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# DECOMPOSITION NUMBERS IN THE PRINCIPAL BLOCK AND SYLOW NORMALISERS

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*To the memory of Gary Seitz*

**If $G$ is a finite group and $p$ is a prime number, we investigate the relationship between the $p$-modular decomposition numbers of characters of height zero in the principal $p$-block of $G$ and the $p$-local structure of $G$. In particular we prove that, under certain conditions on the nonabelian composition factors of $G$, $d_{\chi 1_G} \neq 0$ for all irreducible characters $\chi$ of degree prime to $p$ in the principal $p$-block of $G$ if, and only if, the normaliser of a Sylow $p$-subgroup of $G$ has a normal $p$-complement.**

## 1. Introduction

Let $G$ be a finite group, $p$ a prime number, $\mathrm{Irr}(G)$ the set of irreducible ordinary characters of $G$ and $\mathrm{IBr}(G)$ the set of irreducible $p$-Brauer characters of $G$. Then the restriction $\chi^\circ$ of any $\chi \in \mathrm{Irr}(G)$ to the set of $p$-regular elements of $G$ can be written as

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

where the $d_{\chi\varphi}$ are uniquely determined nonnegative integers. These are called the $p$-modular *decomposition numbers of* $G$, and a great deal of literature is devoted to understanding them.

Navarro and Tiep [2020; 2022] initiated the investigation on relations between $p$-decomposition numbers and properties of Sylow $p$-normalisers, considering two different settings. In [Navarro and Tiep 2020] they conjectured that if $p > 3$ then $d_{\chi 1_G} \neq 0$ for all $\chi \in \mathrm{Irr}_{p'}(G)$, that is, for all irreducible characters of $G$ of degree prime to $p$, if and only if $G$ has self-normalising Sylow $p$-subgroups, and that this happens if and only if $d_{\chi 1_G} = 1$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. Note that irreducible characters

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of degree prime to $p$ lie in $p$-blocks of maximal defect, so this situation can only happen when the principal block is the only Brauer $p$-block of maximal defect of $G$. It is then natural to wonder what can be said when this is not the case and one restricts attention to the principal $p$-block $B_0(G)$. In this sense, in [Navarro and Tiep 2022] they conjectured that $d_{\chi 1_G} \neq 0$ for all $\chi \in \mathrm{Irr}(B_0(G))$ if and only if $G$ has a normal $p$-complement. In this paper we consider the intersection of the two conditions in [Navarro and Tiep 2020] and [2022], namely what happens if $d_{\chi 1_G} = 1$ just for all $\chi \in \mathrm{Irr}_{p'}(B_0(G))$. Our main result is:

**Theorem A.** *Let $G$ be a finite group and $p > 3$ be a prime. Assume that all nonabelian simple composition factors of $G$ of order divisible by $p$ satisfy Property $(*)$ below. The following are equivalent*:

(i) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} \neq 0$.*

(ii) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} = 1$.*

(iii) *For $P \in \mathrm{Syl}_p(G)$ we have $\boldsymbol{N}_G(P) = P \times K$ for some $K \leq G$.*

*Moreover*, *if $G$ is $p$-solvable*, *this equivalence holds for every prime $p$.*

Notice that, as pointed out in [Navarro and Tiep 2020], all irreducible characters of odd degree of the alternating group $\mathfrak{A}_5$ contain the trivial character in their 2-modular reduction, and similarly, all irreducible characters of the Ree group ${}^2G_2(27)$ of nonzero 3-defect contain the trivial character in their 3-modular reduction, while the respective Sylow $p$-normalisers have no normal $p$-complement, so the equivalence in Theorem A fails for nonsolvable groups with $p \leq 3$.

Theorem A involves the following property that a finite nonabelian simple group might, or might not, satisfy:

**Property $(*)$.** Let $S$ be nonabelian simple and $p > 3$ a prime dividing $|S|$. Then for all almost simple groups $H$ with socle $S$ and $|H : S|$ a $p$-power, there exists $\chi \in \mathrm{Irr}_{p'}(B_0(H))$ such that $d_{\chi 1_H} = 0$.

We prove Property $(*)$ does hold for many simple groups: for sporadic groups, alternating groups, and simple groups of Lie type in characteristic different from $p$. We also show it for some groups of Lie type in characteristic $p$. (The general case of groups of Lie type in their defining characteristic was also left open in [Navarro and Tiep 2020; 2022].) As the knowledge on $p$-decomposition numbers for groups of Lie type in their own characteristic is too weak at present we refrain from making a general conjecture and just leave it as a question as to whether Property $(*)$ holds for all nonabelian finite simple groups.

**Structure of the paper.** In Section 2 we prove Property $(*)$ for $S$ a sporadic simple group, an alternating group, a simple group of Lie type in characteristic different from $p$ as well as for some groups of Lie type in characteristic $p$. In Section 3 we show that Theorem A holds for $p$-solvable groups and in Theorem 3.4 we reduce the general case to the validity of Property $(*)$ on composition factors, thus completing the proof of Theorem A.

## 2. Almost simple groups

In this section, we discuss instances of Property $(*)$ from the introduction.

***Alternating and sporadic groups.*** We start out with simple groups not of Lie type.

**Proposition 2.1.** *Property $(*)$ holds for $S$ a sporadic simple group or the Tits group.*

*Proof.* Let $S$ be as in the assumption and $H \geq S$ as in Property $(*)$. Since $p > 2$ this means $H = S$. By [Navarro and Tiep 2022, Proposition 3.2] there exists $\chi \in \mathrm{Irr}(B_0(H))$ such that $d_{\chi 1_H} = 0$. If $S$ has abelian Sylow $p$-subgroups, by one direction of Brauer's height zero conjecture [Kessar and Malle 2013], we then have $\chi \in \mathrm{Irr}_{p'}(B_0(H))$ as required. Now assume Sylow $p$-subgroups are nonabelian. By [Navarro and Tiep 2020, Lemma 3.2], there is $\chi \in \mathrm{Irr}_{p'}(H) = \mathrm{Irr}_{p'}(B_0(H))$ with $d_{\chi 1_H} = 0$, and again we are done since by inspection in [GAP 2020], $S$ has just one $p$-block of maximal defect. □

**Proposition 2.2.** *Property $(*)$ holds for $S$ an alternating group.*

*Proof.* Let $S = \mathfrak{A}_n$ with $n \geq 5$. As $p > 2$ again we have $H = S$. Set $G := \mathfrak{S}_n$. Recall that the irreducible characters of $\mathfrak{S}_n$ are naturally labelled by partitions of $n$. If $p$ divides $n$, then let $\chi = \chi^{(n-2,1^2)} \in \mathrm{Irr}(B_0(G))$, as in [Navarro and Tiep 2022, Proposition 3.1]. Then $\chi^\circ$ is the sum of two irreducible Brauer characters of degrees $n-2$ and $\frac{1}{2}(n-2)(n-3)$ and hence it is of degree prime to $p$ and does not contain any irreducible Brauer character of degree 1. Now, any $\theta \in \mathrm{Irr}(S)$ under $\chi$ lies in $\mathrm{Irr}_{p'}(B_0(S))$ and $d_{\theta 1_S} = 0$, as wanted.

So we may assume that $p$ does not divide $n$. Let $n = a_k p^k + \cdots + a_1 p + a_0$ be the $p$-adic expansion of $n$. Since $p$ does not divide $n$, we have $a_0 > 0$. Suppose first that $a_0 > 1$. Consider $\chi = \chi^{(a_0, 1^{n-a_0})}$. By Peel's theorem (see [James 1978, Theorem 24.1]), $\chi^\circ \in \mathrm{IBr}(G)$. Hence, it is enough to show that $\chi(1) \neq 1$, $\chi(1)$ is $p'$, and $\chi \in \mathrm{Irr}(B_0(G))$. By the hook length formula, $\chi$ has degree

$$\chi(1) = \frac{n!}{n \cdot (a_0 - 1)!(n - a_0)!} = \binom{n-1}{n-a_0}.$$

Then $\chi(1) \neq 1$ since $1 < a_0 < n$. Moreover

$$n - 1 = a_k p^k + \cdots + a_1 p + (a_0 - 1)$$

and

$$n - a_0 = a_k p^k + \cdots + a_1 p,$$

so by Lucas' theorem we have

$$\chi(1) = \binom{n-1}{n-a_0} \equiv \prod_{i=1}^{k} \binom{a_i}{a_i}\binom{a_0-1}{0} \equiv 1 \pmod{p}.$$

Thus, $\chi(1)$ is not divisible by $p$. Finally since the $p$-core of $\lambda = (a_0, 1^{n-a_0})$ is $(a_0)$, we have that $\chi$ lies in the principal $p$-block by the Nakayama conjecture, as desired. Now take $\theta \in \mathrm{Irr}(S)$ under $\chi$, so $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ and $d_{\theta 1_S} = 0$.

Finally, suppose that $a_0 = 1$, so $p$ divides $n-1$. In this case consider $\chi = \chi^{(n-3,2,1)}$, so $\chi$ lies in $B_0(G)$. This is the character in the proof of [Navarro and Tiep 2020, Lemma 3.1(ii)], so we are done in this case as well. □

***Groups of Lie type in nondefining characteristic.*** For groups of Lie type in cross characteristic we consider the following setup. Let $\boldsymbol{G}$ be a simple linear algebraic group of adjoint type over an algebraically closed field of characteristic $r$ and $F : \boldsymbol{G} \to \boldsymbol{G}$ a Steinberg map, with group of fixed points $G := \boldsymbol{G}^F$. It is well known that any simple group of Lie type can be obtained as $S = [G, G]$ for $\boldsymbol{G}$, $F$ chosen suitably. Moreover, if $\boldsymbol{G}_{\mathrm{sc}}$ denotes a simply connected covering of $\boldsymbol{G}$, with corresponding Steinberg map also denoted $F$, then $S \cong \boldsymbol{G}_{\mathrm{sc}}^F/\boldsymbol{Z}(\boldsymbol{G}_{\mathrm{sc}}^F)$, if $S$ is not the Tits simple group, which was already discussed in Proposition 2.1. Let $(\boldsymbol{G}^*, F)$ be dual to $(\boldsymbol{G}, F)$ and $G^* := \boldsymbol{G}^{*F}$.

We let $\boldsymbol{B} \leq \boldsymbol{G}$ denote an $F$-stable Borel subgroup of $\boldsymbol{G}$, and set $B := \boldsymbol{B}^F$.

We recall that outer automorphisms of prime order $p \geq 5$ of simple groups of Lie type are either field automorphisms, or diagonal automorphisms for groups of types $\mathrm{PSL}_n(\epsilon q)$, with $\epsilon \in \{\pm 1\}$.

**Proposition 2.3.** *Property* $(*)$ *holds for* $S$ *as above if* $|B|$ *is prime to* $p$.

*Proof.* By assumption $p$ divides $|S|$, hence also $|G| = |G^*|$. Let $1 \neq s \in G^*$ be a (semisimple) $p$-element in the centre of a Sylow $p$-subgroup of $G^*$, and let $\chi \in \mathrm{Irr}(G)$ be the semisimple character in the Lusztig series $\mathcal{E}(G, s)$, unique since $\boldsymbol{G}$ has connected centre, see [Geck and Malle 2020, Definition 2.6.9]. By the degree formula for Jordan decomposition [Geck and Malle 2020, Corollary 2.6.6], $\chi(1)$ is then prime to $p$, and also $\chi(1) > 1$ as $p$ does not divide $|\boldsymbol{Z}(G^*)|$ and so $C_{G^*}(s) < G^*$. Furthermore, by [Hiss 1990b, Corollary 3.4], the semisimple character $\chi$ lies in the same $p$-block of $G$ as the semisimple character in $\mathcal{E}(G, 1)$, i.e., the trivial character, so in the principal $p$-block. Since $p$ does not divide $|B|$, the permutation module $1_B^G$ is projective, and thus contains the projective cover of the trivial module. But all constituents of $1_B^G$ are unipotent, so lie in $\mathcal{E}(G, 1)$; see [Geck and Malle 2020, Example 3.2.6]. Hence, by Brauer reciprocity, $1_G^\circ$ does not occur as a constituent of $\chi^\circ$. Taking for $\theta$ any character of $S$ below $\chi$ we see that $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ and $d_{\theta 1_S} = 0$.

Next observe that the order of any outer diagonal automorphism of $S$ divides the order of $B$. Thus, $H$ is an extension of $S$ by a $p$-power order field automorphism $\gamma$. Note that any such automorphism of $S$ extends to $G$ and then also induces a dual automorphism on $G^*$, which we denote $\gamma^*$; see, e.g., [Taylor 2018, §5.6]. Now choose $s$ more precisely to be also $\gamma^*$-invariant (which is possible as $\gamma^*$ is a $p$-element, so necessarily has nontrivial fixed points on the centre of a $\gamma^*$-stable Sylow $p$-subgroup of $G^*$). Then $\chi$ is also $\gamma$-invariant by [Taylor 2018, Proposition 7.2]. Since the index $|G:S|$ is a divisor of $|B|$, hence prime to $p$, there exists a $\gamma$-invariant $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ below $\chi$, with $d_{\theta 1_S} = 0$. Let $\tilde{\theta}$ be an extension of $\theta$ to $H = S\langle\gamma\rangle$ in $B_0(H)$. Then $\tilde{\theta}$ is as desired. □

**Proposition 2.4.** *Property* $(*)$ *holds for $S$ as above if $F$ is a Frobenius map with respect to an $\mathbb{F}_q$-structure and $p$ divides $q-1$.*

*Proof.* Let $W$ be the Weyl group of $G$, that is, the $F$-fixed points of the Weyl group of $\boldsymbol{G}$. As $p$ divides $q-1$, the $p$-decomposition matrix of the group algebra of $W$ embeds into the $p$-modular decomposition matrix of $G$; see [Dipper 1990, Corollary 4.10]. Let $\epsilon \in \mathrm{Irr}_{p'}(W)$ be the (linear) sign character. Then $\epsilon^\circ \neq 1_W^\circ$ since $p > 2$, whence $d_{\epsilon 1_W} = 0$. Now $\epsilon$ corresponds to the Steinberg character St of $G$. Then $d_{\mathrm{St}1_G} = d_{\epsilon 1_W} = 0$, St lies in the principal $p$-block of $G$ (e.g., by [Enguehard 2000, Theorem A]) and its degree is a power of the defining characteristic, so prime to $p$. Now note that any $p$-automorphism of $S$ is realised inside the extension of $G$ (which induces all diagonal automorphisms) by a generator $\gamma$ of the cyclic group of $p$-power order field automorphisms, so we may assume $H \leq \tilde{G} := G\langle\gamma\rangle$. By [Geck and Malle 2020, Theorem 4.5.11], St is invariant under $\gamma$. Let $\widetilde{\mathrm{St}}$ be an extension of St to $\tilde{G}$ in $B_0(\tilde{G})$, so $d_{\widetilde{\mathrm{St}}1_{\tilde{G}}} = 0$. Then $\widetilde{\mathrm{St}}|_H$ is irreducible, since St restricts irreducibly to $S$, hence lies in $B_0(H)$ and so is as required. □

**Theorem 2.5.** *Property* $(*)$ *holds for $S$ of Lie type when $p$ is not the defining characteristic.*

*Proof.* By Proposition 2.3 we may assume that $p$ divides the order of a Borel subgroup $B$ of $G$. If $F$ is a Frobenius map with respect to an $\mathbb{F}_q$-structure and $p$ divides $q-1$, we are done by Proposition 2.4. If $G$ is a Suzuki or Ree group and $p$ divides $|B|$, then $p \mid (q^2-1)$ where $F^2$ defines an $\mathbb{F}_{q^2}$-structure, and the exact same arguments as in the proof of Proposition 2.4 apply.

We are reduced to the case that $F$ is a Frobenius map with respect to an $\mathbb{F}_q$-structure and $p$ divides $|B|$ but not $q-1$. Since $p$ is not the defining prime, this implies that $G$ is a twisted group of Lie type ${}^2A_{n-1}$, ${}^2D_n$, or ${}^2E_6$ and $p$ divides $q+1$, respectively of type ${}^3D_4$ and $p$ divides $q^2+q+1$. Let $d = 2, 3$ in the respective cases. Then the centraliser of a Sylow $d$-torus of $G$ is a maximal torus, so has a unique $d$-cuspidal unipotent character. Thus, by [Enguehard 2000, Theorem A] there is a unique unipotent block of $G$ of maximal defect, the principal block, which hence contains the Steinberg character St of $G$. By [Hiss 1990a, Theorem B] we have $d_{\mathrm{St}1_G}=0$ in our case unless $G=\mathrm{PGU}_3(q)$. Except for that latter case, we can now argue as in the proof of Proposition 2.4 to conclude. For $G=\mathrm{PGU}_3(q)$ let $\chi$ be the cuspidal unipotent character of degree $q(q-1)$, prime to $p$. By [Geck 1990, Theorem 4.3(a)] it lies in $B_0(G)$ and satisfies $d_{\chi 1_G}=0$. Again, $\chi$ restricts irreducibly to $S$ and is invariant under all automorphisms, so we can argue as before. □

***Groups of Lie type in defining characteristic.*** We do not see how to approach Property $(*)$ for groups of Lie type in their defining characteristic in general. All characters of positive defect lie in the principal block and decomposition numbers tend to be large and little is known.

**Proposition 2.6** (Navarro and Tiep). *Property* $(*)$ *holds if* $S=\mathrm{PSp}_{2n}(p^f)$, $n\geq 1$, *with either* $p>3$ *or* $p=3$ *and* $n$ *even.*

*Proof.* In this case the principal block of $S$ is the only $p$-block of positive defect. Let $H$ be almost simple with $H/S$ a $p$-group. By [Navarro 1998, Corollary 9.6] there is just one $p$-block of $H$ covering $B_0(S)$, necessarily the principal block $B_0(H)$. Now the irreducible character $\chi\in\mathrm{Irr}_{p'}(H)$ with $d_{\chi 1_H}=0$ constructed in [Navarro and Tiep 2020, Proposition 3.11] lies above a character of $S$ of positive defect, hence in the principal $p$-block of $H$ and we are done. □

Observe that this does not extend to $p=3$ and $n$ odd: the group $H=\mathrm{PSp}_2(3^3).3$ has no irreducible character $\chi\in\mathrm{Irr}_{3'}(H)$ with $d_{\chi 1_H}=0$.

[Navarro and Tiep 2022] also contains results for special linear and unitary groups but these are not applicable here as the considered characters are not of $p'$-degree. Nevertheless, we can follow their general approach.

For $G=\mathrm{SL}_n(q)$ we let $\tau_j$, $j=1,\ldots,q-2$ denote the nonunipotent *Weil characters* of degree $(q^n-1)/(q-1)$, ordered so that $\tau_j$ is trivial on the centre $Z(\mathrm{SL}_n(q))$ of order $z:=\gcd(n,q-1)$ if and only if $z\mid j$.

**Proposition 2.7.** *Let* $S=\mathrm{PSL}_n(q)$ *with* $q=p^f$, $p\neq 2$ *and* $n\geq 3$.

(a) *If either* $\gcd(p-1,(q-1)/\gcd(n,q-1))>1$ *or* $2^f<(q-1)/z-1$ *then there is* $\chi\in\mathrm{Irr}_{p'}(B_0(S))$ *such that* $d_{\chi 1_S}=0$.

(b) *Write* $f=p^af'$ *with* $\gcd(p,f')=1$ *and set* $q':=p^{f'}$. *If*

$$a=0 \quad or \quad 2^f<(q'-1)/\gcd(q'-1,n)-1$$

*then Property* $(*)$ *holds for* $S=\mathrm{PSL}_n(q)$.

*Proof.* Let $G:=\mathrm{SL}_n(q)$ and set $z=\gcd(n,q-1)$. We are interested in the characters $\tau_j$ of $G$ that are trivial on $Z(G)$, that is, for which $j=zj'$ for some integer $1\leq j'\leq(q-1-z)/z$. By [Zalesski and Suprunenko 1990, Theorem 1.11] the decomposition number $d_{\tau_j 1_G}$ equals the number of solutions $x_s \in \{0, 1\}$ of the congruence

$$n(p-1)\sum_{s=0}^{f-1} x_s p^s \equiv j \pmod{(q-1)}.$$

Dividing by $z$, we need to count solutions to

$$\frac{n}{z(p-1)}\sum_{s=0}^{f-1} x_s p^s \equiv j' \pmod{(q-1)/z}.$$

If there is a prime $\ell$ dividing $p-1$ and $(q-1)/z$, then reducing modulo $\ell$ we see there is no solution for $j' = 1$. Also, the left hand side can take at most $2^f$ distinct values. Since there are $(q-1)/z - 1$ admissible values for $j'$, there is $j'$ with no solutions whenever $2^f < (q-1)/z - 1$. Thus, under either of our assumptions we find $j'$ with $\tau_j = \tau_{zj'} \in \mathrm{Irr}_{p'}(G)$ with $d_{\tau_j 1_G} = 0$. Since $G$ has a single $p$-block of positive defect, $\tau_j$ lies in the principal block. Furthermore, by construction $Z(\mathrm{SL}_n(q))$ lies in the kernel of $\tau_j$ and hence $\tau_j$ deflates to a character of $S = \mathrm{PSL}_n(q)$. Thus we get (a).

(b) Since $p > 2$ does not divide $q-1$, the $p$-power order automorphisms of $S$ are field automorphisms, of order dividing $p^a$ where $f = p^a f'$ is as in the statement. If $a = 0$, Property $(*)$ follows from (a) since necessarily $H = S$. For $a > 0$ let $\gamma$ be a field automorphism of $G$ (and hence of $S$) of order $p^a$. There are exactly $q' - 2$ Weil characters of $G$ invariant under $\gamma$, which hence extend to $G\langle\gamma\rangle$. Of these, $(q'-1)/\gcd(q'-1, n) - 1$ are trivial on $Z(G)$, so define characters in $\mathrm{Irr}_{p'}(S)$ invariant under $\gamma$. By the argument above, if this number is bigger than $2^f$ then there exists such a character $\chi$ with $d_{\chi 1_S} = 0$. Hence any character of $H$ in $B_0(H)$ lying above it verifies Property $(*)$. □

Note that the case $n = 2$ is addressed in Proposition 2.6. Observe that the condition in Proposition 2.7(a) is satisfied if $\gcd(n, q-1) = 1$, for example. It also holds when $p > n+1$ (since $p-1$ always divides $q-1$), or if $q > n(2^f+1)$. Thus, Proposition 2.7 extends and complements [Navarro and Tiep 2022, Proposition 3.3(ii)].

**Corollary 2.8.** *Let $S = \mathrm{PSL}_n(q)$ with $q = p^f$, $p \neq 2$ and $3 \le n \le 9$. Then there is $\chi \in \mathrm{Irr}_{p'}(B_0(S))$ with $d_{\chi 1_S} = 0$ unless possibly $S$ is one of*

$$\mathrm{PSL}_4(5),\ \mathrm{PSL}_6(3),\ \mathrm{PSL}_6(7),\ \mathrm{PSL}_8(3),\ \mathrm{PSL}_8(9),\ \mathrm{PSL}_8(5),\ \mathrm{PSL}_8(25).$$

*Proof.* For the groups $\mathrm{PSL}_n(q)$, $n \le 9$, considered here, either the conditions in Proposition 2.7(a) are satisfied, or if not, a direct checking with the Zalesski–Suprunenko formula shows the claim, except for the groups listed in the conclusion and for $\mathrm{PSL}_4(3)$. The decomposition matrix of $\mathrm{PSL}_4(3)$ is available in [GAP 2020] from which the claim can be verified for that group. □

For $G = \mathrm{SU}_n(q)$ we let $\tau_j$, $j = 1, \dots, q$, denote the nonunipotent Weil characters, constructed by Seitz [1975], of degree $(q^n - (-1)^n)/(q+1)$, again ordered so that $\tau_j$ is trivial on the centre $Z(\mathrm{SU}_n(q))$ of order $z := \gcd(n, q+1)$ if and only if $z \mid j$.

**Proposition 2.9.** *Let $S = \mathrm{PSU}_n(q)$ with $q = p^f$, $p \neq 2$ and $n \geq 3$.*

(a) *If $2^f < (q+1)/z - 1$ then there is $\chi \in \mathrm{Irr}_{p'}(B_0(S))$ such that $d_{\chi 1_S} = 0$.*

(b) *Write $f = p^a f'$ with $\gcd(p, f') = 1$ and set $q' := p^{f'}$. If*

$$a = 0 \quad or \quad 2^f < (q'+1)/\gcd(q'+1, n) - 1$$

*then Property $(*)$ holds for $S = \mathrm{PSU}_n(q)$.*

*Proof.* The argument is very similar to the one for the special linear groups. Let $G = \mathrm{SU}_n(q)$ and set $z = \gcd(n, q+1) = |Z(G)|$. Again, we consider characters $\tau_j$ trivial on $Z(G)$, that is, for which $j = zj'$ for some integer $1 \leq j' \leq (q+1-z)/z$. By [Zalesski 1990, Main Theorem] the decomposition number $d_{\tau_j 1_G}$ equals the number of solutions $x_s \in \{0, 1\}$ of the congruence

$$n\left((p-1)\sum_{s=0}^{f-1} x_s p^s - 1\right) \equiv j \pmod{(q+1)}.$$

(In fact, [Zalesski 1990] has an additional summand of $\frac{1}{2}(q+1)$ on the right-hand side, but this disappears here due to a different numbering of the $\tau_j$, see [Navarro and Tiep 2022, p. 612]; in any case, this difference will not matter for our argument here.) Since the left-hand side can take at most $2^f$ distinct values, while there are $(q+1)/z - 1$ admissible values for $j'$ the assertion in (a) follows. For (b) we can argue exactly as in the proof of Proposition 2.7. □

As in [Navarro and Tiep 2020; 2022] we have no general results for orthogonal or exceptional type groups in their defining characteristic.

## 3. The reduction

In this section we prove Theorem A. We will need the following results, which we collect here for the reader's convenience.

**Lemma 3.1** [Murai 1994, Lemma 4.3]. *Let $N \lhd G$ and let $\theta \in \mathrm{Irr}_{p'}(B_0(N))$. Suppose that $\theta$ extends to $PN$, where $P \in \mathrm{Syl}_p(G)$. Then there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ satisfying $[\theta^G, \chi] \neq 0$.*

The following argument is inside the proof of [Navarro and Tiep 2020, Theorem 2.6].

**Lemma 3.2.** *Let $G$ be a finite group and suppose that $d_{\chi 1_G} \neq 0$ for every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$. Let $M \lhd G$ and let $P \in \mathrm{Syl}_p(G)$. Then $d_{\psi 1_{MP}} \neq 0$ for every $\psi \in \mathrm{Irr}_{p'}(B_0(MP))$.*

*Proof.* Since $MP/M$ is a $p$-group, we have that $\psi_M = \tau \in \mathrm{Irr}_{p'}(B_0(M))$. By Lemma 3.1 there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ lying over $\tau$. By hypothesis, $d_{\chi 1_G} \neq 0$, and then $\chi^\circ_M$ contains $1_M$, so $d_{\tau 1_M} \neq 0$. Since $MP/M$ is a $p$-group, we have that $(MP)^\circ = M^\circ$ and then $d_{\psi 1_{MP}} \neq 0$, as wanted. □

We next prove Theorem A for $p$-solvable groups.

**Theorem 3.3.** *Let $G$ be a $p$-solvable group. Then the following are equivalent*:

(i) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} \neq 0$.*

(ii) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} = 1$.*

(iii) *For $P \in \mathrm{Syl}_p(G)$ we have $\boldsymbol{N}_G(P) = P \times K$ for some $K \leq G$.*

*Proof.* We first prove (iii) implies (ii). By [Navarro et al. 2014, Theorem 3.2] we have $K \subseteq \boldsymbol{O}_{p'}(G) =: X$. Now, let $\overline{G} = G/X$ and $\overline{P} = PX/X$, then (iii) implies that $\boldsymbol{N}_{\overline{G}}(\overline{P}) \cong \overline{P}$. Since $\mathrm{Irr}_{p'}(B_0(G)) = \mathrm{Irr}_{p'}(B_0(G/X))$ we know by [Navarro and Tiep 2020, Theorem B] that (i) and (ii) hold.

Since (ii) implies (i) trivially, we just need to show that (i) implies (iii). We proceed by induction on $|G|$. Let $N = \boldsymbol{O}_{p'}(G)$ and use the bar notation. Since $\mathrm{Irr}_{p'}(B_0(G)) = \mathrm{Irr}_{p'}(B_0(\overline{G}))$, if $N > 1$, we have by induction that $\boldsymbol{N}_{\overline{G}}(\overline{P}) = \overline{P} \times \overline{K}$. By [Navarro et al. 2014, Theorem 3.2] we have that $\overline{K} \subseteq \boldsymbol{O}_{p'}(\overline{G}) = 1$ so $\overline{K} = 1$ and hence $\boldsymbol{N}_{\overline{G}}(\overline{P}) = \overline{P}$. This implies that $\boldsymbol{N}_G(P) = P \times \boldsymbol{C}_N(P)$ and we are done. So we may assume that $N = 1$. But in this case the principal $p$-block is the only $p$-block of $G$, and we are done by [Navarro and Tiep 2020, Theorem B]. □

We finally prove Theorem A.

**Theorem 3.4.** *Let $G$ be a finite group. Assume that $p > 3$ and all nonabelian composition factors of $G$ of order divisible by $p$ satisfy Property $(*)$. Then the following are equivalent*:

(i) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} \neq 0$.*

(ii) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} = 1$.*

(iii) *For $P \in \mathrm{Syl}_p(G)$ we have $\boldsymbol{N}_G(P) = P \times K$ for some $K \leq G$.*

*Proof.* Since [Navarro et al. 2014, Theorem 3.2] holds for odd primes, we can argue as in the first part of the proof of Theorem 3.3 to show that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) (notice that, as happens in [Navarro and Tiep 2020], (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is always true if $p > 3$, with no extra conditions on the composition factors of $G$). So we just need to prove that (i) implies (iii). We work by induction on $|G|$. Arguing as in the second paragraph of the proof of Theorem 3.3 we may then assume that $\boldsymbol{O}_{p'}(G) = 1$.

Let $M \lhd G$ be the largest $p$-solvable normal subgroup of $G$. We claim that $M = 1$. Let $\overline{G} = G/M$ and use the bar notation. Suppose $M > 1$. Since $\mathrm{Irr}_{p'}(B_0(\overline{G}))$ is contained in $\mathrm{Irr}_{p'}(B_0(G))$ we have by induction that $\boldsymbol{N}_{\overline{G}}(\bar{P}) = \bar{P} \times \bar{K}$. By [Navarro and Tiep 2020, Theorem 3.2] this implies that $\bar{K} \subseteq \boldsymbol{O}_{p'}(\overline{G})$. Since $M$ is the largest normal $p$-solvable subgroup of $G$, $\boldsymbol{O}_{p'}(\overline{G})$ is trivial and hence $\boldsymbol{N}_{\overline{G}}(\bar{P}) = \bar{P}$. By [Guralnick et al. 2004, Theorem 1.1] this forces $\overline{G}$ to be solvable. Hence $G$ is $p$-solvable and we are done by Theorem 3.3.

Let $N$ be a minimal normal subgroup of $G$; thus $N = S_1 \times \cdots \times S_t$ where $S_i \cong S$ is a nonabelian simple group of order divisible by $p$ and the $S_i$ are transitively permuted by $G$. Let $X/N = \boldsymbol{O}_{p'}(G/N)$ and let $H = \bigcap \boldsymbol{N}_G(S_i)$. We claim that $G = (H \cap X)P$. Write $Y = (H \cap X)P$. We first show that $\boldsymbol{O}_{p'}(Y) = 1$. Indeed, since $\boldsymbol{O}_{p'}(G)$, we have that $\boldsymbol{O}_{p'}(Y) \cap (H \cap X) \subseteq \boldsymbol{O}_{p'}(H \cap X) = 1$. Now, $\boldsymbol{O}_{p'}(Y) \cong (H \cap X)\boldsymbol{O}_{p'}(Y)/(H \cap X)$ is a $p$-group, so $\boldsymbol{O}_{p'}(Y) = 1$ as wanted. By Lemma 3.2 applied to $H \cap X$ in place of $M$, we have that $Y$ satisfies (i). Since the nonabelian composition factors of $Y$ are composition factors of $G$, if $Y < G$, by induction this gives $\boldsymbol{N}_Y(P) = P \times K$ for some $K$. But then $K \subseteq \boldsymbol{O}_{p'}(Y)$ by [Navarro et al. 2014, Theorem 3.2], so $K = 1$. This means that $\boldsymbol{N}_Y(P) = P$ and then by [Guralnick et al. 2004, Theorem 1.1] the group $Y$ is solvable. But then $N$ is solvable, a contradiction. Hence $Y = G$, as wanted.

Since $G = (H \cap X)P$ and $H$ acts trivially on $\{S_1, \ldots, S_t\}$, $P$ must act transitively on the set $\{S_1, \ldots, S_t\}$. Write $S = S_1$ and, for $i = 2, \ldots, t$, write $S_i = S^{x_i}$ with $x_i \in P$. We proceed now as in the proof of [Navarro and Tiep 2020, Theorem 2.6]. Let $R = \boldsymbol{N}_P(S)$. If $SR = G$, then $S = N$ and $\boldsymbol{C}_G(S)$ is a normal $p$-subgroup of $G$. Since there are no nontrivial $p$-solvable normal subgroups of $G$, we conclude that $\boldsymbol{C}_G(S) = 1$ and hence $G$ is almost simple with socle $S$ and $|G : S|$ is a power of $p$. Since $S$ satisfies Property $(*)$ by assumption, we have a contradiction. Hence we may assume that $SR < G$.

Let $Q = P \cap N$ and let $R_1 = R \cap S = Q \cap S = P \cap S \in \mathrm{Syl}_p(S)$. Let $\gamma \in \mathrm{Irr}_{p'}(B_0(SR))$ and notice that $\gamma_S = \psi \in \mathrm{Irr}_{p'}(B_0(S))$ since $SR/S$ is a $p$-group. For $i = 2, \ldots, t$, let $\psi_i = \psi^{x_i} \in \mathrm{Irr}_{p'}(B_0(S_i))$ and let $\eta = \psi \times \psi_2 \times \cdots \times \psi_t \in \mathrm{Irr}_{p'}(B_0(N))$, which is $P$-invariant by [Navarro et al. 2007, Lemma 4.1(ii)]. Then $\eta$ extends to $PN$. By Lemma 3.1 and hypothesis we have $d_{\eta 1_N} \neq 0$. By [Navarro and Tiep 2020, Lemma 2.3] we have that $d_{\psi 1_S} \neq 0$ and then, since $SR/S$ is a $p$-group, we conclude that $d_{\gamma 1_{SR}} \neq 0$. Since $SR < G$ and $S$ is a composition factor of $G$, this implies $\boldsymbol{N}_{SR}(R) = R \times K$ for some $K$. Then $K \subseteq \boldsymbol{O}_{p'}(SR)$. Arguing as before, we have that $\boldsymbol{O}_{p'}(SR) = 1$, so $K = 1$ and then $\boldsymbol{N}_{SR}(R) = R$. Now by [Guralnick et al. 2004, Theorem 1.1], $SR$ is solvable, and hence $S$ is solvable, which is our final contradiction. □

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