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In memory of Gary Seitz

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and $\mathfrak{g} = \mathrm{Lie}(G)$. We show that the nilpotent pieces $\mathrm{LX}(\Delta)$ introduced by Lusztig form a partition of the nilpotent cone of $\mathfrak{g}$ and hence coincide with the Hesselink strata $\mathcal{H}(\Delta)$ where Δ runs through the set of all weighted Dynkin diagrams of G. Thanks to earlier results obtained by Lusztig, Xue and Voggesberger this boils down to describing the pieces $\mathrm{LX}(\Delta)$ for groups of type E_7 in characteristic 2 and for groups of type E_8 in characteristic 2 and 3. Our arguments are computer-free, but rely very heavily on the results of Liebeck and Seitz (2012).

1. Introduction

Let G be a connected reductive algebraic group of rank ℓ over an algebraically closed field $\Bbbk$ and T a maximal torus of G. Let Σ be the root system of G with respect to T and Π a basis of simple roots of Σ. Write $X(T)$ (resp. $X_*(T)$) for the lattice of rational characters (resp. cocharacters) of T and $X_*^+(T)$ for the intersection of $X_*(T)$ with the dual Weyl chamber of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ associated with Π. Each rational cocharacter $\lambda \in X_*(G)$ gives rise to a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\lambda, i), \qquad \mathfrak{g}(\lambda, i) = \{x \in \mathfrak{g} \mid (\mathrm{Ad}\,\lambda(t))x = t^i x \text{ for all } t \in \Bbbk^\times\},$$

of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. For $d \in \mathbb{Z}$, we put $\mathfrak{g}(\lambda, \geq d) := \bigoplus_{i \geq d} \mathfrak{g}(\lambda, i)$ and $\mathfrak{g}(\lambda, < d) := \bigoplus_{i < d} \mathfrak{g}(\lambda, i)$, and denote by $P(\lambda) = L(\lambda)R_u(\lambda)$ the parabolic subgroup of G associated with λ. Here $L(\lambda) = Z_G(\lambda)$ is a Levi subgroup of G. Recall that $\mathrm{Lie}(P(\lambda)) = \mathfrak{g}(\lambda, \geq 0)$ and $\mathrm{Lie}(L(\lambda)) = \mathfrak{g}(\lambda, 0)$.

We let $\mathcal{N}(\mathfrak{g})$ denote the nilpotent cone of $\mathfrak{g}$, the variety of all $(\mathrm{Ad}\,G)$-unstable vectors of $\mathfrak{g}$, and write $\mathfrak{D}_G$ for the set of all *Dynkin labels* attached to the nilpotent orbits of a complex Lie algebra with root system Σ. As explained in [4], the

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Hesselink strata of $\mathcal{N}(\mathfrak{g})$ are parametrised by the set of cocharacters $\tau_\Delta \in X_*^+(T)$ with $\Delta \in \mathfrak{D}_G$ and form a partition of $\mathcal{N}(\mathfrak{g})$, so that

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathcal{H}(\Delta).$$

The cocharacter τ_Δ can be read off the weighted Dynkin diagram $(a_1, \ldots, a_\ell)$ associated with the complex nilpotent orbit with label Δ as follows: if x is a root vector of $\mathfrak{g}$ corresponding to a simple root $\alpha_i \in \Pi$ then $(\operatorname{Ad} \tau_\Delta(t))(x) = t^{a_i} x$ for all $t \in \Bbbk^\times$. The Hesselink stratum attached to τ_Δ has the form

$$\mathcal{H}(\Delta) = (\operatorname{Ad} G)(\mathcal{V}(\tau_\Delta, 2)_{ss} + \mathfrak{g}(\tau_\Delta, \geq 3)),$$

where $\mathcal{V}(\tau_\Delta, 2)_{ss} \neq \varnothing$ is the set of all $(\operatorname{Ad} L^\perp(\tau_\Delta))$-semistable vectors of $\mathfrak{g}(\tau_\Delta, 2)$ (see [4] for more detail).

Given $\Delta \in \mathfrak{D}_G$ we write $\mathfrak{g}_2^{\Delta,!}$ for the set of all $x \in \mathfrak{g}(\tau_\Delta, 2)$ such that $G_x \subset P(\tau_\Delta)$ where $G_x = Z_G(x)$ is the stabiliser of x in G. As explained in [4, Remark 7.3] each set $\mathfrak{g}_2^{\Delta,!}$ contains $\mathcal{V}(\tau_\Delta, 2)_{ss}$, a nonempty Zariski open subset of $\mathfrak{g}(\tau_\Delta, 2)$. The set

$$\mathrm{LX}(\Delta) := (\operatorname{Ad} G)(\mathfrak{g}_2^{\Delta,!} + \mathfrak{g}(\tau_\Delta, \geq 3))$$

containing $\mathcal{H}(\Delta)$ will be referred to as the *Lusztig–Xue piece* of $\mathcal{N}(\mathfrak{g})$ associated with Δ. The pieces $\mathrm{LX}(\Delta)$ and their analogues for $\mathcal{N}(\mathfrak{g}^*)$ and for the unipotent variety of G were introduced by Lusztig. Viability of these pieces has to do with the fact that $\mathfrak{g}_2^{\Delta,!}$ is defined in a more transparent fashion than its elusive subset $\mathcal{V}(\tau_\Delta, 2)_{ss}$.

In [11, Appendix A], Lusztig and Xue proved that the pieces $\mathrm{LX}(\Delta)$ form a partition of $\mathcal{N}(\mathfrak{g})$ in the case where G is a classical group. Very recently, the same property was established by Voggesberger for groups of type G_2, F_4 and E_6 (see our discussion in Section 2.1 for more detail). These results imply that $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ provided that G is not of type E_7 or E_8.

The partition property of the coadjoint analogues of $\mathrm{LX}(\Delta)$ was established by Lusztig [10] and Xue [18] in all cases where G is a simple algebraic group and $p = \operatorname{char}(\Bbbk)$ equals the ratio of the squared lengths of long and short roots in Σ. In all other cases there is a G-equivariant bijection between $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{g}^*)$ which enables one to identify the nilpotent coadjoint orbits and pieces of $\mathfrak{g}^*$ with those of $\mathfrak{g}$; see [13, Section 5.6].

It was conjectured in [17] that $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ should also hold for all $\Delta \in \mathfrak{D}_G$ in the case where G is a group of type E_7 or E_8. Our goal is to prove this conjecture.

The orbits $\mathcal{O}(e) = (\operatorname{Ad} G)e$ with $e \in \mathcal{N}(\mathfrak{g})$ will be denoted by their Dynkin labels Δ or their variants $(\Delta)_p$. The latter are attached to a small number of new nilpotent orbits which appear when $(\Sigma, p) \in \{(\mathrm{G}_2, 3), (\mathrm{F}_4, 2), (\mathrm{E}_7, 2), (\mathrm{E}_8, 2), (\mathrm{E}_8, 3)\}$; see [9] for detail. Combining our results obtained in Section 2 with the results of Lusztig, Xue and Voggesberger mentioned above we obtain the following:

Theorem 1.1. *Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then $\mathcal{H}(\Delta) = \mathrm{LX}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ and hence*

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathrm{LX}(\Delta).$$

Since proving Theorem 1.1 reduces quickly to the case where Σ is an irreducible root system, we may assume without loss of generality that our algebraic group G is simple and simply connected.

We use Steinberg's notation $x_\alpha(t)$ for elements of the unipotent root subgroups U_α of G; see [15, §3]. Simple root vectors e_{α_i} with $\alpha_i \in \Pi$ are denoted by e_i, and we always use Bourbaki's numbering [2] of simple roots. We assume that root vectors $e_\gamma \in \mathfrak{g}_\gamma$ come from a Chevalley basis of an admissible lattice $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}_\mathbb{C}$, where $\mathfrak{g}_\mathbb{C}$ is a complex Lie algebra with root system Σ. Since we mostly work over fields of characteristic 2, the signs of structure constants do not really affect our computations.

The Weyl group of Σ is denoted by W and we write $\langle \cdot\,, \cdot \rangle$ for the canonical pairing between $X(T)$ and $X_*(T)$ with values in $\mathbb{Z}$. We fix a W-invariant $\mathbb{Q}$-valued inner product $(\cdot \mid \cdot)$ on $X_*(T)_\mathbb{Q} = X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$ which enables one to identify $X_*(T)_\mathbb{Q}$ with the dual vector space $X(T)_\mathbb{Q} = X(T) \otimes_\mathbb{Z} \mathbb{Q}$.

When we need to specify a particular root vector, we sometimes follow the conventions of [9]. For example, a root vector e_γ with $\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ is denoted by $e_{234^25^26}$. When confusion is unlikely, we prefer standard conventions for specifying root vectors.

We occasionally use the $(\mathrm{Ad}\, G)$-invariant $\mathbb{Z}$-valued bilinear form κ on a minimal admissible lattice $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}_\mathbb{C}$ introduced in [4, 7.2]. Its reduction modulo p will be denoted by the same symbol. When describing certain cocharacters $\tau \in X_*(T)$ we often specify their effect on the root vectors e_i where $1 \leq i \leq \ell$. More precisely, if $(\mathrm{Ad}\, \tau(t))e_i = t^{r_i} e_i$ for all $t \in \Bbbk^\times$, then we write $\tau = (r_1, \ldots, r_\ell)$. This will cause no confusion since $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is a $\mathbb{Q}$-basis of $X(T)_\mathbb{Q}$.

If $\mathrm{char}(\Bbbk) = p > 0$ then $\mathfrak{g} = \mathrm{Lie}(G)$ carries a canonical restricted Lie algebra structure $\mathfrak{g} \ni x \mapsto x^{[p]} \in \mathfrak{g}$ equivariant under the adjoint action of G. It is well known that the nilpotent cone $\mathcal{N}(\mathfrak{g})$ coincides with the set of all $x \in \mathfrak{g}$ such that $x^{[p]^N} = 0$ for $N \gg 0$. A restricted Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is called $[p]$-*nilpotent* (resp. *toral*) if $\mathfrak{a} \subseteq \mathcal{N}(\mathfrak{g})$ (resp. if the $[p]$-mapping $x \mapsto x^{[p]}$ is one-to-one on $\mathfrak{a}$). Given $x \in \mathfrak{g}$ we write $\mathfrak{g}_x$ for the centraliser of x in the Lie algebra $\mathfrak{g}$, and we often use the fact that $\mathrm{Lie}(G_x) \subseteq \mathfrak{g}_x$. If $x \in \mathfrak{g}(\lambda, r)$ for some $\lambda \in X_*(G)$ and $r \in \mathbb{Z}$ then $\mathfrak{g}_x = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_e(\lambda, i)$ where $\mathfrak{g}_e(\lambda, i) = \mathfrak{g}_e \cap \mathfrak{g}(\lambda, i)$. We say that $x \in \mathfrak{g}$ is *toral* if $x^{[p]} = x$.

2. Hesselink strata and Lusztig–Xue pieces

2.1. Let $\tau = \tau_\Delta \in X_*(T)$ be the cocharacter associated with $\Delta \in \mathfrak{D}_G$ and write $\mathfrak{g}_2^{\Delta,!}$ for the set of all $x \in \mathfrak{g}(\tau, 2)$ such that $G_x \subset P(\tau)$. Since the parabolic subgroup $P(\tau)$

is optimal for all $x \in \mathcal{V}(\tau, 2)_{ss}$ in the sense of Kempf–Rousseau theory, we have the inclusion $\mathcal{V}(\tau, 2)_{ss} \subseteq \mathfrak{g}_2^{\Delta,!}$. From [4, Theorem 6.1(iii)] it follows that $\mathcal{V}(\tau, 2)_{ss}$ is a nonempty Zariski open subset of $\mathfrak{g}(\tau, 2)$. The set

$$\mathrm{LX}(\Delta) := (\mathrm{Ad}\, G)(\mathfrak{g}_2^{\Delta,!} + \mathfrak{g}(\tau, \geq 3))$$

is called the *Lusztig–Xue piece* of $\mathcal{N}(\mathfrak{g})$ associated with Δ. By the above, $\mathrm{LX}(\Delta)$ contains $\mathcal{H}(\Delta)$ for every $\Delta \in \mathfrak{D}_G$.

The pieces $\mathrm{LX}(\Delta)$ were first introduced by Lusztig [11] in the Lie algebra case. In [10], the definition was extended to cover the coadjoint nullcone $\mathcal{N}(\mathfrak{g}^*)$ and the unipotent variety $\mathfrak{U}(G)$ of G. In [11, Appendix A], Lusztig and Xue proved that the decomposition

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathrm{LX}(\Delta) \tag{1}$$

holds for all groups of type A, B, C, D, the key point being that the union is disjoint. Very recently, the same property was established by L. Voggesberger for groups of type G_2, F_4 and E_6 with the help of Magma; see [17, Theorem 1.1].

Note that if (1) holds for G then $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$; see [4, Remark 7.3.1] for detail. It was conjectured in [17, Conjecture 1.2] that (1) should also hold for the $\Bbbk$-groups of type E_7 and E_8. A similar expectation (covering $\mathcal{N}(\mathfrak{g})$, $\mathcal{N}(\mathfrak{g}^*)$ and $\mathfrak{U}(G)$) was expressed by Lusztig in [10, 2.3].

We aim to confirm Voggesberger's conjecture. Lusztig's expectation related to partitioning the unipotent variety of G will be discussed in Section 2.12. For completeness, we also provide a new proof for groups of type G_2, F_4 and E_6.

Our task will become much simpler if we establish the following:

$$\text{If } e \in \mathfrak{g}(\tau_\Delta, 2) \text{ and } G_e \subset P(\tau_\Delta) \text{ then } e \in \mathcal{H}(\Delta). \tag{2}$$

Proving statement (2) will occupy the main body of the paper. From now on we assume that the group G is simple, simply connected, and has type E_6, E_7, E_8, F_4 or G_2. Let κ denote the normalised Killing form introduced in [4, 7.3]. If all roots of Σ have the same length then the radical of κ coincides with the central toral subalgebra of $\mathfrak{g}$; see [4, Lemma 7.3]. We start with a reduction lemma which will also explain why (1) always holds in good characteristic.

Lemma 2.1. *Suppose G is a group of type* E *and let $\tau = \tau_\Delta$, where $\Delta \in \mathfrak{D}_G$. If $e \in \mathfrak{g}(\tau, 2)$ is such that $G_e \subset P(\tau)$ and $\mathfrak{g}_e = \mathrm{Lie}(G_e)$, then $e \in \mathcal{H}(\Delta)$.*

Proof. As G is simply connected, the centre of the Lie algebra $\mathfrak{g}$ is spanned by a toral element $z \in \mathfrak{g}(\tau, 0)$ which is nonzero if and only if (Σ, p) is one of $(\mathrm{E}_6, 3)$ or $(\mathrm{E}_7, 2)$. For $i \geq 0$, we let $[e, \mathfrak{g}(\tau, i)]^\perp$ denote the set of all $x \in \mathfrak{g}(\tau, -i-2)$ such that $\kappa\big(x, [e, \mathfrak{g}(\tau, i)]\big) = 0$. Since κ is $(\operatorname{Ad} G)$-invariant we have that

$$[e, \mathfrak{g}(\tau, i)]^{\perp} = \{x \in \mathfrak{g}(-i-2) \mid [e, x] \in \operatorname{Rad}\kappa\}.$$

As $\operatorname{Rad}\kappa = \Bbbk z \subset \mathfrak{g}(\tau, 0)$, we get $[e, \mathfrak{g}(\tau, i)]^{\perp} = \mathfrak{g}_e(\tau, -i-2)$ for $i \geq 1$ and $[e, \mathfrak{g}(\tau, 0)]^{\perp} = \{x \in \mathfrak{g}(\tau, -2) \mid [e, x] \in \Bbbk z\}$.

By our assumption, $\mathfrak{g}_e = \operatorname{Lie}(G_e) \subset \operatorname{Lie}(P(\tau)) = \bigoplus_{i \geq 0} \mathfrak{g}(\tau, i)$. Since $\mathfrak{g}_e = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_e(\tau, i)$ it follows that $\mathfrak{g}_e(\tau, i) = 0$ for $i \leq -1$, forcing $\mathfrak{g}(\tau, i+2) = [e, \mathfrak{g}(\tau, i)]$ for all $i \geq 1$. Also, $\mathfrak{g}(\tau, 2) = [e, \mathfrak{g}(\tau, 0)]$ when $z = 0$. If $z \neq 0$ we can only say at this point that the subspace $[e, \mathfrak{g}(\tau, 0)]$ has codimension ≤ 1 in $\mathfrak{g}(\tau, 2)$.

Since $\mathfrak{g}_e = \operatorname{Lie}(G_e)$ we have that

$$\begin{aligned}\dim(\operatorname{Ad} G)e &= \dim \mathfrak{g} - \dim \mathfrak{g}_e = \sum_{i \in \mathbb{Z}} \dim \mathfrak{g}(\tau, i) - \sum_{i \geq 0} \dim \mathfrak{g}_e(\tau, i) \\ &= \sum_{i<0} \dim \mathfrak{g}(\tau, i) + \sum_{i \geq 0} \dim\, [e, \mathfrak{g}(\tau, i)] \\ &= \dim\, [e, \mathfrak{g}(\tau, 0)] + \sum_{i \notin \{0,1,2\}} \dim \mathfrak{g}(\tau, i).\end{aligned}$$

Since it follows from [7, Theorem 2] that $\dim\,(\operatorname{Ad} G)e$ and $\sum_{i \notin \{0,1\}} \dim \mathfrak{g}(\tau, i)$ are even numbers, it must be that $\dim\,[e, \mathfrak{g}(\tau, 0)] \equiv \dim \mathfrak{g}(\tau, 2) \bmod 2$. In view of our earlier remarks this yields $\mathfrak{g}(\tau, 2) = [e, \mathfrak{g}(\tau, 0)]$.

Let $L(\tau) = Z_G(\tau)$, a Levi subgroup of $P(\tau)$ with Lie algebra $\mathfrak{g}(\tau, 0)$. Since $[\mathfrak{g}(\tau, 0), e]$ is contained in the tangent space $T_e\big((\operatorname{Ad} L(\tau))e\big)$, the $(\operatorname{Ad} L(\tau))$-orbit of e is Zariski open in $\mathfrak{g}(\tau, 2)$ and hence intersects with $\mathcal{V}(\tau, 2)_{ss} \neq \varnothing$. This yields $e \in \mathcal{H}(\Delta)$. □

2.2. Given a connected reductive $\Bbbk$-group H and a nilpotent element $e \in \operatorname{Lie}(H)$ (i.e., an unstable vector of the $(\operatorname{Ad} H)$-module $\operatorname{Lie}(H)$), we write $\hat{\Lambda}_H(e)$ for the set of all cocharacters in $X_*(H)$ optimal for e in the sense of Kempf–Rousseau theory. By [5, Theorem 7.2], this set of cocharacters does not depend on the choice of an H-invariant $\mathbb{R}_{\geq 0}$-valued norm mapping on $X_*(H)$.

Let $\tau = \tau_\Delta$, where $\Delta \in \mathfrak{D}_G$, and let $e \in \mathfrak{g}(\tau, 2)$ be such that $G_e \subset P(\tau)$. As e is a G-unstable vector of $\mathfrak{g}$ it affords an optimal cocharacter $\tau' \in X_*(G)$ with the property that $e \in \mathfrak{g}(\tau', \geq 2)$; see [4] for detail. Since $\operatorname{Ad} \tau(\Bbbk^\times)$ preserves the line $\Bbbk e$ and the cocharacter τ' is optimal for any nonzero scalar multiple of e, it follows from the main results of Kempf–Rousseau theory that $\operatorname{Ad} \tau(\Bbbk)$ normalises the optimal parabolic subgroup $P(\tau')$ of e; see [12, Theorem 2.1(iv)], for example. Since the latter is self-normalising and contains G_e we have $\tau(\Bbbk^\times)G_e \subset P(\tau')$. Hence $N_e := N_G(\Bbbk e) = \tau(\Bbbk^\times)G_e$ is a subgroup of $P(\tau')$. It follows from [1, 11.14(2)] (applied to tori) that for any maximal torus D of G_e there is a maximal torus $\tilde{D}$ of N_e such that $D \subseteq \tilde{D} \cap G_e$. Since $\tau(\Bbbk^\times)$ is contained in a maximal torus of N_e and all maximal tori of N_e are conjugate, this shows that G_e contains a maximal torus which commutes with $\tau(\Bbbk^\times)$; we call it T_0.

As $N_e \subset P(\tau')$, there is a maximal torus T' of $P(\tau')$ which contains the maximal torus $\tau(\Bbbk^\times)T_0$ of N_e. Since $L := Z_G(T_0)$ is a Levi subgroup of G, there exists $g \in G$ such that gLg^{-1} is a standard Levi subgroup L of G. Replacing e and τ by $(\operatorname{Ad} g)e$ and $g\tau g^{-1}$ we may assume without loss of generality that L is a standard Levi subgroup of G. By the main results of Kempf–Rousseau theory (and the description of Hesselink strata in [4]), there exists a unique cocharacter $\tau'' \in \hat{\Lambda}_G(e) \cap X_*(T')$ conjugate to τ' under $P(\tau')$ and such that $e = \sum_{i \geq 2} e_i$ where $e_i \in \mathfrak{g}(\tau'', i)$ and $e_2 \in \mathcal{V}(\tau, 2)_{ss}$; see [12, Theorem 2.1(iii)].

By construction, T' is a maximal torus of L containing $\tau(\Bbbk^\times)$ and $\tau''(\Bbbk^\times)$. Furthermore, $e \in \mathfrak{g}^{\operatorname{Ad} T_0} = \operatorname{Lie}(L)$. Let $L' = \mathcal{D}L$, the derived subgroup of L, and $\mathfrak{l}' = \operatorname{Lie}(L')$. It is immediate from Jacobson's formula for $[p]$-th powers in the restricted Lie algebra $(\mathfrak{l}, [p])$ that $e \in \mathfrak{l}'$.

2.3. Let $d = \dim T_0$ and $r = \ell - d$ where $\ell = \operatorname{rk} G$. We begin to investigate the case where $d > 0$, that is, the case where e is not distinguished in $\mathfrak{g}$. The group $L' = \mathcal{D}L$ is semisimple and $T' \cap L'$ contains a maximal torus of L'; we call it T_1. The subtorus $T_0 \cdot T_1$ of T' being self-centralising, it must be that $T' = T_0 \cdot T_1$. As the central subgroup $T_0 \cap L'$ of L' is finite, we have a direct sum decomposition

$$X(T')_{\mathbb{Q}} = X(T_0)_{\mathbb{Q}} \oplus X(T_1)_{\mathbb{Q}} \tag{3}$$

of the $\mathbb{Q}$-spans of $X(T_0)$ and $X(T_1)$ in $X(T')_{\mathbb{Q}} = X(T') \otimes_{\mathbb{Z}} \mathbb{Q}$. This shows that

$$\operatorname{rk} L' = \dim T_1 = \ell - \dim_{\mathbb{Q}} X(T_0)_{\mathbb{Q}} = \ell - d = r.$$

Since we identify the dual spaces $X(T')_{\mathbb{Q}}$ and $X_*(T')_{\mathbb{Q}} = X_*(T') \otimes_{\mathbb{Z}} \mathbb{Q}$ by means of a W-invariant inner product $(\,\cdot\,|\,\cdot\,)$ we have that $\eta(\mu(t)) = t^{(\mu|\eta)}$ for all $\eta \in X(T)$, $\mu \in X_*(T')$, and $t \in \Bbbk^\times$. As the subgroup $N_L(T')/T'$ of W acts trivially on $X(T_0)$ and has no nonzero fixed points on $X(T_1)$, the $\mathbb{Q}$-spans $X_*(T_0)_{\mathbb{Q}}$ and $X_*(T_1)_{\mathbb{Q}}$ are orthogonal to each other with respect to $(\,\cdot\,|\,\cdot\,)$.

Lemma 2.2. *Suppose $\nu \in X_*(T')$ is such that $e \in \mathfrak{g}(\nu, 2)$ and $(G_e)^\circ \subset P(\nu)$. If the group $(G_e)^\circ/R_u(G_e)$ is semisimple then $\nu = \tau$.*

Proof. As $T_0 \subseteq T' \cap G_e$ is a maximal torus of G_e it must be that $T_0 = (T' \cap G_e)^\circ$. As $\nu^{-1}(t) \cdot \tau(t) \in T' \cap G_e$ for all $t \in \Bbbk^\times$ we have that $\nu - \tau \in X_*(T_0)$. As $R_u(G_e)$ is a unipotent group, the torus $T_0 \subset G_e$ maps isomorphically onto a maximal torus of the semisimple group $S_e := (G_e)^\circ/R_u(G_e)$. It follows that $\operatorname{rk} S_0 = \dim T_0 = d$. We identify T_0 with its image in S_e. As $\nu(\Bbbk^\times) \subset N_e$ normalises $R_u(G_e)$, it acts on S_e by rational automorphisms.

It is straightforward to see that the connected algebraic group $\tilde{S}_e := \nu(\Bbbk^\times)S_e$ is reductive and $\nu(\Bbbk^\times)T_0$ is a maximal torus of $\tilde{S}_e$. So, if $\bar{\gamma} \in X(\nu(\Bbbk^\times)T_0)$ is a root of $\tilde{S}_e$ then so is $-\bar{\gamma}$. On the other hand, our assumption on ν implies that $\operatorname{Ad}\nu$ has nonnegative weights on $\operatorname{Lie}(S_e)$. This entails that $\nu(\Bbbk^\times)$ is a central torus of $\tilde{S}_e$. Repeating this argument with τ in place of ν we deduce that the torus $\nu(\Bbbk^\times)$ is central in $\tilde{S}_e$ as well.

Since $\operatorname{rk} S_e = d$ there are $\mathbb{Q}$-independent weights $\gamma_1, \dots, \gamma_d \in X(T_0)$ which serve as a basis of simple roots for the root system of $\tilde{S}_e$ with respect to T_0. As $\dim X(T_0)_\mathbb{Q} = d$, they must form a basis of the vector space $X(T_0)_\mathbb{Q}$. On the other hand, the preceding discussion shows that $\gamma_i(\nu(t)) = \gamma_i(\tau(t)) = 1$ for all $t \in \Bbbk^\times$ and $i \le d$. Our identification of $X(T')_\mathbb{Q}$ and $X_*(T')_\mathbb{Q}$ now yields that $\nu - \tau$ is orthogonal to $X(T_0)_\mathbb{Q}$ with respect to $(\cdot \mid \cdot)$. Since $\nu - \tau \in X_*(T_0) \subseteq X(T_0)_\mathbb{Q}$ this forces $\nu - \tau = 0$. □

2.4. Recall from Section 2.2 that our nilpotent element $e \in \mathfrak{l}'$ affords an optimal cocharacter $\tau'' \in \hat{\Lambda}_G(e) \cap X_*(T')$ such that $e = \sum_{i \ge 2} e_i$ with $e_i \in \mathfrak{l}'(\tau'', i)$ and $e_2 \in \mathcal{V}(\tau'', 2)_{ss}$. Identifying $X_*(T')_\mathbb{Q}$ and $X(T')_\mathbb{Q}$ as in Section 2.3 and using (3) we get $\tau'' = \tau''_0 + \tau''_1$ where $\tau''_0 \in X(T_0)_\mathbb{Q}$ and $\tau''_1 \in X(T_1)_\mathbb{Q}$. Since

$$(\tau'' \mid \tau'') = (\tau''_0 \mid \tau''_0) + (\tau''_1, \tau''_1) \ge (\tau''_0 \mid \tau''_0)$$

and $(\tau''_0 \mid \gamma) = 0$ for all $\gamma \in X(T_1)$, it follows from the optimality of τ'' that $\tau''_0 = 0$; see [12, p. 348] for a similar (characteristic-free) argument.

As $\tau'' \in X_*(T')$ we thus obtain that τ'' is an optimal cocharacter for e contained in $X_*(T_1)$. Therefore, the orbit $\mathcal{O}_L(e) := (\operatorname{Ad} L)e$ is contained in the Hesselink stratum $\mathcal{H}_L(\tau'')$ of the nilpotent cone $\mathcal{N}(\mathfrak{l}')$, the variety of all $(\operatorname{Ad} L)$-unstable vectors of $\mathfrak{l}'$. Since T_0 is a maximal torus of G_e the group $(L_e)^\circ$ is unipotent. In other words, e is a distinguished nilpotent element of $\mathfrak{l}'$.

Lemma 2.3. *If the* $(\operatorname{Ad} L)$*-orbit* $\mathcal{O}_L(e)$ *coincides with its stratum* $\mathcal{H}_L(\tau'')$ *and the group* $(G_e)^\circ/R_u(G_e)$ *is semisimple, then* τ *is* L*-conjugate to* τ'' *and* $e \in \mathcal{H}(\Delta)$.

Proof. Recall that $e = \sum_{i \ge 2} e_i$ where $e_i \in \mathfrak{l}'(\tau'', i)$ and $e_2 \in \mathcal{V}(\tau'', 2)_{ss}$. As one of the $(\operatorname{Ad} L)$-orbits of $\mathcal{H}_L(\tau'')$ intersects with $\mathfrak{l}'(\tau'', 2)$, our assumption on $\mathcal{H}_L(\tau'')$ implies that e is $(\operatorname{Ad} L)$-conjugate to e_2. Since $\tau'' \in \hat{\Lambda}_G(e_2)$ we have $G_{e_2} \subset P(\tau'')$. As the group $(G_{e_2})^\circ/R_u(G_{e_2}) \cong (G_e)^\circ/R_u(G_e)$ is semisimple, it follows from Lemma 2.2 that τ is L-conjugate to τ''. As τ'' is G-conjugate to τ_Δ by our discussion in Section 2.2, we deduce that $e_2 \in \mathcal{H}(\Delta)$. But then $e \in \mathcal{H}(\Delta)$ as wanted. □

Let $e \in \mathcal{N}(\mathfrak{g})$ and write S_e for the factor group $(G_e)^\circ/R_u(G_e)$. When $\operatorname{char}(\Bbbk) = 0$, one can use the tables in [3, pp. 401–407] to quickly compile a full list of the nilpotent orbits $\mathcal{O}(e) = (\operatorname{Ad} G)e$ for which S_e is a semisimple group. Then one can use [8] to find out that the same list is still valid in good characteristic. Since we are mainly concerned with the case where $p = \operatorname{char}(\Bbbk)$ is very bad for G, we must rely instead on the following important classification result obtained in [9].

Proposition 2.4 [9]. *Suppose G is exceptional and e is not distinguished in $\mathfrak{g}$. Then either S_e is a semisimple group or $\mathcal{O}(e)$ has one of the following labels*:

Type E_6: A_1^2, $\mathrm{A}_2\mathrm{A}_1$, $\mathrm{A}_2\mathrm{A}_1^2$, A_3, $\mathrm{A}_3\mathrm{A}_1$, $\mathrm{D}_4(\mathrm{a}_1)$, A_4, $\mathrm{A}_4\mathrm{A}_1$, $\mathrm{D}_5(\mathrm{a}_1)$, D_5.

Type E_7: $\mathrm{A}_2\mathrm{A}_1$, $\mathrm{A}_3\mathrm{A}_2$ $(p \neq 2)$, A_4, $\mathrm{A}_4\mathrm{A}_1$, $\mathrm{D}_5(\mathrm{a}_1)$, $\mathrm{E}_6(\mathrm{a}_1)$.

Type E_8: $\mathrm{A}_3\mathrm{A}_2$ $(p \neq 2)$, $\mathrm{A}_4\mathrm{A}_1$, $\mathrm{A}_4\mathrm{A}_1^2$, $\mathrm{D}_5\mathrm{A}_2$ $(p \neq 2)$, $\mathrm{D}_7(\mathrm{a}_2)$, $\mathrm{E}_6(\mathrm{a}_1)\mathrm{A}_1$, $\mathrm{D}_7(\mathrm{a}_1)$ $(p \neq 2)$.

If G is of type G_2 *or* F_4 *then S_e is a semisimple group for any $e \in \mathcal{N}(\mathfrak{g})$.*

Proof. The statement is obtained by examining Tables 22.1.1–22.1.5 in [9]. One also observes in the process that if a nilpotent orbit $\mathcal{O}(e)$ coincides with its Hesselink stratum then the type of S_e is independent of the characteristic of $\Bbbk$. This is curious, but will not be required in what follows. □

2.5. It is well known that $\dim \mathfrak{g}_e \geq \dim G_e$ for any $e \in \mathfrak{g}$, and if $\dim \mathfrak{g}_e = \dim G_e$, the orbit $\mathcal{O}(e)$ is called *smooth*. For all exceptional types, the smooth nilpotent orbits of $\mathfrak{g}$ can be determined from Stewart's tables [16] which record the Jordan blocks of all $\operatorname{ad} e$ with $e \in \mathcal{N}(\mathfrak{g})$. We note that the representatives of nilpotent orbits used in Stewart's tables are compatible with those in [9, Tables 12.1, 13.3 and 14.1].

Remark 2.5. Suppose G is exceptional. Although Liebeck and Seitz do not discuss Hesselink strata in [9], they can be spotted as follows. For each representative $e \in \mathcal{N}(\mathfrak{g})$ listed in the tables of [9] there exists a cocharacter $\mu \in X_*(G)$ conjugate to τ_Δ with $\Delta \in \mathfrak{D}_G$ and such that $e \in \mathfrak{g}(\mu, \geq 2)$ and $G_e \subset P(\mu)$. Of course, in characteristic 2 and 3 there are a few special cases where two representatives of different orbits, say e and $\tilde{e}$, are attached to the same μ (and the same Δ). If there is no $\tilde{e}$, then e is homogeneous (that is, lies in $\mathfrak{g}(\mu, 2)$) and $\overline{(\operatorname{Ad} P(\mu))e} = \mathfrak{g}(\mu, \geq 2)$. Then the orbit $(\operatorname{Ad} Z_G(\mu))e$ is dense in $\mathfrak{g}(\mu, 2)$ and the description of Hesselink strata in [4] implies that e is $Z_G^\perp(\mu)$-semistable. This is an excellent case since $\mathcal{H}(\Delta) = \mathcal{O}(e)$ is a single orbit and $\dim G_e = \dim \mathfrak{g}(\mu, 0) + \dim \mathfrak{g}(\mu, 1)$ is independent of p.

If $\tilde{e}$ does exist then one may assume without loss of generality that $e \in \mathfrak{g}(\mu, 2)$ and $\tilde{e} = e + e_\beta$ for some root vector $e_\beta \in \mathfrak{g}(\mu, d)$ with $d \geq 2$. Moreover, $\tilde{e}$ always lies in the new orbit with label $(\Delta)_p$ and $(\operatorname{Ad} P(\mu))\tilde{e}$ is Zariski dense in $\mathfrak{g}(\mu, \geq 2)$. Then $\dim G_{\tilde{e}} = \dim \mathfrak{g}(\mu, 0) + \dim \mathfrak{g}(\mu, 1)$, but $\dim G_e > \dim G_{\tilde{e}}$.

If $d > 2$ then it is still true that the orbit $(\operatorname{Ad} Z_G(\mu))e$ is dense in $\mathfrak{g}(\mu, 2)$. Therefore, both e and $\tilde{e}$ lie in the stratum $\mathcal{H}(\Delta)$. In fact, it follows from the main results of [9] that $\mathcal{H}(\Delta) = \mathcal{O}(e) \cup \mathcal{O}(\tilde{e})$. So this case is not too bad either.

In order to describe the strata of $\mathcal{H}(\Delta)$ explicitly, one has to clarify the remaining (problematic) case where $\tilde{e} = e + e_\beta$ and $d = 2$. Here $\mathfrak{g}(\mu, 2)$ contains both e and $\tilde{e}$, but the orbit $(\operatorname{Ad} Z_G(\mu))e$ is no longer dense in $\mathfrak{g}(\mu, 2)$. So we cannot conclude at this point that e is $Z_G^\perp(\mu)$-semistable. However, since μ is G-conjugate to τ_Δ

and it is shown in [9] that $G_e \subset P(\mu)$, we see that $e \in \mathrm{LX}(\Delta)$. Therefore, the main results of [9] together with Theorem 1.1 provide a very satisfactory description of the Hesselink strata of $\mathcal{N}(\mathfrak{g})$ for G exceptional. Namely, if there is no $\tilde{e}$ then $\mathcal{H}(\Delta) = \mathcal{O}(e)$ and if $\tilde{e}$ does exist then $\mathcal{H}(\Delta) = \mathcal{O}(e) \cup \mathcal{O}(\tilde{e})$.

Remark 2.6. Suppose e is as in Remark 2.5 and $\mathcal{O}(e) = \mathcal{H}(\Delta)$. Then $\dim G_e = \dim \mathfrak{g}(\mu, 0) + \dim \mathfrak{g}(\mu, 1)$ is independent of the characteristic of $\Bbbk$. Since the representatives e used in [16] agree with those of [9] one can compute $\dim G_e$ by counting the number of Jordan blocks of $\operatorname{ad} e$ under the assumption that $p \gg 0$. If the number obtained coincides with the actual number of Jordan blocks of $\operatorname{ad} e \in \operatorname{End} \mathfrak{g}$ then the orbit $(\operatorname{Ad} G)e$ is smooth.

Thanks to Proposition 2.4 and Lemma 2.1 we can reduce proving statement (2) to a much smaller number cases.

Lemma 2.7. *Suppose $\tau \in X_*(G)$ is conjugate to τ_Δ with $\Delta \in \mathfrak{D}_G$ and $e \in \mathfrak{g}(\tau, 2)$ is such that $G_e \subset P(\tau)$. If the group S_e is not semisimple, then $e \in \mathcal{H}(\Delta)$.*

Proof. Since S_e is not semisimple, e lies in one of the orbits listed in Proposition 2.4. Our discussion in Remark 2.5 (based on results of [9]) shows that any such orbit $\mathcal{O}(\Delta)$ coincides with the corresponding Hesselink stratum $\mathcal{H}(\Delta)$. Using [16, Tables 10, 11, 12] and the method described in Remark 2.6 one now checks directly that $\dim \mathfrak{g}_e = \dim G_e$ unless G is of type E_6, $p = 2$, and $\mathcal{O}(e)$ is one of $\mathcal{O}(\mathrm{A}_3\mathrm{A}_1)$ or $\mathcal{O}(\mathrm{D}_5)$. If $\dim \mathfrak{g}_e = \dim G_e$ then applying Lemma 2.1 gives $e \in \mathcal{H}(\Delta)$.

Now suppose G is of type E_6 and $p = 2$. In the remaining two cases e is a regular nilpotent element of a Levi subalgebra of type $\mathrm{A}_3\mathrm{A}_1$ or D_5, hence no generality will be lost by assuming that e is as in [8, pp. 85, 91].

We first suppose that $e \in \mathcal{O}(\mathrm{A}_3\mathrm{A}_1)$. Then $e = e_1 + e_3 + e_4 + e_6$. Since $e \in \mathfrak{g}(\tau, 2)$ and all maximal tori of N_e are conjugate we may also assume that $\tau = (2, r, 2, 2, s, 2)$ for some $r, s \in \mathbb{Z}$. If $\tilde{\alpha} = 122321$, the highest root of Σ with respect to Π, then $\{x_{\pm\tilde{\alpha}}(t) \mid t \in \Bbbk\}$ generate a subgroup of type A_1 in G_e. Since $G_e \subset P(\tau)$ it must be that $(\tau \mid \tilde{\alpha}) = 0$ forcing $14 + 2(r + s) = 0$. Therefore, $\tau = (2, r, 2, 2, -7 - r, 2)$. Since $x_{-\alpha_5}(t) \in G_e$ for all $t \in \Bbbk$ we also have $7 + r \geq 0$. Let $\beta = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $\beta' = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$. Then $\beta \pm \beta' \notin \Sigma$ and $x_\beta(t)x_{\beta'}(t) \in G_e$ for all $t \in \Bbbk$ yielding $e_\beta + e_{\beta'} \in \operatorname{Lie}(G_e)$. Hence $6 - (7 + r) \geq 0$, i.e., $r \leq -1$. As a result, $r \in \{-7, -6, -5, -4, -3, -2, -1\}$. For r in this set we denote by τ_r the W-conjugate of $\tau = (2, r, 2, 2, -7 - r, 2)$ contained in $X_*^+(T)$.

It is straightforward to check that $\tau_{-7} = (0, 1, 1, 0, 1, 2)$, $\tau_{-6} = (1, 1, 0, 0, 1, 2)$, $\tau_{-5} = (1, 0, 0, 1, 0, 2)$, $\tau_{-4} = (1, 1, 0, 0, 1, 1)$, $\tau_{-3} = (0, 1, 1, 0, 1, 0)$, $\tau_{-2} = (1, 1, 1, 0, 0, 1)$, and $\tau_{-1} = (2, 0, 0, 1, 0, 1)$. Using [3, p. 402] we now observe that only τ_{-3} has form τ_Δ with $\Delta \in \mathfrak{D}_G$. Furthermore, $\Delta = \mathrm{A}_3\mathrm{A}_1$. Since $\mathcal{O}(\mathrm{A}_3\mathrm{A}_1) = \mathcal{H}(\mathrm{A}_3\mathrm{A}_1)$ by Remark 2.5, we get $e \in \mathcal{H}(\Delta)$.

Finally, suppose $e \in \mathcal{O}(\mathrm{D}_5)$. Then $e = e_1 + e_2 + e_3 + e_4 + e_5$. As $e \in \mathfrak{g}(\tau, 2)$ and all maximal tori of N_e are conjugate we may assume that $\tau = (2, 2, 2, 2, 2, r)$ for some $r \in \mathbb{Z}$. Let $\gamma = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $\gamma' = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$. Then $\gamma \pm \gamma' \notin \Sigma$ and $x_{-\gamma}(t)x_{-\gamma'}(t) \in G_e$ for all $t \in \Bbbk$. It follows that $e_{-\gamma} + e_{-\gamma'} \in \mathrm{Lie}(G_e)$. Hence $r \le -6$. Now let $\delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$ and $\delta' = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. Then again $\delta \pm \delta' \notin \Sigma$ and $x_\delta(t)x_{\delta'}(t) \in G_e$ for all $t \in \Bbbk$, so that $e_\delta + e_{\delta'} \in \mathrm{Lie}(G_e)$. It follows that $r \ge -14$.

As a result, $r \in \{-14, -13, -12, -11, -10, -9, -8, -7, -6\}$. Let τ_r be the W-conjugate of $\tau = (2, 2, 2, 2, 2, r)$ contained in $X_*^+(T)$. Direct computations show that $\tau_{-14} = (2, 0, 2, 2, 2, 2)$, $\tau_{-13} = (2, 1, 2, 1, 1, 1)$, $\tau_{-12} = (2, 2, 2, 0, 2, 0)$, $\tau_{-11} = (2, 2, 1, 1, 1, 1)$, $\tau_{-10} = (2, 2, 0, 2, 0, 2)$, $\tau_{-9} = (1, 2, 1, 1, 1, 2)$, $\tau_{-8} = (0, 2, 2, 0, 2, 2)$, $\tau_{-7} = (1, 1, 1, 1, 2, 2)$, $\tau_{-6} = (2, 0, 0, 2, 2, 2)$. Using [3, p. 402] we observe that only τ_{-10} has the form τ_Δ with $\Delta \in \mathfrak{D}_G$. Moreover, $\Delta = \mathrm{A}_5$. Since $\mathcal{O}(\mathrm{D}_5) = \mathcal{H}(\mathrm{D}_5)$ by Remark 2.5, we get $e \in \mathcal{H}(\Delta)$, completing the proof. □

2.6. We are now in a position to reduce proving statement (2) to the case where $e \in \mathfrak{g}(\tau, 2)$ is distinguished in $\mathfrak{g}$ and $\mathcal{O}(e)$ is a proper subset of its Hesselink stratum.

Suppose $e \in \mathfrak{g}(\tau, 2)$ is not distinguished in $\mathfrak{g}$. By Section 2.2, the group G_e contains a maximal torus T_0 commuting with $\tau(\Bbbk^\times)$. We also know that the centraliser $L = Z_G(T_0)$ is a standard Levi subgroup of G containing a maximal torus T' such that $\tau(\Bbbk^\times)T_0 \subseteq T'$. Let $L' = \mathcal{D}L$ and $\mathfrak{l}' = \mathrm{Lie}(L')$. By our discussion in Section 2.2, there exists $\tau'' \in \hat{\Lambda}_G(e) \cap X_*(T')$ such that $e = \sum_{i \ge 2} e_i$ where $e_i \in \mathfrak{l}'(\tau'', i)$ and $e_2 \ne 0$. If e is $(\mathrm{Ad}\, L)$-conjugate to e_2 then $e \in \mathcal{H}(\Delta)$ by Lemma 2.3. Therefore, we may assume that e and e_2 lie in different orbits of $\mathcal{H}_L(\tau'')$. Thanks to Section 2.4 we may also assume that the group S_e is semisimple.

If follows from [6, Table 4] that if H is a group of type D_r with $r \in \{4, 5\}$ then every Hesselink stratum of $\mathrm{Lie}(H)$ is a single $(\mathrm{Ad}\, H)$-orbit. Since the same holds for H of type E_6 and A_r with $r \ge 1$, the proper Levi subgroup L of G must have a component of type D_6, D_7, E_7, B_2, B_3 or C_3, where in the last three cases G is a group of type F_4. Thanks to [9, Tables 22.1.1–22.1.5] we now see that the case we consider can occur only when $p = 2$ and G is not of type E_6 or G_2.

Suppose $p = 2$ and G is of type E_7. As $\mathcal{H}_L(\tau'')$ contains more than one orbit, the group L' must have type D_6. By [6, Table 4], the Lie algebra $\mathfrak{l}'$ has a unique new distinguished nilpotent orbit, and the above discussion shows that this new orbit, labelled $(\mathrm{A}_3\mathrm{A}_2)_2$ in [9], must coincide with $\mathcal{O}_L(e)$. By [9, Lemma 12.6], there exists $\mu \in X_*(L')$ such that $e \in \mathfrak{l}'(\mu, 2)$ and the orbit $((\mathrm{Ad}\, P_L(\mu))e$ is dense in $\mathfrak{l}'(\mu, \ge 2)$. From this it is immediate that e is a $Z_L^\perp(\mu)$-semistable vector of $\mathfrak{l}'(\mu, 2)$, so that $\mu \in \hat{\Lambda}_L(e)$. Since $\hat{\Lambda}_L(e)$ contains $\hat{\Lambda}_G(e) \cap X^*(T')$ and $e_2 \ne 0$, the cocharacters μ and τ'' are L-conjugate. But then $G_e \subset P(\nu)$ and applying Lemma 2.2 with $\nu = \mu$ we get $\mu = \tau$. As a result, $e \in \mathcal{H}(\Delta)$.

Suppose $p = 2$ and G is of type E_8. Then L' has type D_6, D_7 of E_7. If L' has type D_6 we can repeat verbatim the argument from the previous paragraph to conclude that $e \in \mathcal{H}(\Delta)$. Suppose L' is of type D_7. By [6, Table 4], the Lie algebra $\mathfrak{l}'$ has two new orbits, denoted by $(A_3A_2)_2$ and $(D_4A_2)_2$ in [9, Table 15.3]. Since the orbit $\mathcal{O}_{L'}((A_3A_2)_2)$ is not distinguished in $\mathfrak{l}'$ it must be that $e \in \mathcal{O}_{L'}((D_4A_2)_2)$. As [9, Lemma 12.6] is also applicable for $e \in \mathcal{O}_{L'}((D_4A_2)_2)$, we can argue as in the second part of the previous paragraph to conclude that $e \in \mathcal{H}(\Delta)$.

2.7. Retain the assumptions of Section 2.6 and suppose that $p = 2$ and L' is of type E_7. Since e is distinguished in $\mathfrak{l}'$ and $\mathcal{O}_{L'}(e) \neq \mathcal{O}_{L'}(e_2)$, it follows from [9, Tables 22.1.2] that $e \in \mathcal{O}_{L'}((A_6)_2)$. Conjugating e by a suitable element of L' we may assume that

$$e = e_{56} + e_{67} + e_{134} + e_{234} + e_{345} + e_{245} + e_{123^24^25}, \tag{4}$$

where all summands e_γ involved in (4) are root vectors of L' with respect to T'; see [9, Table 14.1]. Let $W' = N_{L'}(T')/T'$, the Weyl group of L'. It is easy to check that the cocharacter $\mu = (-2, -2, -2, 6, -2, 4, -2)$ has the property that $e \in \mathfrak{l}'(\mu, 2)$. Since e is distinguished in $\mathfrak{l}'$, the group L'_e is unipotent and $N_{L'}(\Bbbk e) = \mu(\Bbbk^\times) Z_{L'}(e)$, so that $\mu(\Bbbk^\times)$ is a maximal torus of $N_{L'}(\Bbbk e)$. Since $e \in \mathfrak{g}(\tau, 2)$ and $\tau \in X_*(T')$ this entails that

$$(\operatorname{Ad} \mu(t))x = (\operatorname{Ad} \tau(t))x \quad \text{for all } x \in \mathfrak{l}' \text{ and } t \in \Bbbk^\times.$$

A direct computation shows that μ is W'-conjugate to $\tau_{\Delta'} = (2, 0, 0, 2, 0, 0, 2)$ which corresponds to the distinguished L'-orbit with label $E_7(a_4)$. The latter coincides with its Hesselink stratum $\mathcal{H}_{L'}(\Delta')$. From this it is immediate that $\mathfrak{l}'(\mu, 2)$ contains a Zariski open $Z_{L'}(\mu)$-orbit consisting of $Z^\perp_{L'}(\mu)$-semistable vectors. Since $e \notin \mathcal{H}_{L'}(\Delta')$ the orbit $(\operatorname{Ad} Z_{L'}(\mu)e$ is not dense in $\mathfrak{l}'(\mu, 2)$. Let

$$U_{L'}(\mu) := R_u(P_{L'}(\mu)).$$

Then $P_{L'}(\mu) = Z_{L'}(\mu)U_{L'}(\mu)$. Furthermore, we have $\operatorname{Lie}(U_{L'}(\mu)) = \mathfrak{l}'(\mu, \geq 2)$ and $\operatorname{Lie}(Z_{L'}(\mu)) = \mathfrak{l}'(\mu, 0)$. In type E_7, the present case was investigated in [9, pp. 209]. It was shown there that L'_e is a connected unipotent group of dimension 19 and $A := L'_e \cap Z_{L'}(\mu)$ is a 1-dimensional connected subgroup of L'_e with the property that $\operatorname{Lie}(A) = \mathfrak{l}'_e(\mu, 0)$.

Straightforward computations show that $[e, \mathfrak{l}'(\mu, 4)]$ has codimension 1 in $\mathfrak{l}'(\mu, 6)$ and $[e, \mathfrak{l}'(\mu, 2i)] = \mathfrak{l}'(\mu, 2i+2)$ for $i = 1$ and all $i \geq 3$. Since the group $U_{L'}(\mu)$ is generated by the root elements $x_\alpha(t) \in L'$ with $(\mu \mid \alpha) \geq 2$ we have that $(\operatorname{Ad} U_{L'}(\mu))e \subseteq e + \mathfrak{l}'(\mu, \geq 4)$. Since $[e, \mathfrak{l}'(\mu, 2i)] \subset T_e(\operatorname{Ad} U_{L'}(\mu)e)$ for all $i \geq 1$, the preceding remark yields that $T_e(\operatorname{Ad} U_{L'}(\mu)e)$ has codimension ≤ 1 in $\mathfrak{l}'(\mu, \geq 4)$. Therefore, $\dim U_{L'}(\mu)_e = \dim \operatorname{Lie}(U_{L'}(\mu)) - \dim T_e(\operatorname{Ad} U_{L'}(\mu)e) \leq \dim \mathfrak{l}'(\mu, 2) + 1 = 18$ (one should keep in mind that $P_{L'}(\mu)$ is a distinguished parabolic subgroup of L' and $\mathfrak{l}'(\mu, 0)$ has dimension 17). Hence $A \cdot U_{L'}(\mu)_e \subseteq P_{L'}(\mu)_e$ has dimension 18 or 19.

Since L'_e is a connected group of dimension 19 there are two possibilities one of which would be very bad for us: either $L'_e \subset P_{L'}(\mu)$ or $L'_e \not\subset P_{L'}(\mu)$ and $T_e(\operatorname{Ad} U_{L'}(\mu)e) = \mathfrak{l}'(\mu, \geq 4)$. In the second case we would have $(\operatorname{Ad} U_{L'}(\mu))e = e + \mathfrak{l}'(\mu, \geq 4)$ by Rosenlicht's theorem [14, Theorem 2].

Once again our main source of reference comes to the rescue: it is proved in [9, p. 208] that L'_e contains the 1-parameter unipotent subgroup

$$U = \{x_{\alpha_1}(c)x_{\alpha_1+\alpha_3}(c^2)x_{\alpha_2}(c)x_{\alpha_5}(c)x_{\alpha_7}(c) \mid c \in \Bbbk\}.$$

The Lie algebra of U is spanned by $v = e_1 + e_2 + e_5 + e_7 \in \mathfrak{l}'(\mu, -2)$ (and one can check directly that $[e, v] = 0$). Therefore, $L'_e \not\subset P_{L'}(\mu)$. As a byproduct we obtain that the orbit $(\operatorname{Ad} P_{L'}(\mu))e$ has codimension 1 on $\mathfrak{l}'(\mu, \geq 2)$.

In type E_8 the present case was investigated in [9, pp. 247, 248] where the element in (4) was replaced by its $(\operatorname{Ad} G)$-conjugate

$$e' = e_1 + e_3 + e_4 + e_5 + e_6 + e_7 + e_{12342^52^267}.$$

The cocharacter μ used above was replaced by $\mu' = (2, -14, 2, 2, 2, 2, 2, -3)$. (In [9, p. 248], the torus $\mu'(\Bbbk^\times)$ is denoted by $\widetilde{T}$.) Direct computations show that there is $w \in W(E_8)$ such that $w(\mu') = (0, 0, 0, 1, 0, 1, 0, 2) = \tau_{\Delta'}$ where $\Delta' = E_7(a_4)$. Let $\tilde{\alpha} = 23465432$, the highest root of Σ with respect to Π, and denote by $\tilde{\alpha}^\vee$ the corresponding coroot in $X_*(T)$, so that $(\operatorname{Ad} \tilde{\alpha}^\vee(t))(e_\gamma) = t^{\langle \gamma, \tilde{\alpha}^\vee \rangle} e_\gamma$ for all $t \in \Bbbk^\times$ and $\gamma \in \Sigma$. The adjoint action of $\tilde{\alpha}^\vee(\Bbbk^\times)$ endows $\mathfrak{g}$ with a short $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\dim \mathfrak{g}_{\pm 2} = \Bbbk e_{\pm\tilde{\alpha}}$ and $\mathfrak{g}_0 = \operatorname{Lie}(\tilde{L})$ where $\tilde{L} = Z_G(\tilde{\alpha}^\vee)$.

The derived subgroup $\tilde{L}'$ of $\tilde{L}$ has type E_7 and $\tilde{L} = T_0\tilde{L}'$ where $T_0 = \tilde{\alpha}^\vee(\Bbbk^\times)$ is a 1-dimensional central torus of $\tilde{L}$. The type-A_1 subgroup S generated by $x_{\pm\tilde{\alpha}}(\Bbbk)$ and T_0 commutes with $\tilde{L}'$. We pick $\sigma \in N_S(\tilde{\alpha}^\vee(\Bbbk^\times))$ such that $\sigma(\tilde{\alpha}^\vee) = -\tilde{\alpha}^\vee$. Then $(\operatorname{Ad}\sigma)(\mathfrak{g}_1) = \mathfrak{g}_{-1}$ implying that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic as $(\operatorname{Ad}\tilde{L}')$-modules (both modules have dimension 56 and are irreducible over $\tilde{L}'$). The parabolic subgroup $P(\tilde{\alpha}^\vee) = \tilde{L}Q$, where $Q = R_u(P(\tilde{\alpha}^\vee))$, has the property that $\operatorname{Lie}(Q) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathcal{D}Q = x_{\tilde{\alpha}}(\Bbbk)$.

Computations in [9, p. 248] show that $\widetilde{T} = \mu'(\Bbbk^\times) \subset T \cap \tilde{L}'$, and $Q \cap G_{e'}$ contains an 8-dimensional abelian connected unipotent subgroup $V = \prod_{i=1}^{8} V_i$ such that each V_i is 1-dimensional subgroup of V normalised by the torus $\mu'(\Bbbk^\times)$. The V_i's are described explicitly in [loc. cit.] and it is straightforward to check that $\mu'(\Bbbk^\times)$ acts on V_1, V_2, V_3, V_4, V_5, V_6, V_7 and V_8 with weights 11, 5, 3, 9, 9, 1, 3 and 7, respectively. As $\mu'(\Bbbk^\times) \subset \tilde{L}'$ commutes with S and $e' \in \operatorname{Lie}(\tilde{L}')$, the group $(\operatorname{Ad}\sigma)(V) = \prod_{i=1}^{8}(\operatorname{Ad}\sigma)(V_i) \subset G_{e'}$ has the same properties. As a consequence, both V and $(\operatorname{Ad}\sigma)(V)$ are contained in $P(\mu')_{e'}$.

As $e' \in \mathrm{Lie}(\tilde{L}')$ we also have that $S \subset P(\mu')_{e'}$. Since e and e' are $(\mathrm{Ad}\, G)$-conjugate, our discussion at the beginning of this subsection shows that $P_{\tilde{L}'}(\mu')_{e'}$ has codimension 1 in $\tilde{L}'_{e'}$. In particular, $G_{e'} \not\subset P(\mu')$.

On the other hand, we have that

$$\mathrm{Lie}(V) \oplus (\mathrm{Ad}\,\sigma)(\mathrm{Lie}(V)) \oplus \mathrm{Lie}(S) \oplus \mathrm{Lie}(P_{\tilde{L}'}(\mu')_{e'}) \subseteq \mathrm{Lie}(P(\mu')_{e'}),$$

and the subspace on the left has dimension $8+8+3+18=37$. As $G_{e'}$ is a group of dimension 38 by [9, Table 22.1.1], we have that $P(\mu')_e$ has codimension 1 in $G_{e'}$ This, in turn, implies that the orbit $(\mathrm{Ad}\, P(\mu')e'$ has codimension 1 in $\mathfrak{g}(\mu', \geq 2)$ and $(\mathrm{Ad}\, R_u(P(\mu'))e' = e' + \mathfrak{g}(\mu', \geq 4)$ (since this will not be required in what follows we omit the details).

We mention for completeness that if $\nu \in X_*(T)$ is such that $e' \in \mathfrak{g}(\nu, 2)$ and $G_{e'} \subset P(\nu)$ then the root vectors $e_{\pm\tilde{\alpha}} \in \mathrm{Lie}(G_{e'})$ must have weight 0 with respect to ν. Since $G_{e'} \not\subset P(\mu')$, we have thus excluded the only available option for ν, namely, $\nu = \mu'$. Therefore, $\mathcal{O}((\mathrm{A}_6)_2) \cap \big(\bigcup_{\Delta\in\mathfrak{D}_G} \mathfrak{g}_2^{\Delta,!}\big) = \varnothing$, that is, the present case cannot occur in types E_7 and E_8.

2.8. Retain the assumptions of Section 2.6 and suppose that $p = 2$ and L' is of type F_4. Since this case has already been treated in [17], our goal here is to offer a different proof. In view of our remarks in Section 2.6 we may assume that the Hesselink stratum of $\mathcal{N}(\mathfrak{l}')$ has more than one orbit and L' is of type B_2, B_3 or C_3. The tables in [6] show that in these cases $\mathfrak{l}'$ has a unique new distinguished nilpotent orbit.

The corresponding nilpotent orbits of $\mathfrak{g}$ are denoted in [9, Table 22.1.4] by $(\tilde{\mathrm{A}}_1)_2$, $(\mathrm{B}_2)_2$ and $(\tilde{\mathrm{A}}_2)_2$, respectively. Parts (i) and (iii) of the proof Lemma 16.9 in [9] show that in the first two cases there exists $\mu \in X_*(L')$ such that $e \in \mathfrak{l}'(\mu, 2)$ and the orbit $((\mathrm{Ad}\, P_L(\mu))e$ is dense in $\mathfrak{l}'(\mu, \geq 2)$. As before, this enables us to deduce that e is $Z_L^{\perp}(\mu)$-semistable in $\mathfrak{l}'(\mu, 2)$, so that $\mu \in \hat{\Lambda}_L(e)$. As $\hat{\Lambda}_L(e)$ contains $\hat{\Lambda}_G(e) \cap X^*(T')$ and $e \in \mathfrak{g}(\tau, 2)$, applying Lemma 2.2 gives $\mu = \tau$. Hence $e \in \mathcal{H}(\Delta)$.

Suppose $e \in \mathcal{O}((\tilde{\mathrm{A}}_2)_2)$. Then we may assume that $e = e_{0121} + e_{1111} + e_{2342}$ and T' is the torus used in the proof of part (ii) of [9, Lemma 16.9]. Let $\tau = (a_1, a_2, a_3, a_4)$ where $a_i \in \mathbb{Z}$. As $e \in \mathfrak{g}(\mu, 2)$ we have $a_2 + 2a_3 + a_4 = 2$, $a_1 + a_2 + a_3 + a_4 = 2$ and $2a_1 + 3a_2 + 4a_3 + 2a_4 = 2$. Solving this system of linear equations gives $\tau = (r, -2-r, r, 4)$ where $r \in \mathbb{Z}$. By [9, p. 274], the group G_e contains $x_{\pm\alpha_2}(t)$ and $x_{\alpha_1}(t)x_{\alpha_3}(t)$ for all $t \in \Bbbk$. Therefore, $\mathrm{Lie}(G_e)$ contains $e_{\pm\alpha_2}$ and $e_{\alpha_1} + e_{\alpha_3}$. Since $G_e \subset P(\tau)$, we must have $-2-2r = 0$ and $r \geq 0$. This shows that τ does not exist, that is, the present case cannot occur.

2.9. From now on we may assume that $e \in \mathfrak{g}(\tau_\Delta, 2)$ is a distinguished nilpotent element of $\mathfrak{g}$. If $\mathcal{O}(e) = \mathcal{H}(\Delta')$ then there is $\mu \in \hat{\Lambda}_G(e)$ such that $e \in \mathfrak{g}(\mu, 2)$. As Lemma 2.2 is still applicable in the present case we get $\mu = \tau_\Delta$ forcing $e \in \mathcal{H}(\Delta)$. In other words, statement (2) holds for e.

This shows that we may assume further that $\mathcal{O}(e) \subsetneq \mathcal{H}(\Delta)$. Thanks to the classification results of [9] it remains to consider the case where $e \in \mathcal{N}(\mathfrak{g})$ is distinguished and nonstandard (see Remark 2.5 for more detail). No such orbits exist when G has type E_6 and when G is of type G_2 and $p = 2$.

If G is of type E_7 then $p = 2$ and $\mathcal{O}(e) = \mathcal{O}((A_6)_2)$. In Section 2.7, we have shown that this case cannot occur in our situation.

Lemma 2.8. *Suppose $e \in \mathfrak{g}(\tau_\Delta, 2)$ is distinguished in $\mathcal{N}(\mathfrak{g})$ and $G_e \subset P(\tau_\Delta)$. If $e' \in \mathcal{O}(e) \cap \mathfrak{g}(\mu, 2)$, where $\mu \in X_*(G)$, then there is $g \in G$ such that $e' = (\operatorname{Ad} g)(e)$ and $\mu = g\tau_\Delta g^{-1}$. Consequently, $G_{e'} \subset P(\mu)$.*

Proof. Let $g \in G$ be such that $(\operatorname{Ad} g)e = e'$. Then $e' \in \mathfrak{g}(g\tau_\Delta g^{-1}, 2)$ and $G_{e'} \subset P(g\tau_\Delta g^{-1})$. Since the group $(G_{e'})^\circ$ is unipotent and all maximal tori of $N_G(\Bbbk e')^\circ = \tau_{\Delta'}(\Bbbk^\times)(G_{e'})^\circ$ are conjugate, there is $u \in (G_{e'})^\circ$ such that $ug\tau_\Delta g^{-1}u^{-1} = \mu$. Hence $(\operatorname{Ad} ug)(e) = (\operatorname{Ad} u)(e') = e'$ and $G_{e'} = uG_{e'}u^{-1} \subset P(ug\tau_\Delta g^{-1}u^{-1}) = P(\mu)$. $\square$

It remains to investigate the case where (Σ, p) is one of $(E_8, 3)$, $(E_8, 2)$, $(F_4, 2)$ or $(G_2, 3)$. If $(\Sigma, p) = (E_8, 3)$ then $\mathcal{O}(e) = \mathcal{O}((A_7)_3)$. Recall our standing assumption that $e \in \mathfrak{g}(\tau, 2)$, where $\tau = \tau_\Delta$, and $G_e \subset P(\tau)$. By [9, Table 4.1], we may assume that e is $(\operatorname{Ad} G)$-conjugate to

$$e'' = e_{567} + e_{1234} + e_{1345} + e_{3456} + e_{2456} + e_{234^25} + e_{678} + e_{45678}.$$

The cocharacter $\mu = (0, 2, 2, -2, 2, 0, 0, 2) \in X_*(T)$ has the property that $e \in \mathfrak{g}(\mu, 2)$ and is W-conjugate to $\tau_{\Delta'} = (0, 0, 0, 2, 0, 0, 0, 2)$ with $\Delta' = E_8(b_6)$; see [9, p. 210]. Replacing e'' by its $N_G(T)$-conjugate, e' say, we may assume that $e' \in \mathfrak{g}(\tau_{\Delta'}, 2)$. By Lemma 2.8, there is $g \in G$ such that $e' = (\operatorname{Ad} g)(e)$ and $\tau_{\Delta'} = g\tau_\Delta g^{-1}$. Since both $\tau = \tau_\Delta$ and $\tau_{\Delta'}$ lie in $X_*^+(T)$ it must be that $\Delta = \Delta'$.

We claim that contrary to our standing assumption on τ the group G_e is not contained in $P(\tau) = P(\tau_{\Delta'})$. Indeed, it follows from [9, Theorem 3.2] that the nonempty open subset $\mathcal{V}(\tau, 2)_{ss}$ of $\mathfrak{g}(\tau, 2)$ contains the open $Z_G(\tau)$-orbit of $\mathfrak{g}(\tau, 2)$, we call it V. It has the property that $(\operatorname{Ad} P(\tau))v$ is dense in $\mathfrak{g}(\tau, \geq 2)$ for every $v \in V$. If $x \in V$ then $\dim G_x = 28$, whilst $\dim G_{e'} = 30$ by [9, Table 22.1.1]. Since $e' \in \mathcal{O}((A_7)_3)$ it must be that $e \notin \mathcal{V}(\tau, 2)_{ss}$. But then the orbit $(\operatorname{Ad} Z_G(\tau))e$ is not dense in $\mathfrak{g}(\tau, 2)$ implying that $[e, \mathfrak{g}(\tau, 0)]$ is a proper subspace of $\mathfrak{g}(\tau, 2)$.

In the present case, the normalised Killing form κ is nondegenerate and induces a perfect pairing between $\mathfrak{g}(\tau, 2)$ and $\mathfrak{g}(\tau, -2)$. This yields that $\mathfrak{g}_e(\tau, -2) = [e, \mathfrak{g}(\tau, 0)]^\perp$ is nonzero (a different proof can be found in [9, p. 211]). On the other hand, $\dim \mathfrak{g}_e = \dim G_e = 30$ by [16, Table 10]. If $G_e \subset P(\tau)$ then $\mathfrak{g}_e = \operatorname{Lie}(G_e)$ is contained in $\mathfrak{g}(\tau, \geq 0) = \operatorname{Lie}(P(\tau))$. As $\mathfrak{g}_e(-2) \neq 0$, we reach a contradiction. This shows that the present case cannot occur. In other words, $\mathcal{O}((A_7)_3)$ has no elements contained in the union of $\mathfrak{g}_2^{\Delta,!}$ with $\Delta \in \mathfrak{D}_G$.

2.10. Suppose $(\Sigma, p) = (\mathrm{E}_8, 2)$. In this case we have to consider the new distinguished nilpotent orbits, namely, $\mathcal{O}((\mathrm{D}_5\mathrm{A}_2)_2)$, $\mathcal{O}((\mathrm{D}_7(\mathrm{a}_1)_2)$ and $\mathcal{O}((\mathrm{D}_7)_2)$. Thanks to [9, Table 14.1] we may choose, in the respective cases, the representatives

$$e' = e_{12345} + e_{2342^25} + e_{13456} + e_{23456} + e_{34567} + e_{24567} + e_{78} + e_{678},$$
$$e' = e_5 + e_{45} + e_{2342^2567} + e_{13} + e_{2456} + e_{3456} + e_{78} + e_8,$$
$$e' = e_1 + e_{234} + e_{345} + e_{245} + e_{456} + e_{567} + e_{678} + e_{12345678}.$$

It is easy to check that in the first case $e' \in \mathfrak{g}(\tau_{\Delta'}, 2)$ where $\tau_{\Delta'} = (0, 0, 0, 0, 2, 0, 0, 2)$ is attached to the orbit $\mathcal{O}(\mathrm{D}_5\mathrm{A}_2)$. By Lemma 2.8, there is $g \in G$ be such that $e' = (\mathrm{Ad}\, g)e$ and $\tau_{\Delta'} = g\tau g^{-1}$. Since $\dim G_{e'} = 34 = \dim \mathfrak{g}(\tau_{\Delta'}, 0)$ by [9, Table 22.1.1] and $G_{e'} \subset P(\tau_{\Delta'})$ by Lemma 2.8, the orbit $(\mathrm{Ad}\, P(\tau_{\Delta'}))e$ must be open in $\mathfrak{g}(\tau_{\Delta'}, \geq 2)$. But then e must lie in the open $(\mathrm{Ad}\, Z_G(\tau_{\Delta'}))$-orbit of $\mathfrak{g}(\tau_{\Delta'}, 2)$. As a consequence, e belongs to the nonempty open subset $\mathcal{V}(\tau_{\Delta'}, 2)_{ss}$ of $\mathfrak{g}(\tau_{\Delta'}, 2)$. Since $\tau = \tau_\Delta$ and $\tau_{\Delta'}$ are G-conjugate and lie in $X_*^+(T)$ we conclude that $\Delta' = \Delta$. Hence $e \in \mathcal{H}(\Delta)$.

In the second case, one checks that $e' \in \mathfrak{g}(\tau_{\Delta'}, 2)$ where $\tau_{\Delta'} = (2, 0, 0, 0, 2, 0, 0, 2)$ is attached to the orbit $\mathcal{O}(\mathrm{D}_7(\mathrm{a}_1))$. By [9, Table 22.1.1], we have $\dim G_e = 26 = \dim \mathfrak{g}(\tau_{\Delta'}, 0)$. Since $(G_e)^\circ$ is unipotent, applying Lemma 2.8 and arguing as in the previous case we deduce that $\tau = \tau_\Delta$ and $\tau_{\Delta'}$ are G-conjugate and $e \in \mathcal{V}(\tau_{\Delta'}, 2)_{ss}$. Therefore, $e \in \mathcal{H}(\Delta)$ as wanted.

The case where $e \in \mathcal{O}((\mathrm{D}_7)_2)$ is more complicated. First we note that $e' \in \mathfrak{g}(\mu, 2)$ where $\mu = (2, -4, -4, 10, -4, -4, 10, -4) \in X_*(T)$. One checks directly that μ is W-conjugate to $(2, 0, 0, 2, 0, 0, 2, 2) = \tau_{\Delta''}$, where $\Delta'' = \mathrm{E}_8(\mathrm{b}_4)$. Since both $\tau = \tau_\Delta$ and $\tau_{\Delta''}$ lie in $X_*^+(T)$, it follows from Lemma 2.8 that $\tau = \tau_{\Delta''}$ and there is $v \in \mathcal{O}((\mathrm{D}_7)_2) \cap \mathfrak{g}(\tau, 2)$ such that $G_v \subset P(\tau)$. Let

$$v' = e_{13} + e_{234} + e_{345} + e_{245} + e_{567} + e_{456} + e_7 + e_8,$$

an element of $\mathfrak{g}(\tau, 2)$. By [9, Tables 13.3 and 22.1.1], we have that $v' \in \mathcal{O}(\mathrm{E}_8(\mathrm{b}_4))$ and $\dim G_{v'} = 18 = \mathfrak{g}(\tau, 0)$. Since $G_{v'} \subset P(\tau)$ by [9, Theorem 15.1(ii)] the orbit $(\mathrm{Ad}\, P(\tau)v'$ is open in $\mathfrak{g}(\tau, \geq 2)$. It follows that the orbit $V' := (\mathrm{Ad}\, Z_G(\tau))v'$ is open in $\mathfrak{g}(\tau, 2)$. As $v \notin \mathcal{O}(v')$ we have that $v \notin V'$. Hence $\dim\, (\mathrm{Ad}\, Z_G(\tau))v < \dim \mathfrak{g}(\tau, 2)$ and, as a consequence, $[v, \mathfrak{g}(\tau, 0)]$ is a proper subspace of $\mathfrak{g}(\tau, 2)$.

By [9, Table 13.4], the maps $\mathrm{ad}\, v' : \mathfrak{g}(\tau, 4) \to \mathfrak{g}(\tau, 6)$ and $\mathrm{ad}\, v' : \mathfrak{g}(\tau, 8) \to \mathfrak{g}(\tau, 10)$ are not surjective (this also follows from the fact that $0 \neq (v')^{[2]} \in \mathfrak{g}_{v'}(\tau, 4)$ and $0 \neq (v')^{[4]} \in \mathfrak{g}_{v'}(\tau, 8)$ which is easy to see directly by applying Jacobson's formula for $[p]$-th powers with $p = 2$). Using the perfect pairings between $\mathfrak{g}(\tau, i)$ and $\mathfrak{g}(\tau, -i)$ induced by the normalised Killing form κ (which is nondegenerate in the present case) one observes that $\mathfrak{g}_{v'}(\tau, r) \neq 0$ for $r \in \{-6, -10\}$. As $v \in \mathfrak{g}(\tau, 2)$ lies in the Zariski closure of $(\mathrm{Ad}\, Z_G(\tau))v'$, the semicontinuity of the nullity of a rectangular matrix yields that $\mathfrak{g}_v(\tau, -6) \neq 0$ and $\mathfrak{g}_v(\tau, -10) \neq 0$, whilst our earlier remarks entail that $\mathfrak{g}_v(\tau, -2) = [v, \mathfrak{g}(\tau, 0)]^\perp \neq 0$. Therefore, $\dim \mathfrak{g}_v(\tau, < 0) \geq 3$.

As $v \in \mathcal{O}(e) = \mathcal{O}((\mathrm{D}_7)_2)$ it follows from [16, Table 10] that $\dim \mathfrak{g}_v = 24$. If $G_v \subset P(\tau)$ then [9, Table 22.1.1] shows that $\mathrm{Lie}(G_v)$ is a Lie subalgebra of dimension 22 in $\mathfrak{g}(\tau, \geq 0)$. But then

$$\dim \mathfrak{g}_v = \dim \mathfrak{g}_v(\tau, <0) + \dim \mathfrak{g}_v(\tau, \geq 0) \geq 3 + 22 = 25.$$

This contradiction shows that this case does not occur, that is, $e \in \mathcal{O}((\mathrm{D}_7)_2)$ cannot appear as an element of $\mathfrak{g}_2^{\Delta,!}$ with $\Delta \in \mathfrak{D}$. We thus conclude that (2) holds in type E_8.

2.11. Suppose (Σ, p) is one of $(\mathrm{F}_4, 2)$ or $(\mathrm{G}_2, 3)$. These cases have been treated in [17] by computational methods. The argument below will provide an alternative proof. In type F_4, we only need to consider the nonstandard distinguished orbits with labels $(\tilde{\mathrm{A}}_2\mathrm{A}_1)_2$, $(\mathrm{C}_3(\mathrm{a}_1))_2$ and $(\mathrm{C}_3)_2$. Indeed, [9, Theorem 16.1(ii)] implies that every standard distinguished orbit in $\mathcal{N}(\mathfrak{g})$ has a representative $e \in \mathfrak{g}(\mu, 2)$ such that the orbit $(\mathrm{Ad}\, P(\mu))e$ is open in $\mathfrak{g}(\mu, \geq 2)$, thereby forcing $e \in \mathcal{V}(\mu, 2)_{ss}$. Thanks to Lemma 2.8 we also know that μ is G-conjugate to $\tau = \tau_\Delta$. This yields $e \in \mathcal{H}(\Delta)$.

In view of [9, Table 14.1] we may assume that e is $(\mathrm{Ad}\, G)$-conjugate to one of the elements $e(i)$ with $i \in \{1, 2, 3\}$, where

$$\begin{aligned} e(1) &= e_{234} + e_{1121} + e_{1220} + e_{0122} && \text{in type } (\tilde{\mathrm{A}}_2\mathrm{A}_1)_2,\\ e(2) &= e_{123} + e_{0122} + e_{0120} + e_{1222} && \text{in type } (\mathrm{C}_3(\mathrm{a}_1))_2,\\ e(3) &= e_{123} + e_{0120} + e_4 + e_{1222} && \text{in type } (\mathrm{C}_3)_2. \end{aligned}$$

By Lemma 2.8, there is a unique $\tau_i \in X_*(T)$ conjugate to τ and such that $e(i) \in \mathfrak{g}(\tau_i, 2)$ and $G_{e(i)} \subset P(\tau_i)$. Direct computations show that $\tau_1 = (2, 2, -2, 2)$, $\tau_2 = (2, -2, 2, 0)$ and $\tau_3 = (6, -10, 6, 2)$. Using [2, Planche VIII] one checks directly that in the last two cases $G_{e(i)}$ contains $x_{\alpha_2}(t)$ for every $t \in \Bbbk$. Then the simple root vector e_2 lies in $\mathrm{Lie}(G_e) \cap \mathfrak{g}(\tau, <0)$. As noted in [9, p. 215] we have $x_{-\alpha_1}(t)x_{\alpha_3}(t) \in G_{e(1)}$ for all $t \in \Bbbk$ which yields $\mathrm{Lie}(G_{e(1)}) \cap \mathfrak{g}(\tau, -2) \neq 0$. So none of the three cases can occur in our situation.

Finally, suppose $(\Sigma, p) = (\mathrm{G}_2, 3)$. Thanks to [9, Proposition 13.5] we only need to consider the orbit $\mathcal{O}((\tilde{\mathrm{A}}_1)_3)$ which has a nice representative $e' = e_{21} + e_{32}$; see [9, Table 14.1]. As e is distinguished and $e' \in \mathfrak{g}(\mu, 2)$, where $\mu = (2 - 2) \in X_*(T)$, it follows from Lemma 2.8 that μ is G-conjugate to τ and $G_{e'} \subset P(\mu)$. But $x_{\alpha_2}(t) \in G_{e'}$ for all $t \in \Bbbk$, forcing $\mathrm{Lie}(G_{e'}) \cap \mathfrak{g}(\mu, -2) \neq 0$. This contradiction shows that this case cannot occur either.

Summarising, we have proved that statement (2) holds for all simple algebraic groups of exceptional types over algebraically closed fields of characteristic $p \geq 0$. This means that $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$. Since $\mathcal{N}(\mathfrak{g}) \subset \mathrm{Lie}(\mathcal{D}G)$ and $Z(G)$ acts trivially on $\mathfrak{g}$, proving Theorem 1.1 reduces quickly to the case where G is a simple algebraic group. For G exceptional, the theorem is a direct consequence of statement (2). For G classical, the result is known from [11, Theorem A.2]. The groups of type G_2, F_4 and E_6 were treated earlier in [17].

2.12. We would like to finish this paper by a brief discussion of the unipotent analogues of the Lusztig–Xue pieces, $\mathrm{LX}_u(\Delta)$, introduced by Lusztig in [11, 2.3].

Let $\mathfrak{U}(G)$ denote the unipotent variety of G, the set of all $(\operatorname{Ad} G)$-unstable elements of G. The Hesselink stratification

$$\mathfrak{U}(G) = \bigsqcup_{\Delta\in\mathfrak{D}_G} \mathcal{H}_u(\Delta)$$

is described in [4] as follows. Let $\tau = \tau_\Delta$ and write $P(\tau) = Z_G(\tau)U(\tau)$ where $U(\tau) = R_u(P(\tau))$. Given $k \in \mathbb{Z}_{>0}$ we denote by $U_{\geq k}(\tau)$ the connected normal subgroup of $U(\tau)$ generated by all $x_\gamma(t)$ with $t \in \Bbbk$ and all $\gamma \in \Sigma$ such that $\langle\gamma, \tau\rangle \geq k$. It is well known that the factor group $U_{\geq 2}(\tau)/U_{\geq 3}(\tau)$ is endowed with a natural vector space structure over $\Bbbk$ and $\operatorname{Ad} Z_G(\tau)$ acts $\Bbbk$-linearly on $U_{\geq 2}(\tau)/U_{\geq 3}(\tau)$. Furthermore, $U_{\geq 2}(\tau)/U_{\geq 3}(\tau) \cong \mathfrak{g}(\tau, 2)$ as $(\operatorname{Ad} Z_G(\tau))$-modules, and there is a module isomorphism $\bar{\pi}_\Delta : U_{\geq 2}(\tau)/U_{\geq 3}(\tau) \xrightarrow{\sim} \mathfrak{g}(\tau, 2)$ sending a coset $\prod_{i=1}^r x_{\beta_i}(t_i)U_{\geq 3}(\tau)$ with $\langle\beta_i, \tau\rangle = 2$ to $\sum_{i=1}^r t_i e_{\beta_i}$, where e_{β_i} are root vectors independent of the choice of $(t_1, \ldots, t_r) \in \Bbbk^r$; see [4, 3.6], [9, Lemma 18.1] or [11, 2.2]. Composing $\bar{\pi}_\Delta$ with the canonical homomorphism $U_{\geq 2}(\tau) \to U_{\geq 2}(\tau)/U_{\geq 3}(\tau)$ we obtain a natural surjection $\pi_\Delta : U_{\geq 2}(\tau) \twoheadrightarrow \mathfrak{g}(\tau, 2)$.

It follows from [4, Theorems 3.6 and 5.2] that

$$\mathcal{H}_u(\Delta) = (\operatorname{Ad} G)\big(\pi_\Delta^{-1}(\mathcal{V}(\tau_\Delta, 2)_{ss})\big)$$

for every $\Delta \in \mathfrak{D}_G$. In [11, 2.3], the unipotent pieces $\mathrm{LX}_u(\Delta)$ are defined in a similar fashion except that $\mathcal{V}(\tau_\Delta, 2)_{ss}$ is replaced by an a priori larger set $\mathfrak{g}_2^{\Delta,!}$. More precisely,

$$\mathrm{LX}_u(\Delta) = (\operatorname{Ad} G)(\pi_\Delta^{-1}(\mathfrak{g}_2^{\Delta,!})).$$

In [11, Theorem 2.4], Lusztig proved that

$$\mathfrak{U}(G) = \bigsqcup_{\Delta\in\mathfrak{D}_G} \mathrm{LX}_u(\Delta) \tag{5}$$

when G is a simple algebraic group of type A, B, C or D, and he expected that (5) would continue to hold for all connected reductive groups. Our next result shows that this expectation was correct.

Corollary 2.9. *Let G be a connected reductive group over an algebraically closed field. Then* $\mathrm{LX}_u(\Delta) = \mathcal{H}_u(\Delta)$ *for all* $\Delta \in \mathfrak{D}_G$ *and* (5) *holds for* $\mathfrak{U}(G)$.

Proof. Theorem 1.1 in conjunction with the preceding discussion shows that $\mathrm{LX}_u(\Delta) = \mathcal{H}_u(\Delta)$ for all $\Delta \in \mathfrak{D}_G$. Since the Hesselink strata $\mathcal{H}_u(\Delta)$ with $\Delta \in \mathfrak{D}_G$ form a partition of $\mathfrak{U}(G)$ by [4, Theorem 5.2], we deduce that (5) holds for $\mathfrak{U}(G)$. □

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In memoriam
Gary Seitz