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A REDUCTION THEOREM FOR SIMPLE GROUPS WITH $e(G) = 3$

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In memory of our dear friend, Gary Seitz

This paper highlights the use of transfer to analyze an unbalancing configuration in the GLS classification project for finite simple groups. The configuration occurs when $e(G) = 3$, $m_{2,p}(G) = m_p(G) = 3$ for some odd prime p, and a p-component of p-rank 1 is responsible for the unbalancing.

1. Introduction

An important step in the GLS approach to the classification of the $\mathcal{K}$-proper finite simple groups G of restricted even type with $e(G) \geq 3$, as it was for Michael Aschbacher [1981; 1983] in his classification of such groups of characteristic-2 type with $e(G) = 3$, is to establish a dichotomy quite analogous to the Gorenstein–Walter alternative for generic groups [Gorenstein et al. 2002, Theorem $\mathcal{C}_7^*$: Stage 1, Chapter 3, p. 49].

Throughout this paper, G is a $\mathcal{K}$-proper finite simple group of restricted even type.

For a suitable odd prime divisor p of G such that $m_{2,p}(G) = e(G)$, and a suitable subgroup $A \cong E_{p^n}$ of G, $n \geq 3$, the dichotomy is, roughly speaking:

(1A1) the p-layer $L_{p'}(C_G(a))$ of $C_G(a)$ is semisimple for all $a \in A^{\#}$; or

(1A2) G has a strong p-uniqueness subgroup M.

Without defining the technical term "strong p-uniqueness subgroup", we remark that the archetype of a strong p-uniqueness subgroup is a strongly p-embedded subgroup such that $O_{p'}(M)$ has even order. However, various weaker conditions also suffice to qualify a subgroup M as a strong p-uniqueness subgroup. See, for example, [Gorenstein et al. 1994, Chapter 2, Section 8] for the case $m_{2,p}(G) \geq 4$. And for the case $m_{2,p}(G) = 3 = e(G)$, GLS is using the condition $\Gamma^o_{P,2}(G) \leq M$ for some $P \in \mathrm{Syl}_p(G)$ instead of requiring strong p-embedding.

We shall call this the fundamental dichotomy, and its verification when $e(G) \geq 4$, along with the identification of G in (1A1), forms a large part of [Gorenstein et al.

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2002; 2018a; 2018b]. When $e(G) = 3 = m_{2,p}(G) < m_p(G)$ for some odd prime p, G has been classified in [Capdeboscq et al. 2023] (and without any reference to strong p-uniqueness subgroups).

In this paper, we shall consider the remaining case in which $e(G) = 3$, so that $m_p(G) = 3$ for any odd prime p such that $m_{2,p}(G) = 3$. To avoid complications that are of little interest, we shall assume for simplicity that there exists such a prime p for which $p > 3$. In Section 3, we will reduce the proof of the fundamental dichotomy to one very specific subcase, and in Section 4, we will outline our proof that the fundamental dichotomy holds in this subcase as well. As noted above, when G is of characteristic-2 type (a somewhat more restrictive condition than restricted even type) and $e(G) = 3$, this result was proved by Aschbacher. Our approach to the necessary signalizer analysis, however, is different from his, using elementary p-transfer arguments to reduce to the single subcase of Section 4, where signalizer functors seem to be unavoidable. We thank Michael for his assistance in the treatment of this subcase.

Throughout we shall focus on an odd prime divisor p of $|G|$. For any subgroup H of G such that $H = L_{p'}(H)$, we shall let

$$H^{\dagger} = H/O_{p'}(H).$$

In the Gorenstein–Walter alternative, the second option is that G is $\frac{3}{2}$-balanced with respect to A, and $\Theta_{3/2}(G; A)$ is a nontrivial p'-group. This option is then to be pursued further, to the existence of a strong p-uniqueness subgroup, the first approximation being $N_G(\Theta_{3/2}(G; A))$. A similar connection between $\frac{3}{2}$-balance and strong p-uniqueness subgroups can be seen in the context of this paper. Hence, given our purpose here, we shall assume the following proposition.

Proposition 1.1. *Let G be a $\mathcal{K}$-proper finite group of even type with $e(G) = 3$. Let $p > 3$ be a prime such that $m_{2,p}(G) = m_p(G) = 3$. Let $A \le G$ with $A \cong E_{p^3}$. Then either the fundamental dichotomy holds for G, p, and A, or A can be chosen such that for some $x, a, b \in A^{\#}$ and $L \le C_G(x)$, $A = \langle x, a, b\rangle$, L is an A-invariant p-component of $C_G(x)$, and*

$$[O_{p'}(C_{\mathrm{Aut}_{C_G(x)} L^{\dagger}}(b)), a] \ne 1.$$

The conclusion of this proposition, in the language of [Gorenstein et al. 2002, Chapter 2, 3.2–3.4], is that (x, L) *is a* $\frac{3}{2}$, *A-obstruction*, i.e., an obstruction to A's being $\frac{3}{2}$-balanced in G. In the absence of such obstructions, the fundamental dichotomy follows by applications of [Gorenstein et al. 1996, 20.7, 5.19(i)], and the standard construction (due to Aschbacher) of a $\frac{3}{2}$-balanced functor on A [Gorenstein et al. 1996, 21.9, 21.10]. Accordingly, as this functor is trivial or nontrivial, (1A1) or (1A2) can be proved. (The reader may have noticed that there is a second type of possible obstruction, due to failure of local 2-balance. But such obstructions cannot occur, it turns out, under our hypotheses that $p > 3$, $e(G) = 3$, and G is $\mathcal{K}$-proper.)

2. Statement of results

In view of the foregoing, we shall proceed under the following hypothesis:

(2A1) G is a $\mathcal{K}$-proper finite simple group of restricted even type with $e(G) = 3$.

(2A2) For all odd primes q such that $m_{2,q}(G) = 3$, we have $m_q(G) = 3$.

(2A3) There is a prime $p > 3$ such that $m_p(G) = m_{2,p}(G) = 3$.

(2A4) There is an elementary abelian p-subgroup A of G with $m_p(A) = 3$ and a $\frac{3}{2}$, A-obstruction (x, L) in $C_G(x)$ for some $x \in A^{\#}$.

As G is a $\mathcal{K}$-proper finite simple group, every proper simple section of G is a known simple group. The principal application of the $\mathcal{K}$-proper assumption (in conjunction with the assumption that $p > 3$ and $m_p(G) = 3$) will be via the following $\mathcal{K}$-group fact, which we accept here without proof. The assumption $p > 3$ is critical for this proposition.

Proposition 2.1. *Let G, p and A be as above with (x, L) a $\frac{3}{2}$, A-obstruction. Then $L^{\dagger}$ is a simple group of Lie type in characteristic $s \neq p$ with $m_p(L) = 1$ and $L \lhd N_G(\langle x \rangle)$. Also, $A = \langle x, a, b\rangle$, where $b \in L$ and a induces a field automorphism of order p on $L^{\dagger}$. Moreover, $[O_{p'}(C_{L^{\dagger}}(b)), a] \neq 1$ and $L_{p'}(C_{L^{\dagger}}(a)) \neq 1$.*

We will therefore assume that the following configuration holds (here we let $C_y := C_G(y)$ for all $y \in A^{\#}$):

(2B1) $x \in \mathcal{I}_p(G)$ ($\mathcal{I}_p(X)$ is the set of elements of X of order p).

(2B2) $P \in \mathrm{Syl}_p(G)$ and $T = C_P(x) \in \mathrm{Syl}_p(C_x)$.

(2B3) L is a p-component of C_x and $R = \langle r \rangle = T \cap L$ is cyclic with $\Omega_1(R) = \langle b \rangle$.

(2B4) $Q = C_T(L^{\dagger})$.

(2B5) $\mathcal{I}_p(T - QR) \neq \varnothing$.

(2B6) For any $a \in \mathcal{I}_p(T - QR)$, $Q_a = C_Q(a)$, and $P_a \in \mathrm{Syl}_p(C_a)$ with $C_P(a) \leq P_a$.

(2B7) For any $a \in \mathcal{I}_p(T - QR)$, a induces a nontrivial field automorphism on $L^{\dagger}$, $L_a := L_{p'}(C_L(a)) \neq 1$, and K_a is the subnormal closure of L_a in C_a.

Our main aim in this paper is to prove:

Theorem 2.2. *Assume* (2A1)–(2A4) *and* (2B1)–(2B7). *Then*:

(a) $T = P$.

(b) $Q \cong R$.

(c) *P/QR is cyclic and embeds in* $\mathrm{Out}(L^{\dagger})$; *the image of P/QR consists of images of field automorphisms.*

(d) *There exists $a \in \mathcal{I}_p(P - QR)$ such that $a^P = a\langle x, b\rangle = a\Omega_1(QR)$.*

(e) $L_{p'}(C_a) = L_1L_2$ *with* $L_1^{\dagger} \cong L_2^{\dagger} \cong L_a^{\dagger}$, $Q_a \in \mathrm{Syl}_p(L_1)$, *and* $R_a := C_R(a) \in \mathrm{Syl}_p(L_2)$.

(f) *There exists $t \in N_G(P)$ inverting a and interchanging $\langle x\rangle$ with $\langle b\rangle$, Q with R, and L_1 with L_2.*

As a consequence, GLS will obtain the following theorem in a forthcoming volume. We shall make a few remarks about the proof of this result in Section 4.

Theorem 2.3. *Suppose that (2A1)–(2A4) and (2B1)–(2B7) hold. Then G has a strongly p-embedded subgroup M such that $O_{p'}(M)$ has even order.*

3. Proof of Theorem 2.2

In this section we prove Theorem 2.2. We let

(3A)
$$\mathfrak{A} = \{a_0 \in \mathcal{I}_p(T - QR) \mid K_{a_0} \text{ is a nontrivial pumpup of } L_{a_0}\};$$
$$E = \langle x, b\rangle = A \cap QR.$$

Lemma 3.1. *The following conditions hold*:

(a) $L \lhd C_x$.

(b) *For every $a \in \mathfrak{A}$, K_a is a vertical pumpup of L_a.*

(c) $\Omega_1(T) \le Q\langle a\rangle \times \langle b\rangle$ *for any $a \in \mathcal{I}_p(T - PQ)$.*

(d) *Q_a is cyclic for any $a \in \mathcal{I}_p(T - PQ)$.*

(e) $E = \Omega_1(Z(T))$.

Proof. Proposition 2.1 contains (a). As $m_p(G) = 3$, (d) holds and K_a cannot be a diagonal pumpup of L_a. Then (b) holds by $L_{p'}$-balance and the definition of $\mathfrak{A}$. Since $\Omega_1(T/QR) = \langle QRa\rangle$, we have $\Omega_1(T/Q) \le QR\langle a\rangle/Q$. As R is cyclic, $\Omega_1(R\langle a\rangle) = \langle b, a\rangle$, and (c) follows. As $m_p(G) = 3$ and $L \lhd C_x$, we have $E \le \Omega_1(Z(T)) \le E\langle a\rangle$ for any $a \in \mathcal{I}_p(T - PQ)$. But as a induces a field automorphism of order p on $L^{\dagger}$, $[R, a] \ne 1$. This implies (e). □

Lemma 3.2. *We have that $m_p(Q) = 1$.*

Proof. Suppose not and choose $B \cong E_{p^2}$ with $B \lhd T$ and $B \le Q$.

Suppose first that $x^g \in E - \langle x\rangle$ for some $g \in G$. Then by Lemma 3.1(e) and Burnside's lemma, we may assume that $g \in N_G(T)$. Now, $E = E^g$ and so $E \cap L^g = (E \cap L)^g = \langle b^g\rangle$. Let $Q_0 = C_{Q\langle a\rangle}((L^g)^{\dagger})$. As $Q\langle a\rangle$ normalizes L^g, we have $Q_0 \lhd Q\langle a\rangle$. Hence, if $Q_0 \ne 1$, then $Q_0 \cap \Omega_1(Z(Q\langle a\rangle)) \ne 1$, and so in view of Lemma 3.1(d), $x \in Q_0$. But then $E = \langle x^g, x\rangle \le C_T((L^g)^{\dagger})$, contrary to

$E \cap L^g \neq 1$. Hence, $Q\langle a\rangle$ acts faithfully on $(L^g)^\dagger$, so $Q\langle a\rangle$ embeds in T/Q. But $\Omega_1(T/Q) \cong \Omega_1(R\langle a\rangle) = \langle a, b\rangle$, so $\Omega_1(Q\langle a\rangle) = \langle a, x\rangle$. But $B\langle a\rangle \leq \Omega_1(Q\langle a\rangle)$, a contradiction. Hence $x^G \cap E \subseteq \langle x\rangle$. Thus, $\langle x\rangle \lhd N_G(P)$, so $P = T$.

Since Q_a is cyclic, $[B, a] \neq 1$, so B shears a to $\langle x\rangle$, centralizing $\langle b\rangle$. Likewise R shears a to $\langle b\rangle$, centralizing $\langle x\rangle$.

Suppose that $a^g \in E$ for some $g \in G$. Then $P_a \in \mathrm{Syl}_p(G)$. Then E and $\Omega_1(Z(P_a))$ are commuting E_{p^2}-subgroups of P_a, so $E \cap \Omega_1(Z(P_a))$ contains some $d \neq 1$. As $a \notin E$, $\Omega_1(Z(P_a)) = \langle a, d\rangle$. But then by the previous paragraph, some p-element in $\mathrm{Aut}_G(\langle a, d\rangle)$ shears a to $\langle d\rangle$. As $\langle a, d\rangle$ is contained in a Sylow p-center, however, $\mathrm{Aut}_G(\langle a, d\rangle)$ is a p'-group, a contradiction. Hence $a^G \cap E = \varnothing$. Consequently, $\langle x\rangle$ is weakly closed in $\langle a, b, x\rangle$.

Now let d be an extremal conjugate of a in P. Then we may choose $g \in G$ such that $a^g = d$ and $C_P(a)^g \leq C_P(d)$. As $d \notin E$, we have $\langle b, x, d\rangle = \Omega_1(C_P(d))$, so $\langle a, b, x\rangle^g = \langle b, x, d\rangle$. By the weak closedness of $\langle x\rangle$, $g \in N := N_G(\langle x\rangle)$. As $QR \in \mathrm{Syl}_p(LC_N(L^\dagger))$, $LC_N(L^\dagger) \lhd N$, and the image of P/QR in $\mathrm{Out}(L^\dagger)$ is disjoint from $[\mathrm{Out}(L^\dagger), \mathrm{Out}(L^\dagger)]$, we must have $a^g \in QRa$. But P/QR is cyclic, so by the Thompson transfer lemma [Gorenstein et al. 1996, 15.15], $a \notin O^p(G)$, contradicting the simplicity of G. The proof is complete. □

We immediately deduce:

Lemma 3.3. *We have* $A = \Omega_1(T)$.

Lemma 3.4. *Suppose that* $a \in \mathfrak{A}$, *with* x *inducing a nontrivial field automorphism on* $K_a^\dagger$. *Then*:

(a) $C_Q(a) = \langle x\rangle$ *and* $|Q| \leq p^2$.

(b) *If* $|Q| = p^2$, *then* $T = QR\langle a\rangle$ *and* $\mathcal{E}_1(T) - \mathcal{E}_1(QR) \subseteq \langle a\rangle^{C_x}$.

(c) *There is a complement* F *to* QR *in* T *such that* F *faithfully induces field automorphisms on* $L^\dagger$.

Proof. As $C_Q(a)$ acts faithfully on $K_a^\dagger$ centralizing the image of $L_a = L_{p'}(C_{K_a}(x))$, we have $C_Q(a) = \langle x\rangle$. Therefore $|Q| \leq p^2$, so (a) holds. By the structure of $\mathrm{Aut}(L^\dagger)$, there is a complement F_1 to R in T with $a \in F_1$.

Suppose first that $Q = \langle x\rangle$. Then $F_1/\langle x\rangle$ is cyclic. If F_1 is cyclic, then $\Omega_1(T) = \langle x\rangle \times \Omega_1(R)$, contrary to $a \in \Omega_1(T)$. Hence, F_1 is noncyclic abelian with $F_1/\langle x\rangle$ cyclic, whence $F_1 = F \times \langle x\rangle$ for some F which induces field automorphisms on $L^\dagger$, so (c) holds in this case.

If Q has order p^2, then as $C_Q(a) = \langle x\rangle$, $[Q, a] \neq 1$. But T/QR is cyclic and $[QR, Q] = 1$, so T/QR embeds in $\mathrm{Aut}(Q)$ and hence has order p. Thus $T = QR\langle a\rangle$ and Q shears a to $\langle x\rangle$. As R shears a to $\langle b\rangle$, (b) and (c) hold and the proof is complete. □

Lemma 3.5. *If* $\mathfrak{A} = \varnothing$, *then the conclusions of Theorem 2.2 hold.*

Proof. If $\mathfrak{A} = \varnothing$, then L_a pumps up trivially in C_a for every $a \in A - E$. As $L = L_{p'}(C_x) \ncong L_a$ and $A = \Omega_1(T)$, $x^G \cap T \subseteq E$. If $\langle x \rangle$ is weakly closed in E, then $T = P$ and $\langle x \rangle$ is weakly closed in P; so by [Gorenstein et al. 1996, 16.20], $N_G(\langle x \rangle)$ controls G-fusion, hence G-transfer, in P. As $a \notin [N_G(\langle x \rangle), N_G(\langle x \rangle)]$, we conclude that $a \notin [G, G]$, contradicting the simplicity of G. So there exists $t \in N_G(T)$ such that $x_1 := x^t \in x^G \cap E - \langle x \rangle$. Let $L_0^t = L_{p'}(C_{L^t}(a))$ and let M_a be the (trivial) pumpup of L_0^t in C_a. As $C_E(K_a^\dagger) = \langle x \rangle \neq \langle x_1 \rangle = C_E(M_a^\dagger)$, while K_a and M_a are normal in $L_{p'}(C_a)$, we have $[K_a, M_a] \leq O_{p'}(C_a)$ with $\mho^1(R) \in \mathrm{Syl}_p(K_a)$ and $b^t \in M_a$. Now $\langle b^t \rangle \in C_E(K_a^\dagger) = \langle x \rangle$. But t was arbitrary in $N_G(T) - N_G(\langle x \rangle)$. Hence we must have $\langle x \rangle^{N_G(T)} = \{\langle x \rangle, \langle b \rangle\}$. In particular, $P = T$, as p is odd.

If $Q = Q_a$, then $\langle b \rangle = [T, \Omega_1(T)] \lhd N_G(T)$, contradicting $b^t = x$. Hence, $|Q : Q_a| = p$ and $a^T = Ea$. Also $K_a M_a = L_{p'}(C_G(a))$ as $m_p(G) = 3$. Also $Q \times R = J_a(T)$. In particular, $\langle a \rangle^{N_G(T)} = \langle a \rangle^T$, whence we may modify t by an element of T and assume that $t \in N_G(\langle a \rangle)$.

Indeed, there exists a p'-element $h \in N_L(QR)$ such that $[QR, h] = R$ and $C_{QR}(h) = Q$. As $T^t = T$, $QR = (QR)^t$ induces inner automorphisms on $(L^t)^\dagger = L_{p'}(C_G(b))^\dagger$, and QR is invariant under $h \in N_G(\langle b \rangle)$. Therefore the only two largest h-invariant cyclic subgroups of QR are $\{Q, R\} = \{C_{QR}(h), [QR, h]\} = \{C_{QR}((L^t)^\dagger), QR \cap L^t\}$, so $R = C_{QR}((L^t)^\dagger)$ and $Q = QR \cap L^t$. So t interchanges Q and R.

Now $E = Z(P)$ is the weak closure of $\langle x \rangle$ in P, so by the Hall–Wielandt theorem (see [Gorenstein et al. 1996, 15.27]) and the simplicity of G, we have $N_G(E) = O^p(N_G(E))$. However, $N_G(E) = \big(N_G(E) \cap N_G(\langle x \rangle)\big)\langle t \rangle$ with

$$t^2 \in N_G(\langle x \rangle) \cap N_G(\langle a \rangle) \leq C_G(a).$$

If $[t, a] = 1$, then $a \in N_G(E) - O^p(N_G(E))$, an impossibility. Hence as $t \in N_G(\langle a \rangle)$, t inverts a. All the other conclusions of Theorem 2.2 then follow easily, completing the proof of the lemma. □

Our remaining strategy for Theorem 2.2 is to consider various cases for the possible isomorphism types of $K_a^\dagger$, as a varies over $\mathfrak{A}$. By inspection of the possibilities, we reduce to the following cases.

Lemma 3.6. *Let $a \in \mathfrak{A}$. Then one of the following holds*:

(a) $m_p(K_a) = 1$ *and x induces a nontrivial field automorphism on* $K_a^\dagger$;

(b) $P_a \cap K_a$ *is abelian of rank* 2; *or*

(c) $P_a \cap K_a \cong p^{1+2}$, *and* $(p, K_a^\dagger) = (5, \mathrm{HS})$, $(5, \mathrm{Ru})$, *or* $(7, \mathrm{He})$.

In particular, $O_p(K_a^\dagger) = 1$.

Proof. If $m_p(K_a) = 1$, then as $L_a = L_{p'}(C_{K_a}(x))$, x induces an outer automorphism on $K_a^\dagger$. Then as $p > 3$, certainly $K_a^\dagger \in \mathcal{C}\mathrm{hev}(s)$, $s \neq p$, and (a) follows as $m_p(K_a) = 1$.

Suppose that $m_p(K_a) > 1$. Since $m_p(G) = 3$, $O_p(K_a^\dagger) = 1$ by inspection, and then $m_p(K_a) = 2$ and again $K_a^\dagger \in \mathcal{C}\text{hev}(s) \cup \mathcal{A}\text{lt} \cup \mathcal{S}\text{por}$, $s \neq p$. Again by inspection since $p > 3$, p does not divide the order of the Weyl group of $K_a^\dagger$ if $K_a \in \mathcal{C}\text{hev}(s)$, so $P_a \cap K_a$ is abelian and (b) holds in that case. Finally we may assume that $K_a \in \mathcal{A}\text{lt} \cup \mathcal{S}\text{por}$ and $P_a \cap K_a$ is not abelian, and (c) follows easily by inspection. □

We next eliminate case (b).

Lemma 3.7. *Let $a \in \mathfrak{A}$. Then $P_a \cap K_a$ is not abelian of rank* 2.

Proof. In the case that $K_a^\dagger \in \mathcal{C}\text{hev}(s)$, as $p > 3$, we have that $s \neq p$, p does not divide $|\text{Outdiag}(K_a^\dagger)|$, and $K_a^\dagger$ is simple. If x induces a field automorphism on $K_a^\dagger$, then $m_p(K_a) = m_p(L_a) = 1$, contradiction. Hence x induces an inner automorphism on $K_a^\dagger$, corresponding to an element $x_0 \in P_a \cap K_a$. This conclusion is also obvious if $K_a^\dagger \in \mathcal{A}\text{lt} \cup \mathcal{S}\text{por}$. In any case, since $A = \Omega_1(P) \leq P_a$, $A = A_0 \times \langle a \rangle$ where $A_0 = A \cap K_a$.

In all cases, $N_{K_a}(A)$ is irreducible on A_0 [Gorenstein and Lyons 1983, I-(11.1)]. But also $\text{Aut}_R(A)$ contains a transvection shearing a onto $\langle b \rangle \leq A \cap L_a \leq A_0$ and centralizing x. If $\text{Aut}_G(A)$ is irreducible on A, then by McLaughlin's theorem [1967], it contains $\text{SL}(A)$, so it is transitive on $A^\#$. But $a \notin x^G$. Thus $\text{Aut}_G(A)$ is reducible on A. As the irreducible constituents of $N_{K_a}(A)$ on A are A_0 and $\langle a \rangle$, $\text{Aut}_G(A)$ embed in the maximal parabolic subgroup M of $\text{GL}(A)$ stabilizing A_0, with $O_p(\text{Aut}_G(A)) \cong E_{p^2}$ centralizing A_0 and $\text{Aut}_G(A)$ irreducible on A_0. As the p^2 members of $\mathcal{E}_1(A) - \mathcal{E}_1(A_0)$ are permuted transitively by $\text{Aut}_G(A)$ and $\langle a \rangle \not\leq A_0$, we must have $\langle x \rangle \leq A_0$. Hence, $A_0 = \langle x, b \rangle = E$.

Suppose that $|\text{Aut}_G(A)|_p = p^3$. As $\text{Aut}_G(A)$ is irreducible on E, it follows that $\text{Aut}_G(E)$ covers $O^{p'}(M/O_p(M)) \cong \text{SL}(E)$. In particular as

$$\Omega_1(Z(P)) \leq \Omega_1(Z(T)) = E,$$

every element of $E^\#$ is p-central in G and $T \in \text{Syl}_p(G)$. But then p does not divide $|\text{Aut}_G(E)|$, a contradiction.

Hence, $|\text{Aut}_G(A)|_p = p^2$, so if $T \leq T^* \in \text{Syl}_p(N_G(A))$, then

$$T^* = C_{T^*}(E) = T.$$

As $A = \Omega_1(T)$ char T, $T \in \text{Syl}_p(G)$; and then $N_G(A)$ controls G-fusion in T by [Gorenstein et al. 1996, 16.20].

We have $E = \Omega_1(P_a \cap K_a)$. Suppose that $a^{N_G(A)} \cap \langle a \rangle = \{a\}$. As a is T-conjugate to every element of order p in Ea, i.e., in QRa, it follows that if $g \in G$ with $a^g \in T$, then $a^g \in QRa$. Then $V_{G \to T/QR}(a) \neq 1$, so $a \notin [G, G]$, a contradiction.

We conclude that $a^g \in \langle a \rangle - \{a\}$ for some $g \in N_G(A)$. Since a does not belong to $[N_G(\langle x \rangle), N_G(\langle x \rangle)]$, $N_G(\langle a \rangle) \cap N_G(\langle x \rangle) \leq C_a$, so

$$K_a\big(N_G(\langle a \rangle) \cap N_G(\langle x \rangle)\big) \leq C_a < N_G(\langle a \rangle). \tag{3B}$$

In particular, $N_{C_a}(P_a \cap K_a)$ does not control $N_{N_G(\langle a\rangle)}(P_a \cap K_a)$-fusion in $\mathcal{E}_1(E)$ (otherwise equality would hold in (3B), by a Frattini-type argument). It follows[1] that $K_a^\dagger \cong \mathrm{Sp}_4(2^n)$ for some $n > 1$, and p divides $2^{2n} - 1$. Moreover, $N_{C_a}(P_a \cap K_a)$ maps into the subgroup X of $\mathrm{Out}(K_a^\dagger)$ of index 2 consisting of images of field automorphisms, whereas the image of $N_{N_G(\langle a\rangle)}(P_a \cap K_a)$ in $\mathrm{Out}(K_a^\dagger)$ does not lie in X. Therefore $N_G(\langle a\rangle)/C_a$ has even order, so there exists $g \in N_G(\langle a\rangle)$ inverting a. As $E = Z(T)$ is weakly closed and p-central we may take $g \in N_E := N_G(E)$. Also $N_0 := N_{K_a}(E) \le N_E$. Set $\bar{N}_E = N_E/O_{p'}(N_E)$. By the structure of C_x, $\bar{T} \lhd \bar{N}_E$. Since $[Q, a] \ne 1 \ne [R, a]$, we have $QR = J_a(T)$, so $\overline{QR} \lhd \bar{N}_E$. Now $\bar{N}_0 \cong D_8$ has equivalent absolutely irreducible representations on $\bar{E} = \Omega_1(\overline{QR})$ and $\overline{QR}/\Phi(\overline{QR})$, and one equivalence is the p^m-power mapping, where $Q \cong R \cong Z_{p^{m+1}}$. By absolute irreducibility, the mapping $\overline{QR}/\Phi(\overline{QR}) \to \bar{E}$ induced by commutation with a must also be a power mapping, and so it commutes with the conjugation action of g. That is,

$$[y^g, a] = [y, a]^g \quad \text{for all } y \in QR.$$

Hence $[y^g, a] = [y^g, a^g] = [y^g, a^{-1}]$ for all $y \in QR$, so $a^2 \in C_G(QR)$. As a has order p, $[QR, a] = 1$, a final contradiction. □

Lemma 3.8. *Let $a \in \mathfrak{A}$. Then $\langle b\rangle$ is weakly closed in A.*

Proof. Let $P_a \le S \in \mathrm{Syl}_p(G)$. Since R shears $\langle a\rangle$ to $\langle b\rangle$, $\langle b\rangle = Z(S)$ is weakly closed in E. Suppose that $d \in A - E$ and $\langle d\rangle \in \langle b\rangle^G$. Since d is p-central in G and $K_d \ne 1$, $O_p(K_d^\dagger) \ne 1$. As $O_p(L_d^\dagger) = 1$, we must have $d \in \mathfrak{A}$. But then by Lemma 3.6, $O_p(K_d^\dagger) = 1$, completing the proof. □

Finally, we eliminate Lemma 3.6(c).

Lemma 3.9. *Let $a \in \mathfrak{A}$. Then $m_p(K_a) = 1$, and x induces a nontrivial field automorphism on $K_a^\dagger$.*

Proof. Suppose not. Then $K_a^\dagger \cong \mathrm{Ru}$ or HS (with $p = 5$) or He (with $p = 7$). Let $S_0 = P_a \cap K_a \in \mathrm{Syl}_p(K_a)$, so that $S_0 \cong p^{1+2}$. Let $S_1 = \langle a\rangle \times S_0 \le S \in \mathrm{Syl}_p(G)$, so that $Z(S_1) = \langle a, b\rangle$ and $S_1 = \Omega_1(C_S(a))$, and set $S_2 = N_S(S_1)$. Now a is sheared to $\langle b\rangle$ by R, and that fusion must occur in S_2 as well. So $\langle a\rangle \in \mathrm{Syl}_3(C(a, K_a))$ and $S_1 \in \mathrm{Syl}_p(C_a)$, and we may assume that $S_2 = S_1 R$.

Let $N_0 = N_G(S_1)$ and $N_a = N_{N_0}(\langle a\rangle)$. Let $\bar{S}_1 = S_1/\langle b\rangle \cong E_{p^3}$ and $\widetilde{S}_1 = \bar{S}_1/\langle \bar{a}\rangle \cong S_1/Z(S_1)$. Then as $\langle b\rangle$ is not conjugate to $\langle a\rangle$, but $a^R = a\langle b\rangle$, N_0 acts on $\bar{S}_1$ and $\widetilde{S}_1$. Moreover, $a^{N_0} \supseteq a\langle b\rangle$, so $|N_0 : N_a| = p$ and $N_0 = N_a R_0$ for some $R_0 \in \mathrm{Syl}_p(N_0)$.

[1] By the following $\mathcal{K}$-group lemma, whose proof we omit. Suppose K is a known finite simple group with $e(K) \le 3$, and $R \in \mathrm{Syl}_p(K)$ is abelian of rank 2 for some prime $p > 3$. Suppose also that $L_{p'}(C_K(x)) \ne 1$ for some $x \in P^\#$. Let $g \in N_{\mathrm{Aut}(K)}(R)$ and suppose that for some $X \in \mathcal{E}_1(R)$, $X^g \notin X^{N_K(R)}$. Then for some $q = 2^n > 2$, $K \cong \mathrm{Sp}_4(q)$, p divides $q^2 - 1$, and the image of g in $\mathrm{Out}(K)$ is not the image of a field automorphism.

Let $X_0 = \mathrm{Aut}_{N_0}(\bar{S}_1)$ and $X_a = \mathrm{Aut}_{N_a}(\bar{S}_1)$, and $Y_0 = \mathrm{Aut}_{N_0}(\widetilde{S}_1)$ and $Y_a = \mathrm{Aut}_{N_a}(\widetilde{S}_1)$. Thus $|X_0 : X_a|$ divides p, as does $|Y_0 : Y_a|$. From [Gorenstein et al. 1998, 5.3], $|Y_a| = 16m$, $18m$, or 32 according as $K_a \cong \mathrm{HS}$, He, or Ru. Here $m = 1$ or 2. In any case, $|\mathrm{SL}_2(p)|$ does not divide $p\,|Y_a|$, so Y_0 does not contain $\mathrm{SL}_2(p)$. But Y_a is irreducible on $\widetilde{S}_1$, so p does not divide $|Y_0|$. Therefore $Y_0 = Y_a$.

The image of R_0 in X_0 therefore stabilizes the chain

$$\bar{S}_1 > \langle \bar{a} \rangle > 1.$$

But the stabilizer of this chain in $\mathrm{Aut}(\bar{S}_1) \cong \mathrm{GL}_3(p)$ is of order p^2 and isomorphic to $\widetilde{S}_1$ as $\mathrm{GL}(\widetilde{S}_1) \times \mathrm{GL}(\langle \bar{a} \rangle)$-module. In particular, X_a is irreducible on $O_p(X_0)$. As $|O_p(X_0)| \le p$, $O_p(X_0) = 1$. Hence $X_0 = X_a$ stabilizes $S_0/\langle b \rangle$ and S_0, and it follows that $S_1 R_0$ is extraspecial of order p^{1+4} and exponent p^2, with $S_1 = \Omega_1(S_1 R_0)$. Thus, $N_S(S_1 R_0) \le N_S(S_1)$, forcing $S = S_1 R_0$. As $R_0 \cong Z_{p^2}$, $N_G(S)$ has equivalent representations on S/S_1 and $Z(S)$, so $[S/C_S(R_0), N_G(S)] = 1$. By theorems of Yoshida or Wielandt [Gorenstein et al. 1996, 15.19, 15.20], G is not simple, a final contradiction. □

For the rest of this section, we assume for a contradiction that

(3C1) $\mathfrak{A} \neq \varnothing$;

(3C2) for all $a \in \mathfrak{A}$, $m_p(K_a) = 1$; and

(3C3) x induces a nontrivial field automorphism on $K_a^\dagger$.

We first prove:

Lemma 3.10. *Assume (3C1)–(3C3). Then*:

(a) $\langle x \rangle^{N_G(T)} = \mathcal{E}_1(E) - \{\langle b \rangle\}$.

(b) $T < P$ *and* $\Omega_1(Z(P)) = \langle b \rangle$.

Proof. As $b \in L_a \le K_a$, and $m_p(K_a) = 1$ with x inducing a nontrivial field automorphism on $K_a^\dagger$, we have $\mathcal{E}_1(E) - \langle b \rangle = \langle x \rangle^S$ for some $S \in \mathrm{Syl}_p(C_{K_a}(b))$. In particular, p divides $|\mathrm{Aut}_G(E)|$. As $E = \Omega_1(Z(T))$, we have $T < P$ and $\Omega_1(Z(P)) \le C_{\Omega_1(Z(T))}(S) = \langle b \rangle R$, so (b) holds. But then x is not p-central in G so (a) follows as well. □

Lemma 3.11. *Assume (3C1)–(3C3). Then $\mathfrak{A} = A - E$, and (2B1)–(2B7) are satisfied with any element of $A - \langle b \rangle$ in place of a.*

Let $a_1 \in \mathcal{I}_p(T - \langle b \rangle)$ and set $K_1 = L_{p'}(C_G(a_1))$. Then $K_1^\dagger \cong L^\dagger$. Let $T_1 \in \mathrm{Syl}_p(C_G(a_1))$ and $Q_1 = C_{T_1}(L_1^\dagger)$. Then $|Q_1| \le p^2$. Finally, if $a_1 \notin E$, then $C_{Q_1}(x) = \langle a_1 \rangle$ and x acts as a nontrivial field automorphism on $K_1^\dagger$.

Proof. If $a_1 \in E - \langle b \rangle$, then $\langle a_1 \rangle \in \langle x \rangle^G$ by Lemma 3.10 and the lemma holds by Lemma 3.4(a). Suppose then that the lemma fails for some $a_1 \in \langle a, x \rangle - \langle x \rangle$ with $L_{a_1} := L_{p'}(C_L(a_1))$ and with K_{a_1} the trivial pumpup of L_{a_1} in $C_G(a_1)$.

Let $x_1 \in x\langle b\rangle - \{x\}$. By Lemma 3.10, $x_1 \in x^{C_L(b)}$, and we take $g \in C_L(b)$ with $x_1 = x^g$. Let $L_2 := L_{p'}(C_{L^g}(a_1))$ and let K_2 be the pumpup of L_2 in $C_G(a_1)$. Then $L_2 \le L_{p'}(C_G(a_1))$ with $b \in K_{a_1} \cap K_2$, so $K_2 = [\langle b\rangle, K_{a_1}] \le K_{a_1}$. Thus K_2 is a trivial pumpup of L_2, so $K_2^\dagger$ is centralized by both x_1 and (like $K_{a_1}^\dagger$) x. Hence, $K_2^\dagger$ is centralized by b, a contradiction.

Therefore, K_{a_1} is a vertical pumpup of L_{a_1}, and x acts as a nontrivial field automorphism on $L_{a_1}^\dagger$ by (3C1)–(3C3). We then apply Lemma 3.4 to a_1, L_{a_1}, x, T_1, and Q_1 in place of x, L, a, T, and Q to obtain the conclusions of the lemma. □

We now fix $U \lhd P$ with $U \cong E_{p^2}$, and set $P_0 = C_P(U)$.

Lemma 3.12. *Assume* (3C1)–(3C3). *Then the following conditions hold for some* $x_0 \in A - \langle b\rangle$ *satisfying* (2B1)–(2B7) (*in the role of* x).

(a) *One of the following holds*:

  (1) $\langle a\rangle \in \mathrm{Syl}_p(C(a, K_a))$ *for all* $a \in A - E$; *or*

  (2) $\mathcal{E}_1(A) - \{\langle b\rangle\}$ *is completely fused in* G.

(b) $x_0 \in U \le A$.

(c) $|P : T_{x_0}| = p$ *for some* $T_{x_0} \in \mathrm{Syl}_p(C_G(x_0))$.

Proof. Suppose that (a1) fails, so that for every $x_0 \in A - \langle b\rangle$, there is $y \in A - \langle x_0, b\rangle$ such that $|C(y, K_y)|_p = p^2$. Then we may assume we started with $x_0 \in A - \langle b\rangle$ such that $|C(x_0, K_{x_0})| = p^2$. By Lemma 3.4, y is sheared to $\langle x_0\rangle$. Applying Lemma 3.4 again in $C_G(y)$, we see that x_0 is sheared to $\langle y\rangle$. Hence $\mathcal{E}_1(\langle x_0, y\rangle)$ is completely fused in G. As x_0 and y are each fused to $\langle x_0 y\rangle$ in the centralizer of the other, (a2) holds, and (a) follows.

Next, assume that x_0 has been chosen so that (a) holds. Let $a_0 \in A - \langle x_0, b\rangle$. Since $E_{p^2} \cong U \lhd P$, there exists $1 \ne y \in C_{\langle x_0, a_0\rangle}(U)$. Then $U \le C_G(y)$. If (a2) holds, then we may vary x_0 to satisfy (b). We have $y \in U$, and taking a Sylow p-subgroup T_y of $C_G(y)$ containing $C_P(y) \ge \langle U, A\rangle$, we have $U \le \Omega_1(T_y) = A$, so (b) holds for y. So assume that (a1) holds (for x_0). Let $Q_{x_0} \in \mathrm{Syl}_p(C(x_0, K_{x_0}))$. If $|Q_{x_0}| = p$, then $\langle a\rangle \in \mathrm{Syl}_p(C(a, K_a))$ for all $a \in A - \langle b\rangle$. (We set $K_x = L$ and $K_{x'} = L^g$ for any $x' \in \langle x, b\rangle - \langle x\rangle - \langle b\rangle$, where $g \in N_P(\langle x, b\rangle)$ is such that $x^g = x'$.) Hence we can again vary x_0 to satisfy (b). We again use $y \in C_{\langle x_0, a_0\rangle}(U)$ in place of x_0 to get (b). If, on the other hand, $|Q_{x_0}|_p = p^2$, then $\langle x_0\rangle = \Phi(Q_{x_0}) \le C_P(U)$ so $U \le C_P(x_0) =: T_{x_0}$. If $U \ne \langle x_0, b\rangle$ then by Lemma 3.4 again, $\langle x_0\rangle = [Q_{x_0}, U] \le U$ so $U = \langle x_0, b\rangle$, contradiction. Thus (b) holds in all cases. Finally $\langle b\rangle$ is 3-central and weakly closed in A, so x_0 must be half p-central, proving (c) and the lemma. □

We replace our original x by x_0 as in Lemma 3.12.

Lemma 3.13. *Let* $a \in \mathfrak{A}$. *Then* $Q = \langle x\rangle$ *and* $T = \langle x\rangle \times R\langle a\rangle$.

Proof. Suppose first that $|Q| = p^2$, and let $a \in \mathfrak{A}$. Note that if $y \in P - T$, then $C_Q(y) = 1$. Also $C_{Q\times R}(a) = \langle x\rangle \times \Phi(R)$. So $Q \times R$ is the unique largest abelian subgroup of P. By Lemma 3.4(b), $T = QR\langle a\rangle$. Consider $N := N_G(QR)$, $C = C_G(QR)$, and $\overline{N} = N/C$. Then $C_{\overline{N}}(E) \le O_p(\overline{N})$ and so $C_{\overline{N}}(E) = \langle \bar{a}\rangle$. Let $\widetilde{N} = N/C_N(E)$. Then $\widetilde{N}$ embeds in $\mathrm{GL}(E)$ and $|\widetilde{N} : N_{\widetilde{N}}(\langle x\rangle)| = p$. It follows that $\widetilde{N} = \widetilde{P} N_{\widetilde{N}}(\langle x\rangle)$, and so $\overline{N} = \overline{P} N_{\overline{N}}(\langle x\rangle)$. As $|\overline{P}| = p^2$ and $[\bar{a}, N_{\overline{N}}(\langle x\rangle)] = 1$, we have $\bar{a} \notin [\overline{N}, \overline{N}]$. By the generalized Hall–Wielandt theorem of Yoshida [Gorenstein et al. 1996, 15.27] applied to the weakly closed subgroup QR of P, $a \notin [G, G]$, contrary to the simplicity of G. Therefore, $Q = \langle x\rangle$.

Now, there is a complement F to R in T containing $\langle x\rangle$. Then $F/\langle x\rangle$ is cyclic, so $F = \langle x\rangle \times F_1$ for some F_1 as $m_p(C_x) = 3$. Write $\Omega_1(F_1) = \langle a\rangle$, so that $a \in \mathfrak{A}$. Then $F_1/\langle a\rangle$ acts on K_a and faithfully induces field automorphisms on $L_a^\dagger \cong L_{p'}(C_{K_a^\dagger}(x))$. As x induces a field automorphism of order p on $K_a^\dagger$, $F_1/\langle a\rangle = 1$. Hence $T = \langle x\rangle \times R\langle a\rangle$, as claimed. □

Lemma 3.14. $\mathfrak{A} = \varnothing$.

Proof. Suppose this is false and continue the above argument. Again let $a \in \mathfrak{A}$ and set $N = N_G(A)$, $C = O_{p'}(C_G(A))$, and $\overline{N} = N/C$. Let $Z = C_R(a) = \Phi(R)$. Thus, $C_G(A) = (\langle a, x\rangle \times Z)C$ and $|\mathrm{Aut}_G(A)|_p = p^2$, by Lemma 3.13. Moreover a generator r of R acts as a transvection on A.

Now, $\mathrm{Aut}_G(A)$ is p-closed.[2] As $C_G(A)$ is p-nilpotent, $\overline{P} \lhd \overline{N} = \overline{N}_0$ where $N_0 = N_{N_G(A)}(P)$. As $A \lhd T_1$ for some $T_1 \in \mathrm{Syl}_p(C_a)$, $C_P(a)$ contains some r_1 centralizing $\langle a, b\rangle$ and shearing x to b. Modifying r_1 by an element of $\langle a, x\rangle$, we may choose $r_1 \in K_a$. Then $P \cap K_a \ge \langle r_1, Z\rangle =: R_1$, so $R_1 = P \cap K_a = \langle r_1\rangle$ and $P = \langle Z, a, x, r, r_1\rangle$. Hence, $Z \le Z(P)$.

We consider the structure of

$$P/Z = \langle aZ, xZ, rZ, r_1Z\rangle = \Omega_1(P/Z).$$

Now $|P : C_P(a)| = |P : C_P(x)| = p$, so $[r, r_1] \in C_P(A) = ZA$ and $[P/Z, P/Z] = \langle [r, r_1]Z\rangle \le Z(P/Z)$. Thus, either $[P, P] = \langle b\rangle$ and $P/[P, P] \cong E_{p^4}$, or, by Lemma 3.12(a), we may choose notation so that $[r, r_1] = x$ and $[P, P] = E$.

Suppose first that $[P, P] = E$. Then E char P and so $\overline{E} \lhd \overline{N}_0$. Hence $\langle x\rangle^{N_0} = \langle x\rangle^P$ and so $N_0 = P N_{N_0}(\langle x\rangle)$. Also $P/Z \cong Z_p \times p^{1+2}$ with $[P/Z, P/Z] = \langle xZ\rangle$. Now, $\langle b\rangle \lhd N_0$ and so $\widetilde{N}_0 := N_0/C_{N_0}(E)$ embeds in a Borel subgroup of $\mathrm{GL}(E)$. Also, there is a p'-element $t \in N_{L_a}(\langle b\rangle)$ such that for some integer $\lambda \not\equiv 1 \pmod p$, $b^t = b^\lambda$. Then if $\widetilde{H}$ is a complement to $\widetilde{R}_1$ in $\widetilde{N}_0$ containing $\tilde{t}$, then $\widetilde{H}$ normalizes $C_E(t) = \langle x\rangle$. Let H be a Hall p'-subgroup of N_0 mapping onto $\widetilde{H}$. Then H normalizes $T = C_P(E)$ and as $TH \le N_G(\langle x\rangle)$, we have $[T, H] \le T \cap [N_G(\langle x\rangle), N_G(\langle x\rangle)] \le R\langle x\rangle$. Then $RE \lhd N_0$ and $[H, a] \le RE$. Set $\widehat{N}_0 = N_0/RE$. Then $\widehat{P} = \langle \hat{a}, \hat{r}_1\rangle \lhd \widehat{N}_0$

[2] As is any $H \le \mathrm{GL}_3(p)$ with $|H|_p = p^2$ [Gorenstein et al. 1998, 6.5.3]

with $\widehat{P} \cong E_{p^2}$ and $\hat{a} \notin [\widehat{P}, \widehat{H}]$. Hence $O^p(N_0) \cap P < P$. Now, $Z = Z(P)$ and $Z(P/Z) = \langle aZ, xZ\rangle$. Thus $Z_2(P) = \langle a, x, Z\rangle$ and $A = \Omega_1(Z_2(P))$ char P. So $N_0 = N_G(P)$. By Yoshida's transfer theorem [Gorenstein et al. 1996, 15.19], P has a quotient $P/Y \cong Z_p \wr Z_p$. As $|[P, P]| = p^2$ and $p > 3$, this is impossible. This proves that $[P, P] = \langle b\rangle$.

It follows that $RR_1 \le P$ with RR_1 having distinct cyclic maximal subgroups R and R_1. Thus $RR_1 = R\Omega_1(RR_1) = R\langle e\rangle$ where e has order p. Now the eigenvalues of t on $R/\Phi(R)$, $\langle b\rangle = \Omega_1(R) = \Omega_1(R_1)$, and $R_1/\Phi(R_1)$ are all equal, so $[RR_1, t] = RR_1$. But if RR_1 is not abelian, then $[R, e] = \Omega_1(R)$, and conjugating by t we see that $[e, t] = 1$, whence $[RR_1, t] = R$, a contradiction.

It follows that $RR_1 = R \times \langle e\rangle$ for some $e \in \mathcal{I}_p(P)$. Then $(\langle x\rangle \times R)\langle e\rangle = C_P(R) \in \mathrm{Syl}_p(C_G(R))$. So, $C_P(R) = \langle x, e\rangle * R$ with $\langle x, e\rangle \cong p^{1+2}$. Similarly $C_P(R_1) = \langle a, e\rangle * R_1$ for some $a \in \mathfrak{A}$. Hence e normalizes A and $S := A\langle e\rangle \cong Z_p \times p^{1+2}$. Let $e_1 \in Z(S) - \langle b\rangle$. As $e_1 \in C_P(A)$, $e_1 \in A$ and $[e_1, e] = 1$. Let $S \le P_1 \in \mathrm{Syl}_p(C_G(e_1))$. As $e_1 \in A - \langle b\rangle$, $A = \Omega_1(P_1)$, so $\Omega_1(S) \le A$, contrary to $S = \Omega_1(S)$, a final contradiction. □

Now Lemmas 3.5 and 3.14 prove Theorem 2.2.

4. Remarks on Theorem 2.3

As the hypotheses of Theorem 2.3 yield that $\Omega_1(P) = A \cong E_{p^3}$, the theorem will be proved once we prove that $\Gamma_{A,1}(G) \le M$ for some $M \le G$ such that $O_{p'}(M)$ has even order. Using Theorem 2.2 and the hypothesis that G has even type, we prove that $L^\dagger \in \mathcal{C}\mathrm{hev}(2)$. (Recall that $L := L_{p'}(C_G(x))$.) We are then able to prove that for all $e \in E^\#$, $C_G(e)$ is p-solvable unless $e \in \langle x\rangle \cup \langle b\rangle$, in which case $L_{p'}(C_G(e)) \cong L$. Then, as $e(G) = 3$, we can show that $L/O_2(L)$ is quasisimple, and for every $a \in A - E$, each 2-component of $L_{p'}(C_G(a))/O_2(L_{p'}(C_G(a))$ is quasisimple as well. Next we follow Aschbacher [1981] and use a functor $\Theta^{(2)}_{3/2}$ which is an analogue of the standard $\frac{3}{2}$-balanced functor, but with O_2 replacing $O_{p'}$. The standard $\frac{3}{2}$-balanced functor does not exist in this case because L is not weakly locally balanced with respect to A; but the saving grace is that the obstructing cores are of odd order. We then set $\Sigma = \Theta^{(2)}_{3/2}(G; A)$. If $\Sigma \ne 1$, it is immediate that $\Gamma_{A,2}(G) \le N_G(\Sigma) =: M < G$. On the other hand, if $\Sigma = 1$, then $L_{p'}(C_G(a))$ is semisimple for all $a \in A^\#$, and setting $\Gamma(B) = \langle L_{p'}(C_G(a)) \mid a \in B^\#\rangle$ for any hyperplane B of A, we deduce that $\Gamma(B) = LL^t$ (for t as in Theorem 2.2(f)) and $\Gamma_{A,2}(G) \le N_G(LL^t) =: M < G$. This step uses Gary Seitz's fundamental generation theorem for finite groups of Lie type [Seitz 1982]. Further applications of that result yield in both cases that $\Gamma_{A,1}(G) \le M$. Finally, since $e(G) = 3$, A normalizes a nontrivial 2-subgroup of G, which fairly quickly leads to a contradiction if $O_{p'}(M)$ has odd order, completing the proof of Theorem 2.3.

5. Concluding remarks

We take this opportunity to recognize Gary Seitz's fundamental contributions to the classification of finite simple groups in the 1970s and early 1980s, when he was also a player in the concurrent effort to classify the finite doubly transitive groups. This of course represents only a small fraction of his complete mathematical work. Alone or with collaborators, he proved many landmark results that each served to propel these classification efforts. We give several examples.

On doubly transitive groups, he worked with Christoph Hering, Bill Kantor, and Mike O'Nan [Kantor and Seitz 1971; 1972; Hering et al. 1972; Kantor et al. 1972], in particular completing the classification of finite split BN-pairs of rank 1. Seitz and Paul Fong then classified finite split BN-pairs of rank 2 [1973; 1974].

He made major contributions to the classification of finite simple groups of component type. With Michael Aschbacher he classified simple groups with a known quasisimple standard subgroup centralized by a four-group [1976b; 1981]. Partly with Bob Griess and David Mason he classified most simple groups with a standard subgroup in $\mathcal{C}$hev(2) whose centralizer has a cyclic Sylow 2-subgroup [Seitz 1979a; 1979b; 1979c; Griess et al. 1978].

On top of all these, Gary proved many useful results illuminating the structure of finite groups of Lie type. Outstanding examples are the classification of involutions in groups in $\mathcal{C}$hev(2) [Aschbacher and Seitz 1976a], and the generation properties of r' elementary abelian subgroups of odd order acting on groups in $\mathcal{C}$hev(r) [Seitz 1982]. The latter has been mentioned in the above discussion of Theorem 2.3, and in general is vital for the ultimate success of the signalizer functor method.

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