Pacific Journal of Mathematics

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Volume 336 No. 1-2 May 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 336, No. 1-2, 2025
https://doi.org/10.2140/pjm.2025.336.339

CONSTRUCTIBLE REPRESENTATIONS AND CATALAN NUMBERS

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Dedicated to the memory of Gary Seitz

We establish a connection between constructible representations (arising in the study of left cells in Weyl groups) and Catalan numbers.

0. Introduction

0.1. The sequence of Catalan numbers is the sequence Cat_n, $n = 1, 2, 3, \dots,$ where $\mathrm{Cat}_n = (2n)!/(n!(n+1)!)$. According to [3], Catalan numbers first appeared in the work of Ming Antu (1692–1763). They were rediscovered by Euler (1707–1783). See also [13].

In this paper we give a new way in which Catalan numbers appear in connection with Lie theory.

0.2. Let G be a connected reductive algebraic group of adjoint type over $\mathbb{C}$ whose Weyl group W is assumed to be irreducible. Let $\widehat{W}$ be the set of (isomorphism classes of) irreducible representations (over $\mathbb{Q}$) of W.

In [4], a partition of $\widehat{W}$ into subsets called *families* was defined and in [6] a class of not necessarily irreducible representations (later called *constructible representations*, see [9]) of W with all components in a family c (which we now fix) was defined by an inductive procedure. Let $\mathrm{Con}(c)$ be the set of constructible representations (up to isomorphism) attached to c. In [6] it was conjectured that the representations in $\mathrm{Con}(c)$ are precisely the representations associated in [2] to the various left cells of W contained in the two-sided cell of W defined by c; this conjecture was proved in [7]. It is known that $|c| = 1$ if W is of type A, $|c| = \binom{D+1}{D/2}$ (with $D \in 2\mathbb{N}$) if W is of type B, C or D, and $|c|$ is one of 1, 2, 3, 4, 5, 11, 17 if W is of exceptional type.

0.3. We would like to find an explicit formula for $|\mathrm{Con}(c)|$.

If $|c|$ is one of 1, 2, 3, 4, 5, 11, 17 then $|\mathrm{Con}(c)|$ is 1, 1, 2, 2, 3, 5, 7 respectively.

Lusztig supported by NSF grant DMS-2153741.
MSC2020: 20G99.
Keywords: algebraic group, constructible representation, Catalan number.

In the remainder of this paper we assume that:

(a) $|c| = \binom{D+1}{D/2}$ with $D = 2d \in 2\mathbb{N}$.

In Section 1 we prove the following result.

Theorem 0.4. *We have* $|\mathrm{Con}(c)| = \mathrm{Cat}_{d+1}$.

It is known (see [12, §2.13]) that if W is of type D then $|\mathrm{Con}(c)| = |\mathrm{Con}(c')|$ for some family c' in a Weyl group of type B or C. We will therefore assume in the rest of the paper that W is of type B or C.

0.5. According to [1, Corollary 4], we have

(a)
$$\mathrm{Cat}_n = \sum_{p=1}^{n} N(n, p)$$

where
$$N(n, p) = \frac{1}{n}\binom{n}{p}\binom{n}{p-1}$$

are the Narayana numbers.

We denote by F the field with two elements.

In [8] a bijection between $\mathrm{Con}(c)$ and a certain collection X_c of subgroups of F^d is described. For each p, $1 \le p \le d+1$, let $X_{c,p}$ be the set of subgroups of cardinal 2^{p-1} in X_c. The following refinement of Theorem 0.4 is proved in Section 2.

Theorem 0.6. *We have* $|X_{c,p}| = N_{d+1,p}$.

0.7. In Section 3 we state a conjecture according to which Catalan numbers appear in connection with the study of Springer fibers for G.

0.8. For any $i \le j$ in $\mathbb{Z}$ we set $[i, j] = \{h \in \mathbb{Z}; i \le h \le j\}$.

1. Proof of Theorem 0.4

1.1. Let $D \in 2\mathbb{N}$. Let V_D be an F-vector space with a nondegenerate symplectic form $\langle\cdot,\cdot\rangle : V_D \times V_D \to F$ and with a given subset $\{e_1, e_2, e_3, \dots, e_D\}$ such that $\langle e_i, e_j\rangle = 1$ if $i - j = \pm 1$ and $\langle e_i, e_j\rangle = 0$ otherwise.

Assuming that $D \ge 2$ and $i \in [1, D]$, we define a linear (injective) map $T_i : V_{D-2} \to V_D$ by:

- $e_a \mapsto e_a$ if $a < i-1$.
- $e_{i-1} \mapsto e_{i-1} + e_i + e_{i+1}$.
- $e_a \mapsto e_{a+2}$ if $a \ge i$.

(We regard V_{D-2} as a subspace of V_D in an obvious way.)

Let $\mathcal{F}(V_D)$ be the family of isotropic subspaces associated in [11, §1.17] to V_D and its basis $\{e_1, e_2, \dots, e_D\}$. (The characteristic functions of these subspaces form a basis of the $\mathbb{C}$-vector space of functions $V_D \to \mathbb{C}$.) We have a partition $\mathcal{F}(V_D) = \bigsqcup_{k\geq 0} \mathcal{F}^k(V_D)$. We will only give here the definition of $\mathcal{F}^0(V_D)$ and $\mathcal{F}^1(V_D)$. The definition is by induction on D. When $D=0$, $\mathcal{F}^0(V_D)$ consists of 0 and $\mathcal{F}^1(V_D)$ is empty. Assume now that $D \geq 2$. A subspace E of V_D is said to be in $\mathcal{F}^0(V_D)$ if either $E=0$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}^0(V_{D-2})$ such that $E = T_i(E') + Fe_i$ (this is a direct sum). A subspace E of V_D is said to be in $\mathcal{F}^1(V_D)$ if either $E = F(e_1+e_2+\cdots+e_D)$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}^1(V_{D-2})$ such that $E = T_i(E') + Fe_i$.

For example, if $D=2$, $\mathcal{F}^0(V_D)$ consists of $0, Fe_1, Fe_2$ and $\mathcal{F}^1(V_D)$ consists of $F(e_1+e_2)$. If $D=4$, $\mathcal{F}^0(V_D)$ consists of

$$0,\ Fe_1,\ Fe_2,\ Fe_3,\ Fe_4,\ Fe_1{+}Fe_3,\ Fe_1{+}Fe_4,\ Fe_2{+}Fe_4,\ F(e_1{+}e_2{+}e_3){+}Fe_2,\ F(e_2+e_3+e_4)+Fe_3$$

and $\mathcal{F}^1(V_D)$ consists of

$$F(e_1+e_2+e_3+e_4),\ F(e_1+e_2+e_3+e_4)+Fe_2,\ F(e_1+e_2+e_3+e_4)+Fe_3,\ F(e_1+e_2)+Fe_4,\ Fe_1+F(e_3+e_4).$$

We have

$$\mathcal{F}^0(V_D) = \mathcal{F}^0_{D/2}(V_D) \sqcup \mathcal{F}^0_{<D/2}(V_D)$$

where

$$\mathcal{F}^0_{D/2}(V_D) = \big\{E \in \mathcal{F}^0(V_D); \dim(E) = \tfrac{1}{2}D\big\},$$
$$\mathcal{F}^0_{<D/2}(V_D) = \big\{E \in \mathcal{F}^0(V_D); \dim(E) < \tfrac{1}{2}D\big\}.$$

1.2. Let $\mathcal{G}^0_D$ (resp. $\mathcal{G}^1_D$) be the set of lines in V_D of the form $F(e_a+e_{a+1}+\cdots+e_b)$ where $a \leq b$ in $[1, D]$ satisfy $b-a=1 \pmod 2$ (resp. $b-a=0 \pmod 2$). Let $\mathcal{G}_D = \mathcal{G}^0_D \sqcup \mathcal{G}^1_D$. For $E \in \mathcal{F}(V_D)$ let $B_E = \{L \in \mathcal{G}_D; L \subset E\}$. According to [12, §1.2(e), (f), (g)], if $E \in \mathcal{F}(V_D)$ then

(a) $$E = \bigoplus_{L \in B_E} L;$$

moreover we have $E \in \mathcal{F}^0(V_D)$ if and only if $B_E \subset \mathcal{G}^1_D$; we have $E \in \mathcal{F}^1(V_D)$ if and only if B_E contains a unique line L_E in $\mathcal{G}^0_D$.

It follows that if $E \in \mathcal{F}^1(V_D)$ we can write

(b) $E = E_0 \oplus L_E$ where $E_0 = \bigoplus_{L \in B_E; L \neq L_E} L$.

We show:

(c) $E_0 \in \mathcal{F}^0(V_D)$.

We argue by induction on D. If $D=0$ then $\mathcal{F}^1_D=\varnothing$ and there is nothing to prove. Assume now that $D\geq 2$. If $E=F(e_1+e_2+\cdots+e_D)$, then $E_0=0$ and (c) is obvious. If E is not of this form then there exists $i\in[1,D]$ and $E'\in\mathcal{F}^1_{D-2}$ such that $E=T_i(E')+Fe_i$. By the induction hypothesis we have $E'=E'_0\oplus L_{E'}$ where $E'_0\in\mathcal{F}^0_{D-2}$. We have $E=T_i(E'_0)+Fe_i+T_i(L_{E'})=\tilde{E}_0+\tilde{L}$ where $\tilde{E}_0=T_i(E'_0)+Fe_i\in\mathcal{F}^0(V_D)$ and $\tilde{L}=T_i(L_{E'})\in\mathcal{G}^0_D$ (from the definition of T_i). Since $\tilde{L}\subset E$ we must have $\tilde{L}=L_E$. We have $B_E=B_{\tilde{E}_0}\cup\{L_E\}$ (the union is disjoint since $B_{\tilde{E}_0}\subset\mathcal{G}^1_D,\ L_E\in\mathcal{G}^0_D$). Thus $B_{\tilde{E}_0}=B_E-\{L_E\}$. Since $\tilde{E}_0=\sum_{L\in B_{\tilde{E}_0}}=\sum_{L\in B_E-\{L_E\}}L=E_0$ we see that $E_0=\tilde{E}_0\in\mathcal{F}^0(V_D)$. This proves (c).

Note that in (c) we have $\dim(E)\leq\frac{1}{2}D,\ \dim(L_E)=1$, hence $\dim(E_0)<\frac{1}{2}D$. Thus we can define a map $\Xi_D:\mathcal{F}^1(V_D)\to\mathcal{F}^0_{<D/2}(V_D)$ by $E\mapsto E_0$ (notation of (c)).

We show:

(d) The map Ξ_D is surjective.

We argue by induction on D. If $D=0$ then $\mathcal{F}^0_{<D/2}(V_D)$ is empty and there is nothing to prove. Assume now that $D\geq 2$. Let $E_0\in\mathcal{F}^0_{<D/2}(V_D)$. If $E_0=0$ then $E=F(e_1+e_2+\cdots+e_D)$ is as required. Now assume that $E_0\neq 0$. Then there exists $i\in[1,D]$ and $E'_0\in\mathcal{F}^0(V_{D-2})$ such that $E_0=T_i(E'_0)\oplus Fe_i$. We see that $\dim(E'_0)=\dim(T_i(E'_0))=\dim(E_0)-1<\frac{1}{2}D-1=\frac{1}{2}(D-2)$ so that $E'_0\in\mathcal{F}^0_{<(D-2)/2}(V_{D-2})$. By the induction hypothesis there exists $L\in\mathcal{G}^0_{D-2}$ such that $E'_0+L\in\mathcal{F}^1(V_{D-2})$. Let $E=T_i(E'_0+L)+Fe_i$. We have $E\in\mathcal{F}^1(V_D)$ and $E=E_0+T_i(L)$. Note that $T_i(L)\in\mathcal{G}^0_D$ and is contained in E, hence it is equal to L_E (see (b)). It follows that $E_0=\Xi_D(E)$. This proves (d).

We show:

(e) Ξ_D is injective.

Assume that E,E' in $\mathcal{F}^1(V_D)$ satisfy $\Xi_D(E)=\Xi_D(E')$. We must show that $E=E'$.

We have $E=E_0\oplus L,\ E'=E_0\oplus L'$ where $E_0\in\mathcal{F}^0(V_D)$ and $L=F(e_a+e_{a+1}+\cdots+e_b),\ L'=F(e_{a'}+e_{a'+1}+\cdots+e_{b'})$, where $a<b$ and $a'<b'$ in $[1,D]$ satisfy $b-a=1\ (\mathrm{mod}\ 2),\ b'-a'=1\ (\mathrm{mod}\ 2)$. (In fact, from [11, §1.3(e), see (P_2)] we have that $a=1\ (\mathrm{mod}\ 2),\ b=0\ (\mathrm{mod}\ 2),\ a'=1\ (\mathrm{mod}\ 2),\ b'=0\ (\mathrm{mod}\ 2)$.) Assume first that $a<a'$ so that $a\leq a'-2$. From [11, §1.3(e), see (P_2)] we see that there exist $1\leq c\leq c'\leq D$ such that $c\leq a\leq c'$ and such that the line $\mathcal{L}=F(e_c+e_{c+1}+\cdots+e_{c'})$ is contained in E_0, hence also in $\mathcal{G}^1_D$. But then the pair of distinct lines $\mathcal{L},L$ would violate [11, §1.3(e), see (P_0)]. We see that we must have $a\geq a'$. Similarly we have $a'\geq a$, hence $a'=a$.

Assume next that $b<b'$ so that $b+2\leq b'$. From [11, §1.3(e), see (P_2)] we see that there exist $1\leq c\leq c'\leq D$ such that $c\leq b'\leq c'$ and such that the line

$\mathcal{L} = F(e_c + e_{c+1} + \cdots + e_{c'})$ is contained in E_0 hence also in $\mathcal{G}^1_D$. But then the pair of distinct lines $\mathcal{L}$, L' would violate [11, §1.3(e), see (P_0)]. We see that we must have $b \geq b'$. Similarly we have $b' \geq b$, hence $b' = b$.

We see that $L = L'$, hence $E = E'$. This proves (e).

1.3. From §1.2(c), (d), (e) we see that $|\mathcal{F}^0_{<D/2}(V_D)| = |\mathcal{F}^1(V_D)|$; hence

$$|\mathcal{F}^0(V_D)| - |\mathcal{F}^0_{D/2}(V_D)| = |\mathcal{F}^1(V_D)|,$$

that is, $|\mathcal{F}^0_{D/2}(V_D)| = |\mathcal{F}^0(V_D)| - |\mathcal{F}^1(V_D)|$. According to [11, §1.27] we have

$$|\mathcal{F}^0(V_D)| = \binom{D+1}{D/2}, \quad |\mathcal{F}^1(V_D)| = \binom{D+1}{(D-2)/2}.$$

It follows that

$$|\mathcal{F}^0_{D/2}(V_D)| = \binom{D+1}{D/2} - \binom{D+1}{(D-2)/2} = \frac{(2d+2)!}{(d+1)!(d+2)!} = C_{d+1},$$

where $D = 2d$.

1.4. In [5] the set c is identified with a subset of V_D. Now any object in $\mathrm{Con}(c)$ is multiplicity-free, hence may be identified with a subset of c, hence with a subset of V_D. This subset is a Lagrangian subspace of V_D. Thus $\mathrm{Con}(c)$ is identified with a subset of the set of Lagrangian subspaces of V_D. This subset is the same as $\mathcal{F}^0_{D/2}(V_D)$ (see [10, §2.8(iii)]). We see that $|\mathrm{Con}(c)| = C_{d+1}$ and Theorem 0.4 is proved.

1.5. An alternative proof of Theorem 0.4 can be given using the parametrization of $\mathrm{Con}(c)$ in terms of "admissible arrangements" in [6, p. 220].

2. Proof of Theorem 0.6

2.1. We preserve the notation of V_D. We have $V_D = V^0_D \oplus V^1_D$ where V^0_D has basis $\{e_2, e_4, \ldots, e_D\}$ and V^1_D has basis $\{e_1, e_3, \ldots, e_{D-1}\}$. Assuming that $D \geq 2$ we define for any $i \in [1, D]$ a linear map $\mathcal{T}_i : V^1_{D-2} \to V^1_D$ by:

- $e_k \mapsto e_k$ if $k \leq i - 2$.
- $e_k \mapsto e_{k+2}$ if $k \geq i$.
- $e_{i-1} \mapsto e_{i-1} + e_{i+1}$ if i is even.

Following [10, §2.3] we define a collection $\mathcal{C}(V^1_D)$ of subspaces of V^1_D by induction on D. If $D = 0$, $\mathcal{C}(V^1_D)$ consists of $\{0\}$. Assume now that $D \geq 2$. A subspace $\mathcal{E}$ of V^1_D is said to be in $\mathcal{C}(V^1_D)$ if either $\mathcal{E} = \{0\}$ or there exists $i \in [1, D]$ and $\mathcal{E}' \in \mathcal{C}(V^1_{D-2})$ such that:

- $\mathcal{E} = \mathcal{T}_i(\mathcal{E}') + Fe_i$ if i is odd.
- $\mathcal{E} = \mathcal{T}_i(\mathcal{E}')$ if i is even.

For example, $\mathcal{C}(V_2^1)$ consists of two subspaces: $0, Fe_1$; $\mathcal{C}(V_4^1)$ consists of five subspaces:

$$0,\ Fe_1,\ Fe_3,\ F(e_1+e_3),\ Fe_1+Fe_3;$$

$\mathcal{C}(V_6^1)$ consists of 14 subspaces:

$$0,\ Fe_1,\ Fe_3,\ Fe_5,\ F(e_1+e_3),\ F(e_3+e_5),\ F(e_1+e_3+e_5),\ Fe_1+Fe_3,$$
$$Fe_1+Fe_5,\ Fe_3+Fe_5,\ F(e_1+e_3)+Fe_5,\ Fe_1+F(e_3+e_5),\ F(e_1+e_3+e_5)+Fe_3,$$
$$Fe_1+Fe_3+Fe_5.$$

2.2. If $\mathcal{E}\in\mathcal{C}(V_D^1)$ we set $\mathcal{E}^! = \{x\in V_D^0;\ \langle x,\mathcal{E}\rangle = 0\}$. The following result appears in [10, §2.4].

(a) $\mathcal{E}\mapsto\mathcal{E}\oplus\mathcal{E}^!$ defines a bijection $\mathcal{C}(V_D^1)\xrightarrow{\sim}\mathcal{F}^0_{D/2}(V_D)$. The inverse bijection is given by $E\mapsto E\cap V_D^1$.

2.3. Let $\mathcal{Z}_D^*$ be the set of all elements of V_D^1 of the form

$$e_{a,b} = e_a + e_{a+2} + e_{a+4} + \cdots + e_b$$

for various numbers $a\le b$ in $\{1,3,\dots,D-1\}$.

For any $s\ge 0$, let $\mathcal{Z}_D^s$ be the set of all finite unordered sequences

$$e_{a_1,b_1}, e_{a_2,b_2},\dots,e_{a_s,b_s}$$

in $\mathcal{Z}_D^*$ such that for any $n\ne m$ in $\{1,2,\dots,s\}$ we have either

$$a_n\le b_n < a_m \le b_m \quad\text{or}\quad a_m\le b_m < a_n\le b_n,$$
$$\text{or}\quad a_n < a_m \le b_m < b_n \quad\text{or}\quad a_m < a_n\le b_n < b_m.$$

Let $\mathcal{Z}_D = \bigcup_{s\ge 0}\mathcal{Z}_D^s$ (a disjoint union).

For example, $\mathcal{Z}_2$ consists of the two sequences $\varnothing, \{e_1\}$; $\mathcal{Z}_4$ consists of the five sequences $\varnothing,\ \{e_1\},\ \{e_3\},\ \{e_1+e_3\},\ \{e_1,e_3\}$; and $\mathcal{Z}_6$ consists of 14 sequences:

$$\varnothing,\ \{e_1\},\ \{e_3\},\ \{e_5\},\ \{e_1+e_3\},\ \{e_3+e_5\},\ \{e_1+e_3+e_5\},$$
$$\{e_1,e_3\},\ \{e_1,e_5\},\ \{e_3,e_5\},\ \{e_1+e_3,e_5\},\ \{e_1,e_3+e_5\},\ \{e_1+e_3+e_5,e_3\},\ \{e_1,e_3,e_5\}.$$

Theorem 2.4. *The assignment*

$$\Theta_D : (e_{a_1,b_1}, e_{a_2,b_2},\dots,e_{a_s,b_s})\mapsto Fe_{a_1,b_1}+Fe_{a_2,b_2}+\cdots+Fe_{a_s,b_s}$$

defines a bijection $\mathcal{Z}_D\xrightarrow{\sim}\mathcal{C}(V_D^1)$.

When $D\le 6$ this follows from §2.1, §2.3. Note that Theorem 2.4 gives an order-preserving bijection between the set of noncrossing partitions (see [13]) and $\mathcal{C}(V_D^1)$ (with the order given by inclusion).

2.5. Assuming that $D\ge 2$ we define for any $i\in[1,D]$ a map $\sigma_i:\mathcal{Z}_{D-2}^*\to\mathcal{Z}_D^*$ by:

- $e_{a,b} \mapsto e_{a+2,b+2}$ if $i \leq a$.
- $e_{a,b} \mapsto e_{a,b+2}$ if $a < i \leq b+1$.
- $e_{a,b} \mapsto e_{a,b}$ if $i > b+1$.

Note that

- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b})$ if i is even.
- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b})$ if i is even and $i \leq a$ or $i > b$.
- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b}) + e_i$ if i is odd and $a < i \leq b$.

2.6. Assume that $D \geq 2$ and $i \in [1, D]$. Let $e_{a,b}, e_{a',b'}$ be in $\mathcal{Z}^*_{D-2}$ and let $e_{\tilde{a},\tilde{b}} = \sigma_i(e_{a,b})$, $e_{\tilde{a}',\tilde{b}'} = \sigma_i(e_{a',b'})$. We show:

(i) If $b < a'$ then $\tilde{b} < \tilde{a}'$.

(ii) If $a < a'$ and $b' < b$ then $\tilde{a} < \tilde{a}'$ and $\tilde{b}' < \tilde{b}$.

(iii) If i is odd and $\tilde{a} \leq i \leq \tilde{b}$ then $\tilde{a} < i < \tilde{b}$.

In the setup of (i) assume that $\tilde{a}' \leq \tilde{b}$. Then we have $a' \leq b$ or $a'+2 \leq b$ or $a'+2 \leq b+2$ or $a' \leq b+2$. The first three cases are clearly impossible; in the 4th case we have $b+2 = a'$ (since $b+2 \leq a' \leq b+2$), $b'+1 < i$ and $b+1 \geq i$, so that $b > b' \geq a'$, a contradiction.

In the setup of (ii) assume that $\tilde{a} \geq \tilde{a}'$. Then we have $a \geq a'$ or $a+2 \geq a'+2$ or $a \geq a'+2$ or $a+2 \geq a'$. The first three cases are clearly impossible; in the 4th case we have $a+2 = a'$ (since $a+2 \leq a' \leq a+2$), $a' < i$ and $a \geq i$, so that $a > a'$, a contradiction. Thus, $\tilde{a} < \tilde{a}'$.

Again, in the setup of (ii) assume that $\tilde{b}' \geq \tilde{b}$. Then we have $b' \geq b$ or $b'+2 \geq b+2$ or $b' \geq b+2$ or $b'+2 \geq b$. The first three cases are clearly impossible. In the 4th case we have $b'+2 = b$ (since $b \geq b'+2 \geq b$), $b+1 < i$ and $b'+1 \geq i$ so that $b' > b$, a contradiction. Thus, $\tilde{b}' < \tilde{b}'$.

In the setup of (iii) assume that $\tilde{a} = i$. We have $\tilde{a} = a$ or $\tilde{a} = a+2$. If $\tilde{a} = a$ we have $a = i$ and $b < i$, hence $b < \tilde{b}$ so that $\tilde{b} = b+2$; this implies $i \leq b$, a contradiction. If $\tilde{a} = a+2$ we have $a+2 = i$, $i \leq a$, a contradiction. Thus $\tilde{a} < i$.

In the setup of (iii) assume that $\tilde{b} = i$. We have $\tilde{a} = b$ or $\tilde{b} = b+2$. If $\tilde{b} = b$ we have $b = i$ and $b < i$, a contradiction. If $\tilde{b} = b+2$ we have $b+2 = i$ and either $a \geq i$ or $a < i \leq b$. In the first case we have $a \geq b+2 > b$, a contradiction; in the second case we have $b+2 \leq b$, a contradiction. Thus, $i < \tilde{b}$.

2.7. From §2.6(i)–(iii) we see that when $D \geq 2$ and $i \in [1, D]$, there is a well-defined map $\Sigma_i : \mathcal{Z}_{D-2} \to \mathcal{Z}_D$ given by

$$(e_{a_1,b_1}, e_{a_2,b_2}, \ldots, e_{a_s,b_s}) \mapsto \begin{cases} (\sigma_i(e_{a_1,b_1}), \sigma_i(e_{a_2,b_2}), \ldots, \sigma_i(e_{a_s,b_s}), e_i) & \text{if } i \text{ is odd,} \\ (\sigma_i(e_{a_1,b_1}), \sigma_i(e_{a_2,b_2}), \ldots, \sigma_i(e_{a_s,b_s})) & \text{if } i \text{ is even.} \end{cases}$$

2.8. Let $\epsilon \in \mathcal{Z}_D$, $\epsilon \neq \varnothing$. Let $e_{a,b} \in \epsilon$ be such that $b-a$ is minimum. If $b-a=0$ we set $i=a=b$; we have $i \in [1, D]$ and i is odd. If $b-a>0$ we define $i \in [1, D]$ by $a=i-1<i+1 \leq b$; then i is even. We will show that

(a) ϵ is in the image of $\Sigma_i : \mathcal{Z}_{D-2} \to \mathcal{Z}_D$.

If i is odd we can write $\epsilon = (e_{\tilde{a}_1,\tilde{b}_1}, e_{\tilde{a}_2,\tilde{b}_2}, \dots, e_{\tilde{a}_s,\tilde{b}_s}, e_i)$.

If i is even we can write $\epsilon = (e_{\tilde{a}_1,\tilde{b}_1}, e_{\tilde{a}_2,\tilde{b}_2}, \dots, e_{\tilde{a}_s,\tilde{b}_s})$, where $a_t=a,\ b_t=b$ for some t.

To $e_{\tilde{a}_t,\tilde{b}_t}$, $t=1,2,\dots,s$, we associate the element

$$\begin{aligned} e_{a_t,b_t} &= e_{\tilde{a}_t-2,\tilde{b}_t-2} && \text{if } i \leq \tilde{a}_t - 2,\\ e_{a_t,b_t} &= e_{\tilde{a}_t,\tilde{b}_t-2} && \text{if } \tilde{a}_t < i \leq \tilde{b}_t - 1,\\ e_{a_t,b_t} &= e_{\tilde{a}_t,\tilde{b}_t} && \text{if } \tilde{b}_t < i. \end{aligned}$$

(Note that we cannot have $i=\tilde{a}_t$ or $i=\tilde{b}_t$. Moreover when i is even we see from the definitions that we cannot have $i=\tilde{a}_t-1$.) This element is in $\mathcal{Z}^*_{D-2}$.

Consider $n \neq m$ in $\{1,2,\dots,s\}$. We set

$$(\tilde{a}_n, \tilde{b}_n, \tilde{a}_m, \tilde{b}_m) = (\tilde{a}, \tilde{b}, \tilde{a}', \tilde{b}'), \quad (a_n, b_n, a_m, b_m) = (a, b, a', b').$$

We show:

(i) If $\tilde{b} < \tilde{a}'$, then $b < a'$.

(ii) If $\tilde{a}' < \tilde{a} \leq \tilde{b} < \tilde{b}'$, then $a' < a \leq b < b'$.

In the setup of (i) assume that $a' \leq b$. Then we have $\tilde{a}' \leq \tilde{b}$ or $\tilde{a}'-2 \leq \tilde{b}$ or $\tilde{a}'-2 \leq \tilde{b}-2$ or $\tilde{a}' \leq \tilde{b}-2$. The first three cases are clearly impossible. In the 4th case we have $\tilde{b} < \tilde{a}' \leq \tilde{b}-2$, hence $\tilde{b} < \tilde{b}-2$, a contradiction. Thus $b < a'$.

In the setup of (ii), a', a, b, b' are as follows:

- $\tilde{a}'-2,\ \tilde{a}-2,\ \tilde{b}-2,\ \tilde{b}'-2$ if $i \leq \tilde{a}'-2$.
- $\tilde{a}',\ \tilde{a}-2,\ \tilde{b}-2,\ \tilde{b}'-2$ if $\tilde{a}' < i \leq \tilde{a}-2$ (so that $\tilde{a}' < \tilde{a}-2$).
- $\tilde{a}',\ \tilde{a},\ \tilde{b}-2,\ \tilde{b}'-2$ if $\tilde{a} < i \leq \tilde{b}-1$ (so that $\tilde{a} \leq \tilde{b}-2$).
- $\tilde{a}',\ \tilde{a},\ \tilde{b},\ \tilde{b}'-2$ if $\tilde{b} < i \leq \tilde{b}'-2$ (so that $\tilde{b} < \tilde{b}'-2$).
- $\tilde{a}',\ \tilde{a},\ \tilde{b},\ \tilde{b}'$ if $\tilde{b}' < i$.

Since i is distinct from each of $\tilde{a}',\ \tilde{a}'-1,\ \tilde{a},\ \tilde{a}-1,\ \tilde{b},\ \tilde{b}',\ \tilde{b}'-1$ we see that we must be in one of the five cases above. Note that $a' < a \leq b < b'$ in each case.

From (i), (ii) we see that $\epsilon' := (e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s})$ belongs to $\mathcal{Z}_{D-2}$. From the definitions we see that $\epsilon = \Sigma_i(\epsilon')$. Hence (a) holds.

2.9. We define a subset $\mathcal{Z}'_D$ of $\mathcal{Z}_D$ by induction on D. If $D=0$, $\mathcal{Z}'_D$ consists of the empty sequence. Assume now that $D \geq 2$. A sequence $\epsilon \in \mathcal{Z}_D$ is said to be in $\mathcal{Z}'_D$ if either ϵ is the empty sequence or if there exists $i \in [1, D]$ and $\epsilon' \in \mathcal{Z}'_{D-2}$ such that $\epsilon = \Sigma_i(\epsilon')$. (Note that $\Sigma_i(\epsilon')$ is well defined.) Using §2.8(a) we see by induction on D that

(a) $\mathcal{Z}_D = \mathcal{Z}'_D$.

2.10. Assume that $D \geq 2$ and $i \in [1, D]$. For $\epsilon' \in \mathcal{Z}_{D-2}$ we show:

(a) $\Theta_D(\Sigma_i(\epsilon')) = \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i$ if i is odd.

(b) $\Theta_D(\Sigma_i(\epsilon')) = \mathcal{T}_i(\Theta_{D-2}\epsilon')$ if i is even.

We can write $\epsilon' = (e_{a_1,b_1}, e_{a_2,b_2}, \ldots, e_{a_s,b_s})$. Then

$$\Theta_D(\Sigma_i(\epsilon')) = \begin{cases} F\sigma_i(e_{a_1,b_1}) + F\sigma_i(e_{a_2,b_2}) + \cdots + F\sigma_i(e_{a_s,b_s}) + Fe_i & \text{if } i \text{ is odd,} \\ F\sigma_i(e_{a_1,b_1}) + F\sigma_i(e_{a_2,b_2}) + \cdots + F\sigma_i(e_{a_s,b_s}) & \text{if } i \text{ is even.} \end{cases}$$

Using the definitions we see that

$$\begin{aligned}\Theta_D(\Sigma_i(\epsilon')) &= F\mathcal{T}_i(e_{a_1,b_1}) + F\mathcal{T}_i(e_{a_2,b_2}) + \cdots + F\mathcal{T}_i(e_{a_s,b_s}) + Fe_i \\ &= \mathcal{T}_i(Fe_{a_1,b_1} + Fe_{a_2,b_2} + \cdots + Fe_{a_s,b_s}) + Fe_i = \mathcal{T}_i(X_{D-1}(\epsilon')) + Fe_i\end{aligned}$$

if i is odd,

$$\begin{aligned}\Theta_D(\Sigma_i(\epsilon')) &= F\mathcal{T}_i(e_{a_1,b_1}) + F\mathcal{T}_i(e_{a_2,b_2}) + \cdots + F\mathcal{T}_i(e_{a_s,b_s}) \\ &= \mathcal{T}_i(Fe_{a_1,b_1} + Fe_{a_2,b_2} + \cdots + Fe_{a_s,b_s}) = \mathcal{T}_i(X_{D-1}(\epsilon'))\end{aligned}$$

if i is even. This proves (a), (b).

2.11. We prove the following part of Theorem 2.4.

(a) The map Θ_D in Theorem 2.4 is well defined (it has image contained in $\mathcal{C}(V_D^1)$).

We argue by induction on D. When $D = 0$, (a) is obvious. Assume now that $D \geq 2$. Let $\epsilon \in \mathcal{Z}_D$. If $\epsilon = \varnothing$ then $\Theta_D(\epsilon) = 0 \in \mathcal{F}_D$. Assume now that $\epsilon \neq \varnothing$. Using §2.8, we can find $i \in [1, D]$ and $\epsilon' \in \mathcal{Z}_{D-2}$ such that $\epsilon = \Sigma_i(\epsilon')$ so that $\Theta_D(\epsilon) = \Theta_D(\Sigma_i(\epsilon'))$. By the induction hypothesis we have $\Theta_{D-2}\epsilon' \in \mathcal{C}(V_{D-2}^1)$. By the definition of $\mathcal{C}(V_D^1)$ we then have

$$\begin{aligned} &\mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i \in \mathcal{C}(V_D^1) && \text{if } i \text{ is odd;} \\ &\mathcal{T}_i(\Theta_{D-2}\epsilon') \in \mathcal{C}(V_D^1) && \text{if } i \text{ is even.}\end{aligned}$$

Using §2.10, we can rewrite this as $\Theta_D(\epsilon) \in \mathcal{C}(V_D^1)$. This proves (a).

2.12. We prove the following part of Theorem 2.4.

(a) The map Θ_D in Theorem 2.4 (see §2.11(a)) is surjective.

We argue by induction on D. When $D = 0$, (a) is obvious. Assume now that $D \geq 2$. Let $\mathcal{E} \in \mathcal{C}(V_D^1)$. If $\mathcal{E} = 0$ then $\mathcal{E} = \Theta_D(\varnothing)$. Assume now that $\mathcal{E} \neq 0$. We can find $i \in [1, D]$ and $\mathcal{E}' \in \mathcal{C}(V_{D-2}^1)$ such that $\mathcal{E} = \mathcal{T}_i(\mathcal{E}') + Fe_i$ if i is odd and $\mathcal{E} = \mathcal{T}_i(\mathcal{E}')$ if

i is even. By the induction hypothesis we have $\mathcal{E}' = \Theta_{D-2}(\epsilon')$ for some $\epsilon' \in \mathcal{Z}_{D-2}$. Thus we have $\mathcal{E} = \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i$ if i is odd, $\mathcal{E} = \mathcal{T}_i(\Theta_{D-2}\epsilon')$ if i is even. Using §2.10 we can rewrite this as $\mathcal{E} = \Theta_D(\epsilon)$ where $\epsilon = \Sigma_i(\epsilon') \in \mathcal{Z}_D$. This proves (a).

2.13. We have $\mathcal{C}(V_D^1) = \bigsqcup_{s \in [0,d]} \mathcal{C}^s(V_D^1)$ where $\mathcal{C}^s(V_D^1) = \{\mathcal{E} \in \mathcal{C}(V_D^1); \dim \mathcal{E} = s\}$. Clearly, the map Θ in Theorem 2.4 restricts to a map $\Theta^s : \mathcal{Z}_D^s \to \mathcal{C}^s(V_D^1)$ for any $s \in [0, d]$. From §2.12(a) it follows that Θ^s is surjective for any $s \in [0, d]$. In [1] it is shown that $|\mathcal{Z}_D^s| = N_{d+1,s+1}$ (see §0.5) for any $s \in [0, d]$. Using this and §0.5(a) we see that

$$\mathrm{Cat}_{d+1} = \sum_{s \in [0,d]} N(d+1, s+1) = \sum_{s \in [0,d]} |\mathcal{Z}_D^s| = |\mathcal{Z}_D|.$$

We see that Θ_D is a surjective map from a set with cardinal $|\mathcal{Z}_D| = \mathrm{Cat}_{d+1}$ to a set with the same cardinal $|\mathcal{C}(V_D^1)| = |\mathcal{F}_{D/2}^0(V_D)| = \mathrm{Cat}_{d+1}$ (the first equality holds by §2.2(a); the second equality follows from Theorem 0.4). It follows that Θ is a bijection and Theorem 2.4 is proved.

This implies that $\Theta^s : \mathcal{Z}_D^s \to \mathcal{C}^s(V_D^1)$ is a bijection for any $s \in [0, d]$. We see that Theorem 0.6 holds. (We use that X_c in §0.5 is the same as $\mathcal{C}^s(V_D^1)$ if we identify $V_D^1 = F^d$.)

3. A conjecture

3.1. In this section we fix a unipotent element $u \in G$. We assume that either

- G is of type $C_{d(d+1)}$, $d \geq 1$ and u has Jordan blocks of sizes $2d$, $2d$, $2d-2$, $2d-2, \ldots, 2, 2$ or that
- G is of type $B_{d(d+1)}$, $d \geq 1$ and u has Jordan blocks of sizes $2d+1$, $2d-1$, $2d-1, \ldots, 1, 1$.

Let $\mathcal{B}_u$ be the variety of Borel subgroups of G that contain u and let $[\mathcal{B}_u]$ be the set of irreducible components of $\mathcal{B}_u$. Let $A(u)$ be the group of components of the centralizer of u in G. Note that $A(u)$ acts naturally by permutations on $[\mathcal{B}_u]$. For each $\xi \in [\mathcal{B}_u]$ we denote by $A(u)_\xi$ the stabilizer of ξ in $A(u)$. Let Δ_u be the set of subgroups of $A(u)$ of the form $A(u)_\xi$ for some $\xi \in [\mathcal{B}_u]$.

We assume that c is the family containing the Springer representation of W associated to u and to the unit representation of $A(u)$. We conjecture that

(a) *there exists an isomorphism* $A(u) \xrightarrow{\sim} V_D^1$, $D = 2d$ *which carries* Δ_u *to the collection* $\mathcal{C}(V_D^1)$ (*see* §2.1) *of subspaces of* V_D^1.

(This would imply that $|\Delta_u|$ is a Catalan number.)

We have verified that (a) is true when $d = 1, 2, 3$.

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Received April 7, 2024. Revised July 26, 2024.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 336 No. 1-2 May 2025

In memoriam
Gary Seitz

Gary Seitz (1943–2023): In memoriam 1
MARTIN W. LIEBECK, GERHARD RÖHRLE and DONNA TESTERMAN

Intrinsic components in involution centralizers of fusion systems 7
MICHAEL ASCHBACHER

On good A_1 subgroups, Springer maps, and overgroups of distinguished unipotent elements in reductive groups 29
MICHAEL BATE, SÖREN BÖHM, BENJAMIN MARTIN and GERHARD RÖHRLE

The q-Schur category and polynomial tilting modules for quantum GL_n 63
JONATHAN BRUNDAN

The binary actions of simple groups of Lie type of characteristic 2 113
NICK GILL, PIERRE GUILLOT and MARTIN W. LIEBECK

Finite simple groups have many classes of p-elements 137
MICHAEL GIUDICI, LUKE MORGAN and CHERYL E. PRAEGER

Monogamous subvarieties of the nilpotent cone 161
SIMON M. GOODWIN, RACHEL PENGELLY, DAVID I. STEWART and ADAM R. THOMAS

An extension of Gow's theorem 181
ROBERT M. GURALNICK and PHAM HUU TIEP

On dimensions of RoCK blocks of cyclotomic quiver Hecke superalgebras 191
ALEXANDER KLESHCHEV

Representation growth of Fuchsian groups and modular forms 217
MICHAEL J. LARSEN, JAY TAYLOR and PHAM HUU TIEP

D_4-type subgroups of $F_4(q)$ 249
R. LAWTHER

Constructible representations and Catalan numbers 339
GEORGE LUSZTIG and ERIC SOMMERS

A reduction theorem for simple groups with $e(G) = 3$ 351
RICHARD LYONS and RONALD SOLOMON

Decomposition numbers in the principal block and Sylow normalisers 367
GUNTER MALLE and NOELIA RIZO

Levi decompositions of linear algebraic groups and nonabelian cohomology 379
GEORGE J. MCNINCH

On the intersection of principal blocks 399
GABRIEL NAVARRO, A. A. SCHAEFFER FRY and PHAM HUU TIEP

Hesselink strata in small characteristic and Lusztig–Xue pieces 415
ALEXANDER PREMET

Multiplicity-free representations of the principal A_1-subgroup in a simple algebraic group 433
ALUNA RIZZOLI and DONNA TESTERMAN