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MULTIPLICITY-FREE REPRESENTATIONS OF THE PRINCIPAL A_1-SUBGROUP IN A SIMPLE ALGEBRAIC GROUP

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We dedicate this paper to the memory of the esteemed mathematician, Gary Seitz, whose work and mentorship have a continuing impact on the field and on our lives

Let G be a simple algebraic group defined over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. For $p \geq h$, the Coxeter number of G, any regular unipotent element of G lies in an A_1-subgroup of G; there is a unique G-conjugacy class of such subgroups and any member of this class is a so-called "principal A_1-subgroup of G". Here we classify all irreducible $\Bbbk G$-modules whose restriction to a principal A_1-subgroup of G has no repeated composition factors, extending the work of Liebeck, Seitz and Testerman which treated the same question when $\Bbbk$ is replaced by an algebraically closed field of characteristic zero.

1. Introduction

We consider a question in the representation theory and subgroup structure of simple algebraic groups defined over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The main aim of our work is to generalise the results of [Liebeck et al. 2015; 2022; 2024], where the authors consider so-called "multiplicity-free subgroups" of simple algebraic groups defined over an algebraically closed field K of characteristic zero. More precisely, the authors consider triples (X, Y, V) where X and Y are simple algebraic groups defined over K with X a closed subgroup of Y, and V is an irreducible KY-module such that the KX-module V, obtained by restricting the action of Y to the subgroup X, is a sum of nonisomorphic irreducible KX-modules (a so-called "multiplicity-free" KX-module). The above cited articles provide a complete classification of such triples when either X has rank 1 and does not lie in a

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proper parabolic subgroup of Y, or Y is a classical group with natural module W and X is of type A_ℓ acting irreducibly on W. Note that the case where X acts irreducibly (and hence multiplicity freely) on V was settled by Dynkin [1952] in characteristic zero, and by Seitz [1987] and Testerman [1988] in positive characteristic.

The ultimate far-reaching aim of what we undertake in this paper would be to investigate the "multiplicity-free" triples (X, Y, V) as in [Liebeck et al. 2015; 2022; 2024], described above, replacing the field K by the field $\Bbbk$ of positive characteristic p, and considering composition factors rather than summands. The proofs in [Liebeck et al. 2022; 2024] use induction on the rank of the group X; the case where X is simple of rank 1 is considered in [Liebeck et al. 2015]. Here we treat the rank-1 case for the groups defined over $\Bbbk$, but consider a slightly more general setting than would strictly speaking be required for use in an inductive set-up. Namely, we consider all simple algebraic groups G (classical and exceptional), defined over $\Bbbk$, and A a closed A_1-subgroup of G containing a regular unipotent element of G, which we will call a "principal A_1-subgroup of G". (Such subgroups exist precisely when $p \geq h$, the Coxeter number of G; see [Testerman 1995, Corollary 0.5 and Theorem 0.1]. In addition, there is at most one conjugacy class of principal A_1-subgroups in G; see [Seitz 2000, Theorem 1.1].) We then determine all irreducible $\Bbbk G$-modules V such that the set of composition factors of the $\Bbbk A$-module V consists of nonisomorphic $\Bbbk A$-modules, and obtain a classification analogous to [Liebeck et al. 2015, Theorem 1]. Much of the analysis follows the same line of reasoning as that used in [Liebeck et al. 2015]; the main differences and difficulties arise from the lack of precise knowledge about the dimensions of irreducible $\Bbbk G$-modules and the multiplicities of their weights. In addition, while irreducible $\Bbbk A_1$-modules are completely understood, the description of the set of weights is not as simple as in characteristic zero. In [Liebeck et al. 2022; 2024], another essential ingredient of the proof is the work of Stembridge [2003], where he determines when the tensor product of two irreducible modules for a simple algebraic group defined over the field K is a direct sum of nonisomorphic irreducible modules. There has been recent progress on the analogous question for the simple groups defined over fields of positive characteristic in [Gruber 2021] and [Gruber and Mancini 2024]. The combination of the rank-1 theorem proven here and the work of Gruber and Mancini lays the foundation for the study of multiplicity-free subgroups of higher rank for groups defined over fields of positive characteristic.

In order to state our main result, we introduce some notation; further notation will be set up in Section 2. Fix G a simply connected simple algebraic group of rank $\ell \geq 2$ defined over the algebraically closed field $\Bbbk$. We fix a maximal torus T of G, a

Borel subgroup B of G with $T \subset B$, the root system Φ of G with respect to T, and a base $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of Φ, associated with the choice of Borel subgroup B. Let Φ^+ be the associated set of positive roots. Let $X(T)$ denote the associated weight lattice, with fundamental dominant weights $\{\omega_1, \dots, \omega_\ell\}$ defined by the choice of Π. (We label Dynkin diagrams as in [Bourbaki 2002].) Throughout, we fix $\lambda \in X(T)$ a dominant weight and set $V = L(\lambda)$, the irreducible $\Bbbk G$-module with highest weight λ. Assume that $p \geq h$, so that each regular unipotent element of G lies in an A_1-subgroup of G. Let $A \subset G$ be a principal A_1-subgroup of G. Fix a maximal torus T_A of A with $T_A \subset T$ and $T_A U_\alpha$, a Borel subgroup of A, with root group U_α, lying in B. For α the unique positive root of A (with respect to the given choices), we have $T_A = \alpha^\vee(\Bbbk^*)$, the image of the coroot $\alpha^\vee$. Henceforth, we will write $V{\downarrow}H$ for the kH-module obtained by restricting the action of G to a subgroup H. We say that *$V{\downarrow}A$ is MF* if all composition factors in the restriction are nonisomorphic.

We will also require a notation for the corresponding modules and subgroups for the groups defined over the algebraically closed field K of characteristic zero. We write G_K for a simply connected simple algebraic group defined over the field K, with root system of type Φ, and A_K for a principal A_1-subgroup of G_K (see [Jacobson 1951; Morozov 1942] for the proofs of existence and conjugacy of A_1-subgroups of G_K intersecting the class of regular unipotent elements). For the weight λ as above, we write $\Delta_K(\lambda)$ for the corresponding irreducible G_K-module. We will use the same terminology of "MF" for the action of A_K on $\Delta_K(\lambda)$. Our main result is:

Theorem 1. *Suppose that λ is p-restricted. Then $L(\lambda){\downarrow}A$ is MF if and only if one of the following holds*:

(i) *We have that $p > (\lambda{\downarrow}T_A)$ and $\Delta_K(\lambda){\downarrow}A_K$ is MF.*

(ii) *The group G is of type A_2, $\lambda = \omega_1 + \omega_2$ and $p = 3$.*

(iii) *The group G is of type B_2, $\lambda = 2\omega_1$ and $p = 5$.*

Corollary 2. *Let $\lambda = \sum_{i=0}^{t} p^i \lambda_i$ where each λ_i is a p-restricted dominant weight. Then $L(\lambda){\downarrow}A$ is MF if and only if one of the following holds*:

(i) *The module $\Delta_K(\lambda_i){\downarrow}A_K$ is MF and $p > (\lambda_i{\downarrow}A)$, for all $0 \leq i \leq t$.*

(ii) *The group G is of type A_2, $p = 3$ and there exists $0 \leq i \leq t$ such that $\lambda_i = \omega_1 + \omega_2$. For all $0 \leq j \leq t$ we have $\lambda_j \in \{0, \omega_1 + \omega_2, \omega_1, \omega_2\}$ and if $\lambda_j = \omega_1 + \omega_2$ for some $0 \leq j \leq t-1$, then $\lambda_{j+1} = 0$.*

(iii) *The group G is of type B_2, $p = 5$ and there exists $0 \leq i \leq t$ such that $\lambda_i = 2\omega_1$. For all $0 \leq j \leq t$ we have $\lambda_j \in \{0, 2\omega_1, \omega_1, \omega_2\}$ and if $\lambda_j = 2\omega_1$ for some $0 \leq j \leq t-1$, then $\lambda_{j+1} \in \{0, \omega_2\}$.*

G_K	weight λ
A_ℓ	$\omega_1,\ \omega_2,\ 2\omega_1,\ \omega_1+\omega_\ell$
	$\omega_3\ (5 \le \ell \le 7)$
	$3\omega_1\ (\ell \le 5),\ 4\omega_1\ (\ell \le 3),\ 5\omega_1\ (\ell \le 3)$
A_3	110
A_2	$c1,\ c0$
B_ℓ	$\omega_1,\ \omega_2,\ 2\omega_1$
	$\omega_\ell\ (\ell \le 8)$
B_3	101, 002, 300
B_2	$b0,\ 0b\ (1 \le b \le 5),\ 11,\ 12,\ 21$
C_ℓ	$\omega_1,\ \omega_2,\ 2\omega_1$
	$\omega_3\ (3 \le \ell \le 5)$
	$\omega_\ell\ (\ell = 4,\ 5)$
C_3	300
$D_\ell\ (\ell \ge 4)$	$\omega_1,\ \omega_2\ (\ell = 2m+1),\ 2\omega_1\ (\ell = 2m)$
	$\omega_\ell\ (\ell \le 9)$
E_6	ω_1, ω_2
E_7	ω_1, ω_7
E_8	ω_8
F_4	ω_1, ω_4
G_2	10, 01, 11, 20, 02, 30

Table 1. Multiplicity-free restrictions in characteristic zero.

For the reader's convenience and for completeness, we list in Table 1 the nonzero weights λ for which $\Delta_K(\lambda)\downarrow A_K$ is MF, as obtained in [Liebeck et al. 2015].

We conclude the introduction with a few remarks about the proof. We first note that if $p > (\lambda\downarrow T_A)$, then one can show that the Weyl module with highest weight λ is an irreducible $\Bbbk G$-module (see [Korhonen 2018, Corollary 2.7.6]), and then the considerations of [Liebeck et al. 2015] for the groups defined over K yield the result (see Proposition 2.3). The arguments therefore focus on the cases where $p \le (\lambda\downarrow T_A)$. Many aspects of the proof follow closely the arguments used in [Liebeck et al. 2015]. In particular, we use the fact that all irreducible $\Bbbk A_1$-modules have multiplicity-one weight spaces and therefore considering the set of T_A-weights and their multiplicities in V can directly be used to detect multiplicities of composition factors of $V\downarrow A$. Moreover, there are certain dimension bounds which must be respected by an MF-module. Thus, many of our preliminary lemmas are inspired by the results in [Liebeck et al. 2015, Section 2]. In addition, we rely on a result from [Hague and McNinch 2013] where the authors prove that certain tilting modules for G have a filtration by tilting modules for a principal A_1-subgroup A of G. Since reducible indecomposable tilting modules for groups of type A_1 necessarily have repeated composition factors, this result is quite useful for showing that many $\Bbbk G$-modules are not MF as $\Bbbk A$-modules (see Lemma 2.4).

2. Preliminary lemmas

Let us fix additional notation to be used throughout the paper.

Recall that G is a simply connected simple algebraic group with principal A_1-subgroup A. We assume throughout that $\ell \geq 2$, respectively 2, 3, 4, for G of type A_ℓ, respectively B_ℓ, C_ℓ, D_ℓ. For $1 \leq i \leq \ell$, let s_i denote the simple reflection associated to the root α_i. For $\lambda \in X(T)$, a dominant weight, we write $\Delta_G(\lambda)$ for the Weyl module for G of highest weight λ, and $L_G(\lambda)$ for the irreducible module for G of highest weight λ. We will suppress the G in this notation if there is no ambiguity. For a $\Bbbk G$-module V and $\mu \in X(T)$, we write V_μ for the μ-weight space with respect to T of the module V. When we say that roots are adjacent, or end-nodes, we mean with respect to the Dynkin diagram associated to the root system Φ.

For a group of type A_1, we identify the weight lattice of a fixed maximal torus with the ring $\mathbb{Z}$ and for a nonnegative integer s we write (s) for the irreducible $\Bbbk A_1$-module of highest weight s. If we want to underline that we are talking about the fixed principal A_1-subgroup A, we may write as well $L_A(s)$. Similarly, we write $\Delta(s)$ for the Weyl module of highest weight s and $T(s)$ for the indecomposable tilting module of highest weight s. For a $\Bbbk A$-module (s), we write $(s)^{(p^i)}$ for the module whose structure is induced by the composition of the p^i-Frobenius map on A and the morphism defining the module structure on (s). For A_K, the principal A_1-subgroup of G_K, and s a nonnegative integer, we will write $\Delta_{A_K}(s)$ for the irreducible KA_K-module of highest weight s.

Here and in Sections 3 and 4, we fix a p-restricted dominant weight $\lambda \in X(T)$ and set $V = L_G(\lambda)$. Throughout the paper, set $r = \lambda \downarrow T_A$, that is, $\lambda(\alpha^\vee(c)) = c^r$, for all $c \in \Bbbk^*$. The cocharacter $\alpha^\vee : \mathbb{G}_m \to T$, which defines the maximal torus of A, satisfies $\alpha_i(\alpha^\vee(c)) = c^2$ for all $c \in \Bbbk^*$; that is, $\alpha_i \downarrow T_A = 2$ for all $1 \leq i \leq \ell$. The value for r can then be determined by writing λ as a linear combination of simple roots and then using that each simple root takes value 2 on T_A. We list the values of r in Table 2.

Recall that the existence of a principal A_1-subgroup in G implies that $p \geq h$, the Coxeter number of G, a hypothesis which allows us to apply the following proposition, a consequence of [Premet 1987, Theorem 1].

G	r
A_ℓ	$\sum_1^\ell i(\ell+1-i)c_i$
B_ℓ	$\sum_1^{\ell-1} i(2\ell+1-i)c_i + \frac{\ell(\ell+1)}{2}c_\ell$
C_ℓ	$\sum_1^\ell i(2\ell-i)c_i$
D_ℓ	$\sum_1^{\ell-2} i(2\ell-1-i)c_i + \frac{\ell(\ell-1)}{2}c_{\ell-1} + \frac{\ell(\ell-1)}{2}c_\ell$
G_2	$6c_1+10c_2$
F_4	$22c_1+42c_2+30c_3+16c_4$
E_6	$16c_1+22c_2+30c_3+42c_4+30c_5+16c_6$
E_7	$34c_1+49c_2+66c_3+96c_4+75c_5+52c_6+27c_7$
E_8	$92c_1+136c_2+182c_3+270c_4+220c_5+168c_6+114c_7+58c_8$

Table 2. Values of $r = \lambda{\downarrow}T_A$ for $\lambda = \sum_1^\ell c_i\omega_i$.

Proposition 2.1. *Let $p \geq h$ and let μ be a p-restricted weight for G. Then the irreducible $\Bbbk G$-module $L(\mu)$ has precisely the same set of weights as the $\Bbbk G$-module $\Delta(\mu)$.*

Proof. This follows from [Premet 1987, Theorem 1] since the parameter $e(\Phi)$ appearing in the statement of [loc. cit.] is the maximum of the squares of the ratios of the lengths of the roots in Φ. □

We now introduce a shorthand notation for weights of V. For $\lambda - \sum_{i=1}^\ell a_i\alpha_i$, we write $\lambda - i_1^{a_{i_1}} \cdots i_m^{a_{i_m}}$, where $a_j = 0$ for $j \notin \{i_1, \dots, i_m\}$, and suppress those a_j with $a_j = 1$; for example, the weight $\lambda - \alpha_2 - 2\alpha_3 - \alpha_5$ will be written as $\lambda - 23^25$. For G of rank 2, we write $\lambda - ab$ for the weight $\lambda - a\alpha_1 - b\alpha_2$.

The following result is Corollary 2.7.6 from [Korhonen 2018]; we include a sketch of the proof for completeness.

Lemma 2.2. *If $p > r$, then $\Delta(\lambda)$ is irreducible.*

Proof. By the Jantzen sum formula [2003, Part II, 8.19], it suffices to prove that for all $\alpha \in \Phi^+$, we have $r \geq \langle \lambda+\delta, \alpha\rangle - 1$, where $\delta = \sum_{i=1}^\ell \omega_i$. It is easy to see that $\langle \lambda, \alpha\rangle$ is maximal when α is the highest root of the dual root system $\Phi^\vee$, i.e., when α is the highest short root β of Φ. By [Serre 1994, Proposition 5], we have $\langle \lambda+\delta, \beta\rangle \leq 1 + \sum_{\alpha\in\Phi^+}\langle\lambda, \alpha\rangle$. It is therefore sufficient to show that $r = \sum_{\alpha\in\Phi^+}\langle\lambda,\alpha\rangle$, which is a simple calculation using the fact that for a simple root α_i we have $\sum_{\alpha\in\Phi^+\setminus\{\alpha_i\}}\langle\alpha_i,\alpha\rangle = \sum_{\alpha\in\Phi^+\setminus\{\alpha_i\}}\langle s_i(\alpha_i), s_i(\alpha)\rangle = \sum_{\alpha\in\Phi^+\setminus\{\alpha_i\}}\langle -\alpha_i,\alpha\rangle$ and so $\sum_{\alpha\in\Phi^+\setminus\{\alpha_i\}}\langle\alpha_i,\alpha\rangle = 0$. □

The next proposition establishes Theorem 1 when $p > r$.

Proposition 2.3. *Assume $p > r$. Then $V{\downarrow}A$ is MF if and only if $\Delta_K(\lambda){\downarrow}A_K$ is MF.*

Proof. By Lemma 2.2, the Weyl module is irreducible and therefore $V = \Delta(\lambda)$. We have $\Delta_K(\lambda){\downarrow}A_K = \sum_0^k \Delta_{A_K}(r_i)$ for some integers $r_0 \geq r_1 \geq \cdots \geq r_k \geq 0$ with $r_0 = r$. Since $p > r$, a comparison of characters gives $\Delta(\lambda){\downarrow}A = \sum_0^k \Delta_A(r_i) = \sum_0^k (r_i)$, which then implies the result. □

As the next result shows, in many cases when $\Delta(\lambda)$ is irreducible and $r \geq p$, we can still directly conclude that $V{\downarrow}A$ is not MF.

Lemma 2.4. *Assume that $\Delta(\lambda)$ is irreducible, $r \geq p$, and $r \not\equiv -1 \bmod p$. If G is of type B_ℓ, respectively D_ℓ, and λ does not lie in the root lattice of G, assume that $p > \binom{\ell+1}{2}$, respectively $p > \binom{\ell}{2}$. Then $V{\downarrow}A$ is not MF.*

Proof. Here we use [Hague and McNinch 2013, Theorems 4.1.2, 4.1.4 and 4.2.1] to see that A is a so-called "good filtration" subgroup, which then implies that the irreducible Weyl module $\Delta(\lambda) = V$ has an A-filtration by both Weyl modules and by induced modules. So in particular, $V{\downarrow}A$ is a tilting module. Furthermore, since r is the highest T_A-weight in V, the module $T(r)$ is a summand of $V{\downarrow}A$. The hypotheses on r imply that the indecomposable tilting module $T(r)$ is reducible (see [Carter and Cline 1976, Theorem 1.2]). Since tilting modules for A are self-dual, no reducible indecomposable tilting module is MF, which then concludes the proof. □

We now turn to a sequence of definitions and lemmas which provide tools for studying the set of composition factors of $V{\downarrow}A$ based upon knowing the set of weights of V.

Definition 2.5. For $n \in \mathbb{Z}$, let n_d be the multiplicity of the T_A-weight $r-2d$ in $V{\downarrow}A$ and let m_d be the multiplicity of the composition factor $(r-2d)$ in $V{\downarrow}A$. Also, let $\boldsymbol{S}_d$ denote the multiset of composition factors whose highest weight is greater than $r-2d$ and in which $r-2d$ does not occur as a weight, and let s_d denote the cardinality of $\boldsymbol{S}_d$.

Lemma 2.6. *Assume that $V{\downarrow}A$ is MF. Then $n_d \leq d+1$.*

Proof. Let $\boldsymbol{B}$ be the multiset of composition factors of $V{\downarrow}A$ where $r-2d$ occurs as a weight. Since $V{\downarrow}A$ is MF, we have $\boldsymbol{B} \subseteq \{(r), (r-2), \ldots, (r-2d)\}$. Therefore $|\boldsymbol{B}| \leq d+1$ and we can conclude that $n_d \leq d+1$. □

Lemma 2.7. *For all $0 \leq d \leq r$ we have*

$$m_d = n_d - n_{d-1} + s_d - s_{d-1}. \tag{1}$$

Proof. We prove this by induction on d. If $d = 0$ the statement holds. Indeed, $m_0 = n_0 = 1$ since r is the highest weight and is afforded only by λ, and $n_{-1} = s_{-1} =$

$s_0 = 0$. Assume that (1) holds up to an arbitrary d. In general, the multiplicity m_{d+1} can be determined by taking the difference between n_{d+1} and the number of times the T_A-weight $r - 2(d+1)$ appears in composition factors with greater highest weight. Thus,

$$m_{d+1} = n_{d+1} - \Big(\sum_{0 \le k \le d} m_k - s_{d+1} \Big).$$

By the inductive hypothesis $m_k = n_k - n_{k-1} + s_k - s_{k-1}$ for all $k \le d$. Substituting we get

$$m_{d+1} = n_{d+1} + s_{d+1} - \sum_{0 \le k \le d} (n_k - n_{k-1} + s_k - s_{k-1}) = n_{d+1} - n_d + s_{d+1} - s_d,$$

concluding the proof. □

Lemma 2.8. *For all* $1 \le d < p$ *we have* $\boldsymbol{S}_{d-1} \subseteq \boldsymbol{S}_d$. *In particular* $s_d \ge s_{d-1}$.

Proof. We begin by analysing what weights occur in an arbitrary irreducible module (t). We will write $[a_0, a_1, \dots, a_m]$ to denote the integer $\sum_{i=0}^m a_i p^i$. Then $t = [a_0, a_1, \dots, a_m]$ where the a_i's are the coefficients in the p-adic expansion of t. By Steinberg's tensor product theorem, we have

$$(t) \cong (a_0) \otimes (a_1)^{(p)} \otimes \cdots \otimes (a_m)^{(p^m)}.$$

The weights occurring in (t) are therefore of the form $[a_0 - 2i_0, a_1 - 2i_1, \dots, a_m - 2i_m]$ where $0 \le i_j \le a_j$. Let $t - 2q \ge 0$, with $q \in \mathbb{N}$, be an integer denoting a weight not occurring in (t). Then $t - 2q$ lies in an open interval (δ, γ) with

$$\begin{aligned} \delta &= [a_0, \dots, a_j, a_{j+1} - 2i_{j+1} - 2, \dots, a_m - 2i_m], \\ \gamma &= [-a_0, \dots, -a_j, a_{j+1} - 2i_{j+1}, \dots, a_m - 2i_m], \end{aligned}$$

where $0 \le i_{j+1} < a_{j+1}$ and $0 \le i_k \le a_k$ for $k > j+1$. Conversely, any integer $t - 2q$ lying in such an interval corresponds to a weight not occurring in (t). We call these intervals the *gaps* of (t), so that a composition factor (t) is in $\boldsymbol{S}_d$ if and only if $r - 2d$ is in a gap of (t).

Assume for a contradiction that $(t) \in \boldsymbol{S}_{d-1} \setminus \boldsymbol{S}_d$ for some $t \le r$. Then $t > r - 2d + 2$ and the composition factor (t) has a gap (δ, γ) as above containing $r - 2d + 2$, but not containing $r - 2d$. This means that

$$r - 2d = [a_0, \dots, a_j, a_{j+1} - 2i_{j+1} - 2, \dots, a_m - 2i_m],$$

implying that

$$\begin{aligned} 2d - (r - t) = t - (r - 2d) &= t - [a_0, \dots, a_j, a_{j+1} - 2i_{j+1} - 2, \dots, a_m - 2i_m] \\ &= 2p^{j+1}[i_{j+1} + 1, i_{j+2}, \dots, i_m] \ge 2p. \end{aligned}$$

This contradicts the assumption that $d < p$. □

Lemma 2.9. *Let* $1 \leq d < \min\{\lfloor \frac{r+2}{2} \rfloor, p\}$.

(i) *If* $n_d - n_{d-1} = 1$ *then* $r - 2d$ *is a composition factor of* $V{\downarrow}A$.

(ii) *If* $n_d - n_{d-1} \geq 2$ *then* $m_d \geq 2$ *and* $V{\downarrow}A$ *is not MF.*

(iii) *If* $\lambda = c\omega_i$ *and* $n_d \geq d+1$ *then* $V{\downarrow}A$ *is not MF.*

(iv) *If* $n_d - n_{d-1} = 1$ *and* $\mathbf{S}_{d-1} \neq \mathbf{S}_d$, *then* $m_d \geq 2$ *and* $V{\downarrow}A$ *is not MF.*

Proof. Parts (i), (ii) and (iv) follow directly from combining Lemmas 2.7 and 2.8. If $\lambda = c\omega_i$ then $n_1 = 1$, and since $n_d \geq d+1$, there exists $2 \leq d' \leq d$ such that $n_{d'} - n_{d'-1} \geq 2$, concluding by part (ii). □

We can often deduce the value n_d from the characteristic-zero case.

Lemma 2.10. *Assume that* $V \cong \Delta(\lambda)$. *Then* $n_d = \dim(\Delta_K(\lambda){\downarrow}A_K)_{r-2d}$.

Proof. This follows from [Jantzen 2003, Part II, 5.8], since T_A is uniquely determined by the property $\alpha_i{\downarrow}T_A = 2$ for all $1 \leq i \leq \ell$. □

We now establish two dimension bounds for multiplicity-free $\Bbbk A_1$-modules.

Definition 2.11. Given $r \in \mathbb{N}$, define $B(r)$ and $B_K(r)$ as

$$B(r) = \sum_{r-2k\geq 0} \dim L_A(r-2k) \quad \text{and} \quad B_K(r) = \sum_{r-2k\geq 0} \dim \Delta_{A_K}(r-2k).$$

In particular $B_K(r)$ is either $\left(\frac{r}{2}+1\right)^2$ or $\frac{r+1}{2}\frac{r+3}{2}$ according to whether r is even or odd, respectively.

Lemma 2.12. *We have* $B(r) \leq B_K(r)$ *and if* $V{\downarrow}A$ *is MF*, *then* $\dim V \leq B(r)$.

Proof. We have $B(r) \leq B_K(r)$ immediately since $\dim L_A(r-2k) \leq \dim \Delta_{A_K}(r-2k)$ for all k such that $r-2k \geq 0$. Now if $V{\downarrow}A$ is MF, it can have at most one composition factor $(r-2d)$, i.e., $m_d = 1$, for every $0 \leq d \leq \lfloor \frac{r}{2} \rfloor$. Therefore $\dim V \leq B(r)$. □

Lemma 2.13. *Suppose that* $\lambda = a\omega_i$ *and* $r \not\equiv 0 \bmod p$. *If* $V{\downarrow}A$ *is MF*, *then* $\dim V \leq B(r) - \dim(r-2)$.

Proof. The T_A-weight $r-2$ occurs with multiplicity 1 in V, and since $r \not\equiv 0 \bmod p$, it occurs as a weight in the composition factor (r). Therefore $r-2$ does not afford a composition factor of $V{\downarrow}A$, i.e., $m_1 = 0$. Since $V{\downarrow}A$ is MF, we have $m_d \leq 1$ for all $d \geq 0$ such that $r-2d \geq 0$. This proves that $\dim V \leq B(r) - \dim(r-2)$. □

The following result is our main reduction tool, showing that if $V{\downarrow}A$ is MF, then λ satisfies some highly restrictive conditions. The proof follows closely that of [Liebeck et al. 2024, Lemma 2.6].

Proposition 2.14. *Let $\lambda = \sum_{i=1}^{\ell} c_i\omega_i$. Assume that there exist $i < j$ with $c_i \neq 0 \neq c_j$ and that $V\downarrow A$ is MF. Then*:

(i) *$c_k = 0$ for $k \neq i, j$.*

(ii) *If α_i and α_j are nonadjacent, then $c_i = c_j = 1$.*

(iii) *If α_i and α_j are nonadjacent then they are both end-nodes.*

(iv) *Either α_i or α_j is an end-node.*

(v) *If both $c_i > 1$ and $c_j > 1$, then G has rank 2 and $\lambda - ij$ has multiplicity 1.*

(vi) *If either $c_i > 1$ or $c_j > 1$, then either G has rank 2, or α_i is adjacent to α_j and $\lambda - ij$ has multiplicity 1.*

Proof. We will use Proposition 2.1 throughout the proof, without direct reference.

(i) If $c_k \geq 1$ for $k \neq i, j$, we have $n_1 \geq 3$ as the T_A-weight $r-2$ is afforded by $\lambda - i$, $\lambda - j$ and $\lambda - k$. This contradicts Lemma 2.6.

(ii) Suppose α_i and α_j are not adjacent and that $c_i \geq 2$. Let $k \neq i, j$ such that α_k is adjacent to α_i and let $k' \neq i, j$ such that $\alpha_{k'}$ is adjacent to α_j. Then $n_2 \geq 4$, as the T_A-weight $r-4$ is afforded by $\lambda - i^2$, $\lambda - ik$, $\lambda - jk'$, $\lambda - ij$. This contradicts Lemma 2.6.

(iii) Assume that α_i and α_j are nonadjacent and that α_i is not an end-node. Then there exist distinct simple roots α_k, α_l, both adjacent to α_i, and a simple root $\alpha_m \neq \alpha_i$ adjacent to α_j. Then $n_2 \geq 4$, as the T_A-weight $r-4$ is afforded by $\lambda - ik$, $\lambda - il$, $\lambda - jm$ and $\lambda - ij$. This contradicts Lemma 2.6.

(iv) Assume that neither α_i nor α_j is an end-node. Then by (iii), the roots α_i and α_j are adjacent. Let $1 \leq k, l \leq \ell$ be distinct indices such that $\{i, j\} \cap \{k, l\} = \varnothing$ and such that α_i is adjacent to α_k and α_j is adjacent to α_l. Then the T_A-weight $r-8$ is afforded by $\lambda - kijl$, $\lambda - kij^2$, $\lambda - i^2j^2$, $\lambda - i^2jl$, $\lambda - ki^2j$ and $\lambda - ij^2l$. Therefore $n_4 \geq 6$, contradicting Lemma 2.6.

(v) If both $c_i > 1$ and $c_j > 1$, then by (ii), the roots α_i and α_j are adjacent. If the rank of G is not 2 we can find $k \neq i, j$, such that α_k is adjacent to either α_i or α_j. But then the T_A-weight $r-4$ is afforded by $\lambda - ij$, $\lambda - i^2$, $\lambda - j^2$ and either $\lambda - ik$ or $\lambda - jk$. Therefore $n_2 \geq 4$, contradicting Lemma 2.6. In addition, $\lambda - ij$ has multiplicity 1, else $n_2 \geq 4$, again contradicting Lemma 2.6.

(vi) Assume $c_i \geq 2$ and that G has rank at least 3. Then α_i and α_j are adjacent by (ii), and $r-4$ is afforded by $\lambda - i^2$, $\lambda - ij$ and either $\lambda - ik$ or $\lambda - jk$ for some $k \neq i, j$. Therefore, Lemma 2.6 implies that $\lambda - ij$ has multiplicity 1, as claimed. □

Lemma 2.15. *Assume that $\lambda = \omega_i$ and that there exist $\{\beta_{i-3}, \dots, \beta_{i+3}\} \subseteq \Pi$ such that for $i-3 \le s < t \le i+3$, $(\beta_s, \beta_t) \neq 0$ if and only if $t = s+1$. Then $V{\downarrow}A$ is not MF.*

Proof. Here $p > 7$, as $\operatorname{rank}(G) \ge 7$, and Table 2 shows that $r > 15$. It is now a simple check to see that $n_4 \ge 5$, concluding by Lemma 2.9(iii). □

Lemma 2.16. *Assume that $\lambda = b\omega_i$ with $b \ge 2$. If $V{\downarrow}A$ is MF, then α_i is an end-node.*

Proof. If α_i is not an end-node, it is easy to see that $n_2 \ge 3$. As $\operatorname{rank}(G) \ge 3$ we have $p > 3$, and Table 2 shows that $r > 7$, so Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF. □

Remark 2.17. In the previous two proofs, we have applied Lemma 2.9, and in each case it was straightforward to see that the condition $d < \min\{\lfloor \frac{r+2}{2} \rfloor, p\}$ is satisfied. In what follows, we will apply the lemma without systematically pointing out how we conclude that this hypothesis holds.

The following lemma provides a classification for the second possibility of Proposition 2.14(vi).

Lemma 2.18 [Testerman 1988, 1.35]. *Assume that $\lambda = c_i\omega_i + c_j\omega_j$ with α_i and α_j adjacent and $c_ic_j \neq 0$. Let $d = \dim V_{\lambda-ij}$. Then $1 \le d \le 2$ and the following hold*:

(i) *If $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$, then $d = 1$ if and only if $c_i + c_j = p-1$.*

(ii) *If $(\alpha_i, \alpha_i) = 2(\alpha_j, \alpha_j)$, then $d = 1$ if and only if $2c_i + c_j + 2 \equiv 0 \bmod p$.*

(iii) *If $(\alpha_i, \alpha_i) = 3(\alpha_j, \alpha_j)$, then $d = 1$ if and only if $3c_i + c_j + 3 \equiv 0 \bmod p$.*

Finally, we conclude this section with two further results on the dimensions of certain weight spaces in V.

Lemma 2.19 [Seitz 1987, 8.6]. *Let $G = A_\ell$. Suppose that $\lambda = c_i\omega_i + c_j\omega_j$ and $1 \le s \le i < j \le t \le k$, with $c_ic_j \neq 0$. Let $d = \dim V_{\lambda - s(s+1)\cdots(t-1)t}$. Then*:

(i) *If $a+b+j-i \not\equiv 0 \bmod p$, then $d = j-i+1$.*

(ii) *If $a+b+j-i \equiv 0 \bmod p$, then $d = j-i$.*

Lemma 2.20 [Burness et al. 2016, Lemma 2.2.8]. *Let $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$ and let $\mu = \lambda - \sum_{\beta\in S} c_\beta\beta \in X(T)$ for some subset $S \subseteq \Pi$. Set $X = \langle U_{\pm\beta} \mid \beta \in S\rangle$, where for $\gamma \in \Phi$, U_γ is the T-root subgroup associated to γ, $\lambda' = \lambda{\downarrow}(T \cap X)$ and $\mu' = \mu{\downarrow}(T \cap X)$. Then $\dim V_\mu = V'_{\mu'}$, where $V' = L_X(\lambda')$.*

3. The case where G has rank 2

Here we establish Theorem 1 in the case where G has rank 2. Let us recall our setup. Throughout the section we assume that λ is a p-restricted dominant weight for G, and we let $r = \lambda{\downarrow}T_A$ and $V = L(\lambda)$. We write $\lambda = ab$ as shorthand notation for $\lambda = a\omega_1 + b\omega_2$, and $\lambda - ab$ for $\lambda - a\alpha_1 - b\alpha_2$. We assume that $p \le r$, as Proposition 2.3 settles the case $r < p$, and we have that $p \ge h$ since we are assuming the existence of a principal A_1-subgroup in G.

3.1. *The case where G is A_2.* We begin with the case $G = A_2$, where $p \ge h = 3$. The following is the main result, which we will prove after a sequence of lemmas.

Proposition 3.1. *Let $G = A_2$ and assume $p \le r$. Then $V{\downarrow}A$ is MF if and only if $\lambda = \omega_1 + \omega_2$ and $p = 3$.*

Lemma 3.2. *Let $\lambda = ab$. Then $\lambda - ij$, with $i + j \le a + b$, is a weight of $\Delta(\lambda)$ if and only if one of the following holds*:

(i) *$i \le j$ and $j - i \le b$.*

(ii) *$i \ge j$ and $i - j \le a$.*

Proof. By [Bourbaki 1975, VIII, §7, Proposition 10], the weights in $\Delta(\lambda)$ are precisely the same as those occurring in $\Delta(a0) \otimes \Delta(0b)$. Let $\lambda_1 = a0$ and $\lambda_2 = 0b$ and recall that $\Delta(c_i\omega_i)$, for $i = 1, 2$, is the c_i-th symmetric power of the natural, respectively, dual module for G. Hence, $\lambda_1 - i_1 j_1$ is a weight of $\Delta(\lambda_1)$ if and only if

$$i_1 + j_1 \le 2a \quad \text{and} \quad 0 \le i_1 - j_1 \le a.$$

Similarly, $\lambda_2 - i_2 j_2$ is a weight of $\Delta(\lambda_2)$ if and only if

$$i_2 + j_2 \le 2b \quad \text{and} \quad 0 \le j_2 - i_2 \le b.$$

By symmetry it suffices to show that the statement of the lemma is valid when $i \ge j$. First of all, it is clear that all weights $\lambda - ij$ of $\Delta(\lambda)$ satisfy $i - j \le a$, since a weight $\lambda_2 - i_2 j_2$ of $\Delta(\lambda_2)$ satisfies $i_2 - j_2 \le 0$, and a weight $\lambda_1 - i_1 j_1$ of $\Delta(\lambda_1)$ satisfies $i_1 - j_1 \le a$.

For the converse, consider a pair (i, j), such that $i \ge j$, $i + j = d \le a + b$ and $i - j \le a$. If $d \le a$, then $j \le i \le a$, so $\lambda_1 - ij$ is a weight of $\Delta(\lambda_1)$ and $\lambda_1 + \lambda_2 - ij$ is then a weight of $\Delta(\lambda)$. If $d > a$, write $d = a + k$, where $k \le b$. To conclude we show that we can find (i_1, j_1) with $i_1 + j_1 = a$ and (i_2, j_2) with $i_2 + j_2 = k$, such that $\lambda_1 - i_1 j_1$ is a weight of $\Delta(\lambda_1)$, $\lambda_2 - i_2 j_2$ is a weight of $\Delta(\lambda_2)$ and $i_1 - j_1 + i_2 - j_2 = i - j$. Fix $i_1 + j_1 = a$ and $i_2 + j_2 = k$. Note that we are allowed to pick $i_1 - j_1$ to be any integer between a and 1 if a is odd, and between a

and 0 if a is even. Similarly, we are allowed to choose $i_2 - j_2$ between $-k$ and -1 if k is odd, and between $-k$ and 0 when k is even, concluding easily. □

Lemma 3.3. *Let $\lambda = ab$ with $a \geq b > 0$ and $a + b = p - 1$.*

(i) *For $0 \leq d \leq b$, we have $n_d = d + 1$.*

(ii) *For $b + 1 \leq d \leq a$, we have that n_d increases alternatingly by respectively 0 and 1 with respect to n_{d-1}.*

(iii) *For $a < d \leq a + b$, we have that n_d alternates between $\lceil \frac{a+b}{2} \rceil$ and $\lceil \frac{a+b+1}{2} \rceil$.*

Proof. Here we use the fact that all T-weights in V are of multiplicity 1. (See [Zalesskii and Suprunenko 1987, Proposition 2].) Hence, the proof consists of counting the pairs (i, j) with $i + j = d$ and satisfying the conditions of Lemma 3.2.

(i) Let $0 \leq d \leq b$. The statement then follows immediately from noting that $\lambda - i(d - i)$ is a weight for $0 \leq i \leq d$.

(ii) Let us start from $d = b + 1$, where the weights are given by $\lambda - (b - i + 1)i$ for $0 \leq i \leq b$. This means that $n_{b+1} = b + 1 = n_b$ by part (i). For $d = b + 2$, still assuming that $d \leq a$, we find weights of the form $\lambda - (b - i + 2)i$ for $0 \leq i \leq b + 1$. The same reasoning continues until $d = a$, proving the statement.

(iii) Let $a < d \leq a + b$. We must count the weights of the form $\lambda - i(d - i)$ where $a \geq 2i - d$ and $b \geq d - 2i$. The conditions on i are equivalent to the inequalities $\frac{d-b}{2} \leq i \leq \frac{a+d}{2}$. Considering the various possibilities for the evenness of the terms in the inequality gives the result. □

Lemma 3.4. *Let $\lambda = ab$ with $a \geq b > 0$ and $a + b = p - 1$. Then $V{\downarrow}A$ is MF if and only if $a = b = 1$.*

Proof. Note that $r = 2(a + b) < 2p$ and that $a - b$ is an even number. For clarity we split the proof into four cases, depending on whether $a - b \geq 6$, $a - b = 4$, $a - b = 2$ or $a = b$. Suppose first that $a - b \geq 6$. By Lemma 3.3, all weights of the form $r - 2d$ with $b + 1 \leq d \leq a$ follow the pattern in (ii) of the same lemma. Since $r - 2(b + 1) = 2a - 2 \geq a + b + 4 = p + 3$, and $r - 2a = 2b \leq b + a - 6 = p - 7$, this includes weights that restrict to $p + 3$, $p + 1$, $p - 1$, $p - 3$, $p - 5$. Therefore by Lemma 2.9(i) either $(p + 3)$ or $(p + 1)$ is a composition factor for $V{\downarrow}A$. In the first case $p - 5$ occurs with multiplicity 1 more than $p - 3$, and does not occur as a weight in the composition factor $(p + 3)$, while $p - 3$ does. Therefore by Lemma 2.9(iv) the module $V{\downarrow}A$ is not MF. In the second case $(p - 3)$ is similarly a repeated composition factor.

Now suppose that $a - b = 4$. We have $p + 3 = a + b + 4 = r - 2\big(\frac{a+b}{2} - 2\big) = r - 2b$. Therefore by Lemma 3.3(i), we have that $(p + 3)$ is a composition factor for $V{\downarrow}A$.

The weights $p+1$, $p-1$, $p-3$, $p-5$ follow the pattern described in Lemma 2.9(ii). Therefore we can conclude like in the previous case.

Now suppose that $a=b+2$. Then by Lemma 3.3 we know that $r-2k$ occurs with multiplicity $k+1$ for k ranging between 0 and b. In particular $(r-2b)=(p+1)$ is a composition factor by Lemma 2.9(i). Again by Lemma 3.3, the T_A-weight $r-2(b+1)=2b+2$ occurs with multiplicity $b+1$ and $r-2a=2b$ occurs with multiplicity $b+2$. Since $2b=p-5$ does not occur as a weight in the composition factor $(p+1)$, while $p-3=2b+2$ does, Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF.

Finally assume that $a=b$. Then by Lemmas 3.3 and 2.9(i), the weights $r=4a, 4a-2, \dots, 2a$ afford composition factors for $V{\downarrow}A$, with the last weight occurring with multiplicity $a+1$. If $a \geq 2$ we find that $2a-2$ occurs with multiplicity $\lceil \frac{a+b}{2} \rceil = a$ and $2a-4$ occurs with multiplicity $\lceil \frac{a+b+1}{2} \rceil = a+1$. Since $2a-4=p-5$ does not occur as a weight in the composition factor $(p+3)$, while $p-3$ does, Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF. On the other hand if $a=b=1$ we find that $V{\downarrow}A=(4)\oplus(2)$. □

Proof of Proposition 3.1. Suppose that $V{\downarrow}A$ is MF, with $\lambda = ab$ and $a \geq b$. Since the Weyl module $\Delta(c0)$ is irreducible, the assumption that $r=2a+2b \geq p > a$, together with Lemma 2.4, implies that $b \geq 1$. If $\dim V_{\lambda-11}=2$, then $a+b \neq p-1$ by Lemma 2.18, and $b=1$ by Proposition 2.14(v). In this case, using the Jantzen p-sum formula [2003, Part II, 8.19] (for example), one sees that $\Delta(\lambda)$ is irreducible, a contradiction by Lemma 2.4. If $\dim V_{\lambda-11}=1$, then by Lemma 2.18 we have $a+b=p-1$, and we conclude by Lemma 3.4. □

3.2. ***The case where G is B_2.*** We proceed with the case $G=B_2$, where $p \geq h = 4$. The main result is the following, which we shall prove after a series of lemmas.

Proposition 3.5. *Let $G=B_2$ and assume that $p \leq r$. Then $V{\downarrow}A$ is MF if and only if $\lambda = 2\omega_1$ and $p=5$.*

We begin by recalling some information about the structure of B_2 Weyl modules (with p-restricted highest weights). Let $\lambda = ab$ be a p-restricted dominant weight; here α_1 is long.

We consider the following alcoves in which a p-restricted weight can lie:

- $C_0 = \{\lambda \mid 2a+b+3 < p\}$;
- $C_1 = \{\lambda \mid a+b+2 < p < 2a+b+3\}$;
- $C_2 = \{\lambda \mid b+1 < p < a+b+2 \text{ and } 2a+b+3 < 2p\}$;
- $C_3 = \{\lambda \mid 2a+b+3 > 2p \text{ and } \max\{b+1, a+1\} < p\}$.

Lemma 3.6. (i) *If $\lambda \in C_i$ for $i = 1, 2, 3$, then $\Delta(\lambda)$ has exactly two composition factors, namely V and $L(\mu)$, where $\mu = (p-a-b-3)\omega_1 + b\omega_2$, respectively $a\omega_1 + (2p-2a-b-4)\omega_2$, $(2p-a-b-3)\omega_1 + b\omega_2$, when $i = 1, 2, 3$.*

(ii) *For $\lambda = a\omega_1 + (p-1)\omega_2$ with $2a + (p-1) + 3 > 2p$ and $a < p-1$, we have that the module $\Delta(\lambda)$ has exactly two composition factors, V and $L(\mu)$ for $\mu = (p-a-2)\omega_1 + (p-1)\omega_2$.*

For λ a p-restricted dominant weight not lying in $\bigcup_{i=1}^{3} C_i$ and not of the form described in (ii) above, $\Delta(\lambda)$ is irreducible.

Proof. This follows from the Jantzen p-sum formula [2003, Part II, 8.19]. □

Remark 3.7. Recall that here $\omega_1 = \alpha_1 + \alpha_2$. It follows from Lemma 3.6 that for a p-restricted weight $\lambda = ab$, if $\Delta(\lambda)$ is reducible then the module $\Delta(\lambda)$ has exactly one composition factor in addition to the composition factor $L(\lambda)$. The highest weight of the second composition factor is of the form $(a-k)\omega_1 + b\omega_2$ or $a\omega_1 + (b-k)\omega_2$, for some $k \geq 1$. More precisely, for $\lambda \in C_i$, $i = 1, 2, 3$, and μ as in the statement of the lemma, we have $\mu = \lambda - (2a+b+3-p)(\alpha_1+\alpha_2)$, respectively $\lambda - (a+b+2-p)(\alpha_1+2\alpha_2)$, $\lambda - (2a+b+3-2p)(\alpha_1+\alpha_2)$. And in case (ii) of the lemma, $\mu = \lambda - (2a-p+2)(\alpha_1+\alpha_2)$.

We record for convenience the dimension of the Weyl module $\Delta(ab)$, namely

$$\dim \Delta(ab) = \tfrac{1}{6}(a+1)(b+1)(a+b+2)(2a+b+3).$$

Lemma 3.8. *Let $\lambda = c0$. Then $\lambda - ij$, with $i + j \leq 2c$, is a weight of V if and only if one of the following holds*:

(i) *$i \leq j$ and $j - i \leq i$.*

(ii) *$i \geq j$ and $i - j \leq c - \lfloor \frac{j+1}{2} \rfloor$.*

Proof. By Proposition 2.1, the set of weights of V is precisely the same as the set of weights of the corresponding KG_K-module $\Delta_K(\lambda)$. First we show that all weights satisfying either (i) or (ii) are weights of $\Delta_K(\lambda)$.

Suppose $i \leq j \leq 2i$. Since $i + j \leq 2c$, we have that $i \leq c$. In particular, $\lambda - i0$ is a weight of $\Delta_K(\lambda)$. Now the weight $s_{\alpha_2}(\lambda - i0) = \lambda - i(2i)$ is also a weight of $\Delta_K(\lambda)$ and by [Bourbaki 1975, VIII, §7, Proposition 3] for all $0 \leq m \leq 2i$ we have that $\lambda - im$ is a weight of $\Delta_K(\lambda)$. So in particular, $\lambda - ij$ is a weight of V.

Suppose now that $j \leq i \leq c + j - \lfloor \frac{1}{2}(j+1) \rfloor$. As $i + j \leq 2c$, we have that $j \leq c$ and so $\lfloor \frac{1}{2}(j+1) \rfloor \leq c$ and $\mu = \lambda - (\lfloor \frac{1}{2}(j+1) \rfloor)\alpha_1$ is a weight of V. Hence, $s_{\alpha_2}(\mu) = \mu - 2\lfloor \frac{1}{2}(j+1) \rfloor \alpha_2$ is a weight of $\Delta_K(\lambda)$, which again by [Bourbaki 1975, VIII, §7, Proposition 3] implies that $\nu = \lambda - (\lfloor \frac{1}{2}(j+1) \rfloor) j$ is a weight of $\Delta_K(\lambda)$. Further, we have that $\langle \nu, \alpha_1 \rangle = c + j - 2\lfloor \frac{1}{2}(j+1) \rfloor$ and using again [loc. cit.] for all $0 \leq m \leq c+j-2\lfloor \frac{1}{2}(j+1)\rfloor$, we have that $\nu - m\alpha_1 = \lambda - \bigl(m + \lfloor \frac{1}{2}(j+1)\rfloor\bigr)j$ is a weight of $\Delta_K(\lambda)$, giving that $\lambda - ij$ is a weight of V.

We now show that any weight $\lambda - ij$ with $i+j \leq 2c$ satisfies either (i) or (ii). To this end, we use the fact that the set of weights of $\Delta_K(\lambda)$ is the same as the set of weights of the module $V_1^{\otimes^c}$, the c-fold tensor product of the module V_1 with itself, where V_1 is the KG_K-module with highest weight ω_1. (See [Bourbaki 1975, VIII, §7, Proposition 10].) Let μ be a weight of $V_1^{\otimes^c}$, so that

$$\begin{aligned}\mu &= c\omega_1 - a_1\alpha_1 - a_2(\alpha_1+\alpha_2) - a_3(\alpha_1+2\alpha_2) - a_4(2\alpha_1+2\alpha_2)\\ &= \lambda - (a_1+a_2+a_3+2a_4)\alpha_1 - (a_2+2a_3+2a_4)\alpha_2,\end{aligned}$$

with $a_i \in \mathbb{N}$ such that $a_1+a_2+a_3+a_4 \leq c$.

There are two cases to consider; suppose first that $a_1 \leq a_3$, so that

$$a_1+a_2+a_3+2a_4 \leq a_2+2a_3+2a_4.$$

Then $a_2+2a_3+2a_4 \leq 2a_1+2a_2+2a_3+4a_4 = 2(a_1+a_2+a_3+2a_4)$ and the weight μ satisfies the conditions of (i).

Now suppose $a_1 \geq a_3$, so that $a_1+a_2+a_3+2a_4 \geq a_2+2a_3+2a_4$ and as usual

$$a_1+a_2+a_3+2a_4+a_2+2a_3+2a_4 = a_1+2a_2+3a_3+4a_4 \leq 2c, \tag{2}$$

and

$$a_1+a_2+a_3+a_4 \leq c. \tag{3}$$

Note that $j - \lfloor \frac{j+1}{2}\rfloor = \lfloor \frac{j}{2}\rfloor$. If

$$a_1+a_2+a_3+2a_4 > c + \lfloor \tfrac{1}{2}(a_2+2a_3+2a_4)\rfloor = c+a_3+a_4+\lfloor \tfrac{1}{2}a_2\rfloor,$$

then $a_1+a_2-\lfloor \frac{a_2}{2}\rfloor+a_4 > c$ and $2a_1+a_2+1+2a_4 > 2c$. If $2a_1+2a_2+2a_3+2a_4 \geq 2a_1+a_2+1+2a_4$ we obtain a contradiction to (3). Hence we may now assume $2a_2+2a_3 < a_2+1$, that is, $a_2 = 0 = a_3$. Now (3) becomes $a_1+a_4 \leq c$ and so $a_1+2a_4 \leq c+\lfloor \frac{2a_4}{2}\rfloor$ and the weight satisfies condition (ii). □

Lemma 3.9. *Let $\lambda = c1$, $c < p$. Then $\lambda - ij$, with $i+j \leq 2c$, is a weight of V if and only if one of the following holds*:

(i) $i \leq j$ *and* $j-i \leq i+1$.

(ii) $i \geq j$ *and* $i-j \leq c - \lfloor \frac{j}{2}\rfloor$.

Proof. By [Bourbaki 1975, VIII, §7, Proposition 10] and Proposition 2.1, the weights occurring in V are the same as the weights occurring in $c0 \otimes 01$. The statement then follows from Lemma 3.8. □

Lemma 3.10. *Let $\lambda = ab$ with $p > a \geq 1$, $p > b \geq 2$ and $2a+b+2 \equiv 0 \bmod p$. Then $V{\downarrow}A$ is not MF.*

Proof. Since $2a+b+2 \equiv 0 \bmod p$, by Lemma 3.6 we have

$$\dim V = \dim L(ab) = \dim \Delta(ab) - \dim L((a-1)b) \geq \dim \Delta(ab) - \dim \Delta((a-1)b).$$

Using the Weyl character formula, we have that

$$\dim V \geq \tfrac{1}{6}(1+b)(6+6a^2+5b+b^2+12a+6ab). \tag{4}$$

Since $p > a$ and $p > b$, there are exactly two possibilities for p, either $p = 2a+b+2$ and b is odd, or $p = a+1+\frac{b}{2}$ and b is even. Let us start with the first case, namely $p = 2a+b+2$. Assume that $b = 3$ and $a \geq 2$. Then $r = 4a+3b = 2p-1$ and

$$B(r) = 2\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + \sum_{k=1}^{\frac{p-1}{2}} 2k = \tfrac{1}{4}(3p^2+4p+1). \tag{5}$$

Plugging in $p = 2a+5$ and combining (5) with (4) gives $\dim V > B(r)$ and Lemma 2.12 implies that $V{\downarrow}A$ is not MF. The case $b = 3$, $a = 1$ and $p = 7$ can be handled directly; we observe that $n_1 = n_2 = 2$, while $n_3 = 4$, as the weight space $\lambda - 12$ is 2-dimensional (see [Lübeck 2018]). Then Lemma 2.9(ii) implies that $V{\downarrow}A$ is not MF.

Next assume that $b \geq 5$, in which case $r = 2p+(b-4) < 3p$. Then

$$B(r) = 3\sum_{k=1}^{\frac{b-3}{2}} 2k + 2\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + \sum_{k=1}^{\frac{p-1}{2}} 2k = \tfrac{1}{4}(3p^2+4p+10-12b+3b^2). \tag{6}$$

Plugging in $p = 2a+b+2$ and combining (6) with (4) gives

$$\dim V - B(r) \geq -39-36a-12a^2+5b+6a^2b-3b^2+6ab^2+b^3.$$

As $b \geq 5$ and $a \geq 1$, this means that $\dim V - B(r) > 0$, and Lemma 2.12 implies that $V{\downarrow}A$ is not MF.

We now consider the second case, where $p = a+1+\frac{b}{2}$. Here we have $r = 4p+b-4$. Suppose that $b = 2$, so that $a = p-2 \geq 3$ and $r = 3p+a < 4p$. If a is even,

$$\begin{aligned} B(r) &= 4\sum_{k=1}^{\frac{a+2}{2}}(2k-1) + 3\sum_{k=1}^{\frac{p-1}{2}} 2k + 2\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + \sum_{k=1}^{\frac{p-1}{2}} 2k \\ &= \tfrac{1}{2}(3p^2+2p+7+8a+2a^2). \end{aligned} \tag{7}$$

Plugging in $p = a+2$ and combining (7) with (4) gives

$$\dim V - B(r) \geq \tfrac{1}{2}(a^2+2a-3).$$

Therefore $\dim V - B(r) > 0$, and Lemma 2.12 implies that $V{\downarrow}A$ is not MF. If a is odd, we have

$$(8)\qquad \begin{aligned} B(r) &= 4\sum_{k=1}^{\frac{a+1}{2}} 2k + 3\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + 2\sum_{k=1}^{\frac{p-1}{2}} 2k + \sum_{k=1}^{\frac{p+1}{2}}(2k-1) \\ &= \tfrac{1}{2}(3p^2+4p+7+8a+2a^2). \end{aligned}$$

Plugging in $p = a+2$ and combining (8) with (4) gives

$$\dim V - B(r) \geq \tfrac{1}{2}(a^2-7).$$

As $a \geq 3$, Lemma 2.12 implies that $V{\downarrow}A$ is not MF. Now suppose that $b \geq 4$, in which case $r = 4p+b-4 < 5p$. Then

$$(9)\quad \begin{aligned} B(r) &= 5\sum_{k=1}^{\frac{b-2}{2}}(2k-1) + 4\sum_{k=1}^{\frac{p-1}{2}} 2k + 3\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + 2\sum_{k=1}^{\frac{p-1}{2}} 2k + \sum_{k=1}^{\frac{p+1}{2}}(2k-1) \\ &= \tfrac{1}{4}(10p^2+8p+5b^2-20b+18). \end{aligned}$$

Plugging in $p = a+1+\frac{b}{2}$ and combining (9) with (4) gives

$$\dim V - B(r) \geq \tfrac{1}{24}(-192-120a-36a^2+80b+12ab+24a^2b-21b^2+24ab^2+4b^3).$$

We can write this as

$$\dim V - B(r) \geq \tfrac{1}{24}(-192+80b-21b^2+4b^3+12a^2(-3+2b)+12a(-10+b+2b^2)).$$

Treating the right-hand side as a quadratic polynomial in a, it is easy to see that since $b \geq 4$, we must again have $\dim V - B(r) > 0$, concluding by Lemma 2.12. □

Lemma 3.11. *Let $\lambda = 1b$ with $b \geq 2$. Then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 3.6, we have that one of the following holds:

(i) $p > b+5$ and $V = \Delta(1b)$.

(ii) $b = p-4$.

(iii) $b = p-2$ and $\dim V \geq \dim \Delta(1b) - \dim \Delta(1(b-2))$.

In the first case, for $b \geq 3$, $\dim V$ exceeds $B_K(r)$ and Lemma 2.12 then implies that $V{\downarrow}A$ is not MF. For $b = 2$, we have $p \geq 11 > r$, contradicting our assumption that $p \leq r$.

The second case is covered by Lemma 3.10.

Finally, we consider the third case. Here $\dim V \geq 2(3+4b+b^2) = 2(p-1)(p+1)$, $r = 3p-2 > p$ and $b \geq 3$. In addition, we have $B(r) = \frac{1}{2}(p+1)(3p-1)$ so that $\dim V > B(r)$ and $V{\downarrow}A$ is not MF by Lemma 2.12. □

Lemma 3.12. *Let $\lambda = c1$ with $c \geq 1$ and $p = 2c+3$. Then $V{\downarrow}A$ is not MF.*

Proof. Note that all T-weight spaces of V are 1-dimensional, see [Zalesskii and Suprunenko 1987, Proposition 2]. By Lemma 3.9, for $1 \leq d \leq c$ and $0 \leq k \leq d$, we have that $\lambda - (d-k)k$ is a weight if and only if $0 \leq k \leq 2d-2k+1$. We then find that $n_d = \lfloor \frac{2d+1}{3} \rfloor + 1$, for $1 \leq d \leq c$. Since $\lambda - (c+1-k)k$ is a weight if and only if $1 \leq k \leq 2c+2-2k+1$, we find that $n_{c+1} = \lfloor \frac{2c}{3} \rfloor + 1$. Similarly $n_{c+2} = \lfloor \frac{2c+2}{3} \rfloor + 1$. There are now two cases to consider: either $c \equiv 1 \bmod 3$ or $c \equiv -1 \bmod 3$. In the first case $n_{c-1} > n_{c-2}$ (note that $n_{c-2} = 0$ if $c = 1$), and therefore by Lemma 2.9(i) we know that $(p+2)$ is a composition factor of $V{\downarrow}A$. We have $n_{c+2} = n_{c+1} + 1$, and the T_A-weight $r - 2(c+2) = p-4$ does not occur in the composition factor $(p+2)$, but the T_A-weight $p-2$ does. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF. The second case, when $c \equiv -1 \bmod 3$, follows similarly. □

Lemma 3.13. *Let $\lambda = c1$ with $2c+3 \neq p$ and $c \geq 1$. Then $V{\downarrow}A$ is not MF.*

Proof. Note that $r = 4c+3$. First assume $c = 1$, so that $r = 7$; hence $p = 7$ and the result follows from Lemma 2.4. When $c = 2$ we have $p \neq 7$ and by Lemma 3.6 $\Delta(\lambda)$ is irreducible. Since $p = 5$ or $p = 11$, the hypotheses of Lemma 2.4 are satisfied, and $V{\downarrow}A$ is not MF.

We henceforth assume that $c \geq 3$ and will show that $V{\downarrow}A$ is not MF. Suppose first that $p > 2c+4$, so that by Lemma 3.6 $\Delta(\lambda)$ is irreducible. Since $p \geq 5$ and $r = 4c+3$, the hypotheses of Lemma 2.4 are satisfied and $V{\downarrow}A$ is not MF.

Assume now that $p \leq 2c+4$, so that in fact $p \leq 2c+1$. Then by Lemma 3.6 and Remark 3.7, either $p-3 \leq c \leq p-1$ and $\Delta(\lambda)$ is irreducible which implies $\dim V = \dim \Delta(\lambda)$; or $c \leq p-4$ and $\dim V \geq \dim \Delta(c1) - \dim \Delta((c-k)1)$, for $k = 2c+4-p \geq 3$. In particular, $\dim V \geq \dim \Delta(c1) - \dim \Delta((c-3)1) = 4+6c+6c^2$. In both cases, one checks that $\dim V > B_K(r)$ so that $V{\downarrow}A$ is not MF by Lemma 2.12. □

Lemma 3.14. *Let $\lambda = c0$ with $c > 1$ and $p = 2c+1$. Then $V{\downarrow}A$ is MF if and only if $c = 2$.*

Proof. First note that $L(20){\downarrow}A = (8) + (4)$, since here $p = 5$. Now assume that $c \geq 3$, so that $p \geq 7$. By [Zalesskii and Suprunenko 1987, Proposition 2], all T-weight spaces of V are 1-dimensional. Let $1 \leq l \leq c$. Then $n_l = \lfloor \frac{2l}{3} \rfloor + 1$, since by Lemma 3.9 we have that $\lambda - (l-k)k$ is a weight if and only if $0 \leq k \leq 2l-2k$. Similarly $n_{c+1} = \lfloor \frac{2c+2}{3} \rfloor$, $n_{c+2} = \lfloor \frac{2c+1}{3} \rfloor$, $n_{c+3} = n_c$. Therefore by Lemma 2.9(i) either both $(p+1)$ and $(p+5)$ or both $(p+3)$ and $(p+5)$ are composition factors of $V{\downarrow}A$, and by Lemma 2.9(iii), the composition factor $(p-7)$ is repeated. □

Lemma 3.15. *Let $\lambda = c0$ with $c > 1$ and $p \neq 2c+1$. Then $V{\downarrow}A$ is not MF.*

Proof. First assume that $p > 2c+3$, so that Lemma 3.6 implies that $\Delta(c0)$ is irreducible. Since $r = 4c$ and by hypothesis $r \geq p$ and $p \geq 2c+4$, we have that p does not divide $r+1$ and Lemma 2.4 gives the result.

So we now assume that $p \leq 2c+3$. If $\Delta(\lambda)$ is irreducible, then one checks that $\dim V = \dim \Delta(\lambda)$ exceeds $B_K(r)$ for all $c \geq 9$. For $c \leq 8$ we combine the information from the tables in [Lübeck 2001] with the criteria of Lemma 2.4 to reduce to the case $c = 5$ and $p = 7$. But then we have $\dim V = 91$ and $B(r) = 88$, so we conclude by applying Lemma 2.12.

Now assume that $p \leq 2c+3$ and $\Delta(\lambda)$ is reducible. Then by Lemma 3.6, $c+2 < p$ and $2c+3 > p$, and so in particular, $4c = r > p$. By Remark 3.7, $\dim V = \dim \Delta(c0) - \dim \Delta((c-k)0)$ for $k = 2c+3-p$, so k is even. Assume that $c \geq k \geq 6$. We find that

$$\dim V - B_K(r) = c^2(-4+k) - c(4-3k+k^2) + \tfrac{1}{6}(-6+13k-9k^2+2k^3).$$

Treating this as a quadratic polynomial in c, we find that $\dim V - B_K(r)$ is certainly strictly positive if $-44+47k-16k^2+k^3 > 0$. Therefore if $k > 12$, by Lemma 2.12 we have that $V{\downarrow}A$ is not MF. If $k = 6$, 8, 10 or 12, we have $\dim V - B_K(r) > 0$ when $c \geq 10$. Therefore the only possibilities for (c, k), with $k \in \{6, 8, 10, 12\}$, are $(8, 6)$ with $p = 13$, $(7, 6)$ with $p = 11$, $(8, 8)$ with $p = 11$ or $(9, 6)$ with $p = 13$. In each of these cases, we find that $\dim V - B(r) > 0$, concluding by Lemma 2.12.

Note that $k \neq 2$ as $p \neq 2c+1$. So finally we consider the case $k = 4$ and $p = 2c-1$. Now $r = 4c = 2p+2$ and a direct computation shows that $\dim V = 4c^2 - 4c + 6$ while $B(r) = 3c^2 - 2c + 12$. Now we have $c \geq 4$ (since $k = 4$) and hence $\dim V$ exceeds $B(r)$, showing as before that $V{\downarrow}A$ is not MF. □

Lemma 3.16. *Let $\lambda = 0c$ with $c > 1$ and $p \leq r$. Then $V{\downarrow}A$ is not MF.*

Proof. We have $V = \Delta(0c)$ (see [Seitz 1987, Table 1]). One checks that $\dim V = \frac{1}{6}(1+c)(2+c)(3+c) > B_K(r)$ for $c \geq 9$; by Lemma 2.12, the module $V{\downarrow}A$ is not MF in these cases. For $2 \leq c \leq 8$, we may apply Lemma 2.4 to conclude that $V{\downarrow}A$ is not MF except for the pairs $(c, p) = (3, 5)$ and $(c, p) = (7, 11)$. Here we apply Lemma 2.13 to again conclude that $V{\downarrow}A$ is not MF. □

Proof of Proposition 3.5. Suppose that $V{\downarrow}A$ is MF, with $\lambda = ab$ and $r \geq p$. By Proposition 2.14 and Lemma 2.18, if $a, b \geq 2$ we must have $2a+b+2 \equiv 0 \bmod p$. Therefore Lemma 3.10 implies that either $a \leq 1$ or $b \leq 1$ and Lemmas 3.12 and 3.13 show that $\lambda \neq 11$. By Lemma 3.11 we conclude that if $a = 1$, then $b = 0$ contrary to our assumption that $r \geq p$, and another application of Lemmas 3.12 and 3.13 shows that if $b = 1$ then $a = 0$, again contrary to our assumption on p and r. We therefore reduce to the case $a = 0$ or $b = 0$, the first being ruled out by Lemma 3.16. If $b = 0$, by Lemmas 3.14 and 3.15, and the above remarks, we conclude that $a = 2$ with $p = 5$, in which case $V{\downarrow}A$ is MF by Lemma 3.14. □

3.3. *The case where G is G_2.* We now move on to the final case where G has rank 2, i.e., $G = G_2$. Our main result, to be proven in a sequence of lemmas, is the following proposition.

Proposition 3.17. *Let $G = G_2$ and $\lambda = ab$ with $p \le r$. Then $V{\downarrow}A$ is not MF.*

Set $\lambda = ab$, with $0 \le a, b < p$, where we take α_1 to be short, $(\alpha_2, \alpha_2) = 1$. (This choice of root lengths is required for using the result [Seitz 1987, (6.2)] stated below in Lemma 3.18.) Here we have $r = 6a + 10b$, and $p \ge 7$ since $p \ge h$. We set $\mu = \lambda - 11$ throughout the entire section and note that $\mu = (a+1)\omega_1 + (b-1)\omega_2$. For $\alpha \in \Phi$, we let e_α, f_α denote the T-weight vectors in the Lie algebra of G associated with the root α, respectively $-\alpha$.

We will use a result from [Seitz 1987], which we state here only for the group G_2:

Lemma 3.18 [Seitz 1987, (6.2)]. *Assume $p > 3$. Let ν be a dominant weight such that $L(\nu)$ affords a composition factor of $\Delta(\lambda)$. Then*

$$2(\lambda + \rho, \lambda - \nu) - (\lambda - \nu, \lambda - \nu) \in \tfrac{p}{6}\mathbb{Z}.$$

In view of applying Lemma 3.18, we record the results of some computations for particular subdominant weights in $\Delta(\lambda)$ in Table 3.

We note that since λ is p-restricted, V is irreducible as a module for the Lie algebra of G (see [Curtis 1960, Chapter II]). For the following lemmas, we let $v^+ \in V_\lambda$, that is, v^+ is a highest weight vector in V. Then by [Testerman 1988, 1.29] we have that, for $\nu \le \lambda$, the weight space V_ν is spanned by vectors of the form $f_{\gamma_1}^{m_1} \cdots f_{\gamma_r}^{m_r} v^+$, where $\gamma_j \in \Phi^+$ and $m_j \in \mathbb{N}$ with $\lambda - \nu = \sum m_j \gamma_j$.

(a,b)	ν	$\dim \Delta(\lambda)_\nu$	$2(\lambda+\rho, \lambda-\nu) - (\lambda-\nu, \lambda-\nu)$
$a \ge 1, b \ge 1$	$\lambda - 21$	$3 - \delta_{a,1}$	$\frac{2a+3b+4}{3}$
$a \ge 1, b \ge 2$	$\lambda - 12$	2	$\frac{a+6b}{3}$
$a \ge 1, b \ge 1$	$\lambda - 22$	$4 - \delta_{a,1} - \delta_{b,1}$	$\frac{2a+6b+4}{3}$
$a = 0, b \ge 2$	$\lambda - 22$	2	$\frac{6b+4}{3}$
$a \ge 1, b \ge 2$	$\lambda - 13$	$2 - \delta_{b,2}$	$\frac{a+9b-9}{3}$
$a \ge 1, b \ge 2$	$\lambda - 32$	$7 - 2\delta_{a,1} - \delta_{a,2}$	$a + 2b + 2$
$a \ge 1, b \ge 3$	$\lambda - 23$	$4 - \delta_{a,1}$	$\frac{2a+9b-2}{3}$
$a = 1, b \ge 3$	$\lambda - 14$	$2 - \delta_{b,3}$	$\frac{a+12b-24}{3}$

Table 3. Weight multiplicities for G_2-modules.

Lemma 3.19. *Assume $a \geq 2$, $b \geq 1$ and set $\nu = \lambda - 21$.*

(i) *If $a + 3b + 3 \not\equiv 0 \bmod p$ and $2a + 3b + 4 \not\equiv 0 \bmod p$, then* $\dim V_\nu = 3$.

(ii) *If $a + 3b + 3 \equiv 0 \bmod p$, then* $\dim V_\nu = 2$.

(iii) *If $2a + 3b + 4 \equiv 0 \bmod p$, then* $\dim V_\nu = 2$.

Proof. Using [Lübeck 2018] and [Cavallin 2017, Proposition A], one checks that $\dim \Delta(\lambda)_\nu = 3$. Note as well that if η is a dominant weight satisfying $\nu \prec \eta \prec \lambda$, then $\eta \in \{\lambda - 10, \lambda - 01 \text{ (if } b \geq 2), \lambda - 20 \text{ (if } a \geq 4), \mu\}$. The weight η does not afford a composition factor of $\Delta(\lambda)$, for $\eta \in X(T)$, $\eta \neq \mu$.

First consider the case where $a + 3b + 3 \not\equiv 0 \bmod p$. In this case, μ does not afford a composition factor of $\Delta(\lambda)$ (see Lemma 2.18) and the vectors $f_{\alpha_1+\alpha_2} v^+$ and $f_{\alpha_2} f_{\alpha_1} v^+$ are linearly independent. The weight space V_ν is spanned by the vectors $v_1 = f_{2\alpha_1+\alpha_2} v^+$, $v_2 = f_{\alpha_1+\alpha_2} f_{\alpha_1} v^+$ and $v_3 = f_{\alpha_2} f_{\alpha_1}^2 v^+$. Suppose $\sum_{i=1}^3 a_i v_i = 0$, for $a_i \in k$. Then applying e_{α_1} and e_{α_2} respectively, and using the fact that $f_{\alpha_1}^2 v^+ \neq 0$, we obtain the following system of equations:

$$2a_1 + a a_2 = 0, \quad 3a_2 + a_3(2a - 2) = 0, \quad a_3(b + 2) - a_2 = 0.$$

(These computations depend on a choice of structure constants; we have used those given in [Carter 1989, §12.5].) We then have that v_1, v_2, v_3 are linearly dependent if and only if $a_3 \neq 0$. If $a_3 \neq 0$, then we deduce that $2a+3b+4 \equiv 0 \bmod p$. Moreover, if $2a + 3b + 4 \equiv 0 \bmod p$ the three vectors are linearly dependent and it is easy to check that v_1 and v_2 are linearly independent. This gives (i).

Now consider the case where $a+3b+3 \equiv 0 \bmod p$, so that μ affords a composition factor of $\Delta(\lambda)$ and one checks that $b f_{\alpha_1+\alpha_2} v^+ + f_{\alpha_1} f_{\alpha_2} v^+ = 0$. Now if $a = p - 1$ (so that $2a + 3b + 4 \equiv 0 \bmod p$), then ν does not occur in the composition factor afforded by μ. In addition, arguing as above, we see that $v_1 \in \langle v_2, v_3 \rangle$ and v_2 and v_3 are linearly independent, so that $\dim V_\nu = 2$. While if $a \neq p - 1$, then ν occurs in the composition factor afforded by μ, with multiplicity 1. Moreover, $2a + 3b + 4 \not\equiv 0 \bmod p$, and Lemma 3.18 implies that the weight ν does not afford a composition factor of $\Delta(\lambda)$ and so $\dim V_\nu = 2$. These arguments give the conclusions of (ii) and (iii). □

Lemma 3.20. *Let $a = 1$, $b \geq 1$, and set $\nu = \lambda - 21$. Then* $\dim V_\nu = 1$ *if $3b + 4 \equiv 0 \bmod p$ and* $\dim V_\nu = 2$ *otherwise.*

Proof. By [Lübeck 2018] and [Cavallin 2017, Proposition A], we have $\dim \Delta(\lambda)_\nu = 2$ and V_ν is spanned by $v_1 = f_{2\alpha_1+\alpha_2} v^+$ and $v_2 = f_{\alpha_1+\alpha_2} f_{\alpha_1} v^+$. If $3b+4 \equiv 0 \bmod p$, then by Lemma 2.18, μ affords a composition factor of $\Delta(\lambda)$ and ν occurs with multiplicity 1 there. So by Proposition 2.1, we have $\dim V_\nu = 1$.

If $3b+4 \not\equiv 0 \bmod p$, then μ does not afford a composition factor and $f_{\alpha_1+\alpha_2}v^+$ and $f_{\alpha_2}f_{\alpha_1}v^+$ are linearly independent. If $a_1v_1+a_2v_2=0$ for $a_i \in k$, then applying e_{α_1} and e_{α_2}, we deduce that $2a_1+a_2=0=3a_2$. Hence the two vectors are linearly independent and $\dim V_\nu = 2$. □

Lemma 3.21. *Assume* $b \geq 2$, $a \geq 1$, *and set* $\nu = \lambda - 12$. *Then* $\dim V_\nu = 1$ *if* $a+3b+3 \equiv 0 \bmod p$ *and* $\dim V_\nu = 2$ *otherwise.*

Proof. As in the preceding lemmas, we find that $\dim \Delta(\lambda)_\nu = 2$. If $a+3b+3 \equiv 0 \bmod p$, then μ affords a composition factor of $\Delta(\lambda)$ and using Proposition 2.1 we deduce that $\dim V_\nu = 1$.

So assume that $a+3b+3 \not\equiv 0 \bmod p$ and then $f_{\alpha_1+\alpha_2}v^+$ and $f_{\alpha_2}f_{\alpha_1}v^+$ are linearly independent. The ν weight space is spanned by $v_1 = f_{\alpha_1+\alpha_2}f_{\alpha_2}v^+$ and $v_2 = f_{\alpha_2}^2 f_{\alpha_1}v^+$. Suppose $a_1v_1+a_2v_2=0$ for $a_i \in k$. Applying e_{α_1} and e_{α_2} and using that $f_{\alpha_2}^2 v^+ \neq 0$, we deduce that $3a_1+aa_2=0$ and $a_1b=0$. Hence the two vectors are linearly independent, giving the result. □

We are now ready to prove the main proposition.

Proof of Proposition 3.17. We treat various cases separately below. In Cases 1 to 4, we use Proposition 2.14(v) and Lemma 2.18 to reduce to the case where $a+3b+3 \equiv 0 \bmod p$ (as otherwise $V{\downarrow}A$ is not MF and neither is $\Delta_K(\lambda){\downarrow}A_K$). Throughout we rely on the tables in [Lübeck 2018].

Case 1: $a \geq 3$ and $b \geq 3$. The T_A-weight $r-6$ is afforded by $\lambda-30$, $\lambda-03$, $\lambda-21$ and $\lambda-12$. An application of Lemma 3.19 then shows that $n_3 \geq 5$ and Lemma 2.6 then shows that $V{\downarrow}A$ is not MF.

Case 2: $a=2$ and $b \geq 4$. Here $3b+5 \equiv 0 \bmod p$, so Lemma 3.18 implies that neither of the weights $\lambda-21$ and $\lambda-12$ affords a composition factor of $\Delta(\lambda)$. Now, the T_A-weight $r-8$ is afforded by $\lambda-31$, $\lambda-22$, $\lambda-13$ and $\lambda-04$. By Lemma 3.18, none of these weights affords a composition factor of $\Delta(\lambda)$. Therefore $\dim V_{\lambda-31} = \dim V_{\lambda-22} = 2$, Indeed, for the weight $\lambda-31$ we note that the only dominant weights μ' with $\lambda-31 \prec \mu'$ are μ, $\lambda-01$, $\lambda-10$, $\lambda-21$. We have assumed that μ affords a composition factor of $\Delta(\lambda)$, but the second and third weights occur with multiplicity 1 in $\Delta(\lambda)$ and so do not afford a composition factor of $\Delta(\lambda)$ (and as mentioned above, neither does the fourth weight). This then allows us to determine $\dim V_{\lambda-31}$ and similarly for $V_{\lambda-22}$. We conclude that $n_4 \geq 6$ and apply Lemma 2.6 to see that $V{\downarrow}A$ is not MF.

Case 3: $a \geq 3$ and $b=2$. Here we have $a+9 \equiv 0 \bmod p$. Lemma 3.18 implies that neither of the weights $\lambda-21$, $\lambda-12$ affords a composition factor of $\Delta(\lambda)$. Now consider the T_A-weight $r-8$, afforded by $\lambda-31$, $\lambda-22$ and $\lambda-13$, none of which affords a composition factor of $\Delta(\lambda)$. Counting the occurrences of these weights in the irreducible $L(\mu)$, we see that $n_4 \geq 6$ and then use Lemma 2.6 to see that $V{\downarrow}A$ is not MF.

Case 4: $a = 2$ and $b \in \{2, 3\}$. Here, we have $a + 3b + 3 \equiv 0 \bmod p$. Consider first the weight $\lambda = 2\omega_1 + 2\omega_2$ with $p = 11$; here $\dim V = 295$ and $r = 32$. One then checks that $B(r) = 204$. For the weight $\lambda = 2\omega_1 + 3\omega_2$, with $p = 7$, we have $r = 42$, $\dim V = 532$ and $B(r) = 295$. In both cases, Lemma 2.12 then implies that $V{\downarrow}A$ is not MF.

We now turn to the cases where one or both of a and b is less than 2, in which case we no longer deduce that $\dim V_\mu = 1$.

Case 5: $a \geq 3$ and $b = 1$. If $a = p - 6$, then $n_2 = 2$, while the T_A-weight $r - 6$ is afforded by $\lambda - 21$, $\lambda - 12$ and $\lambda - 30$; using Lemma 3.18 we have that $n_3 = 4$ and so $V{\downarrow}A$ is not MF by Lemma 2.9.

Now suppose $a \neq p - 6$ so that $n_2 = 3$ and μ does not afford a composition factor of $\Delta(\lambda)$. Let $\nu = \lambda - 21$. Suppose first that $2a + 7 \equiv 0 \bmod p$; then by Lemma 3.19 we have $\dim V_\nu = 2$, which implies that the composition multiplicity $[\Delta(\lambda) : L(\nu)]$ is equal to 1 and $n_3 = 4$. Now count the occurrences of the T_A-weight $r - 8$ which is afforded by $\lambda - 31$, $\lambda - 22$ and $\lambda - 40$, the latter only if $a \geq 4$. If $a \geq 4$, Lemma 3.18 implies that $n_4 \geq 6$, giving the usual contradiction. The case where $2a + 7 \not\equiv 0 \bmod p$ is easier; here ν does not afford a composition factor of $\Delta(\lambda)$ and $n_3 = 5 = n_2 + 3$ (even if $a = 3$).

So we are left with the case $a = 3$, $b = 1$ and $p = 13$, where $\dim V = 259$ and $r = 28$. But as above, one checks that $\dim V > B(r)$, and Lemma 2.12 implies that $V{\downarrow}A$ is not MF.

Case 6: $a = 1$ and $b \geq 3$. Consider first the case where $3b + 4 \equiv 0 \bmod p$, when μ affords a composition factor of $\Delta(\lambda)$. Moreover, we note that $b \neq 4$. We claim that $n_4 = 4 - \delta_{b,3}$ and $n_5 \geq 6 - \delta_{b,3}$, which then shows that $V{\downarrow}A$ is not MF.

The T_A-weight $r - 8$ is afforded by $\lambda - 31$, $\lambda - 22$, $\lambda - 13$ and $\lambda - 04$ (the latter only if $b \geq 4$). The first of these is conjugate to μ and so has multiplicity 1 in V and the last of these has multiplicity $1 - \delta_{b,3}$. For the remaining two weights, we use repeatedly Lemma 3.18 and note that

(i) $\lambda - 21$ and $\lambda - 12$ do not afford composition factors of $\Delta(\lambda)$;

(ii) $\lambda - 22$ does not afford a composition factor of $\Delta(\mu)$ and so occurs with multiplicity 2 in $L(\mu)$; and

(iii) $\lambda - 13$ and $\lambda - 22$ do not afford composition factors of $\Delta(\lambda)$.

We then deduce that the weights $\lambda - 22$ and $\lambda - 13$ each occur with multiplicity 1 in V. Hence $n_4 = 4 - \delta_{b,3}$ as claimed.

Now we turn to n_5; the T_A-weight $r-10$ is afforded by $\lambda-41$, $\lambda-32$, $\lambda-23$, $\lambda-14$ and $\lambda-05$ (the latter only if $b \geq 5$). The first of these is conjugate to λ. We now argue that $\nu=\lambda-32$ has multiplicity 2 in V, which establishes the claim on n_5. Note that ν does not afford a composition factor of $\Delta(\lambda)$ nor of $\Delta(\mu)$. Applying Lemma 3.19, we deduce that $\nu=\mu-21$ has multiplicity 3 in $L(\mu)$ and so has multiplicity 2 in V, as claimed.

Now consider the case where $3b+4 \not\equiv 0 \bmod p$ and so $n_2=3$. By Lemmas 3.20 and 3.21 we have

$$\dim V_{\lambda-21} = \dim V_{\lambda-12} = 2,$$

which means that $n_3=5$, so that $V{\downarrow}A$ is not MF.

Case 7: $(a, b) \in \{(1, 1), (1, 2), (2, 1)\}$. Here we have $r=16$, respectively 26, 22. If $p \neq 7$, respectively $p \neq 7$, $p \neq 11$, the Weyl modules are irreducible and we may apply Lemma 2.4. For the primes $p=7, 7, 11$, respectively, an application of Lemma 2.13 shows that $V{\downarrow}A$ is not MF in the second and third cases. Now for the case $\lambda=\omega_1+\omega_2$ and $p=7$, we must argue more carefully. Here, one checks that the weights $r, r-2, r-4, r-6$ occur with multiplicities 1, 2, 1, 2 respectively. Since $r-6$ does not occur as a weight in (r), while $r-4$ does, by Lemma 2.9 we conclude that $V{\downarrow}A$ is not MF.

Case 8: $b=0$. Here we view G as a subgroup of B_3 via the 7-dimensional irreducible representation afforded by $L(\omega_1)$. Then we have that $A \subset G$ is the principal A_1-subgroup of B_3 and moreover the B_3-module $L_{B_3}(a\omega_1)$ remains irreducible upon restriction to G, and affords the module V. (See [Seitz 1987, Table 1].) Hence, we can use the B_3 analysis, which is given in Proposition 4.16, to conclude.

Case 9: $a=0$, $b \geq 4$. Here the T_A-weight $r-6$ is afforded by $\lambda-21$, $\lambda-12$ and $\lambda-03$, each of which has multiplicity 1 in $\Delta(\lambda)$ and so $n_3=3$. In particular, none of the listed weights affords a composition factor of $\Delta(\lambda)$, nor does $\lambda-11$. Now we separate into two cases. First suppose that $\lambda-22$ does not afford a composition factor of $\Delta(\lambda)$; then $n_4 \geq 5$ and $V{\downarrow}A$ is not MF.

Now suppose that $\nu=\lambda-22$ affords a composition factor of $\Delta(\lambda)$ and so by Lemma 3.18 we have $3b+2 \equiv 0 \bmod p$. We first treat the case where $b \geq 6$. We claim that $n_5=4$. The T_A-weight $r-10$ is afforded by $\lambda-23$, $\lambda-32$, $\lambda-14$, and $\lambda-05$. The first two occur in the composition factor afforded by ν, each with multiplicity 1, and using the multiplicities in the Weyl module and Proposition 2.1, we see that each of the four listed weights occurs with multiplicity 1 in V, establishing the claim. The T_A-weight $r-12$ is afforded by $\lambda-42$, $\lambda-33$, $\lambda-24$, $\lambda-15$ and $\lambda-06$. The second of these weights has multiplicity 4 in $\Delta(\lambda)$ and occurs with multiplicity 2 in $L(\nu)$. Moreover, this weight does not afford a composition factor of $\Delta(\lambda)$ (nor does any dominant weight $\eta \neq \nu$ with $\lambda - 33 \prec \eta \prec \lambda$) and so occurs with multiplicity 2 in V. Hence, $n_6 \geq 6$ and $V{\downarrow}A$ is not MF.

It remains to consider the cases $b = 4$ and $b = 5$ with $p = 7$, respectively $p = 17$ and $\dim V = 267$, respectively 546. In both cases, an application of Lemma 2.13 shows that $V{\downarrow}A$ is not MF.

Case 10: $a = 0$ and $1 \leq b \leq 3$. When $b = 1$, the Weyl module is irreducible and the result follows from Lemma 2.4. If $b = 2$ and $p \neq 7$, we may apply Lemma 2.4 to conclude. When $(b, p) = (2, 7)$, we use Lemma 2.10 and the proof of [Liebeck et al. 2015, Lemma 4.5] to deduce that $n_0 = 1, n_1 = 1, n_2 = 2, n_3 = 2, n_4 = 3, n_5 = 4$, so that $V{\downarrow}A$ has composition factors (20), (16) and (12). Since the T_A-weight 12 lies in the composition factor (16) but the T_A-weight 10 does not, Lemma 2.9 implies that $V{\downarrow}A$ is not MF.

Finally, we consider the case $b = 3$, where $r = 30$. Here the Weyl module is irreducible unless $p = 11$. If $p \neq 11$, the result follows from Lemma 2.4. If $p = 11$, we use [Lübeck 2018] to see that $n_0 = 1, n_2 = 1, n_2 = 2, n_3 = 3, n_4 = 3, n_5 = 3$. We then deduce that $V{\downarrow}A$ has no composition factor (22), nor (20). But then $\dim V = 148 > B(30) - \dim(20) - \dim(22)$, so that $V{\downarrow}A$ is not MF. □

4. The case where G has rank at least 3

We handle the case where G has rank at least 3, establishing the next proposition.

Proposition 4.1. *Suppose that G has rank at least 3 and $p \leq r$. Then $V{\downarrow}A$ is not MF.*

We assume throughout Section 4 that $p \leq r$. By Proposition 2.14(i) we only need to consider the case $\lambda = c_i\omega_i + c_j\omega_j$ (with c_i or c_j possibly 0), i.e., the weight λ has support on at most two nodes.

4.1. *The case $c_i c_j$ not 0.*

We treat the case where $\lambda = c_i\omega_i + c_j\omega_j$ with $c_i c_j \neq 0$ in a sequence of lemmas.

Lemma 4.2. *Suppose that G has rank at least 4 and $\lambda = c_i\omega_i + c_j\omega_j$ with α_i and α_j adjacent and $c_i, c_j \geq 1$. Then $V{\downarrow}A$ is not MF.*

Proof. Since $p \geq h$ we have $p \geq 5, 11, 11, 7$ respectively for $G = A_\ell, B_\ell, C_\ell, D_\ell$ and $p \geq 13, 13, 19, 31$ respectively for $G = F_4, E_6, E_7, E_8$. By Proposition 2.14(iv) and (v), we can assume that $c_i = 1$ or $c_j = 1$, and α_i or α_j is an end-node. Recall that by Proposition 2.1, the set of weights of V is the same as the set of weights of $\Delta(\lambda)$. Using this, it is straightforward to see that if $c_i \geq 3$ or $c_j \geq 3$, then $V{\downarrow}A$ is not MF.

For example if $\lambda = c_1\omega_1 + \omega_2$ (so G is not of type E_ℓ) and $c_1 \geq 3$, the T_A-weight $r-6$ is afforded by $\lambda-123$, $\lambda-234$, $\lambda-12^2$, $\lambda-1^22$ and $\lambda-1^3$. Therefore $n_3 \geq 5$, and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. Similarly, if $\lambda = \omega_1 + c_2\omega_2$, with $c_2 \geq 3$ (so again G is not of type E_ℓ), then the T_A-weight $r-6$ is afforded by $\lambda-123$, $\lambda-234$, $\lambda-12^2$, $\lambda-2^23$ and $\lambda-2^3$. As before, $n_3 \geq 5$, and $V{\downarrow}A$ is not MF. If $c_i = 2$ or $c_j = 2$, by Lemma 2.18 we have $\dim V_{\lambda-\alpha_i-\alpha_j} > 1$, and by Proposition 2.14(vi) the module $V{\downarrow}A$ is not MF. Thus, we reduce to $c_i = c_j = 1$.

Consider the weight $\lambda = \omega_1 + \omega_2$. For G classical, the weights $\lambda-123 = (\lambda-12)^{s_3}$, $\lambda-234$, $\lambda-1^22 = (\lambda-2)^{s_1}$, $\lambda-12^2 = (\lambda-1)^{s_2}$ occur with multiplicities 2, 1, 1, 1 respectively by Lemma 2.18. Therefore $n_3 \geq 5$ and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. The same argument, with the appropriate relabelling of indices, handles all remaining cases where $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$, including the cases $G = E_\ell$ and $\lambda \in \{\omega_1+\omega_3, \omega_2+\omega_4, \omega_{\ell-1}+\omega_\ell\}$. For the group of type F_4 and the weights $\omega_1+\omega_2$ and $\omega_3+\omega_4$, we use the weight space dimensions provided in [Lübeck 2018] to conclude again that $n_3 \geq 5$.

Therefore we reduce to $G = B_\ell$ or $G = C_\ell$ with $\lambda = \omega_{\ell-1}+\omega_\ell$. Suppose $G = B_\ell$. Since $p \geq h$, by Lemma 2.18 we have $\dim V_{\lambda-(\ell-1)\ell} = 2$. The T_A-weight $r-6$ is afforded by $\lambda-(\ell-1)\ell^2 = (\lambda-(\ell-1)\ell)^{s_\ell}$, $\lambda-(\ell-2)(\ell-1)\ell$ and $\lambda-(\ell-1)^2\ell$. If $\ell = 4$, the first two weight spaces have dimension 2 by [Lübeck 2018]. By Lemma 2.20, for any $\ell \geq 4$, we have $n_3 \geq 5$, and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. The C_ℓ case is handled similarly. □

Lemma 4.3. *Let $G = A_3$ and $\lambda = c\omega_1 + \omega_2$ or $\omega_1 + c\omega_2$, with $c > 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Proposition 2.14(vi) we can assume that the weight space $\lambda-12$ is 1-dimensional. In particular we must have $c = p-2$ by Lemma 2.18. Let us start with $\lambda = \omega_1+(p-2)\omega_2$. Since $p \geq 5$, the T_A-weight $r-6$ is afforded by $\lambda-123$, $\lambda-12^2$, $\lambda-2^23$, $\lambda-2^3$ and $\lambda-1^22$. Therefore Lemma 2.6 implies that $V{\downarrow}A$ is not MF.

For the case $\lambda = (p-2)\omega_1 + \omega_2$, we will use a dimension argument. We refer to the discussion in [Jantzen 2003, Part II, 8.20], where the weight λ satisfies the conditions of the weight λ_2, with $s = 1 = t$ and $r = p-3$. Then one has

$$\dim V = \dim \Delta(\lambda) - \dim \Delta(\lambda-\alpha_1-\alpha_2) + \dim \Delta(\lambda-2\alpha_1-2\alpha_2-\alpha_3).$$

Using the Weyl degree formula we find that $\dim V = \frac{(p-1)}{6}(p^2+7p+18)$. We have $r = 3p-2$, and a simple calculation shows that $B(r) = \frac{(p+1)(3p-1)}{2}$. Since $B(r) < \dim V$ for all $p > 3$, by Lemma 2.12 we conclude that $V{\downarrow}A$ is not MF. □

Lemma 4.4. *Let $G = B_3$ or C_3 and let $\lambda \in \{c\omega_1+\omega_2, \omega_1+c\omega_2, c\omega_2+\omega_3, \omega_2+c\omega_3\}$, with $c > 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Proposition 2.14(vii) we can assume that the weight space $\lambda - ij$ is 1-dimensional, where $\lambda = c_i\omega_i + c_j\omega_j$. Note that $p \geq 7$ as $p \geq h$.

Case 1: $\lambda = c\omega_1 + \omega_2$. As the weight space $\lambda - ij$ is 1-dimensional, we have $c = p - 2$ by Lemma 2.18. In particular $c \geq 5$. The T_A-weight $r - 6$ is afforded by $\lambda - 1^3$, $\lambda - 1^2 2$, $\lambda - 12^2$, and $\lambda - 123$; in addition, for $G = B_3$, $r - 6$ is afforded by $\lambda - 23^2$ and if $G = C_3$, by $\lambda - 2^2 3$. Hence $n_3 \geq 5$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 2: $\lambda = \omega_1 + c\omega_2$. As in the previous case, we reduce to $c = p - 2$, so $c \geq 5$. The T_A-weight $r - 6$ is afforded by $\lambda - 12^2$, $\lambda - 1^2 2$, $\lambda - 2^2 3$, $\lambda - 123$, and $\lambda - 2^3$. Therefore $n_3 \geq 5$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 3: $G = B_3$ and $\lambda = c\omega_2 + \omega_3$. The T_A-weight $r - 8$ is afforded by $\lambda - 12^2 3$, $\lambda - 1^2 2^2$, $\lambda - 2^3 3$, $\lambda - 23^3$, $\lambda - 123^2$, $\lambda - 2^2 3^2$ which implies $n_4 \geq 6$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 4: $G = C_3$ and $\lambda = c\omega_2 + \omega_3$. By Lemma 2.18 we may assume that $c + 4 \equiv 0 \bmod p$, implying $c \geq 3$. The T_A-weight $r - 6$ is afforded by $\lambda - 2^3$, $\lambda - 12^2$, $\lambda - 2^2 3$, $\lambda - 123$, $\lambda - 23^2$, which implies $n_3 \geq 5$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 5: $G = B_3$ and $\lambda = \omega_2 + c\omega_3$. As above we reduce to $c + 4 \equiv 0 \bmod p$ and so $c \geq 3$. The T_A-weight $r - 6$ is afforded by $\lambda - 123$, $\lambda - 2^2 3$, $\lambda - 3^3$ and $\lambda - 23^2$. By Lemma 3.6, the Weyl module $\Delta_{B_2}(1c)$ has exactly two composition factors $L_{B_2}(1c)$ and $L_{B_2}(0c)$, the latter afforded by $\lambda - 11$. Therefore the multiplicity of the weight $\lambda - 23^2$ in V is 2 and so $n_3 \geq 5$, which by Lemma 2.6 implies that $V{\downarrow}A$ is not MF.

Case 6: $G = C_3$ and $\lambda = \omega_2 + c\omega_3$. This is entirely similar. Here we may assume $2c + 3 \equiv 0 \bmod p$. If $c \geq 4$, the T_A-weight $r - 8$ is afforded by $\lambda - 12^2 3$, $\lambda - 2^3 3$, $\lambda - 23^3$, $\lambda - 2^2 3^2$, $\lambda - 123^2$, and $\lambda - 3^4$. Therefore $n_4 \geq 6$, and $V{\downarrow}A$ is not MF by Lemma 2.6. If $c \leq 3$, we must have $c = 2$ and $p = 7$. By [Lübeck 2018], all weight spaces of V are 1-dimensional. The T_A-weight $r - 8$ is again afforded by the first five weights listed above. On the other hand, the T_A-weight $r - 6$ is afforded precisely by $\lambda - 123$, $\lambda - 23^2$ and $\lambda - 2^2 3$, implying that $n_3 = 3$. Therefore $n_4 - n_3 \geq 2$, and by Lemma 2.9 we conclude that $V{\downarrow}A$ is not MF. □

Lemma 4.5. *Let $G = A_3$, B_3 or C_3 and $\lambda = \omega_1 + \omega_2$ or $\omega_2 + \omega_3$. Then $V{\downarrow}A$ is not MF.*

Proof. Consider $G = A_3$. Then $r = 7$ and $p = 5$ or $p = 7$ as $r \geq p \geq h$. In both cases the Weyl module is irreducible and the conditions for Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF.

Now consider $G = B_3$. Since $p \geq 7$, the Weyl module is irreducible. The conditions for Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF.

Finally consider $G = C_3$. If $(\lambda, p) \neq (\omega_1 + \omega_2, 7)$ we can conclude as for B_3. Therefore assume that $\lambda = \omega_1 + \omega_2$ with $p = 7$. By Lemma 2.18 we have $\dim V_{\lambda-123} = 2$, and it is straightforward to see that $n_3 \geq 5$, which by Lemma 2.6 implies that $V{\downarrow}A$ is not MF. □

Lemma 4.6. *Assume that G has rank at least* 3 *and $\lambda = \omega_i + \omega_j$ where α_i and α_j are end-nodes. Then $V{\downarrow}A$ is not MF.*

Proof. Consider first the case where $G = A_\ell$, where $p \geq h = \ell + 1$. By [Lübeck 2001] the Weyl module $\Delta(\lambda)$ is irreducible if and only if $p \neq \ell+1$. If $p \neq \ell+1$, the conditions of Lemma 2.4 are satisfied and therefore $V{\downarrow}A$ is not MF. We therefore reduce to the case $p = \ell + 1$, where V is isomorphic to the quotient of $\Delta(\lambda)$ by a 1-dimensional trivial submodule. For $d < \ell$, it is straightforward to see that $n_d = d + 1$ (where we use that $r = 2\ell = 2(p-1)$). Therefore by Lemma 2.9(i) we know that $(p+1)$ is a composition factor of $V{\downarrow}A$. Now the T_A-weight $p-3$ occurs with multiplicity one more than the T_A-weight $p-1$, and it does not occur as a weight in $(p+1)$, while $p-1$ does. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF.

If $G = B_\ell$ or C_ℓ and $\ell \geq 4$, the first paragraph of the proof of [Liebeck et al. 2015, Lemma 3.5] shows that $n_3 \geq 5$, so $V{\downarrow}A$ is not MF by Lemma 2.6. If $G = C_3$, we can apply Lemma 2.4 to conclude that $V{\downarrow}A$ is not MF.

Now assume $G = B_3$. We have $p = 7$ or $p = 11$, as $r = 12$. If $p = 11$, the Weyl module is irreducible by [Lübeck 2001], and the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. When $p = 7$, using [Lübeck 2018], we find that $n_2 = 3$ and $(r-4)$ is therefore a composition factor by Lemma 2.9(i). Furthermore, we have $n_3 = 3$, $n_4 = 4$ and the T_A-weight $r-8$ does not occur as a weight in $(r-4)$, while the T_A-weight $r-6$ does. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF.

Now consider $G = D_\ell$, with $\ell \geq 4$. If $\lambda = \omega_1 + \omega_{\ell-1}$, the T_A-weight $r - 2(\ell-1)$ is afforded by $\lambda - 1 \cdots (\ell-1)$, $\lambda - 2 \cdots \ell$, and $\lambda - 1 \cdots (\ell-2)\ell$. Since $p \geq h$ we have $p > \ell$, and therefore by Lemmas 2.19 and 2.20 we have $\dim V_{\lambda-1\cdots(\ell-1)} = \ell - 1$. Therefore $n_{\ell-1} \geq \ell + 1$ and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. It is also easy to see that if $\lambda = \omega_{\ell-1} + \omega_\ell$, we have $n_3 \geq 5$. Again Lemma 2.6 implies that $V{\downarrow}A$ is not MF.

Finally, if G is exceptional, the arguments used in the proof of [Liebeck et al. 2015, Lemma 3.6] in characteristic zero allow us to conclude, as [Lübeck 2018] shows that the relevant weight spaces in V have the same dimension as the corresponding weight spaces in $\Delta_K(\lambda)$. □

Proposition 4.7. *Suppose that G has rank at least 3 and $\lambda = c_i\omega_i + c_j\omega_j$ with $c_i, c_j \geq 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 4.2, if α_i and α_j are adjacent and G has rank at least 4, the module $V{\downarrow}A$ is not MF. If α_i and α_j are adjacent and G has rank 3, Lemmas 4.3–4.5 combine to imply that $V{\downarrow}A$ is not MF.

Now assume that α_i and α_j are not adjacent, in which case by Proposition 2.14 we can assume that $c_i = c_j = 1$ and α_i and α_j are both end-nodes. In this case, by Lemma 4.6 we conclude that $V{\downarrow}A$ is not MF. □

4.2. ***The case where $\lambda = b\omega_i$.*** We now consider the case $\lambda = b\omega_i$. Note that if G is classical, then $\lambda \neq \omega_1$, as we are assuming that $p \leq r$, and necessarily $p \geq h$.

Lemma 4.8. *Assume that $G = A_\ell, B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$. Let $\lambda = b\omega_1$, with $b \geq 2$. Then $V{\downarrow}A$ is not MF.*

Proof. We first consider the case $b = 2$ and start by assuming that $(G, p) \neq (B_\ell, 2\ell + 1)$. By [Lübeck 2001] and since $p \geq h$, the Weyl module is irreducible. A simple check shows that the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. Consider now the case $G = B_\ell$ and $p = 2\ell + 1$, where V is isomorphic to the quotient of $\Delta(\lambda)$ by a 1-dimensional trivial submodule. For all strictly positive weights $r - 2d$, we have $n_d = \dim(\Delta_K(\lambda){\downarrow}A_K)_{r-2d}$. By [Liebeck et al. 2015, Lemma 4.2], we have $\Delta_K(\lambda){\downarrow}A_K = (4\ell) + (4\ell - 4) + \cdots$, which implies $n_d = d+1$ for d even with $d < 2\ell$, and $n_{d+1} = n_d$ for d odd with $d+1 < 2\ell$. By Lemma 2.9(i), for all $0 \leq d < 2\ell$ we have that $(r - 2d)$ is a composition factor of $V{\downarrow}A$. In particular, either $(p + 1)$ or $(p + 3)$ is a composition factor of $V{\downarrow}A$. In the first case, the T_A-weight $p - 3$ occurs with multiplicity one more than the T_A-weight $p-1$, but does not occur as a weight in $(p+1)$. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF. Similarly, if $(p + 3)$ is a composition factor of $V{\downarrow}A$, then the T_A-weight $p - 5$ (note that $p > 5$ since $\ell \geq 3$) occurs with multiplicity one more than the T_A-weight $p - 3$, but does not occur as a weight in $(p + 3)$, concluding in the same way.

Now consider the case $b \geq 3$. Start with $G = A_\ell$. Here $V = \Delta(\lambda)$ by [Seitz 1987, 1.14]. If $\ell \geq 6$, the first paragraph of the proof of [Liebeck et al. 2015, Lemma 4.4] shows that $n_6 \geq 7$, which by Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF. If $b = 4$ and $\ell = 4$ or $\ell = 5$, we similarly have $n_4 \geq 5$. If $b \geq 5$ and $(b, \ell) \neq (5, 3)$, we have $n_6 - n_5 \geq 2$. Therefore by Lemma 2.9(iii) and (ii), we reduce down to the cases $(b, \ell) = (4, 3), (5, 3), (3, 3), (3, 4), (3, 5)$. For these cases we can conclude using Lemma 2.4, unless $\ell = b = 3$ and $p = 5$. In this case $r = 9$ and the weights 9, 7, 5, 3 occur respectively with multiplicities 1, 1, 2, 3. By Lemma 2.9(i), we have that

(5) is a composition factor, and in addition $r-6$ does not occur as weight in this composition factor. Therefore by Lemma 2.9(iv) the module $V{\downarrow}A$ is not MF.

The C_ℓ case follows from the $A_{2\ell-1}$ case since $A < C_\ell < A_{2\ell-1}$ is a principal A_1-subgroup of $A_{2\ell-1}$ and $V = S^b(L_{C_\ell}(\omega_1)) = L_{A_{2\ell-1}}(b\omega_1){\downarrow}C_\ell$. Now consider $G = B_\ell$. If $b \geq 4$, it is straightforward to check that $n_4 \geq 5$, which by Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF. If $b = 3$ and $\ell \geq 4$, by the proof of [Liebeck et al. 2015, Lemma 4.4], we have $n_6 \geq 7$, concluding in the same way. If $\ell = b = 3$ we have $p \leq r = 18$ and by Lemma 2.4 the module $V{\downarrow}A$ is not MF. Finally, we consider the case where $G = D_\ell$ where we have $A \leq B_{\ell-1} < G$. Since $\Delta_{B_{\ell-1}}(b\omega_1)$ is a composition factor of $\Delta_{D_\ell}(b\omega_1)$, if $\Delta_{D_\ell}(b\omega_1){\downarrow}A$ is MF, so is $\Delta_{B_{\ell-1}}(b\omega_1)$. Therefore by the B_ℓ result, we conclude that $V{\downarrow}A$ is not MF. □

Lemma 4.9. *Assume that $G = B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$. Let $\lambda = b\omega_i$, with $i > 1$ and $b > 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 2.16 we can assume that $\lambda = b\omega_\ell$. We will treat the case $G = D_\ell$ at the end of the proof.

Assume for now that $b \geq 3$. If $G = C_\ell$, the T_A-weight $r-6$ is afforded by $\lambda-\ell^3$, $\lambda-(\ell-2)(\ell-1)\ell$, $\lambda-(\ell-1)^2\ell$, $\lambda-(\ell-1)\ell^2$. If $G = B_\ell$, the T_A-weight $r-6$ is afforded by $\lambda-(\ell-1)\ell^2$, $\lambda-(\ell-2)(\ell-1)\ell$ and $\lambda-\ell^3$, and using the fact that the B_2-module $\Delta(b\omega_2)$ is irreducible, by Lemma 2.20, we have that the first of these weights has multiplicity 2. Hence for both of the groups C_ℓ and B_ℓ, we have $n_3 \geq 4$. By Lemma 2.9(iii), the module $V{\downarrow}A$ is not MF.

We now consider the case $b = 2$ when $G = C_\ell$ and first assume that $\ell \geq 5$. The T_A-weight $r-10$ is afforded by $\lambda-(\ell-1)^3\ell^2$, $\lambda-(\ell-2)(\ell-1)^2\ell^2$, $\lambda-(\ell-1)^2\ell^3$, $\lambda-(\ell-2)^2(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)(\ell-1)\ell^2$ and $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)\ell$. Again, by Lemma 2.9(iii) the module $V{\downarrow}A$ is not MF.

Now consider the cases C_ℓ, for $\lambda = 2\omega_\ell$ and $\ell = 3, 4$ where $r = 18$, respectively 32, and $p \geq 7$, respectively 11. In both cases we have that $\Delta(2\omega_\ell)$ is irreducible by [Lübeck 2001]. For $\ell = 3$, the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. If $\ell = 4$, by the first paragraph of [Liebeck et al. 2015, Lemma 4.3] we have $\dim(\Delta_K(\lambda){\downarrow}A_K)_{r-12} \geq \dim(\Delta_K(\lambda){\downarrow}A_K)_{r-10}+2$. Therefore by Lemma 2.10 we find that $n_6 - n_5 \geq 2$, concluding by Lemma 2.9(ii).

Turn now to the case $G = B_\ell$ and $b = 2$. Here the T_A-weight $r-8$ is afforded by $\lambda-(\ell-2)(\ell-1)\ell^2$, $\lambda-(\ell-1)^2\ell^2$, $\lambda-(\ell-1)\ell^3$ and if $\ell \geq 4$, $\lambda-(\ell-3)(\ell-2)(\ell-1)\ell$. The first of these is conjugate to $\lambda-(\ell-1)\ell^2$ and so has multiplicity 2 by the first paragraph of this proof. Thus, if $\ell \geq 4$, $V{\downarrow}A$ is not MF by Lemma 2.9(iii). So finally, we reduce to $b = 2$ and $\ell = 3$, where $p \geq 7$ and $r = 12$. The Weyl module is irreducible by [Lübeck 2001]. The conditions of Lemma 2.4 are satisfied, so $V{\downarrow}A$ is not MF.

Finally suppose that $G = D_\ell$ and $\lambda = b\omega_\ell$. Since $A \le B_{\ell-1} \le D_\ell$ and $V\downarrow B_{\ell-1} \cong L_{B_{\ell-1}}(b\omega_{\ell-1})$, we may use the $B_{\ell-1}$ result to conclude. □

Lemma 4.10. *If $G = E_\ell$ and $\lambda = b\omega_i$ with $b > 1$, then $V\downarrow A$ is not MF.*

Proof. This follows verbatim from the proof of [Liebeck et al. 2015, Lemma 4.6], unless $i = \ell$ and $G = E_7$ or E_8 with $b = 2$ or $b = 3$. In these remaining cases, it is not difficult to check that we have $n_6 \ge n_5 + 2$ (as stated in the proof of [Liebeck et al. 2015, Lemma 4.6]), as this count relies on 1-dimensional weight spaces. By Proposition 2.14 the module $V\downarrow A$ is not MF. □

Lemma 4.11. *If $G = F_4$ and $\lambda = b\omega_i$ with $b > 1$, then $V\downarrow A$ is not MF.*

Proof. By Lemma 2.16 the simple root α_i corresponds to an end-node of the Dynkin diagram. If $i = 1$ we can conclude as in the first paragraph of the proof of [Liebeck et al. 2015, Lemma 4.7].

Assume $i = 4$. If $b \ge 3$, like in [Liebeck et al. 2015, Lemma 4.7] we have $n_4 \ge 5$, concluding by Lemma 2.9(iii). If $b = 2$ we have $V = \Delta(\lambda)$ by [Lübeck 2001], and since $r = 32$ and $r \ge p > 11$, the conditions of Lemma 2.4 are satisfied. Thus, $V\downarrow A$ is not MF. □

It remains to consider the case $\lambda = \omega_i$. Recall that for G classical, we have $\lambda \ne \omega_1$.

Lemma 4.12. *Assume that G has rank at least 3, $\lambda = \omega_i$ and that one of the following holds:*

(i) *$G = A_\ell, B_\ell, C_\ell$ with $\ell \ge 3$ or $G = D_\ell$ with $\ell \ge 4$, and $4 \le i \le \ell - 3$.*

(ii) *$G = A_\ell$, $i = 3$, and $\ell \ge 5$.*

(iii) *$G = A_\ell, B_\ell, C_\ell$ with $\ell \ge 3$ or $G = D_\ell$ with $\ell \ge 4$ and $i = 2$.*

Then $V\downarrow A$ is not MF.

Proof. (i) Lemma 2.15 applies, except when $G = D_7$, and implies that the module $V\downarrow A$ is not MF. For the case $G = D_7$, where $i = 4$, it is straightforward to see that $n_4 \ge 5$ and then Lemma 2.9(iii) implies that $V\downarrow A$ is not MF.

(ii) Here $V = \bigwedge^3(L(\omega_1))$. Assume for now that $l \ge 8$. The T_A-weight $r - 12$ is afforded by the wedge of weight vectors in $L(\omega_1)$ for each of the following triples of T_A-weights: $\ell(\ell-2)(\ell-16)$, $\ell(\ell-4)(\ell-14)$, $\ell(\ell-6)(\ell-12)$, $\ell(\ell-8)(\ell-10)$, $(\ell-2)(\ell-4)(\ell-12)$, $(\ell-2)(\ell-6)(\ell-10)$, $(\ell-4)(\ell-6)(\ell-8)$. Therefore $n_6 \ge 7$, and Lemma 2.9(iii) implies that $V\downarrow A$ is not MF.

For the remaining cases, when $5 \le l \le 7$, we have $\Delta(\lambda) = V$ and a quick check shows that the conditions of Lemma 2.4 are satisfied, implying that $V\downarrow A$ is not MF.

(iii) Here $\lambda = \omega_2$, and as $p > \ell$ we have $V = \Delta(\lambda)$ [Lübeck 2001, Table 2]. We have $r = 2\ell-2, 4\ell-2, 4\ell-4$ or $r = 4\ell-6$ according to whether $G = A_\ell, B_\ell, C_\ell$ or $G = D_\ell$. Furthermore we have p greater than $\ell, 2\ell-1, 2\ell, 2\ell-2$ respectively. It is then an easy check to see that the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. □

Lemma 4.13. *Assume that $G = B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$, and that $\lambda = \omega_i$ for $i \geq 3$ and V is not a spin module for B_ℓ or D_ℓ. Then $V{\downarrow}A$ is not MF.*

Proof. If $G = B_\ell$ or D_ℓ, then $V = \bigwedge^i(\omega_1)$ by [Seitz 1987] and the result follows from Lemma 4.12(i)(ii) for $G = A_{2\ell}$ or $A_{2\ell-1}$. Indeed, if $G = B_\ell$, then A is regular in $A_{2\ell}$ and $V = L_{A_{2\ell}}(\omega_i){\downarrow}G$, while if $G = D_\ell$, then $A < B_{\ell-1} < D_\ell$ and by the B_ℓ case there is a $B_{\ell-1}$-composition factor of V (namely $L_{B_{\ell-1}}(\omega_i)$) that is not multiplicity-free in its restriction to A, implying that $V{\downarrow}A$ is not MF.

We now consider the case $G = C_\ell$. By part (i) of Lemma 4.12 we can furthermore assume that $i = 3$ or $i > \ell-3$. If $i = \ell-2 > 3$, the T_A-weight $r-8$ has multiplicity at least 5 as it is afforded by five different weights as in the proof of [Liebeck et al. 2015, Lemma 5.3]. Therefore Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF.

Assume $i = \ell-1 > 3$. Because $\ell \geq 5$, the T_A-weight $r-12$ is afforded by $\lambda-(\ell-3)(\ell-2)(\ell-1)^3\ell$, $\lambda-(\ell-2)(\ell-1)^3\ell^2$, $\lambda-(\ell-2)^2(\ell-1)^3\ell$, $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)^2(\ell-1)^2\ell$. When $\ell = 5$, the last two weights have multiplicity 2 in V by [Lübeck 2018], and therefore the same holds for $\ell \geq 5$ by Lemma 2.20. Thus, $n_6 \geq 7$, and by Lemma 2.9(iii) the module $V{\downarrow}A$ is not MF.

Now assume $i = \ell > 3$. Start with $\ell = 4$ or 5. In both cases the Weyl module is irreducible. We have $r = 16$ if $\ell = 4$, and $r = 25$ if $\ell = 5$. If $(\ell, p) \neq (5, 13)$, the conditions of Lemma 2.4 are satisfied, showing that $V{\downarrow}A$ is not MF. In the remaining case (when $(\ell, p) = (5, 13)$), we find that $B(r) - \dim(r-2) = 118 < \dim V = 132$. Therefore by Lemma 2.13, the module $V{\downarrow}A$ is not MF.

Now suppose $\ell \geq 6$ with $\lambda = \omega_\ell$. Here the T_A-weight $r-10$ has multiplicity 4 as it is afforded precisely by $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)\ell$, $\lambda-(\ell-2)^2(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)(\ell-1)^2\ell$, $\lambda-(\ell-2)(\ell-1)^2\ell^2$. The T_A-weight $r-12$ has multiplicity at least 6 as it is afforded by $\lambda-(\ell-5)(\ell-4)(\ell-3)(\ell-2)(\ell-1)\ell$, $\lambda-(\ell-2)^2(\ell-1)^2\ell^2$, $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)^2(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)(\ell-1)^2\ell^2$, $\lambda-(\ell-2)(\ell-1)^3\ell^2$. By Lemma 2.9(ii) $V{\downarrow}A$ is not MF.

Finally, assume $i = 3$. If $\ell \geq 6$, we have $n_6 \geq 7$, since by the last paragraph of the proof of [Liebeck et al. 2015, Lemma 5.3] there are seven distinct weights of V affording the T_A-weight $r-12$. Therefore $V{\downarrow}A$ is not MF by Lemma 2.9(iii).

In the remaining cases, when $\ell \in \{3, 4, 5\}$, the Weyl module is irreducible and we can apply Lemma 2.4, unless $\ell = 5$ and $p = 11$, in which case $r = 21 \equiv -1 \mod p$. In this case, we find that $B(r) - \dim(r-2) = 84 < \dim V = 110$. Therefore by Lemma 2.13, the module $V{\downarrow}A$ is not MF. □

Lemma 4.14. *Assume that V is a spin module for B_ℓ with $\ell \geq 3$ or D_ℓ with $\ell \geq 4$. Then $V{\downarrow}A$ is not MF.*

Proof. We have $V = \Delta(\lambda)$. If $G = D_\ell$, then $A \leq B_{\ell-1} < G$ and $V{\downarrow}B_{\ell-1}$ is the spin module for $B_{\ell-1}$; therefore it suffices to prove the lemma for $G = B_\ell$, where $r = \ell(\ell+1)/2$ and $\dim V = 2^\ell$. If $V{\downarrow}A$ is MF the dimension of V is at most $B_K(r)$, by Lemma 2.12. This implies that if $\ell \geq 10$, the module $V{\downarrow}A$ is not MF.

Now assume $\ell \leq 9$. Since $p > h = 2\ell$ we know that $p \nmid r$. Therefore if $V{\downarrow}A$ is MF the dimension of V is at most $B(r) - \dim(r-2)$, by Lemma 2.13. This then reduces our considerations to the pairs (n, p) in the list (5, 11), (5, 13), (6, 13), (6, 17), (6, 19), (7, 17), (7, 19), (7, 23), (8, 31). For every $3 \leq \ell \leq 8$, by Lemma 2.10 we can read the dimension of the T_A-weight space $r - 2k$ off the table in the proof of [Liebeck et al. 2015, Lemma 5.4]. In each case we apply part (iii) of Lemma 2.9 to find that $V{\downarrow}A$ is not MF. The first repeated composition factors are of highest weight respectively 5, 9, 9, 11, 15, 14, 14, 16, 24. □

Lemma 4.15. *Assume $G = E_\ell$ or F_4 and $\lambda = \omega_i$. Then $V{\downarrow}A$ is not MF.*

Proof. If $G = E_\ell$ and $\lambda = \omega_4$, the T_A-weight $r - 4$ is afforded by $\lambda - 34$, $\lambda - 24$, $\lambda - 45$. Therefore $n_2 \geq 3$, and by Lemma 2.9(iii), the module $V{\downarrow}A$ is not MF.

If $G = E_8$ and $\lambda = \omega_3$ or ω_6, then $r = 182$ respectively $r = 168$, giving $B_K(r) =$ 8464 and 7225 respectively. By [Lübeck 2001], we have $\dim V > B_K(r)$ and therefore by Lemma 2.12 the module $V{\downarrow}A$ is not MF. If $G = E_8$ and $\lambda = \omega_5$, by Lemma 2.15 the module $V{\downarrow}A$ is not MF.

In all remaining cases, by [Lübeck 2001] we have that $V = \Delta(\lambda)$. Lemma 2.12 then allows us to reduce to the case where V is the minimal module for G, or the adjoint module for E_6, E_7 or F_4. The conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. □

Proposition 4.16. *Suppose that G has rank at least 3 and $\lambda = b\omega_i$, with $b \geq 2$ for G classical and $b \geq 1$ for G of exceptional type. Then $V{\downarrow}A$ is not MF.*

Proof. If G is classical, this is a direct consequence of Lemmas 4.8 and 4.9. If G is exceptional and $b \geq 2$, we similarly conclude by Lemmas 4.10 and 4.11.

If $b = 1$, where G is exceptional, we reach the same conclusion by Lemmas 4.12, 4.13, 4.14, 4.15. □

Proof of Proposition 4.1. If V is MF, then by Proposition 2.14, the weight λ is of the form $c_i\omega_i + c_j\omega_j$. If $c_ic_j \neq 0$, then the conclusion follows from Proposition 4.7. If $c_ic_j = 0$, then the conclusion follows from Proposition 4.16. □

5. Proof of Corollary 2

We prove Corollary 2, thereby extending Theorem 1 to the case where λ is not p-restricted. The following lemma serves as an inductive tool.

Lemma 5.1. *Let $\lambda = \sum_{i=0}^{t} p^i\lambda_i$ where λ_i is a p-restricted dominant weight for all $0 \le i \le t$. Assume that for some s with $0 \le s < t$, we have $\big(\sum_{i=0}^{s} p^i\lambda_i\big){\downarrow}T_A < p^{s+1}$. Then $V{\downarrow}A$ is MF if and only if*

(i) *$L\big(\sum_{i=0}^{s} p^i\lambda_i\big){\downarrow}A$ is MF, and*

(ii) *$L\big(\sum_{i=s+1}^{t} p^i\lambda_i\big){\downarrow}A$ is MF.*

Proof. Let $V_1 = L\big(\sum_{i=0}^{s} p^i\lambda_i\big)$ and $V_2 = L\big(\sum_{i=s+1}^{t} p^i\lambda_i\big)$, so that $V = V_1 \otimes V_2$. If $V_2 = L(0)$, the statement is trivial. Thus, assume $V_2 \neq L(0)$.

One direction is clear. If either $V_1{\downarrow}A$ or $V_2{\downarrow}A$ is not MF, then $V{\downarrow}A$ is not MF. Assume now that both $V_1{\downarrow}A$ and $V_2{\downarrow}A$ are MF, and let $V_1{\downarrow}A$ have composition factors $(r_0), (r_1), \ldots, (r_m)$, so that by the assumption on s, we have $p^{s+1} > r_0 > r_1 > \cdots > r_m$. Similarly let $V_2{\downarrow}A$ have composition factors $(v_0), (v_1), \ldots, (v_n)$ where $v_0 > v_1 > \cdots > v_n \ge p^{s+1}$. Then for all $0 \le i \le m$ and $0 \le j \le n$ we have

$$(r_i) \otimes (v_j) \cong (r_i + v_j),$$

since $r_i < p^{s+1}$ and $v_j \ge p^{s+1}$. This implies that the composition factors of $V_1 \otimes V_2$ are precisely of the form $(r_i + v_j)$, which are all clearly distinct. Therefore $V{\downarrow}A$ is MF. □

Let us restate, and prove, Corollary 2.

Corollary 5.2. *Let $\lambda = \sum_{i=0}^{t} p^i\lambda_i$ where each λ_i is a p-restricted dominant weight and set $r_i = \lambda_i{\downarrow}T_A$, for $0 \le i \le t$. Then $V{\downarrow}A$ is MF if and only if one of the following holds*:

(i) *We have $p > r_i$ and $\Delta_K(\lambda_i){\downarrow}A_K$ is MF for all $0 \le i \le t$.*

(ii) *The group G is of type A_2, $p = 3$ and there exists $0 \le i \le t$ such that $\lambda_i = \omega_1 + \omega_2$. For all $0 \le j \le t$ we have $\lambda_j \in \{0, \omega_1 + \omega_2, \omega_1, \omega_2\}$ and if $\lambda_j = \omega_1 + \omega_2$ for some $0 \le j \le t-1$, then $\lambda_{j+1} = 0$.*

(iii) *The group G is of type B_2, $p = 5$ and there exists $0 \le i \le t$ such that $\lambda_i = 2\omega_1$. For all $0 \le j \le t$ we have $\lambda_j \in \{0, 2\omega_1, \omega_1, \omega_2\}$ and if $\lambda_j = 2\omega_1$ for some $0 \le j \le t-1$, then $\lambda_{j+1} \in \{0, \omega_2\}$.*

Proof. We use induction on t. If $t = 0$, then λ is p-restricted and the statement follows from Theorem 1. Suppose now that $t > 0$ and that the statement is valid for all $0 \le t \le N$ for some $N \in \mathbb{N}$. Let $t = N + 1$ and $V_1 = L(\lambda_0)$, $V_2 = L\big(\sum_{i=1}^{t} p^i \lambda_i\big)$. If V_1 or V_2, is the trivial $\Bbbk G$-module, then we can conclude by the inductive assumption (since the Frobenius twist of a module M is MF if and only if the module M is MF). Therefore we can assume that V_1 and V_2 are nontrivial.

Suppose first that $V{\downarrow}A$ is MF. Then certainly $V_1{\downarrow}A$ and $V_2{\downarrow}A$ are both MF. If $r_0 < p$, by Lemma 5.1 and the inductive assumption, we conclude that (G, λ, p) is as in one of the three conclusions of the statement. Therefore assume that $r_0 \ge p$. By Theorem 1 we have $G = A_2$, $p = 3$ and $\lambda_0 = 11$, or $G = B_2$, $p = 5$ and $\lambda_0 = 20$.

Consider first the case $G = A_2$. By Theorem 1 and Table 1, we must have $\lambda_i \in \{0, 11, 10, 01\}$ for all $0 \le i \le t$. If $\lambda_1 = 0$, we conclude by the inductive assumption combined with Lemma 5.1 for $s = 1$. If $\lambda_1 \in \{\omega_1 + \omega_2, \omega_1, \omega_2\}$ then $V_1{\downarrow}A \otimes L(p\lambda_1){\downarrow}A$ has (4) as a repeated composition factor and so $V{\downarrow}A$ is not MF. For $G = B_2$, by Theorem 1 and Table 1, we have $\lambda_1 \in \{0, \omega_1, 2\omega_1, \omega_2\}$ and a straightforward computation shows that $V_1{\downarrow}A \otimes L(p\lambda_1){\downarrow}A$ has a repeated composition factor for $\lambda_1 \in \{\omega_1, 2\omega_1\}$.

Suppose now that (i) holds. Then $V{\downarrow}A$ is MF by the inductive assumption combined with Lemma 5.1.

If (ii) or (iii) holds, it is easy to verify that the conditions of Lemma 5.1 with $s = 1$ are satisfied, concluding again by the inductive assumption. □

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In memoriam
Gary Seitz