

*Pacific  
Journal of  
Mathematics*

**LEVI DECOMPOSITIONS  
OF LINEAR ALGEBRAIC GROUPS  
AND NONABELIAN COHOMOLOGY**

GEORGE J. MCNINCH

# LEVI DECOMPOSITIONS OF LINEAR ALGEBRAIC GROUPS AND NONABELIAN COHOMOLOGY

GEORGE J. MCNINCH

*To the memory of Gary Seitz (1943–2023)*

Let  $k$  be a field, and let  $G$  be a linear algebraic group over  $k$  for which the unipotent radical  $U$  of  $G$  is defined and split over  $k$ . Consider a finite, separable field extension  $\ell$  of  $k$  and suppose that the group  $G_\ell$  obtained by base-change has a *Levi decomposition* (over  $\ell$ ). We continue here our study of the question previously investigated (*Arch. Math.* 100:1 (2013), 7–24): does  $G$  have a *Levi decomposition* (over  $k$ )?

Using nonabelian cohomology we give some condition under which this question has an affirmative answer. On the other hand, we provide an (other) example of a group  $G$  as above which has no Levi decomposition over  $k$ .

1. Introduction	379
2. Preliminaries	381
3. Nonabelian cohomology	385
4. Descent of Levi factors, I	387
5. Automorphisms of extensions	390
6. Automorphisms and cohomology	391
7. Descent of Levi factors, II	392
8. An example	394
Acknowledgments	396
References	396

## 1. Introduction

Let  $k$  be a field, and let  $G$  be a linear algebraic group over  $k$ . Thus  $G$  is a group scheme which is smooth and affine over  $k$ .

If  $k_{\text{alg}}$  denotes an algebraic closure of  $k$ , the *unipotent radical* of  $G_{k_{\text{alg}}}$  is the maximal connected, unipotent, normal subgroup. The unipotent radical of  $G$  is defined over  $k$  if  $G$  has a  $k$ -subgroup  $U$  such that  $U_{k_{\text{alg}}}$  is the unipotent radical of  $G_{k_{\text{alg}}}$ .

---

*MSC2020:* primary 20G15; secondary 20G07, 20G10.

*Keywords:* linear algebraic groups, cohomology, Levi factors, reductive groups, unipotent groups.

**Definition 1.1.** We say that  $G$  satisfies condition **(R)** if the unipotent radical  $U$  of  $G$  is defined and split over  $k$ . (See [Definition 2.1](#) for the notion of split unipotent group). Write  $\pi : G \rightarrow G/U$  for the quotient morphism; we say that  $G/U$  is the *reductive quotient* of  $G$ . Thus  $G/U$  is a (not necessarily connected) reductive algebraic group.

**Remark 1.2.** If  $k$  is perfect then **(R)** holds for any linear algebraic group  $G$  over  $k$ . Indeed, the unipotent radical  $U$  is defined over  $k$  by Galois descent. Every (smooth) connected unipotent group over a perfect field  $k$  is  $k$ -split; see [Remark 2.2](#).

**Definition 1.3.** Suppose that  $G$  satisfies condition **(R)**. The group  $G$  has a *Levi decomposition* (over  $k$ ) if there is a closed  $k$ -subgroup scheme  $M$  of  $G$  such that the restriction of the quotient mapping determines an isomorphism

$$\pi|_M : M \xrightarrow{\sim} G/R.$$

The subgroup  $M$  is then a *Levi factor* of  $G$ .

If  $G$  satisfies **(R)** and if  $M$  is a Levi factor of  $G$  then [Proposition 2.7](#) below shows that  $G$  may be identified with the semidirect product  $U \rtimes M$  as algebraic groups.

**Remark 1.4.** When  $k$  has characteristic 0, Mostow showed that  $G$  always has a Levi decomposition; see, e.g., [[McNinch 2010](#), §3.1]. For any field  $k$  of characteristic  $p > 0$ , there are linear algebraic groups  $G$  over  $k$  with no Levi factor; see, e.g., Conrad, Gabber, and Prasad's work [[Conrad et al. 2015](#), A.6] for a construction.

We now fix a linear algebraic  $k$ -group  $G$  satisfying **(R)**. Suppose that  $\ell$  is a finite, separable field extension of  $k$ , and suppose that  $G_\ell$  has a Levi decomposition. We pose this question:

( $\diamond$ ) If  $G_\ell$  has a Levi decomposition (over  $\ell$ ), does  $G$  have a Levi decomposition (over  $k$ )?

This question about descent of Levi factors was already considered in [[McNinch 2013](#)] whose main result gave the following partial answer:

**Theorem 1.5.** Assume that  $\ell$  is a finite, Galois field extension of  $k$  with Galois group  $\Gamma = \text{Gal}(\ell/k)$ , and assume that  $G_\ell$  has a Levi decomposition. If  $|\Gamma|$  is invertible in  $k$  then  $G$  has a Levi decomposition.

We introduce the nonabelian cohomology set  $H_{\text{coc}}^1(M, U)$  in [Section 3](#), and in [Section 4](#) we prove the following result providing a different partial answer to ( $\diamond$ ):

**Theorem 1.6.** If  $\ell$  is a finite separable extension of  $k$ , suppose

- (a)  $G_\ell$  has a Levi decomposition,
- (b) the group scheme  $U_\ell^{M_\ell}$  is trivial, and
- (c)  $H_{\text{coc}}^1(M_\ell, U_\ell) = 1$ .

Then  $G$  has a Levi decomposition.

We also prove [Corollary 4.5](#) which gives a reformulation of [Theorem 1.6](#) using a filtration of  $U$ . After some preliminaries in [Sections 5](#) and [6](#), we prove the following related result in [Section 7](#):

**Theorem 1.7.** *Suppose*

- (a)  $G_\ell$  has a Levi decomposition,
- (b)  $\text{Inn}(U_\ell)^{M_\ell}$  is trivial,
- (c) the center  $Z$  of  $U$  is a vector group on which  $G$  acts linearly, and
- (d)  $H_{\text{coc}}^1(M_\ell, \text{Inn}(U_\ell)) = 1$ .

*Then  $G$  has a Levi decomposition.*

The reader should compare these results with [[McNinch 2010](#), Theorem 5.2]. This older result shows that a certain condition involving the vanishing of second cohomology  $H^2$  unconditionally guarantees the existence of a Levi factor. These newer results — [Theorems 1.6](#), [1.7](#) and [Corollary 4.5](#) — instead give conditions using vanishing of (some form of) first cohomology to descend Levi factors over finite separable field extensions.

We note that *some* additional hypotheses are required to answer the question ( $\diamond$ ). Indeed, [Section 8](#) provides an example of an algebraic group  $G$  satisfying condition **(R)** for which  $G_\ell$  has a Levi factor for some cyclic Galois extension  $\ell$  of degree  $p$  over  $k$ , but  $G$  has no Levi factor over  $k$ .

Every example currently known to the author of a group  $G$  satisfying **(R)** for which ( $\diamond$ ) has a negative answer is *not connected*. This suggests the following natural problem for which a solution would be desirable:

**Problem 1.8.** Let  $\ell$  be a finite, separable field extension of  $k$  and  $G$  a connected linear algebraic group over  $k$  satisfying **(R)**. Either find a proof of the assertion “ $G_\ell$  has a Levi factor implies that  $G$  has a Levi factor” or find an example of a group for which this condition fails.

## 2. Preliminaries

We fix an arbitrary field  $k$ . Throughout the paper,  $G$  will denote a linear algebraic group over  $k$ . Thus  $G$  is a group scheme which is smooth, affine, and of finite type over  $k$ .

If  $V$  is a linear representation of  $G$ , then for  $i \geq 0$ ,  $H^i(G, V)$  denotes the  $i$ -th (Hochschild) cohomology group of  $V$ ; see, for instance, [[Jantzen 2003](#), I.4].

**Automorphism group functors.** By a  $k$ -group functor, we mean a functor from the category of commutative  $k$ -algebras to the category of groups. Of course, any group scheme — and in particular, any linear algebraic group — over  $k$  is a fortiori

a  $k$ -group functor, but we will consider a few group functors which are in general not representable (i.e., which fail to be group schemes).

For a linear algebraic group  $G$  over  $k$ , we write  $\text{Aut}(G)$  for the  $k$ -group functor which assigns to a commutative  $k$ -algebra  $\Lambda$  the group  $\text{Aut}(G)(\Lambda) = \text{Aut}(G(\Lambda))$ .

If  $Z$  denotes the (scheme-theoretic) center of  $G$ , there is a natural homomorphism of  $k$ -group functors  $\text{Inn} : G/Z \rightarrow \text{Aut}(G)$  whose image determines a normal  $k$ -subgroup functor  $\text{Inn}(G)$  of  $\text{Aut}(G)$ ; see [SGA 3<sub>III</sub> 2011, XXIV §1.1].

Now, the  $k$ -group functor  $\text{Out}(G)$  is defined for each  $\Lambda$  by the rule

$$\text{Out}(G)(\Lambda) = \text{Aut}(G)(\Lambda) / \text{Inn}(G)(\Lambda).$$

The quotient mappings  $\text{Aut}(G(\Lambda)) \rightarrow \text{Aut}(G(\Lambda)) / \text{Inn}(G(\Lambda))$  determine a homomorphism of  $k$ -group functors

$$(2-1) \quad \Psi : \text{Aut}(G) \rightarrow \text{Out}(G).$$

**Unipotent groups.** Recall from [Borel 1991, §15.1] the following:

**Definition 2.1.** A connected, unipotent linear algebraic group  $U$  over  $k$  is said to be  $k$ -split provided that there is a sequence

$$1 = U_0 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

of closed, connected, normal  $k$ -subgroups of  $U$  such that  $U_{i+1}/U_i \simeq \mathbf{G}_{a/k}$  for  $i = 0, \dots, m-1$ , where  $\mathbf{G}_a = \mathbf{G}_{a/k}$  is the additive group.

**Remark 2.2.** When  $k$  is not a perfect field, there are connected unipotent  $k$ -groups which are not  $k$ -split; see, for example, [Serre 2002, III. §2.1, Exercise 3]. On the other hand, if  $k$  is perfect, every connected unipotent  $k$ -group is  $k$ -split; see [Borel 1991, Corollary 15.5(ii)].

**Proposition 2.3.** *Let  $U$  be a  $k$ -split unipotent group. If  $V$  is a normal  $k$ -subgroup of  $U$ , then  $U/V$  is again a  $k$ -split unipotent group.*

*Proof.* The assertion follows from [Borel 1991, Theorem 15.4(i)]. □

A substantial reason for our focus on split unipotent groups is the following result of Rosenlicht:

**Proposition 2.4.** *Suppose that  $U$  is a connected,  $k$ -split unipotent subgroup of  $G$  and write  $\pi : G \rightarrow G/U$  for the quotient morphism. Then there is a morphism of  $k$ -varieties*

$$\sigma : G/U \rightarrow G,$$

*which is a section to  $\pi$  — that is,  $\pi \circ \sigma$  is the identity. In particular, the mapping  $\pi : G(k) \rightarrow (G/U)(k)$  on  $k$ -points is surjective.*

*Proof.* See [Springer 2009, Theorem 14.2.6]. □

**Extensions, group actions and semidirect products.** Let  $A$  and  $M$  be linear algebraic  $k$ -groups.

**Definition 2.5.** An *extension* of  $M$  by  $A$  is a linear algebraic  $k$ -group  $E$  together with a sequence

$$(2-2) \quad 1 \rightarrow A \xrightarrow{i} E \xrightarrow{\pi} M \rightarrow 1,$$

where  $i$  and  $\pi$  are morphisms of algebraic groups over  $k$ ,  $i$  determines an isomorphism of  $A$  onto  $\ker \pi$ , and the homomorphism  $\pi$  is faithfully flat.

**Definition 2.6.** If  $A$  and  $M$  are linear algebraic groups, we say that  $A$  is an  $M$ -group provided that there is a morphism of  $k$ -group functors  $M \rightarrow \text{Aut}(A)$ .

If  $A$  is an  $M$ -group via the homomorphism of  $k$ -group functors

$$\alpha : M \rightarrow \text{Aut}(A)$$

then we can form the semidirect product  $A \rtimes_{\alpha} M$ ; it is an extension of  $M$  by  $A$ . (We omit the subscript  $\alpha$  from  $\rtimes_{\alpha}$  when it is clear from context).

If  $E$  is an extension (2-2), observe that the conjugation action of  $E$  determines a morphism of group functors  $\text{Inn} : E \rightarrow \text{Aut}(A)$ .

We record the following two results; their proofs are straightforward and left to the reader:

**Proposition 2.7.** Let  $A$  and  $M$  be linear algebraic  $k$ -groups and consider an extension (2-2)

$$1 \rightarrow A \rightarrow G \xrightarrow{\pi} M \rightarrow 1.$$

If  $s : M \rightarrow G$  is a group homomorphism that is a section to  $\pi$  then the multiplication mapping  $(x, m) \mapsto xm$  induces an isomorphism

$$A \rtimes_{\phi} M \xrightarrow{\sim} G$$

of algebraic  $k$ -groups, where  $\phi : M \rightarrow \text{Aut}(A)$  is the composite  $\text{Inn} \circ s$ .

**Proposition 2.8.** Let  $A$  and  $M$  be linear algebraic  $k$ -groups and consider an extension (2-2)

$$1 \rightarrow A \rightarrow G \xrightarrow{\pi} M \rightarrow 1.$$

There is a unique homomorphism of  $k$ -group functors  $\phi : M \rightarrow \text{Out}(A)$  such that for any section  $s_0 : M \rightarrow G$  to  $\pi$  as in Proposition 2.4, for any commutative  $k$ -algebra  $\Lambda$ , and for any  $m \in M(\Lambda)$ ,  $\phi(m)$  is the class of the inner automorphism  $\text{Inn}(s_0(m))$  in  $\text{Out}(A)$ .

**Remark 2.9.** A unipotent  $k$ -group  $U$  is *wound* if every mapping  $A^1 \rightarrow U$  of  $k$ -schemes is constant. A connected, wound unipotent group of positive dimension is

not  $k$ -split. If  $M$  is a connected and reductive  $k$ -group and if  $U$  is a wound unipotent  $k$ -group, then

(\*) any homomorphism of  $k$ -group functors  $M \rightarrow \text{Aut}(U)$  is trivial.

If  $M$  is a torus then (\*) follows from [Conrad et al. 2015, Corollary B.44]. Now (\*) follows in general since the connected reductive group  $M$  is generated by its maximal  $k$ -tori — see [Springer 2009, Theorem 13.3.6].

Observation (\*) provides some partial justification for our focus on groups satisfying **(R)**.

**Linear actions.** Let  $G$  and  $U$  be linear algebraic groups, suppose that  $U$  is connected and unipotent, and suppose that  $U$  is a  $G$ -group.

**Definition 2.10.** If  $U$  is a vector group, the action of  $G$  on  $U$  is said to be *linear* if there is a  $G$ -equivariant isomorphism of algebraic groups  $U \simeq \text{Lie}(U)$ .

**Definition 2.11.** A filtration

$$1 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

by  $G$ -invariant closed  $k$ -subgroups  $U_i$  with  $U_i$  normal in  $U_{i+1}$  for each  $i$  is a *linear filtration* for the action of  $G$  if  $U_{i+1}/U_i$  is a vector group on which  $G$  acts linearly for each  $i = 0, \dots, m-1$ .

A linear filtration is a *central linear filtration* if  $U_{i+1}/U_i$  is central in  $U/U_i$  for each  $i \geq 0$ .

The following result was proved already in [Stewart 2013] under the assumption that  $k$  is algebraically closed.

**Theorem 2.12.** *Assume that the unipotent radical  $U$  of  $G$  is defined and split over  $k$ .*

- (a) *If  $G$  is connected, there is a linear filtration of  $U$  for the action of  $G$ .*
- (b) *If  $U$  has a linear filtration for the action of  $U \rtimes G$  then it has a central linear filtration.*

*Proof.* (a) is the main result of [McNinch 2014].

To see (b), suppose that the subgroups  $U_i$  form a linear filtration of  $U$  for the action of  $U \rtimes G$ . We may clearly refine this filtration to arrange that  $\text{Lie}(U_i)/\text{Lie}(U_{i+1})$  is an irreducible representation of  $U \rtimes G$  for each  $i$ .<sup>1</sup> We claim that this refined filtration is central. We proceed by induction on the length  $m$  of the linear filtration. If  $m = 1$  then  $U$  is abelian and the result is immediate.

Suppose now that  $m > 1$  and that one knows that any linear filtration of  $U$  for the action of  $U \rtimes G$  of length  $< m$  for which the factors of consecutive terms form irreducible  $U \rtimes G$ -representations is central.

<sup>1</sup>Since  $U$  is unipotent, an irreducible representation of  $U \rtimes G$  amounts to an irreducible representation of  $G$ .

Now, the conjugation action of  $U$  on  $U_1$  is a *linear* action; thus, the fixed points for the conjugation action of  $U$  on  $U_1$  form a  $G$ -invariant subgroup scheme which is smooth over  $k$ . Since  $U_1 \simeq \text{Lie}(U_1)$  is an irreducible  $G$ -representation, it follows that  $U$  acts trivially on  $U_1$ ; thus  $U_1$  is central in  $U$ . Now, it is clear that

$$(2-3) \quad 1 \subset U_2/U_1 \subset \cdots \subset U_m/U_1 = U/U_1$$

forms a linear filtration of  $U/U_1$  for the action of  $G$  for which the factors of consecutive terms form irreducible  $U \rtimes G$ -representations. Thus by induction (2-3) is a central linear filtration; this completes the proof.  $\square$

**Remark 2.13.** In the proof of [Theorem 2.12](#), we constructed a central linear filtration by arranging that the action of  $U \rtimes G$  on each quotient  $U_{i+1}/U_i$  is irreducible. This condition is sufficient, but not necessary — in general, there are central linear filtrations for which  $\text{Lie}(U_{i+1})/\text{Lie}(U_i)$  is a reducible  $G$ -representation for some  $i$ .

**Galois cohomology.** Write  $\Gamma = \text{Gal}(k_{\text{sep}}/k)$  for the absolute Galois group of  $k$  where  $k_{\text{sep}}$  is a separable closure of  $k$ .

Let  $G$  be a  $k$ -group functor satisfying the conditions spelled out in [\[Serre 2002, II.§1.1\]](#). Then  $\Gamma$  acts continuously on the group  $G(k_{\text{sep}})$  and we may consider the Galois cohomology set  $H^1(k, G) := H^1(\Gamma, G(k_{\text{sep}}))$  [\[Serre 2002, II.§5.1\]](#).

**Proposition 2.14.** *Let  $U$  be a connected, split unipotent algebraic group over  $k$ . Then the Galois cohomology set satisfies  $H^1(k, U) = 1$ .*

*Proof.* The necessary tools are recalled in [\[McNinch 2004, Proposition 30\]](#).  $\square$

### 3. Nonabelian cohomology

Let  $A$  and  $M$  be linear algebraic  $k$ -groups and suppose that  $A$  is an  $M$ -group. Following [\[Demarche 2015, §2.1\]](#), we introduce the cohomology set  $H_{\text{coc}}^1(M, A)$  as follows. Let  $Z_{\text{coc}}^1(M, A)$  denote the set of regular maps  $f : M \rightarrow A$  such that for each commutative  $k$ -algebra  $\Lambda$  and each  $x, y \in M(\Lambda)$ , the 1-cocycle condition

$$(3-1) \quad f(xy) = f(x) \cdot {}^x f(y)$$

holds. Two cocycles  $f, f' \in Z_{\text{coc}}^1(M, A)$  are *cohomologous* provided there is  $u \in U(k)$  such that for each  $\Lambda$  and each  $x \in M(\Lambda)$  we have

$$f(x) = u^{-1} \cdot f'(x) \cdot {}^x u.$$

This defines an equivalence relation on  $Z_{\text{coc}}^1(M, A)$  and we write  $H_{\text{coc}}^1(M, A)$  for the quotient set.

We view  $H_{\text{coc}}^1(M, A)$  as a *pointed set*; the marked point  $1 \in H_{\text{coc}}^1(M, A)$  is the class of the cocycle in  $Z_{\text{coc}}^1(M, A)$  which takes the constant value 1. The pointed

set  $H_{\text{coc}}^1(M, A)$  is trivial if  $H_{\text{coc}}^1(M, A) = \{1\}$ ; we often indicate this condition by the shorthand  $H_{\text{coc}}^1(M, A) = 1$ .

One interpretation or application of this cohomology set arises from examination of a semidirect product  $G = A \rtimes M$ . Consider a linear algebraic group  $G$  with normal subgroup  $A$  and a quotient mapping  $\pi : G \rightarrow M = G/A$ . We suppose that there is a group homomorphism  $s_0 : M \rightarrow G$  which is a section to  $\pi$ . According to [Proposition 2.7](#),  $s_0$  determines an isomorphism  $G \simeq A \rtimes M$ .

**Definition 3.1.** Consider the set of all homomorphisms of  $k$ -groups  $M \rightarrow G$  which are sections to  $\pi$ ; two such homomorphisms  $s, s'$  will be considered *equivalent* if there is  $a \in A(k)$  such that  $s = as'a^{-1}$ . Then  $\text{Sect}(G \xrightarrow{\pi} M)$  denotes the quotient of the set of all such homomorphisms by this equivalence relation.

**Proposition 3.2.** Write  $\mu : G \times G \rightarrow G$  for the multiplication mapping. For a given homomorphism  $s_0 : M \rightarrow G$  which is a section to  $\pi$ , the assignment

$$f \mapsto \mu \circ (f, s_0),$$

— where  $(f, s_0) : M \rightarrow M \times G$  is the mapping  $m \mapsto (f(m), s_0(m))$  — determines a bijection

$$A_{s_0} : H_{\text{coc}}^1(M, A) \rightarrow \text{Sect}(G \xrightarrow{\pi} M).$$

*Proof.* As already observed above, the choice of  $s_0$  determines an isomorphism of linear algebraic groups  $G \simeq A \rtimes M$ ; see [Proposition 2.7](#). Now the result follows from [[Demarche 2015](#), Proposition 2.2.2]. □

**Remark 3.3.**  $H_{\text{coc}}^1(M, A)$  is a pointed set — i.e., a set with a distinguished element. That distinguished element is the class of the trivial mapping  $(x \mapsto 1) : G \rightarrow A$ . In the bijection of [Proposition 3.2](#) the section corresponding to the trivial class is  $s_0$ .

**Remark 3.4.** When  $Z$  is a vector group with a linear action of  $M$ ,  $H_{\text{coc}}^1(M, Z)$  coincides with the usual Hochschild cohomology group  $H^1(M, Z) \simeq H^1(M, \text{Lie}(Z))$ . In that case  $H_{\text{coc}}^1(M, Z)$  is a  $k$ -vector space.

Suppose now that  $A = U$  is a split unipotent  $M$ -group and that  $Z \subset U$  is a central  $k$ -subgroup that is  $M$ -invariant. Then  $U/Z$  is a split unipotent  $M$ -group, and there is a mapping

$$(3-2) \quad \Delta : H_{\text{coc}}^1(M, U/Z) \rightarrow H^2(M, Z),$$

where  $H^2(M, Z)$  denotes the second Hochschild cohomology; it is defined as follows. First, use Rosenlicht’s result [Proposition 2.4](#) to choose a regular mapping  $s : U/Z \rightarrow U$  which is a section to the quotient homomorphism  $U \rightarrow U/Z$ . Let  $\alpha = [f] \in H_{\text{coc}}^1(M, A/Z)$  with  $f \in Z_{\text{coc}}^1(M, A/Z)$ .

As in [Demazure and Gabriel 1970, II, Subsection 3.2.3]—see also [McNinch 2010, §4.4]—the rule  $(g, h) \mapsto s(f(g))s(f(h))s(f(gh))^{-1}$  determines a Hochschild 2-cocycle whose class in  $H^2(G, Z)$  we denote by  $\Delta(\alpha)$ .

**Proposition 3.5.** *Let  $U$  be a split unipotent  $M$ -group, and let  $Z$  be a central, closed and smooth  $k$ -subgroup of  $U$  that is  $M$ -invariant. Write  $i : Z \rightarrow U$  and  $\pi : U \rightarrow U/Z$  for the inclusion and quotient mappings, respectively.*

(a) *The sequence of pointed sets*

$$H^1(M, Z) \xrightarrow{i_*} H^1_{\text{coc}}(M, U) \xrightarrow{\pi_*} H^1_{\text{coc}}(M, U/Z) \xrightarrow{\Delta} H^2(M, Z)$$

*is exact.*

(b) *If  $(U/Z)^M = 1$  then  $i_*$  is injective.*

*Sketch.* (a) The proof of the corresponding statement for cohomology of pro-finite groups given in [Serre 2002, I.§5.7] may be applied here mutatis mutandum. The main required adaptation is the definition (given above) of the mapping  $\Delta$  (which required the existence of a regular section  $U/Z \rightarrow U$ ).

(b) Suppose that  $f_1, f_2 : M \rightarrow Z$  are 1-cocycles and that  $i_*([f_1]) = i_*([f_2])$ . Thus  $f_1, f_2$  are cohomologous in  $Z^1_{\text{coc}}(M, U)$ , so there is  $u \in U(k)$  such that

$$f_1(x) = u^{-1} \cdot f_2(x) \cdot xu$$

for every commutative  $k$ -algebra  $\Lambda$  and every  $x \in M(\Lambda)$ . Passing to the quotient  $U/Z$  we see that  $1 = u^{-1}xu$ , so that the class of  $u$  lies in  $(U/Z)^M(\Lambda)$ .  $\square$

**Remark 3.6.** Assume that  $\ell$  is a finite, Galois extension of  $k$  with Galois group  $\Gamma = \text{Gal}(\ell/k)$ . Then  $\Gamma$  acts on the Galois cohomology  $H^1(M_\ell, A_\ell)$  through its action on regular mappings  $M_\ell \rightarrow A_\ell$ .

If  $A$  is a vector group on which  $M$  acts linearly, then  $H^1(M_\ell, A_\ell)$  may be identified with  $H^1(M, A) \otimes_k \ell$ . In that case  $H^1(M, A)$  may be identified with  $H^1(M_\ell, A_\ell)^\Gamma$ .

This observation prompts several questions. Suppose  $U$  is a split unipotent  $M$ -group and that  $U$  has a central linear filtration for the action of  $M$ .

- (a) Under what conditions is it true that  $H^1_{\text{coc}}(M, U) = H^1_{\text{coc}}(M_\ell, U_\ell)^\Gamma$ ?
- (b) Under what conditions is it true that the condition  $H^1_{\text{coc}}(M, U) = 1$  is equivalent to the condition  $H^1_{\text{coc}}(M_\ell, U_\ell) = 1$ ?

### 4. Descent of Levi factors, I

*Proof of Theorem 1.6.* Recall that  $G$  is a linear algebraic group satisfying condition **(R)**,  $U$  is the unipotent radical and  $M = G/U$  is the reductive quotient. Moreover,

$\ell$  is a finite, separable field extension of  $k$ . We must show that under assumptions (a), (b), and (c), the group  $G$  has a Levi decomposition.

First, note that the assumptions are unaffected if we pass to a finite separable extension of  $\ell$ . Thus, we may and will suppose that  $\ell$  is Galois over  $k$ ; write  $\Gamma = \text{Gal}(\ell/k)$  for the Galois group.

According to (a),  $G_\ell$  has a Levi decomposition. Thus we may choose a homomorphism  $s : M_\ell \rightarrow G_\ell$  which is a section to  $\pi$ . According to (c), we have  $H_{\text{coc}}^1(M_\ell, U_\ell) = 1$ . Together with [Proposition 3.2](#), this shows that  $\text{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$  contains a single element. In particular, every homomorphism  $u : M_\ell \rightarrow G_\ell$  which is a section to  $\pi$  differs from  $s$  by conjugation with an element of  $U(\ell)$ .

There is a natural action of  $\Gamma$  on homomorphisms  $M_\ell \rightarrow G_\ell$  which determines in turn an action of  $\Gamma$  on  $\text{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$ . For each  $\gamma \in \Gamma$ , we thus find an element  $u_\gamma \in U(\ell)$  such that  ${}^\gamma s = u_\gamma^{-1} \cdot s \cdot u_\gamma$ .

We now contend that ():  $u_\gamma$  is a 1-cocycle on  $\Gamma$  with values in  $U(\ell)$ . Well, for  $\gamma, \tau \in \Gamma$  we see that

$$(4-1) \quad {}^{\gamma\tau} s = u_{\gamma\tau}^{-1} \cdot s \cdot u_{\gamma\tau},$$

while on the other hand

$$(4-2) \quad {}^{\gamma\tau} s = {}^\gamma(u_\tau^{-1} \cdot s \cdot u_\tau) = {}^\gamma u_\tau^{-1} \cdot {}^\gamma s \cdot {}^\gamma u_\tau = {}^\gamma u_\tau^{-1} \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot {}^\gamma u_\tau.$$

Now, assumption (b) guarantees that  $U_\ell^{M_\ell}$  is trivial, and it follows that the stabilizer in  $U_\ell$  of the section  $s$  is trivial. Thus together (4-1) and (4-2) imply that

$$u_{\gamma\tau} = u_\gamma \cdot {}^\gamma u_\tau.$$

This confirms (). Since  $U$  is a split unipotent  $k$ -group,  $H^1(k, U) = 1$ ; see [Proposition 2.14](#). Thus there is  $u \in U(\ell)$  such that

$$(4-3) \quad u_\gamma = u^{-1} \cdot {}^\gamma u$$

for each  $\gamma \in \Gamma$ ; that is,  ${}^\gamma u = uu_\gamma$ .

Now set  $s_0 = u \cdot s \cdot u^{-1} \in \text{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$ . We claim that  $s_0$  is a  $k$ -homomorphism. It is enough to argue that  $s$  is fixed by the Galois group  $\Gamma$ . For  $\gamma \in \Gamma$  we note that

$${}^\gamma s_0 = {}^\gamma u \cdot s \cdot u^{-1} = {}^\gamma u \cdot {}^\gamma s \cdot {}^\gamma u^{-1} = u \cdot u_\gamma \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot u_\gamma^{-1} \cdot u = usu^{-1} = s_0.$$

Thus  $s_0 : M \rightarrow G$  is a  $k$ -morphism which is a section to  $\pi$ ; this shows that  $G$  has a Levi factor as required. □

In the remainder of this section, we are going to formulate a variant of [Theorem 1.6](#) using a filtration of  $U$ . We are going to *assume that  $U$  has a central linear filtration*

$$1 = Z_0 \subset Z_1 \subset \cdots \subset Z_m = U$$

for the action of  $G$ ; see [Definition 2.11](#). Note that such a filtration always exists in the case  $G$  is connected; see [Theorem 2.12](#).

**Proposition 4.1.** *For each  $n \geq 0$  the homomorphism of  $k$ -group functors*

$$\phi_0 : M \rightarrow \text{Out}(U)$$

*of Proposition 2.8 determines an action of  $M$  on the quotient  $Z_{n+1}/Z_n$ .*

*Proof.* Since  $Z_{n+1}/Z_n$  is abelian,  $\text{Out}(Z_{n+1}/Z_n) = \text{Aut}(Z_{n+1}/Z_n)$ . For each natural number  $n$ ,  $\phi_0$  determines by restriction and passage to the quotient a homomorphism of  $k$ -group functors

$$\phi_{0|Z_{n+1}} : M \rightarrow \text{Out}(Z_{n+1}/Z_n) = \text{Aut}(Z_{n+1}/Z_n),$$

i.e., an action of  $M$  on  $Z_{n+1}/Z_n$ . □

**Lemma 4.2.** *Suppose that  $H^1(M, Z_{i+1}/Z_i) = 0$  for each  $i = 0, \dots, m - 1$ . Then*

$$H^1_{\text{coc}}(M, U) = 1 \quad \text{and} \quad H^1_{\text{coc}}(M_\ell, U_\ell) = 1.$$

*Proof.* First observe that for a linear representation  $V$  of  $G$ ,  $H^1(G, V) = 0$  if and only if  $H^1(G_\ell, V_\ell) = 0$ . Now the result follows from Proposition 3.5. □

**Remark 4.3.** Viewing a finite-dimensional linear representation  $V$  of  $M$  as an algebraic group, the scheme-theoretic fixed-point subgroup  $V^M$  coincides with the vector group given by the  $M$ -fixed points on the linear representation  $V$ . In particular, if  $V$  is an irreducible representation of  $M$ , the group scheme  $V^M$  is equal to  $\{0\}$ .

**Lemma 4.4.** *Suppose that  $(Z_{i+1}/Z_i)^M = \{1\}$  for each  $i = 0, \dots, m - 1$ . Then  $U^M = \{1\}$  is the trivial group scheme.*

*Proof.* We proceed by induction on  $m$ , the length of the central linear filtration of  $U$ . If  $m = 0$ ,  $U = 1$  and the result is immediate.

Now suppose that  $m > 0$  and that the result is known for connected and split unipotent  $M$ -groups having a central linear filtration of length  $< m$ . Thus by induction we know  $(U/Z_1)^M = \{1\}$ . Thus  $U^M$  is contained in the kernel of the quotient mapping  $U \rightarrow U/Z_1$ , that is,  $U^M$  is contained in  $Z_1$ . Since  $(Z_1)^M$  is the trivial group scheme, the proof is complete. □

We now obtain a corollary to Theorem 1.6:

**Corollary 4.5.** *Assume that  $U$  has a central linear filtration for the action of  $G$  and suppose*

- (a)  $G_\ell$  has a Levi decomposition (over  $\ell$ ),
- (bb) the group scheme  $(Z_{i+1}/Z_i)^M$  is trivial for  $i = 0, \dots, m - 1$ , and
- (cc)  $H^1(M, Z_{i+1}/Z_i) = 0$  for  $i = 0, \dots, m - 1$ .

*Then  $G$  has a Levi decomposition.*

*Proof.* Note that according to Lemma 4.4, condition (bb) implies hypothesis (b) of Theorem 1.6. Similarly, according to Lemma 4.2 (cc) implies hypothesis (c) of Theorem 1.6. Thus the result follows from Theorem 1.6. □

### 5. Automorphisms of extensions

Let  $A$  and  $M$  be linear algebraic groups over  $k$ , and let  $E$  and  $E'$  be extensions of  $M$  by  $A$  as in [Definition 2.5](#).

**Definition 5.1.** A morphism of extensions  $\phi : E \rightarrow E'$  is a morphism of algebraic groups for which the following diagram is commutative:

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{\pi} & M & \longrightarrow & 1 \\
 & & \parallel & & \downarrow & & \parallel & & \\
 1 & \longrightarrow & A & \xrightarrow{i'} & E' & \xrightarrow{\pi} & M & \longrightarrow & 1
 \end{array}$$

**Remark 5.2.** If  $\phi : E \rightarrow E'$  is a morphism of extensions, then  $\phi$  is necessarily an isomorphism of algebraic groups  $E \xrightarrow{\sim} E'$ .

Write  $\text{Autext}(E)$  for the group of automorphisms of  $E$ . Let  $Z$  be the (schematic) center of  $A$ . Since  $Z$  is characteristic in  $A$ ,  $E$  acts on  $Z$  by conjugation. Since  $A$  acts trivially on  $Z$ , the action of  $E$  on  $Z$  factors through  $M \simeq E/A$ .

Write  $Z^1_{\text{coc}}(M, Z)$  for the Hochschild 1-cocycles as in [Section 3](#). Since  $Z$  is commutative,  $Z^1_{\text{coc}}(M, Z)$  is a group. The following result is a consequence of [\[Florence and Lucchini Arteché 2019, Proposition 2.3\]](#).

**Proposition 5.3.** *There is a canonical isomorphism of groups  $Z^1_{\text{coc}}(M, Z) \xrightarrow{\sim} \text{Autext}(E)$ .*

Now suppose that  $\ell$  is a finite, separable field extension of  $k$ .

**Theorem 5.4.** *Assume that the center  $Z$  of  $A$  is a vector group and that the action of  $M$  on  $Z$  is linear. If the extensions  $E_\ell$  and  $E'_\ell$  of  $M_\ell$  by  $A_\ell$  are isomorphic, then  $E$  and  $E'$  are isomorphic extensions of  $M$  by  $A$ .*

*Proof.* Write  $k_{\text{sep}}$  for a separable closure of  $k$  containing  $\ell$  and write  $\mathcal{E}$  for the set of isomorphism classes of extensions of  $M$  by  $A$  over  $k$  which after scalar extension to  $k_{\text{sep}}$  become isomorphic to the extension  $E_{k_{\text{sep}}}$  of  $M_{k_{\text{sep}}}$  by  $A_{k_{\text{sep}}}$ .

As in [\[Serre 2002, III.§1\]](#), one knows that there is a bijection

$$(5-1) \quad \mathcal{E} \xrightarrow{\sim} H^1(k, \text{Autext}(E)) := H^1(\text{Gal}(k_{\text{sep}}/k), \text{Autext}(E_{k_{\text{sep}}}))$$

Thus, the theorem will follow if we argue that the Galois cohomology set  $H^1(k, \text{Autext}(E))$  is trivial — i.e., contains a unique element.

By assumption,  $Z$  is a vector group with linear action of  $M$ , so that  $Z^1(M, Z)$  is a  $k$ -vector space (possibly of infinite dimension). Now [Proposition 5.3](#) shows that  $\text{Autext}(E) = Z^1(M, Z)$  is a  $k$ -vector space; it follows from “additive Hilbert 90” that

$$H^1(k, \text{Aut}(E)) \simeq H^1(k, Z^1(A, Z))$$

is trivial; see, for example, [\[McNinch 2013, \(4.1.2\)\]](#). □

### 6. Automorphisms and cohomology

Let  $A$  and  $M$  be a linear algebraic  $k$ -groups, and suppose that  $A$  is a  $M$ -group via the mapping

$$\phi : M \rightarrow \text{Aut}(A).$$

Let  $Z$  denote the center of  $A$  as a group scheme. Then  $\text{Inn}(A) \simeq A/Z$  is also an  $M$ -group via  $\phi$ ; for  $h \in \text{Inn}(A)(\Lambda)$  and  $g \in M(\Lambda)$ , we have  ${}^g h = \phi(g)h\phi(g)^{-1}$ .

Denote by  $\phi_0 = \Psi \circ \phi$  the homomorphism of group functors

$$M \xrightarrow{\phi} \text{Aut}(A) \xrightarrow{\Psi} \text{Out}(A),$$

where  $\Psi : \text{Aut}(A) \rightarrow \text{Out}(A)$  is the natural map of (2-1).

Consider those homomorphisms of  $k$ -group functors  $\theta : M \rightarrow \text{Aut}(A)$  satisfying

$$(*) \quad \Psi \circ \theta_1 = \phi_0.$$

We say that two such homomorphisms  $\theta_1$  and  $\theta_2$  are equivalent if they are conjugate by  $\text{Inn}(A)(k)$ ; that is, if there is  $h \in \text{Inn}(A)(k)$  for which

$$\theta_1(g) = h^{-1}\theta_2(g)h$$

for each commutative  $k$ -algebra  $\Lambda$  and each  $g \in M(\Lambda)$ . We write  $\text{Lift}(\phi_0)$  for the quotient of the set of all homomorphisms  $M \rightarrow \text{Aut}(A)$  satisfying  $(*)$  by the equivalence relation just described.

**Proposition 6.1.** *Write  $\mu : \text{Aut}(A) \times \text{Aut}(A) \rightarrow \text{Aut}(A)$  for the group operation. For  $f \in Z_{\text{coc}}^1(M, A)$ , define  $\Phi_f : M \rightarrow \text{Aut}(A)$  by the rule*

$$\Phi_f = \mu \circ (f, \phi) : M \rightarrow \text{Aut}(A) \times \text{Aut}(A) \rightarrow \text{Aut}(A).$$

*Then the assignment  $f \mapsto \Phi_f$  determines a bijection*

$$\Phi : H_{\text{coc}}^1(G, \text{Inn}(A)) \rightarrow \text{Lift}(\phi_0).$$

*Proof.* For any 1-cocycle  $f \in Z_{\text{coc}}^1(G, M)$ , one checks that the mapping  $\Phi_f : G \rightarrow \text{Aut}(A)$  is homomorphism of  $k$ -group functors contained in  $\text{Lift}(f\phi)$ .

We now claim for  $f_1, f_2 \in Z_{\text{coc}}^1(M, A)$  that  $f_1$  and  $f_2$  are cohomologous if and only if  $\Phi_{f_1}$  and  $\Phi_{f_2}$  are equivalent.

( $\Rightarrow$ ): By assumption there is  $h \in \text{Inn}(U)(k)$  such that for each commutative  $k$ -algebra  $\Lambda$  and each  $g \in M(\Lambda)$ , the following identity holds:

$$f_1(g) = h^{-1}f_2(g)h.$$

Now observe that

$$\begin{aligned} \Phi_{f_1}(g) &= f_1(g)\phi(g) = h^{-1}f_2(g)^g h \cdot \phi(g) \\ &= h^{-1}f_2(g)\phi(g)h\phi(g)^{-1}\phi(g) = h^{-1}f_2(g)\phi(g)h = h^{-1}\Phi_{f_2}(g)h, \end{aligned}$$

so that  $\Phi_{f_1}$  and  $\Phi_{f_2}$  are equivalent.

( $\Leftarrow$ ): By assumption there is  $h \in \text{Inn}(A)(k)$  for which

$$\Phi_{f_1} = h^{-1}\Phi_{f_2}h.$$

Then for each commutative  $k$ -algebra  $\Lambda$  and each  $g \in M(\Lambda)$  we have

$$\begin{aligned} f_1(g) &= \Phi_{f_1}(g) \cdot \phi(g)^{-1} = h^{-1}\Phi_{f_2}(g)h \cdot \phi(g)^{-1} \\ &= h^{-1}\Phi_{f_2}(g)\phi(g)^{-1}g h = h^{-1}f_2(g)^g h, \end{aligned}$$

so that  $f_1$  and  $f_2$  are cohomologous.

It now follows that  $f \mapsto \Phi_f$  determines a well-defined injective mapping

$$\Phi : H_{\text{coc}}^1(M, \text{Inn}(A)) \rightarrow \text{Lift}(\phi_0).$$

To see that  $\Phi$  is surjective, suppose  $\theta : M \rightarrow \text{Aut}(A)$  represents a class in  $\text{Lift}(\phi_0)$ . For each commutative  $k$ -algebra  $\Lambda$  and each  $g \in M(\Lambda)$ , we have  $\theta(g)\phi(g)^{-1} \in \text{Inn}(A)(\Lambda)$ . Thus we have a morphism of  $k$ -functors  $f : M \rightarrow \text{Inn}(A)$  given by

$$f(g) = \theta(g)\phi(g)^{-1}.$$

By the Yoneda lemma, the assignment  $f$  is a morphism of varieties, and a calculation confirms that  $f$  is a 1-cocycle for the action of  $M$  on  $\text{Inn}(A)$  determined by  $\phi$ . Then  $[\theta] = [\Phi_f] = \Phi([f])$  which proves that  $\Phi$  is surjective.  $\square$

### 7. Descent of Levi factors, II

We are going to prove [Theorem 1.7](#). We first prove the following:

**Lemma 7.1.** *Let  $M, A$  be linear algebraic groups, and suppose that  $A$  is an  $M$ -group via the homomorphism  $\phi : M \rightarrow \text{Aut}(A)$  of  $k$ -group functors. Let  $x \in A(k)$  and consider the mapping  $\phi_1 : M \rightarrow \text{Aut}(A)$  given for each commutative  $k$ -algebra  $\Lambda$  and each  $g \in M(\Lambda)$  by the rule  $\phi_1(g) = \text{Inn}(x)\phi(g)\text{Inn}(x)^{-1}$ . Then there is a  $k$ -isomorphism of extensions of  $M$  by  $A$ :*

$$A \rtimes_{\phi} M \simeq A \rtimes_{\phi_1} M.$$

*Proof.* Write  $G = A \rtimes_{\phi} M$  for the semidirect product constructed using the action defined by  $\phi$ . Now, the mapping  $\phi : M \rightarrow \text{Aut}(A)$  may be identified with the composite

$$M \xrightarrow{m \mapsto (1,m)} G = A \rtimes_{\phi} M \xrightarrow{\text{Inn}} \text{Aut}(A)$$

and  $\phi_1 : M \rightarrow \text{Aut}(A)$  identifies with the composite

$$M \xrightarrow{m \mapsto (x,1)(1,m)(x,1)^{-1}} A \rtimes_{\phi} M \xrightarrow{\text{Inn}} \text{Aut}(A).$$

Write  $s_1 : M \rightarrow G = A \rtimes_{\phi} M$  for the section given by the rule

$$s_1(m) = (x, 1)(1, m)(x, 1)^{-1}.$$

It now follows from [Proposition 2.7](#) that the product mapping

$$((a, m) \mapsto a \cdot s_1(m)) : A \times M \rightarrow G$$

determines an isomorphism  $A \rtimes_{\phi_1} M \xrightarrow{\sim} G = A \rtimes_{\phi} M$  of extensions, as required.  $\square$

We now prove [Theorem 1.7](#) from [Section 1](#):

*Proof.* By assumption (a),  $G_{\ell}$  has a Levi factor  $M_{\ell}$ ; this choice determines a homomorphism

$$\phi : M_{\ell} \rightarrow \text{Aut}(U_{\ell})$$

such that  $\phi_{0,\ell} = \Psi \circ \phi$  where  $\phi_0 : M \rightarrow \text{Out}(U)$  is the mapping determined by [Proposition 2.8](#) and  $\Psi : \text{Aut}(U) \rightarrow \text{Out}(U)$  is the natural mapping of (2-1).

There is a natural action of the Galois group  $\Gamma$  on  $\text{Aut}(U_{\ell})$  and on  $\text{Out}(U_{\ell})$  for which  $\Psi$  is equivariant. For any  $\gamma \in \Gamma$  it follows that

$$\Psi \circ {}^{\gamma}\phi = \phi_0,$$

i.e., in the notation of [Proposition 6.1](#),  ${}^{\gamma}\phi$  determines a class in  $\text{Lift}(\phi_{0,\ell})$ .

According to [Proposition 6.1](#) there is a bijection  $H_{\text{coc}}^1(M_{\ell}, \text{Inn}(U_{\ell})) \xrightarrow{\sim} \text{Lift}(\phi_0)$ . Since  $H_{\text{coc}}^1(M_{\ell}, \text{Inn}(U_{\ell})) = 1$  it follows that classes of the automorphisms  ${}^{\gamma}\phi$  in  $\text{Lift}(\phi_0)$  all coincide; that is, all  ${}^{\gamma}\phi$  are equivalent.

By the definition of the equivalence relation defining  $\text{Lift}(\phi_0)$ , we find for each  $\gamma \in \Gamma$  an element  $h_{\gamma} \in \text{Inn}(U)(\ell)$  such that

$${}^{\gamma}\phi = h_{\gamma}^{-1} \cdot \phi \cdot h_{\gamma}.$$

If  $\gamma, \tau \in \Gamma$  we see that

$$(7-1) \quad {}^{\gamma\tau}\phi = h_{\gamma\tau}^{-1} \cdot \phi \cdot h_{\gamma\tau},$$

while on the other hand

$$(7-2) \quad {}^{\gamma}(\tau\phi) = {}^{\gamma}(h_{\tau}^{-1} \cdot \phi \cdot h_{\tau}) = {}^{\gamma}h_{\tau}^{-1} \cdot {}^{\gamma}\phi \cdot {}^{\gamma}h_{\tau} = {}^{\gamma}h_{\tau}^{-1} \cdot h_{\gamma}^{-1} \phi \cdot h_{\gamma} \cdot {}^{\gamma}h_{\tau}.$$

By assumption (b) we know that the stabilizer in  $\text{Inn}(U)$  of the automorphism  $\phi$  is trivial. Thus taken together (7-1) and (7-2) imply that

$$h_{\gamma\tau} = h_{\gamma} {}^{\gamma}h_{\tau};$$

i.e.,  $h_\gamma$  is a 1-cocycle on  $\Gamma$  with values in  $\text{Inn}(U)(\ell)$ . Since  $U$  is connected and split unipotent, so is  $\text{Inn}(U)$ ; see [Proposition 2.3](#). Thus  $H_{\text{coc}}^1(M_\ell, \text{Inn}(U_\ell)) = 1$  by [Proposition 2.14](#).

It follows that the cocycle  $h_\gamma$  is trivial. Thus there is  $h \in \text{Inn}(U)(\ell)$  such that for each  $\gamma \in \Gamma$  we have

$$h_\gamma = h^{-1} \cdot {}^\gamma h.$$

We now claim that the mapping  $\phi_1 : M_\ell \rightarrow \text{Aut}(U_\ell)$  defined by

$$\phi_1 = h \cdot \phi \cdot h^{-1}$$

is  $\Gamma$ -stable. For  $\gamma \in \Gamma$  we have

$${}^\gamma \phi_1 = {}^\gamma (h \cdot \phi \cdot h^{-1}) = {}^\gamma h \cdot {}^\gamma \phi \cdot {}^\gamma h^{-1} = hh_\gamma \cdot h_\gamma^{-1} \phi h_\gamma \cdot h_\gamma^{-1} h^{-1} = \phi_1.$$

Thus  $\phi_1$  is  $\Gamma$ -stable and hence defines a morphism  $\phi_1 : M \rightarrow \text{Aut}(U)$  of  $k$ -group functors which we may use to define a semidirect product  $G_1 = U \rtimes_{\phi_1} M$  over  $k$ .

The center  $Z$  of  $U$  is a connected and split unipotent group; thus  $H^1(\ell, Z) = 1$ . It follows that the mapping  $U(\ell) \rightarrow \text{Inn}(U)(\ell)$  is surjective, so we may choose an element  $u \in U(\ell)$  for which  $\text{Inn}(u) = h \in \text{Inn}(U)(\ell)$ .

Thus we have

$$\phi_1 = \text{Inn}(u) \cdot \phi \cdot \text{Inn}(u)^{-1}.$$

It now follows from [Lemma 7.1](#) that there is an isomorphism of extensions

$$G_\ell = U_\ell \rtimes_\phi M_\ell \simeq G_{1,\ell} = U_\ell \rtimes_{\phi_1} M_\ell$$

of  $M_\ell$  by  $U_\ell$ .

According to [Theorem 5.4](#), assumption (c) implies that the extension  $G_\ell$  has a unique  $k$ -form. Since  $G$  and  $G_1$  are both  $k$ -forms of this extension, it follows that  $G \simeq G_1$  are  $k$ -isomorphic extensions and in particular are  $k$ -isomorphic algebraic groups; since  $G_1$  has a Levi factor over  $k$ , we conclude that  $G$  has a Levi factor over  $k$  as well. □

### 8. An example

In [[McNinch 2013](#), §5] we gave an example of an extension

$$1 \rightarrow W \rightarrow E \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 1$$

with  $E$  commutative and  $W$  a connected, commutative unipotent group of exponent  $p^2$ . The group  $E$  was constructed by *twisting*, and it provided a negative answer to the question ( $\diamond$ ) from [Section 1](#). Namely, for a suitable finite Galois extension  $\ell$  of  $k$  the group  $E_\ell$  has a Levi factor, but  $E$  had no Levi factor.

We conclude with another example of a linear algebraic group over  $k$  which provides a negative answer to the question ( $\diamond$ ).

The example below gives a noncommutative extension of a finite abelian  $p$ -group by a connected, noncommutative unipotent group; in this case, the construction of the extension is perhaps slightly more straightforward.

Suppose that the characteristic of  $k$  is  $p > 2$ . Consider the additive polynomial  $X^p - X \in k[X]$  defining the Artin–Schreier mapping  $\mathcal{P}$ : for any commutative  $k$ -algebra  $\Lambda$ , this mapping  $\mathcal{P} : \Lambda \rightarrow \Lambda$  is given by the rule  $x \mapsto x^p - x$ .

Recall that if  $s \in k$  is not in the image of  $\mathcal{P} : k \rightarrow k$  then the polynomial  $F(X) = X^p - X - s \in k[X]$  is irreducible. If  $\alpha$  is a root of  $F(X)$  in an extension field of  $k$  then  $\ell = k(\alpha)$  is a Galois extension of  $k$  with  $\text{Gal}(\ell/k) \simeq \mathbb{Z}/p\mathbb{Z}$ .

Let  $V$  be a vector space of dimension 2 over  $k$  with a basis  $e, f$ , and write  $\beta : V \times V \rightarrow k$  for the unique nondegenerate symplectic form satisfying  $\beta(e, f) = 1 = -\beta(f, e)$ . Viewing  $\mathcal{P} \circ \beta$  as a factor system, we define a unipotent group  $H$  as an extension of  $V$  by  $\mathbf{G}_a$ ; see [Serre 1988, VII. §1.4]. Explicitly, for a commutative  $k$ -algebra  $\Lambda$  we have

$$H(\Lambda) = \Lambda \times V \otimes_k \Lambda$$

with operation

$$\begin{aligned} (t, v) \cdot (s, w) &= (t + s + \mathcal{P}(\beta(v, w)), v + w) \\ &= (t + s + \beta(v, w)^p - \beta(v, w), v + w) \end{aligned}$$

for  $v, w \in V \otimes \Lambda$  and  $s, t \in \Lambda$ .

Thus  $H$  is the nonabelian central extension

$$(8-1) \quad 0 \rightarrow \mathbf{G}_a \xrightarrow{i} H \xrightarrow{(v,t) \mapsto v} V \rightarrow 0.$$

Write  $Z$  for the center of  $H$ ; then  $Z \simeq \mathbf{G}_a$  is the image of the mapping  $i$  of (8-1).

Fix  $t \in k$  and let  $V_{0,t} = \langle te, f \rangle \subset V$ , so that  $V_{0,t} \simeq (\mathbb{Z}/p\mathbb{Z})^2$ . Let  $\mu_t$  be the central extension of  $V_{0,t}$  by  $Z \simeq \mathbf{G}_a$  defined by  $\beta$  (not by  $\mathcal{P} \circ \beta$ ). Thus there is an exact sequence

$$0 \rightarrow \mathbf{G}_a \rightarrow \mu_t \rightarrow V_{0,t} = (\mathbb{Z}/p\mathbb{Z})^2 \rightarrow 0$$

and the group operation is given by  $(a, v) \cdot (b, w) = (a + b + \beta(v, w), v + w)$ , for  $v, w \in V_{0,t} \otimes \Lambda = V_{0,t}$  and  $a, b \in \Lambda$ .

Write  $E$  for the fiber product  $E = H \times_{\mathbf{G}_a} \mu_t$ ; thus  $E$  is an extension of  $V_{0,t} \simeq (\mathbb{Z}/p\mathbb{Z})^2$  by  $H$ . By the definition of the fiber product, there is a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{G}_a & \longrightarrow & \mu_t & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & H \times_{\mathbf{G}_a} \mu_t & \xrightarrow{\pi} & (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow 0 \end{array}$$

**Proposition 8.1.** *If  $X^p - X + t$  has no root in  $k$ , then the group  $E = H \times_{\mathbf{G}_a} \mu_t$  has no Levi factor over  $k$ . If  $\alpha$  is a root  $X^p - X + t$  and  $\ell = k(\alpha)$  then  $E_\ell$  has a Levi factor.*

*Sketch.* We may represent elements of  $E(k)$  as tuples  $(a, v, w)$  where  $v \in V_{0,t}$ ,  $w \in V$  and  $a \in k$ . We have

$$(a, v, w) \cdot (a', v', w') = (a + a' + \beta(v, v') + \mathcal{P}\beta(w, w)', v + v', w + w').$$

Now, any elements  $\tilde{e}, \tilde{f}$  of  $E(k)$  mapping to  $te, f \in V_{0,t}$  via  $\pi$  must have the form  $\tilde{e} = (a, te, v)$  for some  $v \in V$  and  $a \in k$  and  $\tilde{f} = (b, f, w)$  for some  $w \in V$  and  $b \in k$ .

We see that

$$\tilde{e} \cdot \tilde{f} = (a, te, v) \cdot (b, f, w) = (a + b + t + \mathcal{P}\beta(v, w), te + f, v + w)$$

while

$$\tilde{f} \cdot \tilde{e} = (b, f, w) \cdot (a, te, v) = (a + b + -t - \mathcal{P}\beta(v, w), te + f, v + w)$$

Since the characteristic of  $k$  is not 2,  $\tilde{e} \cdot \tilde{f} = \tilde{f} \cdot \tilde{e}$  if and only if

$$0 = \mathcal{P}\beta(v, w) + t = \beta(v, w)^p - \beta(v, w) + t.$$

If  $X^p - X + t$  has no root in  $k$ , it follows that the group  $\langle \tilde{e}, \tilde{f} \rangle$  is nonabelian for any choice of  $\tilde{e}, \tilde{f}$ . This shows that  $E$  has no Levi factor.

On the other hand,  $E_\ell$  always has a Levi factor since we may take  $\tilde{e} = (0, te, \alpha e)$  and  $\tilde{f} = (0, f, f)$ ; then  $\langle \tilde{e}, \tilde{f} \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^2$ , so that  $\langle \tilde{e}, \tilde{f} \rangle$  provides a Levi factor.  $\square$

**Remark 8.2.** The group  $E$  of Proposition 8.1 fails to satisfy hypotheses (b) and (c) of Theorem 1.6: Let  $M = E/H \simeq (\mathbb{Z}/p\mathbb{Z})^2$  be the reductive quotient of  $E$ . Then:

- $M_\ell$  acts trivially on  $H_\ell$ . Thus,  $H_\ell^{M_\ell} = H_\ell \neq \{1\}$ , so that condition (b) fails.
- The cohomology group  $H_{\text{coc}}^1(\mathbb{Z}/p\mathbb{Z}, \mathbf{G}_a)$  is nontrivial. Using a Künneth formula, we see that  $H_{\text{coc}}^1(M, \mathbf{G}_a) \neq 1$ . Now use Proposition 3.5 to conclude that  $H_{\text{coc}}^1(M, H) \neq 1$  and  $H_{\text{coc}}^1(M_\ell, H_\ell) \neq 1$ . Thus condition (c) fails.

### Acknowledgments

I would like to thank Brian Conrad, Skip Garibaldi, and an anonymous referee for making a number of useful suggestions and observations on the manuscript.

### References

[Borel 1991] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, 1991. [MR](#) [Zbl](#)

[Conrad et al. 2015] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, 2nd ed., New Mathematical Monographs **26**, Cambridge University Press, 2015. [MR](#) [Zbl](#)

- [Demarche 2015] C. Demarche, “Cohomologie de Hochschild non Abélienne et extensions de Faisceaux en groupes”, pp. 255–292 in *Autours des schémas en groupes, II*, edited by B. Edixhoven et al., Panor. Synthèses **46**, Soc. Math. France, Paris, 2015. [MR](#) [Zbl](#)
- [Demazure and Gabriel 1970] M. Demazure and P. Gabriel, *Groupes algébriques, I: Géométrie algébrique, généralités, groupes commutatifs*, Masson, Paris, 1970. [MR](#) [Zbl](#)
- [Florence and Lucchini Arteché 2019] M. Florence and G. Lucchini Arteché, “On extensions of algebraic groups”, *Enseign. Math.* **65**:3-4 (2019), 441–455. [MR](#) [Zbl](#)
- [Jantzen 2003] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs **107**, American Mathematical Society, Providence, RI, 2003. [MR](#) [Zbl](#)
- [McNinch 2004] G. J. McNinch, “Nilpotent orbits over ground fields of good characteristic”, *Math. Ann.* **329**:1 (2004), 49–85. [MR](#) [Zbl](#)
- [McNinch 2010] G. J. McNinch, “Levi decompositions of a linear algebraic group”, *Transformation Groups* **15**:4 (2010), 937–964. [Zbl](#)
- [McNinch 2013] G. McNinch, “On the descent of Levi factors”, *Arch. Math.* **100**:1 (2013), 7–24. [MR](#) [Zbl](#)
- [McNinch 2014] G. McNinch, “Linearity for actions on vector groups”, *Journal of Algebra* **397** (2014), 666–688. [Zbl](#)
- [Serre 1988] J.-P. Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics **117**, Springer, 1988. [MR](#) [Zbl](#)
- [Serre 2002] J.-P. Serre, *Galois cohomology*, Springer, 2002. [MR](#) [Zbl](#)
- [SGA 3<sub>III</sub> 2011] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome III: Structure des schémas en groupes réductifs, Exposés XIX–XXVI* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), revised ed., Doc. Math **8**, Soc. Math. France, Paris, 2011. [MR](#) [Zbl](#)
- [Springer 2009] T. A. Springer, *Linear algebraic groups*, 2nd ed., Birkhäuser, Boston, 2009. [MR](#) [Zbl](#)
- [Stewart 2013] D. I. Stewart, “On unipotent algebraic  $G$ -groups and 1-cohomology”, *Trans. Amer. Math. Soc.* **365**:12 (2013), 6343–6365. [MR](#) [Zbl](#)

Received July 1, 2024. Revised August 25, 2024.

GEORGE J. MCNINCH  
 DEPARTMENT OF MATHEMATICS  
 TUFTS UNIVERSITY  
 MEDFORD, MA  
 UNITED STATES  
[george.mcninch@tufts.edu](mailto:george.mcninch@tufts.edu)

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

[msp.org/pjm](http://msp.org/pjm)

## EDITORS

Don Blasius (Managing Editor)  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[blasius@math.ucla.edu](mailto:blasius@math.ucla.edu)

Matthias Aschenbrenner  
Fakultät für Mathematik  
Universität Wien  
Vienna, Austria  
[matthias.aschenbrenner@univie.ac.at](mailto:matthias.aschenbrenner@univie.ac.at)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Atsushi Ichino  
Department of Mathematics  
Kyoto University  
Riverside, CA 92521-0135  
[atsushi.ichino@gmail.com](mailto:atsushi.ichino@gmail.com)

Robert Lipshitz  
Department of Mathematics  
University of Oregon  
Eugene, OR 97403  
[lipshitz@uoregon.edu](mailto:lipshitz@uoregon.edu)

Kefeng Liu  
School of Sciences  
Chongqing University of Technology  
Chongqing 400054, China  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Sucharit Sarkar  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[sucharit@math.ucla.edu](mailto:sucharit@math.ucla.edu)

Dimitri Shlyakhtenko  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[shlyakht@ipam.ucla.edu](mailto:shlyakht@ipam.ucla.edu)

Ruixiang Zhang  
Department of Mathematics  
University of California  
Berkeley, CA 94720-3840  
[ruixiang@berkeley.edu](mailto:ruixiang@berkeley.edu)

## PRODUCTION

Silvio Levy, Scientific Editor, [production@msp.org](mailto:production@msp.org)

---

See inside back cover or [msp.org/pjm](http://msp.org/pjm) for submission instructions.

---

The subscription price for 2025 is US \$677/year for the electronic version, and \$917/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).


---

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

---

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2025 Mathematical Sciences Publishers

# PACIFIC JOURNAL OF MATHEMATICS

Volume 336    No. 1-2    May 2025

---

## In memoriam Gary Seitz

Gary Seitz (1943–2023): In memoriam	1
MARTIN W. LIEBECK, GERHARD RÖHRLE and DONNA TESTERMAN	
Intrinsic components in involution centralizers of fusion systems	7
MICHAEL ASCHBACHER	
On good $A_1$ subgroups, Springer maps, and overgroups of distinguished unipotent elements in reductive groups	29
MICHAEL BATE, SÖREN BÖHM, BENJAMIN MARTIN and GERHARD RÖHRLE	
The $q$ -Schur category and polynomial tilting modules for quantum $GL_n$	63
JONATHAN BRUNDAN	
The binary actions of simple groups of Lie type of characteristic 2	113
NICK GILL, PIERRE GUILLOT and MARTIN W. LIEBECK	
Finite simple groups have many classes of $p$ -elements	137
MICHAEL GIUDICI, LUKE MORGAN and CHERYL E. PRAEGER	
Monogamous subvarieties of the nilpotent cone	161
SIMON M. GOODWIN, RACHEL PENGELLY, DAVID I. STEWART and ADAM R. THOMAS	
An extension of Gow's theorem	181
ROBERT M. GURALNICK and PHAM HUU TIEP	
On dimensions of RoCK blocks of cyclotomic quiver Hecke superalgebras	191
ALEXANDER KLESHCHEV	
Representation growth of Fuchsian groups and modular forms	217
MICHAEL J. LARSEN, JAY TAYLOR and PHAM HUU TIEP	
$D_4$ -type subgroups of $F_4(q)$	249
R. LAWThER	
Constructible representations and Catalan numbers	339
GEORGE LUSZTIG and ERIC SOMMERS	
A reduction theorem for simple groups with $e(G) = 3$	351
RICHARD LYONS and RONALD SOLOMON	
Decomposition numbers in the principal block and Sylow normalisers	367
GUNTER MALLE and NOELIA RIZO	
Levi decompositions of linear algebraic groups and nonabelian cohomology	379
GEORGE J. MCNINCH	
On the intersection of principal blocks	399
GABRIEL NAVARRO, A. A. SCHAEFFER FRY and PHAM HUU TIEP	
Hesselink strata in small characteristic and Lusztig–Xue pieces	415
ALEXANDER PREMÉT	
Multiplicity-free representations of the principal $A_1$ -subgroup in a simple algebraic group	433
ALUNA RIZZOLI and DONNA TESTERMAN	