,1} + (q_2+f_2-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1}) \\
&\quad + \sum_{[i]\in\bar{S}_{10}}(\epsilon_i^2\chi_{\kappa_{10,[i]}}^{1,1} + (q_2-f_2+1-\epsilon_i^2)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1}) \\
&\quad + \sum_{[(i,j)]\in\bar{S}_{31}}(q_2+f_2+1-2\epsilon_{i,j}^2)\chi_{\kappa_{31,[(i,j)]}}^{1} + \sum_{[(i,j)]\in\bar{S}_{32}}(q_2-f_2+1-2\epsilon_{i,j}^2)\chi_{\kappa_{32,[(i,j)]}}^{1} \\
&\quad + \sum_{[i]\in\bar{S}_{33}}(q_2+f_2-1-\epsilon_i^2)\chi_{\kappa_{33,[i]}}^{1} + \sum_{[i]\in\bar{S}_{34}}(q_2-f_2+1-\epsilon_i^2)\chi_{\kappa_{34,[i]}}^{1} \\
&\quad + \sum_{[i]\in\bar{S}_{35}}(q_2+f_2-2)\chi_{\kappa_{35,[i]}}^{1} + \sum_{[i]\in\bar{S}_{36}}(q_2-f_2+1)\chi_{\kappa_{36,[i]}}^{1}.
\end{aligned}$$

Again we may now calculate the ranks of the actions; we obtain the following.

Corollary 4.2. *The rank of the action of $G = F_4(q)$ on cosets of $H_m.2 = {}^mD_4(q).2$ for $m = 1, 2$ is*

$$\begin{aligned}
&\tfrac{1}{4}(q^4+q^3+12q^2+20q+28) && \text{if } m=1 \text{ and } d=0,\\
&\tfrac{1}{4}(q^4+q^3+12q^2+27q+39) && \text{if } m=1 \text{ and } d=1,\\
&\tfrac{1}{4}(q^4+q^3+4q^2+4q+8) && \text{if } m=2 \text{ and } d=0,\\
&\tfrac{1}{4}(q^4+q^3+4q^2+7q+11) && \text{if } m=2 \text{ and } d=1.
\end{aligned}$$

4.3. ***The characters*** $\mathbf{1}_{H_m.3}{}^G$ ***for*** $m = 1, 3$. Recall that we take $e \in \{0, \pm 1\}$ with $q \equiv e \bmod 3$. We define

$$q_3 = \tfrac{1}{3}(q + 2e), \qquad f_3 = \tfrac{1}{3} f = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 3. \end{cases}$$

Again we proceed as in Section 3. This time contributions from classes lying in H_m are as already calculated, except for a factor of $\frac{1}{3}$; it remains to consider the classes in $H_m\tau_3$ and $H_m\tau_3{}^2$ (clearly the contributions from the two outer cosets will be complex conjugates of each other).

4.3.1. *Unipotent characters.* We must consider the H_m-classes of type $h_\ell\tau_3$ or $h_\ell{}'\tau_3$ with $\ell \in \{31, \dots, 36\}$; for example, we again take $\ell = 31$, so that the elements concerned have semisimple parts lying in tori $T_{(n)}$ for $n = 1, 3$ and 7. In all cases the elements are regular, so the Green function value is 1. Thus if $e = 0$ the contributions to the scalar product with $R_{(n)}$ are

$$\frac{192 \cdot 1 \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{3(q^2 + \cdots)}, \qquad \frac{48 \cdot 1 \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{3(q^2 + \cdots)}, \qquad \frac{12 \cdot 1 \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{3(q^2 + \cdots)},$$

having nonnegligible parts $\frac{16}{3}$, $\frac{4}{3}$ and $\frac{1}{3}$, respectively; if $e = 1$ the contributions to the scalar product with $R_{(1)}$ and $R_{(7)}$ are

$$\frac{1152 \cdot 1 \cdot \left(\frac{1}{36}q^2 + \cdots\right)}{3(q^2 + \cdots)}, \qquad \frac{36 \cdot 1 \cdot \left(\frac{1}{18}q^2 + \cdots\right)}{3(q^2 + \cdots)}$$

(the latter from $h_{31}{}'\tau_3$), having nonnegligible parts $\frac{32}{3}$ and $\frac{2}{3}$, respectively; if $e = -1$ the contribution to the scalar product with $R_{(3)}$ is

$$\frac{96 \cdot 1 \cdot \frac{1}{12}(q^2 - \cdots)}{3(q^2 + \cdots)},$$

having nonnegligible part $\frac{8}{3}$. The other instances are precisely similar.

Combining the three possibilities for e gives the following table of nonnegligible parts.

$h_{31}\tau_3$ or $h_{31}{}'\tau_3$	$R_{(1)}$	$\frac{16}{3}(1+e)$	$R_{(3)}$	$\frac{4}{3}(1-e)$	$R_{(7)}$	$\frac{1}{3}(1+e)$
$h_{32}\tau_3$ or $h_{32}{}'\tau_3$	$R_{(17)}$	$\frac{16}{3}(1-e)$	$R_{(10)}$	$\frac{4}{3}(1+e)$	$R_{(20)}$	$\frac{1}{3}(1-e)$
$h_{33}\tau_3$ or $h_{33}{}'\tau_3$	$R_{(2)}$	$\frac{4}{3}(1+e)$	$R_{(5)}$	$\frac{2}{3}(1-e)$	$R_{(15)}$	$\frac{1}{3}(1+e)$
$h_{34}\tau_3$ or $h_{34}{}'\tau_3$	$R_{(9)}$	$\frac{4}{3}(1-e)$	$R_{(5)}$	$\frac{2}{3}(1+e)$	$R_{(13)}$	$\frac{1}{3}(1-e)$
$h_{35}\tau_3$ or $h_{35}{}'\tau_3$	$R_{(6)}$	$\frac{1}{3}(1+e)$	$R_{(16)}$	$\frac{1}{3}(1-e)$	$R_{(18)}$	$\frac{4}{3}(1+e)$
$h_{36}\tau_3$ or $h_{36}{}'\tau_3$	$R_{(21)}$	$\frac{1}{3}(1-e)$	$R_{(14)}$	$\frac{1}{3}(1+e)$	$R_{(25)}$	$\frac{4}{3}(1-e)$

	H_1	H_3
$R_{(1)}$	$16q_3+46$	$16q_3+16$
$R_{(2)}$	$4q_3+16$	$4q_3+4$
$R_{(3)}$	$-4q_3+2$	$-4q_3+4$
$R_{(4)}$	2	0
$R_{(5)}$	0	4
$R_{(6)}$	q_3+7	q_3+1
$R_{(7)}$	q_3+1	q_3+1
$R_{(8)}$	2	0
$R_{(9)}$	$-4q_3+4$	$-4q_3+4$
$R_{(10)}$	$4q_3-2$	$4q_3+4$
$R_{(11)}$	2	0
$R_{(12)}$	0	0
$R_{(13)}$	$-q_3+1$	$-q_3+1$

	H_1	H_3
$R_{(14)}$	q_3+1	q_3+1
$R_{(15)}$	q_3+1	q_3+1
$R_{(16)}$	$-q_3-1$	$-q_3+1$
$R_{(17)}$	$-16q_3+22$	$-16q_3+16$
$R_{(18)}$	$4q_3+4$	$4q_3+1$
$R_{(19)}$	-2	0
$R_{(20)}$	$-q_3+1$	$-q_3+1$
$R_{(21)}$	$-q_3+1$	$-q_3+1$
$R_{(22)}$	2	0
$R_{(23)}$	0	0
$R_{(24)}$	0	1
$R_{(25)}$	$-4q_3+4$	$-4q_3+1$

Table 17. Scalar products $({1_{H_m.3}}^G, R_{(n)})_G$.

Much as before, note that the coefficients of e above are precisely the same, apart from the factor of $\frac{1}{3}$ already mentioned, as those of q obtained from consideration of types $h_{31}, \dots, h_{36}$ in Section 3.2. This means that when the sets of contributions are added and q is replaced by $3q_3 - 2e$, the terms in e cancel to leave linear polynomials in q_3.

Summing the nonnegligible parts gives the values in Table 17 for the scalar products $({1_{H_m.3}}^G, R_{(n)})_G$.

We may now proceed as before to find the scalar products of ${1_{H_m.3}}^G$ with irreducible unipotent characters. On taking linear combinations given by the character table of W, we find that the only scalar products of ${1_{H_m.3}}^G$ with almost characters R_ϕ which are nonzero for some m are as follows.

ϕ	$({1_{H_m.3}}^G, R_\phi)_G$
$\phi_{1,0}, \phi_{9,2}$	1
$\phi''_{8,3}$	q_3+f_3
$\phi''_{8,9}$	q_3
$\phi_{4,1}, \phi''_{2,4}, \phi''_{9,6}, \phi''_{1,12}$	f_3
$\phi_{12,4}$	$\frac{1}{3}$
$\phi''_{6,6}$	$f_3-\frac{1}{3}$
$\phi'_{6,6}$	$\frac{2}{3}$

For $\phi = \phi_{1,0}$, $\phi_{9,2}$, $\phi''_{8,3}$ and $\phi''_{8,9}$ we again have irreducible characters χ_ϕ, appearing in ${1_{H_m.3}}^G$ with multiplicities 1, 1, $q_3 + f_3$ and q_3, respectively. For $\phi = \phi_{4,1}$ and $\phi''_{2,4}$, in the family of size four, as in Section 3.2 we obtain two irreducible characters $\chi_{\phi_{4,1}}$ and $\chi_{\phi''_{2,4}}$, each appearing with multiplicity f_3. Separate analyses of the family of size 21 for the two values of m lead to the following: if $m = 1$ we have constituents $\chi_{\phi_{12,4}}$, $\chi_{\phi''_{9,6}}$, $\chi_{\phi''_{1,12}}$ and $\chi_{F_4{}^{\mathrm{II}}[1]}$, each with multiplicity 1; if $m = 3$ we have a constituent $\chi_{\phi'_{6,6}}$ with multiplicity 1. This completes the treatment of unipotent characters.

4.3.2. *Other geometric conjugacy classes.* We begin again with geometric conjugacy classes of type κ_{31}; we consider H_m-classes of type $h_{31}\tau_3$, containing elements $s\tau_3$ with $s = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$. These contribute if $e = 0$ or 1, but not if $e = -1$. As in Section 4.2.2 we recall that there are three distinct characters $R_{\boldsymbol{T},\theta}$ lying in the geometric conjugacy class $\kappa_{31,[(i,j)]}$, and the one with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ (in which s lies) has $\theta = \theta^{(1)}_{i00j}$.

If $e = 0$, we saw in Section 3.3.1 that 12 of the conjugates of s lying in $T_{(1)}$ were of the form $(1, *, *, *; 1)$ and thus sent to 1 by $\theta^{(1)}_{i00j}$, with the values taken at other elements producing cancellation. Thus the contribution to the scalar product from these elements is

$$\frac{12 \cdot 1 \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{3(q^2 + \cdots)} = \tfrac{1}{3} + \cdots .$$

If $e = 1$, we have $(s\tau_2)^{y_2} = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, \omega^2; \omega{\xi_1}^{a+b})$; of the conjugates of this element in $T_{(1)}$, we need only consider those of the form $(\omega^{\pm 1}, *, *, *; \omega^{\pm 1})$, since the values obtained from the others will again produce cancellation. If we write $\bar{\zeta}$ for the cube root of unity in $\mathbb{C}$ corresponding to ω in k, there are 36 conjugates of the form $(\omega^2, *, *, *; \omega)$, where the value taken by $\theta^{(1)}_{i00j}$ is $\bar{\zeta}^{-i+j}$, and 36 of the form $(\omega, *, *, *; \omega^2)$, where the value taken is $\bar{\zeta}^{i-j}$. Since

$$\bar{\zeta}^{-i+j} + \bar{\zeta}^{i-j} = 2 - 3\epsilon^{3,+}_{i-j},$$

we obtain a contribution to the scalar product of

$$\frac{36(2 - 3\epsilon^{3,+}_{i-j}) \cdot 1 \cdot \left(\frac{1}{36}q^2 + \cdots\right)}{3(q^2 + \cdots)} = \tfrac{2}{3} - \epsilon^{3,+}_{i-j} + \cdots .$$

Thus for all three values of e the extra nonnegligible part is $\frac{1}{3}(1 + e) - \epsilon^{3,+}_{i-j}$. We saw in Section 3.3.1 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ is $q + f + 1$; dividing by three and adding twice the nonnegligible part just found gives $q_3 + f_3 + 1 - 2\epsilon^{3,+}_{i-j}$.

We find that the additional contributions from H_m-classes of type $h_{33}\tau_3$ and $h_{35}\tau_3$ to the scalar product with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T} = \boldsymbol{T}_{(2)}$ and $\boldsymbol{T}_{(6)}$, respectively, are the same as those just calculated; thus the scalar product of ${1_{H_m.3}}^G$ with each of the three characters $R_{\boldsymbol{T},\theta}$ treated here is $q_3+f_3+1-2\epsilon_{i-j}^{3,+}$, which is therefore the multiplicity of the semisimple character $\chi^1_{\kappa_{31,[(i,j)]}}$ in ${1_{H_m.3}}^G$.

The geometric conjugacy classes of types $\kappa_{32}, \dots, \kappa_{36}$ behave very similarly; we obtain the following multiplicities in ${1_{H_m.3}}^G$:

$$\begin{aligned}
\chi^1_{\kappa_{32,[(i,j)]}} &: q_3 - f_3 - 1 + 2\epsilon_{i-j}^{3,-},\\
\chi^1_{\kappa_{33,[i]}} &: q_3 + f_3 - 1 + 2\epsilon_{i}^{3,-},\\
\chi^1_{\kappa_{34,[i]}} &: q_3 - f_3 + 1 - 2\epsilon_{i}^{3,+},\\
\chi^1_{\kappa_{35,[i]}} &: q_3 + f_3 - 2\epsilon_{i}^{3,+},\\
\chi^1_{\kappa_{36,[i]}} &: q_3 - f_3 + 2\epsilon_{i}^{3,-}.
\end{aligned}$$

Next we turn to the geometric conjugacy classes of types κ_9, κ_{10}, κ_3 and κ_4. As before, these may loosely be regarded as the unions of certain of those of types $\kappa_{31}, \dots, \kappa_{36}$ just considered, for appropriate values of the parameters. The details of the calculations of additional nonnegligible parts are just as above; we obtain the following multiplicities in ${1_{H_m.3}}^G$:

$$\begin{aligned}
\chi^{\mathrm{St},1}_{\kappa_{9,[i]}} &: q_3 + f_3, & \chi^{1,1}_{\kappa_{9,[i]}} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_{10,[i]}} &: q_3 - f_3, & \chi^{1,1}_{\kappa_{10,[i]}} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_{3}} &: q_3 + f_3 - 1, & \chi^{\rho,1}_{\kappa_{3}} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_{4}} &: q_3 - f_3 + 1, & \chi^{\rho,1}_{\kappa_{4}} &: 1.
\end{aligned}$$

For the geometric conjugacy classes of type κ_7, we again treat H_m-classes of type $h_{31}\tau_3$, containing elements $s\tau_2$ with $s = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$. These contribute if $e = 0$ or 1, but not if $e = -1$; the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ has $\theta = \theta^{(1)}_{i000}$.

If $e = 0$, we saw in Section 3.3.5 that 48 of the conjugates of s lying in $T_{(1)}$ were of the form $(1, *, *, *; *)$ and thus sent to 1 by $\theta^{(1)}_{i000}$, with the values taken at other elements producing cancellation. Thus the contribution to the scalar product from these elements is

$$\frac{48 \cdot 1 \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{3(q^2 + \cdots)} = \frac{4}{3} + \cdots .$$

If instead $e = 1$, we have $(s\tau_2)^{y_2} = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, \omega^2; \omega{\xi_1}^{a+b})$; there are 144 conjugates of the form $(\omega^2, *, *, *; *)$, where the value taken by $\theta^{(1)}_{i000}$ is $\bar{\zeta}^{-i}$, and 144 of the form $(\omega, *, *, *; *)$, where the value taken is $\bar{\zeta}^i$. Since

$$\bar{\zeta}^{-i} + \bar{\zeta}^i = 2 - 3\epsilon_i^{3,+},$$

we obtain a contribution to the scalar product of

$$\frac{144(2-3\epsilon_i^{3,+})\cdot 1\cdot\left(\frac{1}{36}q^2+\cdots\right)}{3(q^2+\cdots)}=\tfrac{8}{3}-4\epsilon_i^{3,+}+\cdots.$$

Thus for all three values of e the extra nonnegligible part is $\frac{4}{3}(1+e)-4\epsilon_i^{3,+}$. We saw in Section 3.3.5 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(1)}$ is $4q+5f+4$; dividing by three and adding twice the nonnegligible part just found gives $4q_3+5f_3+4-8\epsilon_i^{3,+}$.

There are seven other pairs (ℓ,n) such that the geometric conjugacy class contains a character $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(n)}$ and the classes of type $h_\ell\tau_3$ contain elements whose semisimple parts lie in the torus $T_{(n)}$. For $(\ell,n)=(31,3)$ or $(33,5)$ we find that all roots of unity concerned produce cancellation, so the nonnegligible part is 0 (these are the two pairs where the classes meet the torus if $e=-1$). For the other five pairs the calculations are very similar to the above: the extra nonnegligible part is $\frac{1}{3}(1+e)-\epsilon_i^{3,+}$ times the coefficient of $q+1$ in the value obtained in Section 3.3.5, and it follows that the scalar product of ${1_{H_m.3}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ is obtained from that value by replacing $q+1$ by $q_3+1-2\epsilon_i^{3,+}$ and f by f_3. Accordingly the irreducible characters $\chi^1_{\kappa_{7,[i]}}$ and $\chi^{\chi_{1,2}}_{\kappa_{7,[i]}}$ have multiplicities in ${1_{H_m.3}}^G$ equal to $q_3+f_3+1-2\epsilon_i^{3,+}$ and $q_3+1-2\epsilon_i^{3,+}$, respectively, while both $\chi^{\chi_{2,1}}_{\kappa_{7,[i]}}$ and $\chi^{\chi_{-,3}}_{\kappa_{7,[i]}}$ have multiplicity f_3.

Again the geometric conjugacy classes of type κ_8 behave entirely similarly; here we find that the irreducible characters $\chi^1_{\kappa_{8,[i]}}$ and $\chi^{\chi_{1,2}}_{\kappa_{8,[i]}}$ have multiplicities in ${1_{H_m.3}}^G$ equal to $q_3+f_3-1+2\epsilon_i^{3,-}$ and $q_3-1+2\epsilon_i^{3,-}$, respectively, while both $\chi^{\chi_{2,1}}_{\kappa_{8,[i]}}$ and $\chi^{\chi_{-,3}}_{\kappa_{8,[i]}}$ have multiplicity f_3.

Finally the geometric conjugacy class κ_1 may again be treated in similar fashion; we find that the irreducible characters $\chi^{1,1}_{\kappa_1}$ and $\chi^{\chi_{1,2},1}_{\kappa_1}$ both have multiplicity 1 in ${1_{H_m.3}}^G$, while $\chi^{1,\mathrm{St}}_{\kappa_1}$, $\chi^{\chi_{1,2},\mathrm{St}}_{\kappa_1}$, $\chi^{\chi_{2,1},\mathrm{St}}_{\kappa_1}$ and $\chi^{\chi_{-,3},\mathrm{St}}_{\kappa_1}$ have multiplicities q_3+f_3, q_3, f_3 and f_3, respectively.

4.3.3. *The complete decomposition of* ${1_{H_m.3}}^G$ *for* $m=1,3$. Combining the multiplicities obtained above gives the complete decomposition of ${1_{H_m.3}}^G$ for $m=1,3$ as follows.

Proposition 4.3. *If* $G=F_4(q)$ *and* $H_m={}^mD_4(q)$ *for* $m=1,3$, *the decomposition of* ${1_{H_m.3}}^G$ *into irreducible characters is*

$$\chi_{\phi_{1,0}}+\chi_{\phi_{9,2}}+(q_3+f_3)\chi_{\phi''_{8,3}}+q_3\chi_{\phi''_{8,9}}+f_3\chi_{\phi_{4,1}}+f_3\chi_{\phi''_{2,4}}$$
$$+\begin{Bmatrix}\chi_{\phi_{12,4}}+\chi_{\phi''_{9,6}}+\chi_{\phi''_{1,12}}+\chi_{F_4^{\mathrm{II}}[1]} & \textit{if } m=1\\ \chi_{\phi'_{6,6}} & \textit{if } m=3\end{Bmatrix}$$
$$+\chi^{1,1}_{\kappa_1}+\chi^{\chi_{1,2},1}_{\kappa_1}+(q_3+f_3)\chi^{1,\mathrm{St}}_{\kappa_1}+q_3\chi^{\chi_{1,2},\mathrm{St}}_{\kappa_1}+f_3\chi^{\chi_{2,1},\mathrm{St}}_{\kappa_1}+f_3\chi^{\chi_{-,3},\mathrm{St}}_{\kappa_1}\quad\text{(continues)}$$

$$+\ (q_3+f_3-1)\chi_{\kappa_3}^{\mathrm{St},1}+\chi_{\kappa_3}^{\rho,1}+(q_3-f_3+1)\chi_{\kappa_4}^{\mathrm{St},1}+\chi_{\kappa_4}^{\rho,1}$$

$$+\sum_{[i]\in\bar{S}_7}\big((q_3+f_3+1-2\epsilon_i^{3,+})\chi_{\kappa_{7,[i]}}^{1}+(q_3+1-2\epsilon_i^{3,+})\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}+f_3\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}+f_3\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}\big)$$

$$+\sum_{[i]\in\bar{S}_8}\big((q_3+f_3-1+2\epsilon_i^{3,-})\chi_{\kappa_{8,[i]}}^{1}+(q_3-1+2\epsilon_i^{3,-})\chi_{\kappa_{8,[i]}}^{\chi_{1,2}}+f_3\chi_{\kappa_{8,[i]}}^{\chi_{2,1}}+f_3\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}\big)$$

$$+\sum_{[i]\in\bar{S}_9}(\chi_{\kappa_{9,[i]}}^{1,1}+(q_3+f_3)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1})+\sum_{[i]\in\bar{S}_{10}}(\chi_{\kappa_{10,[i]}}^{1,1}+(q_3-f_3)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1})$$

$$+\sum_{[(i,j)]\in\bar{S}_{31}}(q_3+f_3+1-2\epsilon_{i-j}^{3,+})\chi_{\kappa_{31,[(i,j)]}}^{1}+\sum_{[(i,j)]\in\bar{S}_{32}}(q_3-f_3-1+2\epsilon_{i-j}^{3,-})\chi_{\kappa_{32,[(i,j)]}}^{1}$$

$$+\sum_{[i]\in\bar{S}_{33}}(q_3+f_3-1+2\epsilon_i^{3,-})\chi_{\kappa_{33,[i]}}^{1}+\sum_{[i]\in\bar{S}_{34}}(q_3-f_3+1-2\epsilon_i^{3,+})\chi_{\kappa_{34,[i]}}^{1}$$

$$+\sum_{[i]\in\bar{S}_{35}}(q_3+f_3-2\epsilon_i^{3,+})\chi_{\kappa_{35,[i]}}^{1}+\sum_{[i]\in\bar{S}_{36}}(q_3-f_3+2\epsilon_i^{3,-})\chi_{\kappa_{36,[i]}}^{1}.$$

Again we may now calculate the ranks of the actions; we obtain the following.

Corollary 4.4. *The rank of the action of $G=F_4(q)$ on cosets of $H_m.3={}^mD_4(q).3$ for $m=1,3$ is*

$$\begin{array}{ll}\frac{1}{9}(q^4+q^3+13q^2+21q+36) & \text{if } m=1 \text{ and } e=0 \text{ or } 1,\\ \frac{1}{9}(q^4+q^3+13q^2+21q+44) & \text{if } m=1 \text{ and } e=-1,\\ \frac{1}{9}(q^4+q^3+4q^2+3q+9) & \text{if } m=3 \text{ and } e=0 \text{ or } 1,\\ \frac{1}{9}(q^4+q^3+4q^2+3q+17) & \text{if } m=3 \text{ and } e=-1.\end{array}$$

4.4. ***The character* $\mathbf{1}_{H_1.S_3}{}^G$.** We define

$$q_6=\tfrac{1}{6}(q+3d+2e)=\tfrac{1}{3}(q_2+d+e)=\tfrac{1}{2}(q_3+d).$$

If we write $1_{H_1}{}^{H_1.S_3}=1+\epsilon+2\rho$ where ϵ is linear and ρ has dimension 2, then $1_{H_1.2}{}^{H_1.S_3}=1+\rho$ and $1_{H_1.3}{}^{H_1.S_3}=1+\epsilon$, whence

$$1_{H_1.S_3}{}^{H_1.S_3}=1_{H_1.2}{}^{H_1.S_3}-\tfrac{1}{2}(1_{H_1}{}^{H_1.S_3}-1_{H_1.3}{}^{H_1.S_3});$$

inducing up from $H_1.S_3$ to G gives

$$1_{H_1.S_3}{}^G=1_{H_1.2}{}^G-\tfrac{1}{2}(1_{H_1}{}^G-1_{H_1.3}{}^G).$$

It is therefore now easy to calculate the decomposition of $1_{H_1.S_3}{}^G$ to be as follows.

Proposition 4.5. *If $G = F_4(q)$ and $H_1 = D_4(q)$, the decomposition of ${1_{H_1.S_3}}^G$ into irreducible characters is*

$$\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + (q_6+1)\chi_{\phi''_{8,3}} + q_6\chi_{\phi''_{8,9}} + \chi_{\phi_{4,1}} + \chi_{\phi''_{2,4}} + \chi_{\phi_{12,4}} + \chi_{\phi''_{9,6}}$$
$$+ \epsilon^4\chi_{\kappa_1}^{1,1} + \epsilon^4\chi_{\kappa_1}^{\chi_{1,2},1} + (q_6+\epsilon^4)\chi_{\kappa_1}^{1,\mathrm{St}} + q_6\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}} + \epsilon^4\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}} + \epsilon^4\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}}$$
$$+ q_6\chi_{\kappa_3}^{\mathrm{St},1} + \chi_{\kappa_3}^{\rho,1} + q_6\chi_{\kappa_4}^{\mathrm{St},1}$$
$$+ \sum_{[i]\in\bar{S}_7}\Big((q_6+2-\epsilon_i^{3,+}-2\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{1} + (q_6+1-\epsilon_i^{3,+}-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}$$
$$+ (1-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{2,1}} + (1-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}\Big)$$
$$+ \sum_{[i]\in\bar{S}_8}((q_6+\epsilon_i^{3,-})\chi_{\kappa_{8,[i]}}^{1} + (q_6+\epsilon_i^{3,-}-\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{1,2}} + \epsilon_i^2\chi_{\kappa_{8,[i]}}^{\chi_{2,1}} + \epsilon_i^2\chi_{\kappa_{8,[i]}}^{\chi_{-,3}})$$
$$+ \sum_{[i]\in\bar{S}_9}((1-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{1,1} + (q_6+1-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1}) + \sum_{[i]\in\bar{S}_{10}}(\epsilon_i^2\chi_{\kappa_{10,[i]}}^{1,1} + (q_6-\epsilon_i^2)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1})$$
$$+ \sum_{[(i,j)]\in\bar{S}_{31}}(q_6+2-\epsilon_{i-j}^{3,+}-2\epsilon_{i,j}^2)\chi_{\kappa_{31,[(i,j)]}}^{1} + \sum_{[(i,j)]\in\bar{S}_{32}}(q_6+\epsilon_{i-j}^{3,-}-2\epsilon_{i,j}^2)\chi_{\kappa_{32,[(i,j)]}}^{1}$$
$$+ \sum_{[i]\in\bar{S}_{33}}(q_6+\epsilon_i^{3,-}-\epsilon_i^2)\chi_{\kappa_{33,[i]}}^{1} + \sum_{[i]\in\bar{S}_{34}}(q_6-\epsilon_i^{3,+}-\epsilon_i^2)\chi_{\kappa_{34,[i]}}^{1}$$
$$+ \sum_{[i]\in\bar{S}_{35}}(q_6-\epsilon_i^{3,+})\chi_{\kappa_{35,[i]}}^{1} + \sum_{[i]\in\bar{S}_{36}}(q_6-1+\epsilon_i^{3,-})\chi_{\kappa_{36,[i]}}^{1}.$$

Yet again we may now calculate the rank of the action; we obtain the following.

Corollary 4.6. *The rank of the action of $G = F_4(q)$ on cosets of $H_1.S_3 = D_4(q).S_3$ is*

$$\begin{array}{ll} \frac{1}{36}(q^4+q^3+28q^2+48q+84) & \text{if } d=0 \text{ and } e=1,\\ \frac{1}{36}(q^4+q^3+28q^2+48q+92) & \text{if } d=0 \text{ and } e=-1,\\ \frac{1}{36}(q^4+q^3+28q^2+75q+99) & \text{if } d=1 \text{ and } e=0,\\ \frac{1}{36}(q^4+q^3+28q^2+75q+111) & \text{if } d=1 \text{ and } e=1,\\ \frac{1}{36}(q^4+q^3+28q^2+75q+119) & \text{if } d=1 \text{ and } e=-1. \end{array}$$

5. Contribution to classification programs

In this final section we consider the part played by D_4-type subgroups of $F_4(q)$ in the classification programs mentioned in Section 1, namely those of primitive actions which are multiplicity-free, or have all suborbits self-paired, or arise from a distance-transitive graph. Recall that the primitive actions (in which the action is on the cosets of a subgroup which is maximal) are those where the subgroup is either $D_4(q).S_3$ or ${}^3D_4(q).3$.

5.1. ***Primitive multiplicity-free actions.*** From the decompositions of Section 4 we can see that the permutation character ${1_H}^{F_4(q)}$, where H is either $D_4(q).S_3$ or ${}^3D_4(q).3$, is multiplicity-free if and only if $q = 2$. Indeed, the multiplicity of the constituent $\chi_{\phi''_{8,3}}$ in the former case is $q_6 + 1$, which is greater than 1 for all q apart from 2, and in the latter case is q_3, which is greater than 1 for all q apart from 2, 3 and 5; if q is 3 or 5 the multiplicity of the constituent $\chi^1_{\kappa_{34,[1]}}$ in the latter case is $q_3 + 1$, which is greater than 1.

However, if $q = p^a$ for some $a > 1$, the possibility arises of extending both $F_4(q)$ and H by field automorphisms. Write ϕ for the field automorphism which for each $\alpha \in \Phi$ and $\lambda \in k$ sends $x_\alpha(\lambda)$ to $x_\alpha(\lambda^p)$; then $F_4(q).\langle\phi\rangle$ acts on cosets of $H.\langle\phi\rangle$, and we have the corresponding permutation character ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$. Given an irreducible character χ of $F_4(q)$, there exist $a', a'' \geq 1$ with $a = a'a''$ such that applying ϕ fuses together a' irreducible characters of $F_4(q)$ including χ and the resulting character extends to a'' distinct characters of the group $F_4(q).\langle\phi\rangle$. If the multiplicity of χ as a constituent of the permutation character ${1_H}^{F_4(q)}$ is greater than a'' (which in particular is true if it is greater than a), then at least one of the extensions to $F_4(q).\langle\phi\rangle$ must have multiplicity greater than 1 in ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$.

We claim that, if $q = p^a$ with $a > 1$, then ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$ is not multiplicity-free. For most such values of q we may see this by again considering the multiplicity of the constituent $\chi_{\phi''_{8,3}}$. Indeed, if $H = D_4(q).S_3$ we have $q_6 + 1 > a$ for all q apart from 4, 8 and 16, while if $H = {}^3D_4(q).3$ we have $q_3 > a$ for all q apart from 4 and 8. We are therefore left with five pairs (H, q) to treat.

Two of these five pairs may be settled by considering a single constituent. If $H = D_4(q).S_3$ and $q = 16$, the multiplicity in ${1_H}^{F_4(q)}$ of the constituent $\chi^1_{\kappa_{7,[3]}}$ is equal to $q_6 + 2 = 5 > 4 = a$. Likewise if $H = {}^3D_4(q).3$ and $q = 8$, the multiplicity in ${1_H}^{F_4(q)}$ of the constituent $\chi^1_{\kappa_{36,[1]}}$ is equal to $q_3 + 2 = 4 > 3 = a$. Thus in neither case is ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$ multiplicity-free.

Another two may be settled by considering how ϕ fuses geometric conjugacy classes. Suppose $H = D_4(q).S_3$ and $q = 8$; here the constituents $\chi^1_{\kappa_{8,[i]}}$ for $i = 1, 2, 4$ all have multiplicity $q_6 + 1 = 2$. However, these three characters are fused by ϕ, because the semisimple classes corresponding to the geometric conjugacy classes $\kappa_{8,[i]}$ for $i = 1, 2, 4$ contain elements $(1, {\xi_2}^i, {\xi_2}^{-i}, 1; 1)$, and as ϕ squares entries in root elements $x_\alpha(\lambda)$, and therefore in torus elements $(\mu_1, \mu_2, \mu_3, \mu_4; \nu)$, it fuses these semisimple classes; so $a' = 3$ and hence $a'' = 1$. Likewise suppose $H = {}^3D_4(q).3$ and $q = 4$; here the constituents $\chi^1_{\kappa_{36,[i]}}$ for $i = 1, 2$ both have multiplicity $q_3 = 2$. However, these two characters are fused by ϕ, because the semisimple classes corresponding to the geometric conjugacy classes $\kappa_{36,[i]}$ for $i = 1, 2$ contain elements $({\xi_6}^i, {\xi_6}^{qi}, {\xi_6}^{-(q-1)i}, 1; {\xi_6}^i)$, and ϕ similarly fuses these semisimple classes; so $a' = 2$ and hence $a'' = 1$. Thus in each case the single extension of the fused character has multiplicity 2 in the permutation character ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$.

This leaves the pair where $H = D_4(q).S_3$ and $q = 4$. Here the rank is 29; there are two constituents of multiplicity 2, namely $\chi_{\phi''_{8,3}}$ and $\chi^1_{\kappa_{7,[1]}}$, and all other constituents have multiplicity 1. There is therefore no fusion among the constituents of multiplicity greater than 1; consequently the methods used until now are insufficient to determine whether or not either multiplicity persists in ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$. In this case we refer to [17], which calculates $(P, D_4(q))$-double cosets in $F_4(q)$ (where P is the maximal parabolic subgroup whose Levi subgroup has derived group $B_3(q)$), and in fact concludes that if $q > 2$ then the action of $F_4(q)$ on the cosets of $D_4(q).S_3$ is never multiplicity-free, even if field automorphisms are applied.

(A word is in order regarding the relationship of the current article to [17]. As stated in the very first paragraph here, the project on D_4-type subgroups of $F_4(q)$ is essentially a continuation of [15], which was published at the end of the previous millennium. Almost all of the work on D_4-type subgroups was completed more than twenty years ago; in particular it became clear that something beyond the character decompositions presented here was required to settle the case of the preceding paragraph, and the double coset calculations of [17] were performed to do this. However, some issues remained unresolved and the material ended up being set aside. A few years ago the opportunity arose of publishing the double coset material as a paper in its own right; but there seemed no prospect of applying similar methods to treat the action on cosets of ${}^3D_4(q).3$. More recently the issues which had prevented publication of the work as a whole were finally resolved, and the present article is the result; but [17] should really be regarded as an addendum to it. The author apologizes for the inordinate delay in completing the project, especially to those who have waited patiently for decades to see the results appear.)

Thus the only two primitive multiplicity-free actions here are those of $F_4(2)$ on cosets of $D_4(2).S_3$ and ${}^3D_4(2).3$. The first permutation character has rank 9 and decomposition

$$\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + \chi_{\phi''_{8,3}} + \chi_{\phi_{4,1}} + \chi_{\phi''_{2,4}} + \chi_{\phi_{12,4}} + \chi_{\phi''_{9,6}} + \chi^1_{\kappa_{8,[1]}} + \chi^{\chi_{1,2}}_{\kappa_{8,[1]}};$$

the constituents have degrees 1, 22932, 44200, 1377, 1105, 584766, 541450, 23205 and 1949220, respectively. The second has rank 7 and decomposition

$$\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + \chi_{\phi'_{6,6}} + {\chi_{\kappa_4}^{\mathrm{St},1}} + {\chi_{\kappa_4}^{\rho,1}} + + \chi^1_{\kappa_{8,[1]}} + \chi^{\chi_{1,2}}_{\kappa_{8,[1]}};$$

the constituents have degrees 1, 22932, 519792, 2165800, 541450, 23205 and 1949220, respectively.

For the remainder of the paper we take $q = 2$; we have $G = F_4(2)$, and we write $H = D_4(2).S_3$ or ${}^3D_4(2).3$. As $T_0 = \{1\}$ we may identify W with N.

5.2. ***Subdegrees and pairing of suborbits.*** We recall that in the action of G on the left cosets of H, the suborbit containing the left coset gH is its orbit under the stabilizer H, and thus is the set of left cosets whose union is the (H, H)-double coset HgH. The size of the suborbit is called the subdegree, and is equal to $|HgH|/|H| = |H|/|H \cap {}^gH|$; the subgroup $H \cap {}^gH$ consists of the elements of G fixing both H and gH, and is known as the 2-point stabilizer. The sum of the subdegrees is the index $|G : H|$. For any double coset HgH, the set of the inverses of its elements is the double coset $Hg^{-1}H$; the corresponding suborbits are said to be paired, and if they are equal the suborbit is called self-paired.

Our goal here is to compute the subdegrees and show that all suborbits are self-paired (for which the multiplicity-freeness of the permutation character is a necessary condition).

As we shall be performing explicit calculations here, we need to know the exact location of the subgroup H of G. In the case where $H = D_4(2).S_3$ this is immediate, but in the other case where $H = {}^3D_4(2).3$ it depends on the choice of the element g_3, which we recall was chosen to lie in $\boldsymbol{A}$, commute with $\tau_3 = n_4 n_{+---}$ and satisfy ${g_3}^F.{g_3}^{-1} = \tau_3$. We begin with some comments which apply for any such choice of g_3, and in fact for all values of q.

Label the simple roots of G as

$$\alpha_1 = \epsilon_2 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_4, \quad \alpha_3 = \epsilon_4, \quad \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4);$$

then any root in Φ has the form $\sum_{i=1}^4 c_i \alpha_i$ with all $c_i \in \mathbb{Z}$. We may partition Φ according to the pair of values (c_1, c_2), and write the corresponding equivalence class as $[c_1 c_2]$. Roots in the equivalence class [00] lie in the $\tilde{A}_2$ subsystem $\Phi(\boldsymbol{A})$. There are two other types of equivalence class containing positive roots: each of [10], [13] and [23] is a singleton class containing a long root; by contrast each of [01], [11] and [12] is of size six, containing three short roots and three long roots, with the union of the class with $\Phi(\boldsymbol{A}) \cap \Phi^+$ being the set of positive roots of a C_3 subsystem (and the corresponding six root subgroups of G all commute with each other) — indeed in the case of [01] the C_3 subsystem is $\Phi(\boldsymbol{C})$. Taking similarly negative roots (and writing $-[c_1 c_2]$ for $[(-c_1)(-c_2)]$), we see that the 'nonzero' equivalence classes effectively form a root system $\overline{\Phi}$ of type G_2, with positive roots $\mathsf{a} = [01]$, $\mathsf{b} = [10]$, $\mathsf{a} + \mathsf{b} = [11]$, $2\mathsf{a} + \mathsf{b} = [12]$, $3\mathsf{a} + \mathsf{b} = [13]$ and $3\mathsf{a} + 2\mathsf{b} = [23]$. For each such $\mathsf{r} = [c_1 c_2] \in \overline{\Phi}$ there is a root subgroup U_{r} in ${}^3D_4(q)$, of order q^3 or q according as r is short or long.

Now return to the case where $q = 2$. Take $\lambda \in \mathbb{F}_8$ satisfying $\lambda^3 = \lambda + 1$. We choose

$$g_3 = x_4(\lambda) x_{+---}(\lambda^6) x_{+--+}(\lambda^5) h_4(\lambda^2) h_{+---}(\lambda) n_4 n_{+---} n_4 x_4(\lambda^6) x_{+---}(\lambda) x_{+--+}(\lambda^5);$$

a straightforward calculation in $\boldsymbol{A}$ shows that

$${g_3}^F = \tau_3 g_3 = g_3 \tau_3,$$

so that g_3 commutes with τ_3 and satisfies ${g_3}^F.{g_3}^{-1} = \tau_3$. We then have

$$^3D_4(2) = (\boldsymbol{H}^{g_3})^F = (\boldsymbol{H}^{F\tau_3})^{g_3},$$

and the group H is obtained by adjoining $\langle \tau_3 \rangle$ to $^3D_4(2)$.

For each short root $\mathsf{r} \in \bar{\Phi}$, we see that τ_3 cycles both the three short roots and the three long roots in r; if $\alpha \in \Phi$ is one of the long roots, the corresponding elements of $\boldsymbol{H}^{F\tau_3}$ are $x_\alpha(\mu)x_{{\tau_3}^2(\alpha)}(\mu^2)x_{\tau_3(\alpha)}(\mu^4)$ for $\mu \in \mathbb{F}_8$. Conjugation by g_3 gives the root subgroup $U_\mathsf{r} = \{x_\mathsf{r}(t_1, t_2, t_3) : t_1, t_2, t_3 \in \mathbb{F}_2\}$ in $^3D_4(q)$, where

$$x_\mathsf{a}(t_1, t_2, t_3) = x_{3-4}(t_1)x_{3+4}(t_2)x_{1-2}(t_3)x_{+-+-}(t_1+t_2)x_{+-++}(t_1+t_3)x_3(t_2+t_3),$$
$$x_{\mathsf{a}+\mathsf{b}}(t_1, t_2, t_3) = x_{2-4}(t_1)x_{2+4}(t_2)x_{1-3}(t_3)x_{++--}(t_1+t_2)x_{++-+}(t_1+t_3)x_2(t_2+t_3),$$
$$x_{2\mathsf{a}+\mathsf{b}}(t_1, t_2, t_3) = x_{1+4}(t_1)x_{1-4}(t_2)x_{2+3}(t_3)x_{++++}(t_1+t_2)x_{+++-}(t_1+t_3)x_1(t_2+t_3),$$

and $x_{-\mathsf{r}}(t_1, t_2, t_3)$ is obtained from $x_\mathsf{r}(t_1, t_2, t_3)$ by negating all roots. For each long root $\mathsf{r} \in \overline{\Phi}$, the corresponding root $\alpha \in \Phi$ is orthogonal to $\Phi(A)$, whence the root subgroup $\boldsymbol{U}_\alpha < \boldsymbol{H}$ commutes with both g_3 and τ_3; we therefore have the root subgroup $U_\mathsf{r} = \{x_\mathsf{r}(t) : t \in \mathbb{F}_2\}$ in $^3D_4(q)$, where

$$x_\mathsf{b}(t) = x_{2-3}(t), \quad x_{3\mathsf{a}+\mathsf{b}}(t) = x_{1+3}(t), \quad x_{3\mathsf{a}+2\mathsf{b}}(t) = x_{1+2}(t),$$

and $x_{-\mathsf{r}}(t)$ is obtained from $x_\mathsf{r}(t)$ by negating the root.

We may proceed as usual to obtain elements of the maximal torus of $^3D_4(2)$. This is a cyclic subgroup $\langle s \rangle$ of A of order 7, where

$$\begin{aligned}
s &= x_4(1)x_{+--+}(1)n_{+---}n_4x_4(1),\\
s^2 &= x_{+---}(1)x_4(1)n_4n_{+---}n_4,\\
s^3 &= x_{+---}(1)x_{+--+}(1)n_{+---}n_4x_{+--+}(1),\\
s^4 &= x_{+--+}(1)n_4n_{+---}x_{+---}(1)x_{+--+}(1),\\
s^5 &= n_4n_{+---}n_4x_4(1)x_{+---}(1),\\
s^6 &= x_4(1)n_4n_{+---}x_4(1)x_{+--+}(1);
\end{aligned}$$

according as $\mathsf{r} \in \overline{\Phi}$ is short or long, the A_1 subgroup $\langle U_\mathsf{r}, U_{-\mathsf{r}} \rangle$ either contains $\langle s \rangle$ or intersects it trivially, and either

$${}^s x_\mathsf{r}(t_1, t_2, t_3) = x_\mathsf{r}(t_1+t_2, t_1+t_2+t_3, t_2) \quad \text{or} \quad {}^s x_\mathsf{r}(t) = x_\mathsf{r}(t).$$

We also obtain elements $n_\mathsf{r} \in {}^3D_4(q)$ for $\mathsf{r} \in \overline{\Phi}$, where

$$\begin{aligned}
n_\mathsf{a} &= n_{3-4}n_{3+4}n_{1-2}, & n_\mathsf{b} &= n_{2-3},\\
n_{\mathsf{a}+\mathsf{b}} &= n_{2-4}n_{2+4}n_{1-3}, & n_{3\mathsf{a}+\mathsf{b}} &= n_{1+3},\\
n_{2\mathsf{a}+\mathsf{b}} &= n_{1+4}n_{1-4}n_{2+3}, & n_{3\mathsf{a}+2\mathsf{b}} &= n_{1+2};
\end{aligned}$$

the subgroup $N^\dagger = \langle n_{\mathsf{r}} : \mathsf{r} \in \overline{\Phi} \rangle = \langle n_{\mathsf{a}}, n_{\mathsf{b}} \rangle$ of N is dihedral of order 12 and is the Weyl group of ${}^3D_4(q)$, and according as $\mathsf{r} \in \overline{\Phi}$ is short or long we have

$$ {}^{n_{\mathsf{r}}}s = s^{-1} \quad \text{or} \quad {}^{n_{\mathsf{r}}}s = s. $$

Finally the effect of τ_3 on all of these elements of ${}^3D_4(q)$ is as follows: it commutes with each n_{r}, and with $x_{\mathsf{r}}(t)$ if $\mathsf{r} \in \overline{\Phi}$ is long; if $\mathsf{r} \in \overline{\Phi}$ is short then

$$ {}^{\tau_3}x_{\mathsf{r}}(t_1, t_2, t_3) = x_{\mathsf{r}}(t_3, t_1, t_2); $$

and

$$ {}^{\tau_3}s = s^2. $$

5.2.1. *The action of $F_4(2)$ on cosets of $D_4(2).S_3$.* Here we take $H = D_4(2).S_3$. There are 9 subdegrees $|HgH|/|H|$, and they sum to $|G : H| = 3168256$; we shall prove the following.

Proposition 5.1. *In the action of $G = F_4(2)$ on cosets of $H = D_4(2).S_3$, the subdegrees are as given in Table 18, and all suborbits are self-paired.*

In the calculations in this section we shall frequently use Bruhat decomposition (see [3, Corollary 8.4.4]). Given $n \in N$, write $U_n = \prod \{U_\alpha : \alpha \in \Phi^+,\ n(\alpha) \notin \Phi^+\}$ (recall that we identify W with N). Then each element of G has a unique expression in the form unv, where $u \in U$, $n \in N$ and $v \in U_n$; and $unv \in H \iff u, v \in H$.

We shall work through the rows of Table 18 in turn. Taking $g = 1$ clearly gives the suborbit $HgH = H$, and the subdegree is 1.

Take $g = x_1(1)$, $x_{++++}(1)x_1(1)$ or $x_{+++-}(1)x_{++++}(1)x_1(1)$; then $g^2 = 1$, so the suborbit is self-paired. As g centralizes $U \cap H$, given $h = unv \in H$ we have $h \in {}^gH \iff (unv)^g \in H \iff n^g \in H \iff g^{-1}.{}^ng \in H$. We have ${}^ng = x_{n(1)}(1)$, $x_{n(++++)}(1)x_{n(1)}(1)$

g	$\lvert H \cap {}^gH \rvert$	$\frac{\lvert HgH \rvert}{\lvert H \rvert}$
1	1045094400	1
$x_1(1)$	2580480	405
$x_{++++}(1)x_1(1)$	172032	6075
$x_{+++-}(1)x_{++++}(1)x_1(1)$	73728	14175
$x_{++--}(1)x_{+-++}(1)$	10752	97200
$x_{++--}(1)x_{+-++}(1)x_2(1)$	1536	680400
$x_3(1)x_{++--}(1)x_{+-++}(1)x_2(1)$	1536	680400
$x_4(1)x_{+--+}(1)n_{+--+}x_4(1)x_{+---}(1)$	672	1555200
$x_{3+4}(1)x_{+---}(1)x_4(1)x_3(1)x_{+-+-}(1)$ $\times\, n_{3+4}n_{+--+}x_4(1)x_{+--+}(1)x_{3+4}(1)$	7776	134400
		3168256

Table 18. Suborbits and subdegrees for the action of $F_4(2)$ on cosets of $D_4(2).S_3$.

or $x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1)$, respectively; thus $g^{-1}.{}^{n}g \in H \iff {}^{n}g = g \iff n$ preserves the set $\{1\}$, $\{++++, 1\}$ or $\{+++-, ++++, 1\}$, respectively. In the first case this gives $n \in \langle n_{2-3}, n_{3-4}, n_4 \rangle$; so

$$H \cap {}^{g}H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4} \rangle \langle U_{\pm(2-3)}, U_{\pm(3-4)}, U_{\pm(3+4)} \rangle \langle n_4 \rangle,$$

of order $2q^{12}(q^2-1)(q^3-1)(q^4-1) = 2580480$, and the subdegree is 405. In the second case it gives $n \in \langle n_{2-3}, n_{3-4}, n_{+---} \rangle$; so

$$H \cap {}^{g}H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4}, U_{2+3}, U_{2+4}, U_{3+4} \rangle \langle U_{\pm(2-3)}, U_{\pm(3-4)} \rangle \langle n_{+---} \rangle,$$

of order $2q^{12}(q^2-1)(q^3-1) = 172032$, and the subdegree is 6075. In the third case it gives $n \in \langle n_{2-3}, n_4, n_{+---} \rangle$; so

$$H \cap {}^{g}H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4}, U_{2\pm 4}, U_{3\pm 4}, U_{2+3} \rangle \langle U_{\pm(2-3)} \rangle \langle n_4, n_{+---} \rangle,$$

of order $6q^{12}(q^2-1) = 73728$, and the subdegree is 14175.

In the next few cases we shall find it helpful to define a symmetric relation on the set of short roots in Φ^+: given such roots α and β, we say that α is related to β, and write $\alpha \sim \beta$, if $\alpha + \beta$ is another short root in Φ^+.

Now take $g = x_{++--}(1)x_{+-++}(1)$; then $g^{-1} = g^{n_{2-3}n_4}$, so the suborbit is self-paired. Given $h = unv \in H$ we have $h \in {}^{g}H \iff (unv)^g \in H \iff (g^u)^{-1}.{}^{n}({}^{v}g) \in H$. Here g centralizes the root groups $U_{1\pm 2}$, $U_{1\pm 3}$, $U_{1\pm 4}$, U_{2+3}, U_{2+4}, U_{3-4}; so conjugating g by an element of $U \cap H$ gives an element of

$$x_{++--}(1)x_{+-++}(1)U_{++-+}U_{+++-}U_{++++}U_{1+4}U_{1+3}U_{1+2}.$$

Thus if we write

$$S = \{++--, +-++, ++-+, +++-, ++++\},$$

the condition $(g^u)^{-1}.{}^{n}({}^{v}g) \in H$ forces $n(++--), n(+-++) \in S$. As the only instance of roots in S being related is

$$++-- \sim +-++,$$

we see that n must either fix or interchange $++--$ and $+-++$; but if it interchanged them then $(g^u)^{-1}.{}^{n}({}^{v}g)$ would involve $x_1(1)$ which is not in H, so n must fix them, whence $n \in \langle n_{3-4}, n_{2+4} \rangle$. As $v \in U_n$, any short root elements appearing in ${}^{n}({}^{v}g)$ apart from those in g must lie in root subgroups U_α for α of height less than that of $++--$ or $+-++$, whereas any short root elements appearing in $(g^u)^{-1}$ apart from those in g must lie in root subgroups U_α for α of height greater than that of $++--$ or $+-++$; so both u and v must centralize g. Therefore we have

$$H \cap {}^{g}H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4} \rangle \langle U_{\pm(3-4)}, U_{\pm(2+4)} \rangle,$$

of order $q^9(q^2-1)(q^3-1) = 10752$, and the subdegree is 97200.

Now take $g = x_{++--}(1)x_{+-++}(1)x_2(1)$; then $g^{-1} = g^{n_{+---}}$, so the suborbit is self-paired. Given $h = unv \in H$ we again have $h \in {}^gH \iff (unv)^g \in H \iff (g^u)^{-1}.{}^n({}^vg) \in H$. Here g centralizes the root groups U_{1+2}, $U_{1\pm3}$, $U_{1\pm4}$, U_{2+3}, U_{2+4}, U_{3-4}; so conjugating g by an element of $U \cap H$ gives an element of

$$x_{++--}(1)x_{+-++}(1)x_2(1)U_{++-+}U_{+++-}U_{++++}U_1U_{1+4}U_{1+3}U_{1+2}.$$

Thus if we write

$$S = \{++--, +-++, 2, ++-+, +++-, ++++, 1\},$$

the condition $(g^u)^{-1}.{}^n({}^vg) \in H$ forces $n(++--), n(+-++), n(2) \in S$. As the only instances of roots in S being related are

$$++-- \sim +-++ \sim 2,$$

we see that n must fix $+-++$, and either fix or interchange $++--$ and 2; thus $n \in \langle n_{3-4}, n_{+---}\rangle$, and as $v \in U_n \cap H$ we must have $v \in U_{3-4}$ so that v centralizes g. If $n \in \langle n_{3-4}\rangle$ then n also centralizes g, as then must u; if instead $n \in n_{+---}\langle n_{3-4}\rangle$ then as $x_{1-2}(1)x_{3+4}(1)n_{+---}$ centralizes g we see that $x_{1-2}(1)x_{3+4}(1)u$ must. Therefore we have

$$H \cap {}^gH = \langle U_{1+2}, U_{1\pm3}, U_{1\pm4}, U_{2+3}, U_{2+4}, x_{1-2}(1)x_{3+4}(1)n_{+---}\rangle\langle U_{\pm(3-4)}\rangle,$$

of order $q^9(q^2-1) = 1536$, and the subdegree is 680400.

Now take $g = x_3(1)x_{++--}(1)x_{+-++}(1)x_2(1)$; then $g^{-1} = g^{n_{2-3}n_4}$, so the suborbit is self-paired. Given $h = unv \in H$ we again have $h \in {}^gH \iff (unv)^g \in H \iff (g^u)^{-1}.{}^n({}^vg) \in H$. Here g centralizes the root groups U_{1+2}, U_{1+3}, $U_{1\pm4}$, U_{2+3}, U_{2+4}, U_{3-4}, while its commutator with $x_{1-2}(1)x_{1-3}(1)$ is $x_{1+3}(1)x_{1+2}(1)$; so conjugating g by an element of $U \cap H$ gives an element of

$$x_3(1)x_{++--}(1)x_{+-++}(1)x_2(1+t)x_{++-+}(t)U_{+++-}U_{++++}U_1U_{2+3}U_{1+4}U_{1+3}U_{1+2},$$

where the projection of the element of $U \cap H$ on the root group U_{2-3} is $x_{2-3}(t)$. Thus if we write

$$S = \{3, ++--, +-++, 2, ++-+, +++-, ++++, 1\},$$

the condition $(g^u)^{-1}.{}^n({}^vg) \in H$ forces $n(3), n(++--), n(+-++), n(\alpha) \in S$, where α is 2 or $++-+$ according as the projection of v on the root group U_{2-3} is trivial or not. As the only instances of roots in S being related are

$$++-+ \sim 3 \sim ++-- \sim +-++ \sim 2,$$

if α is 2 the chain $n(3), n(++--), n(+-++), n(2)$ must be $3, ++--, +-++, 2$ or its reverse, or $++-+, 3, ++--, +-++$ or its reverse, while if α is $++-+$ the same must be true of the chain $n(++-+), n(3), n(++--), n(+-++)$. In each case this uniquely determines n, which we find lies in $\langle n_{2-3}, n_4, n_{+---}\rangle$; as $v \in U_n$ this forces $v \in U_{2-3}$. If α is 2 we must then have $v = 1$; if however α is $++-+$ we must then have $v = x_{2-3}(1)$, which eliminates two of the four possibilities since they have $n \in \langle n_4, n_{+---}\rangle$. There are therefore six possibilities for n, and for each we may find an element u giving $(g^u)^{-1}.{}^n({}^v g) \in H$: for example, if $n = n_{2-3}n_4$ then $v = 1$ and we may take $u = x_{1-3}(1)x_{2-4}(1)x_{3+4}(1)$, while if $n = n_{2-3}n_{+---}n_4$ then $v = x_{2-3}(1)$ and we may take $u = 1$. Therefore we have

$$H \cap {}^g H = \langle U_{1+2}, U_{1+3}, U_{1\pm4}, U_{2+3}, U_{2+4}, U_{3-4}, x_{1-2}(1)x_{1-3}(1)\rangle \\ .\langle x_{1-3}(1)x_{2-4}(1)x_{3+4}(1)n_{2-3}n_4, n_{2-3}n_{+---}n_4x_{2-3}(1)\rangle,$$

of order $6q^8 = 1536$, and the subdegree is 680400. Note that the center of the 2-point stabilizer here is trivial, while that of the 2-point stabilizer in the previous case is U_{1+2}; so the two suborbits must be distinct.

At this point we note that the remaining two subdegrees sum to 1689600, and of course each divides $|H| = 2^{13}3^65^27$. It is now a simple matter to determine the pairs of factors of $|H|$ with the correct sum (the larger must lie in the range [844800, 1689600), so given each of the 42 possibilities for the powers of 3, 5 and 7 there is at most one for the power of 2): we find they are (1658880, 30720), (1382400, 307200), (1555200, 134400), (860160, 829440), (1075200, 614400) and (1612800, 76800). (In fact by using [29, Theorem 30.1(C)] we could reduce this list of six pairs to the third and fifth, but we shall see that this is unnecessary.)

Now take $g = x_4(1)x_{+--+}(1)n_{+--+}x_4(1)x_{+---}(1)$; then $g^{-1} = gn_4$, so the suborbit is self-paired. Since $g \in A$ it is immediate that g commutes with $\langle U_{\pm(2-3)}, U_{\pm(1+3)}\rangle$, and it certainly commutes with its square n_4; moreover calculation in C shows that $(n_{1-2}n_{3-4}n_{3+4})^g = n_{1-2}n_3$. Thus we have

$$H \cap {}^g H \geq \langle U_{\pm(2-3)}, U_{\pm(1+3)}\rangle\langle n_{1-2}n_{3-4}n_{3+4}, n_4\rangle,$$

of order $4q^3(q^2-1)(q^3-1) = 672$, and the subdegree divides 1555200. Since none of the (nontrivial) 2-point stabilizers found to date contains a group $A_2(q)$ with a graph automorphism, the suborbit is one of the remaining two; as the only one of the twelve possible subdegrees given in the previous paragraph which divides 1555200 is 1555200 itself, we have equality in the previous sentence in both the 2-point stabilizer and the subdegree; moreover the remaining subdegree must be 134400, so the 2-point stabilizer must be of order 7776.

For the final suborbit take

$$g = x_{3+4}(1)x_{+---}(1)x_4(1)x_3(1)x_{+-+-}(1)n_{3+4}n_{+--+}x_4(1)x_{+--+}(1)x_{3+4}(1);$$

then $g^2 = 1$, so the suborbit is self-paired. Clearly g commutes with $\langle U_{\pm(1+2)}\rangle$; calculation shows that g also commutes with n_4, and that conjugation by g multiplies $x_{1-2}(1)x_{3-4}(1)x_{3+4}(1)$ by n_4, and it sends n_{+---} to $x_{3-4}(1)n_{3-4}n_{3+4}x_{3+4}(1)$, and

$x_{2-3}(1)x_{2-4}(1)x_{2+4}(1)x_{1-3}(1)x_{1+3}(1)$ to $x_{2-3}(1)x_{2+3}(1)x_{1-4}(1)x_{1+4}(1)x_{1+3}(1)$. As g has order 2, it follows that each of these elements lies in $H \cap {}^gH$. The last two of them generate a quaternion group of order 8 with center U_{1+2}; adding n_{1+2} extends it to a group $\mathrm{PSU}_3(2)$ of order 72. The group generated by $x_{3-4}(1)n_{3-4}n_{3+4}x_{3+4}(1)$ and its images under $\langle n_4, n_{+---}\rangle$ is $\mathbb{Z}_3 \times \mathbb{Z}_3$ of order 9; adding $x_{1-2}(1)x_{3-4}(1)x_{3+4}(1)$ extends it to $(\mathbb{Z}_3 \times \mathbb{Z}_3).2$ of order 18, and this group normalizes $\mathrm{PSU}_3(2)$. The group $\langle n_4, n_{+---}\rangle$ commutes with $\mathrm{PSU}_3(2)$ and normalizes $(\mathbb{Z}_3 \times \mathbb{Z}_3).2$, so we have a group of order $72 \cdot 18 \cdot 6 = 7776$. Since none of the (nontrivial) 2-point stabilizers found to date has order divisible by 7776, the suborbit is indeed the remaining one. As conjugation by $x_{1-2}(1)x_{3-4}(1)x_{3+4}(1)$ takes the first generator of the quaternion group to the inverse of the second, we have

$$H \cap {}^gH = \langle x_{2-3}(1)x_{2-4}(1)x_{2+4}(1)x_{1-3}(1)x_{1+3}(1), n_{1+2},$$
$$x_{3-4}(1)n_{3-4}n_{3+4}x_{3+4}(1), x_{1-2}(1)x_{3-4}(1)x_{3+4}(1), n_4, n_{+---}\rangle,$$

of order 7776, and the subdegree is 134400.

This concludes the proof of Proposition 5.1.

5.2.2. *The action of $F_4(2)$ on cosets of ${}^3D_4(2).3$.* Here we take $H = {}^3D_4(2).3$. There are 7 subdegrees $|HgH|/|H|$, and they sum to $|G : H| = 5222400$; we shall prove the following.

Proposition 5.2. *In the action of $G = F_4(2)$ on cosets of $H = {}^3D_4(2).3$, the subdegrees are as given in Table 19, and all suborbits are self-paired.*

In the calculations in this section we shall again use Bruhat decomposition, but here we shall require a slightly different form. First set

$$U_A = U \cap A = \prod\{U_\alpha : \alpha \in \Phi^+ \cap \Phi(\boldsymbol{A})\}, \quad U' = \prod\{U_\alpha : \alpha \in \Phi^+ \setminus \Phi(\boldsymbol{A})\};$$

then $U = U_A U' = U' U_A$ and $U_A \cap U' = \{1\}$. Next recall that we have the Weyl groups $W_{\boldsymbol{A}}$ and $W_{\boldsymbol{H}}$, with $W = W_{\boldsymbol{A}} W_{\boldsymbol{H}} = W_{\boldsymbol{H}} W_{\boldsymbol{A}}$ and $W_{\boldsymbol{A}} \cap W_{\boldsymbol{H}} = \{1\}$. Now given

g	$\lvert H \cap {}^gH\rvert$	$\frac{\lvert HgH\rvert}{\lvert H\rvert}$
1	634023936	1
$x_{+++-}(1)x_{++++}(1)x_1(1)$	36864	17199
n_{+--+}	36288	17472
$x_{+++-}(1)x_{++++}(1)x_1(1)n_{+--+}$	576	1100736
$x_{++--}(1)x_{+-++}(1)$	768	825552
$x_4(1)n_{3-4}$	216	2935296
$x_{1-2}(1)x_4(1)n_{3-4}$	1944	326144
		5222400

Table 19. Suborbits and subdegrees for the action of $F_4(2)$ on cosets of ${}^3D_4(2).S_3$.

an element of $N = W$ we may write it uniquely as $n_A n$ with $n_A \in W_{\boldsymbol{A}}$, $n \in W_{\boldsymbol{H}}$; as before we have $U_{n_A n} = \prod\{U_\alpha : \alpha \in \Phi^+,\ n_A n(\alpha) \notin \Phi^+\}$. Set

$$(U_{n_A n})_A = \prod\{U_\alpha : \alpha \in \Phi^+,\ n_A n(\alpha) \notin \Phi^+,\ n(\alpha) \in \Phi(\boldsymbol{A})\},$$
$$(U_n)' = \prod\{U_\alpha : \alpha \in \Phi^+,\ n_A n(\alpha) \notin \Phi^+,\ n(\alpha) \notin \Phi(\boldsymbol{A})\}$$
$$= \prod\{U_\alpha : \alpha \in \Phi^+,\ n(\alpha) \notin \Phi^+ \cup \Phi(\boldsymbol{A})\}$$

(where the final line justifies the notation $(U_n)'$); then $U_{n_A n} = (U_{n_A n})_A (U_n)' = (U_n)'(U_{n_A n})_A$ and $(U_{n_A n})_A \cap (U_n)' = \{1\}$. Standard Bruhat decomposition now shows that any element of G may be uniquely written as $(uu_A)(n_A n)(v_A v)$ with $u \in U'$, $u_A \in U_A$, $n_A \in W_{\boldsymbol{A}}$, $n \in W_{\boldsymbol{H}}$, $v_A \in (U_{n_A n})_A$ and $v \in (U_n)'$; setting $a = u_A n_A{}^n v_A \in A$ we see that the element may be written as $uanv$ with $u \in U'$, $a \in A$, $n \in W_{\boldsymbol{H}}$ and $v \in (U_n)'$.

To see that this last expression is unique, suppose we have $uanv = \hat{u}\hat{a}\hat{n}\hat{v}$ with $u, \hat{u} \in U'$, $a, \hat{a} \in A$, $n, \hat{n} \in W_{\boldsymbol{H}}$ and $v, \hat{v} \in (U_n)'$. Writing the left side in standard Bruhat decomposition gives an expression $(uu_A)(n_A n)(v_A v)$, where $u_A \in U \cap A$, $n_A \in W_{\boldsymbol{A}}$ and $v_A \in U_{n_A n}$; doing the same with the right side and using uniqueness of standard Bruhat decomposition we see that $uu_A = \hat{u}\hat{u}_A \in U$ and $n_A n = \hat{n}_A \hat{n} \in N$, and then the factorizations of the previous paragraph give $u = \hat{u}$ and $n = \hat{n}$. Therefore $anv = \hat{a}n\hat{v}$, so $\hat{a}^{-1}a = n\hat{v}v^{-1}n^{-1}$; the left side here is in A and the right side is in ${}^n(U_n)'$, which is a product of negative root subgroups each lying outside A, so both sides must be the identity, whence $a = \hat{a}$ and $v = \hat{v}$. We therefore have uniqueness in the expression $uanv$. Since by [3, Proposition 13.5.3] each element of H may be written in the form $u_1 s^i n_1 v_1 \tau_3{}^j = u_1 s^i \tau_3{}^j n_1 v_1{}^{\tau_3{}^j}$, where $u_1 \in U \cap H < U'$, $s^i \tau_3{}^j \in A \cap H$, $n_1 \in N^\dagger < W_{\boldsymbol{H}}$ and $v_1{}^{\tau_3{}^j} \in U_{n_1} \cap H < (U_{n_1})'$, we see that $uanv \in H \iff u, a, n, v \in H$.

In some cases our approach will require the following. We have the Weyl group $W_{\boldsymbol{C}}$ of $\boldsymbol{C}$, and $|W : W_{\boldsymbol{C}}| = 24$. Recall from [3, Theorem 2.5.8, Corollary 2.5.9] that there is a set of right coset representatives of $W_{\boldsymbol{C}}$ in W, each of which is of minimal length in its coset; we shall call these $n_{(1)}, \dots, n_{(24)}$. (Note that [3] uses left cosets instead of right cosets, so our elements are the inverses of those there.) Likewise we have $|W_{\boldsymbol{H}} : W_{\boldsymbol{H}} \cap W_{\boldsymbol{C}}| = 24$, where $W_{\boldsymbol{H}} \cap W_{\boldsymbol{C}}$ is the Weyl group of $\boldsymbol{H} \cap \boldsymbol{C}$, which is a Levi subgroup of $\boldsymbol{H}$ of type $A_1{}^3$; if we regard $\epsilon_1 - \epsilon_2$, $\epsilon_2 - \epsilon_3$, $\epsilon_3 - \epsilon_4$, $\epsilon_3 + \epsilon_4$ as simple roots of $\boldsymbol{H}$ then again we have coset representatives of $W_{\boldsymbol{H}} \cap W_{\boldsymbol{C}}$ in $W_{\boldsymbol{H}}$ of minimal length, which we shall call $n_{(1)}{}', \dots, n_{(24)}{}'$. We choose notation such that for all j we have $n_{(j)}{}' \in W_{\boldsymbol{C}} n_{(j)}$.

Now as $N^\dagger \cap W_{\boldsymbol{C}} = \langle n_{\mathsf{a}} \rangle$ we have $|N^\dagger : N^\dagger \cap W_{\boldsymbol{C}}| = 6$, so that just six of the cosets $W_{\boldsymbol{C}} n_{(j)}$ contain elements of $N^\dagger$; write $J = \{j \leq 24 : N^\dagger \cap W_{\boldsymbol{C}} n_{(j)} \neq \varnothing\}$. The elements $n_{(j)}{}'$ for $j \in J$ are $1, n_{\mathsf{b}}, n_{\mathsf{b}} n_{\mathsf{a}}, n_{\mathsf{a+b}}, n_{\mathsf{a+b}} n_{\mathsf{a}}, n_{\mathsf{3a+2b}}$, each of which in fact lies in $N^\dagger$; we have $n_{(j)}{}' = n_{(j)}$ in the first, second, fifth and sixth of these cases and ${n_{(j)}}' = n_{+--+}n_{(j)}$ in the third and fourth. As roots in $\Phi(\boldsymbol{A})$ are either fixed or negated by elements of $N^\dagger$, we see that for $j \in J$ we have either $U_{{n_{(j)}}'} = U_{n_{(j)}}$ or $U_{{n_{(j)}}'} = U_A U_{n_{(j)}}$, whence in either case $(U_{{n_{(j)}}'})' = U_{n_{(j)}}$.

Write $P = UC$, so that P is a maximal parabolic subgroup of G. For each $j \le 24$ we have $Pn_{(j)} = P{n_{(j)}}'$; moreover G is the disjoint union of the double cosets $Pn_{(j)}U$ for $j \le 24$, and for fixed j the double coset $Pn_{(j)}U$ is the disjoint union of the cosets $Pn_{(j)}v$ as v runs through $U_{n_{(j)}}$. Given such a coset $Pn_{(j)}v$, if $j \in J$ and $v \in H$ then evidently the coset meets H; we claim that the converse is also true.

Thus suppose we have $h \in Pn_{(j)}v \cap H$, and as above write $h = uanv_0$ with $u \in U' \cap H$, $a \in A \cap H$, $n \in N^\dagger$ and $v_0 \in (U_n)' \cap H$; then $Pn_{(j)}v = Ph = Pnv_0$, so $n \in W_{\boldsymbol{C}}n_{(j)} = W_{\boldsymbol{C}}{n_{(j)}}'$, whence $j \in J$ and n is either ${n_{(j)}}'$ or $n_{\mathsf{a}}{n_{(j)}}'$. In the former case we have $v_0 \in (U_{{n_{(j)}}'})' = U_{n_{(j)}}$, so as $Pn_{(j)}v = P{n_{(j)}}'v_0 = Pn_{(j)}v_0$ we must have $v = v_0 \in H$ as required. In the latter case we have $v_0 \in (U_{n_{\mathsf{a}}{n_{(j)}}'})'$, and we may write $nv_0 = n_{\mathsf{a}}{n_{(j)}}'v_0 = n_{\mathsf{a}}v_1.{n_{(j)}}'v_2$, where $v_1 \in \prod_{\alpha\in[01]} U_\alpha < P$ and $v_2 \in (U_{{n_{(j)}}'})' = U_{n_{(j)}}$; as ${v_1}^{{n_{(j)}}'}.v_2 = v_0 \in H$ and the sets of roots involved in ${v_1}^{{n_{(j)}}'}$ and v_2 are disjoint, both v_1 and v_2 must lie in H. Thus

$$Pn_{(j)}v = Pnv_0 = P{n_{(j)}}'v_2 = Pn_{(j)}v_2$$

with $v_2 \in U_{n_{(j)}}$, so we must have $v = v_2 \in H$ as required. We have thus shown that the converse is indeed true; we shall use this in some of the arguments in this section.

We shall work through the rows of Table 19 in turn. Taking $g = 1$ clearly gives the suborbit $HgH = H$, and the subdegree is 1.

Take $g = x_{+++-}(1)x_{++++}(1)x_1(1)$; then $g^2 = 1$, so the suborbit is self-paired. As g centralizes $U \cap H$, given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g \in H \iff (an)^g \in H \iff (g^a)^{-1}.{}^ng \in H$. We have ${}^ng = x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1)$, while τ_3 and g commute and $\langle s\rangle$ acts simply transitively on $U_{+++-}U_{++++}U_1 \setminus \{1\}$; thus $(g^a)^{-1}.{}^ng \in H \iff g^a = {}^ng = g \iff a \in \langle\tau_3\rangle$ and n preserves the set $\{+++-, ++++, 1\}$, which forces $n \in \langle n_{\mathsf{b}}\rangle$. So

$$H \cap {}^gH = \langle U \cap H, n_{\mathsf{b}}\rangle\langle\tau_3\rangle,$$

of order $3q^{12}(q^2-1) = 36864$, and the subdegree is 17199.

Take $g = n_{+--+}$; then $g^2 = 1$, so the suborbit is self-paired. As g preserves U', A and $N^\dagger$, given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g = u^ga^gn^gv^g \in H \iff u^g, a^g, n^g, v^g \in H$. For all $n \in N^\dagger$ we have $n^g = n$; calculation shows that $s^g \notin H$, while $(\tau_3)^g = {\tau_3}^{-1}$; and given $\mathsf{r} \in \overline{\Phi}$, if r is long then g commutes with U_{r}, while if r is short then g preserves the set of roots α lying in r, and ${U_{\mathsf{r}}}^g \cap H = \langle x_{\mathsf{r}}(1,1,1)\rangle$. So

$$H \cap {}^gH = \langle U_{\mathsf{b}}, U_{-\mathsf{b}}, x_{\mathsf{a}}(1,1,1), x_{-\mathsf{a}}(1,1,1)\rangle\langle\tau_3\rangle,$$

of order $3q^6(q^2-1)(q^6-1) = 36288$, and the subdegree is 17472.

Now take $g = x_{+++-}(1)x_{++++}(1)x_1(1)n_{+--+}$; then $g^2 = 1$, so the suborbit is self-paired. Write $g' = x_{+++-}(1)x_{++++}(1)x_1(1)$; using the comments of the previous two paragraphs about g' and n_{+--+} centralizing and inverting certain elements, we see that given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g \in H \iff (u^{n_{+--+}})g'(a^{n_{+--+}})ng'v^{n_{+--+}} \in H$, with $(u^{n_{+--+}})g' \in U'$ and $a^{n_{+--+}} \in A$. If $n(1) \notin \Phi^+$ then as $n \in N^\dagger$ and $g'v^{n_{+--+}} \in (U_n)'$ the expression is in the desired form; if instead $n(1) \in \Phi^+$ we rewrite it as

$$(u^{n_{+--+}})g'(x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1))^{a^{n_{+--+}}}.a^{n_{+--+}}.n.v^{n_{+--+}}.$$

In either case the element of A is $a^{n_{+--+}}$; for this to be in H we require $a \in \langle \tau_3 \rangle$, so $a^{n_{+--+}} \in \langle \tau_3 \rangle$. Now observe that if $\mathsf{r} \in \overline{\Phi}$ is a short root, then in each element of U_r the sum of the coefficients in the three short root subgroups is 0, and n_{+--+} permutes these coefficients. Thus $g'v^{n_{+--+}}$ cannot lie in H, so if $n(1) \notin \Phi^+$ we cannot have $(uanv)^g \in H$; assuming $n(1) \in \Phi^+$ we have $(uanv)^g \in H \iff u^{n_{+--+}}g'(x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1))^{a^{n_{+--+}}}$, $v^{n_{+--+}} \in H$, and the first of these implies that n preserves the set $\{+++-, ++++, 1\}$, which forces $n \in \langle n_\mathsf{b} \rangle$. It now follows that

$$H \cap {}^gH = \langle x_\mathsf{a}(1,1,1), x_{\mathsf{a+b}}(1,1,1), x_{\mathsf{2a+b}}(1,1,1)\rangle U_{\mathsf{3a+b}}U_{\mathsf{3a+2b}}\langle U_{\pm\mathsf{a}}\rangle\langle\tau_3\rangle,$$

of order $3q^6(q^2-1) = 576$, and the subdegree is 1100736.

Now take $g = x_{++--}(1)x_{+-++}(1)$; then $g^{-1} = g^{x_{\mathsf{a+b}}(0,1,1)}$, so the suborbit is self-paired. Given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g \in H \iff (g^{ua})^{-1}({}^n({}^vg)) \in H$. Here g centralizes the root groups $U_{\mathsf{2a+b}}, U_{\mathsf{3a+b}}, U_{\mathsf{3a+2b}}$; so conjugating g by an element of $U \cap H$ gives an element of

$$x_{++--}(1)x_{+-++}(1)U_{++-+}U_{+++-}U_{++++}U_1U_{1+4}U_{1+3}U_{1+2}.$$

Thus the projection of ${}^vg$ on the product of the root groups U_α for $\alpha \in \mathsf{a}$ or for $\alpha \in \mathsf{a+b}$ involves short root elements but no long root elements; so the condition $(g^{ua})^{-1}({}^n({}^vg)) \in H$ forces $n \in N^\dagger$ to send both a and $\mathsf{a+b}$ to positive roots in $\overline{\Phi}$, whence $n \in \langle n_\mathsf{b}\rangle$ and so $v \in U_\mathsf{b}$. A straightforward calculation shows that $h' = x_{\mathsf{a+b}}(1,0,1)x_\mathsf{b}(1)n_\mathsf{b}s^2\tau_3 \in H \cap {}^gH$. Thus if $n = n_\mathsf{b}$, according as $v = 1$ or $v = x_\mathsf{b}(1)$ we may multiply h on the right by h'^{-1} or h' to reduce to the case where $n = 1$; so we may assume $(g^{ua})^{-1}g \in H$. The projection of g^u on the product of the root groups whose roots lie in a is $x_{+-++}(1)$, which thus must be centralized by a; this forces $a \in \{1, s^3\tau_3, s^2{\tau_3}^2\}$. Likewise the projection of g^u on the product of the root groups whose roots lie in $\mathsf{a+b}$ is either $x_{++--}(1)$ or $x_{++--}(1)x_{++-+}(1)$; since conjugating each of these by either $s^3\tau_3$ or $s^2{\tau_3}^2$ gives a term $x_2(1)$, we must have $a = 1$, so $(g^u)^{-1}g \in H$. After calculation in $U \cap H$ it now follows that

$$H \cap {}^gH = \{x_\mathsf{a}(t_1,t_2,t_3)x_{\mathsf{a+b}}(t_2+t_3, t_1+t_2, t_1) : t_i \in \mathbb{F}_2\}U_{\mathsf{2a+b}}U_{\mathsf{3a+b}}U_{\mathsf{3a+2b}}$$
$$.\langle x_{\mathsf{a+b}}(1,0,1)x_\mathsf{b}(1)n_\mathsf{b}s^2\tau_3\rangle,$$

of order $3q^8 = 768$, and the subdegree is 825552.

Finally take $g = x_4(1)n_{3-4}$ or $x_{1-2}(1)x_4(1)n_{3-4}$; then $g^{-1} = g^{x_{\mathsf{a}}(0,0,1)}$, so the suborbit is self-paired. Here we shall use the approach involving cosets of P. Take $h = uanv \in H$ as usual and consider $(uanv)^g \in (Pnv)^g = Pnvn_{3-4}x_3(1)x_{1-2}(\delta)$ where $\delta = 0$ or 1. If v involves the term $x_{3-4}(1)$ we must have either $n \in \langle n_{\mathsf{a}}\rangle n_{\mathsf{b}}n_{\mathsf{a}}$ or $n \in \langle n_{\mathsf{a}}\rangle n_{\mathsf{a+b}}n_{\mathsf{a}}$. In the former possibility we have $v = x_{\mathsf{a}}(1, t_2, t_3)x_{3\mathsf{a+b}}(t_4)$ for some $t_i \in \mathbb{F}_2$: conjugating each term in v other than $x_{3-4}(1)$ by n_{3-4} and using the relation $n_{3-4}x_{3-4}(1)n_{3-4} = x_{3-4}(1)n_{3-4}x_{3-4}(1)$ gives the coset

$$Pn_{2-3}n_{1-2}n_{3+4}x_{3-4}(1)n_{3-4}x_{3-4}(1)x_{3+4}(t_2)x_{1-2}(t_3)$$
$$\times\, x_{+--+}(1+t_2)x_{+-++}(1+t_3)x_4(t_2+t_3)x_{1+4}(t_4)x_3(1)x_{1-2}(\delta);$$

now moving all possible terms to the left gives

$$Pn_{2-3}n_{1-2}n_{3+4}n_{3-4}x_{3-4}(1)x_{3+4}(t_3)x_{1-2}(1+t_2+t_3+\delta)$$
$$\times\, x_{+-+-}(1+t_2)x_{+-++}(1+t_2t_3)x_3(1+t_2+t_3)x_{1+3}(t_4).$$

The sum of the coefficients in the three short root subgroups is $1+t_2t_3+t_3$, which is 0 only when $t_3 = 1$ and $t_2 = 0$; but then the sum of the coefficients in the three root subgroups U_{3-4}, U_{3+4} and U_{+-+-} is not 0. Thus the product of the root elements with roots in a does not lie in H, so the coset $Pnvn_{3-4}x_3(1)$ contains no elements of H. In the latter possibility we have $v = x_{\mathsf{a}}(1, t_2, t_3)x_{2\mathsf{a+b}}(t_4, t_5, t_6)x_{3\mathsf{a+b}}(t_7)x_{3\mathsf{a+2b}}(t_8)$ for some $t_i \in \mathbb{F}_2$; although the expression is more complicated, the above approach gives on the right the same root elements with roots in a, and thus yields the same conclusion. Thus we may assume that v does not involve the term $x_{3-4}(1)$, so the coset is $Pnn_{3-4}(v^{n_{3-4}})x_3(1)x_{1-2}(\delta)$. Unless $n \in \langle n_{\mathsf{a}}\rangle$ or $\langle n_{\mathsf{a}}\rangle n_{3\mathsf{a+2b}}$ we see that nn_{3-4} is an element $n_{(j)}{}'$ for some $j \notin J$, so again the coset contains no elements of H; we have therefore reduced to the possibilities where $n \in \{1, n_{\mathsf{a}}, n_{3\mathsf{a+2b}}, n_{\mathsf{a}}n_{3\mathsf{a+2b}}\}$. From now on we treat the two cases $\delta = 0$ and $\delta = 1$ separately, although we shall see that there are considerable similarities between them.

First assume $\delta = 0$, so that $g = x_4(1)n_{3-4}$. If $n = 1$ then $v = 1$; a straightforward calculation shows that $\{ua : (ua)^g \in H\}$ is

$$Q = \langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(1,1,1)x_{2\mathsf{a+b}}(0,0,1),\ x_{\mathsf{a+b}}(1,0,0)x_{2\mathsf{a+b}}(1,0,1)x_{3\mathsf{a+b}}(1)\rangle,$$

a quaternion group with center $U_{3\mathsf{a+2b}}$. If $n = n_{\mathsf{a}}$ we obtain two further cosets of Q, containing the element $h_0 = x_{\mathsf{a}}(1,1,1)sn_{\mathsf{a}}x_{\mathsf{a}}(0,1,1)$ and its inverse (note that h_0 has order 3, and centralizes Q). If $n = n_{3\mathsf{a+2b}}$ then as $g \in C$ it commutes with n, so we obtain all elements q_1nq_2 with $q_1, q_2 \in Q$. Finally if $n = n_{\mathsf{a}}n_{3\mathsf{a+2b}}$ we obtain all elements $h_0{}^{\pm 1}q_1nq_2$. Thus

$$H \cap {}^gH = \big\langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(1,1,1)x_{2\mathsf{a+b}}(0,0,1),\ x_{\mathsf{a+b}}(1,0,0)x_{2\mathsf{a+b}}(1,0,1)x_{3\mathsf{a+b}}(1),$$
$$x_{\mathsf{a}}(1,1,1)sn_{\mathsf{a}}x_{\mathsf{a}}(0,1,1),\ n_{3\mathsf{a+2b}}\big\rangle,$$

of order $q^3(q^2-1)(q^3+1) = 216$, and the subdegree is 2935296.

Now assume $\delta = 1$, so that $g = x_{1-2}(1)x_4(1)n_{3-4}$. Here we first note that $(x_{\mathsf{a}}(0,1,1)s^3\tau_3)^g = s^3\tau_3$, so it suffices to work in ${}^3D_4(2)$. If $n = 1$ then $v = 1$; a straightforward calculation shows that $\{us^i : (us^i)^g \in H\}$ is

$$Q = \langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(0,0,1)x_{\mathsf{2a+b}}(0,0,1),\, x_{\mathsf{a+b}}(1,1,0)x_{\mathsf{2a+b}}(0,1,0)x_{\mathsf{3a+b}}(1)\rangle,$$

a quaternion group with center $U_{\mathsf{3a+2b}}$. If $n = n_{\mathsf{a}}$ we obtain eight further cosets of Q, containing the element $h_0 = x_{\mathsf{a}}(1,0,0)n_{\mathsf{a}}x_{\mathsf{a}}(0,1,0)$ and its powers (note that h_0 has order 9, and normalizes Q). If $n = n_{\mathsf{3a+2b}}$ then as $g \in C$ it commutes with n, so we obtain all elements q_1nq_2 with $q_1, q_2 \in Q$. Finally if $n = n_{\mathsf{a}}n_{\mathsf{3a+2b}}$ we obtain all elements ${h_0}^i q_1 n q_2$ for $1 \le i \le 8$. Thus

$$H \cap {}^gH = \big\langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(0,0,1)x_{\mathsf{2a+b}}(0,0,1),\, x_{\mathsf{a+b}}(1,1,0)x_{\mathsf{2a+b}}(0,1,0)x_{\mathsf{3a+b}}(1),\\ x_{\mathsf{a}}(1,0,0)n_{\mathsf{a}}x_{\mathsf{a}}(0,1,0),\, n_{\mathsf{3a+2b}}\big\rangle\langle x_{\mathsf{a}}(0,1,1)s^3\tau_3\rangle,$$

of order $9q^3(q^2-1)(q^3+1) = 1944$, and the subdegree is 326144.

This concludes the proof of Proposition 5.2.

5.3. *Distance-transitive graphs.* We recall further that in the action of G on the left cosets of a maximal subgroup H, given a self-paired suborbit corresponding to a double coset HgH which is not simply H itself, we may obtain a graph as follows: the vertices are the left cosets $g'H$ for $g' \in G$, and there is an edge between the vertices $g'H$ and $g''H$ if and only if $g'^{-1}g'' \in HgH$ (note that this makes sense because the suborbit is self-paired). The graph is regular, of valency $|HgH|/|H|$; it is connected as H is a maximal subgroup of G; and G acts transitively on it.

If we consider the vertex H itself, the vertices at distance 1 from H are the cosets lying in HgH, those at distance 2 from H are those lying in $HgHgH$ which are not at distance 0 or 1, and so on. Writing r for the rank of the action, the graph is distance-transitive if, for each $i < r$, the left cosets at distance i from H form a single suborbit; in this case we may order the subdegrees $k_0, k_1, \dots, k_{r-1}$ so that the number of left cosets at distance i from H is k_i (so that $k_0 = 1$, and $k_1 = |HgH|/|H|$).

Our goal here is to show that for no choice of suborbit HgH the graph is distance-transitive. To do this we shall make use of [2, Proposition 5.1.1], which among other things implies the following: if the graph as above is distance-transitive with $r \ge 4$,

(i) there exist h, l with $1 \le h \le l \le r-1$ such that $1 < k_1 < \cdots < k_h = \cdots = k_l > \cdots > k_{r-1}$, and

(ii) if $i < j$ and $i + j \le r-1$ then $k_i \le k_j$.

5.3.1. *The action of $F_4(2)$ on cosets of $D_4(2).S_3$.* Here we take $H = D_4(2).S_3$. We have $r = 9$; our result is the following.

Proposition 5.3. *The action of $G = F_4(2)$ on cosets of $H = D_4(2).S_3$ gives rise to no distance-transitive graph.*

Proof. Suppose the statement is false. The smallest nontrivial subdegree is 405, corresponding to the suborbit HgH where $g = x_1(1)$. If we had $k_1 = 405$ then $HgHgH$ would contain only one double coset other than H or HgH; but we have

$$x_1(1)x_{++++}(1) = g.n_{+---}.g.n_{+---} \in HgHgH$$

and

$$x_{++--}(1)x_{+-++}(1) = n_{+-++}.g.n_{+-++}n_{++--}.g.n_{++--} \in HgHgH,$$

and the two left-hand sides lie in the double cosets corresponding to subdegrees 6075 and 97200. Thus by (i) above we must have $k_8 = 405$. The next smallest subdegree is 6075, corresponding to the suborbit HgH where $g = x_1(1)x_{++++}(1)$. By (ii) above we cannot have $k_7 = 6075$ as this would force $k_1 > k_7$, so by (i) above we must have $k_1 = 6075$; but

$$x_1(1) = n_{+-++}x_{-3-4}(1).g.x_{-3-4}(1).g.x_{1+2}(1)n_{+-++} \in HgHgH,$$

so that $x_1(1)H$ would be at distance 2 from H instead of 8. This contradiction proves the result. □

5.3.2. *The action of $F_4(2)$ on cosets of ${}^3D_4(2).3$.* Here we take ${}^3D_4(2).3$. We have $r = 7$; our result is the following.

Proposition 5.4. *The action of $G = F_4(2)$ on cosets of $H = {}^3D_4(2).3$ gives rise to no distance-transitive graph.*

Proof. Suppose the statement is false. The smallest nontrivial subdegree is 17199, corresponding to the suborbit HgH where $g = x_{+++-}(1)x_{++++}(1)x_1(1)$; as already mentioned, $\langle s\rangle$ acts simply transitively on $U_{+++-}U_{++++}U_1 \setminus \{1\}$, and indeed $g^{s^2} = x_1(1)$. We have $x_3(1) = x_1(1)^{n_{\mathsf{a}}n_{\mathsf{b}}} \in HgH$, and then

$$n_{3-4} = x_{-\mathsf{a}}(1,1,1)s.x_3(1).x_{\mathsf{a}}(1,1,1)n_{\mathsf{a}}sx_{\mathsf{a}}(1,1,1) \in HgH;$$

thus $n_4(1)n_{3-4} = n_{3-4}n_3(1) \in HgHgH$. It follows that if we had $k_1 = 17199$ then we would have $k_2 = 2935296$; but this is the largest subdegree, so we would have $k_2 > k_3$, contrary to (ii) above. Thus by (i) above we must have $k_6 = 17199$. The next smallest subdegree is 17472, corresponding to the suborbit HgH where $g = n_{+--+}$. By (ii) above we cannot have $k_5 = 17472$ as this would force $k_1 > k_5$, so by (i) above we must have $k_1 = 17472$; but

$$x_{+++-}(1)x_{++++}(1)x_1(1) = s^5\tau_3.g.x_{2\mathsf{a}+\mathsf{b}}(0,0,1).g.x_{2\mathsf{a}+\mathsf{b}}(1,0,0)\tau_3{}^2s^2 \in HgHgH,$$

so that $x_{+++-}(1)x_{++++}(1)x_1(1)H$ would be at distance 2 from H instead of 6. This contradiction proves the result. □

Acknowledgement

The author wishes to express his gratitude to the anonymous referee, attention to whose comments has substantially improved the paper.

References

[1] R. W. Baddeley, "Multiplicity-free and self-paired primitive permutation groups", *J. Algebra* **162**:2 (1993), 482–530. MR Zbl

[2] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Ergebnisse der Math. (3) **18**, Springer, 1989. MR Zbl

[3] R. W. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics **28**, Wiley, London, 1972. MR Zbl

[4] R. W. Carter, *Finite groups of Lie type*: *conjugacy classes and complex characters*, Wiley, New York, 1985. MR Zbl

[5] D. I. Deriziotis and G. O. Michler, "Character table and blocks of finite simple triality groups ${}^3D_4(q)$", *Trans. Amer. Math. Soc.* **303**:1 (1987), 39–70. MR Zbl

[6] F. Digne and J. Michel, *Fonctions L des variétés de Deligne–Lusztig et descente de Shintani*, Mém. Soc. Math. France (N.S.) **20**, 1985. MR Zbl

[7] M. Geck, "Computing Green functions in small characteristic", *J. Algebra* **561** (2020), 163–199. MR Zbl

[8] J. A. Green, "The characters of the finite general linear groups", *Trans. Amer. Math. Soc.* **80** (1955), 402–447. MR Zbl

[9] B. Hartley and M. Kuzucuoḡlu, "Centralizers of elements in locally finite simple groups", *Proc. London Math. Soc.* (3) **62**:2 (1991), 301–324. MR Zbl

[10] T. Kondo, "The characters of the Weyl group of type F_4", *J. Fac. Sci. Univ. Tokyo Sect. I* **11** (1965), 145–153. MR Zbl

[11] L. Lambe and B. Srinivasan, "A computation of Green functions for some classical groups", *Comm. Algebra* **18**:10 (1990), 3507–3545. MR Zbl

[12] R. Lawther, *On certain coset actions in finite groups of Lie type*, Ph.D. thesis, University of Cambridge, 1990.

[13] R. Lawther, "Double cosets involving involutions in algebraic groups", *Proc. London Math. Soc.* (3) **70**:1 (1995), 115–145. MR Zbl

[14] R. Lawther, "Jordan block sizes of unipotent elements in exceptional algebraic groups", *Comm. Algebra* **23**:11 (1995), 4125–4156. MR Zbl

[15] R. Lawther, "The action of $F_4(q)$ on cosets of $B_4(q)$", *J. Algebra* **212**:1 (1999), 79–118. MR Zbl

[16] R. Lawther, "Finiteness of double coset spaces", *Proc. London Math. Soc.* (3) **79**:3 (1999), 605–625. MR Zbl

[17] R. Lawther, "Double cosets in F_4", *J. Algebra* **607** (2022), 499–530. MR Zbl

[18] M. W. Liebeck and G. M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs **180**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[19] F. Lübeck, "Green functions in small characteristic", preprint, 2024. arXiv 2403.18190v1

[20] G. Lusztig, "Character sheaves on disconnected groups, I", *Represent. Theory* **7** (2003), 374–403. MR Zbl

[21] G. Malle, "Generalized Deligne–Lusztig characters", *J. Algebra* **159**:1 (1993), 64–97. MR Zbl

[22] G. Malle, "Green functions for groups of types E_6 and F_4 in characteristic 2", *Comm. Algebra* **21**:3 (1993), 747–798. MR Zbl

[23] C. E. Praeger, J. Saxl, and K. Yokoyama, "Distance transitive graphs and finite simple groups", *Proc. London Math. Soc.* (3) **55**:1 (1987), 1–21. MR Zbl

[24] K.-i. Shinoda, "The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic 2", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **21** (1974), 133–159. MR Zbl

[25] T. Shoji, "The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic $p \neq 2$", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **21** (1974), 1–17. MR Zbl

[26] T. Shoji, "On the Green polynomials of a Chevalley group of type F_4", *Comm. Algebra* **10**:5 (1982), 505–543. MR Zbl

[27] T. Shoji, "Generalized Green functions and unipotent classes for finite reductive groups, II", *Nagoya Math. J.* **188** (2007), 133–170. MR Zbl

[28] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. **80**, Amer. Math. Soc., Providence, RI, 1968. MR Zbl

[29] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964. MR Zbl

Received July 31, 2024. Revised April 18, 2025.

R. LAWTHER
FACULTY OF MATHEMATICS
UNIVERSITY OF CAMBRIDGE
CAMBRIDGE
UNITED KINGDOM

ril10@cam.ac.uk

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 336 No. 1-2 May 2025

In memoriam
Gary Seitz