Pacific Journal of Mathematics

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Volume 336 No. 1-2 May 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 336, No. 1-2, 2025
https://doi.org/10.2140/pjm.2025.336.399

ON THE INTERSECTION OF PRINCIPAL BLOCKS

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To the memory of Gary Seitz

If p and q are two primes, G is a finite group, and $B_p(G)$ is the set of complex irreducible characters in the principal p-block of G, we study $|B_p(G) \cap B_q(G)|$ and its possible relation with some local subgroup.

1. Introduction

It is not often the case that one finds interactions between the representation theory of finite groups with respect to different primes p and q. If G is a finite group and $B_p(G)$ is the set of the complex irreducible characters of G in the principal p-block of G, the p-block containing the principal character 1_G, relations between the sets $B_p(G)$ and $B_q(G)$ are an exception. For instance, it was proved in [Bessenrodt et al. 2007] that $B_p(G) = B_q(G)$ implies that $p = q$ (following pioneering work in [Navarro and Willems 1997]). The study of the other end case, when $B_p(G) \cap B_q(G) = \{1_G\}$, has led to the main conjecture in [Liu et al. 2020] (and its strengthening in [Navarro et al. 2022]): G should have a Sylow p-subgroup P and a Sylow q-subgroup Q such that $xy = yx$ for all $x \in P$ and $y \in Q$ (in other words, $[P, Q] = 1$). This conjecture, which in turn would generalize the main theorem in [Bessenrodt and Zhang 2008], has been reduced to almost simple groups in [Liu et al. 2020] but, as surprising as it may seem, the values of the characters of the almost simple groups are not yet understood well enough in order to solve this problem. The local condition $[P, Q] = 1$ is not isolated in character theory and has already appeared in the so-called Brauer's height zero conjecture for two primes,

The research of Navarro is supported by Grant PID2022-137612NB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and ERDF "A way of making Europe". Schaeffer Fry gratefully acknowledges support from the NSF (grant DMS-2100912), and her former institution, MSU Denver, which holds the award and allows her to serve as PI. Tiep gratefully acknowledges the support of the NSF (grants DMS-1840702 and DMS-2200850), the Simons Foundation, and the Joshua Barlaz Chair in Mathematics at Rutgers University.
The authors are grateful to Gunter Malle for helpful comments on the paper.
The authors are grateful to the referee for a careful reading and helpful comments on the paper.
MSC2020: primary 20C20; secondary 20C15, 20C33, 20C34, 20D20.
Keywords: modular representations, Brauer blocks, Sylow subgroups.

formulated in [Malle and Navarro 2020], and solved recently in [Liu et al. 2024]. (See also [Beltrán et al. 2016].) The global condition $B_p(G) \cap B_q(G) = \{1_G\}$ is definitely less transparent. (For a recent group-theoretical characterization of this condition, see [Robinson 2024].)

Our aim is to propose a new counting global/local conjecture which at the same time implies a characterization of the trivial intersection block property.

Conjecture A. *Let p and q be primes, and let G be a finite group.*

(a) *Assume that there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$, and let $N = \mathbf{N}_G(PQ)$. Then*

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|.$$

(b) *We have that $B_p(G) \cap B_q(G) = \{1_G\}$ if and only if there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ and $B_p(N) \cap B_q(N) = \{1_N\}$, where $N = \mathbf{N}_G(PQ)$.*

As we will prove below (see Lemma 2.2(ii)), we have that $B_p(N) \cap B_q(N) = \{1_N\}$ (for $N = \mathbf{N}_G(PQ)$) if and only if $N = KL$, where K and L are normal subgroups of N of order not divisible by p and q, respectively. This would give, together with Conjecture A(b), another purely group-theoretical characterization of the trivial intersection block property.

We will give solid evidence that Conjecture A is true, but we cannot be optimistic about a general proof of it at the present time. As we have already mentioned, the "only if" implication of Conjecture A(b) seems out of reach. On the other hand, notice that Conjecture A(a) for $p = q$ implies the Alperin–McKay conjecture for principal blocks and abelian Sylow p-subgroups, a conjecture which is still open (for odd primes) in its full generality. Interestingly enough, Conjecture A(a) does not seem implied by it or by the inductive Alperin–McKay condition [Späth 2013]. A reduction theorem of Conjecture A along the lines of the global/local inductive conditions will appear elsewhere.

The main result is the following.

Theorem B. *Conjecture A is true for quasisimple groups, symmetric groups and p-solvable groups.*

We also prove Conjecture A for several almost (quasi)simple groups, including $\mathrm{GL}_n(q_0)$ and $\mathrm{GU}_n(q_0)$ for any prime power q_0. (See Theorem 4.2 and Corollary 4.3.)

2. Preliminaries

Our notation for blocks follows [Navarro 1998], and for characters [Navarro 2018]. To simplify notation, we have chosen to write $B_p(G)$ for what we usually write $\mathrm{Irr}(B_p(G))$. Also, when there is no possible confusion, we write $1 = 1_X$ to be the principal character of any finite group X, and we write $B_p(X) \cap B_q(X) = 1_X$ when $B_p(X) \cap B_q(X) = \{1_X\}$.

As usual, we identify the irreducible characters of G/N with the irreducible characters of G having N in their kernel. Although the following is well-known, it seems convenient to write it down.

Lemma 2.1. *Let G be a finite group, $N \trianglelefteq G$, and suppose that $\chi \in \mathrm{Irr}(G)$ has N in its kernel. Let $\hat{\chi} \in \mathrm{Irr}(G/N)$ be the character of G/N naturally associated to χ.*

(i) *If $\hat{\chi} \in B_p(G/N)$, then $\chi \in B_p(G)$. Therefore $B_p(G/N) \subseteq B_p(G)$.*

(ii) *If N is a central p-subgroup of G, then $\chi \in B_p(G)$ if and only if $\hat{\chi} \in B_p(G/N)$.*

(iii) *If N is a p'-subgroup of G, then $\chi \in B_p(G)$ if and only if $\hat{\chi} \in B_p(G/N)$. Therefore $B_p(G/N) = B_p(G)$.*

(iv) *G is a p'-group if and only if $B_p(G) = 1$.*

Proof. By the comments in pages 198 and 199 of [Navarro 1998], we have that $\bar{B} = B_p(G/N)$ is contained in a unique block B of G. Also, $\bar{B}$ is contained in B if and only if $\mathrm{Irr}(\bar{B}) \cap \mathrm{Irr}(B) \neq \varnothing$. Since the principal character 1_G belongs to $\mathrm{Irr}(\bar{B}) \cap \mathrm{Irr}(B_p(G))$, then (i) is done. For the second part, recall that the map $G^0 \to (G/N)^0$ given by $x \mapsto xN$ is a bijection. Then $\sum_{x \in G^0} \hat{\chi}(xN) = \sum_{x \in G^0} \chi(x)$. The proof is complete using Corollary 3.25 of [Navarro 1998]. The third part follows from Theorem 9.9(c) of [Navarro 1998].

If G is a p'-group, then $B_p(G) = 1$ by part (iii). For the converse, apply, for instance, weak block orthogonality Corollary 3.7 of [Navarro 1998]. □

Of course, the converse of Lemma 2.1(i) does not hold: if $\chi \in \mathrm{Irr}(B_p(G))$ and $N \subseteq \ker(\chi)$, it is not true that $\hat{\chi} \in B_p(G/N)$. (For instance, the sign character in S_3 for $p = 3$.) In what follows, $\boldsymbol{O}_{p'}(G)$ denotes the largest normal subgroup of G of order not divisible by p.

Lemma 2.2. *Let G be a finite group, and let p and q be primes.*

(i) *If $B_p(G) \cap B_q(G) = 1$ and $N \trianglelefteq G$, then $B_p(G/N) \cap B_q(G/N) = 1$.*

(ii) *If $\boldsymbol{O}_{p'}(G)\boldsymbol{O}_{q'}(G) = G$, then $B_p(G) \cap B_q(G) = 1$. If G is p-solvable and q-solvable, then the converse holds.*

(iii) *Let $Q \in \mathrm{Syl}_q(G)$, $N \trianglelefteq G$, and let $M = N\boldsymbol{C}_G(Q \cap N)$. Then $B_q(G)$ is the only block covering $B_q(M)$, and $\mathrm{Irr}(G/M) \subseteq B_q(G)$. Therefore, if $B_p(G) \cap B_q(G) = 1$, then $B_p(M) \cap B_q(M) = 1$ and G/M is a p'-group.*

(iv) *Suppose that $N \trianglelefteq G$, and that $|N|$ is not divisible by p nor q. Then*

$$B_p(G) \cap B_q(G) = B_p(G/N) \cap B_q(G/N).$$

(v) *Suppose that Z is a central subgroup of G and $p \neq q$. Then*

$$B_p(G) \cap B_q(G) = B_p(G/Z) \cap B_q(G/Z).$$

If $p \neq q$ and Conjecture A(a) holds for G/Z, then it also holds for G, and similarly for Conjecture A(b).

Proof. Part (i) follows from Lemma 2.1(i).

We prove (ii). Recall that if $N, M \trianglelefteq G$ and $G = NM$, then the only irreducible character of G lying over 1_N and 1_M is 1_G. If $\chi \in B_p(G)$, then $\boldsymbol{O}_{p'}(G) \subseteq \ker(\chi)$ by Lemma 2.1(iii), and the first part easily follows. For the second, recall that $B_p(G) = \mathrm{Irr}(G/\boldsymbol{O}_{p'}(G))$ if G is p-solvable, by Theorem 10.20 of [Navarro 1998]. Let $L = \boldsymbol{O}_{p'}(G)\boldsymbol{O}_{q'}(G)$. Hence, $\mathrm{Irr}(G/L) \subseteq B_p(G) \cap B_q(G)$, and (ii) is easily completed.

For (iii), notice that $M \trianglelefteq G$, using the Frattini argument. We have that $Q \cap N \subseteq Q \cap M$, and therefore $\boldsymbol{C}_G(Q \cap M) \subseteq M$. Then the first part follows from Lemma 9.20 and Theorem 9.19 of [Navarro 1998]. Now assume that $B_p(G) \cap B_q(G) = 1$. Let $\tau \in B_p(M) \cap B_q(M)$. Since $B_p(G)$ covers $B_p(M)$, there is some $\chi \in B_p(G)$ over τ (by Theorem 9.4 of [Navarro 1998]). Now χ lies in some q-block that covers $B_q(M)$, by Theorem 9.2 of [Navarro 1998]. By the previous part, $\chi \in B_q(G)$, and then $\chi = 1$. Thus $\tau = 1$. Finally, let $\gamma \in B_p(G/M) \subseteq B_p(G)$. Then γ lies over 1_M, and therefore the q-block of γ covers the q-block of M. It follows that γ lies in the principal q-block of G, and therefore $\gamma = 1$, by hypothesis. Thus G/M is a group with $B_p(G/M) = 1$, and this implies that G/M is a p'-group by Lemma 2.1(iv). This finishes (iii).

Part (iv) is obvious since $B_p(G) = B_p(G/N)$ if $|N|$ is not divisible by p.

Now, we prove part (v). Arguing by induction on $|G : Z|$, we may assume that $|Z|$ is a prime. Using part (iv), we may assume that $|Z| = p$. Let $\chi \in B_p(G) \cap B_q(G)$. Since $q \neq p$ and $\chi \in B_q(G)$, we have that $Z \subseteq \ker(\chi)$. Also, $\chi \in B_q(G/Z)$, by Lemma 2.1(iii). By Lemma 2.1(ii), we have that $\chi \in B_p(G/Z)$. This finishes the proof of the first statement.

For the second statement, let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, so that $PZ/Z \in \mathrm{Syl}_p(G/Z)$ and $QZ/Z \in \mathrm{Syl}_q(G/Z)$. Observe that $[P, Q] = 1$ if and only if $[PZ/Z, QZ/Z] = 1$. (Indeed, if $x \in P$ and $y \in Q$ commute modulo Z, then $xyx^{-1}y^{-1} = z \in Z$. Now $xyx^{-1} = yz$ is a q-element, and so z is a q-element. Similarly, z is a p-element, and thus $[x, y] = 1$.) If $[P, Q] = 1$ then $\boldsymbol{N}_{G/Z}(PQZ/Z) = N/Z$ for $N = \boldsymbol{N}_G(PQ)$. (Indeed, writing $Z = \boldsymbol{O}_p(Z) \times \boldsymbol{O}_q(Z) \times Z_1$ and $Z_1 = \boldsymbol{O}_{\{p,q\}'}(Z)$, we have $PQZ = PQ \times Z_1$ and $PQ = \boldsymbol{O}_{\{p,q\}}(PQZ)$. Hence if $h \in G$ normalizes PQZ, then it normalizes PQ.) Now apply the first statement. □

Remark 2.3. Suppose that G is a finite group and $N \trianglelefteq G$. If $B_p(N) \cap B_q(N) = 1$ and $B_p(G/N) \cap B_q(G/N) = 1$, then it is false in general that $B_p(G) \cap B_q(G) = 1$, as shown by A_4, with $p = 2$ and $q = 3$. Also, if $B_p(G) \cap B_q(G) = 1$, then it is false in general that $B_p(N) \cap B_q(N) = 1$, as shown by $G = \mathsf{D}_{10} \times \mathsf{S}_3$ for $p = 3$ and $q = 5$.

We will use the Alperin–Dade theorem on isomorphic blocks several times.

Theorem 2.4. (Alperin–Dade) *Let p be a prime. Suppose that S is a normal subgroup of a finite group G such that p does not divide $|G/S|$, and let $P \in \mathrm{Syl}_p(S)$. If $G = S\boldsymbol{C}_G(P)$, then restriction from G to S defines a bijection $B_p(G) \to B_p(S)$.*

Proof. See Theorem 2.4 of [Navarro et al. 2022], for instance. □

Now we work towards proving that Conjecture A holds if G is p-solvable.

Lemma 2.5. *Suppose that G has a normal p'-subgroup K containing $Q \in \mathrm{Syl}_q(G)$. Then restriction of characters defines a natural bijection*

$$B_p(G) \cap B_q(G) \to B_p(\boldsymbol{N}_G(Q)) \cap B_q(\boldsymbol{N}_G(Q)).$$

Also,

$$B_p(G) \cap B_q(G) = \mathrm{Irr}(G/K\boldsymbol{C}_G(Q)),$$

and therefore $B_p(G) \cap B_q(G) = 1$ if and only if $G = K\boldsymbol{C}_G(Q)$.

Proof. Let $H = \boldsymbol{N}_G(Q)$. By the Frattini argument, we have that $G = KH$. Thus restriction defines a bijection $\mathrm{Irr}(G/K) \to \mathrm{Irr}(H/\boldsymbol{N}_K(Q))$.

Let $M = K\boldsymbol{C}_G(Q) \trianglelefteq G$. Let $V = \boldsymbol{O}_{q'}(H) \subseteq \boldsymbol{C}_G(Q)$. Since $Q \in \mathrm{Syl}_q(G)$, we have that $\boldsymbol{C}_G(Q) = \boldsymbol{Z}(Q) \times V$, using the Schur–Zassenhaus theorem. Since H is q-solvable, we have that $B_q(H) = \mathrm{Irr}(H/V)$, by Theorem 10.20 of [Navarro 1998]. Also, $M = KV$. By Alperin–Dade Theorem 2.4, we have that restriction defines a bijection $B_q(M) \to B_q(K)$.

We claim that $B_p(G) \cap B_q(G) = \mathrm{Irr}(G/M)$. Let $\chi \in B_p(G) \cap B_q(G)$. Then $\chi \in \mathrm{Irr}(G/K)$ by Lemma 2.1(iii). Let $\eta \in \mathrm{Irr}(M)$ be under χ. Then $\eta \in B_q(M)$ and since η lies over 1_K, we have that $\eta = 1_M$, by Alperin–Dade. Hence $\chi \in \mathrm{Irr}(G/M)$. Conversely, if $\chi \in \mathrm{Irr}(G/M)$, then $\chi \in B_p(G) \cap B_q(G)$, using Lemma 2.2(iii) and Lemma 2.1(iii). This proves the claim. The claim, applied now in H (with respect to its normal subgroup $\boldsymbol{N}_K(Q)$), proves that

$$B_p(H) \cap B_q(H) = \mathrm{Irr}(H/\boldsymbol{N}_K(Q)V).$$

The proof of the lemma now easily follows. □

Lemma 2.6. *Suppose that $P \in \mathrm{Syl}_p(G)$ is normal in G. Let q be any prime.*

(i) *Suppose that $Q \in \mathrm{Syl}_q(G)$ centralizes P. Then*

$$|B_p(G) \cap B_q(G)| = \left|B_p(\boldsymbol{N}_G(Q)) \cap B_q(\boldsymbol{N}_G(Q))\right|.$$

(ii) *We have that $B_p(G) \cap B_q(G) = 1$ if and only if there is $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ and $B_p(\boldsymbol{N}_G(Q)) \cap B_q(\boldsymbol{N}_G(Q)) = 1$.*

Proof. Both parts of the lemma are trivial if $p = q$. So we may assume that $p \neq q$. Let $C = \boldsymbol{C}_G(P)$, $Z = \boldsymbol{Z}(P)$ and $K = \boldsymbol{O}_{p'}(G)$. Then $C = Z \times K$.

If we assume that $[Q, P] = 1$, then $Q \subseteq K$ and (i) follows from Lemma 2.5. So we prove (ii). Now, we assume that $B_p(G) \cap B_q(G) = 1$ and we prove that there exists $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ by induction on $|G|$. Since G is p-solvable, we know that $B_p(G) = \mathrm{Irr}(G/K)$. Assume that $K > 1$. Then $B_p(G/K) \cap B_q(G/K) = 1$ and we have that there is $Q \in \mathrm{Syl}_p(G)$ such that $[P, Q] \subseteq K$. Since $P \trianglelefteq G$, $[P, Q] \subseteq P$, and thus $[P, Q] = 1$. Hence, we may assume that $K = 1$. Therefore $B_p(G) = \mathrm{Irr}(G)$ and therefore $B_q(G) = 1$. Thus q does not divide $|G|$ by Lemma 2.1(iv). Hence, in (ii), we may assume that $[P, Q] = 1$, where $Q \in \mathrm{Syl}_q(G)$. Then $Q \subseteq K$, and we apply Lemma 2.5. □

Theorem 2.7. *Suppose that G is p-solvable.*

(i) *Suppose that there are $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then $|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|$, where $N = \boldsymbol{N}_G(PQ)$.*

(ii) *We have that $B_p(G) \cap B_q(G) = 1$ if and only if there are $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ and $B_p(N) \cap B_q(N) = 1$, where $N = \boldsymbol{N}_G(PQ)$.*

Proof. Suppose first that $p = q$. In this case, the second part is trivial, using Lemma 2.1(iv). For the first part, this is the Alperin–McKay conjecture for p-solvable groups with abelian Sylow p-subgroups, which is nearly trivial. (By the Hall–Higman 1.2.3 lemma, we have that $KP \trianglelefteq G$, where $K = \boldsymbol{O}_{p'}(G)$. Since we may also assume that $K = 1$ by Lemma 2.1(iii), the result easily follows.)

We may assume that $p \neq q$. We start with the first part, which we argue by induction on $|G|$. Let $K = \boldsymbol{O}_{p'}(G)$. Since $[P, Q] = 1$, then we have that $Q \subseteq K$ by the Hall–Higman 1.2.3 lemma. Let $H = \boldsymbol{N}_G(Q)$. By Lemma 2.5, we have that $|B_p(G) \cap B_q(G)| = |B_p(H) \cap B_q(H)|$. Since $\boldsymbol{N}_G(PQ) = \boldsymbol{N}_G(P) \cap H$, by induction, we may assume that $Q \trianglelefteq G$. By Lemma 2.6, we conclude that

$$|B_p(G) \cap B_q(G)| = |B_p(\boldsymbol{N}_G(P)) \cap B_q(\boldsymbol{N}_G(P))|$$

and this proves (ii).

Finally we prove (ii). The "if" part is immediate from (i). Assume now that G is p-solvable and that $B_p(G) \cap B_q(G) = 1$. By Theorem 1.4 of [Liu et al. 2020] we know that there exists $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then we apply part (i). □

3. (Quasi)simple groups

For a finite group G and a prime p, continue to let $B_p(G)$ denote the irreducible complex characters in the principal p-block of G. The goal of this section is to show that for quasisimple groups, the following holds:

Theorem 3.1. *Let p and q be two primes (not necessarily distinct) and let S be a finite quasisimple group. Let $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_q(S)$. Assume that $[P, Q] = 1$. Then*

$$|B_p(S) \cap B_q(S)| = |B_p(N) \cap B_q(N)|,$$

where $N := \mathbf{N}_S(PQ)$.

We remark that when $p = q$, the statement is the Alperin–McKay conjecture together with Brauer's height zero conjecture for principal blocks of quasisimple groups with abelian Sylow p-subgroups.

As a corollary, we obtain Theorem B for quasisimple groups:

Corollary 3.2. *Conjecture A holds for finite quasisimple groups.*

Proof. Thanks to Theorem 3.1, it remains to show that if S is a quasisimple group such that $B_p(S) \cap B_q(S) = 1$, then $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_q(S)$. We assume $p \neq q$, as otherwise this is trivially true. Let $\bar{S} = S/\mathbf{Z}(S)$ be the corresponding simple group, and note that $B_p(\bar{S}) \subseteq B_p(S)$ and $B_q(\bar{S}) \subseteq B_q(S)$. Then $|B_p(\bar{S}) \cap B_q(\bar{S})| = 1$, which implies $(\bar{S}, p, q)$ is one of very few possibilities thanks to [Brough et al. 2021], and these satisfy $[\bar{P}, \bar{Q}] = 1$ for some $\bar{P} \in \mathrm{Syl}_p(\bar{S})$ and $\bar{Q} \in \mathrm{Syl}_q(\bar{S})$ (see the discussion after Conjecture 1.3 in [Liu et al. 2020]). Hence by [Malle and Navarro 2020, Lemma 3.1] (or Lemma 2.2(v)), we also have $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_q(S)$, completing the proof. □

Given a connected reductive group $\mathbf{G}$ and Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$, we write $\mathbf{G}^F$ for the corresponding finite group of Lie type obtained as the fixed points under F. If p is a prime and n an integer prime to p, we write $d_p(n)$ for the order of n modulo p if p is odd and modulo 4 if $p = 2$.

The main results of [Malle and Navarro 2020, Section 3] yield:

Proposition 3.3 (Malle–Navarro). *Let G be a finite quasisimple group and $p \neq q$ two primes dividing $|G|$. Assume that there are $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then one of the following holds:*

(i) *$G = J_1$ and $\{p, q\} = \{3, 5\}$.*

(ii) *$G = J_4$ and $\{p, q\} = \{5, 7\}$.*

(iii) *$S = G/\mathbf{Z}(G)$ is a simple group of Lie type defined in characteristic r distinct from p and q. In this case, if $G = \mathbf{G}^F$ is a quasisimple group of Lie type, then:*

- *$d_p(r^f) = d_q(r^f) =: d$, where G is defined over $\mathbb{F}_{r^f}$.*
- *p and q are odd.*
- *P and Q are abelian, and $PQ \leq \mathbf{S}$ for an (F-stable) Sylow d-torus $\mathbf{S}$ of $\mathbf{G}$.*
- *p and q are good for $\mathbf{G}$, larger than 3 if G is of type ${}^3\mathrm{D}_4$, and do not divide $|\mathbf{Z}(\mathbf{G})^F : (\mathbf{Z}(\mathbf{G})^\circ)^F| \cdot |\mathbf{Z}(\mathbf{G}^*)^F : (\mathbf{Z}(\mathbf{G}^*)^\circ)^F|$, where $(\mathbf{G}^*, F)$ is dual to $(\mathbf{G}, F)$.*

Proof. This is [Malle and Navarro 2020, Propositions 3.2–3.5]. The last item of (iii) follows from [Malle 2014, Lemma 2.1 and Proposition 2.2]. □

Before working on simple groups of Lie type, we settle Conjecture A (when $p \neq q$) for sporadic groups:

Lemma 3.4. *If $p \neq q$, then Conjecture A holds for sporadic quasisimple groups.*

Proof. By Lemma 2.2(v), we need only consider sporadic simple groups G.

On the one hand, by [Brough et al. 2021, Theorem 1.2], $B_p(G) \cap B_q(G) = 1$ precisely when (G, p, q) occurs in Proposition 3.3(i) and (ii). On the other hand, assume G admits commuting Sylow p- and q-subgroups P and Q. Then we are in (i) or (ii) of Proposition 3.3. Using the information in [GAP 2018] on centralizer size and power fusion of elements of order p, q, and pq in G, one can show that $N = \boldsymbol{N}_G(PQ) = PQ \rtimes \boldsymbol{O}_{\{p,q\}'}(N)$ is $\mathsf{S}_3 \times \mathsf{D}_{10}$ in the case of Proposition 3.3(i), and $(\mathsf{C}_5 \rtimes \mathsf{C}_4) \times (\mathsf{C}_7 \rtimes \mathsf{C}_3)$ in the case of Proposition 3.3(ii). It follows from Lemma 2.2(ii) that $B_p(N) \cap B_p(N) = 1$. □

Proposition 3.5. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group such that $G = \boldsymbol{G}^F$ is a quasisimple group of Lie type. Assume $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ for primes p and q (not necessarily distinct) satisfying assertion* (iii) *of Proposition 3.3 and let $\boldsymbol{L} := \boldsymbol{C}_{\boldsymbol{G}}(\boldsymbol{S})$. Then $\boldsymbol{L} = \boldsymbol{C}_{\boldsymbol{G}}(P) = \boldsymbol{C}_{\boldsymbol{G}}(Q)$ and $\boldsymbol{N}_G(PQ) = \boldsymbol{N}_G(Q) = \boldsymbol{N}_G(P) = \boldsymbol{N}_G(\boldsymbol{L})$. Further, $\boldsymbol{L}^F = P \times O^p(\boldsymbol{L}^F) = Q \times O^q(\boldsymbol{L}^F)$.*

Proof. Note that $\boldsymbol{N}_G(PQ) \leq \boldsymbol{N}_G(P) \cap \boldsymbol{N}_G(Q)$. Then the statements follow from [Malle 2014, Propositions 2.3 and 2.4]. □

Lemma 3.6. *Keep the situation of Proposition 3.5. Write $N := \boldsymbol{N}_G(PQ)$ and $L := \boldsymbol{L}^F \lhd N$. Then $B_q(N)$ is the unique q-block of N covering $B_q(L)$ and $B_p(N)$ is the unique p-block of N covering $B_p(L)$.*

Proof. This follows from Proposition 3.5 and [Navarro 1998, Corollary (9.21)]. □

For $G = \boldsymbol{G}^F$ a finite reductive group, let $(\boldsymbol{G}^*, F^*)$ be dual to $(\boldsymbol{G}, F)$ and write $G^* := (\boldsymbol{G}^*)^{F^*}$. Then $\mathrm{Irr}(G)$ is partitioned into rational Lusztig series $\mathcal{E}(G, s)$, where s ranges over semisimple elements of G^*, up to G^*-conjugacy. (See, for example, [Cabanes and Enguehard 2004, Theorem 8.24].) We will write $\mathrm{UCh}(G) := \mathcal{E}(G, 1)$ for the set of unipotent characters, and will similarly denote by $\mathrm{UCh}(B_p(G))$ the set $\mathcal{E}(G, 1) \cap B_p(G)$ of unipotent characters in the principal p-block for a given prime p.

Proposition 3.7. *Keep the situation of Proposition 3.5, but now assume $p \neq q$ are distinct primes. Then we have a bijection between the sets $B_p(G) \cap B_q(G)$ and $\mathrm{Irr}(N/L)$ and between the sets $B_p(N) \cap B_q(N)$ and $\mathrm{Irr}(N/L)$.*

In particular, we have

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|.$$

Proof. First, by [Cabanes and Enguehard 2004, Theorem 9.12], and using that neither p nor q is the defining prime for G, we have $B_p(G)$ contains only characters in rational series $\mathcal{E}(G, s)$ with $|s|$ a power of p. Similarly, $B_q(G)$ is comprised only of characters in series with s a power of q. This means that $B_p(G) \cap B_q(G) = \mathrm{UCh}(B_p(G)) \cap \mathrm{UCh}(B_q(G))$ contains only unipotent characters.

Recall that P is abelian, so every character in $B_p(G)$ has degree prime to p by the "if" direction of Brauer's height zero conjecture [Kessar and Malle 2013]. Further, note that the last item of Proposition 3.3 implies that both p and q satisfy the necessary hypotheses for [Cabanes and Enguehard 1994, Theorem]. Now, by [Cabanes and Enguehard 1994, Theorem], $\mathrm{UCh}(B_p(G)) = \mathrm{UCh}(B_q(G))$ is comprised of those characters $\chi \in \mathrm{Irr}(G)$ in the d-Harish–Chandra series of $(\boldsymbol{L}, 1_L)$, with $\boldsymbol{L}$ as in Proposition 3.5 and $L := \boldsymbol{L}^F$. Note that $\boldsymbol{L}$ is as stated due to [Malle 2007, Corollary 6.6].

Then we have $B_p(G) \cap B_q(G) = \mathrm{UCh}(B_p(G)) = \mathrm{UCh}(B_q(G))$. Further, this set is in bijection with the irreducible characters of the so-called relative Weyl group $W_G(\boldsymbol{L}) = N/L$ by [Broué et al. 1993, Theorem 3.2], completing the claim for G.

Now, note that every character in $B_p(N) \cap B_q(N)$ lies above some character in $B_p(L) \cap B_q(L)$. Recall from Proposition 3.5 that $L = P \times X = Q \times Y$ for some p', respectively q', subgroups $X, Y \lhd L$, so $\mathrm{Irr}(B_p(L)) = \mathrm{Irr}(P) \otimes \{1_X\}$ and similar for q. Then this forces $B_p(L) \cap B_q(L) = \{1_L\}$, and hence every character of $B_p(N) \cap B_q(N)$ lies above the trivial character of L.

Conversely, by Lemma 3.6, $B_p(N)$, respectively $B_q(N)$, is the unique block above $B_p(L)$, respectively $B_q(L)$. Hence, the characters of N above 1_L are exactly the members of $B_p(N) \cap B_q(N)$. By Gallagher's theorem, this set is in bijection with $\mathrm{Irr}(N/L)$, completing the proof. □

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $S = G/Z$ with $Z \leq \boldsymbol{Z}(G)$, where G is the Schur cover of the simple group $\bar{S} := S/\boldsymbol{Z}(S)$, and write P, Q as $P = \widehat{P}Z/Z$ and $Q = \widehat{Q}Z/Z$ for $\widehat{P} \in \mathrm{Syl}_p(G)$ and $\widehat{Q} \in \mathrm{Syl}_q(G)$.

First, assume that $p \neq q$. Then [Malle and Navarro 2020, Lemma 3.1] yields that $[P, Q] = 1$ implies $[\widehat{P}, \widehat{Q}] = 1$. It follows that G is of one of the forms in Proposition 3.3. In cases (i) and (ii), the result is Lemma 3.4. In case (iii), we may apply Lemma 2.2(v) to replace the Schur cover with a group of Lie type G of the form in Proposition 3.5. Then every character in $B_p(\boldsymbol{N}_G(\widehat{P}\widehat{Q})) \cap B_q(\boldsymbol{N}_G(\widehat{P}\widehat{Q}))$ is trivial on $\boldsymbol{Z}(G) \leq L$ by the second-to-last paragraph of the proof of Proposition 3.7. Further, we have $N = \boldsymbol{N}_S(PQ) = \boldsymbol{N}_G(\widehat{P}\widehat{Q})/Z$. Using, e.g., [Cabanes and Enguehard 2004, Lemma 17.2], we have $B_p(N)$ is the set of characters in $B_p(\boldsymbol{N}_G(\widehat{P}\widehat{Q}))$ lying above 1_Z and similar for q. Then we have, by Proposition 3.7,

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|.$$

For the same reason, we have $B_p(S)$ is the set of characters in $B_p(G)$ lying above 1_Z and similar for $B_q(S)$. But recall from the proof of Proposition 3.7 that $B_p(G) \cap B_q(G) = \mathrm{UCh}(B_p(G))$. Since unipotent characters are trivial on $\boldsymbol{Z}(G)$,

$$|B_p(S) \cap B_q(S)| = |B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|,$$

which completes the proof when $p \neq q$.

Now let $p = q$ and assume that $P \in \mathrm{Syl}_p(S)$ is abelian. Since the Alperin–McKay conjecture is known when $p = 2$ by [Ruhstorfer $\geq$ 2025] (alternatively, we can use Walter's classification of simple groups with abelian Sylow 2-subgroups [1969]), we assume p is odd. By [Koshitani and Späth 2016], we may further assume that P is not cyclic.

If S is a covering group of an alternating group A_n, then the Alperin–McKay conjecture holds for S by the main result of [Michler and Olsson 1990]. If $\bar{S}$ is a sporadic simple group or the Tits group, the Alperin–McKay conjecture has been checked by T. Breuer [$\geq$ 2025]. In the cases where a Sylow p-subgroup of $\bar{S}$ is abelian, we note that the Schur multiplier is a p'-group, so we are done by Lemma 2.1(iii). Similarly, when $\bar{S}$ is a group of Lie type with an exceptional Schur multiplier and an abelian noncyclic Sylow p-subgroup, we have that the Schur multiplier has size not divisible by p, except for the case $\mathrm{PSL}_4(3)$, which can be checked in [GAP 2018]. Hence in these cases, we may work with a quasisimple group of Lie type G such that $G/\boldsymbol{Z}(G) = \bar{S}$, rather than the full Schur multiplier.

So we now assume $G = \boldsymbol{G}^F$ is a quasisimple group of Lie type and $\bar{S} = S/\boldsymbol{Z}(S) = G/\boldsymbol{Z}(G)$ is not isomorphic to any A_n. If p is the defining prime for G, then the assumption that P is abelian yields that $\bar{S} = \mathrm{PSL}_2(p^a)$ for some positive integer a. In this case, $|\boldsymbol{Z}(G)|$ is prime to p, and $B_p(S) = B_p(\bar{S}) = \mathrm{Irr}_{p'}(\bar{S}) = \mathrm{Irr}(\bar{S}) \setminus \{\mathrm{St}_S\}$ and the result follows from the considerations in [Isaacs et al. 2007, Section (15F)] (see also the proof of [Späth 2013, Theorem 8.4]). Hence, we may assume that p is not the defining characteristic for G.

If $p \geq 5$, we have that $\widehat{P}$ is also abelian (see the discussion after [Malle 2014, Proposition 2.2]), and we are again in the situation of Proposition 3.3(iii). As $p \nmid |\boldsymbol{Z}(G)|$, we then have $B_p(G) = B_p(S)$ and $B_p(\boldsymbol{N}_G(\widehat{P})) = B_p(N)$, by [Navarro 1998, Theorem 9.9]. Recall that $B_p(L) = \mathrm{Irr}(\widehat{P}) \otimes \{1_X\}$ where X is the p'-group such that $L = \boldsymbol{C}_G(\widehat{P}) = \widehat{P} \times X$ guaranteed by Proposition 3.5. Note that every $\psi \in \mathrm{Irr}(\widehat{P})$ extends to its inertia group in $\boldsymbol{N}_G(\widehat{P}) = \boldsymbol{N}_G(\boldsymbol{L})$ by [Isaacs 2006, Theorem (6.26)]. Then [Malle 2014, Theorem 2.9] yields a bijection between $B_p(G)$ and $B_p(\boldsymbol{N}_G(\widehat{P}))$, and hence between $B_p(S)$ and $B_p(N)$, as long as $p \geq 5$.

Suppose $p = 3$. Then the assumption that P is abelian implies that $\bar{S} = \mathrm{PSL}_2(q_0)$, $\bar{S} = \mathrm{PSL}_n^{\epsilon}(q_0)$ with $3 \leq n \leq 5$ and $\epsilon \in \{\pm 1\}$, or $\bar{S} = \mathrm{PSp}_4(q_0)$, considering the order polynomials and using [Malle 2014, Proposition 2.2] and the discussion after. If $\widehat{P}$ is abelian, then the same considerations from before hold, noting that in this situation for the principal blocks, the conclusions of [Malle 2014, Proposition 2.7 and Theorem 2.8] continue to hold. That is, $B_p(G)$ and $B_p(\boldsymbol{N}_G(\widehat{P}))$ are both in bijection with pairs (ψ, ϕ) with $\psi \in \mathrm{Irr}(P)$ and $\phi \in \mathrm{Irr}(\boldsymbol{N}_G(\boldsymbol{L})_\psi/L)$.

Then we assume $\widehat{P}$ is nonabelian, and hence $S = \mathrm{PSL}_3^\epsilon(q_0)$ with $(q_0-\epsilon)_3 = 3$. Recall that by the "if" direction of BHZ [Kessar and Malle 2013], we have $B_3(S)$ and $B_3(N)$ consist only of $3'$-characters. From here, the proof of [Malle 2008, Corollary 3.9 and Theorem 3.12] yields the result. Indeed, note that the group N is described in loc. cit., and its construction in GAP shows that it contains a unique block of maximal defect. On the other hand, the six members of $\mathrm{Irr}_{p'}(S)$ are the deflations of three unipotent characters of G and the three irreducible constituents of the restriction to G of a semisimple character of $\mathrm{GL}_3^\epsilon(q_0)$ corresponding to a semisimple element with eigenvalues $\{\omega, \omega^{-1}, 1\}$, where $|\omega| = 3$. The three unipotent characters of $\mathrm{GL}_3^\epsilon(q_0)$ and the stated semisimple character all lie in $B_3(\mathrm{GL}_3^\epsilon(q_0))$, using [Fong and Srinivasan 1982, Theorem 7A]. Hence these six characters all lie in $B_3(S)$. □

4. Almost simple groups

Proposition 4.1. (a) *Let $p \neq q$ be primes. Then Conjecture A holds for any finite group G with a central subgroup Z such that $G/Z \cong \mathsf{A}_n$ or S_n with $n \geq 5$.*

(b) *Let $p = q$ be any prime. Then Conjecture A holds for covering groups G of A_n and S_n with $n \geq 5$.*

Proof. (a) By Lemma 2.2(v), we may assume that $G \cong \mathsf{A}_n$ or S_n, and that $n \geq p > q$. By [Beltrán et al. 2016, Lemma 2.4], see also Proposition 3.3, G has no Hall $\{p, q\}$-subgroup if $q > 2$. Furthermore, by [Beltrán et al. 2016, Theorem 2.2], A_n contains a p-element x such that $\boldsymbol{C}_{\mathsf{A}_n}(x)$ contains no Sylow 2-subgroup of A_n. It follows that G has no Hall $\{p, 2\}$-subgroup. Thus G does not admit any pair (P, Q) of commuting Sylow p-subgroup P and Sylow q-subgroup Q. Finally, [Bessenrodt and Zhang 2008, Proposition 3.2] and its proof show that $B_p(G) \cap B_q(G) \neq 1_G$.

(b) If $p \neq 2$ then the statement follows from [Michler and Olsson 1990]. If $p = 2$, then $B_p(G) \neq 1_G$, and G can have abelian Sylow 2-subgroups P only when $G = \mathsf{A}_5$, in which case $|B_p(G)| = 4 = |B_p(\boldsymbol{N}_G(P))|$. □

Our next result proves Conjecture A for almost simple Lie-type groups of adjoint type.

Theorem 4.2. *Let p and q be (not necessarily distinct) primes, and let $\boldsymbol{G}_{\mathrm{ad}}$ be a simple algebraic group of adjoint type over a field of positive characteristic, with a Steinberg endomorphism F such that $\boldsymbol{G}_{\mathrm{ad}}^F$ is almost simple. Then Conjecture A holds for any G with $[\boldsymbol{G}_{\mathrm{ad}}^F, \boldsymbol{G}_{\mathrm{ad}}^F] \lhd G \leq \boldsymbol{G}_{\mathrm{ad}}^F$.*

Proof. Since Conjecture A holds trivially if p or q does not divide $|G|$, we will assume p and q both divide $|G|$. Now, as shown in the proof of [Brough et al. 2021, Lemma 5.4], $B_p(G) \cap B_q(G) \neq 1_G$ if $p \neq q$. If $p = q$, then $B_p(G) \neq 1_G$ because $p \mid |G|$.

It remains to show that if G admits $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with $[P, Q] = 1$, then

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|. \tag{4-1}$$

Inspecting the orders of G and the simple subgroup $S = [G, G]$, we see that p and q both divide $|S|$.

(a) Here we consider the case $p \neq q$. As S admits the Hall $\{p, q\}$-subgroup $(P \cap S)(Q \cap S)$, we can apply Proposition 3.3 and Theorem 3.1 to S. In particular, $\boldsymbol{G}_{\mathrm{ad}}$ is defined over a field of characteristic $r \neq p, q$.

Assume first that

$$G = \boldsymbol{G}_{\mathrm{ad}}^F.$$

To work on the local side, we identify $\boldsymbol{G}_{\mathrm{ad}} = \boldsymbol{G}/Z$ where $\boldsymbol{G}$ is a simple, simply connected algebraic group and Z a finite central subgroup, and $S = \boldsymbol{G}^F Z/Z \cong \boldsymbol{G}^F/Z^F$ for a suitable Steinberg endomorphism $F : \boldsymbol{G} \to \boldsymbol{G}$. As in the proof of Lemma 2.2(v), $P \cap S = P_1 Z/Z$ and $Q \cap S = Q_1 Z/Z$ for some $P_1 \in \mathrm{Syl}_p(\boldsymbol{G}^F)$ and $Q_1 \in \mathrm{Syl}_q(\boldsymbol{G}^F)$ with $[P_1, Q_1] = 1$. By Proposition 3.3, we have $p, q > 2$ and $P_1 Q_1 \leq \boldsymbol{T}^F$ for an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$. We also observe that

$$\gcd(pq, |G/S|) = 1. \tag{4-2}$$

Indeed, since $p, q > 2$, the claim follows unless $G = \mathrm{PGL}_n^\epsilon(q_0)$ or $G = E_6^\epsilon(q_0)_{\mathrm{ad}}$ with $\epsilon = \pm$. In the former case, it was shown in the proof of [Beltrán et al. 2016, Proposition 2.7] that the existence of the commuting pair (P_1, Q_1) in $\boldsymbol{G}^F = \mathrm{SL}_n^\epsilon(q_0)$ in the case $\gcd(pq, q_0 - \epsilon) > 1$ implies that $n < \min(p, q)$, and so (4-2) follows as $|G/S| = \gcd(n, q_0 - \epsilon)$. In the latter case, [Beltrán et al. 2016, Proposition 2.12(a)] shows that $p, q > 3$, and so (4-2) follows as $|G/S| = \gcd(3, q_0 - \epsilon)$.

Thus $P, Q \leq S$, and so $P, Q \leq T := (\boldsymbol{T}/Z)^F$. Next we note that

$$G = ST. \tag{4-3}$$

(Indeed, suppose $gZ \in G = (\boldsymbol{G}/Z)^F$ for $g \in \boldsymbol{G}$. Then $g^{-1}F(g) \in Z$. Since $Z \leq \boldsymbol{T}$, by the Lang–Steinberg theorem, there is $t \in \boldsymbol{T}$ such that $t^{-1}F(t) = g^{-1}F(g)$. It follows that $tZ \in (\boldsymbol{T}/Z)^F$, $gt^{-1} \in \boldsymbol{G}^F$, and $gZ = (gt^{-1}Z)(tZ) \in (\boldsymbol{G}^F Z/Z)(\boldsymbol{T}/Z)^F$.) Let

$$N := \boldsymbol{N}_G(PQ), \quad C := \boldsymbol{C}_G(PQ) = PQ \times \boldsymbol{O}_{\{p,q\}'}(C), \quad N_1 := \boldsymbol{N}_S(PQ).$$

As T centralizes PQ, (4-3) shows that $G = SC$ and $N = N_1 C$; in particular,

$$G = S\boldsymbol{C}_G(P) = S\boldsymbol{C}_G(Q), \quad N = N_1\boldsymbol{C}_N(P) = N_1\boldsymbol{C}_N(Q). \tag{4-4}$$

It therefore follows from Theorem 2.4 that

$$|B_p(G)\cap B_q(G)| = |B_p(S)\cap B_q(S)|, \quad |B_p(N)\cap B_q(N)| = |B_p(N_1)\cap B_q(N_1)|.$$

Together with Theorem 3.1 applied to S, these equalities yield (4-1).

In the general case $S \lhd G \leq \boldsymbol{G}_{\mathrm{ad}}^F$, note that (4-2) and (4-4) still hold for any such G, and hence we are again done by using Theorems 2.4 and 3.1.

(b) Now we consider the case $p = q$. Arguing as in the proof of Theorem 3.1, we may assume that $p > 2$ and that $\boldsymbol{G}_{\mathrm{ad}}$ is defined over a field of characteristic $r \neq p, q$. Suppose in addition that both assertion (iii) of Proposition 3.3 (with $p = q$) and (4-2) hold. (Note that $|G_1| = |G|$ for $G_1 = \boldsymbol{G}^F$, so $p \nmid |G_1|/|S|$ in this case as well.) Then the arguments in (a) apply to show that (4-4) holds, and so we are done again by using Theorems 2.4 and 3.1.

The same considerations as in the proof of Theorem 3.1 show that the only remaining case is when $p = 3$, $S = \mathrm{PSL}_3^\epsilon(q_0)$, $G = \mathrm{PGL}_3^\epsilon(q_0)$, $\epsilon = \pm$, and $(q_0 - \epsilon)_3 = 3$. In this case, $H \cong \boldsymbol{O}_{3'}(\boldsymbol{Z}(H)) \times G$ for $H := \mathrm{GL}_3^\epsilon(q_0)$. Hence, by Lemma 2.1(iii), (4-1) follows from the main result of [Michler and Olsson 1983] applied to H. □

Corollary 4.3. *Conjecture A holds for* $\mathrm{GL}_n(q_0)$ *and* $\mathrm{GU}_n(q_0)$, *for any* $n \geq 2$ *and any prime power* q_0.

Proof. It suffices to consider the case $G = \mathrm{GL}_n^\epsilon(q_0)$ is nonsolvable. In this case, $G/\boldsymbol{Z}(G) = \mathrm{PGL}_n^\epsilon(q_0)$ satisfies Conjecture A by Theorem 4.2 when $p \neq q$. Hence we are done using Lemma 2.2(v) if $p \neq q$, and by the main results of [Michler and Olsson 1983], which prove the Alperin–McKay conjecture for these groups, when $p = q$. □

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Received April 5, 2024. Revised August 12, 2024.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 336 No. 1-2 May 2025

In memoriam
Gary Seitz