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**CONSTRUCTIBLE REPRESENTATIONS
AND CATALAN NUMBERS**

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Dedicated to the memory of Gary Seitz

We establish a connection between constructible representations (arising in the study of left cells in Weyl groups) and Catalan numbers.

0. Introduction

0.1. The sequence of Catalan numbers is the sequence Cat_n , $n = 1, 2, 3, \dots$, where $\text{Cat}_n = (2n)!/(n!(n+1)!)$. According to [3], Catalan numbers first appeared in the work of Ming Antu (1692–1763). They were rediscovered by Euler (1707–1783). See also [13].

In this paper we give a new way in which Catalan numbers appear in connection with Lie theory.

0.2. Let G be a connected reductive algebraic group of adjoint type over \mathbb{C} whose Weyl group W is assumed to be irreducible. Let \widehat{W} be the set of (isomorphism classes of) irreducible representations (over \mathbb{Q}) of W .

In [4], a partition of \widehat{W} into subsets called *families* was defined and in [6] a class of not necessarily irreducible representations (later called *constructible representations*, see [9]) of W with all components in a family c (which we now fix) was defined by an inductive procedure. Let $\text{Con}(c)$ be the set of constructible representations (up to isomorphism) attached to c . In [6] it was conjectured that the representations in $\text{Con}(c)$ are precisely the representations associated in [2] to the various left cells of W contained in the two-sided cell of W defined by c ; this conjecture was proved in [7]. It is known that $|c| = 1$ if W is of type A , $|c| = \binom{D+1}{D/2}$ (with $D \in 2\mathbb{N}$) if W is of type B , C or D , and $|c|$ is one of 1, 2, 3, 4, 5, 11, 17 if W is of exceptional type.

0.3. We would like to find an explicit formula for $|\text{Con}(c)|$.

If $|c|$ is one of 1, 2, 3, 4, 5, 11, 17 then $|\text{Con}(c)|$ is 1, 1, 2, 2, 3, 5, 7 respectively.

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In the remainder of this paper we assume that:

(a) $|c| = \binom{D+1}{D/2}$ with $D = 2d \in 2\mathbb{N}$.

In Section 1 we prove the following result.

Theorem 0.4. *We have $|\text{Con}(c)| = \text{Cat}_{d+1}$.*

It is known (see [12, §2.13]) that if W is of type D then $|\text{Con}(c)| = |\text{Con}(c')|$ for some family c' in a Weyl group of type B or C . We will therefore assume in the rest of the paper that W is of type B or C .

0.5. According to [1, Corollary 4], we have

(a)
$$\text{Cat}_n = \sum_{p=1}^n N(n, p)$$

where

$$N(n, p) = \frac{1}{n} \binom{n}{p} \binom{n}{p-1}$$

are the Narayana numbers.

We denote by F the field with two elements.

In [8] a bijection between $\text{Con}(c)$ and a certain collection X_c of subgroups of F^d is described. For each p , $1 \leq p \leq d+1$, let $X_{c,p}$ be the set of subgroups of cardinal 2^{p-1} in X_c . The following refinement of Theorem 0.4 is proved in Section 2.

Theorem 0.6. *We have $|X_{c,p}| = N_{d+1,p}$.*

0.7. In Section 3 we state a conjecture according to which Catalan numbers appear in connection with the study of Springer fibers for G .

0.8. For any $i \leq j$ in \mathbb{Z} we set $[i, j] = \{h \in \mathbb{Z}; i \leq h \leq j\}$.

1. Proof of Theorem 0.4

1.1. Let $D \in 2\mathbb{N}$. Let V_D be an F -vector space with a nondegenerate symplectic form $\langle \cdot, \cdot \rangle : V_D \times V_D \rightarrow F$ and with a given subset $\{e_1, e_2, e_3, \dots, e_D\}$ such that $\langle e_i, e_j \rangle = 1$ if $i - j = \pm 1$ and $\langle e_i, e_j \rangle = 0$ otherwise.

Assuming that $D \geq 2$ and $i \in [1, D]$, we define a linear (injective) map $T_i : V_{D-2} \rightarrow V_D$ by:

- $e_a \mapsto e_a$ if $a < i - 1$.
- $e_{i-1} \mapsto e_{i-1} + e_i + e_{i+1}$.
- $e_a \mapsto e_{a+2}$ if $a \geq i$.

(We regard V_{D-2} as a subspace of V_D in an obvious way.)

Let $\mathcal{F}(V_D)$ be the family of isotropic subspaces associated in [11, §1.17] to V_D and its basis $\{e_1, e_2, \dots, e_D\}$. (The characteristic functions of these subspaces form a basis of the \mathbb{C} -vector space of functions $V_D \rightarrow \mathbb{C}$.) We have a partition $\mathcal{F}(V_D) = \bigsqcup_{k \geq 0} \mathcal{F}^k(V_D)$. We will only give here the definition of $\mathcal{F}^0(V_D)$ and $\mathcal{F}^1(V_D)$. The definition is by induction on D . When $D = 0$, $\mathcal{F}^0(V_D)$ consists of 0 and $\mathcal{F}^1(V_D)$ is empty. Assume now that $D \geq 2$. A subspace E of V_D is said to be in $\mathcal{F}^0(V_D)$ if either $E = 0$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}^0(V_{D-2})$ such that $E = T_i(E') + Fe_i$ (this is a direct sum). A subspace E of V_D is said to be in $\mathcal{F}^1(V_D)$ if either $E = F(e_1 + e_2 + \dots + e_D)$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}^1(V_{D-2})$ such that $E = T_i(E') + Fe_i$.

For example, if $D = 2$, $\mathcal{F}^0(V_D)$ consists of 0, Fe_1, Fe_2 and $\mathcal{F}^1(V_D)$ consists of $F(e_1 + e_2)$. If $D = 4$, $\mathcal{F}^0(V_D)$ consists of

$$0, Fe_1, Fe_2, Fe_3, Fe_4, Fe_1+Fe_3, Fe_1+Fe_4, Fe_2+Fe_4, F(e_1+e_2+e_3)+Fe_2, \\ F(e_2+e_3+e_4) + Fe_3$$

and $\mathcal{F}^1(V_D)$ consists of

$$F(e_1 + e_2 + e_3 + e_4), F(e_1 + e_2 + e_3 + e_4) + Fe_2, F(e_1 + e_2 + e_3 + e_4) + Fe_3, \\ F(e_1 + e_2) + Fe_4, Fe_1 + F(e_3 + e_4).$$

We have

$$\mathcal{F}^0(V_D) = \mathcal{F}_{D/2}^0(V_D) \sqcup \mathcal{F}_{<D/2}^0(V_D)$$

where

$$\mathcal{F}_{D/2}^0(V_D) = \{E \in \mathcal{F}^0(V_D); \dim(E) = \frac{1}{2}D\}, \\ \mathcal{F}_{<D/2}^0(V_D) = \{E \in \mathcal{F}^0(V_D); \dim(E) < \frac{1}{2}D\}.$$

1.2. Let \mathcal{G}_D^0 (resp. \mathcal{G}_D^1) be the set of lines in V_D of the form $F(e_a + e_{a+1} + \dots + e_b)$ where $a \leq b$ in $[1, D]$ satisfy $b - a = 1 \pmod{2}$ (resp. $b - a = 0 \pmod{2}$). Let $\mathcal{G}_D = \mathcal{G}_D^0 \sqcup \mathcal{G}_D^1$. For $E \in \mathcal{F}(V_D)$ let $B_E = \{L \in \mathcal{G}_D; L \subset E\}$. According to [12, §1.2(e), (f), (g)], if $E \in \mathcal{F}(V_D)$ then

(a)
$$E = \bigoplus_{L \in B_E} L;$$

moreover we have $E \in \mathcal{F}^0(V_D)$ if and only if $B_E \subset \mathcal{G}_D^1$; we have $E \in \mathcal{F}^1(V_D)$ if and only if B_E contains a unique line L_E in \mathcal{G}_D^0 .

It follows that if $E \in \mathcal{F}^1(V_D)$ we can write

(b) $E = E_0 \oplus L_E$ where $E_0 = \bigoplus_{L \in B_E; L \neq L_E} L$.

We show:

(c) $E_0 \in \mathcal{F}^0(V_D)$.

We argue by induction on D . If $D = 0$ then $\mathcal{F}_D^1 = \emptyset$ and there is nothing to prove. Assume now that $D \geq 2$. If $E = F(e_1 + e_2 + \cdots + e_D)$, then $E_0 = 0$ and (c) is obvious. If E is not of this form then there exists $i \in [1, D]$ and $E' \in \mathcal{F}_{D-2}^1$ such that $E = T_i(E') + Fe_i$. By the induction hypothesis we have $E' = E'_0 \oplus_{L_{E'}} L_{E'}$ where $E'_0 \in \mathcal{F}_{D-2}^0$. We have $E = T_i(E'_0) + Fe_i + T_i(L_{E'}) = \tilde{E}_0 + \tilde{L}$ where $\tilde{E}_0 = T_i(E'_0) + Fe_i \in \mathcal{F}^0(V_D)$ and $\tilde{L} = T_i(L_{E'}) \in \mathcal{G}_D^0$ (from the definition of T_i). Since $\tilde{L} \subset E$ we must have $\tilde{L} = L_E$. We have $B_E = B_{\tilde{E}_0} \cup \{L_E\}$ (the union is disjoint since $B_{\tilde{E}_0} \subset \mathcal{G}_D^1$, $L_E \in \mathcal{G}_D^0$). Thus $B_{\tilde{E}_0} = B_E - \{L_E\}$. Since $\tilde{E}_0 = \sum_{L \in B_{\tilde{E}_0}} L = \sum_{L \in B_E - \{L_E\}} L = E_0$ we see that $E_0 = \tilde{E}_0 \in \mathcal{F}^0(V_D)$. This proves (c).

Note that in (c) we have $\dim(E) \leq \frac{1}{2}D$, $\dim(L_E) = 1$, hence $\dim(E_0) < \frac{1}{2}D$. Thus we can define a map $\Xi_D : \mathcal{F}^1(V_D) \rightarrow \mathcal{F}_{<D/2}^0(V_D)$ by $E \mapsto E_0$ (notation of (c)).

We show:

(d) The map Ξ_D is surjective.

We argue by induction on D . If $D = 0$ then $\mathcal{F}_{<D/2}^0(V_D)$ is empty and there is nothing to prove. Assume now that $D \geq 2$. Let $E_0 \in \mathcal{F}_{<D/2}^0(V_D)$. If $E_0 = 0$ then $E = F(e_1 + e_2 + \cdots + e_D)$ is as required. Now assume that $E_0 \neq 0$. Then there exists $i \in [1, D]$ and $E'_0 \in \mathcal{F}^0(V_{D-2})$ such that $E_0 = T_i(E'_0) \oplus Fe_i$. We see that $\dim(E'_0) = \dim(T_i(E'_0)) = \dim(E_0) - 1 < \frac{1}{2}D - 1 = \frac{1}{2}(D - 2)$ so that $E'_0 \in \mathcal{F}_{<(D-2)/2}^0(V_{D-2})$. By the induction hypothesis there exists $L \in \mathcal{G}_{D-2}^0$ such that $E'_0 + L \in \mathcal{F}^1(V_{D-2})$. Let $E = T_i(E'_0 + L) + Fe_i$. We have $E \in \mathcal{F}^1(V_D)$ and $E = E_0 + T_i(L)$. Note that $T_i(L) \in \mathcal{G}_D^0$ and is contained in E , hence it is equal to L_E (see (b)). It follows that $E_0 = \Xi_D(E)$. This proves (d).

We show:

(e) Ξ_D is injective.

Assume that E, E' in $\mathcal{F}^1(V_D)$ satisfy $\Xi_D(E) = \Xi_D(E')$. We must show that $E = E'$.

We have $E = E_0 \oplus L$, $E' = E_0 \oplus L'$ where $E_0 \in \mathcal{F}^0(V_D)$ and $L = F(e_a + e_{a+1} + \cdots + e_b)$, $L' = F(e_{a'} + e_{a'+1} + \cdots + e_{b'})$, where $a < b$ and $a' < b'$ in $[1, D]$ satisfy $b - a = 1 \pmod{2}$, $b' - a' = 1 \pmod{2}$. (In fact, from [11, § 1.3(e), see (P_2)] we have that $a = 1 \pmod{2}$, $b = 0 \pmod{2}$, $a' = 1 \pmod{2}$, $b' = 0 \pmod{2}$.) Assume first that $a < a'$ so that $a \leq a' - 2$. From [11, § 1.3(e), see (P_2)] we see that there exist $1 \leq c \leq c' \leq D$ such that $c \leq a \leq c'$ and such that the line $\mathcal{L} = F(e_c + e_{c+1} + \cdots + e_{c'})$ is contained in E_0 , hence also in \mathcal{G}_D^1 . But then the pair of distinct lines \mathcal{L}, L would violate [11, § 1.3(e), see (P_0)]. We see that we must have $a \geq a'$. Similarly we have $a' \geq a$, hence $a' = a$.

Assume next that $b < b'$ so that $b + 2 \leq b'$. From [11, § 1.3(e), see (P_2)] we see that there exist $1 \leq c \leq c' \leq D$ such that $c \leq b' \leq c'$ and such that the line

$\mathcal{L} = F(e_c + e_{c+1} + \dots + e_{c'})$ is contained in E_0 hence also in \mathcal{G}_D^1 . But then the pair of distinct lines \mathcal{L}, L' would violate [11, § 1.3(e), see (P_0)]. We see that we must have $b \geq b'$. Similarly we have $b' \geq b$, hence $b' = b$.

We see that $L = L'$, hence $E = E'$. This proves (e).

1.3. From § 1.2(c), (d), (e) we see that $|\mathcal{F}_{<D/2}^0(V_D)| = |\mathcal{F}^1(V_D)|$; hence

$$|\mathcal{F}^0(V_D)| - |\mathcal{F}_{D/2}^0(V_D)| = |\mathcal{F}^1(V_D)|,$$

that is, $|\mathcal{F}_{D/2}^0(V_D)| = |\mathcal{F}^0(V_D)| - |\mathcal{F}^1(V_D)|$. According to [11, § 1.27] we have

$$|\mathcal{F}^0(V_D)| = \binom{D+1}{D/2}, \quad |\mathcal{F}^1(V_D)| = \binom{D+1}{(D-2)/2}.$$

It follows that

$$|\mathcal{F}_{D/2}^0(V_D)| = \binom{D+1}{D/2} - \binom{D+1}{(D-2)/2} = \frac{(2d+2)!}{(d+1)!(d+2)!} = C_{d+1},$$

where $D = 2d$.

1.4. In [5] the set c is identified with a subset of V_D . Now any object in $\text{Con}(c)$ is multiplicity-free, hence may be identified with a subset of c , hence with a subset of V_D . This subset is a Lagrangian subspace of V_D . Thus $\text{Con}(c)$ is identified with a subset of the set of Lagrangian subspaces of V_D . This subset is the same as $\mathcal{F}_{D/2}^0(V_D)$ (see [10, § 2.8(iii)]). We see that $|\text{Con}(c)| = C_{d+1}$ and Theorem 0.4 is proved.

1.5. An alternative proof of Theorem 0.4 can be given using the parametrization of $\text{Con}(c)$ in terms of “admissible arrangements” in [6, p. 220].

2. Proof of Theorem 0.6

2.1. We preserve the notation of V_D . We have $V_D = V_D^0 \oplus V_D^1$ where V_D^0 has basis $\{e_2, e_4, \dots, e_D\}$ and V_D^1 has basis $\{e_1, e_3, \dots, e_{D-1}\}$. Assuming that $D \geq 2$ we define for any $i \in [1, D]$ a linear map $\mathcal{T}_i : V_{D-2}^1 \rightarrow V_D^1$ by:

- $e_k \mapsto e_k$ if $k \leq i - 2$.
- $e_k \mapsto e_{k+2}$ if $k \geq i$.
- $e_{i-1} \mapsto e_{i-1} + e_{i+1}$ if i is even.

Following [10, § 2.3] we define a collection $\mathcal{C}(V_D^1)$ of subspaces of V_D^1 by induction on D . If $D = 0$, $\mathcal{C}(V_D^1)$ consists of $\{0\}$. Assume now that $D \geq 2$. A subspace \mathcal{E} of V_D^1 is said to be in $\mathcal{C}(V_D^1)$ if either $\mathcal{E} = \{0\}$ or there exists $i \in [1, D]$ and $\mathcal{E}' \in \mathcal{C}(V_{D-2}^1)$ such that:

- $\mathcal{E} = \mathcal{T}_i(\mathcal{E}') + Fe_i$ if i is odd.
- $\mathcal{E} = \mathcal{T}_i(\mathcal{E}')$ if i is even.

For example, $\mathcal{C}(V_2^1)$ consists of two subspaces: $0, Fe_1$; $\mathcal{C}(V_4^1)$ consists of five subspaces:

$$0, Fe_1, Fe_3, F(e_1 + e_3), Fe_1 + Fe_3;$$

$\mathcal{C}(V_6^1)$ consists of 14 subspaces:

$$0, Fe_1, Fe_3, Fe_5, F(e_1 + e_3), F(e_3 + e_5), F(e_1 + e_3 + e_5), Fe_1 + Fe_3, \\ Fe_1 + Fe_5, Fe_3 + Fe_5, F(e_1 + e_3) + Fe_5, Fe_1 + F(e_3 + e_5), F(e_1 + e_3 + e_5) + Fe_3, \\ Fe_1 + Fe_3 + Fe_5.$$

2.2. If $\mathcal{E} \in \mathcal{C}(V_D^1)$ we set $\mathcal{E}^! = \{x \in V_D^0; \langle x, \mathcal{E} \rangle = 0\}$. The following result appears in [10, §2.4].

(a) $\mathcal{E} \mapsto \mathcal{E} \oplus \mathcal{E}^!$ defines a bijection $\mathcal{C}(V_D^1) \xrightarrow{\sim} \mathcal{F}_{D/2}^0(V_D)$. The inverse bijection is given by $E \mapsto E \cap V_D^1$.

2.3. Let \mathcal{Z}_D^* be the set of all elements of V_D^1 of the form

$$e_{a,b} = e_a + e_{a+2} + e_{a+4} + \dots + e_b$$

for various numbers $a \leq b$ in $\{1, 3, \dots, D - 1\}$.

For any $s \geq 0$, let \mathcal{Z}_D^s be the set of all finite unordered sequences

$$e_{a_1, b_1}, e_{a_2, b_2}, \dots, e_{a_s, b_s}$$

in \mathcal{Z}_D^* such that for any $n \neq m$ in $\{1, 2, \dots, s\}$ we have either

$$a_n \leq b_n < a_m \leq b_m \quad \text{or} \quad a_m \leq b_m < a_n \leq b_n, \\ \text{or} \quad a_n < a_m \leq b_m < b_n \quad \text{or} \quad a_m < a_n \leq b_n < b_m.$$

Let $\mathcal{Z}_D = \bigcup_{s \geq 0} \mathcal{Z}_D^s$ (a disjoint union).

For example, \mathcal{Z}_2 consists of the two sequences $\emptyset, \{e_1\}$; \mathcal{Z}_4 consists of the five sequences $\emptyset, \{e_1\}, \{e_3\}, \{e_1 + e_3\}, \{e_1, e_3\}$; and \mathcal{Z}_6 consists of 14 sequences:

$$\emptyset, \{e_1\}, \{e_3\}, \{e_5\}, \{e_1 + e_3\}, \{e_3 + e_5\}, \{e_1 + e_3 + e_5\}, \\ \{e_1, e_3\}, \{e_1, e_5\}, \{e_3, e_5\}, \{e_1 + e_3, e_5\}, \{e_1, e_3 + e_5\}, \{e_1 + e_3 + e_5, e_3\}, \{e_1, e_3, e_5\}.$$

Theorem 2.4. *The assignment*

$$\Theta_D : (e_{a_1, b_1}, e_{a_2, b_2}, \dots, e_{a_s, b_s}) \mapsto Fe_{a_1, b_1} + Fe_{a_2, b_2} + \dots + Fe_{a_s, b_s}$$

defines a bijection $\mathcal{Z}_D \xrightarrow{\sim} \mathcal{C}(V_D^1)$.

When $D \leq 6$ this follows from §2.1, §2.3. Note that Theorem 2.4 gives an order-preserving bijection between the set of noncrossing partitions (see [13]) and $\mathcal{C}(V_D^1)$ (with the order given by inclusion).

2.5. Assuming that $D \geq 2$ we define for any $i \in [1, D]$ a map $\sigma_i : \mathcal{Z}_{D-2}^* \rightarrow \mathcal{Z}_D^*$ by:

- $e_{a,b} \mapsto e_{a+2,b+2}$ if $i \leq a$.
- $e_{a,b} \mapsto e_{a,b+2}$ if $a < i \leq b + 1$.
- $e_{a,b} \mapsto e_{a,b}$ if $i > b + 1$.

Note that

- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b})$ if i is even.
- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b})$ if i is even and $i \leq a$ or $i > b$.
- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b}) + e_i$ if i is odd and $a < i \leq b$.

2.6. Assume that $D \geq 2$ and $i \in [1, D]$. Let $e_{a,b}, e_{a',b'}$ be in \mathcal{Z}_{D-2}^* and let $e_{\tilde{a},\tilde{b}} = \sigma_i(e_{a,b}), e_{\tilde{a}',\tilde{b}'} = \sigma_i(e_{a',b'})$. We show:

- (i) If $b < a'$ then $\tilde{b} < \tilde{a}'$.
- (ii) If $a < a'$ and $b' < b$ then $\tilde{a} < \tilde{a}'$ and $\tilde{b}' < \tilde{b}$.
- (iii) If i is odd and $\tilde{a} \leq i \leq \tilde{b}$ then $\tilde{a} < i < \tilde{b}$.

In the setup of (i) assume that $\tilde{a}' \leq \tilde{b}$. Then we have $a' \leq b$ or $a' + 2 \leq b$ or $a' + 2 \leq b + 2$ or $a' \leq b + 2$. The first three cases are clearly impossible; in the 4th case we have $b + 2 = a'$ (since $b + 2 \leq a' \leq b + 2$), $b' + 1 < i$ and $b + 1 \geq i$, so that $b > b' \geq a'$, a contradiction.

In the setup of (ii) assume that $\tilde{a} \geq \tilde{a}'$. Then we have $a \geq a'$ or $a + 2 \geq a' + 2$ or $a \geq a' + 2$ or $a + 2 \geq a'$. The first three cases are clearly impossible; in the 4th case we have $a + 2 = a'$ (since $a + 2 \leq a' \leq a + 2$), $a' < i$ and $a \geq i$, so that $a > a'$, a contradiction. Thus, $\tilde{a} < \tilde{a}'$.

Again, in the setup of (ii) assume that $\tilde{b}' \geq \tilde{b}$. Then we have $b' \geq b$ or $b' + 2 \geq b + 2$ or $b' \geq b + 2$ or $b' + 2 \geq b$. The first three cases are clearly impossible. In the 4th case we have $b' + 2 = b$ (since $b \geq b' + 2 \geq b$), $b + 1 < i$ and $b' + 1 \geq i$ so that $b' > b$, a contradiction. Thus, $\tilde{b}' < \tilde{b}$.

In the setup of (iii) assume that $\tilde{a} = i$. We have $\tilde{a} = a$ or $\tilde{a} = a + 2$. If $\tilde{a} = a$ we have $a = i$ and $b < i$, hence $b < \tilde{b}$ so that $\tilde{b} = b + 2$; this implies $i \leq b$, a contradiction. If $\tilde{a} = a + 2$ we have $a + 2 = i, i \leq a$, a contradiction. Thus $\tilde{a} < i$.

In the setup of (iii) assume that $\tilde{b} = i$. We have $\tilde{a} = b$ or $\tilde{b} = b + 2$. If $\tilde{b} = b$ we have $b = i$ and $b < i$, a contradiction. If $\tilde{b} = b + 2$ we have $b + 2 = i$ and either $a \geq i$ or $a < i \leq b$. In the first case we have $a \geq b + 2 > b$, a contradiction; in the second case we have $b + 2 \leq b$, a contradiction. Thus, $i < \tilde{b}$.

2.7. From §2.6(i)–(iii) we see that when $D \geq 2$ and $i \in [1, D]$, there is a well-defined map $\Sigma_i : \mathcal{Z}_{D-2} \rightarrow \mathcal{Z}_D$ given by

$$(e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s}) \mapsto \begin{cases} (\sigma_i(e_{a_1,b_1}), \sigma_i(e_{a_2,b_2}), \dots, \sigma_i(e_{a_s,b_s}), e_i) & \text{if } i \text{ is odd,} \\ (\sigma_i(e_{a_1,b_1}), \sigma_i(e_{a_2,b_2}), \dots, \sigma_i(e_{a_s,b_s})) & \text{if } i \text{ is even.} \end{cases}$$

2.8. Let $\epsilon \in \mathcal{Z}_D$, $\epsilon \neq \emptyset$. Let $e_{a,b} \in \epsilon$ be such that $b - a$ is minimum. If $b - a = 0$ we set $i = a = b$; we have $i \in [1, D]$ and i is odd. If $b - a > 0$ we define $i \in [1, D]$ by $a = i - 1 < i + 1 \leq b$; then i is even. We will show that

(a) ϵ is in the image of $\Sigma_i : \mathcal{Z}_{D-2} \rightarrow \mathcal{Z}_D$.

If i is odd we can write $\epsilon = (e_{\tilde{a}_1, \tilde{b}_1}, e_{\tilde{a}_2, \tilde{b}_2}, \dots, e_{\tilde{a}_s, \tilde{b}_s}, e_i)$.

If i is even we can write $\epsilon = (e_{\tilde{a}_1, \tilde{b}_1}, e_{\tilde{a}_2, \tilde{b}_2}, \dots, e_{\tilde{a}_s, \tilde{b}_s})$, where $a_t = a$, $b_t = b$ for some t .

To $e_{\tilde{a}_t, \tilde{b}_t}$, $t = 1, 2, \dots, s$, we associate the element

$$\begin{aligned} e_{a_t, b_t} &= e_{\tilde{a}_t-2, \tilde{b}_t-2} && \text{if } i \leq \tilde{a}_t - 2, \\ e_{a_t, b_t} &= e_{\tilde{a}_t, \tilde{b}_t-2} && \text{if } \tilde{a}_t < i \leq \tilde{b}_t - 1, \\ e_{a_t, b_t} &= e_{\tilde{a}_t, \tilde{b}_t} && \text{if } \tilde{b}_t < i. \end{aligned}$$

(Note that we cannot have $i = \tilde{a}_t$ or $i = \tilde{b}_t$. Moreover when i is even we see from the definitions that we cannot have $i = \tilde{a}_t - 1$.) This element is in \mathcal{Z}_{D-2}^* .

Consider $n \neq m$ in $\{1, 2, \dots, s\}$. We set

$$(\tilde{a}_n, \tilde{b}_n, \tilde{a}_m, \tilde{b}_m) = (\tilde{a}, \tilde{b}, \tilde{a}', \tilde{b}'), \quad (a_n, b_n, a_m, b_m) = (a, b, a', b').$$

We show:

- (i) If $\tilde{b} < \tilde{a}'$, then $b < a'$.
- (ii) If $\tilde{a}' < \tilde{a} \leq \tilde{b} < \tilde{b}'$, then $a' < a \leq b < b'$.

In the setup of (i) assume that $a' \leq b$. Then we have $\tilde{a}' \leq \tilde{b}$ or $\tilde{a}' - 2 \leq \tilde{b}$ or $\tilde{a}' - 2 \leq \tilde{b} - 2$ or $\tilde{a}' \leq \tilde{b} - 2$. The first three cases are clearly impossible. In the 4th case we have $\tilde{b} < \tilde{a}' \leq \tilde{b} - 2$, hence $\tilde{b} < \tilde{b} - 2$, a contradiction. Thus $b < a'$.

In the setup of (ii), a', a, b, b' are as follows:

- $\tilde{a}' - 2, \tilde{a} - 2, \tilde{b} - 2, \tilde{b}' - 2$ if $i \leq \tilde{a}' - 2$.
- $\tilde{a}', \tilde{a} - 2, \tilde{b} - 2, \tilde{b}' - 2$ if $\tilde{a}' < i \leq \tilde{a} - 2$ (so that $\tilde{a}' < \tilde{a} - 2$).
- $\tilde{a}', \tilde{a}, \tilde{b} - 2, \tilde{b}' - 2$ if $\tilde{a} < i \leq \tilde{b} - 1$ (so that $\tilde{a} \leq \tilde{b} - 2$).
- $\tilde{a}', \tilde{a}, \tilde{b}, \tilde{b}' - 2$ if $\tilde{b} < i \leq \tilde{b}' - 2$ (so that $\tilde{b} < \tilde{b}' - 2$).
- $\tilde{a}', \tilde{a}, \tilde{b}, \tilde{b}'$ if $\tilde{b}' < i$.

Since i is distinct from each of $\tilde{a}', \tilde{a}' - 1, \tilde{a}, \tilde{a} - 1, \tilde{b}, \tilde{b}', \tilde{b}' - 1$ we see that we must be in one of the five cases above. Note that $a' < a \leq b < b'$ in each case.

From (i), (ii) we see that $\epsilon' := (e_{a_1, b_1}, e_{a_2, b_2}, \dots, e_{a_s, b_s})$ belongs to \mathcal{Z}_{D-2} . From the definitions we see that $\epsilon = \Sigma_i(\epsilon')$. Hence (a) holds.

2.9. We define a subset \mathcal{Z}'_D of \mathcal{Z}_D by induction on D . If $D = 0$, \mathcal{Z}'_D consists of the empty sequence. Assume now that $D \geq 2$. A sequence $\epsilon \in \mathcal{Z}_D$ is said to be

in \mathcal{Z}'_D if either ϵ is the empty sequence or if there exists $i \in [1, D]$ and $\epsilon' \in \mathcal{Z}'_{D-2}$ such that $\epsilon = \Sigma_i(\epsilon')$. (Note that $\Sigma_i(\epsilon')$ is well defined.) Using §2.8(a) we see by induction on D that

(a) $\mathcal{Z}_D = \mathcal{Z}'_D$.

2.10. Assume that $D \geq 2$ and $i \in [1, D]$. For $\epsilon' \in \mathcal{Z}_{D-2}$ we show:

(a) $\Theta_D(\Sigma_i(\epsilon')) = \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i$ if i is odd.

(b) $\Theta_D(\Sigma_i(\epsilon')) = \mathcal{T}_i(\Theta_{D-2}\epsilon')$ if i is even.

We can write $\epsilon' = (e_{a_1, b_1}, e_{a_2, b_2}, \dots, e_{a_s, b_s})$. Then

$$\Theta_D(\Sigma_i(\epsilon')) = \begin{cases} F\sigma_i(e_{a_1, b_1}) + F\sigma_i(e_{a_2, b_2}) + \dots + F\sigma_i(e_{a_s, b_s}) + Fe_i & \text{if } i \text{ is odd,} \\ F\sigma_i(e_{a_1, b_1}) + F\sigma_i(e_{a_2, b_2}) + \dots + F\sigma_i(e_{a_s, b_s}) & \text{if } i \text{ is even.} \end{cases}$$

Using the definitions we see that

$$\begin{aligned} \Theta_D(\Sigma_i(\epsilon')) &= F\mathcal{T}_i(e_{a_1, b_1}) + F\mathcal{T}_i(e_{a_2, b_2}) + \dots + F\mathcal{T}_i(e_{a_s, b_s}) + Fe_i \\ &= \mathcal{T}_i(Fe_{a_1, b_1} + Fe_{a_2, b_2} + \dots + Fe_{a_s, b_s}) + Fe_i = \mathcal{T}_i(X_{D-1}(\epsilon')) + Fe_i \end{aligned}$$

if i is odd,

$$\begin{aligned} \Theta_D(\Sigma_i(\epsilon')) &= F\mathcal{T}_i(e_{a_1, b_1}) + F\mathcal{T}_i(e_{a_2, b_2}) + \dots + F\mathcal{T}_i(e_{a_s, b_s}) \\ &= \mathcal{T}_i(Fe_{a_1, b_1} + Fe_{a_2, b_2} + \dots + Fe_{a_s, b_s}) = \mathcal{T}_i(X_{D-1}(\epsilon')) \end{aligned}$$

if i is even. This proves (a), (b).

2.11. We prove the following part of Theorem 2.4.

(a) The map Θ_D in Theorem 2.4 is well defined (it has image contained in $\mathcal{C}(V_D^1)$).

We argue by induction on D . When $D = 0$, (a) is obvious. Assume now that $D \geq 2$. Let $\epsilon \in \mathcal{Z}_D$. If $\epsilon = \emptyset$ then $\Theta_D(\epsilon) = 0 \in \mathcal{F}_D$. Assume now that $\epsilon \neq \emptyset$. Using §2.8, we can find $i \in [1, D]$ and $\epsilon' \in \mathcal{Z}_{D-2}$ such that $\epsilon = \Sigma_i(\epsilon')$ so that $\Theta_D(\epsilon) = \Theta_D(\Sigma_i(\epsilon'))$. By the induction hypothesis we have $\Theta_{D-2}\epsilon' \in \mathcal{C}(V_{D-2}^1)$. By the definition of $\mathcal{C}(V_D^1)$ we then have

$$\begin{aligned} \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i &\in \mathcal{C}(V_D^1) && \text{if } i \text{ is odd;} \\ \mathcal{T}_i(\Theta_{D-2}\epsilon') &\in \mathcal{C}(V_D^1) && \text{if } i \text{ is even.} \end{aligned}$$

Using §2.10, we can rewrite this as $\Theta_D(\epsilon) \in \mathcal{C}(V_D^1)$. This proves (a).

2.12. We prove the following part of Theorem 2.4.

(a) The map Θ_D in Theorem 2.4 (see §2.11(a)) is surjective.

We argue by induction on D . When $D = 0$, (a) is obvious. Assume now that $D \geq 2$. Let $\mathcal{E} \in \mathcal{C}(V_D^1)$. If $\mathcal{E} = 0$ then $\mathcal{E} = \Theta_D(\emptyset)$. Assume now that $\mathcal{E} \neq 0$. We can find $i \in [1, D]$ and $\mathcal{E}' \in \mathcal{C}(V_{D-2}^1)$ such that $\mathcal{E} = \mathcal{T}_i(\mathcal{E}') + Fe_i$ if i is odd and $\mathcal{E} = \mathcal{T}_i(\mathcal{E}')$ if

i is even. By the induction hypothesis we have $\mathcal{E}' = \Theta_{D-2}(\epsilon')$ for some $\epsilon' \in \mathcal{Z}_{D-2}$. Thus we have $\mathcal{E} = \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i$ if i is odd, $\mathcal{E} = \mathcal{T}_i(\Theta_{D-2}\epsilon')$ if i is even. Using §2.10 we can rewrite this as $\mathcal{E} = \Theta_D(\epsilon)$ where $\epsilon = \Sigma_i(\epsilon') \in \mathcal{Z}_D$. This proves (a).

2.13. We have $\mathcal{C}(V_D^1) = \bigsqcup_{s \in [0, d]} \mathcal{C}^s(V_D^1)$ where $\mathcal{C}^s(V_D^1) = \{\mathcal{E} \in \mathcal{C}(V_D^1); \dim \mathcal{E} = s\}$. Clearly, the map Θ in Theorem 2.4 restricts to a map $\Theta^s : \mathcal{Z}_D^s \rightarrow \mathcal{C}^s(V_D^1)$ for any $s \in [0, d]$. From §2.12(a) it follows that Θ^s is surjective for any $s \in [0, d]$. In [1] it is shown that $|\mathcal{Z}_D^s| = N_{d+1, s+1}$ (see §0.5) for any $s \in [0, d]$. Using this and §0.5(a) we see that

$$\text{Cat}_{d+1} = \sum_{s \in [0, d]} N(d+1, s+1) = \sum_{s \in [0, d]} |\mathcal{Z}_D^s| = |\mathcal{Z}_D|.$$

We see that Θ_D is a surjective map from a set with cardinal $|\mathcal{Z}_D| = \text{Cat}_{d+1}$ to a set with the same cardinal $|\mathcal{C}(V_D^1)| = |\mathcal{F}_{D/2}^0(V_D)| = \text{Cat}_{d+1}$ (the first equality holds by §2.2(a); the second equality follows from Theorem 0.4). It follows that Θ is a bijection and Theorem 2.4 is proved.

This implies that $\Theta^s : \mathcal{Z}_D^s \rightarrow \mathcal{C}^s(V_D^1)$ is a bijection for any $s \in [0, d]$. We see that Theorem 0.6 holds. (We use that X_c in §0.5 is the same as $\mathcal{C}^s(V_D^1)$ if we identify $V_D^1 = F^d$.)

3. A conjecture

3.1. In this section we fix a unipotent element $u \in G$. We assume that either

- G is of type $C_{d(d+1)}$, $d \geq 1$ and u has Jordan blocks of sizes $2d, 2d, 2d - 2, 2d - 2, \dots, 2, 2$ or that
- G is of type $B_{d(d+1)}$, $d \geq 1$ and u has Jordan blocks of sizes $2d + 1, 2d - 1, 2d - 1, \dots, 1, 1$.

Let \mathcal{B}_u be the variety of Borel subgroups of G that contain u and let $[\mathcal{B}_u]$ be the set of irreducible components of \mathcal{B}_u . Let $A(u)$ be the group of components of the centralizer of u in G . Note that $A(u)$ acts naturally by permutations on $[\mathcal{B}_u]$. For each $\xi \in [\mathcal{B}_u]$ we denote by $A(u)_\xi$ the stabilizer of ξ in $A(u)$. Let Δ_u be the set of subgroups of $A(u)$ of the form $A(u)_\xi$ for some $\xi \in [\mathcal{B}_u]$.

We assume that c is the family containing the Springer representation of W associated to u and to the unit representation of $A(u)$. We conjecture that

(a) *there exists an isomorphism $A(u) \xrightarrow{\sim} V_D^1$, $D = 2d$ which carries Δ_u to the collection $\mathcal{C}(V_D^1)$ (see §2.1) of subspaces of V_D^1 .*

(This would imply that $|\Delta_u|$ is a Catalan number.)

We have verified that (a) is true when $d = 1, 2, 3$.

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
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