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Special Issue

In memoriam

Gary Seitz

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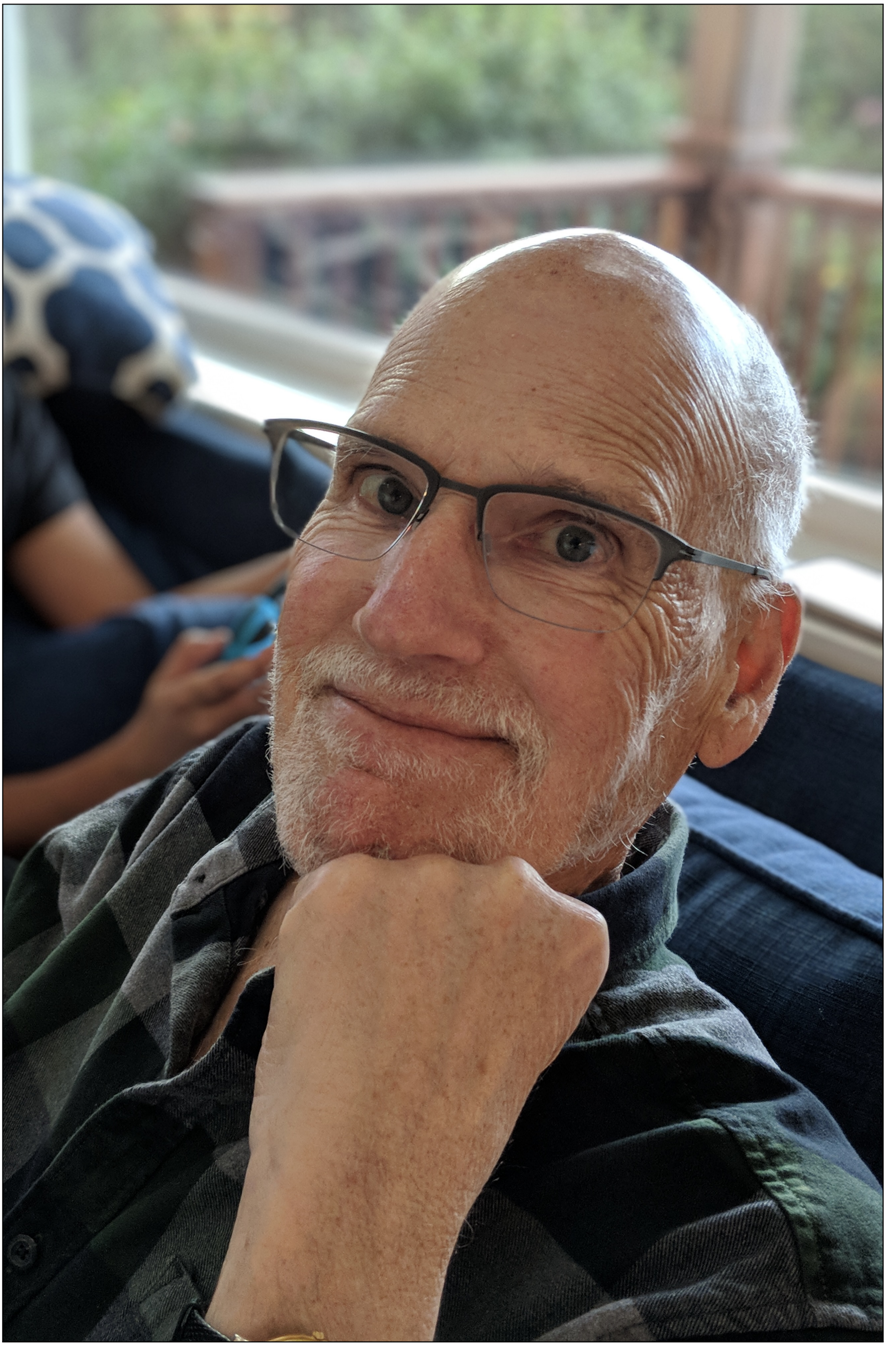

Gary Seitz (1943–2023)

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GARY SEITZ (1943–2023): IN MEMORIAM

Gary Seitz was born in Santa Monica, California, and grew up in Los Angeles. His father, David Seitz, was in the casino business, and his mother Sarah worked in a hair salon. As a youth, Gary was into body-building and bowling, and at one point had to make a career decision between going into mathematics or becoming a professional bowler. Fortunately for us, he decided on the former, and did his Bachelor's and Master's degrees at Berkeley. While at Berkeley he met Sheila Coutin, and they married while still undergraduates in 1964. They had two sons, Aaron and Steve, both of whom went on to have academic careers.

After Berkeley, Gary moved to the University of Oregon in Eugene for his PhD with advisor Charles Curtis, which he completed in 1968. He then held a postdoctoral research position at the University of Illinois at Chicago Circle until 1970, when he returned as a faculty member to Eugene, where he remained until his retirement. He served the University of Oregon with enormous distinction, both academically and administratively, as Head of Department 1994–2001, CAS Distinguished Professor from 2000, and Associate Dean of Natural Sciences 2002–2005.

Gary was a leading figure in algebra for over 50 years, publishing about 100 articles and books on a wide variety of topics, mainly centering around group theory: finite groups, algebraic groups, representation theory, maximal subgroups, and applications to other areas such as number theory and algebraic combinatorics. He was extremely collaborative in his research, publishing with 29 different coauthors, and holding visiting appointments at Caltech, Notre Dame, IHES, Bar Ilan, Tel Aviv, the Technion, IAS Princeton, Aarhus, Utrecht, Essen, Tokyo, Warwick and Imperial College London. He was named a Fellow of the American Mathematical Society in 2013.

Let us discuss some of the themes of Gary's research in a little more detail. In his PhD thesis and several subsequent papers, he proved deep results about a wide class of finite solvable groups known as M-groups. He then moved to Chicago, which at the time was a tremendous centre for finite group theory, particularly surrounding the finite simple groups and the attempt to classify them. Jacques Tits had recently introduced his theory of BN-pairs for finite groups, and their associated buildings, and

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he had shown that any simple group with a BN-pair of rank at least 3 is necessarily a group of Lie type, providing a geometric setting for these families of simple groups, and also a powerful method for identifying them. Tits' methods were largely geometric, and for ranks 1 and 2 the geometric structure was not strong enough — for example, the groups with a BN-pair of rank 1 are just the 2-transitive permutation groups. It was important to fill this gap, and together with Christoph Hering and Bill Kantor, also based in Chicago at the time, Gary succeeded in classifying the split BN-pairs of rank 1; soon thereafter, with Paul Fong, Gary classified the split BN-pairs of rank 2.

These results provided an essential tool in the ongoing programme to classify the finite simple groups, which was proceeding apace at the time. Gary made further key contributions to this programme with his work, partly with Michael Aschbacher, on standard subgroups. A standard subgroup of a finite group G is a quasisimple subgroup that is embedded in a very specific way in an involution centralizer in G. One key part of the classification programme was to determine the finite groups G that have a standard subgroup A belonging to one of the known families of quasisimple groups. Many authors were involved in this project; Gary and Michael handled the case where $C_G(A)$ has 2-rank at least 2, and in several further papers Gary dealt with the case of 2-rank 1 when A is a group in the family Lie(2) of groups of Lie type in characteristic 2. Along the way, Gary and Michael found it necessary to develop a complete theory of involution classes and centralizers for groups in Lie(2), and wrote a much-cited paper on this that proved to be a precursor of Gary's later fundamental work on unipotent elements in algebraic groups, on which more later.

The completion of the classification of finite simple groups was first announced in the early 1980s. However, not much was known about the subgroup structure of these groups, and in particular their maximal subgroups. The study of these maximal subgroups formed one of the themes of Gary's work for the next 30 years. The finite groups of Lie type are intimately related to the corresponding simple algebraic groups over algebraically closed fields, and in the 1950s Dynkin had solved the maximal subgroup problem for classical groups over $\mathbb{C}$; a major part of his solution was the determination of all triples (G, H, V) with V a finite-dimensional complex vector space and $G < H < \mathrm{Cl}(V)$, where $\mathrm{Cl}(V)$ is a classical group on V and G, H are connected algebraic groups acting irreducibly on V. Gary took on the formidable project of generalizing this result to algebraically closed fields of arbitrary characteristic. He gave part of this problem to his then PhD student Donna Testerman, and between them they solved it completely, publishing their results in two *Memoirs of the AMS*, totalling about 500 pages. This work has been used many times both within finite and algebraic group theory, and in its applications.

For the simple algebraic groups of exceptional Lie types ($G_2, \ldots, E_8$), one can hope to determine all the connected maximal subgroups. In further pioneering work, Gary achieved this in another *Memoir* published in 1991, assuming the characteristic

p of the underlying field is not too small ($p > 7$ suffices in all cases). In a later *Memoir* with Martin Liebeck, he extended the result to all characteristics, and also to disconnected subgroups. These results had implications for maximal subgroups of the finite exceptional groups of Lie type, and in a long series of papers with Martin, and also some with Arjeh Cohen, Jan Saxl and others, Gary built an edifice of theory on the subgroup structure of these families of simple groups.

We mentioned before Gary's work with Michael Aschbacher classifying involution classes in the groups in Lie(2). He continued to work on many different aspects of unipotent elements in finite and algebraic groups G of Lie type. In a 1983 paper, he built a theory of root groups relative to arbitrary maximal tori of the finite groups, and used this to determine the subgroups containing such tori. In other papers published in the 1990s, he classified the subgroups generated by root elements, and at the other extreme, by regular or semiregular unipotent elements.

In 2000 Gary achieved a breakthrough, solving the "saturation" problem of J-P. Serre for arbitrary unipotent elements of order p (the characteristic of the underlying field, assumed good for the simple algebraic group G): he proved that any such unipotent element u is contained in a unique 1-dimensional unipotent subgroup of a particularly nice A_1 subgroup (called a "good" A_1 by Gary), unique up to conjugacy in G by $C_G(u)$. This enables one to answer many questions about unipotents by studying the good A_1's, a beautiful class of subgroups. Gary published numerous further papers on this topic, culminating in his book with Martin Liebeck, which presents a complete theory of unipotent classes and centralizers in simple algebraic groups, and nilpotent classes in the corresponding Lie algebras.

Of course much of Gary's work already discussed involves heavy use of the representation theory of finite and algebraic groups. Gary also published many articles that are purely on this topic. An early one was a much-cited 1974 paper with Vicente Landazuri giving lower bounds for the dimensions of irreducible representations of groups in Lie(p) over fields of characteristic coprime to p; this work has been built on by many authors to classify all the low-dimensional representations of these groups, an important theory with many applications. Another highlight is Gary's 1992 paper with Jens Jantzen on the innocent-looking problem of determining which irreducible representations (in arbitrary characteristic) of the symmetric group S_n remain irreducible on restriction to S_{n-1}. The results and conjectures posed in this paper formed the first step in a theory of modular branching rules for representations of S_n developed by Alexander Kleshchev and others, now a fundamental tool of representation theory.

Another topic on which Gary made decisive contributions is the theory of G-complete reducibility (G-cr): this was introduced by Serre as a way of interpreting concepts of representation theory in the more general setting of maps between algebraic groups. In a 1996 *Memoir* with Martin Liebeck, Gary proved that arbitrary reductive subgroups of exceptional algebraic groups are completely reducible provided the underlying characteristic p is not too small ($p > 7$ suffices in all cases). Together with results of Jantzen and McNinch, and Serre himself, on classical groups, these results formed the basis of G-cr theory, which has since been taken much further by many authors.

Gary was involved in several projects applying group theory to other areas of mathematics. The most striking of these was his proof with Yoav Segev and Andrei Rapinchuk that all finite quotients of the multiplicative group of a finite dimensional division algebra are solvable. A consequence was the solution of the Margulis–Platonov conjecture on the normal subgroup structure of algebraic groups over number fields. The method behind their proof was based on the remarkable idea, pioneered by Segev, that the commuting graph of such a finite quotient must have strong connectivity properties; in several substantial papers, they proved that commuting graphs of nonsolvable groups could not have such properties.

Let us finally mention the topic on which Gary was working for most of the last ten years of his life: the theory of multiplicity-free representations. The project was to classify the triples (G, H, V), where $H < G < \mathrm{GL}(V)$ are connected reductive algebraic groups over an algebraically closed field of characteristic zero, and V is an irreducible G-module whose restriction to H is multiplicity-free (i.e., each composition factor appears with multiplicity 1). A great deal of classical work, going back to Dynkin, Howe, Kac, Stembridge, Weyl and others, can be set in this context. In his final *Memoir* with Martin and Donna, Gary determined all such triples in cases where H and G are both simple algebraic groups of type A, showing that there are many beautiful families of such representations.

There are large parts of Gary's output that we have not mentioned, but we hope we have conveyed some of the profound influence of his work across many areas and over many years.

Gary was the advisor of eleven PhD students, almost all of whom continued into the academic profession. Three of them, George McNinch, Gerhard Röhrle and Donna Testerman, have contributed articles to this volume.

We three had the privilege of learning continually from Gary's enormous depth of knowledge and ideas, as well as collaborating with him over many years. But we valued above all his warm, generous friendship; his wisdom in matters mathematical and non-mathematical; his boundless energy; and his wonderful company, full of laughter and fun. We miss him deeply.

Martin Liebeck
Gerhard Röhrle
Donna Testerman

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INTRINSIC COMPONENTS IN INVOLUTION CENTRALIZERS OF FUSION SYSTEMS

MICHAEL ASCHBACHER

This paper lays the foundation for the study of the saturated 2-fusion systems in which the centralizer of some fully centralized involution has a component whose center is nontrivial.

The results in this paper are part of a program to, first, classify a large subclass of the class of simple 2-fusion systems of component type, and then, second, to use the theorem on fusion systems to simplify the proof of the theorem classifying the finite simple groups. See [4; 5] for a description of the program; there is also a bit of discussion of the program below.

Let p be a prime and S a finite p-group. A *fusion system* on S is a category $\mathcal{F}$ whose objects are the subgroups of S and, for subgroups P, Q of S, the set $\hom_{\mathcal{F}}(P, Q)$ of morphisms from P to Q is a set of injective group homomorphisms from P to Q, and that set satisfies two weak axioms. The standard example is the fusion system $\mathcal{F}_S(G)$ for G a finite group and $S \in \mathrm{Syl}_p(G)$, whose morphisms are those induced via conjugation in G. A fusion system is *saturated* if it satisfies two more axioms easily seen to hold in the standard example using Sylow's theorem. See [12] for notation, terminology, and basic definitions and results on fusion systems.

Let $\mathcal{F}$ be a saturated fusion system on a finite 2-group S. Proceeding by analogy with finite groups, one can define the notion of a *normal subsystem* of $\mathcal{F}$, which can then be used to define the notions of *simple* and *quasisimple* systems, *subnormal subsystems* of $\mathcal{F}$, and the set $\mathrm{Comp}(\mathcal{F})$ of *components* of $\mathcal{F}$. For t an involution in S the *centralizer* $C_{\mathcal{F}}(t)$ of t in $\mathcal{F}$ is defined, and if t is *fully centralized* (i.e., $|C_S(t)| \geq |C_S(x)|$ for each conjugate x of t) then $C_{\mathcal{F}}(t)$ is saturated, so we can define $\mathrm{Comp}(C_{\mathcal{F}}(t))$.

Define $\mathfrak{C}(\mathcal{F})$ to be the set of *components of centralizers of involutions* in $\mathcal{F}$; that is, $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ if there exists some involution $t \in S$ and a conjugate $(\bar{t}, \bar{\mathcal{C}})$ of $(t, \mathcal{C})$ such that $\bar{t}$ is fully centralized and $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{F}}(\bar{t}))$; we write $\mathcal{I}(\mathcal{C})$ for the set of such involutions t. We say that $\mathcal{F}$ is of *component type* if $\mathfrak{C}(\mathcal{F})$ is nonempty.

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Our main theorem is a contribution to the case where some $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is *intrinsic*: that is, $Z(\mathcal{C})$ is nontrivial.

Theorem 1. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ has a fully normalized Sylow group. In addition assume the following*:

(a) *$\mathcal{C}$ is realized by some $K \in \mathcal{K}_{\mathrm{qs}}$.*

(b) *$\mathcal{C}$ is maximal in $\mathfrak{C}(\mathcal{F})$.*

(c) *$\mathcal{I}(\mathcal{C}) \cap Z(\mathcal{C}) \neq \varnothing$.*

Then one of the following holds:

(1) *$\mathcal{C}$ is a component of $\mathcal{F}$.*

(2) *$\mathcal{C}$ is nearly standard, so $\widetilde{\mathcal{X}}(\mathcal{C})$ has a unique maximal member Q, and one of the following holds*:
   - (i) *$Q = Z(K)$ is of order 2.*
   - (ii) *$K \cong \Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $\epsilon \equiv q \bmod 4$, $\mathcal{C}$ is standard, and $Q \cong \mathbb{Z}_4$.*
   - (iii) *$K/Z(K) \cong L_3(4)$ and $Q = Z(K) \cong E_4$.*

The definition of a *maximal member* of $\mathfrak{C}(\mathcal{F})$ appears in Notation 6.1.12 in [5]. The definition of a *standard or nearly standard subsystem* appears in Section 9.1 of [5]. The definition of $\widetilde{\mathcal{X}}(\mathcal{C})$ appears in Notation 6.1.2 of [5]. $\mathcal{K}_{\mathrm{qs}}$ is a class of quasisimple groups with centers of even order, defined in Definition 1.3 and listed in Definition 1.3, 1.4, and 1.5; the remaining quasisimple groups with centers of even order have been treated elsewhere.

1. Intrinsic components

Definition 1.1. Let K be a quasisimple group with center of even order, $\bar{K} = K/Z(K)$, and $\bar{x}$ an involution in $\bar{K}$. Following Definition 5.5.1 in [14], define $\bar{x}$ to *split relative to* K if the coset $\bar{x}$ contains an involution. Define $\bar{x}$ to be *stable relative to* K if $\bar{x}$ splits and for some involution y in $\bar{x}$, $C_{\mathrm{Aut}(K)}(\bar{x}) = C_{\mathrm{Aut}(K)}(y)$.

Definition 1.2. We extend a notion in [8] to quasisimple groups: define K to be 2-*small* if for $T \in \mathrm{Syl}_2(K)$ and $\overline{T} \leq S \in \mathrm{Syl}_2(\mathrm{Aut}(K))$, we have $C_S(T) = \overline{Z(T)}$ with $|Z(T) : Z(K)| = 2$.

Definition 1.3. Define $\mathcal{K}_{\mathrm{qs}}$ to be the collection of known quasisimple groups K with $Z(K)$ of even order and $K/Z(K)$ not Goldschmidt, other than

(a) $\hat{A}_n$, $n \geq 5$,

(b) K of Lie type of odd characteristic other than $\Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$,

(c) $\widehat{\mathrm{Sp}_6(2)}$,

(d) $K/Z(K) \cong L_3(4)$ with $\Phi(Z(K)) \neq 1$.

We recall that, from [6], the 2-fusion systems of all the groups $\Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$ appearing in Definition 1.3(b) are isomorphic. Thus in this case we may take K to be $\Omega_6^{-}(3)$.

1.4. *A quasisimple group K with $K/Z(K)$ sporadic is in $\mathcal{K}_{\mathrm{qs}}$ precisely when*

(1) *$K/Z(K)$ is M_{12}, M_{22}, J_2, Co_1, HS, Suz, Ru, F_{22}, or F_2, and*

(2) *either $|Z(K)| = 2$ or $K/Z(K) \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$.*

Proof. See 6.4.2 in [5]. □

1.5. *A quasisimple group with $K/Z(K) \in \mathrm{Chev}(2)$ is in $\mathcal{K}_{\mathrm{qs}}$ precisely when one of the following holds:*

(1) *$K/Z(K) \cong G_2(4)$ or $F_4(2)$ and $|Z(K)| = 2$.*

(2) *$K/Z(K)$ is $U_6(2)$, $\Omega_8^{+}(2)$, or ${}^2E_6(2)$ and $Z(K) \cong \mathbb{Z}_2$ or E_4.*

(3) *$K/Z(K) \cong L_3(4)$ and $Z(K) \cong \mathbb{Z}_2$ or E_4.*

Proof. See 6.4.3 in [5]. □

1.6. *Let $K \in \mathcal{K}_{\mathrm{qs}}$, $\bar{K} = K/Z(K)$, and $\bar{z}$ a 2-central involution in $\bar{K}$. Then one of the following holds:*

(1) *$\bar{z}$ is stable relative to K.*

(2) *$\bar{K} \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$.*

(3) *$\bar{K} \cong \Omega_8^{+}(2)$.*

Proof. This follows from Proposition 6.4.2 in [14]. □

1.7. *Let $K \in \mathcal{K}_{\mathrm{qs}}$ satisfy one of the following:*

(1) *$\bar{K} \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$.*

(2) *$\bar{K} \cong \Omega_8^{+}(2)$.*

Then for $T \in \mathrm{Syl}_2(K)$, $Z(T) = Z(K)$.

Proof. In each case, $Z(\bar{T}) = \langle \bar{z} \rangle$ is of order 2. Therefore either $Z(T) = Z(K)$ or $\overline{Z(T)} = Z(\bar{T})$. But no involution in $Z(\bar{T})$ is stable from the remark following Proposition 6.4.2 in [14], so the lemma follows. □

1.8. *Let $K \in \mathcal{K}_{\mathrm{qs}}$ and assume*

(1) *neither of the exceptional cases in 1.7 hold, and*

(2) *$K/Z(K)$ is not $G_2(4)$, $F_4(2)$, or $L_3(4)$.*

Then K is 2-small.

Proof. Let $T \in \mathrm{Syl}_2(K)$ and Z be the preimage of $Z(\overline{T})$ in K. By (1) and 1.6, all involutions in $Z(\overline{T})$ are stable, so $Z = Z(T)$. Further $\mathcal{K}_{\mathrm{qs}}$ can be retrieved from Definition 1.3, 1.4, and 1.5, and, for groups K on that list but not in (2) we check using Definition 7.1, 7.2, and 7.3 in [8] that $\overline{K}$ is 2-small. Hence $|Z(\overline{T})| = 2$, so $|Z(T) : Z(K)| = 2$, and for $\overline{T} \leq S \in \mathrm{Syl}_2(\mathrm{Aut}(K))$ we have $C_S(T) = \overline{Z(T)}$, completing the proof. □

2. Terminal components

See Definition 8.1.1 in [5] for the definition of a *terminal component*.

2.1. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ has a fully normalized Sylow group T. In addition assume the following*:

(a) *$\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$.*

(b) *$\mathcal{C}$ is maximal in $\mathfrak{C}(\mathcal{F})$.*

(c) $m(T) > 1$.

Then one of the following holds:

(1) *$\mathcal{C}$ is a component of $\mathcal{F}$.*

(2) *$\mathcal{C}$ is a terminal member of $\mathfrak{C}(\mathcal{F})$.*

Proof. By (a) there is an involution $j \in \mathcal{C}^f$ and $\mathcal{L} \in \mathrm{Comp}(C_{\mathcal{C}}(j))$ such that $j \in Z(\mathcal{L})$ and $\mathcal{L} \in \mathfrak{C}(\mathcal{F})$ with $j \in \mathcal{I}(\mathcal{L})$. See Definition 6.1.17 in [5] for the definition of $\mathcal{C}^{\perp}$, and Definition 6.2.7 in [5] for the definition of $\rho(\mathcal{C})$ and $\rho_0(\mathcal{C})$.

Suppose first that $\mathcal{C}^{\perp} \neq \{\mathcal{C}\}$. Then (1) holds by (c) and Theorem 7.4.14 in [5]. Thus we may assume $\mathcal{C}^{\perp} = \{\mathcal{C}\}$. Therefore if $\rho(\mathcal{C}) = \rho_0(\mathcal{C})$, then (2) holds by Definition 8.1.1 in [5], so we may assume otherwise. Thus by Definition 6.2.7 in [5], there is $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C})$ and an involution $a \in Q_{t_1} - \widetilde{\mathcal{X}}(\mathcal{C}_1)$. Without loss of generality $(t_1, \mathcal{C}_1) = (t, \mathcal{C})$. Let $\alpha \in \mathfrak{A}(a)$ and adopt the bar convention of Notation 6.1.12 in [5]. Now there is a conjugate $(i, \mathcal{E}, \bar{\mathcal{C}})$ of $(j, \mathcal{L}, \mathcal{C})$ under α. As $j \in \mathcal{I}(L) \cap Z(\mathcal{L})$, $i \in \mathcal{I}(\mathcal{E}) \cap Z(\mathcal{E})$, so $\bar{\mathcal{C}}$ pumps up to a component $\mathcal{D}$ of $\mathcal{F}_{\bar{a}}$ by 1.9 in [7]. Thus by (b), $\bar{\mathcal{C}} = \mathcal{D}$, contradicting $a \notin \widetilde{\mathcal{X}}(\mathcal{C}_1)$. □

See Section 9.1 in [5] for the definition of a *standard subsystem* and a *nearly standard subsystem* of $\mathcal{F}$. In particular if $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is nearly standard, then $\widetilde{\mathcal{X}}(\mathcal{C})$ has a unique maximal member Q.

Often we assume the following hypothesis:

Hypothesis 2.2. (1) $\mathcal{F}$ is a saturated fusion system over a finite 2-group S.

(2) $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ has fully normalized Sylow group T and is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$.

(3) $\mathcal{C}$ is terminal in $\mathfrak{C}(\mathcal{F})$.

2.3. *Assume Hypothesis 2.2. Then*:

(1) *$\mathcal{C}$ is nearly standard. Let Q be the unique maximal member of $\widetilde{\mathcal{X}}(\mathcal{C})$ and $Q_0 = C_S(T)$.*

(2) *$Z(K) = Z(\mathcal{C}) \leq Q$.*

(3) *For $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$, $\mathcal{C}\alpha^* \trianglelefteq N_{\mathcal{F}}(X\alpha)$.*

(4) *Set $\Sigma = N_{\mathrm{Aut}_{\mathcal{F}}(Q_0T)}(T)$. Then $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Aut}_{\Sigma}(T)$.*

(5) *If $\mathrm{Aut}_{\mathcal{F}}(T) \leq \mathrm{Aut}(\mathcal{C})$ then $\mathcal{C}$ is standard.*

(6) *If $\sigma \in \Sigma$ with X and $X\sigma$ in $\widetilde{\mathcal{X}}$ then $\sigma_{|T} \in \mathrm{Aut}(\mathcal{C})$.*

(7) *If $Z(T) = Z(K)$ then $\mathcal{C}$ is standard.*

Proof. By Hypothesis 2.2(2), $Z(\mathcal{C}) = Z(K) \neq 1$, so part (1) follows from 9.2.4 in [5] and Hypothesis 2.2(3). Part (2) follows from 6.2.10(3) in [5] and Hypothesis 2.2(3). Part (3) follows from (1) and (S2) in the definition of "nearly standard" in Section 9.1 in [5]. Part (4) follows from 6.1.8(1) in [5]. Part (5) follows from (1) and the definition of "standard". Part (6) is 6.1.8(3) in [5]. Assume the setup of (7). By (2), $X = Z(K) = Z(\mathcal{C}) \leq Q$, so by 8.1.3 in [5], $X \in \widetilde{X}$. By assumption $Z(T) = Z(K) = X$, so Σ acts on X; hence $\Sigma_{|T} \leq \mathrm{Aut}(\mathcal{C})$ by (6). Now (7) follows from (4) and (5). □

In this section Σ is defined as in 2.3(4).

2.4. *Assume Hypothesis 2.2 and $\mathcal{C}$ is not standard. Then*:

(1) *Q_0 is abelian.*

(2) *$Q_0 = QZ(T)$, so $|Q_0 : Q| = |Z(\overline{T})|$.*

(3) *There exists $\sigma \in \Sigma$ with $Q \cap Q\sigma = 1$. Hence $|Q| \leq |Z(\overline{T})|$.*

(4) *If K is 2-small then $Q = Z(K)$ is of order 2.*

(5) *If $K/Z(K)$ is $G_2(4)$, $F_4(2)$, or $L_3(4)$ then either $Q = Z(K)$ is of order 2 or $Q \cong E_4$.*

(6) *Assume the setup of (5). Then for each $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$, $F^*(N_{\mathcal{F}}(X\alpha))$ equals $(Q\alpha) * \mathcal{C}\alpha^*$ and $\alpha : Q_0T \to Q_0T$ induces an isomorphism of $Q * \mathcal{C}$ with $F^*(N_{\mathcal{F}}(X\alpha))$.*

Proof. Part (1) follows from 2.3.5 and 9.2.3(9) in [5].

Choose $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$. By 8.1.4(1) in [5], we may take $T\alpha = T$. Set $\mathcal{B} = \mathcal{C}\alpha^*$; by 2.3(3), $\mathcal{B} \trianglelefteq N_{\mathcal{F}}(X\alpha)$.

As $Q_0 = C_S(T)$, $1 = [Q_0, T]$, and by (1), $Q_0T \leq C_S(X)$, so $(Q_0T)\alpha \leq C_S(X\alpha)$ and hence $1 = [Q_0\alpha, T\alpha] = [Q_0\alpha, T]$. Thus $Q_0\alpha \leq C_S(T) = Q_0$, so $Q_0\alpha = Q_0$.

Define Q_X as in Notation 9.2.2 in [5]; that is,

$$Q_X\alpha = \theta \cap N_S(X)\alpha, \quad \text{where } \theta = C_{N_S(X\alpha)}(\mathcal{B}). \qquad (*)$$

By 9.2.3(6) in [5], $Q_X = N_Q(X)$, so $Q_X = Q$ by (1). As $\mathcal{B} \trianglelefteq N_{\mathcal{F}}(X\alpha)$ and $Q_0 = Q_0\alpha$, we can form $Q_0\mathcal{B}$ in $N_{\mathcal{F}}(X\alpha)$. By Lemma 2.22 in [9], $Q_0\mathcal{B}$ is realized by a group Q_0K_1 with $K_1 \cong K$. As T is Sylow in $\mathcal{B}$, θ centralizes T, and hence is contained in Q_0, so $\theta = C_{Q_0}(\mathcal{B}) = C_{Q_0}(K_1)$. Define $\pi : Q_0K_1 \to \mathrm{Aut}(K_1)$ by $x\pi = c_x$ to be conjugation of K_1 by x. Observe that $\ker(\pi) = \theta$ so $Q_0K_1\pi \cong Q_0\pi\overline{K}_1$. Also $Q_0\pi = \overline{Z(T)}$; this follows from 1.8 unless K is one of the five exceptional groups appearing in 1.8, where the claim can be verified using 2-local facts about the five groups such as in [14]. Hence $Q_0/\theta \cong \overline{Z(T)}$ and $Q_0 = \theta Z(T)$. Now

$$(**) \qquad Q\alpha = Q_X\alpha = \theta \cap N_S(X)\alpha.$$

Also $\theta \leq Q_0 = Q_0\alpha \leq N_S(X)\alpha$ so

$$(***) \qquad Q\alpha = \theta.$$

Thus $Q_0 = \theta Z(T) = Q\alpha Z(T) = Q\alpha Z(T)\alpha$, so (2) holds. Also by $(*)$ and $(***)$, $Q\alpha = C_{N_S(X\alpha)}(\mathcal{B})$, so the first statement in (6) holds. We have seen that α acts on Q_0 and T, so α is an automorphism of the group $Q_0T = QT$, and induces an isomorphism of the fusion systems $Q\mathcal{C}$ and $Q\alpha\mathcal{B}$. This completes the proof of (6). As $\mathcal{C}$ is not standard by hypothesis, 2.3(4)–(5) say there exists $\sigma \in \Sigma$ with $\sigma \notin \mathrm{Aut}(\mathcal{C})$. Then $Q \cap Q\sigma = 1$ by 2.3(6). Then $|Q| \leq |Z(\overline{T})|$ by the second statement in (2), completing the proof of (3). Part (4) follows from (3). Similarly if K is as in (5) then $Z(\overline{T}) \cong E_4$, so (3) implies (5). □

In the next lemma for a saturated 2-fusion system $\mathcal{E}$, $\mathcal{E}^\infty$ is the last term in the Puig series for $\mathcal{E}$, defined in Definition 2.18 of [9].

2.5. *Assume Hypothesis 2.2 with* $K/Z(K) \cong F_4(2)$ *and* $|Q| > 2$. *Then* $\mathcal{C}$ *is standard.*

Proof. Assume $\mathcal{C}$ is not standard. Then $Q \cong E_4$ by 2.4(5). By 2.4(3) there is $\sigma \in \Sigma$ with $Q_0 = Q \times Q\sigma$. Now $Z(\overline{T}) = \langle \bar{z}_l, \bar{z}_s \rangle$ where $\bar{z}_c$ is a root involution. Then there is $y \in Q\sigma$ such that the image of y in $\overline{K}$ is a root involution and $\mathcal{D} = C_{\mathcal{C}}(y)$ is the 2-fusion system of $C_K(y)$. By 8.5 in [3], $\overline{\mathcal{D}}$ is the fusion system of a maximal parabolic of $\overline{K}$ with $\overline{\mathcal{D}}^\infty = \overline{\mathcal{D}}$. As K is quasisimple, the parabolic does not split over $Z(K)$, so $\mathcal{D} = \mathcal{D}^\infty$. Let $\tau = \sigma^{-1} \in \mathrm{Aut}_{\mathcal{F}}(Q_0T)$, so that $Q\sigma\tau = Q$. Now y centralizes $\mathcal{D}$, so $x = y\tau$ centralizes $\mathcal{D}\tau^*$. Choose X and α as in the proof of 2.4 with $X = \langle x \rangle$. Then $x\alpha$ centralizes $\mathcal{E} = \mathcal{D}\tau^*\alpha^*$. As $\mathcal{D} = \mathcal{D}^\infty$, $\mathcal{E} = \mathcal{E}^\infty$ and hence $\mathcal{E} \leq \mathcal{F}_{x\alpha}^\infty = \mathcal{B} = \mathcal{C}\alpha^*$ by 2.4(6). Thus as $Q\alpha$ centralizes $\mathcal{B}$, it centralizes $\mathcal{E}$. By 2.4(6), α induces an isomorphism of the fusion systems $Q\mathcal{C}$ and $Q\alpha\mathcal{B}$. Thus applying α^{-1}, Q centralizes $\mathcal{D}\tau^*$. Then applying σ, $Q\sigma$ centralizes $\mathcal{D}$, contradicting $C_{Q_0}(\mathcal{D}) = Q\langle y \rangle$. □

2.6. *Assume* $K/Z(K) \cong L_3(4)$ *and* $|Z(K)| = 2$. *Then*:

(1) $\mathrm{Aut}(T)$ *acts on* $Z(K)$.

(2) *If Hypothesis 2.2 holds for* K *then* $\mathcal{C}$ *is standard.*

Proof. Let $Z(K) = \langle k \rangle$; we show k is the unique member of $Z(T)$ that is not a commutator in T, hence establishing (1). One way to see this is to consider a maximal parabolic $\widehat{Y}$ of the covering group $\widehat{K}$ of K with $Z(\widehat{K}) \cong E_4$. Using 1.6, $\widehat{Y}$ is of the form $A \cong E_{64}$ extended by $L_2(4)$, with $A/Z(\widehat{T})$ the natural module for $\widehat{Y}/A$ and $\widehat{Y}$ indecomposable on A as $\widehat{K}$ is quasisimple. Regarding A as an $\mathbb{F}_4$-module for $\widehat{Y}/A$ and taking $\widehat{T} \in \mathrm{Syl}_2(\widehat{Y})$, we claim that $\widehat{Z} = C_A(\widehat{T})$ has five points $Z(\widehat{K})$, $[\widehat{Z}, \widehat{B}]$ for $\mathbb{Z}_3 \cong \widehat{B} \leq N_{\widehat{Y}}(\widehat{T})$, and $[A, t_i]$, $1 \leq i \leq 3$, for $t_i A$ the involutions in $\widehat{T}/A$. If so, the only members of $\widehat{Z}$ that are commutators in $\widehat{T}$ are in $[A, t_i]$ for some i. Now $K = \widehat{K}/\Pi$ for some Π of order 2 in $Z(\widehat{K})$, and thus k is the unique noncommutator in $Z(T)$. Thus to complete the proof of (1) it remains to prove the claim. But if the claim fails, then $[A, \widehat{T}]$ is the point $[\widehat{Z}, \widehat{B}]$, so $[A, \widehat{Y}] = [A, \widehat{T}][A, \widehat{T}_1]$ is of rank 2, where $\widehat{T}_1$ is a second Sylow group of $\widehat{Y}$, contradicting $\widehat{Y}$ indecomposable on A. So (1) is established.

Assume Hypothesis 2.2. By 2.3(2), $X = Z(K) \leq Q$, so by (1) and 2.3(6), $\Sigma \leq \mathrm{Aut}(\mathcal{C})$. Then (2) follows from 2.3(4)–(5). □

2.7. *Assume $K/Z(K) \cong G_2(4)$ and $|Z(K)| = 2$. Let $M = C_K(Z(T))$. Then*:

(1) $\mathrm{Aut}(M)$ *acts on* $Z(K)$.

(2) *If Hypothesis 2.2 holds for K then $\mathcal{C}$ is standard.*

Proof. Let $Z(K) = \langle k \rangle$ and $Z = Z(T)$. By 18.4 in [10], $\bar{K}$ has two classes of involutions — the long and short root involutions — and $Z(\bar{T})$ is a long root subgroup. By 1.6, $Z(\bar{T}) = \bar{Z}$. Again by [10], $\bar{M} = \bar{L}\bar{R}$, where $\bar{R}$ is the radical of $\bar{M}$ and $\bar{L} \cong L_2(4)$ contains a short root subgroup. From the Atlas, short root involutions lift to elements of order 4 in K, so the preimage L of $\bar{L}$ in K is isomorphic to $\mathrm{SL}_2(5)$. From Example 3.2.4 in [14], R/Z is the sum of two orthogonal modules for $\bar{L}$, so $\bar{R}$ is transitive on complements $\bar{J}$ to $\bar{R}$ in $\bar{M}$, so $Z(J) = \langle k \rangle$ for each such J. This proves (1).

Assume the setup of (2). Then, using 2.4(6), $QC_{\mathcal{C}}(Z) = C_{\mathcal{F}}(Z) \trianglelefteq N_{\mathcal{F}}(Z) = \mathcal{B}$, with $QR \trianglelefteq \mathcal{B}$ and $C_{\mathcal{B}}(QR) \leq QR$, so $\mathcal{B}$ is constrained with model B with $C_B(Z) \cong QC_K(Z) = QM$. Also $M = O^2(QM) \trianglelefteq B$. Then $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Aut}_B(T)$ as $Z = Z(T)$ and $B = N_{\mathcal{F}}(Z)$. Then as $M \trianglelefteq \mathrm{Aut}_B(T)$ centralizes k by (1), (2) follows from 2.4(3). □

Theorem 2.8. *Assume Hypothesis 2.2. Then one of the following holds*:

(1) $Q = Z(K)$ *is of order* 2.

(2) $\mathcal{C}$ *is standard.*

(3) $K/Z(K) \cong L_3(4)$ *and* $Q = Z(K) \cong E_4$.

Proof. If $|Q| = 2$ then (1) follows from 2.3(2), so we may assume $|Q| > 2$. By 2.3(7), we may assume $Z(T) \neq Z(K)$, while K is not 2-small by 2.4(4). Hence by 1.7 and 1.8, $K/Z(K)$ appears in case (2) of 1.8. Next $K/Z(K) \cong L_3(4)$ by 2.5 and 2.7, while $Z(K) \cong E_4$ by 1.5(3) and 2.6. Finally (3) holds by 2.4(3). □

3. Tight embedding

Given a saturated fusion system on a 2-group S, a fully normalized abelian subgroup P of S is *tightly embedded* in $\mathcal{F}$ if

(T1) For each $1 \neq X \leq P$ and $\alpha \in \mathfrak{A}(X)$, $P\alpha \trianglelefteq N_{\mathcal{F}}(X\alpha)$, and

(T2) for each $1 \neq X \leq P$, $X^{\mathcal{F}} \cap P = X^{N_{\mathcal{F}}(P)}$.

We can also define tight embedding for groups. Given a finite group G, an abelian 2-subgroup P of G is *tightly embedded* in G if distinct conjugates of P intersect trivially. Equivalently,

(TE1) For each $1 \neq X \leq P$, $P \trianglelefteq N_G(X)$, and

(TE2) for each $1 \neq X \leq P$, $X^G \cap P = X^{N_G(P)}$.

Another equivalent definition is obtained by quantifying over all involutions in P rather than over all nontrivial subgroups of P.

In the remainder of this section we assume the following hypothesis:

Hypothesis 3.1. $\mathcal{F}$ is a saturated fusion system on a finite 2-group S such that $\mathcal{F} = P\mathcal{C}$ where

(1) $\mathcal{C} = O^2(\mathcal{F})$ is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$, and

(2) P is an abelian 2-subgroup of S tightly embedded and fully normalized in $\mathcal{F}$.

3.2. (1) *There exists a finite group G with $S \in \mathrm{Syl}_2(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, $K = E(G)$, $F^*(G) = O_2(G)K$, and $G = PK$.*

(2) *$G/K \cong P/(P \cap K)$ is an abelian 2-group.*

Proof. As K tamely realizes $\mathcal{C}$, (1) follows from Lemma 2.22 in [9]. As $G = PK$ and P is an abelian 2-group, (2) holds. □

Set $U = O_2(G)$ and $G^* = G/U$; thus $U = C_G(K)$ and $G^* \leq \mathrm{Aut}(K^*)$ with $K^* \cong K/Z(K)$ simple.

3.3. *Let $1 \neq X \leq P$ and $G_X = \langle P^{N_G(X)} \rangle$. Then*:

(1) $N_G(X) = PN_K(X)$.

(2) $O^2(C_K(X))^* = O^2(C_{K^*}(X^*))$.

(3) *P is strongly closed in S_X with respect to $N_G(X)$ for $S_X \in \mathrm{Syl}_2(N_G(X))$.*

(4) *$G_X^+ = G_X/O(G_X) = H_0^+ \times H_1^+ \times \cdots \times H_n^+$ where $H_0 \leq P$ and for $1 \leq i \leq n$, H_i^+ is a Goldschmidt group with $P \cap H_i = \Omega_1(S \cap H_i)$.*

(5) *If $N_K(X)/O(N_K(X))$ has no Goldschmidt components then $O(N_K(X))P$ is a normal subgroup of $N_G(X)$.*

Proof. As P is abelian and $G = PK$, (1) holds. Next $k \in C_K(X) = \Delta$ if and only if $[k, X] = 1$ while $k^* \in C_{K^*}(X^*)$ if and only if $1 = [k^*, X^*] = [k, X]^*$ if and only if $[k, X] \leq U$. Thus $O^2(\Delta^*) \leq O^2(C_{K^*}(X^*))$. Let $j^* \in C_{K^*}(X^*)$ be of odd order m; we may choose $|j| = m$. Then $j \in O^2(G) = K$, so $[j, X] \leq U \cap K = Z(K)$ and hence j centralizes $XZ(K)/Z(K)$ and $Z(K)$ so j centralizes X by 24.5 in [2]. Thus $O^2(C_{K^*}(X^*)) \leq O^2(\Delta^*)$, completing the proof of (2).

As P is abelian and tightly embedded in $\mathcal{F}$, from condition (T1) above for $\alpha \in \mathfrak{A}(X)$ we have $P\alpha \trianglelefteq N_{\mathcal{F}}(X\alpha)$. As $P \in \mathcal{F}^f$ there is $\beta \in \mathfrak{A}(P\alpha)$ with $P\alpha\beta = P$; set $\zeta = \alpha\beta$ and observe that $P\zeta = P$ and as $P\alpha \trianglelefteq N_{\mathcal{F}}(X\alpha)$, $\zeta \in \mathfrak{A}(X)$, so replacing α by ζ we may take $P\alpha = P$. Thus $P \trianglelefteq N_{\mathcal{F}}(X\alpha)$, so P is strongly closed in $N_S(X\alpha)$ with respect to $N_{\mathcal{F}}(X\alpha)$. Next by (T2), there is $\gamma \in N_{\mathcal{F}}(P)$ with $(X\alpha)\gamma = X$, so there is $g \in N_G(P)$ with $(X\alpha)^g = X$, so (3) holds with $S_X = N_S(X\alpha)^g$.

Part (4) follows from Goldschmidt's fusion theorem [13], determining groups generated by conjugates of a strongly closed abelian 2-subgroup. As $O^2(G) = K$, $O^2(G_X) \leq N_K(X)$. Then (4) implies (5). □

3.4. *Let $\mathfrak{X}$ be the set of involutions $x \in P$ such that $O(C_{K^*}(x^*)) = 1$ and $C_{K^*}(x^*)$ has no Goldschmidt components. Then*:

(1) *For each $x \in \mathfrak{X}$, $P \trianglelefteq C_G(x)$.*

(2) *If each involution in P is in $\mathfrak{X}$ then P is tightly embedded in G.*

Proof. By 3.3(2), $O(C_G(x))^* = O(C_{K^*}(x^*))$ and if $C_{K^*}(x^*)$ has no Goldschmidt components then, using 3.3(5), neither does $C_G(x)$. Hence (1) follows from 3.3(5). From Hypothesis 3.1(2) and condition (T2) in the definition of tight embedding, $x^{\mathcal{F}} \cap P = x^{N_{\mathcal{F}}(P)}$ so $x^G \cap P = x^{N_G(P)}$, and, together with (1) and the third of the equivalent definitions of tight embedding in groups at the beginning of this section, (2) follows. □

3.5. *Either P is faithful on K or $G = P * K$ is a central product of P with K.*

Proof. Suppose x is an involution in $C_P(K)$. As K is not Goldschmidt and $F^*(G) = UK$ by 3.2(1), it follows from 3.4 that $P \trianglelefteq C_G(x)$. Thus $[P, K] \leq P \cap K \leq Z(K)$ so K centralizes P. The lemma follows as $G = PK$ by 3.2(1). □

3.6. *Assume $|P| = 4$ or P is cyclic. Then*:

(1) *For each involution $x \in P$, $PO(C_K(x)) \trianglelefteq C_G(x)$.*

(2) *If $O(C_{K^*}(x^*)) = 1$ for each involution x in P then P is tightly embedded in G.*

Proof. If $|P| = 4$ then $|P : \langle x \rangle| = 2$, so as P is strongly closed in $C_G(x)$, (1) follows from the Z^*-theorem. Similarly (1) holds when P is cyclic. Then (1), 3.3(2) and 3.4(2) imply (2). □

3.7. *Assume $K/Z(K) \in$* Chev(2). *Then*:

(1) *P is tightly embedded in G.*

(2) *If P is faithful on K, $\Phi(P) = 1$, and $|P| > 2$ then $P \leq UK$.*

Proof. We may assume P is not normal in G, so P is faithful on K by 3.5. We first prove (1), where by 3.4 it suffices to verify the two conditions defining $\mathfrak{X}$. If $x^* \in K^*$ then $F^*(C_{K^*}(x^*)) = O_2(C_{K^*}(x^*))$, so the conditions hold; thus we may assume x^* induces an outer automorphism on K^*, so x^* and its centralizer are listed in [10]. As K appears in 1.5, we find that the conditions are satisfied, unless $K^* \cong L_3(4)$ and x^* induces a graph or graph-field automorphism. In either case we conclude from 3.3(4) that $P^* \cap K^* \neq 1$, so there is an involution $z \in P$ with $z^* \in K^*$. But as we just saw, $F^*(C_{K^*}(z^*)) = O_2(C_{K^*}(z^*))$, so $P \trianglelefteq C_G(z)$ by 3.3(5). Next z^* is stable by 1.6 and P is fully normalized by Hypothesis 3.1(2), so $z \in Z(T)$, where $T = S \cap K$. As $\langle x^T \rangle$ is nonabelian, this contradicts $P \trianglelefteq C_G(z)$ and P abelian. This completes the proof of (1).

Assume the setup of (2). Using (1), the hypotheses of 20.1 in [10] are satisfied; then (2) follows from that lemma. □

3.8. *Assume $K/Z(K)$ is sporadic and either $|P| = 4$ or P is cyclic. Then P is tightly embedded in G.*

Proof. By 3.6(2) it suffices to show $O(C_{K^*}(x^*)) = 1$ for each involution x in P. This follows by inspection of Table 5.3 in [14] for each of the groups listed in 1.4. □

4. Splitting

In this section we assume $\mathcal{C}$ is a quasisimple fusion system on a 2-group T tamely realized by some $K \in \mathcal{K}_{\text{qs}}$.

Recall from Subsection 0.13 in [5] that a *critical split extension* of $\mathcal{C}$ is a pair $(\mathcal{F}, P)$, where $\mathcal{F}$ is a saturated fusion system on a 2-group S, $\mathcal{C} = O^2(\mathcal{F})$, P is a complement in S to a Sylow group T of $\mathcal{C}$, P is isomorphic to E_4, and P is tightly embedded in $\mathcal{F}$. Further $\mathcal{C}$ is said to *split* if there is no nontrivial critical extension $(\mathcal{F}, P)$ of $\mathcal{C}$; that is, for each such pair we have $\mathcal{F} = C_S(\mathcal{C}) * \mathcal{C}$.

Throughout this section we assume $(\mathcal{F}, P)$ is a critical split extension of $\mathcal{C}$.

4.1. (1) *Hypothesis 3.1 is satisfied.*

(2) *There exists a finite group G with $S \in \mathrm{Syl}_2(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, $K = E(G)$, $F^*(G) = O_2(G)K$, and $G = PK$.*

(3) *If $(\mathcal{F}, P)$ is nontrivial then P is faithful on K.*

Proof. Part (1) is immediate; then (1) and 3.2(1) imply (2), and 3.5 implies (3). □

Set $U = O_2(G)$ and $G^* = G/U$.

4.2. *Assume $(\mathcal{F}, P)$ is nontrivial. Then*:

(1) *For some involution* $x \in P$, $x^* \notin K^*$.

(2) $|\mathrm{Out}(K/Z(K))|$ *is even.*

Proof. If (1) fails then $G^* = K^*$, so $G = UK$ and hence $\mathcal{F} = U * \mathcal{C}$, contradicting $(\mathcal{F}, P)$ nontrivial. Thus (1) holds and (1) and 4.1(2) imply (2). □

4.3. *If* $K/Z(K) \in \mathrm{Chev}(2)$ *then* $\mathcal{C}$ *splits.*

Proof. This follows from 3.7(2). □

4.4. *If* $K/Z(K)$ *is sporadic then* $\mathcal{C}$ *splits.*

Proof. Assume $(\mathcal{F}, P)$ is nontrivial. By 3.8, P is tightly embedded in G. Hence, using 3.3(2), for each involution $x \in P$, $O^2(C_{K^*}(x^*))$ acts on $P^* \cong E_4$.

By 4.2 there is an involution x in $P - KU$, and $|\mathrm{Out}(K^*)|$ is even. Inspecting Table 5.3 in [14] in those cases where $|\mathrm{Out}(K^*)|$ is even, we find that K^*, $C_{K^*}(x^*)$ is among the pairs listed in 14.5 in [8]. In each case $O^2(C_K(x))$ is irreducible on $V^* = O_2(C_{K^*}(x^*))$, so $V^* = P^* \cap K^*$ is of order 2, and hence $K^* \cong M_{12}$ and the preimage V of V^* in K is generated by v of order 4. Next by Table 5.3b in [14], there exists an involution $i^* \in C_{K^*}(v^*)$ with $[x^*, i^*] = v^*$. It follows that x inverts v and $P = \langle x, y\rangle$ where $y = uv$ for some $u \in U$ with $u^2 = v^2$. Now $v \in C_K(y)$, so v acts on P, a contradiction as then $[x, v] = v^2 \in P$, whereas P is faithful on K by 3.5. □

We recall that, from [6], the 2-fusion systems of all the groups $\Omega_6^{\epsilon}(q)$ with $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$ appearing in Definition 1.3(b) are isomorphic. Thus in this case we may take K to be $\Omega_6^-(3)$.

4.5. *If* K *is* $\Omega_6^{\epsilon}(q)$ *with* $q \equiv \pm 3 \bmod 8$ *and* $q \equiv \epsilon \bmod 4$ *then* $\mathcal{C}$ *splits.*

Proof. Assume $(\mathcal{F}, P)$ is nontrivial. We may take K to be $\Omega_6^-(3)$. Then the classes of involutions in $\mathrm{Aut}(K)$ and their centralizers are listed in Table 2.10 in [1]. Inspecting that list, we find that for x^* an involution in G^*, we have $O(C_{K^*}(x^*)) = 1$ and $C_{K^*}(x^*)$ has no Goldschmidt components. Therefore by 3.4, P is tightly embedded in G and normal in $C_G(x)$ for each involution x in P. As $E_4 \cong P^*$ is $O^2(C_{K^*}(x^*))$-invariant we conclude from Table 2.10 in [1] that x^* is of type $i(4, \delta)$ for $\delta \in \{1, -1\}$. By 4.2(1), we may choose $x^* \notin K^*$, so $x^* \in i(4, -1)$. Thus the remaining two involutions in $P^* = O_2(C_{G^*}(x^*))$ are reflections, whereas we just saw they are in $i(4, \delta)$. □

Theorem 4.6. *Assume* $\mathcal{C}$ *is a quasisimple* 2*-fusion system tamely realized by some* $K \in \mathcal{K}_{\mathrm{qs}}$. *Then* $\mathcal{C}$ *splits.*

Proof. The possibilities for K are listed in Definition 1.3, 1.4, 1.5. Now appeal to 4.3–4.5. □

5. Standard subsystems

In this section we assume the following hypothesis:

Hypothesis 5.1. (1) $\mathcal{F}$ is a saturated fusion system on a finite 2-group S.

(2) $\mathcal{C}$ is a standard subsystem of $\mathcal{F}$ on $T \in \mathcal{F}^f$. Let Q be the unique maximal member of $\widetilde{\mathcal{X}}(\mathcal{C})$ and write $\mathcal{Q}$ for the centralizer in $\mathcal{F}$ of $\mathcal{C}$.

(3) $\mathcal{C}$ is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$.

(4) $\mathcal{C}$ is not a component of $\mathcal{F}$.

5.2. *$\mathcal{F}$ is almost simple.*

Proof. Observe Hypothesis 9.4.1 of [5] is satisfied by (1) and (2) of Hypothesis 5.1, so the result follows from Hypothesis 5.1(4) and 9.4.6 in [5]. □

5.3. *Either $\Phi(Q) = 1$ or Q is cyclic.*

Proof. By Theorem 4.6 and 9.4.10 in [5], either $\Phi(Q) = 1$ or $m(Q) = 1$. Thus we may assume Q is quaternion and it remains to exhibit a contradiction. Let z be the involution in Q.

As $\mathcal{C}$ is standard, $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$, so $\tau = (\mathcal{F}, \Omega)$ is a quaternion fusion packet, where $\Omega = Q^{\mathcal{F}}$. By 5.2, $\mathcal{L} = F^*(\mathcal{F})$ is simple, so by Theorem 1 in [6], and as $\mathcal{C}$ is a component of $\mathcal{F}_z$, we conclude that $\mathcal{L}$ is the 2-fusion system of a group L of Lie type over $\mathbb{F}_q$ for some odd q. Then as $\mathcal{C}$ is a component of $\mathcal{L}_z$, $K \cong \Omega_6^{\epsilon}(q)$. As $\mathcal{Q} \trianglelefteq \mathcal{F}_z$, $\Omega(z) = \{Q\}$, where $\Omega(z) = \{P \in \Omega : z \in P\}$. As $\mathcal{C} = E(C_{\mathcal{F}_z}(\mathcal{Q}))$, $|\Omega| > 1$ from Theorem 1 in [6], and then $\rho = (\mathcal{C}, \Gamma)$ is a quaternion fusion packet, where $\Gamma = \Omega - \{Q\}$. As $K \cong \Omega_6^{\epsilon}(q)$ it follows that $\Gamma = \Omega(t)$ is of order 2, for some $t \in z^{\mathcal{F}}$, contradicting $|\Omega(t)| = |\Omega(z)| = 1$. □

In the remainder of the section we assume:

Hypothesis 5.4. Hypothesis 5.1 holds with $|Q| > 2$.

5.5. (1) *$N_{\mathcal{F}}(Q)$ is tamely realized by a group M with $F^*(M) = Q * K$.*

(2) *Q is tightly embedded in $\mathcal{F}$.*

Proof. By 5.3, Q is abelian, so $Q = O_2(\mathcal{Q})$ and $F^*(N_{\mathcal{F}}(Q)) = Q\mathcal{C}$. Then (1) follows from Lemma 2.22 in [9]. As $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$ and $Q = O_2(\mathcal{Q}) = O^{2'}(\mathcal{Q})$, (2) follows. □

5.6. *It is not the case that Q is weakly closed in $N_S(Q)$ with respect to $\mathcal{F}$.*

Proof. If Q is cyclic the lemma follows from 9.4.7(3) in [5]. If $\Phi(Q) = 1$ it follows from 9.4.11 in [5]. □

Notation 5.7. Set $\Delta = Q^{\mathcal{F}} \cap N_S(Q) - \{Q\}$.

By 5.6, $\Delta \neq \varnothing$. Recall from Definition 3.1.9 in [5] that $\mathcal{P}(Q)$ is the set $\{1 \neq P \leq S : \hom_{\mathcal{F}}(P, Q) \neq \varnothing\}$ and $\mathcal{P}^*(Q)$ is the set of maximal members of $\mathcal{P}(Q)$. Let $\mathcal{P}_Q = \{P \in \mathcal{P}(Q) : P \leq N_S(Q)\}$ and $\mathcal{P}^*_Q$ be the maximal members of $\mathcal{P}_Q$. For example, $\Delta \subseteq \mathcal{P}^*_Q$.

Choose $Q \neq P \in \mathcal{P}^*_Q \cap N_{\mathcal{F}}(Q)^f$, and set $H = PK \leq M$ and $H^* = H/(H \cap Q)$.

5.8. (1) *P is tightly embedded in $N_{\mathcal{F}}(Q)$ and in $P\mathcal{C}$.*

(2) *P is faithful on K.*

Proof. Part (1) is a consequence of 13.1 in [8]. As $Q = C_S(K)$, $C_P(K) \leq Q$; but $P \cap Q = 1$, by, for example, 3.1.12(2) in [5]. Thus (2) holds. □

5.9. (1) *$Z(K)$ is strongly closed in Q with respect to $\mathcal{F}$.*

(2) *Let $P \in \Delta$ and $\phi \in \hom_{\mathcal{F}}(Q, P)$. Then $Z(K)\phi$ is strongly closed in P with respect to $\mathcal{F}$.*

Proof. Let $x \in Z(K)$ and $y \in x^{\mathcal{F}} \cap Q$. As $\mathrm{Aut}_{\mathcal{F}}(Q)$ controls fusion in Q there is $m \in M$ with $x^m = y$. Then as $K \trianglelefteq M$, $y \in Z(K)$, proving (1). Now (1) implies (2). □

6. Sporadic components

We prove:

Theorem 6.1. *Assume Hypothesis 5.1 with $K/Z(K)$ sporadic. Then $|Q| = 2$.*

Assume $\mathcal{F}$ is a counterexample to the theorem. Adopt the notation from Section 5; in particular choose $P \in \Delta$ as in Notation 5.7.

6.2. $\Phi(Q) = 1$.

Proof. Assume otherwise; by 5.3, Q is cyclic, so $P \cong Q$ is also cyclic. Let x be the involution in P. By 5.8(1) and 3.8, P is tightly embedded in H, so $P \trianglelefteq C_H(x)$. Hence by 3.3(2), $O^2(C_{K^*}(x^*))$ centralizes P^*. Inspecting Table 5.3 in [14] for involution centralizers with this property, we conclude $K^* \cong HS$, $C_{K^*}(x^*) \cong S_5/(\mathbb{Z}_4 * Q_8^2)$ and $P^* = Z(O_2(C_{K^*}(x^*)))$.

Let $W = O^2(C_K(x))$ and $R = O_2(W)$. Then $R = \langle r^W \rangle$ for $r^* \in R^* - P^*$ of order 4, so $\Phi(R) = \langle r^2 \rangle$ is of order 2 with $\Phi(R)^* = \langle x^* \rangle$.

Let y be a preimage of x^* in K; by 1.6, x^* is stable, so y is an involution and $x = qy$ for some $q \in Q$ with $q^2 = 1$. Then as Q is cyclic, $q \in Z(K)$ and $x \in K$. Let P_0 be the preimage of P^* in K; then $P_0 = P \times Z(K)$ with $\langle x \rangle = \Phi(P_0)$. Thus $\langle x \rangle = \Phi(R) \leq W$.

Let $\alpha \in \mathfrak{A}(x)$ with $z = x\alpha \in Z(\mathcal{C})$. Let $\mathcal{W} = \mathcal{F}_{S \cap W}(W)$; then $\mathcal{W} = O^2(C_{\mathcal{F}_z}(x))$. Next $\mathcal{U} = \mathcal{W}\alpha^* \leq O^2(\mathcal{F}_z) = \mathcal{C}$. As $W/\langle x, z \rangle \cong A_5/E_{16}$, it follows from Table 5.3m in [14] that $z\alpha \in \{x, xz\}$ and $\mathcal{U} = \mathcal{W}$. This is a contradiction as $Z(K) = \langle x\alpha \rangle = \Phi(R\alpha)$, so $O_2(W/Z(K)) = R\alpha/Z(K)$ is elementary abelian. □

6.3. $|Z(K)| = 2$.

Proof. If not then by 1.4(2), $K/Z(K) \cong M_{22}$ and $Z(K) \cong \mathbb{Z}_4$, contrary to 6.2. □

Notation 6.4. By 6.3, $Z(K) = \langle t \rangle$ is of order 2. Let $\phi \in \hom_{\mathcal{F}}(Q, P)$ and $x_0 = t\phi$. Let $x \in P - \langle x_0 \rangle$ and $X = \langle x, x_0 \rangle$.

6.5. (1) $x_0 \in Z(C_H(x))$.

(2) x_0^* *centralizes* $O^2(C_{K^*}(x^*))$.

Proof. By 5.9(2), x_0 is strongly closed in P with respect to $\mathcal{F}$. Then as P is strongly closed in S_x with respect to $C_H(x)$ for $S_x \in \mathrm{Syl}_2(C_H(x))$ by 3.3(3), x_0 is strongly closed in S_x. Hence, by the Z^*-theorem, $O(C_H(x))\langle x_0 \rangle \trianglelefteq C_H(x)$. But as we saw during the proof of 3.8, $O(C_H(x)) = 1$, so (1) holds. Then (1) and 3.3(2) imply (2). □

6.6. $X^* = O_2(C_{H^*}(x^*))$, $C_{H^*}(X^*) = X^* \times L^*$, *and one of the following holds*:

(1) $K^* \cong M_{12}$, $X^* \not\leq K^*$ *and* $L^* \cong A_5$.

(2) $X^* \leq K^* \cong J_2$ *and* $L^* \cong A_5$.

(3) $X^* \leq K^* \cong \mathrm{Co}_1$ *and* $L^* \cong G_2(4)$.

(4) $X^* \leq K^* \cong \mathrm{Suz}$ *and* $L^* \cong L_3(4)$.

(5) $X^* \leq K^* \cong \mathrm{Ru}$ *and* $L^* \cong \mathrm{Sz}(8)$.

Proof. We appeal to 6.5 and inspect the tables in Section 5.3 of [14] for involutions x^* and a 4-group X^* centralizing $O^2(C_{K^*}(x^*))$. □

We are now in a position to derive a contradiction that will establish Theorem 6.1. From 6.6 there is an involution $y \in X$ with $y^* \in K^*$. Thus $y = ck$ for some $c \in Q$ and $k \in K$ with $k^* = y^*$. But from Section 5.3 in [14], $|k| = 4$. Then as $\Phi(Q) = 1$ by 6.2, $y = ck$ is also of order 4, contradicting $\Phi(P) = 1$. This completes the proof of Theorem 6.1.

7. $\Omega_6^-(3)$

We prove:

Theorem 7.1. *Assume Hypothesis 5.1 with* $K \cong \Omega_6^\epsilon(q)$ *and* $|Q| > 2$. *Then we may take* $q = 3$ *and we have*:

(1) $Q \cong \mathbb{Z}_4$.

(2) *Let x be the involution in P; then $x \in K$ is in $i(2, -1)$ in $K = \Omega_6^-(3)$, so* $O^2(C_{K^*}(x^*)) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$.

Assume $\mathcal{F}$ is a counterexample to the theorem. Adopt the notation from Section 5; in particular choose $P \in \Delta$ as in Notation 5.7. Note that by Definition 1.3(b), $q \equiv \pm 3 \bmod 8$ and $q \equiv \epsilon \bmod 4$; indeed the 2-fusion systems of any two groups satisfying these congruences are isomorphic, so, for example, we may take $(q, \epsilon) = (3, -1)$.

7.2. *Q is cyclic.*

Proof. Assume otherwise; by 5.3, $\Phi(Q) = 1$. Adopt Notation 6.4; applying 5.9(2) as in the proof of 6.5, x_0^* centralizes $O^2(C_{K^*}(x^*))$. We may take $K \cong \Omega_6^-(3)$. Then inspecting the list of involution centralizers in Table 2.10 of [1] for involutions x^* and a 4-group X^* centralizing $O^2(C_{K^*}(x^*))$ we conclude that $x^* \in i(4, \delta)$ for some $\delta \in \{1, -1\}$. But then for $y^* \in X^* - \langle x^* \rangle$, y^* is a reflection, a contradiction as $(xx_0)^*$ also serves in the role of x^*. □

Let x be the involution in P. Inspecting the list of centralizers in Table 2.10 of [1] we find that $O(C_{K^*}(i^*)) = 1$ for each involution $i^* \in H^*$, so by 3.6(2):

7.3. *P is tightly embedded in H, so $P \trianglelefteq C_H(x)$ and P^* centralizes $O^2(C_{K^*}(x^*))$.*

7.4. $|P^*| = 4$ *and either*

(1) $x^* \in i(4, +)$ *and* $O^2(C_{K^*}(x^*)) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$, *or*

(2) x^* *is a projective involution and* $C_{K^*}(x^*) \cong U_3(3)$.

Proof. This time we inspect Table 2.10 in [1] for centralizers in which some cyclic group of order at least 4 centralizes $O^2(C_{K^*}(x^*))$. □

7.5. *Case* (1) *of 7.4 holds.*

Proof. Assume instead that 7.4(2) holds. Let $F = \mathbb{F}_9$, V be a 4-dimensional unitary space over F, and $G = \mathrm{GU}(V)$. Let $B = \{v_1, \ldots, v_4\}$ be an orthonormal basis for V, $\lambda \in F$ of order 4, $w_i \in C_G(v_i^\perp)$ with $v_i w_i = \lambda v_i$, $Q_i = \langle w_i \rangle$, and $t_i = w_i^2$. Then $C_G(t_i) = Q_i \times L_i$ where $L_i = C_G(v_i) \cong \mathrm{GU}_3(3)$. Therefore Q_i is tightly embedded in G.

Next, setting $Z = \langle -\mathrm{id}_V \rangle \leq Z(G)$ and $G^+ = G/Z$, we have $\mathrm{SU}(V)^+ \cong K$ and $\mathrm{SU}(V)^+ Q_1^+$ is a split extension with w_1 inducing an automorphism on $\mathrm{SU}(V)^+$ quasiequivalent to that of a generator of P on K; so $H = PK \cong \mathrm{SU}(V)^+ Q_1^+$.

Let $\Sigma = \{Q_i : 1 \leq i \leq 4\}$ and $W_0 = \langle \Sigma \rangle$. Then $\Sigma = Q_1^G \cap C_G(W_0)$ as the weight spaces of W_0 are the Fv_i. Hence as the weak closure of Q_1 in a Sylow 2-subgroup S_G of G is abelian (since $Q_1 \cong \mathbb{Z}_4$ is tightly embedded), W_0 is that weak closure and $\Sigma = Q_1^G \cap S_G$ is of order 4. Moreover $\mathrm{Aut}_G(W_0)$ acts as $\mathrm{Sym}(\Sigma)$ on Σ and $U_0 = \langle t_1^{N_G(W_0)} \rangle$ is of rank 4 with $t_1^{+N_G(W_0)} = U_0^+ - U_1^+$ for a unique hyperplane U_1 of U_0, and a generator t^+ for $Z(\mathrm{SU}(V)^+)$ is not contained in U_0^+ as $|t| = 4$. In particular $\prod_i t_i^+ = 1$.

As $H \cong \mathrm{SU}(V)^+ Q_1^+$ it follows that $P^H \cap N_S(Q) = \theta$ is of order 4 and $\Theta = Q^{\mathcal{F}} = \theta \cup \{Q\}$ is of order 5. Moreover $W = \langle \Theta \rangle$ is abelian with $\mathrm{Aut}_{\mathcal{F}}(W)$ transitive on Θ, so as $\mathrm{Aut}_{N_{\mathcal{F}}(Q)}(W_0)$ induces $\mathrm{Sym}(\theta)$ on θ, $\mathrm{Aut}_{\mathcal{F}}(W)$ induces $\mathrm{Sym}(\Theta)$ on Θ. Let x_i be the involution in $P_i \in \theta$ and y the involution in Q. As $\prod_i t_i^+ = 1$ we have $\prod_i x_i = 1$. Then as $\prod_i x_i = 1$ and $\mathrm{Aut}_{\mathcal{F}}(\Theta)$ induces $\mathrm{Sym}(\Theta)$ on Θ, we conclude that $y = x_1x_2x_3 \in U = \langle x_i : 1 \le i \le 4\rangle$, contradicting $t^+ \notin U_0^+$. □

We are now in a position to complete the proof of Theorem 7.1. By 7.4 and 7.5, 7.4(1) holds. Thus $x^* \in i(4, +)$ induces an inner automorphism on K, so $x = ck$ for some $c \in Q$ and $k \in K$ with $k^* = x^*$. Then k is of type $i(4, +)$ or $i(2, -)$ in $K \cong \Omega_6^-(3)$. Further $1 = x^2 = c^2k^2 = c^2$, so as $Q \cong \mathbb{Z}_4$ by 7.4, $c \in Z(K)$ so $x \in K$ is in $i(4, +)$ or $i(2, -)$. Let t be the involution in Q, $X = \langle t, x\rangle$, and $\alpha \in \mathfrak{A}(x)$ with $x\alpha = t$. As P centralizes $O^2(C_K(x)) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$ and $x \in P$, we may take $X\alpha = X$. Then as $Z(O^2(C_K(x)))$ is generated by an element in $i(4, +)$ and $x\alpha = t \notin O^2(C_K(x))$, we conclude that x is in $i(2, -)$. This completes the proof of Theorem 7.1.

8. Chev(2)

In this section we assume the following hypothesis:

Hypothesis 8.1. Hypothesis 5.1 is satisfied with $K/Z(K) \in \mathrm{Chev}(2)$.

We continue to adopt the notation from Section 5, and choose $P \in \Delta$ as in Notation 5.7.

8.2. *P is tightly embedded in H.*

Proof. This is 3.7(1). □

In the remainder of the section we assume:

Hypothesis 8.3. Hypothesis 8.1 holds with $\Phi(Q) = 1$ and $|Q| > 2$.

8.4. *$P \le QK$.*

Proof. This is 3.7(2). □

8.5. *Assume $m(Q) > 2$. Then for each $h \in H$, $N_{P^h}(P) \le C_H(P)$.*

Proof. See the proof of 15.18 and 21.2 in [8]. □

8.6. *Assume $K/Z(K) \cong L_3(4)$. Then:*

(1) $Q = Z(K) \cong E_4$.

(2) $P^* = Z(T^*)$.

Proof. By choice of P in Notation 5.7, $P \in N_{\mathcal{F}}(Q)^f$ and by 8.4, $P^* \le T^*$, so as K^* has one class of involutions there exists an involution $z \in P$ with $z^* \in Z(T^*)$, and hence $z \in Z(T)Q$. Let $J = TQ$ and suppose $m(Q) > 2$. Then $z \in Z(J)$ so for $h \in H$, $P^h \cap J \le C_{P^h}(z) \le N_{P^h}(P)$ by 8.2, so $P^h \cap J \le C_H(P)$ by 8.5. But

$J \leq \langle z^H \cap J \rangle Q$, so $P \leq Z(J) = Z(T)Q$. Therefore as $|Z(J) : Q| = 4 < |P|$, $1 \neq P \cap Q$, a contradiction.

We've shown $m(Q) \leq 2$, so as $|Q| > 2$ by Hypothesis 8.3, we have $Q \cong E_4$. As $J \leq C_H(z) \leq N_H(P)$, we have $[P, T] \leq P \cap T$. Indeed if $P^* \neq Z(T^*)$ then $[P^*, T^*] = Z(T^*)$, so $Z(T)Q \leq PQ$, contradicting $|P| = |Q| = 4$. Hence $PQ = Z(J)$, so the lemma holds if $Q = Z(K)$. Thus we may assume $|Z(K)| = 2$ and it remains to produce a contradiction.

Next $J = J(N_S(Q))$, so $\mathcal{N} = N_{\mathcal{F}}(J)$ controls fusion of conjugates of members of Q in $Z(J)$. Therefore a generator k of $Z(K)$ is fused to a member of P in $\mathcal{N}$. This is a contradiction as k is the only member of $[J, J]$ which is not a commutator in J, which we saw during the proof of 2.6. In any event this is a contradiction. □

8.7. *$K/Z(K)$ is not $U_6(2)$ or $\Omega_8^+(2)$.*

Proof. Assume otherwise. By 8.2, P is tightly embedded in H, and by 8.5, either

(a) $|P| = 4$, or

(b) for each $h \in H$, $N_{P^h}(P) \leq C_H(P)$.

Therefore Hypothesis 22.1 in [10] is satisfied, so as $Z(K) \neq 1$, 22.2 in [10] supplies a contradiction. Note that the case $K/Z(K) \cong L_3(4)$ should have been excluded in Hypothesis 21.1, and it was not treated in 21.3 in [10]. □

8.8. *$K/Z(K)$ is not $G_2(4)$, $F_4(2)$, or ${}^2E_6(2)$.*

Proof. We can repeat the proof of 8.3 in [11]. Note the appeal to 3.9 in paragraph four of that proof should be an appeal to 2.7. □

Theorem 8.9. *Assume Hypothesis 8.1 with $\Phi(Q) = 1$. Then either $|Q| = 2$ or $K/Z(K) \cong L_3(4)$ and $Q = Z(K) \cong E_4$.*

Proof. Assume otherwise; then Hypothesis 8.3 is satisfied. Then $K \in \mathcal{K}_{qs}$, with $K/Z(K) \in \mathrm{Chev}(2)$. In particular K appears in 1.5. But we have eliminated the cases in 1.5(1)–(2) in 8.7 and 8.8, and 8.6 completes the proof when 1.5(3) holds. □

9. Mopping up

In this section we assume the following hypothesis:

Hypothesis 9.1. Hypothesis 8.1 is satisfied with Q cyclic and $|Q| > 2$.

We continue to adopt the notation from Section 5, and choose $P \in \Delta$ as in Notation 5.7. Let x be the involution in P and $\langle u \rangle = U = \Omega_2(P)$. Let $S_P = PT$, $\mathcal{H} = P\mathcal{C}$, and $W = \langle U^{\mathcal{H}} \rangle$.

9.2. *$\Phi(P) \leq QK$, so $x^* \in K^*$.*

Proof. From [14] a Sylow 2-subgroup of $\mathrm{Out}(K)$ is of exponent at most 2. □

9.3. *$K/Z(K)$ is not $G_2(4)$.*

Proof. Assume otherwise; from 9.2, $x^* \in K^*$, so x^* is in one of the two classes of involutions of K^* described in 18.2 of [10], and $C_{K^*}(x^*)$ is described in 18.4 of [10]. In particular $C_{H^*}(O^2(C_{K^*}(x^*))) = X^*$ is the root subgroup of the root involution x^*, whereas from 3.3(2) and 8.2, $P^* \leq X^*$ and P is cyclic by Hypothesis 9.1. □

9.4. *$K/Z(K)$ is not $L_3(4)$.*

Proof. Assume otherwise; then K^* has one class of involutions, so as $P \in N_{\mathcal{F}}(Q)^f$, $x^* \in Z(T^*)$ and then by 1.6, $x \in Z(QT)$. Hence by 8.2, $[QT, U] \leq U$, whereas T^* normalizes no $\mathbb{Z}_4$-subgroup of H^*. □

Given a finite group G, a *near transposition* in G is an involution t such that whenever $s \in t^G$ with $\langle s, t\rangle$ a 2-group, we have $[s, t] = 1$.

9.5. *Let $R \in \mathcal{H}^{\mathrm{frc}}$ with $R \trianglelefteq S_P$, let Y be a model for $N_{\mathcal{H}}(R)$, and set $Y^+ = Y/R$. Then*:

(1) $\Phi(W) \leq R$.

(2) *If $u \notin R$ then u^+ is a near transposition in Y^+.*

Proof. This follows from 18.4 in [8]. □

9.6. *$K/Z(K)$ is not $U_6(2)$.*

Proof. Assume otherwise; regard K^* as the image of $\widehat{K} = \mathrm{SU}(V)$ for a 6-dimensional unitary space V over $\mathbb{F}_4$. Let V_3 be a T-invariant 3-dimensional totally singular subspace of V, $Y = N_H(V_3)$, and $R = O_2(Y)$. Then $R^* \cong E_{2^9}$ and $Y^+ \cong L_3(4)$ is irreducible on R^*. By 9.5, $x \in \Phi(W) \leq R$ and as $\Phi(R^*) = 1$, $u \notin R$. Thus u^+ is a near transposition in Y^+ by 9.5(2). This contradicts 17.2 in [8], which says that $L_3(4)$ has no near transpositions. □

9.7. *$K/Z(K)$ is not $\Omega_8^+(2)$.*

Proof. Assume otherwise; then K^* has three maximal parabolics Y_i^*, $1 \leq i \leq 3$, such that $R_i^* = O_2(Y_i^*)$ is the orthogonal module for $Y_i^*/R_i^* \cong \Omega_6^+(2)$. As Q is cyclic, $|Z(K)| = 2$, so there is a unique i with $\Phi(R_i) = 1$, say $i = 1$. By 9.5(1), $x \in R_1$, and as $\Phi(R_1^*) = 1$, $u \notin R_1$. Hence by 9.5(2), u^+ is a near transposition in Y^+, where $Y = N_H(R_1^*)$. It follows that $Y^+ \cong O_6^+(2)$ and u^+ is a transvection on the orthogonal space R_1^*. Then $[R_1, u] = \langle x\rangle$ with x^* a nonsingular point in R_1^*, so $X^* = O^2(C_{K^*}(x^*))$ is $[R_1^*, X^*]$ extended by $\mathrm{Sp}_6(2)$. This is a contradiction as such an X^* centralizes no element of Y^* of order 4. □

9.8. *$K/Z(K)$ is not ${}^2E_6(2)$.*

Proof. Assume otherwise; we repeat the argument in 18.15, 18.6, and 18.19 in [8].

Let z^* be the long root involution in $Z(T^*)$ and $Y^* = C_{K^*}(z^*)$. Then $R^* = O_2(Y^*) \cong 2^{1+20}$ is extraspecial with $Y^+ \cong U_6(2)$ irreducible on $\widetilde{R} = R^*/\langle z^* \rangle$. In particular Y^* centralizes no element of order 4, so $x^* \neq z^*$. Then as $\Phi(R^*) = \langle z^* \rangle$, $u^* \notin R^*$, so u^+ is a near transposition in Y^+. Hence by 17.7(1) in [8], u^+ is a transvection in Y^+. As $\Phi(R) \leq Z(K)\langle z \rangle$, $|R : C_R(x)| \leq 4$, so as $[C_R(x), u] \leq \langle x \rangle$ we have $m([\widetilde{R}, u]) \leq 3$. Then by 17.7(2) in [8], $m(\widetilde{R}) \leq 18$, a contradiction. □

9.9. *$K/Z(K)$ is not $F_4(2)$.*

Proof. Assume otherwise and for $c \in \{l, s\}$ let z_c^* be the c-root involution in $Z(T^*)$. By 1.6, $z_c \in Z(T)$.

Suppose $u^* \notin K^*$; then $z_l^{*u} = z_s^*$ so $[z_l, u] = z_l z_s = x' \in Z(T)$, and hence $x = x'$. But then $T = C_T(u)\langle z_l \rangle$, a contradiction.

Hence $u^* \in K^*$, so $u \in Y = C_{PK}(z_l)$. Now from Section 8 in [3], $R^* = O_2(Y^*) = R_1^* Z(R^*)$ where $R_1^* \cong 2^{1+8}$ and $Z(R^*) \cong E_{2^7}$, and $Y^*/R^* \cong \mathrm{Sp}_6(2)$ with $\langle z_l^* \rangle = Z(Y^*)$. Hence $x^* \neq z_l^*$ and as $\Phi(R^*) = \langle z_l^* \rangle$, we conclude that $u^* \notin R^*$. Therefore u^+ is a near transposition in $Y^+ \cong \mathrm{Sp}_6(2)$, so u^+ is a transvection. Arguing as in the proof of the previous lemma, $m([\widetilde{R}, u]) \leq 3$. This contradicts $m([R/Z(R), u]) = 4$. □

Theorem 9.10. *Assume Hypothesis 8.1 holds with Q cyclic. Then $|Q| = 2$.*

Proof. By Hypothesis 8.1, $K \in \mathcal{K}_{\mathrm{qs}}$ with $K/Z(K) \in \mathrm{Chev}(2)$. Thus K is described in 1.5. But now the various cases arising in 1.5 are treated in this section, establishing the theorem. □

Theorem 2. *Assume Hypothesis 5.1. Then one of the following holds*:

(1) $|Q| = 2$.

(2) $K \cong \Omega_6^\epsilon(q)$ *with* $q \equiv \pm 3 \bmod 8$ *and* $\epsilon \equiv q \bmod 4$, *and* $Q \cong \mathbb{Z}_4$.

(3) $K/Z(K) \cong L_3(4)$ *and* $Q = Z(K) \cong E_4$.

Proof. By Hypothesis 5.1(3), $\mathcal{C}$ is tamely realized by $K \in \mathcal{K}_{\mathrm{qs}}$. The class $\mathcal{K}_{\mathrm{qs}}$ is defined in Definition 1.3; in particular one of the following holds:

(i) $K/Z(K)$ is sporadic.

(ii) $K/Z(K) \in \mathrm{Chev}(2)$.

(iii) $K \cong \Omega_6^\epsilon(q)$ with $q \equiv \pm 3 \bmod 8$ and $\epsilon \equiv q \bmod 4$.

This follows as $K/Z(K)$ is a known finite simple group that is not Goldschmidt, $Z(K)$ is of even order, coverings of alternating groups are excluded in Definition 1.3(a), and coverings of groups of Lie type and odd characteristic, with exception of the orthogonal group in (iii), are excluded in Definition 1.3(b).

In case (i) conclusion (1) of Theorem 2 holds by Theorem 6.1. In case (iii) conclusion (1) or (2) of Theorem 2 holds by Theorem 7.1. This leaves case (ii), where Hypothesis 8.1 is satisfied.

By 5.3 either $\Phi(Q) = 1$ or Q is cyclic. In the first case conclusion (1) or (3) of Theorem 2 holds by Theorem 8.9. Thus we may assume Hypothesis 9.1 is satisfied. Here conclusion (1) of Theorem 2 holds by Theorem 9.10, completing the proof. □

Finally we supply a proof of Theorem 1. So assume the hypothesis of that theorem. First we check that the hypotheses of 2.1 are satisfied. Condition 2.1(a) holds by hypothesis (c) of Theorem 1. Condition 2.1(b) holds by hypothesis (b) of Theorem 1. And condition 2.1(c) holds by hypothesis (a) of Theorem 1.

By 2.1, either conclusion (1) of Theorem 1 is satisfied, or $\mathcal{C}$ is terminal, and we may assume the latter. Thus Hypothesis 2.2 is satisfied. Hence by 2.3(1), $\mathcal{C}$ is nearly standard, so Q exists. We may assume conclusion (2)(i) of Theorem 1 does not hold, so $|Q| > 2$. Hence by Theorem 2.8 either conclusion (2)(iii) holds or $\mathcal{C}$ is standard, and we may assume the latter. Therefore Hypothesis 5.1 is satisfied. But now Theorem 2 completes the proof.

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ON GOOD A_1 SUBGROUPS, SPRINGER MAPS, AND OVERGROUPS OF DISTINGUISHED UNIPOTENT ELEMENTS IN REDUCTIVE GROUPS

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Dedicated to the fond memory of Gary Seitz

Suppose G is a simple algebraic group defined over an algebraically closed field of good characteristic p. In 2018 Korhonen showed that if H is a connected reductive subgroup of G which contains a distinguished unipotent element u of G of order p, then H is G-irreducible in the sense of Serre. We present a short and uniform proof of this result under an extra hypothesis using so-called *good* A_1 subgroups of G, introduced by Seitz. In the process we prove some new results about good A_1 subgroups of G and their properties. We also formulate a counterpart of Korhonen's theorem for overgroups of u which are finite groups of Lie type. Moreover, we generalize both results above by removing the restriction on the order of u under a mild condition on p depending on the rank of G, and we present an analogue of Korhonen's theorem for Lie algebras.

1. Introduction and main results

Throughout, G is a connected reductive linear algebraic group defined over an algebraically closed field k of characteristic p and H is a closed subgroup of G.

Following Serre [35], we say that H is *G-completely reducible* (G-cr for short) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P, and that H is *G-irreducible* (G-ir for short) provided H is not contained in any proper parabolic subgroup of G at all. Clearly, if H is G-irreducible, it is trivially G-completely reducible, and an overgroup of a G-irreducible subgroup is again G-irreducible; for an overview of this concept see [4], [34] and [35]. Note that in case $G = \mathrm{GL}(V)$ a subgroup H is G-completely reducible exactly when V is a semisimple H-module and it is G-irreducible precisely

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when V is an irreducible H-module. Recall that if H is G-completely reducible, then the identity component H° of H is reductive [35, Proposition 4.1].

A unipotent element u of G is *distinguished* provided any torus in the centralizer $C_G(u)$ of u in G is central in G. Likewise, a nilpotent element X of the Lie algebra $\mathfrak{g}$ of G is *distinguished* provided any torus in the centralizer $C_G(X)$ of X in G is central in G, see [10, Section 5.9] and [16, Section 4.1]. For instance, regular unipotent elements in G are distinguished, and so are regular nilpotent elements in $\mathfrak{g}$ [39, III, 1.14] (or [10, Proposition 5.1.5]). The converse is true in type A, since a distinguished unipotent or nilpotent element must clearly consist of a single Jordan block. Overgroups of regular unipotent elements have attracted much attention in the literature, e.g., see [22; 32; 42; 44] and [6].

The following remarkable result was proved by Korhonen.

Theorem 1.1 [17, Theorem 6.5]. *Suppose G is simple and p is good for G. Let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G of order p. Then H is G-irreducible.*

One can easily extend this theorem to arbitrary connected reductive G by reducing to the simple case, see Remark 6.3.

Korhonen's proof of Theorem 1.1 depends on checks for the various possible Dynkin types for simple G. E.g., for G simple of exceptional type, Korhonen's argument relies on long exhaustive case-by-case investigations from [20], where all connected reductive non-G-cr subgroups are classified in the exceptional type groups in good characteristic. For classical G, Korhonen requires an intricate classification of all SL_2-representations on which a nontrivial unipotent element of SL_2 acts with at most one Jordan block of size p. Our main aim is to give a short uniform proof of Theorem 1.1 in Section 6 without resorting to further case-by-case checks, but imposing an extra hypothesis which allows us to use a landmark result by Seitz (see Section 5.1).

Theorem 1.2. *Suppose p is good for G. Let H be a connected reductive subgroup of G. Suppose H contains a distinguished unipotent element of G of order p. Suppose also that*:

(†) *There exists a Springer map ϕ for H such that $\phi(u)$ is a distinguished element of $\mathfrak{g}$.*

Then H is G-irreducible.

For a discussion of Springer maps, see Section 4.1.

Remark 1.3. Suppose as in Theorem 1.1, that G is simple classical with natural module V, and $p \geq \dim V > 2$. Then, thanks to [15, Proposition 3.2], V is semisimple as an H°-module, and by [35, (3.2.2(b))], this is equivalent to H° being

G-cr. Then H is G-ir, by Lemma 3.1. This gives a short uniform proof of the conclusion of Theorem 1.1 in this case, as the bound $p \geq \dim V > 2$ ensures that every distinguished unipotent element (including the regular ones) is of order p. The conclusion can fail if the bound is not satisfied, see Theorem 1.5.

We say that a subgroup of G is of *type* A_1 if it is isomorphic to SL_2 or PGL_2. Our proof of Theorem 1.2 involves the notion of a *good* A_1 subgroup, which was introduced by Seitz in [33]. We consider the interaction of good A_1 subgroups with associated cocharacters and Springer maps; we identify a useful class of Springer maps (Definition 5.16), which we call *logarithmic* Springer maps, and we prove some results that are of interest in their own right (see Corollary 5.20 and Lemma 5.30). Our main result on good A_1 subgroups is the following (see Section 5.2 for definitions).

Theorem 1.4. *Suppose p is good for G and let A be an A_1 subgroup of G. The following are equivalent.*

(i) *A is subprincipal.*

(ii) *A is optimal.*

(iii) *A is good.*

Theorem 1.1 covers the situation when p is good for G. There are only a few cases when G is simple, p is bad for G, and G admits a distinguished unipotent element of order p, by work of Proud, Saxl, and Testerman [31, Lemmas 4.1, 4.2] (see Lemmas 2.5 and 2.7). In this case the conclusion of Theorem 1.1 fails precisely in one instance, as observed in [17, Proposition 1.2] (Example 2.6), else it is valid (Example 2.8). Combining the cases when p is bad for G with Theorem 1.2, we recover Korhonen's main theorem [17, Theorem 1.3] (assuming that (†) from Theorem 1.2 holds).

Theorem 1.5. *Suppose G is simple and let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G of order p, and suppose that (†) holds. Then H is G-irreducible, unless $p = 2$, G is of type C_2, and H is a type A_1 subgroup of G.*

Our next goal is an extension of Theorem 1.2 to finite groups of Lie type in G. Let $\sigma : G \to G$ be a Steinberg endomorphism of G, so that the finite fixed point subgroup $G_\sigma = G(q)$ is a finite group of Lie type over the field $\mathbb{F}_q$ of q elements. For a Steinberg endomorphism σ of G and a connected reductive σ-stable subgroup H of G, σ is also a Steinberg endomorphism for H with finite fixed point subgroup $H_\sigma = H \cap G_\sigma$ [40, 7.1(b)]. Obviously, one cannot directly appeal to Theorem 1.2 to deduce anything about H_σ, because $(H_\sigma)^\circ$ is trivial. For the notion of a q-Frobenius endomorphism, see Section 2.3.

Theorem 1.6. *Let H be a connected reductive subgroup of G and suppose p is good for G. Let $\sigma : G \to G$ be a Steinberg endomorphism stabilizing H such that $\sigma|_H$ is a q-Frobenius endomorphism of H. If G admits components of exceptional type, then assume $q > 7$. Suppose H_σ contains a distinguished unipotent element of G of order p, and suppose that (†) holds. Then H_σ is G-irreducible.*

Combining Theorem 1.6 with the aforementioned results from [31], we are able to deduce the following analogue of Theorem 1.5 for finite subgroups of Lie type in G.

Theorem 1.7. *Let H be a connected reductive subgroup of G. Let $\sigma : G \to G$ be a Steinberg endomorphism stabilizing H such that $\sigma|_H$ is a q-Frobenius endomorphism of H. If G is of exceptional type, then assume $q > 7$. Suppose H_σ contains a distinguished unipotent element of G of order p, and suppose that (†) holds. Then H_σ is G-irreducible, unless $p = 2$, G is of type C_2, and H is a type A_1 subgroup of G.*

In the special instance in Theorems 1.6 and 1.7 when H_σ contains a regular unipotent element u from G, the conclusion of both theorems holds without any restriction on the order of u and without any restriction on q (and without any exceptions of the type seen in Theorem 1.7), see [6, Theorem 1.3].

In our final main result we show that we can remove condition (†) and the condition that u has order p from Theorem 1.2, at the cost of increasing our bound on p. We also obtain an analogue under the hypothesis that $\mathrm{Lie}(H)$ contains a distinguished nilpotent element of $\mathfrak{g}$. For a unipotent element $u \in G$ to be distinguished is a mere condition on the structure of the centralizer $C_G(u)$ of u in G. The extra condition for u to have order p is thus somewhat artificial. This restriction in Theorems 1.1 and 1.2 is due to the methods used in [17] and in our proofs in Section 6, which require the unipotent element to lie in a subgroup of type A_1; such an element must obviously have order p.

To state our theorem, we need to introduce an invariant $a(G)$ of G from [35, Section 5.2]: for G simple, set $a(G) = \mathrm{rk}(G) + 1$, where $\mathrm{rk}(G)$ is the rank of G. For reductive G, let $a(G) = \max\{1, a(G_1), \ldots, a(G_r)\}$, where $G_1, \ldots, G_r$ are the simple components of G.

Theorem 1.8. *Suppose $p \geq a(G)$. Let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G or* $\mathrm{Lie}(H)$ *contains a distinguished nilpotent element of* $\mathfrak{g}$. *Then H is G-irreducible.*

Section 2 contains background material. In Section 3 we prove Theorem 1.8, along with some analogues for finite subgroups of Lie type. In Section 4 we discuss Springer maps and associated cocharacters. We recall Seitz's notion of good A_1 subgroups in Section 5 and we prove Theorem 1.4 in Section 5.2 (see Theorem 5.24). Theorems 1.2 and 1.5–1.7 are proved in Section 6.

2. Preliminaries

2.1. ***Notation.*** Throughout, we work over an algebraically closed field k of characteristic p. For convenience we assume that $p > 0$ unless otherwise stated; most of our results hold for $p = 0$ with obvious modifications and in many cases the proof is much easier (see Remarks 3.3(vi), for example). All affine varieties are considered over k and are identified with their sets of k-points. A linear algebraic group H over k has identity component H°; if $H = H^\circ$, then we say that H is *connected*. We denote by $R_u(H)$ the *unipotent radical* of H; if $R_u(H)$ is trivial, then we say H is *reductive*.

Throughout, G denotes a connected reductive linear algebraic group over k. All subgroups of G considered are closed. By $\mathscr{D}G$ we denote the derived subgroup of G, and likewise for subgroups of G. We denote the Lie algebra of G by $\mathrm{Lie}(G)$ or by $\mathfrak{g}$. If $p > 0$ then we denote the p-power map on $\mathfrak{g}$ by $X \mapsto X^{[p]}$. By a Levi subgroup of G we mean a Levi subgroup of some parabolic subgroup of G. Recall that a homomorphism $f : G_1 \to G_2$ of connected algebraic groups is a *central isogeny* if f is surjective, $\ker(f)$ is finite and the kernel of the derivative df is central in $\mathrm{Lie}(G_1)$.

Let $Y(G) = \mathrm{Hom}(\mathbb{G}_m, G)$ denote the set of cocharacters of G. For $\mu \in Y(G)$ and $g \in G$ we define the *conjugate cocharacter* $g \cdot \mu \in Y(G)$ by $(g \cdot \mu)(t) = g\mu(t)g^{-1}$ for $t \in \mathbb{G}_m$; this gives a left action of G on $Y(G)$. For H a subgroup of G, let $Y(H) := Y(H^\circ) = \mathrm{Hom}(\mathbb{G}_m, H)$ denote the set of cocharacters of H. There is an obvious inclusion $Y(H) \subseteq Y(G)$.

Fix a Borel subgroup B of G containing a maximal torus T. Let $\Phi = \Phi(G, T)$ be the root system of G with respect to T, let $\Phi^+ = \Phi(B, T)$ be the set of positive roots of G, and let $\Sigma = \Sigma(G, T)$ be the set of simple roots of Φ^+. For each $\alpha \in \Phi$ we have a root subgroup U_α of G. For α in Φ, let $x_\alpha : \mathbb{G}_a \to U_\alpha$ be a parametrization of the root subgroup U_α of G.

We denote the unipotent variety of G by $\mathscr{U}_G$ and the nilpotent cone of $\mathfrak{g}$ by $\mathscr{N}_G$. We define

$$\mathscr{U}_G^{(1)} = \{u \in \mathscr{U}_G \mid u^p = 1\}$$

and

$$\mathscr{N}_G^{(1)} = \{X \in \mathscr{N}_G \mid X^{[p]} = 0\}.$$

If $u \in \mathscr{U}_G$ then we have a unique decomposition $u = u_1 \cdots u_r$, where $u_i \in G_i$ and the G_i are the simple factors of $\mathscr{D}G$; we call u_i the *projection of u onto G_i*. Clearly u is distinguished in G if and only if u_i is distinguished in G_i for each i.

2.2. ***Good primes.*** A prime p is said to be *good* for G if it does not divide any coefficient of any positive root when expressed as a linear combination of simple ones. Else p is called *bad* for G [39, Section 4]. Explicitly, if G is simple, p is *good* for G provided $p > 2$ in case G is of Dynkin type B_n, C_n, or D_n; $p > 3$ in case G is of Dynkin type E_6, E_7, F_4 or G_2 and $p > 5$ in case G is of type E_8. If G is semisimple then we say that p is *separably good* for G if p is good for G and the canonical map from G_{sc} to G is separable, where G_{sc} is the simply connected cover of G. For arbitrary connected reductive G we say that p is *separably good* for G if it is separably good for $[G, G]$. We observe that if L is a Levi subgroup of G and p is good for G, then it is also good for L.

2.3. *Steinberg endomorphisms of G.* Recall that there is a basic dichotomy for endomorphisms of a simple algebraic group: either the endomorphism is an automorphism, or its set of fixed points is finite [40, Theorem 10.13]. Given a reductive group G, a *Steinberg endomorphism of* G is a surjective homomorphism $\sigma : G \to G$ such that the corresponding fixed point subgroup $G_\sigma := \{g \in G \mid \sigma(g) = g\}$ of G is finite. If $\mathscr{S}$ is a σ-stable set of closed subgroups of G, then $\mathscr{S}_\sigma$ denotes the subset consisting of all σ-stable members of $\mathscr{S}$.

If G is a reductive group defined over the finite field $\mathbb{F}_q$ (i.e., with some $\mathbb{F}_q$-structure), then the corresponding standard Frobenius endomorphism $\sigma_q : G \to G$ is an example of a Steinberg endomorphism, and in this case we also write $G_\sigma = G(q)$. In this situation, there exist a σ_q-stable maximal torus T and Borel subgroup $B \supseteq T$, and with respect to a chosen parametrization of the root groups as above, we have $\sigma_q(x_\alpha(t)) = x_\alpha(t^q)$ for each $\alpha \in \Phi$ and $t \in \mathbb{G}_a$, see [13, Theorem 1.15.4(a)].

Recall that a *generalized Frobenius endomorphism* of a reductive group G is an endomorphism of G for which some power is a standard Frobenius endomorphism σ_q. If G is simple, then *every* Steinberg endomorphism of G is actually a generalized Frobenius morphism [13, Theorem 2.1.11]. Further, when G is simple and p is good for G, every such endomorphism has the form $\sigma = \tau\sigma_q$, where τ is an algebraic automorphism of G of finite order, σ_q is a standard q-power Frobenius endomorphism of G, and σ_q and τ commute, see [40, Section 11]. Conversely, it is clear (for arbitrary reductive G) that any endomorphism σ which factorizes in this way is a generalized Frobenius endomorphism. However, if G is not simple and p is bad for G, then a generalized Frobenius map may fail to factor into a field and algebraic automorphism of G, e.g., see [14, Example 1.3].

Following [31], we call a generalized Frobenius endomorphism σ a *q-Frobenius endomorphism* provided $\sigma = \tau\sigma_q$, where τ is an algebraic automorphism of G of finite order, σ_q is a standard q-power Frobenius endomorphism of G, and σ_q and τ commute.

2.4. *Bala–Carter theory.* We recall some relevant results and concepts from Bala–Carter theory. Suppose p is good for G. A parabolic subgroup P of G admits a dense open orbit on its unipotent radical $R_u(P)$, the so-called *Richardson orbit*, see [10, Theorem 5.2.1]. A parabolic subgroup P of G is called *distinguished* provided

$\dim(\mathscr{D}P/R_u(P)) = \dim(R_u(P)/\mathscr{D}R_u(P))$, see [30, Section 2.1]. For G simple, the distinguished parabolic subgroups of G (up to G-conjugacy) were worked out in [2] and [3], see [10, pp. 174–177]. The notion of a distinguished parabolic subgroup of G also makes sense in case p is bad for G, see [16, Section 4.10].

The following is the celebrated Bala–Carter theorem [10, Theorems 5.9.5, 5.9.6], which is valid in good characteristic, thanks to work of Pommerening [28; 29]. For the Lie algebra versions see also [16, Proposition 4.7, Theorem 4.13].

Theorem 2.1. *Suppose p is good for G.*

(i) *There is a bijective map between the G-conjugacy classes of distinguished unipotent elements of G and conjugacy classes of distinguished parabolic subgroups of G. The unipotent class corresponding to a given parabolic subgroup P contains the dense P-orbit on $R_u(P)$.*

(ii) *There is a bijective map between the G-conjugacy classes of unipotent elements of G and conjugacy classes of pairs (L, P), where L is a Levi subgroup of G and P is a distinguished parabolic subgroup of $\mathscr{D}L$. The unipotent class corresponding to the pair (L, P) contains the dense P-orbit on $R_u(P)$.*

Remark 2.2. (i) Let $1 \neq u \in \mathscr{U}_G$. Let S be a maximal torus of $C_G(u)$. Then u is distinguished in the Levi subgroup $C_G(S)$ of G, since S is the unique maximal torus of $C_{C_G(S)}(u)$. Conversely, if L is a Levi subgroup of G with u distinguished in L, then the connected center of L is a maximal torus of $C_G(u)^\circ$, see [16, Remark 4.7].

(ii) Let $\sigma : G \to G$ be a Steinberg endomorphism of G and let $1 \neq u \in G_\sigma$ be unipotent. Then $C_G(u)^\circ$ is σ-stable. The set of all maximal tori of $C_G(u)^\circ$ is σ-stable and $C_G(u)^\circ$ is transitive on that set [38, Theorem 6.4.1]. Thus the Lang–Steinberg theorem, see [39, I, 2.7], provides a σ-stable maximal torus, say S, of $C_G(u)^\circ$. Then, by part (i), $L = C_G(S)$ is a σ-stable Levi subgroup of G and u is distinguished in L.

2.5. ***Cocharacters and parabolic subgroups of G.*** Let $\lambda \in Y(G)$. Recall that λ affords a $\mathbb{Z}$-grading on $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j, \lambda)$, where

$$\mathfrak{g}(j, \lambda) := \{X \in \mathfrak{g} \mid \operatorname{Ad}(\lambda(t))X = t^j X \text{ for every } t \in \mathbb{G}_m\}$$

is the j-weight space of $\operatorname{Ad}(\lambda(\mathbb{G}_m))$ on $\mathfrak{g}$, see [10, Section 5.5] or [16, Section 5.1]. Let $\mathfrak{p}_\lambda := \bigoplus_{j\geq 0} \mathfrak{g}(j, \lambda)$. Then there is a unique parabolic subgroup P_λ with $\operatorname{Lie}(P_\lambda) = \mathfrak{p}_\lambda$ and $C_G(\lambda) := C_G(\lambda(\mathbb{G}_m))$ is a Levi subgroup of P_λ. Since all maximal tori in G are conjugate, it suffices to describe these subgroups and subalgebras when $\lambda \in Y(T)$ for our fixed maximal torus T. In this case, letting $X(T) = \operatorname{Hom}(T, \mathbb{G}_m)$ denote the character group of T, we have $U_\alpha \subseteq P_\lambda$ if and only if $\langle \lambda, \alpha\rangle \geq 0$, where $\langle\, ,\, \rangle : Y(T) \times X(T) \to \mathbb{Z}$ is the usual pairing between cocharacters and characters.

We have $U_\alpha \subseteq C_G(\lambda)$ if and only if $\langle \lambda, \alpha \rangle = 0$, and $R_u(P_\lambda)$ is generated by the U_α with $\langle \lambda, \alpha \rangle > 0$, see the proof of [38, Proposition 8.4.5].

Set $J := \{\alpha \in \Sigma \mid \langle \alpha, \lambda \rangle = 0\}$. Then $P_\lambda = P_J = \langle T, U_\alpha \mid \langle \alpha, \lambda \rangle \geq 0 \rangle$ is the *standard parabolic subgroup* of G associated with $J \subseteq \Sigma$.

Let $\rho = \sum_{\alpha \in \Sigma} c_{\alpha\rho}\, \alpha$ be the highest root in Φ^+. Define $\mathrm{ht}_J(\rho) := \sum_{\alpha \in \Sigma \setminus J} c_{\alpha\rho}$. In view of Theorem 2.1, the following gives the order of a distinguished unipotent element in good characteristic.

Lemma 2.3 [43, order formula (0.4)]. *Suppose p is good for G. Let $P = P_J$ be a distinguished parabolic subgroup of G and let u be in the Richardson orbit of P on $R_u(P)$. Then the order of u is* $\min\{p^a \mid p^a > \mathrm{ht}_J(\rho)\}$.

2.6. ***Overgroups of type A_1.*** It has been understood for some time now that if p is good for G then one can study unipotent elements of G having order p by embedding them in A_1 subgroups of G. The existence of A_1 overgroups for unipotent elements of order p is guaranteed by the following fundamental results of Testerman [43, Theorem 0.1] if p is good for G and else by Proud, Saxl, and Testerman [31]; these results were originally proved for semisimple G but the extension to arbitrary connected reductive G is immediate.

Theorem 2.4 [43, Theorems 0.1, 0.2]. *Suppose p is good for G. Let σ be* id_G *or a Steinberg endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. Then there exists a σ-stable subgroup of G of type A_1 containing u.*

The proof of Theorem 2.4 is based on case-by-case checks and depends in part on computer calculations involving explicit unipotent class representatives. For a uniform proof of the theorem, we refer the reader to McNinch [23]. Conditions to ensure G-complete reducibility of such a subgroup were given in [26].

We now consider A_1 overgroups of distinguished unipotent elements in arbitrary characteristic. There are only a few instances when G is simple, p is bad for G, and G admits a distinguished unipotent element of order p. We recall the relevant results concerning the existence of A_1 overgroups of such elements.

Lemma 2.5 [31, Lemma 4.1]. *Let G be simple classical of type B_l, C_l, or D_l and suppose $p = 2$. Then G admits a distinguished involution u if and only if G is of type C_2 and u belongs to the subregular class $\mathscr{C}$ of G. If σ is* id_G *or a q-Frobenius endomorphism of G and $u \in \mathscr{C} \cap G_\sigma$, then there exists a σ-stable subgroup A of G of type A_1 containing u.*

Example 2.6. Let G be simple of type C_2 and let $p = 2$. Let σ be id_G or a q-Frobenius endomorphism of G, and suppose $u \in G_\sigma$ is a distinguished unipotent element of order 2. Then Lemma 2.5 provides a σ-stable subgroup A of type A_1 containing u. Thanks to [17, Proposition 1.2], there are such subgroups A which are not G-ir. In fact, according to the same reference, there are two G-conjugacy classes of such A_1 subgroups in G, see Example 5.15 below. Since A is contained in a proper parabolic subgroup of G, so is A_σ. So the latter is also not G-ir. By Lemma 3.1 below, A and A_σ are not G-cr, either.

Lemma 2.7 [31, Lemmas 3.3, 4.2]. *Let G be simple of exceptional type and suppose p is bad for G. Then G admits a distinguished unipotent element u of order p if and only if G is of type G_2, $p = 3$, and u belongs either to the subregular class $G_2(a_1)$*[1] *or to the class $A_1^{(3)}$ of G. Moreover, if σ is id_G or a q-Frobenius endomorphism of G and $u \in G_2(a_1) \cap G_\sigma$, then there exists a σ-stable subgroup A of G of type A_1 containing u. In case $u \in A_1^{(3)}$, there is no overgroup of u in G of type A_1.*

Example 2.8. Let G be simple of type G_2 and $p = 3$. Let H be a reductive subgroup of G containing a distinguished unipotent element u from G. Then, as $p = 3 = a(G_2)$, it follows from Theorem 1.8 that H° is G-ir, and so is H. This applies in particular to the subgroup A of G of type A_1 containing u when $u \in G_2(a_1)$. Since 3 is not a good prime for G, Theorem 1.1 does not apply in this case. See also [41, Corollary 2].

In case of the presence of a q-Frobenius endomorphism of G stabilizing H, we show in our proof of Theorem 1.7 that H_σ is also G-ir.

Theorem 2.9 [31, Theorem 5.1]. *Let G be semisimple and suppose p is bad for G. Let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. If $p = 3$, and G has a simple component of type G_2, assume that the projection of u into this component does not lie in the class $A_1^{(3)}$. Then there exists a σ-stable subgroup of G of type A_1 containing u.*

Corollary 2.10. *Let G be simple of type G_2, $p = 3$ and let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in A_1^{(3)} \cap G_\sigma$. Then there is no proper semisimple subgroup H of G containing u. In particular, any such u is* ***semiregular****, that is, $C_G(u)$ does not contain a noncentral semisimple element of G.*

Proof. By way of contradiction, suppose H is a proper semisimple subgroup of G containing u. Since $p = 3$ is good for H (e.g., see [41, Corollary 3]), there is a σ-stable A_1 subgroup A in H containing u, by Theorem 2.4. It follows from Lemma 2.7 that $u \in G_2(a_1)$ which contradicts the hypothesis that $u \in A_1^{(3)}$. □

The following result is needed in the proof of Theorem 1.2 below.

Lemma 2.11. *Suppose G is semisimple and $p = 3$ is good for G. Let H be a connected reductive subgroup of G. Let $u \in H$ be a unipotent element of order 3 which is distinguished in G. Then H does not admit a simple component of type G_2.*

[1] Throughout, we use the Bala–Carter notation for distinguished classes in the exceptional groups, see [10, Section 5.9].

Proof (see [17, p. 387]). Since p is good for G, every simple component of G is of classical type. Let V' be the natural module of the simple component G' of G, and let H' be the projection of H into G'. Since the projection u' of u into G' has order 3, the largest Jordan block size of u' on V' is at most 3. Since u' is distinguished in G', the Jordan block sizes of u' are distinct and of the same parity. Hence $\dim V' \leq 4$. Since a nontrivial representation of a simple algebraic group of type G_2 has dimension at least 5, H' does not have a simple component of type G_2. Hence H has no simple component of type G_2. □

In summary, we see that if $1 \neq u \in \mathscr{U}_G^{(1)}$ then u is contained in an A_1 subgroup of G unless $p = 3$ and G has a simple G_2 factor such that the projection of u onto this factor lies in the class $A_1^{(3)}$.

3. Variations on Theorems 1.2 and 1.6

In this section we prove Theorem 1.8. We also state and prove some related results for finite subgroups of Lie type. We need the following analogue of [6, Corollary 4.6], which shows that in order to derive the G-irreducibility of H in Theorem 1.8, it suffices to show that H is G-cr, see also [17, Lemma 6.1]. This also applies to Theorems 1.2 and 1.6.

Lemma 3.1. *Let H be a G-completely reducible subgroup of G. Suppose that H contains a distinguished unipotent element u of G or* $\operatorname{Lie}(H)$ *contains a distinguished nilpotent element X of $\mathfrak{g}$. Then H is G-irreducible.*

Proof. Suppose H is contained in a parabolic subgroup P of G. Then, by hypothesis, H is contained in a Levi subgroup L of P. As the latter is the centralizer of a torus S in G, S centralizes u (resp., X) and so S is central in G. Hence $L = G$, which implies $P = G$. □

Along with Lemma 3.1, the following theorem of Serre immediately yields Theorem 1.8.

Theorem 3.2 [35, Theorem 4.4]. *Suppose $p \geq a(G)$ and $(H : H^\circ)$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

Proof of Theorem 1.8. Since $p \geq a(G)$, Theorem 3.2 applied to H° shows the latter is G-cr. Thus H° is G-ir by Lemma 3.1, and so is H. □

Remarks 3.3. (i) The characteristic restriction in Theorem 1.8 (and Theorem 3.2) is needed, see Theorem 1.5.

(ii) The condition in Theorem 1.8 that the distinguished unipotent element of G belongs to H° (as opposed to H) is also necessary, as for instance the finite unipotent subgroup of G generated by a given distinguished unipotent element of G is not G-cr [35, Proposition 4.1].

(iii) Under the given hypotheses, Theorem 1.8 applies to an arbitrary distinguished unipotent element of G, irrespective of its order. For Theorem 1.1 to achieve the same uniform result, p has to be sufficiently large to guarantee that the chosen element has order p. For G simple classical with natural module V, this requires the bound $p \geq \dim V$, see Remark 1.3. For G simple of exceptional type, this requires the following bounds: $p > 11$ for E_6, $p > 17$ for E_7, $p > 29$ for E_8, $p > 11$ for F_4, and $p > 5$ for G_2, see [43, Proposition 2.2]. So in many cases the bound $p \geq a(G)$ from Theorem 1.8 is better.

(iv) For an instance when p is bad for G so that Theorem 1.1 does not apply, but Theorem 1.8 does, see Example 2.8.

(v) Theorem 1.8 generalizes [6, Theorem 3.2] which consists of the analogue in the special instance when the distinguished element is regular in G (or $\mathfrak{g}$). Note that in this case no restriction on p is needed, see [6, Theorem 3.2; 22, Theorem 1; 44, Theorem 1.2].

(vi) In characteristic 0, a subgroup H of G is G-cr if and only if it is reductive [35, Proposition 4.1]. So in that case the conclusion of Theorem 1.8 follows directly from Lemma 3.1.

Once again, in the presence of a Steinberg endomorphism σ of G, one cannot appeal to Theorem 1.8 directly to deduce anything about H_σ, because $(H_\sigma)^\circ$ is trivial. In Corollary 3.5 we present an analogue of Theorem 1.8 for the finite groups of Lie type H_σ under an additional condition stemming from [7].

Note that for S a torus in G, we have $C_G(S) = C_G(s)$ for some $s \in S$, see [8, III, Proposition 8.18].

Proposition 3.4 [7, Proposition 3.2]. *Let $H \subseteq G$ be connected reductive groups. Let $\sigma : G \to G$ be a Steinberg endomorphism that stabilizes H and a maximal torus T of H. Suppose*:

(i) $C_G(T) = C_G(t)$ *for some* $t \in T_\sigma$.

(ii) H_σ *meets every T-root subgroup of H nontrivially.*

Then H_σ and H belong to the same parabolic and the same Levi subgroups of G. In particular, H is G-completely reducible if and only if H_σ is G-completely reducible; *similarly, H is G-irreducible if and only if H_σ is G-irreducible.*

Without condition (i), the proposition is false in general, see [7, Example 3.2]. The following is an immediate consequence of Theorem 1.8 and Proposition 3.4.

Corollary 3.5. *Suppose G, H and σ satisfy the hypotheses of* Proposition 3.4. *Suppose in addition that $p \geq a(G)$. If H_σ contains a distinguished unipotent element of G, then H_σ is G-irreducible.*

Corollary 3.5 generalizes [6, Theorem 1.3] which consists of the analogue in the special instance when the distinguished element is regular in G. Note that in this case no restriction on p is needed.

Example 3.6 below shows that the conditions in Corollary 3.5 hold generically.

Example 3.6. Let $\sigma_q : \mathrm{GL}(V) \to \mathrm{GL}(V)$ be a standard Frobenius endomorphism which stabilizes a connected reductive subgroup H of $\mathrm{GL}(V)$ and a maximal torus T of H. Pick $l \in \mathbb{N}$ such that firstly all the different T-weights of V are still distinct when restricted to $T_{\sigma_q^l}$ and secondly there is a $t \in T_{\sigma_q^l}$, such that $C_{\mathrm{GL}(V)}(T) = C_{\mathrm{GL}(V)}(t)$. Then for every $n \geq l$, both conditions in Corollary 3.5 are satisfied for $\sigma = \sigma_q^n$. Thus there are only finitely many powers of σ_q for which the conditions in Corollary 3.5 can fail. The argument here readily generalizes to a Steinberg endomorphism of a connected reductive G which induces a generalized Frobenius morphism on H.

4. Springer maps and associated cocharacters

4.1. ***Springer maps.*** The notion of a Springer isomorphism was introduced in [37]. A *Springer isomorphism* is a G-equivariant isomorphism of varieties $\phi : \mathscr{U}_G \to \mathscr{N}_G$. It follows from work of Springer [37, Theorem 3.1] that a Springer isomorphism ϕ exists if p is good and G is simple and simply connected. We follow Springer and consider G-equivariant maps from $\mathscr{U}_G$ to $\mathscr{N}_G$, but note that several other authors consider G-equivariant maps from $\mathscr{N}_G$ to $\mathscr{U}_G$ instead (see, e.g., [36]).

We wish to consider versions of Springer maps for arbitrary connected reductive G. To prove existence, we need to weaken the definition slightly.

Definition 4.1. A *Springer map* (*for* G) is a G-equivariant homeomorphism of varieties $\phi : \mathscr{U}_G \to \mathscr{N}_G$.

Remark 4.2. It follows from G-equivariance that if ϕ is a Springer map then $\phi(1) = 0$ and for any $u \in \mathscr{U}_G$, u is distinguished if and only if $\phi(u)$ is distinguished.

Remark 4.3. If p is good for G then there exists a Springer map ϕ for G, see [25, Proposition 5]. Below we sketch the argument briefly, following [25, Proposition 5] and [36, Section 1.2]. Note first that a Springer map is uniquely determined by its value on a single regular unipotent element u of G: this follows from G-equivariance, and because the orbit $G \cdot u$ is dense in $\mathscr{U}_G$. If G is simple and p is separably good for G then we can prove existence of a Springer isomorphism by reversing this argument. Fix a regular unipotent element $u \in G$, and choose $X \in \mathscr{N}_G$ such that $C_G(u) = C_G(X)$. We have an obvious isomorphism from $G \cdot u$ to $G \cdot X$. Because $\mathscr{U}_G$ and $\mathscr{N}_G$ are normal (for references, see [34, Lecture 2]), one can show that this map extends to a unique G-equivariant isomorphism from $\mathscr{U}_G$ to $\mathscr{N}_G$. Let us say that G is *of separable type* if it is of the form $G = G_1 \times \cdots \times G_r$, where each G_i

is simple and p is separably good for G. A similar argument to the above works for G of separable type: for $\mathscr{U}_G = \mathscr{U}_{G_1} \times \cdots \times \mathscr{U}_{G_r}$ is normal since each $\mathscr{U}_{G_i}$ is, and likewise $\mathscr{N}_G$ is normal.

Now let G be an arbitrary connected reductive group and assume p is good for G. Since $\mathscr{U}_G \subseteq \mathscr{D}G$ and $\mathscr{N}_G \subseteq \operatorname{Lie}(\mathscr{D}G)$, there is no harm in assuming that G is semisimple. Choose a central isogeny π from $\tilde{G}$ to G, where $\tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_r$ with each $\tilde{G}_i$ simple and p separably good for $\tilde{G}$. Then π (resp., $d\pi$) gives a homeomorphism from $\mathscr{U}_{\tilde{G}}$ to $\mathscr{U}_G$ (resp., from $\mathscr{N}_{\tilde{G}}$ to $\mathscr{N}_G$) [23, Lemma 27]. If $\tilde{\phi}$ is a Springer map for $\tilde{G}$ then the composition

$$\mathscr{U}_G \to \mathscr{U}_{\tilde{G}} \xrightarrow{\tilde{\phi}} \mathscr{N}_{\tilde{G}} \to \mathscr{N}_G$$

is a Springer map for G. This gives a bijection between the set of Springer maps for $\tilde{G}$ and the set of Springer maps for G. Since $\tilde{G}$ admits a Springer isomorphism, it follows that G admits a Springer map.

Note that if G is of separable type then any Springer map ϕ for G is an isomorphism. For fix a regular unipotent element $u \in G$ and let $X = \phi(u)$. By the above discussion, there is a unique Springer isomorphism ϕ' for G such that $\phi'(u) = X$; the uniqueness implies that $\phi' = \phi$. It also follows from the construction in the previous paragraph that if G is an arbitrary connected reductive group and p is good for G then the restriction of ϕ to any maximal unipotent subgroup U of G gives an isomorphism of varieties from U to $\operatorname{Lie}(U)$.

Remark 4.4. Let G_1, G_2 be connected reductive groups and let ϕ_i be a Springer map for G_i for $i = 1, 2$. We claim that the map $\phi_1 \times \phi_2 : \mathscr{U}_{G_1 \times G_2} \to \mathscr{N}_{G_1 \times G_2}$ given by $(\phi_1 \times \phi_2)((u_1, u_2)) = (\phi_1(u_1), \phi_2(u_2))$ is a Springer map for $G_1 \times G_2$. It is clear that $\phi_1 \times \phi_2$ is a $(G_1 \times G_2)$-equivariant bijection. The Zariski topology on the product of varieties is not the product topology, so it is not immediately clear that $\phi_1 \times \phi_2$ is a homeomorphism. To see this, we can pass to the case when G_1 and G_2 are of separable type, by Remark 4.3. Then ϕ_1 and ϕ_2 are isomorphisms, so $\phi_1 \times \phi_2$ is an isomorphism, and the claim follows. We show in Lemma 4.14 that every Springer map for $G_1 \times G_2$ arises in this way.

Remark 4.5. It follows from G-equivariance that a Springer map ϕ gives rise to a bijective map from the set of unipotent conjugacy classes of G to the set of nilpotent conjugacy classes of $\mathfrak{g}$. Serre [24, Section 10, Corollary] shows that this map does not depend on the choice of Springer map (the proof given in the same work is for simple G, but the extension to arbitrary G follows easily from Remarks 4.3 and 4.4). In particular, the condition in (†) does not depend on the choice of Springer map for H.

Remark 4.6. Springer maps need not exist in bad characteristic. For instance, a simple group G of type F_4 with $p = 2$ does not admit a Springer map, because the numbers of unipotent classes in G and nilpotent G-orbits in $\mathrm{Lie}(G)$ are different (see [10, Section 5.11]).

Lemma 4.7 [36, Section 1.2, Remark 1]. *Let ϕ be a Springer map for G. Then $\phi(u^p) = \phi(u)^{[p]}$ for any $u \in \mathscr{U}_G$.*

Remark 4.8. It follows from Lemma 4.7 that any Springer map for G induces a homeomorphism from $\mathscr{U}_G^{(1)}$ to $\mathscr{N}_G^{(1)}$.

In Section 4.2 we define the notion of an associated cocharacter for an element $u \in \mathscr{U}_G$, using a fixed Springer map to give a correspondence between $\mathscr{U}_G$ and $\mathscr{N}_G$. In many contexts one can fix a single Springer map once and for all. We need, however, to consider the interaction of Springer maps with subgroups of G. This motivates the following definition.

Definition 4.9. Let M be a connected subgroup of G. We say that a Springer map ϕ for G is *M-compatible* if $\phi(\mathscr{U}_M) \subseteq \mathscr{N}_M$, and we say that M is *Springer-compatible* if there exists an M-compatible Springer map for G.

If ϕ is M-compatible then in fact $\phi(\mathscr{U}_M) = \mathscr{N}_M$, since $\dim(\mathscr{U}_M) = \dim(\mathscr{N}_M)$; note that dimension can be defined in a purely topological way (via Krull dimension), so it is preserved by homeomorphisms. Note also that when M is reductive and ϕ is an M-compatible Springer map, the restriction of ϕ to $\mathscr{U}_M$ gives a Springer map for M, which we denote by ϕ_M.

Example 4.10 [27, (3.3.1)(a)]. Let M be a connected reductive subgroup of the form $C_G(S)^\circ$, where $S \subseteq G$. It follows from G-equivariance that any Springer map for G is M-compatible, so M is Springer-compatible.

Example 4.11. The arguments in Remark 4.3 show that if G_i is a simple factor of G then any Springer map for G is G_i-compatible, so G_i is Springer-compatible.

Example 4.12. Assume $p > h(G)$, where $h(G)$ denotes the Coxeter number of G. The map $\log : \mathscr{U}_G \to \mathscr{N}_G$ from [34, Theorem 3] is a Springer map. Let H be a connected reductive subgroup of G. We see that log is H-compatible if and only if H is *saturated* in the sense of [34, Lecture 3]. For some properties of saturated subgroups, see [7] and [34].

Example 4.13. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$. For q a positive power of p, let H_q be SL_2 diagonally embedded in G with a q-Frobenius twist in one of the factors: say, the second factor. Note that $\mathrm{Lie}(H_q) = \mathrm{Lie}(\mathrm{SL}_2) \oplus 0$, so $\mathrm{Lie}(H_q)$ contains no nilpotent elements that are distinguished in $\mathfrak{g}$. It follows from Remark 4.2 that no Springer map for G is H_q-compatible, so H_q is not Springer-compatible.

We can find a similar example for G simple. Let G be a simple group of type G_2 and assume $p > 2$. Define H_q to be SL_2 diagonally embedded in the $A_1\tilde{A}_1$ regular subgroup of G with a q-Frobenius twist in one of the factors, and let $1 \neq u \in H_q$

be unipotent. Then u is a distinguished unipotent element of G by [19, Table 10, Section 4.1], but $\mathrm{Lie}(H_q)$ contains no nilpotent elements that are distinguished in $\mathfrak{g}$, so H_q is not Springer-compatible. We are grateful to Adam Thomas for this example.

Lemma 4.14. *Let G_1, G_2 be connected reductive groups and let ϕ be a Springer map for $G_1 \times G_2$. Then ϕ is G_1-compatible and G_2-compatible. Also, $\phi = \phi_1 \times \phi_2$, where ϕ_i is the restriction of ϕ to G_i.*

Proof. By Remark 4.4, we can reduce to the case when $G_1 \times G_2$ is of separable type. The G_i-compatibility of ϕ follows easily from the $(G_1 \times G_2)$-equivariance. Now fix regular $u_1 \in \mathscr{U}_{G_1}$ and $u_2 \in \mathscr{U}_{G_2}$, and set $X = (X_1, X_2)$, where $X_i = \phi_i(u_i)$ for $1 \leq i \leq 2$. Then X_i is a regular element of $\mathrm{Lie}(G_i)$ for $1 \leq i \leq 2$, $u = (u_1, u_2)$ is a regular element of G and X is a regular element of $\mathrm{Lie}(G)$. Clearly, we have $C_{G_i}(u_i) = C_{G_i}(X_i)$ for $1 \leq i \leq 2$.

Let ϕ_i' be the unique Springer isomorphism for G_i such that $\phi_i'(u_i) = X_i$. We have

$$(\phi_1' \times \phi_2')((u_1, u_2)) = (\phi_1'(u_1), \phi_2'(u_2)) = (X_1, X_2) = \phi((u_1, u_2)),$$

so $\phi = \phi_1' \times \phi_2'$. Moreover,

$$(\phi_1(u_1), 0) = \phi((u_1, 0)) = (\phi_1' \times \phi_2')((u_1, 0)) = (\phi_1'(u_1), 0),$$

so $\phi_1(u_1) = \phi_1'(u_1)$, so $\phi_1 = \phi_1'$. Likewise $\phi_2 = \phi_2'$, and the result follows. □

4.2. ***Cocharacters associated to nilpotent and unipotent elements.*** The Jacobson–Morozov theorem allows one to associate an $\mathfrak{sl}(2)$-triple to any given nonzero element of $\mathscr{N}_G$ in characteristic zero or large positive characteristic. This is an indispensable tool in the Dynkin–Kostant classification of the nilpotent orbits in characteristic zero as well as in the Bala–Carter classification of unipotent conjugacy classes of G in large prime characteristic, see [10, Section 5.9]. In good characteristic there is a replacement for $\mathfrak{sl}(2)$-triples, so-called *associated cocharacters*, see Definition 4.15 below. These cocharacters are important tools in the classification theory of unipotent classes and nilpotent orbits of reductive algebraic groups in good characteristic, see, for instance, [16, Section 5] and [30]. We recall the relevant concept of cocharacters associated to a nilpotent element following [16, Section 5.3].

Definition 4.15. Let $X \in \mathscr{N}_G$. A cocharacter $\lambda \in Y(G)$ of G is *associated* to X (in G) provided $X \in \mathfrak{g}(2, \lambda)$ and there exists a Levi subgroup L of G such that X is distinguished nilpotent in $\mathrm{Lie}(L)$ and $\lambda(\mathbb{G}_m) \leq \mathscr{D}L$. Following [12, Definition 2.13], we write

$$\Omega_G^a(X) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } X\}$$

for the set of cocharacters of G associated to X. Likewise, for M a connected reductive subgroup of G such that $X \in \mathrm{Lie}(M)$, we write $\Omega^a_M(X)$ for the set of cocharacters of M that are associated to X. This notation stems from the fact that associated cocharacters are destabilizing cocharacters of G for X in the sense of Kempf–Rousseau theory, see [24] and [30].

Let $u \in \mathscr{U}_G$. A cocharacter $\lambda \in Y(G)$ of G is *associated* to u (in G) provided it is associated to $\phi(u)$, where $\phi : \mathscr{U}_G \to \mathscr{N}_G$ is a fixed Springer map as in Section 4.1, see [25, Section 3]. We write

$$\Omega^a_{G,\phi}(u) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } u\}$$

for the set of cocharacters of G associated to u. Likewise, for M a connected reductive subgroup of G containing u and ϕ' a Springer map for M, we write $\Omega^a_{M,\phi'}(u)$ for the set of cocharacters of M that are associated to u. If ϕ is understood then we sometimes write $\Omega^a_G(u)$ instead of $\Omega^a_{G,\phi}(u)$.

Remark 4.16. Let $u \in \mathscr{U}_G$, $\lambda \in \Omega^a_G(u)$, and $g \in C_G(u)$. Then $g \cdot \lambda$ is also associated to u, see [16, Section 5.3]. Proposition 4.19(ii) gives a converse to this property.

Remark 4.17. Suppose that $G_1, \ldots, G_r$ are connected reductive groups and set $G = G_1 \times \cdots \times G_r$. Let $u_i \in \mathscr{U}_{G_i}$ for each $1 \le i \le r$ and let L be a Levi subgroup of G. Then $L = L_1 \times \cdots \times L_r$ for some Levi subgroups L_i of G_i. Set $u = (u_1, \ldots, u_r) \in L$. It is clear that u is distinguished in L if and only if u_i is distinguished in L_i for each i. Likewise, if $X = (X_1, \ldots, X_r) \in \mathscr{N}_L$ then X is distinguished in $\mathrm{Lie}(L)$ if and only if X_i is distinguished in $\mathrm{Lie}(L_i)$ for each i.

Fix a Springer map for G. Let $\lambda \in Y(G)$. We can write $\lambda = \lambda_1 \times \cdots \times \lambda_r$ for some $\lambda_i \in Y(G_i)$. It follows from the previous paragraph that λ is associated to X in $\mathrm{Lie}(G)$ if and only if λ_i is associated to X_i in $\mathrm{Lie}(G_i)$ for each i [16, Section 5.6]. We deduce the analogous statement for u from Remark 4.4: if $\phi = \phi_1 \times \cdots \times \phi_r$ is a Springer map for G then λ is associated to u in G if and only if λ_i is associated to u_i in G_i for each i.

Let $\psi : \tilde{G} \to G$ be an epimorphism of connected reductive groups such that $\ker(d\psi)$ is central in $\mathrm{Lie}(\tilde{G})$. Let $\tilde{u} \in \mathscr{U}_{\tilde{G}}$, let $\tilde{X} \in \mathscr{N}_{\tilde{G}}$, let $\tilde{L}$ be a Levi subgroup of $\tilde{G}$ and let $\tilde{\lambda} \in Y(\tilde{G})$. Set $u = \psi(\tilde{u})$, $X = d\psi(\tilde{X})$, $L = \psi(\tilde{L})$ and $\lambda = \psi \circ \tilde{\lambda}$. Let $\tilde{\phi}$ be a Springer map for $\tilde{G}$ and let ϕ be the corresponding Springer map for G as described in Remark 4.3. Using [16, Section 4.3] and Remark 4.4 we get analogues of the above statements: u is distinguished in L if and only if $\tilde{u}$ is distinguished in $\tilde{L}$, X is distinguished in $\mathrm{Lie}(L)$ if and only if $\tilde{X}$ is distinguished in $\mathrm{Lie}(\tilde{L})$ and λ is associated to X (resp., to u) if and only if $\tilde{\lambda}$ is associated to $\tilde{X}$ (resp., to $\tilde{u}$).

Remark 4.18. The notion of an associated cocharacter for an element $u \in \mathscr{U}_G$ depends on the choice of the Springer map for G; see [24, Remark 23]. We do, however, have the following. Let ϕ_1 and ϕ_2 be Springer maps for G. Let $1 \neq u_1 \in \mathscr{U}_G$ and let $\lambda \in \Omega^a_{G,\phi_1}(u_1)$. Then $\lambda \in \Omega^a_{G,\phi_2}(u_2)$, where $u_2 = \phi_2^{-1}(\phi_1(u))$. Note that u_2 is conjugate to u_1 by Remark 4.5.

We require some basic facts about cocharacters associated to unipotent elements. The following results are [16, Lemma 5.3, Proposition 5.9] for nilpotent elements (see also [30, Theorem 2.3, Proposition 2.5]); the versions for unipotent elements follow immediately.

Proposition 4.19. *Suppose p is good for G. Let $1 \neq u \in \mathscr{U}_G$.*

(i) $\Omega^a_G(u) \neq \varnothing$, *i.e., cocharacters of G associated to u exist.*

(ii) $C_G(u)^\circ$ *acts transitively on* $\Omega^a_G(u)$.

(iii) *Let $\lambda \in \Omega^a_G(u)$ and let P_λ be the parabolic subgroup of G defined by λ as in* Section 2.5. *Then P_λ depends only on u and not on the choice of λ.*

(iv) *Let $\lambda \in \Omega^a_G(u)$ and let $P(u) := P_\lambda$ be as in* (iii). *Then $C_G(u) \subseteq P(u)$.*

If u is distinguished in G, then the parabolic subgroup $P(u)$ of G from Proposition 4.19(iii) is a distinguished parabolic subgroup of G and u belongs to the Richardson orbit of $P(u)$ on its unipotent radical, see Theorem 2.1(i); see also [24, Proposition 22].

Remark 4.20. Let $p > 0$ and suppose $1 \neq u \in \mathscr{U}_G^{(1)}$ is contained in a subgroup A of G of type A_1. Such a subgroup A always exists when p is good, and when p is bad there is essentially only one exception, due to Testerman [43] and Proud, Saxl, and Testerman [31], see Theorems 2.4 and 2.9. Then, since p is good for A, by Proposition 4.19(i) there exists a cocharacter $\lambda \in \Omega^a_A(u)$. Note that $\lambda(\mathbb{G}_m)$ is a maximal torus in A.

It follows from the work of Pommerening [28; 29] that the description of the unipotent classes in characteristic 0 is identical to the one for G when p is good for G. In both instances these are described by so-called *weighted Dynkin diagrams*. As a result, a cocharacter associated to a unipotent element in good characteristic acts with the same weights on the Lie algebra of G as its counterpart does in characteristic 0. This fact is used in the proof of the following result by Lawther [18, Theorem 1]; see also the proof of [33, Proposition 4.2] and [24, Remark 31]. The result is stated in [18, Theorem 1] for G simple, but the extension to arbitrary connected reductive G is immediate, using arguments like those in Remark 4.17; note that if $\psi : \tilde{G} \to G$ is an epimorphism of connected reductive groups such that $\ker(d\psi)$ is central in $\mathrm{Lie}(G)$ then $d\psi$ gives an isomorphism from $\mathrm{Lie}(\tilde{U})$ onto $\mathrm{Lie}(\psi(\tilde{U}))$, where $\tilde{U}$ is any maximal unipotent subgroup of $\tilde{G}$, so the weights of $\tilde{\lambda} \in Y(\tilde{G})$ on $\mathrm{Lie}(\tilde{G})$ are the same as the weights of $\psi \circ \tilde{\lambda}$ on $\mathrm{Lie}(G)$.

Lemma 4.21. *Let $u \in \mathscr{U}_G$. Suppose p is good for G. Let $\lambda \in \Omega_G^a(u)$. Denote by ω_G the highest weight of $\lambda(\mathbb{G}_m)$ on $\mathfrak{g}$. Then u has order p if and only if $\omega_G \leq 2p-2$.*

The concept of associated cocharacters is not only a convenient replacement for $\mathfrak{sl}(2)$-triples from the Jacobson–Morozov theory, it is a very powerful tool in the classification theory of unipotent conjugacy classes and nilpotent orbits. Specifically, in [30] Premet showcases a conceptual and uniform proof of Pommerening's extension of the Bala–Carter Theorem 2.1 to good characteristic. His proof uses the fact that associated cocharacters are *optimal* in the geometric invariant theory sense of Kempf–Rousseau–Hesselink.

4.3. *Cocharacters associated to distinguished elements.* The linchpin of our proofs of Theorems 1.2 and 1.6 is the following collection of facts.

Lemma 4.22 [12, Lemma 3.1]. *Suppose p is good for G. Let M be a connected reductive subgroup of G. Let $X \in \mathrm{Lie}(M)$ be a distinguished nilpotent element of $\mathfrak{g}$. Then $\Omega_M^a(X) = \Omega_G^a(X) \cap Y(M)$.*

The assertion of the lemma fails in general if X is not distinguished in $\mathfrak{g}$, even when p is good for both M and G, e.g., see [16, Remark 5.12]. However, we do have the following result for all nilpotent elements in good characteristic.

Lemma 4.23 [12, Corollary 3.22]. *Suppose p is good for G. Let $L \subseteq G$ be a Levi subgroup of G. Let $X \in \mathscr{N}_L$. Then $\Omega_L^a(X) = \Omega_G^a(X) \cap Y(L)$.*

We need group-theoretic analogues of Lemmas 4.22 and 4.23. For the former we need an extra Springer compatibility assumption, otherwise the result can fail (see Remark 6.1).

Lemma 4.24. *Suppose p is good for G. Let M be a connected reductive subgroup of G. Suppose M is Springer-compatible and let ϕ be an M-compatible Springer map. Let $u \in M$ be a distinguished unipotent element of G. Then*

$$\Omega_{M,\phi_M}^a(u) = \Omega_{G,\phi}^a(u) \cap Y(M).$$

Proof. Let $X = \phi(u) = \phi_M(u)$. Then

$$\Omega_{M,\phi_M}^a(u) = \Omega_M^a(X) = \Omega_G^a(X) \cap Y(M) = \Omega_{G,\phi}^a(u) \cap Y(M),$$

where the middle equality is from Lemma 4.22. □

Lemma 4.25. *Suppose p is good for G. Let $L \subseteq G$ be a Levi subgroup of G and let ϕ be a Springer map for G. Let $u \in \mathscr{U}_L$. Then $\Omega_{L,\phi_L}^a(u) = \Omega_{G,\phi}^a(u) \cap Y(L)$.*

Proof. Since $L = C_G(S)$ for some torus S, ϕ is L-compatible by Example 4.10. The result now follows by the same argument as in Lemma 4.24. □

5. Good A_1 subgroups

5.1. *Good A_1 overgroups.* In his seminal work [33], Seitz defines an important class of A_1 overgroups of an element $1 \neq u \in \mathscr{U}_G^{(1)}$ for G simple (see [33, Section 1]). He establishes the existence and fundamental properties of these overgroups provided p is good for G. We recall some of these results and generalize them to arbitrary connected reductive G.

Definition 5.1. Following [23, Section 1], we say that a homomorphism $\beta : \mathrm{SL}_2 \to G$ is *good* if each weight of the corresponding representation of SL_2 on $\mathfrak{g}$ is at most $2p-2$. We say that a subgroup A of G of type A_1 is a *good* A_1 subgroup of G, or is *good for* G, if it is the image of a good homomorphism. Else we call A a *bad* A_1 subgroup of G. This is of course independent of the choice of a maximal torus of A. For $1 \neq u \in \mathscr{U}_G^{(1)}$, we define

$$\mathscr{A}(u) := \mathscr{A}_G(u) := \{A \subseteq G \mid A \text{ is a good } A_1 \text{ subgroup of } G \text{ containing } u\}$$

and analogously, for a connected reductive subgroup M of G we write $\mathscr{A}_M(u)$ for the set of all good A_1 subgroups of M containing u.

Clearly any conjugate of a good A_1 homomorphism (resp., subgroup) is good. If $A \subseteq H \subseteq G$ are connected reductive groups such that A is a good A_1 subgroup of G, then A is obviously also a good A_1 subgroup of H. We see in Lemma 5.30 that the converse holds under some extra hypotheses. The converse is false in general, however, e.g., just take $A = H$ to be a bad A_1 subgroup of G.

Example 5.2. Let V be an SL_2-module such that weights of a maximal torus T of SL_2 on V are less than p. Then the weights of T in the induced action on $\mathrm{Lie}(\mathrm{GL}(V)) \cong V \otimes V^*$ are at most $2p-2$. Thus the induced subgroup A in $\mathrm{GL}(V)$ is a good A_1. In this situation the highest weights of T on each composition factor of V are restricted, so V is a semisimple SL_2-module; see [1, Corollary 3.9]. Hence A is $\mathrm{GL}(V)$-cr; this is a special case of Theorem 5.4(iii) below.

We record parts of the main theorems from [33] for our purposes, using the notation above. These were formulated and proved in [33] for simple G, but we need extensions to arbitrary connected reductive G. To obtain this, we need the following lemma.

Lemma 5.3. *Let G be a connected reductive group. Let $\beta_1, \beta_2 : \mathrm{SL}_2 \to G$ be good homomorphisms with the same image A. Then β_1 and β_2 are conjugate by an element of A.*

Proof. Assume first that $A \cong \mathrm{SL}_2$. Let $1 \leq i \leq 2$. Then we can regard β_i as an element of $\mathrm{End}(\mathrm{SL}_2)$, so it is an inner endomorphism followed by a Frobenius q-th power map for $q = p^r$ for some $r \geq 0$. Let T be a maximal torus of SL_2. If $r \geq 1$ then the highest weight of T is at least $2q$, since SL_2 acts on $\mathrm{Lie}(A)$ with highest weight 2, which contradicts the goodness assumption. Therefore, $\beta_i \in \mathrm{Aut}(\mathrm{SL}_2)$. But all automorphisms of SL_2 are inner. The result follows.

For $A \cong \mathrm{PGL}_2$, we can factor β_i as

$$\mathrm{SL}_2 \to \mathrm{PGL}_2 \xrightarrow{\beta_i'} \mathrm{PGL}_2,$$

where the first map is the canonical projection. One can now apply an argument like the above one to the maps $\beta_i' : \mathrm{PGL}_2 \to A$. □

Theorem 5.4. *Let G be connected reductive. Suppose p is good for G and let $1 \neq u \in \mathscr{U}_G^{(1)}$. Then the following hold*:

(i) $\mathscr{A}(u) \neq \varnothing$.

(ii) $R_u(C_G(u))$ *acts transitively on* $\mathscr{A}(u)$.

(iii) *Let $A \in \mathscr{A}(u)$. Then A is G-completely reducible.*

(iv) *There is a unique 1-dimensional unipotent subgroup U of G such that $u \in U$ and U is contained in a good A_1 subgroup of G.*

Notation 5.5. We denote the subgroup U from Theorem 5.4(iv) by $\mathcal{U}(u)$.

Proof of Theorem 5.4. For G simple see [33, Theorems 1.1–1.3]. Now let G be connected reductive. Since SL_2 and PGL_2 are perfect, any A_1 subgroup of G is contained in $\mathscr{D}G$. Hence without loss we can assume that G is semisimple; note for (iii) that a subgroup of $\mathscr{D}G$ is $\mathscr{D}G$-cr if and only if it is G-cr [5, Proposition 2.8]. Moreover, let $\psi : \tilde{G} \to G$ be a central isogeny of connected reductive groups. If $\tilde{A}$ is an A_1 subgroup of $\tilde{G}$ then $\tilde{A}$ is good for $\tilde{G}$ if and only if $\psi(\tilde{A})$ is good for G: see the argument of the paragraph preceding Lemma 4.21. Note also for (iii) that if $\tilde{H} \subseteq \tilde{G}$ then $\tilde{H}$ is $\tilde{G}$-cr if and only if $\psi(\tilde{H})$ is G-cr [4, Lemma 2.12]. Hence we can assume without loss that $G = G_1 \times \cdots \times G_r$, where each G_i is simple.

We need a description of good A_1 subgroups of G in terms of good A_1 subgroups of the G_i. Let T be a maximal torus of SL_2. Denote by π_i the projection from G to G_i. Let $\beta : \mathrm{SL}_2 \to G$ be a homomorphism and define $\beta_i := \pi_i \circ \beta$. For notational convenience, we assume that each β_i is nontrivial. The weights of T on $\mathrm{Lie}(G_i)$ form a subset of the set of weights of T on $\mathrm{Lie}(G)$, since $\mathrm{Lie}(G) = \oplus \mathrm{Lie}(G_i)$. Therefore, if β is a good homomorphism for G, then β_i is a good homomorphism for G_i or trivial. Conversely, if $\beta_i : \mathrm{SL}_2 \to G_i$ is a nontrivial homomorphism for each i, define $\beta := \beta_1 \times \cdots \times \beta_r$ to be the diagonal embedding into G. Then the maximal weight ω_G of T on $\mathrm{Lie}(G)$ is given by $\max\{\omega_{G_i}\}$, where ω_{G_i} is the maximal weight of T on $\mathrm{Lie}(G_i)$. Thus, β is good if and only if the β_i are good. Now (i) and (iii) are immediate from the above observations, [4, Lemma 2.12] and the results for G simple.

For (ii), let A^1 and A^2 be good A_1 subgroups of $G = G_1 \times \cdots \times G_r$ containing $u = (u_1, \ldots, u_r)$ with $u_i \neq 1$ for each i. Choose two good homomorphisms

$\beta^1, \beta^2 : \mathrm{SL}_2 \to G$ such that $\mathrm{Im}(\beta^i) = A^i$. By the observations above, there are good homomorphisms $\beta_i^1, \beta_i^2 : \mathrm{SL}_2 \to G_i$ with images A_i^1, A_i^2 containing u_i. Now [33, Theorem 1.1(ii)] implies that $A_i^2 = g_i A_i^1 g_i^{-1}$ for some $g_i \in R_u(C_{G_i}(u_i))$. Lemma 5.3 (applied to G_i) implies that $h_i g_i \cdot \beta_i^1 = \beta_i^2$ for some $h_i \in A_i^2$. Hence $\beta^2 = hg \cdot \beta^1$, where $g = (g_1, \ldots, g_r) \in R_u(C_G(u))$ and $h = (h_1, \ldots, h_r) \in A^2$. It follows that $A^2 = g \cdot A^1$.

For (iv), let $G = G_1 \times \cdots \times G_r$ and let $u = (u_1, \ldots, u_r) \in \mathscr{U}_G^{(1)}$ with $u_i \neq 1$ for each i. Choose an $A \in \mathscr{A}_G(u)$ which is the image of the good homomorphism β. We get good homomorphisms β_i with images $A_i \in \mathscr{A}_{G_i}(u_i)$, and $\beta = \beta_1 \times \cdots \times \beta_r$, as before. Without loss we can assume the β_i are nontrivial. Fix a 1-dimensional unipotent subgroup V of SL_2. After conjugating β by an element of A, we can assume that $\mathcal{U}(u_i) = \beta_i(V)$ for each i. Define $\mathcal{U}(u) = (\beta_1 \times \cdots \times \beta_r)(V)$. This is a 1-dimensional unipotent subgroup of G containing u and is contained in the good A_1 subgroup A. This proves the existence. For the uniqueness, let U' be another 1-dimensional unipotent subgroup of G such that $u \in U' \subseteq A'$ for some $A' \in \mathscr{A}_G(u)$. By (ii), $A = gA'g^{-1}$ for some $g \in C_G(u)$, and so $gU'g^{-1} = \mathcal{U}(u)$. Write $g = (g_1, \ldots, g_r)$ with $g_i \in C_{G_i}(u_i)$. By [33, Theorem 1.2(i)] g_i centralizes $\mathcal{U}(u_i)$, and hence g centralizes $\mathcal{U}(u)$. Thus, $U' = \mathcal{U}(u)$. □

Example 5.6. Let H_q be the bad A_1 subgroup of $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ from Example 4.13. Here $\beta_1 = \mathrm{id}_{\mathrm{SL}_2}$, while $\beta_2 : \mathrm{SL}_2 \to \mathrm{SL}_2$ is the q-th power map, which is not a good homomorphism. On the other hand, the projection of H_q onto each factor is just SL_2, which is a good A_1 subgroup of SL_2, so we cannot detect the badness of H_q just by looking at its images in the simple factors of G.

Remark 5.7. Let $1 \neq u \in \mathscr{U}_G^{(1)}$. We claim that

$$C_G(\mathcal{U}(u)) = C_G(u) = C_G\big(\mathrm{Lie}(\mathcal{U}(u))\big). \tag{5.8}$$

To see this, suppose first that $\tilde{G}$ is of the form $G_1 \times \cdots \times G_r$, where each G_i is simple, and let $\pi_i : \tilde{G} \to G_i$ be the canonical projection. Let $\tilde{u} = (u_1, \ldots, u_r) \in \mathscr{U}_{\tilde{G}}^{(1)}$ with $u_i \neq 1$ for each i. Choose a good homomorphism $\tilde{\beta} : \mathrm{SL}_2 \to \tilde{G}$ such that $\mathcal{U}(\tilde{u}) \subseteq \tilde{A} := \mathrm{Im}(\mathrm{SL}_2)$, and set $\beta_i = \pi_i \circ \tilde{\beta}$ and $A_i = \beta_i(\mathrm{SL}_2)$. It follows from [33, Theorem 1.2(i)] that

$$C_{G_i}(\mathcal{U}(u_i)) = C_{G_i}(u_i) = C_{G_i}\big(\mathrm{Lie}(\mathcal{U}(u_i))\big)$$

for each i. We deduce from the arguments in the proof of Theorem 5.4 that $C_{\tilde{G}}(\mathcal{U}(\tilde{u})) = C_{\tilde{G}}(\tilde{u}) = C_{\tilde{G}}\big(\mathrm{Lie}(\mathcal{U}(\tilde{u}))\big)$; note that $d\beta_i : \mathrm{Lie}(\mathrm{SL}_2) \to \mathrm{Lie}(A_i)$ is surjective for each i because β_i does not involve a Frobenius twist.

If $\psi : \tilde{G} \to G$ is a central isogeny and $1 \neq \tilde{u} \in \mathscr{U}_{\tilde{G}}^{(1)}$, then it is clear that $\mathcal{U}(u) = \psi(\mathcal{U}(\tilde{u}))$, where $u = \psi(\tilde{u})$, and we deduce that

$$C_G(\mathcal{U}(u)) = C_G(u) = C_G\big(\mathrm{Lie}(\mathcal{U}(u))\big). \tag{5.9}$$

Now let G be an arbitrary connected reductive group and let $1 \neq u \in \mathscr{U}_G^{(1)}$. Then $\mathcal{U}(u) \subseteq \mathscr{D}G$. Now (5.8) follows easily from (5.9) applied to the semisimple group $\mathscr{D}G$.

We deduce from (5.8) and Theorem 5.4(ii) that $\mathcal{U}(u)$ is contained in *every* good A_1 overgroup of u.

Lemma 5.10. *Suppose p is good for G. Let $1 \neq u \in \mathscr{U}_G^{(1)}$ and let A be an A_1 subgroup of G containing $\mathcal{U}(u)$. Then A is good in G.*

Proof. Let A' be a good A_1 subgroup containing $\mathcal{U}(u)$. Then A and A' have a common maximal unipotent subgroup $\mathcal{U}(u)$. By [21, Theorem 1.1], A and A' are G-conjugate. Hence A is good, because A' is. □

Lemma 5.11. *Suppose p is good for G. Let A be an A_1 subgroup of G and let $\lambda \in Y(A)$. Suppose that*

(i) *$\lambda \in \Omega_G^a(X)$ for some $0 \neq X \in \mathscr{N}_G^{(1)}$, or*

(ii) *$\lambda \in \Omega_{G,\phi}^a(u)$ for some $1 \neq u \in \mathscr{U}_G^{(1)}$ and some Springer map ϕ for G.*

Then A is a good A_1 subgroup of G.

Proof. Let ϕ be a Springer map for G, and let $\lambda \in \Omega_{G,\phi}^a(u)$ for some $1 \neq u \in \mathscr{U}_G^{(1)}$. It follows from Lemma 4.21 that the weights of λ on $\mathfrak{g}$ are at most $2p-2$. Define $\beta : \mathrm{SL}_2 \to A$ to be an isomorphism if $A \cong \mathrm{SL}_2$, and the usual central isogeny $\mathrm{SL}_2 \to \mathrm{PGL}_2$ followed by an isomorphism from PGL_2 onto A if $A \cong \mathrm{PGL}_2$. Then there exists $\mu : \mathbb{G}_m \to \mathrm{SL}_2$ such that μ is an isomorphism onto a maximal torus of SL_2 and $\lambda = \beta \circ \mu$. The weights of μ on $\mathfrak{g}$ are at most $2p-2$ by construction, so A is good. Hence A is good if (ii) holds.

If (i) holds then $\lambda \in \Omega_{G,\phi}^a(u)$, where $u := \phi^{-1}(X)$. But $u \in \mathscr{U}_G^{(1)}$, by Lemma 4.7, so (ii) holds, so A is good by the argument above. □

In the next theorem we recall parts of the analogue of Theorem 5.4 for finite overgroups of type A_1.

Theorem 5.12. *Let G be connected reductive. Suppose p is good for G. Let $\sigma : G \to G$ be a Steinberg endomorphism of G. Suppose $u \in G_\sigma$ is unipotent of order p.*

(i) *$\mathscr{A}(u)_\sigma \neq \varnothing$.*

(ii) *$R_u(C_G(u))_\sigma$ acts transitively on $\mathscr{A}(u)_\sigma$.*

(iii) *Let $A \in \mathscr{A}(u)_\sigma$. Suppose that $q > 7$ if G is of exceptional type. Then A_σ is σ-completely reducible.*

(iv) *There is a unique σ-stable 1-dimensional unipotent subgroup U of G such that $u \in U$ and U is contained in a good A_1 subgroup of G.*

Proof. (i)–(iii) The simple case is proved by Seitz in [33, Theorem 1.4]. For connected reductive groups we use an argument similar to the one in the proof of Theorem 5.4.

(iv) By (i) we can choose some $A \in \mathscr{A}(u)_\sigma$. Now $\mathcal{U}(u) \subseteq A$ by Remark 5.7. Clearly $\mathcal{U}(u)$ is the unique 1-dimensional unipotent subgroup of A that contains u, so $\mathcal{U}(u)$ must be σ-stable. Hence $U := \mathcal{U}(u)$ has the desired properties. □

Remark 5.13. Parts (i) and (ii) of Theorem 5.12 follow from parts (i) and (ii) of Theorem 5.4 and the Lang–Steinberg theorem, see [33, Proposition 9.1].

Remark 5.14. (i) Concerning the terminology in Theorem 5.12(iii), following [14], a subgroup H of G is said to be *σ-completely reducible*, provided that whenever H lies in a σ-stable parabolic subgroup P of G, it lies in a σ-stable Levi subgroup of P. This notion is motivated by certain rationality questions concerning G-complete reducibility, see [14] for details. For a σ-stable subgroup H of G, this property is equivalent to H being G-cr, thanks to [14, Theorem 1.4].

(ii) Apart from the special conjugacy class of good A_1 subgroups in G asserted in Theorem 5.4, there might be a plethora of conjugacy classes of bad A_1 subgroups in G even when p is good for G. Just take a nonsemisimple representation

$$\beta : \mathrm{SL}_2 \to \mathrm{SL}(V) = G$$

in characteristic $p > 0$. Then the A_1 subgroup $\beta(\mathrm{SL}_2)$ is bad in G, while p is good for G. For a concrete example, see [16, Remark 5.12]. This can only happen if p is sufficiently small compared to the rank of G, thanks to Theorem 3.2.

The subgroups H_q of $\mathrm{SL}_2 \times \mathrm{SL}_2$ in Example 4.13 are also bad A_1 subgroups, see Remark 6.1.

(iii) The proofs of Theorems 5.4 and 5.12 for G simple by Seitz in [33] depend on separate considerations for each Dynkin type and involve in part intricate arguments for the component groups of centralizers of unipotent elements. In [24], McNinch presents uniform proofs of Seitz's theorems for G strongly standard reductive, which are almost entirely free of any case-by-case checks, utilizing methods from geometric invariant theory. However, McNinch's argument (see [24, Theorem 44]) of the conjugacy result in Theorem 5.4(ii) depends on the fact that for a good A_1 subgroup A of G, the A-module $\mathfrak{g}$ is a tilting module. The latter is established by Seitz in [33, Theorem 1.1].

In [33, Section 9], Seitz exhibits instances when there is no good A_1 overgroup of an element of order p when p is bad for G. As we explain next, Example 2.6 gives a counterexample to Theorem 5.4(iii) in case p is bad for G: that is, it gives a good A_1 subgroup A such that A is not G-cr. Specifically, we show that some of the A_1 subgroups in that example are good A_1 subgroups of G, but thanks to Example 2.6, they are not G-cr.

Example 5.15 (Example 2.6 continued). Let G be simple of type C_2 and $p = 2$. Let σ be id_G or a q-Frobenius endomorphism of G. Let $\mathscr{C}$ denote the subregular unipotent class of G. Suppose $u \in \mathscr{C} \cap G_\sigma$. Then by Example 2.6 there are σ-stable subgroups A of G of type A_1 containing u that are not G-cr. Specifically, let E be the natural module for SL_2. Consider the two conjugacy classes of embeddings of SL_2 into $G = \mathrm{Sp}(V)$, where we take either $V \cong E \perp E$ or $V \cong E \otimes E$, as an SL_2-module. The images of both embeddings meet the class $\mathscr{C}$ nontrivially. One checks that the highest weight of a maximal torus of SL_2 on $\mathfrak{g}$ is 4 in the second instance. So in this case the image of SL_2 in G is not a good A_1. In contrast, in the first instance the highest weight of a maximal torus of SL_2 on $\mathfrak{g}$ is $2 = 2p - 2$, by Example 5.2. So the image of SL_2 in G is a good A_1 in $\mathrm{SL}(V)$, and so it is a good A_1 in G as well.

5.2. *Characterizations of good A_1 subgroups.* In this section we investigate some other types of A_1 subgroup which were introduced by McNinch. We prove that these other notions are all equivalent to goodness (Theorem 5.24). The key ingredient we need is work of Sobaje, who proved the existence of a Springer map for G with especially nice properties. We assume throughout the section that p is good for G.

We recall a construction from [33, Proposition 5.2] (see also [36]). Let P be a parabolic subgroup of G, and set $U = R_u(P)$. It can be shown that any Springer map for G maps U to $\mathrm{Lie}(U)$. Suppose U has nilpotency class less than p; in this case we say that P is *restricted*. In particular, any distinguished parabolic subgroup of G corresponding to a distinguished unipotent element of order p is restricted [24, Proposition 24]. We endow $\mathrm{Lie}(U)$ with the structure of an algebraic group using the Baker–Campbell–Hausdorff formula. There is a unique P-equivariant isomorphism of algebraic groups $\exp_P : \mathrm{Lie}(U) \to U$ such that the derivative of $\exp_P$ is the identity on $\mathrm{Lie}(U)$ (this is established in [33, Proposition 5.2] for semisimple G, but the extension to connected reductive G is immediate). We denote the inverse of $\exp_P$ by $\log_P : \mathrm{Lie}(U) \to U$.

Definition 5.16. We say that a Springer map ϕ for G is *logarithmic* if the following holds: for any $1 \neq u \in \mathscr{U}_G^{(1)}$, the restriction of ϕ gives an isomorphism ϕ_u of algebraic groups from $\mathcal{U}(u)$ to $\mathrm{Lie}(\mathcal{U}(u))$, and $d\phi_u$ is the identity on $\mathrm{Lie}(\mathcal{U}(u))$.

Proposition 5.17. (i) *There exists a logarithmic Springer map for G.*

(ii) *Let ϕ be a logarithmic Springer map for G. Then for every restricted parabolic subgroup P, the restriction of ϕ to $R_u(P)$ is* $\log_P$.

(iii) *Any two logarithmic Springer maps induce the same map from $\mathscr{U}_G^{(1)}$ to $\mathscr{N}_G^{(1)}$.*

Proof. First assume that G is simple and p is separably good for G. Part (ii) follows from [36, Proposition 2.1]. For part (i), let $\varphi : \mathcal{N}_G \to \mathcal{U}_G$ be a G-equivariant isomorphism of varieties as in [36, Theorem 4.1]. Fix a maximal unipotent subgroup U of G. By [36, Theorem 1.1], $d\varphi : \mathrm{Lie}(U) \to \mathrm{Lie}(U)$ is a scalar multiple of the identity. Condition (1) of [36, Theorem 4.1] implies that this scalar is 1, so $d\varphi$ is the identity map. Let $1 \neq u \in \mathcal{U}_G^{(1)}$ and set $X = \varphi^{-1}(u)$. Then $X \in \mathcal{N}_G^{(1)}$ by Remark 4.8, so by [36, Corollary 4.3(1)], φ gives an isomorphism from kX onto a 1-dimensional unipotent subgroup U' of G which is contained in a good A_1 subgroup of G. By construction, $U' = \mathcal{U}(u)$. Since $d\varphi$ is the identity map, X belongs to $\mathrm{Lie}(\mathcal{U}(u))$, so φ gives an isomorphism of algebraic groups from $\mathrm{Lie}(\mathcal{U}(u))$ to $\mathcal{U}(u)$. It follows that φ^{-1} is a logarithmic Springer map for G, so (i) is proved.

Now let $1 \neq u \in \mathcal{U}_G^{(1)}$. Choose a good A_1 overgroup A of u in G. Choose a maximal torus T of A such that T normalizes $\mathcal{U}(u)$. Definition 5.16 and the T-equivariance of φ^{-1} imply that the map from $\mathcal{U}(u)$ to $\mathrm{Lie}(\mathcal{U}(u))$ induced by φ^{-1} does not depend on the choice of φ^{-1}. This proves part (iii).

The result now follows for arbitrary connected reductive G using Remark 4.3. □

Remark 5.18. If $p > h(G)$ then the map log from Example 4.12 is a logarithmic Springer map (see [34, Theorem 3] and [24, Remark 27]). In this case any Borel subgroup of G is a restricted parabolic, so the restriction of any logarithmic Springer map for G to $R_u(B)$ is $\log_B$ by Proposition 5.17(ii). Hence log is the unique logarithmic Springer map for G.

Remark 5.19. We saw above that the condition on ϕ in Definition 5.16 implies part (ii) of Proposition 5.17. Sobaje observes at the beginning of [36, Section 2] that the converse also holds. The reason is that every $1 \neq u \in \mathcal{U}_G^{(1)}$ belongs to $R_u(P)$ for some restricted parabolic subgroup P of G: this follows from [9, Theorem 2.4]. We also deduce that the restriction of $\log_P$ to $\mathcal{U}(u)$ is ϕ_u for every restricted parabolic subgroup P of G and every $u \in R_u(P)$ such that u has order p.

Corollary 5.20. *Let ϕ be a logarithmic Springer map for G. Then for any A_1 subgroup A of G, A is good for G if and only if ϕ is A-compatible.*

Proof. Suppose A is good. Let $1 \neq u \in \mathcal{U}_A$. Then $\mathcal{U}(u) \subseteq A$ and

$$\phi(\mathcal{U}(u)) = \mathrm{Lie}(\mathcal{U}(u)) \subseteq \mathrm{Lie}(A),$$

so ϕ is A-compatible. Conversely, suppose ϕ is A-compatible. Let $1 \neq u \in \mathcal{U}_A$ and set $X = \phi(u) \in \mathrm{Lie}(\mathcal{U}(u))$. Now $X \in \mathrm{Lie}(A)$ by the A-compatibility, so $kX \subseteq \mathrm{Lie}(A)$. Hence $\mathcal{U}(u) = \phi^{-1}(kX) \subseteq A$ by the A-compatibility. We deduce from Lemma 5.10 that A is good for G. □

We now recall the other types of A_1 subgroup that we need, namely optimal and subprincipal A_1 subgroups. These were introduced by McNinch in [23] and [24].

Definition 5.21. We call a homomorphism $\beta : \mathrm{SL}_2 \to G$ *optimal* if there is a maximal torus T of SL_2 such that the restriction λ of β to $T \cong \mathbb{G}_m$ is a cocharacter associated in G to some nilpotent $0 \neq X \in \mathrm{Im}(d\beta)$. We call an A_1 subgroup of G *optimal* if it is the image of an optimal homomorphism.

Remark 5.22. This is equivalent to the definition in [24, Section 1]: for it is clear that if T is the standard maximal torus of SL_2 and λ is associated to some nilpotent $0 \neq X \in \mathrm{Lie}(\mathrm{SL}_2)$ then X is a scalar multiple of $d\phi\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$.

Definition 5.23. Fix a Springer map ϕ for G. We call a homomorphism $\beta : \mathrm{SL}_2 \to G$ *subprincipal* if there is a maximal torus T of SL_2 such that the restriction λ of β to $T \cong \mathbb{G}_m$ is a cocharacter associated in G to some nilpotent $0 \neq X \in \mathrm{Im}(d\beta)$ and $\phi^{-1}(X)$ is G-conjugate to an element of $\mathrm{Im}(\beta)$. Note that the latter condition does not depend on the choice of ϕ, by Remark 4.5. We call an A_1 subgroup of G *subprincipal* if it is the image of a subprincipal homomorphism.

The next result implies Theorem 1.4.

Theorem 5.24. *Let A be an A_1 subgroup of G. Let ϕ be a logarithmic Springer map for G. The following conditions are equivalent.*

(i) *A is subprincipal.*

(ii) *A is optimal.*

(iii) *There exist $u \in \mathscr{U}_G^{(1)}$ and $\lambda \in Y(A)$ such that $\lambda \in \Omega^a_{G,\phi}(u)$.*

(iv) *A is good.*

Proof. The implication (i) $\Longrightarrow$ (ii) is immediate from the definitions, and (iii) $\Longrightarrow$ (iv) follows from Lemma 5.11. If A is optimal then there exist $0 \neq X \in \mathscr{N}_A$ and $\lambda \in Y(A)$ such that $\lambda \in \Omega^a_G(X)$. Then $\lambda \in \Omega^a_{G,\phi}(u')$, where $u' = \phi^{-1}(X)$, and u' has order p by Lemma 4.7. Hence (ii) $\Longrightarrow$ (iii).

By [23, Remark 21], there exists at least one subprincipal A_1 subgroup A of G such that $u \in A$, and A is good by the arguments above. It is clear from the definition that any $C_G(u)$-conjugate of A is subprincipal. Since $C_G(u)$ acts transitively on $\mathscr{A}(u)$ (Theorem 5.4(ii)), it follows that any good A_1 subgroup of G that contains u is subprincipal. This shows that (iv) $\Longrightarrow$ (i). Hence (i)–(iv) are all equivalent. □

Remark 5.25. If the equivalent conditions from Theorem 5.24 hold then there exist $\lambda \in Y(A)$ and $0 \neq X \in \mathscr{N}_A$ such that $\lambda \in \Omega^a_G(X)$. Then $\lambda \in \Omega^a_G(u)$, where $u = \phi^{-1}(X)$, which belongs to A by Corollary 5.20. Hence we can take the element u from Theorem 5.24(iii) to belong to A if we wish.

Remark 5.26. It is implicit in the discussion in [23, Section 1] that a subprincipal A_1 subgroup of G is good. McNinch also proved that goodness and optimality are equivalent for A_1 subgroups under the extra assumption that G is strongly standard (see [24, Proposition 53]).

Proposition 5.27. *Let L be a Levi subgroup of G and let A be a good A_1 subgroup of L. Then A is a good A_1 subgroup of G.*

Proof. Since p is good for G, p is good for L. By Theorem 5.24, A is optimal in L, so there exist $0 \neq X \in \mathcal{N}_A$ and $\lambda \in Y(A)$ such that $\lambda \in \Omega_L^a(X)$. Lemma 4.23 implies that $\lambda \in \Omega_G^a(X)$, so A is optimal in G. Hence A is a good A_1 subgroup of G by Theorem 5.24. □

Corollary 5.28. *Let A be a good A_1 subgroup of G and let $1 \neq u \in \mathcal{U}_A$. Then there is a Levi subgroup L of G such that $A \subseteq L$ and u is a distinguished unipotent element of L.*

Proof. Pick a Levi subgroup L' of G such that u is a distinguished unipotent element of L'. By Theorem 5.4(i) we can choose a good A_1 subgroup A' of L' such that $u \in A'$. Now A' is a good A_1 subgroup of G by Proposition 5.27, so there exists $g \in C_G(u)$ such that $gA'g^{-1} = A$ (Theorem 5.4(ii)). Then $A \subseteq L$, where $L := gL'g^{-1}$. Clearly, L is a Levi subgroup of G and u is a distinguished unipotent element of L. □

Corollary 5.29. *Let ϕ be a logarithmic Springer map for G and let L be a Levi subgroup of G. Then ϕ_L is a logarithmic Springer map for L.*

Proof. Let $1 \neq u \in \mathcal{U}_L^{(1)}$. Choose a good A_1 overgroup A of u in L. Then A is good for G by Proposition 5.27, so $\mathcal{U}(u) \subseteq A$. We see that $\mathcal{U}(u)$ is both the unique 1-dimensional overgroup of u that is contained in a good A_1 subgroup of L, and the unique 1-dimensional overgroup of u that is contained in a good A_1 subgroup of G. The result now follows from the definition of a logarithmic Springer map. □

Lemma 5.30. *Let H be a connected reductive subgroup of G, and assume p is good for H. Let $u \in \mathcal{U}_H^{(1)}$ such that u is distinguished in G. Let $A \in \mathscr{A}_H(u)$. Suppose there is a Springer map ϕ for H such that $\phi(u)$ is a distinguished element of $\mathfrak{g}$. Then A is good for G.*

Proof. By Theorem 5.24, A is a subprincipal A_1 subgroup of H, so there exist $\lambda \in Y(A)$ and $0 \neq X \in \mathcal{N}_A$ such that λ is associated to X in H and $\phi(u)$ is H-conjugate to X. Since by hypothesis $\phi(u)$ is a distinguished element of $\mathfrak{g}$, X is also a distinguished element of $\mathfrak{g}$. Lemma 4.22 implies that λ is associated to X in G. Hence A is an optimal A_1 subgroup of G, so A is a good A_1 subgroup of G by Theorem 5.24. □

Remark 5.31. Let H be a connected reductive subgroup of G and assume p is good for H. Suppose H is Springer-compatible. Let A be an A_1 subgroup of H containing a distinguished unipotent element u of G. Let ϕ be the restriction to H

of any H-compatible Springer map for G. Then $\phi(u)$ is a distinguished element of $\mathfrak{g}$ by Remark 4.2, so the hypotheses of Lemma 5.30 hold. Hence if A is good for H then A is good for G.

The following relates the set of cocharacters of G that are associated to some $1 \neq u \in \mathscr{U}_G^{(1)}$ to those stemming from good A_1 overgroups of u in G.

Corollary 5.32. *Let $1 \neq u \in \mathscr{U}_G^{(1)}$. Let ϕ be a logarithmic Springer map for G. We have a disjoint union*

$$\Omega^a_{G,\phi}(u) = \dot{\bigcup_{A \in \mathscr{A}(u)}} \Omega^a_{A,\phi_A}(u),$$

where ϕ_A denotes the restriction of ϕ to A.

Proof. Note that it makes sense to speak of the restriction of ϕ to a good A_1 subgroup A of G, by Corollary 5.20. We first prove that the union above is disjoint. Let $A, \tilde{A} \in \mathscr{A}(u)$ and suppose there exists some

$$\lambda \in \Omega^a_{A,\phi_A}(u) \cap \Omega^a_{\tilde{A},\phi_{\tilde{A}}}(u).$$

Then A and $\tilde{A}$ share the common Borel subgroup $\lambda(\mathbb{G}_m)\mathcal{U}(u)$. It follows from [21, Lemma 2.4] that $A = \tilde{A}$.

Let $A \in \mathscr{A}(u)$ and let $\lambda \in \Omega^a_{A,\phi_A}(u)$. By Corollary 5.28 there is a Levi subgroup L of G such that $A \subseteq L$ and u is a distinguished unipotent element of L. It follows from Lemma 4.24 (applied to the inclusion $A \subseteq L$) and Lemma 4.25 that $\lambda \in \Omega^a_{G,\phi}(u)$. Hence

$$\Omega^a_{G,\phi}(u) \supseteq \dot{\bigcup}_{A \in \mathscr{A}(u)} \Omega^a_{A,\phi_A}(u).$$

Since we have $\mathscr{A}(u) \neq \varnothing$ (Theorem 5.4(i)), $C_G(u)$ acts transitively on both $\mathscr{A}(u)$ (Theorem 5.4(ii)) and $\Omega^a_{G,\phi}(u)$ (Proposition 4.19(ii)), and we see that the reverse inclusion follows. □

6. Proofs of Theorems 1.2 and 1.5–1.7

Armed with the results from above, we prove Theorems 1.2 and 1.6 simultaneously.

Proof of Theorems 1.2 and 1.6. We may assume that G is semisimple, since any unipotent element of G is contained in the derived subgroup $\mathscr{D}G^\circ$. Likewise, we may also assume that H is connected and semisimple, as any unipotent element of H° is contained in the derived subgroup $\mathscr{D}H^\circ$, and H is G-ir if $\mathscr{D}H^\circ$ is. Let $u \in \mathscr{U}_H^{(1)}$ be distinguished in G.

First suppose p is bad for H. If $p > 2$ then H admits a simple component H' of exceptional type. If $u \in H$ is a distinguished unipotent element of G then the projection u' of u onto H' is a distinguished unipotent element of H', so $p = 3$ and H' is of type G_2, by Lemma 2.7. But this is impossible by Lemma 2.11 since p is good for G. Hence $p = 2$. It follows that each simple component of G is of type A. Now distinguished unipotent elements are regular in type A, so u is a regular element of G. It follows from [6, Theorem 1.1] (resp., [6, Theorem 1.3]) that H (resp., H_σ) is G-ir.

Therefore we can assume that p is good for H. By Theorem 5.4(i) (resp., Theorem 5.12(i)) there is a good A_1 subgroup (resp., good σ-stable A_1 subgroup) A of H such that $u \in A$. By Lemma 5.30 and hypothesis (†), A is a good A_1 subgroup of G. Hence A (resp., A_σ) is G-cr by Theorem 5.4(iii)) (resp., Theorem 5.12)(iii)), so A (resp., A_σ) is G-ir by Lemma 3.1. We conclude that H (resp., H_σ) is G-ir. □

Remark 6.1. If we remove hypothesis (†) from Lemma 5.30, Theorem 1.2, etc., then our arguments break down. For instance, let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$, q and H_q be as in Example 4.13. Let u be any unipotent element of H_q such that the projection of u onto each SL_2-factor of H_q is nontrivial; then u is distinguished (in fact, regular) in G. It is easy to see that H_q is not a good A_1 subgroup of G and there does not exist $\lambda \in Y(H_q)$ such that λ is associated to u in G; in particular, the conclusion of Lemma 5.30 does not hold for H_q. Of course Theorem 1.1 still applies, alternately so does Theorem 1.8, so H_q is G-ir.

As a consequence of Theorems 1.2 and 1.6 we obtain the following.

Corollary 6.2. *Let G be a connected reductive group. Suppose p is good for G. Let σ be* id_G *or a q-Frobenius endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. Suppose u is distinguished in the σ-stable Levi subgroup L of G (see* Remark 2.2(ii)). *Let H be a σ-stable connected reductive subgroup of L containing u, and suppose there is a Springer map ϕ for L such that $\phi(u)$ is a distinguished element of* $\mathrm{Lie}(L)$. *Then H_σ is G-completely reducible.*

Proof. As p is also good for L (see Section 2.2), it follows from Theorem 1.2 (resp., Theorem 1.6) applied to L that H_σ is L-ir and so is L-cr. Thus, H_σ is G-cr, by [35, Proposition 3.2]. □

Remark 6.3. In the setting of Theorem 1.1 the following argument allows us to reduce the case when G is connected reductive to the simple case. As in the proof of Theorem 1.2 above, we can assume that G is semisimple. Let $G_1, \dots, G_r$ be the simple factors of G. Multiplication gives an isogeny from $G_1 \times \cdots \times G_r$ to G. Thus, by [4, Lemma 2.12(ii)(b)] and [16, Section 4.3], we can replace G with $G_1 \times \cdots \times G_r$, so we can assume G is the product of its simple factors. Finally, thanks to [4, Lemma 2.12(i)] and [16, Section 4.3], we can reduce to the case when G is simple.

Finally, we address Theorems 1.5 and 1.7.

Proof of Theorems 1.5 and 1.7. By Theorems 1.2 and 1.6, the only cases we need to consider are when p is bad for G. If G is classical, then we are in the situation of Lemma 2.5 and Example 2.6, so we are done.

We are left to consider the case when G is of exceptional type. Then owing to Lemma 2.7, G is of type G_2 and $p = 3$. There is no harm in assuming that H is semisimple. It follows from Example 2.8 that H is G-ir. Thus Theorem 1.5 follows. So consider the setting of Theorem 1.7 when $\sigma|_H$ is a q-Frobenius endomorphism of H in this case. By Corollary 2.10, u belongs to the subregular class of G_2. It follows from the proof of Lemma 2.7 in [31] that u is contained in a σ-stable maximal rank subgroup of G of type $A_1\tilde{A}_1$ and this type is unique. Since H is proper and semisimple, $H \subseteq M$, where M is a σ-stable maximal rank subgroup of G of type $A_1\tilde{A}_1$. Since p is good for H, there is a σ-stable subgroup A of H of type A_1 containing u, by Theorem 2.4. Thus $A \subseteq H \subseteq M$. Since u is also distinguished in M and $p = 3$ is good for M, Theorem 1.6 shows that A_σ is M-ir. Note that M is the centralizer of a semisimple element of G of order 2 (by Deriziotis' criterion, see [11, 2.3]). Since A_σ is M-cr, it is G-cr, owing to [4, Corollary 3.21]. Once again, by Lemma 3.1, A_σ is G-ir and so is H_σ. Theorem 1.7 follows. □

Remark 6.4. In [17, Section 7], Korhonen gives counterexamples to Theorem 1.1 when the order of the distinguished unipotent element of G is greater than p (even when p is good for G [17, Proposition 7.1]). Theorem 1.8 implies that this can only happen when $p < a(G)$. For instances of overgroups of distinguished unipotent elements of G of order greater than p for $p \geq a(G)$ (and p good for G), so that Theorem 1.8 applies, see Examples 6.6 and 6.7.

Remark 6.5. In view of Remark 6.4, it is natural to ask for instances of G, u and H when the conclusion of Theorem 1.8 holds even when $p < a(G)$ but p is still good for G. If p is good for G and G is simple classical, nonregular distinguished unipotent elements always belong to a maximal rank semisimple subgroup H of G, by [43, Propositions 3.1, 3.2]. For G simple of exceptional type this is also the case in almost all instances of nonregular distinguished unipotent elements, see [43, Lemma 2.1]. Each such H is obviously G-irreducible. This is independent of p of course and thus applies in particular when $p < a(G)$. For instance, let G be of type E_7, $p = 5$, and suppose u belongs to the distinguished class $E_7(a_3)$ (resp., $E_7(a_4)$, $E_7(a_5)$). Then $\mathrm{ht}_J(\rho) = 9$ (resp., 7, 5), so u has order 5^2, by Lemma 2.3 in each case. Since u does not have order 5, Theorem 1.1 does not apply, and since $5 < 8 = a(G)$ neither does Theorem 1.8. Nevertheless, in each case u is contained in a maximal rank subgroup H of type A_1D_6, see [43, p. 52], and each such H is G-ir.

We close the section with several additional higher order examples in good characteristic when Theorem 1.1 does not apply but Theorem 1.8 does.

Example 6.6. Let G be of type E_6. Suppose p is good for G. In [43, Lemma 2.7], Testerman exhibits the existence of a simple subgroup H of G of type C_4 whose regular unipotent class belongs to the subregular class $E_6(a_1)$ of G. Let u be regular unipotent in H. For $p = 7$, the order of u is 7^2, by Lemma 2.3, so Theorem 1.1 can't be invoked to say anything about H. However, for $p = 7 = a(G)$, we infer from Theorem 1.8 that H is G-ir.

Example 6.7. Let G be of type E_8. Suppose $p = 11$. Let u be in the distinguished class $E_8(a_3)$ (resp., $E_8(a_4)$, $E_8(b_4)$, $E_8(a_5)$, or $E_8(b_5)$). From the corresponding weighted Dynkin diagram corresponding to u we get $\mathrm{ht}_J(\rho) = 17$ (resp., 14, 13, 11, or 11), see [10, p. 177]. It follows from Lemma 2.3 that in each of these instances u has order 11^2. So we can't appeal to Theorem 1.1 to deduce anything about reductive overgroups of u. But as $11 = p \geq a(G) = 9$, Theorem 1.8 applies and allows us to conclude that each such overgroup is G-ir. For example, in each instance above, u is contained in a maximal rank subgroup H of G of type A_1E_7 or D_8, see [43, p. 52].

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THE q-SCHUR CATEGORY AND POLYNOMIAL TILTING MODULES FOR QUANTUM GL_n

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The *q-Schur category* is a $\mathbb{Z}[q, q^{-1}]$-linear monoidal category closely related to the q-Schur algebra. We explain how to construct it from coordinate algebras of quantum GL_n for all $n \geq 0$. Then we use Donkin's work on Ringel duality for q-Schur algebras to make precise the relationship between the q-Schur category and a $\mathbb{Z}[q, q^{-1}]$-form for the $U_q\mathfrak{gl}_n$-web category of Cautis, Kamnitzer and Morrison. We construct explicit integral bases for morphism spaces in the latter category, and extend the Cautis–Kamnitzer–Morrison theorem to polynomial representations of quantum GL_n at a root of unity over a field of any characteristic.

1. Introduction

We revisit some algebra from the 1990s using the diagrammatic technique of string calculus for strict monoidal categories which has become ubiquitous in this area since then. The initial goal is to give a self-contained construction of a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category, the *q-Schur category*, together with three important bases for its morphism spaces. The path algebra of this category is Morita equivalent to the direct sum of the q-Schur algebras $S_q(n, n)$ of Dipper and James [1989] for all $n \geq 0$. In that context, all three bases were studied in detail already 30 years ago, and this part of the article is mainly expository. There are already many generalizations in the literature — cyclotomic [Dipper et al. 1998], affine [Green 1999; Miemietz and Stroppel 2019; Maksimau and Stroppel 2021], and 2-categorical [Williamson 2011; Mackaay et al. 2013; Webster 2017], to name but a few.

Once the general framework is in place, we use the q-Schur category to define a $\mathbb{Z}[q, q^{-1}]$-form for the positive half of the $U_q\mathfrak{gl}_n$-web category of Cautis, Kamnitzer and Morrison [2014], complete with bases for its morphism spaces as free $\mathbb{Z}[q, q^{-1}]$-modules. Integral bases in the latter category have previously been constructed in

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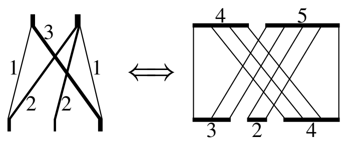
an unpublished paper of Elias [2015], and their existence also follows theoretically from [Andersen et al. 2018], but the relationship to the known bases for the q-Schur algebra is not apparent from that work. We also explain how the canonical basis fits into this picture, something which is not mentioned at all in [Elias 2015].

Our starting point is the definition of a strict $\mathbb{Z}$-linear monoidal category called the *Schur category*, denoted simply by **Schur**, from [Brundan et al. 2020, Definition 4.2]. The object set of **Schur** is the set Λ_s of all *strict compositions*, that is, sequences $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of positive integers for $\ell \geq 0$, with tensor product of objects defined by concatenation. For strict compositions λ and μ, the morphism space $\mathrm{Hom}_{\mathbf{Schur}}(\mu, \lambda)$ is zero unless $r := \sum_i \lambda_i = \sum_i \mu_i$, in which case this morphism space is a free $\mathbb{Z}$-module with a distinguished *standard basis* parametrized by the set $(S_\lambda \backslash S_r / S_\mu)_{\min}$ of minimal length representatives for the double cosets of the parabolic subgroups S_λ and S_μ in the symmetric group S_r. Vertical composition making **Schur** into a $\mathbb{Z}$-linear category is defined by *Schur's product rule* as in the classical Schur algebra (see [Green 2007, 2.3b]), and the horizontal composition making it into a monoidal category is induced by the natural embeddings $S_a \times S_b \hookrightarrow S_{a+b}$.

As usual with strict monoidal categories, it is convenient to represent morphisms in **Schur** by certain string diagrams; the vertical composition $f \circ g$ of morphisms f and g is obtained by stacking the string diagram for f on top of the one for g, and their horizontal composition $f \star g$ is obtained by stacking f to the left of g. We represent the standard basis elements for $\mathrm{Hom}_{\mathbf{Schur}}(\mu, \lambda)$ by $\lambda \times \mu$ *double coset diagrams*,[1] such as the diagram on the left:



$$\iff (2\,5\,8\,4\,7\,3\,6) \in (S_{(4,5)} \backslash S_9 / S_{(3,2,4)})_{\min} \iff A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \end{bmatrix}.$$

In this double coset diagram, there are strings of various thicknesses indicated by the numerical labels. Thick strings at the bottom split into thinner strings, which are allowed to cross each other forming a *reduced* diagram for a permutation in the middle of the picture, before merging back into thick strings at the top. Subsequently, we will index $S_\lambda \backslash S_r / S_\mu$-double cosets also by the set $\mathrm{Mat}(\lambda, \mu)$ consisting of matrices of nonnegative integers whose row and column sums are the entries of the compositions λ and μ, respectively. The ij-entry $a_{i,j}$ of the matrix A

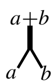
[1] Called "chicken foot diagrams" in [Brundan et al. 2020].

records the thickness of the string that connects the i-th thick string at the top to the j-th thick string at the bottom of the corresponding double coset diagram.

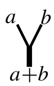
The q-analog of the Schur category is a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category denoted by q-**Schur** whose specialization at $q = 1$ recovers **Schur**. In our approach, q-**Schur** is defined from the outset to be the $\mathbb{Z}[q, q^{-1}]$-linear category with the same objects as **Schur**, tensor product of objects being by concatenation as before. Its morphism spaces are defined so that $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ is the free $\mathbb{Z}[q, q^{-1}]$-module with a *standard basis* $\{\xi_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$, which we represent graphically by almost the same double coset diagrams as above, except that we replace each singular crossing [singular crossing] with a positive crossing [positive crossing]. Then we need rules for computing vertical and horizontal compositions of standard basis vectors. Horizontal composition is defined by horizontally stacking diagrams just as in **Schur**. Vertical composition is defined by the q-analog of Schur's product rule; see (4-8) and (4-9). Although there is no simple closed formula for this in general, it can be computed algorithmically using relations in Manin's quantized coordinate algebra $\mathcal{O}_q(n)$ of $n \times n$ matrices [1988].

Our first theorem gives a presentation for q-**Schur** which incorporates the positive crossings as one of three types of generating morphism. Setting $q = 1$ in this recovers the presentation for **Schur** derived in [Brundan et al. 2020].

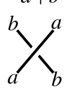
Theorem 1. *As a strict* $\mathbb{Z}[q, q^{-1}]$*-linear monoidal category,* q-**Schur** *is generated by the objects* (r) *for* $r > 0$ *and morphisms called* ***merges***, ***splits*** *and* ***positive crossings*** *represented by*



$: (a) \star (b) \to (a+b),$



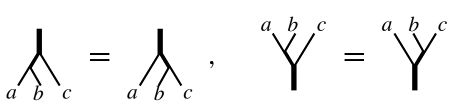
$: (a+b) \to (a) \star (b),$



$: (a) \star (b) \to (b) \star (a)$

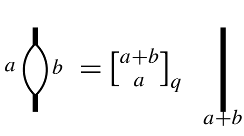
for $a, b > 0$, *subject to the* ***associativity*** *and* ***coassociativity relations***

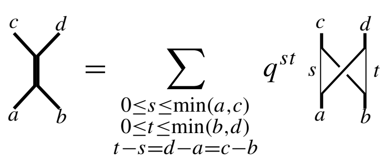
(1-1)



for $a, b, c > 0$, *together with*

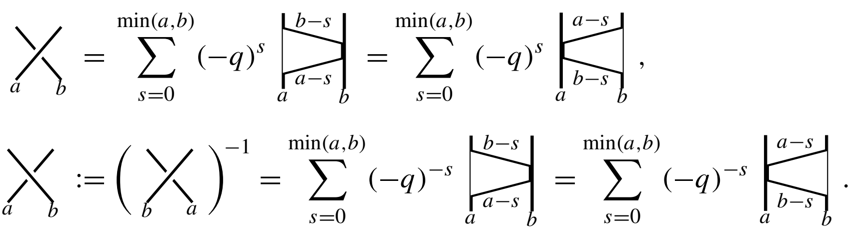
(1-2)





for $a, b, c, d > 0$ *with* $a + b = c + d$. *Here,* $\begin{bmatrix} n \\ s \end{bmatrix}_q$ *is the* q*-binomial coefficient* (3-2), *and splits/merges with a string of thickness zero should be interpreted as identities.*

The positive crossings are important because they define a braiding making q-**Schur** into a braided monoidal category. In fact, positive crossings and their inverses, the *negative crossings*, can be written in terms of merges and splits:

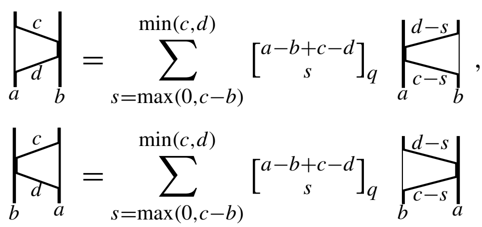


The following gives a slightly more efficient presentation for q-**Schur** using only the merges and splits as generating morphisms.

Theorem 2. *The monoidal category* q-**Schur** *is generated by the objects* (r) *for* $r > 0$ *and the morphisms*

for $a, b > 0$, *subject only to the relations* (1-1) *together with one of the equivalent* ***square-switch relations***

(1-3)



for $a, b, c, d \geq 0$ *with* $d \leq a$ *and* $c \leq b + d$.

The presentations for q-**Schur** in Theorems 1 and 2 are not new, e.g., the relations can be found in [Latifi and Tubbenhauer 2021] (with a different choice of normalization for the positive crossings coming from quantum SL_n rather than quantum GL_n). We give complete proofs here, rather than attempting to adapt related results already in the literature such as [Doty 2003]. Our general approach to the definition of q-**Schur**, equipping each of its morphism spaces with a standard basis over $\mathbb{Z}[q, q^{-1}]$ from the outset with structure constants which can computed algorithmically, facilitates calculations which seem quite awkward otherwise; for example, see Corollary 6.2 for a formula for the composition of two positive crossings. The ability to compute products effectively is also exploited in the proof of the straightening formula in Lemma 7.4.

This straightening formula is the key ingredient in the proof of Theorem 3, which constructs a second basis for morphism spaces in q-**Schur**. We formulate this in terms of the path algebra

$$H := \bigoplus_{\lambda,\mu\in\Lambda_s} \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda) \tag{1-4}$$

viewed as a locally unital algebra with distinguished idempotents $\{1_\lambda \mid \lambda \in \Lambda_s\}$ arising from the identity endomorphisms of the objects of q-**Schur**. Multiplication in H is induced by composition. Let Λ^+ be the subset of Λ_s consisting of all *partitions*, that is, ordered sequences $\kappa = (\kappa_1 \geq \cdots \geq \kappa_\ell)$ of positive integers for $\ell \geq 0$. For $\lambda \in \Lambda_s$ and $\kappa \in \Lambda^+$, we denote the usual set of all semistandard tableaux of shape κ and content λ by $\mathrm{Std}(\lambda,\kappa)$. For $P \in \mathrm{Std}(\lambda,\kappa)$, let $A(P) \in \mathrm{Mat}(\lambda,\kappa)$ be the matrix whose ij-entry records the number of times i appears on row j of P. For the definition of "symmetrically based quasihereditary algebra" used in the statement of the theorem, see Definition 7.1. The triangular bases in this definition are *cellular bases* in the sense of [Graham and Lehrer 1996]. However, the axioms are simpler than the ones for a cellular algebra; they are also more restrictive since it follows automatically that the underlying algebra is a split quasihereditary algebra with duality in the sense of [Cline et al. 1990].

Theorem 3. *The locally unital algebra* $H = \bigoplus_{\lambda,\mu\in\Lambda_s} 1_\lambda H 1_\mu$ *is a symmetrically based quasihereditary algebra with weight poset* Λ^+ *ordered by the dominance ordering* $\leq$, *anti-involution* $\mathtt{T} : H \to H$, $\xi_A \mapsto \xi_{A^{\mathtt{T}}}$, *and triangular basis consisting of the* ***codeterminants*** $\xi_{A(P)}\xi_{A(Q)^{\mathtt{T}}}$ *for* $(P,Q) \in \bigcup_{\lambda,\mu\in\Lambda_s,\kappa\in\Lambda^+} \mathrm{Std}(\lambda,\kappa) \times \mathrm{Std}(\mu,\kappa)$.

There is a third remarkable basis in this subject, the canonical basis, which appeared originally in the context of q-Schur algebras in [Beilinson et al. 1990] and was studied in detail by Du [1992a; 1992b; 1995] from the perspective of Hecke algebras; see also [Deng et al. 2008, Chapter 9]. To define it, take $\lambda, \mu \in \Lambda_s$ such that $r := \sum_i \lambda_i = \sum_i \mu_i$, and $A \in \mathrm{Mat}(\lambda,\mu)$. Writing $d_A^+ \in (S_\lambda \backslash S_r / S_\mu)_{\max}$ for the maximal length double coset representative indexed by A, let

$$\theta_A := \sum_{B\in\mathrm{Mat}(\lambda,\mu)} q^{\ell(d_A^+)-\ell(d_B^+)} P_{d_A^+,d_B^+}(q^{-2})\xi_B,$$

where $P_{x,y}(t) \in \mathbb{Z}[t]$ is the Kazhdan–Lusztig polynomial for $x, y \in S_r$. Then the *canonical basis* for $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda)$ is $\{\theta_A \mid A \in \mathrm{Mat}(\lambda,\mu)\}$. The canonical basis can also be defined in terms of the *bar involution* $- : q$-**Schur** $\to q$-**Schur**, the antilinear strict monoidal functor which fixes objects and the generating merge and split morphisms: θ_A is the unique morphism in $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ such that $\overline{\theta}_A = \theta_A$ and $\theta_A \equiv \xi_A \pmod{\sum_{B \in \mathrm{Mat}(\lambda,\mu)} q\mathbb{Z}[q]\xi_B}$. Note also that the bar involution interchanges positive and negative crossings.

The canonical basis makes the path algebra H into a "standardly full-based algebra" using the language of [Du and Rui 1998], with the same weight poset and cell ideals as the ones arising from the codeterminant basis in Theorem 3. This follows from the results in [Du and Rui 1998, §5.3], which imply that the canonical basis is cellular, hence, equivalent to a triangular basis; see Remark 7.6 for a precise statement. Theorem 3 could be deduced as a consequence of this like in [Du and Rui 1998, §5.5]. It could also be deduced from R. M. Green's construction [1996] of the q-analog of J. A. Green's codeterminant basis for the Schur algebra. The short self-contained proof of Theorem 3 given here is similar to the one in [Green 1996] (and in [Woodcock 1993] when $q = 1$), but incorporates simplifications made possible by working in the less constrained setting of the q-Schur category. Analogous bases for cyclotomic q-Schur algebras of all levels (not merely level one) have been constructed in [Dipper et al. 1998, Theorem 6.6] by a different method.

At least one of these new bases (codeterminant or canonical) is needed in order to understand a certain truncation q-$\mathbf{Schur}_n$ of the q-Schur category. By definition, this is the quotient of q-$\mathbf{Schur}$ by the two-sided tensor ideal $\mathbf{I}_n$ generated by the identity endomorphisms $1_{(r)}$ for $r > n$. The presentation for q-$\mathbf{Schur}_n$ arising from Theorem 2 makes it clear that it is a version of Cautis, Kamnitzer and Morrison's $U_q\mathfrak{gl}_n$-web category, or rather, its positive half involving only upward-pointing strings. The ideal $\mathbf{I}_n$ is compatible with the basis from Theorem 3. Consequently, the path algebra of q-$\mathbf{Schur}_n$ is also a symmetrically based quasihereditary algebra with triangular basis given by the images of the codeterminants $\xi_{A(P)}\xi_{A(Q)^{\mathrm{T}}}$ for all pairs (P, Q) of semistandard tableaux whose shape κ satisfies $\kappa_1 \leq n$. This basis is similar to the integral bases for morphism spaces in this category constructed in [Elias 2015]. The canonical basis also induces a cellular basis for the path algebra of q-$\mathbf{Schur}_n$.

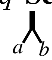
Let $\Bbbk$ be a field viewed as a $\mathbb{Z}[q, q^{-1}]$-algebra in some way, and consider the specialization $q\text{-}\mathbf{Schur}_n(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}_n$. Also let U_n be Lusztig's $\mathbb{Z}[q, q^{-1}]$-form for the quantized enveloping algebra $U_q\mathfrak{gl}_n$ with Chevalley generators E_i, F_i $(1 \leq i \leq n-1)$ and $D_i^{\pm 1}$ $(1 \leq i \leq n)$. Let $U_n(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} U_n$. We view it as a Hopf algebra with comultiplication Δ satisfying

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes D_i^{-1} D_{i+1},$$
$$\Delta(F_i) = F_i \otimes 1 + D_i D_{i+1}^{-1} \otimes F_i,$$
$$\Delta(D_i) = D_i \otimes D_i.$$

The natural $U_n(\Bbbk)$-module V is the vector space with basis $v_1, \dots, v_n$ such that $E_i v_j = \delta_{i+1,j} v_i$, $F_i v_j = \delta_{i,j} v_{i+1}$, $D_i v_j = q^{\delta_{i,j}} v_j$. Its r-th quantum exterior power $\bigwedge^r V$ is a certain quotient of the r-th tensor power $V^{\otimes r}$ with a basis given by the monomials $v_{i_1} \wedge \cdots \wedge v_{i_r}$ that are images of the pure tensors $v_{i_1} \otimes \cdots \otimes v_{i_r}$ for $1 \le i_1 < \cdots < i_r \le n$. Let q-$\mathbf{Tilt}_n^+(\Bbbk)$, the category of *polynomial tilting modules*, be the full additive Karoubian monoidal subcategory of $U_n(\Bbbk)$-mod generated by the exterior powers $\bigwedge^r V$ for all $r \ge 0$. This is a braided (but not rigid) monoidal category with braiding $c : - \otimes - \stackrel{\sim}{\Longrightarrow} - \otimes^{\mathrm{rev}} -$ defined so that

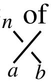
$$(1\text{-}5)\quad c_{V,V} : V \otimes V \to V \otimes V, \qquad v_i \otimes v_j \mapsto \begin{cases} v_j \otimes v_i & \text{if } i < j, \\ q^{-1} v_j \otimes v_i & \text{if } i = j, \\ v_j \otimes v_i - (q - q^{-1}) v_i \otimes v_j & \text{if } i > j. \end{cases}$$

If $\Bbbk$ is of characteristic zero and the image of q in $\Bbbk$ is not a root of unity, q-$\mathbf{Tilt}_n^+(\Bbbk)$ is a semisimple abelian category, and the following theorem can be deduced from [Cautis et al. 2014].

Theorem 4. *There is a $\Bbbk$-linear monoidal functor $\Sigma_n : q$-$\mathbf{Schur}_n(\Bbbk) \to q$-$\mathbf{Tilt}_n^+(\Bbbk)$ taking the generating object (r) to $\bigwedge^r V$, the merge*  *to the natural surjection $\bigwedge^a V \otimes \bigwedge^b V \twoheadrightarrow \bigwedge^{a+b} V$, and the split*  *to the inclusion $\bigwedge^{a+b} V \hookrightarrow \bigwedge^a V \otimes \bigwedge^b V$ defined by*

$$v_{i_1} \wedge \cdots \wedge v_{i_{a+b}} \mapsto q^{-ab} \sum_{w \in (S_{a+b}/S_a \times S_b)_{\min}} (-q)^{\ell(w)} v_{i_{w(1)}} \wedge \cdots \wedge v_{i_{w(a)}} \otimes v_{i_{w(a+1)}} \wedge \cdots \wedge v_{i_{w(a+b)}}$$

for $1 \le i_1 < \cdots < i_{a+b} \le n$. This functor is full and faithful, and it induces a monoidal equivalence between the additive Karoubi envelope of q-$\mathbf{Schur}_n(\Bbbk)$ and q-$\mathbf{Tilt}_n^+(\Bbbk)$.

The monoidal functor Σ_n of Theorem 4 is not a braided monoidal functor — it takes the positive crossing  to

$$(-1)^{ab} c^{-1}_{\bigwedge^b V, \bigwedge^a V}$$

rather than to $c_{\bigwedge^a V, \bigwedge^b V}$. This twist, which may at first seem inconvenient, is reasonable since the proof involves some Ringel duality — the generating object (r) of the q-Schur category corresponds more naturally to the r-th quantum symmetric power of the natural module rather than its exterior power.

There is one more important explanation to be made: subsequently, the notation q-**Schur** will be used to denote a slightly larger version of the q-Schur category than appears in this introduction, with objects that are indexed by *all* compositions, not just strict ones. In other words, we adjoin an additional generating object (0) which is isomorphic but not equal to the strict identity object $\mathbb{1}$. We prefer to use the same notation for both versions — it should be clear from context whether we are working with or without strings of thickness zero. The natural inclusion of the q-Schur category as defined in the introduction into the one with 0-strings is a monoidal equivalence, making it easy to go back and forth between the two versions. One advantage of q-Schur category *with* 0-strings is that there is a surjective algebra homomorphism from U_n to the path algebra of the full subcategory whose objects are compositions with exactly n parts. Actually, it is more convenient to work with Lusztig's modified form $\dot{U}_n$ here; see (8-1). Using this connection, the approach to q-Schur algebras taken in [Doty 2003], exploiting Lusztig's refined Peter–Weyl theorem for $\dot{U}_n$ [Lusztig 2010, Section 29.3], could be adapted to give yet another approach to the results here.

2. Double coset combinatorics

A *composition* $\lambda \vDash r$ is a finite sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of nonnegative integers summing to r. We write $\ell(\lambda)$ for the total number ℓ of parts, which is allowed to be zero, and $|\lambda|$ for the sum of the parts. We emphasize that we treat compositions of different lengths as being different, e.g., $() \neq (0) \neq (0, 0)$. A *partition* $\lambda \vdash r$ is a composition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vDash r$ whose parts satisfy

$$\lambda_1 \geq \cdots \geq \lambda_\ell > 0.$$

For partitions, we allow ourselves to write λ_r even if $r > \ell(\lambda)$, in which case $\lambda_r = 0$. We denote the sets of all compositions and all partitions by Λ and Λ^+, respectively. Let $\leq$ be the usual dominance ordering on Λ^+.

We denote the transposition $(i\ i{+}1)$ in the symmetric group S_r by s_i, $\ell : S_r \to \mathbb{N}$ is the length function, and $w_r \in S_r$ is the longest element. Elements of S_r act on the *left* on the set $\{1, \dots, r\}$. There is also a *right* action of S_r on $\mathbb{Z}^r$ by place permutation: for $\boldsymbol{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$ and $w \in S_r$, the r-tuple $\boldsymbol{i} \cdot w$ has j-th entry $i_{w(j)}$. For $\lambda = (\lambda_1, \dots, \lambda_\ell) \vDash r$, the set

$$\mathrm{I}_\lambda := \left\{ \boldsymbol{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r \mid \#\{k = 1, \dots, r \mid i_k = i\} = \lambda_i \text{ for all } i \in \{1, \dots, \ell(\lambda)\} \right\}$$

is a single orbit under this action. Also let $\boldsymbol{i}^\lambda = (i_1^\lambda, \dots, i_r^\lambda)$ denote the unique element of I_λ whose entries are in weakly increasing order. Its stabilizer in S_r is the parabolic subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$.

For $\lambda, \mu \vDash r$, the symmetric group S_r acts diagonally on the right on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$. The orbits are parametrized by the set $\mathrm{Mat}(\lambda, \mu)$ of all $\ell(\lambda) \times \ell(\mu)$ matrices with nonnegative integer entries such that the entries in the i-th row sum to λ_i and the entries in the j-th column sum to μ_j for all $i \in \{1, \dots, \ell(\lambda)\}$ and $j \in \{1, \dots, \ell(\mu)\}$. For $A = (a_{i,j}) \in \mathrm{Mat}(\lambda, \mu)$, the corresponding S_r-orbit on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$ is

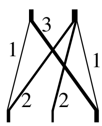
$$\Pi_A := \big\{(\boldsymbol{i}, \boldsymbol{j}) \in \mathrm{I}_\lambda \times \mathrm{I}_\mu \mid \#\{k = 1, \dots, r \mid (i_k, j_k) = (i, j)\} = a_{i,j} \text{ for all } i \in \{1, \dots, \ell(\lambda)\},\ j \in \{1, \dots, \ell(\mu)\}\big\}. \tag{2-1}$$

The set $\mathrm{Mat}(\lambda, \mu)$ is actually just one of many different sets used in the literature to parametrize the orbits of S_r on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$. Another is by the set $\mathrm{Row}(\lambda, \mu)$ of *row tableaux* of shape μ and content λ, that is, left justified arrays with μ_1 boxes in row 1 (the top row), μ_2 boxes in row 2, and so on, with boxes filled with integers so that entries are weakly increasing in order from left to right along each row, and there are a total of λ_1 entries equal to 1, λ_2 equal to 2, and so on. We use the explicit bijection

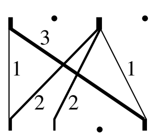
$$A : \mathrm{Row}(\lambda, \mu) \to \mathrm{Mat}(\lambda, \mu) \tag{2-2}$$

taking $P \in \mathrm{Row}(\lambda, \mu)$ to the matrix $A(P) \in \mathrm{Mat}(\lambda, \mu)$ whose ij-entry records the number of times i appears on row j of P. The inverse bijection maps $A \in \mathrm{Mat}(\lambda, \mu)$ to the row tableau $P \in \mathrm{Row}(\lambda, \mu)$ whose j-th row is equal to $1^{a_{1,j}}\ 2^{a_{2,j}}\ \cdots\ \ell^{a_{\ell(\lambda),j}}$.

A third way to parametrize orbits is by the double coset diagrams introduced already in the introduction. We gave already there an example in which $\lambda = (4, 5)$, $\mu = (3, 2, 4)$, for which the matrix $A \in \mathrm{Mat}(\lambda, \mu)$, the corresponding double coset diagram, and the corresponding row tableau $P \in \mathrm{Row}(\lambda, \mu)$ are

(2-3) $\quad A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \end{bmatrix} \iff$  $\iff P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 2 \\ \hline 1 & 1 & 1 & 2 \\ \hline \end{array}$.

Unlike in the introduction, we are now allowing compositions with parts equal to 0, so double coset diagrams can also have strings labeled by 0. In fact, it is harmless to omit these zero thickness strings from the diagram entirely, but one should mark their endpoints. Here is an example with $\lambda = (4, 0, 5, 0)$ and $\mu = (3, 2, 0, 4)$:

(2-4) $\quad A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \iff$  $\iff P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 \\ \hline 3 & 3 \\ \hline \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array}$.

Two other sets in bijection with $\mathrm{Mat}(\lambda, \mu)$ are the sets of minimal length and maximal length double coset representatives, which we denote by $(S_\lambda \backslash S_r / S_\mu)_{\min}$ and $(S_\lambda \backslash S_r / S_\mu)_{\max}$, respectively. For $A \in \mathrm{Mat}(\lambda, \mu)$, we denote the corresponding elements of $(S_\lambda \backslash S_r / S_\mu)_{\min}$ and $(S_\lambda \backslash S_r / S_\mu)_{\max}$ by d_A and d_A^+, respectively.

Lemma 2.1. *Given* $\lambda, \mu \vDash r$ *and* $A \in \mathrm{Mat}(\lambda, \mu)$, *let* $\lambda^- \vDash r$ (*resp.*, $\mu^+ \vDash r$) *be obtained by reading the entries of* A *in order along rows starting with the top row* (*resp.*, *in order down columns starting with the leftmost column*). *We have*

$$(S_\lambda d_A) \cap (d_A S_\mu) = d_A S_{\mu^+} = S_{\lambda^-} d_A.$$

Every element $w \in S_\lambda d_A S_\mu$ *can be written uniquely as* $x d_A y$ *for* $x \in (S_\lambda / S_{\lambda^-})_{\min}$, $y \in S_\mu$ (*resp.*, $x d_A y$ *for* $x \in S_\lambda$, $y \in (S_{\mu^+} \backslash S_\mu)_{\min}$), *and we have that*

$$\ell(x d_A y) = \ell(x) + \ell(d_A) + \ell(y).$$

Proof. This follows from [Dipper and James 1989, Lemma 1.6]. □

The double coset diagram gives a convenient visual way to translate an element $A \in \mathrm{Mat}(\lambda, \mu)$ into the minimal length double coset representatives d_A. Alternatively, to obtain d_A, let $(i_1, \dots, i_r) \in I_\lambda$ be the sequence $\overrightarrow{Z}(P)$ obtained by reading the entries of the corresponding row tableau P from left to right along rows, starting with the top row. Then replace the λ_1 entries equal to 1 in this sequence by $1, \dots, \lambda_1$ in increasing order, the λ_2 entries equal to 2 by $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ in increasing order, and so on. The result is d_A written in one-line notation. To compute d_A^+, we instead start from the sequence $\overleftarrow{Z}(P)$ obtained by reading entries of P from right to left along rows, starting with the top row. Then we replace the entries 1 by $\lambda_1, \dots, 1$ in decreasing order, the entries 2 by $\lambda_1 + \lambda_2, \dots, \lambda_1 + 1$ in decreasing order, and so on. In the example (2-3), $\overrightarrow{Z}(P) = (1, 2, 2, 2, 2, 1, 1, 1, 2)$ so $d_A = (1, 5, 6, 7, 8, 2, 3, 4, 9)$, and $\overleftarrow{Z}(P) = (2, 2, 1, 2, 2, 2, 1, 1, 1)$ so $d_A^+ = (9, 8, 4, 7, 6, 5, 3, 2, 1)$.

Let $\le$ be the *Bruhat ordering* on the symmetric group (so the identity element is *minimal*). This restricts to partial orders on the sets $(S_\lambda \backslash S_r / S_\mu)_{\min}$ and $(S_\lambda \backslash S_r / S_\mu)_{\max}$, such that

$$d_A \le d_B \iff d_A^+ \le d_B^+ \tag{2-5}$$

if d_A and d_B are minimal length double coset representatives and d_A^+ and d_B^+ are the corresponding maximal ones (this coincidence is proved in [Hohlweg and Skandera 2005]). Using the bijections between these sets, we transport the Bruhat order to partial orders on $\mathrm{Row}(\lambda, \mu)$ and $\mathrm{Mat}(\lambda, \mu)$. The resulting partial order on $\mathrm{Mat}(\lambda, \mu)$ is given explicitly in terms of matrices by

$$A \le B \iff \left(\sum_{i=1}^{s} \sum_{j=1}^{t} a_{i,j} \ge \sum_{i=1}^{s} \sum_{j=1}^{t} b_{i,j} \text{ for all } s \in \{1, \dots, \ell(\lambda)\}, t \in \{1, \dots, \ell(\mu)\} \right). \tag{2-6}$$

One finds this elementary combinatorial observation in many places in the literature; for example, see [Beilinson et al. 1990] which also explains the geometric origin of this ordering.

3. The quantized coordinate algebra

The ring $\mathbb{Z}[q, q^{-1}]$ has a bar involution $-$ which sends q to q^{-1}. We will use the term "antilinear map" for a $\mathbb{Z}$-module homomorphism between $\mathbb{Z}[q, q^{-1}]$-modules which intertwines q and q^{-1} in this way. For $\mathbb{Z}[q, q^{-1}]$-modules, $V \otimes W$ means tensor product over $\mathbb{Z}[q, q^{-1}]$ and V^* denotes $\mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(V, \mathbb{Z}[q, q^{-1}])$. We will need the quantum integer

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \tag{3-1}$$

and quantum binomial coefficient

$$\begin{bmatrix} n \\ s \end{bmatrix}_q := \frac{[n]_q[n-1]_q \cdots [n-s+1]_q}{[s]_q[s-1]_q \cdots [1]_q}, \tag{3-2}$$

which we interpret as zero in the case $s < 0$. These satisfy the Pascal-type recurrence relation

$$\begin{bmatrix} n \\ s \end{bmatrix}_q = q^s \begin{bmatrix} n-1 \\ s \end{bmatrix}_q + q^{s-n} \begin{bmatrix} n-1 \\ s-1 \end{bmatrix}_q = q^{-s} \begin{bmatrix} n-1 \\ s \end{bmatrix}_q + q^{n-s} \begin{bmatrix} n-1 \\ s-1 \end{bmatrix}_q. \tag{3-3}$$

The following play the role of the binomial theorem for positive and negative exponents:

$$\prod_{s=1}^{n}(1 + q^{2s-n-1}x) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q x^s, \quad \prod_{s=1}^{n} \frac{1}{(1 + q^{2s-n-1}x)} = \sum_{s=0}^{n} \begin{bmatrix} -n \\ s \end{bmatrix}_q x^i. \tag{3-4}$$

Here are some more identities that will be needed later.

Lemma 3.1. *For $n \geq 0$, we have $\sum_{s=0}^{n}(-1)^s q^{s(n-1)} \begin{bmatrix} n \\ s \end{bmatrix}_q = \delta_{n,0}$.*

Proof. Set $x = -q^{-n-1}$ in the first identity from (3-4). □

Lemma 3.2. *For $m, n \in \mathbb{Z}$ and $s \geq 0$, we have*

$$\sum_{a+b=s} q^{mb-na} \begin{bmatrix} m \\ a \end{bmatrix}_q \begin{bmatrix} n \\ b \end{bmatrix}_q = \begin{bmatrix} m+n \\ s \end{bmatrix}_q.$$

Proof. This is proved by a standard argument using (3-4). See also [Fiebig 2023, Proposition 4.1(5)] (where this is called the Chu–Vandermonde convolution formula). □

Lemma 3.3. *For $m \in \mathbb{Z}$ and $s \geq 0$, we have $\sum_{a+b=s} (-q)^{-b} \begin{bmatrix} m+a \\ a \end{bmatrix}_q \begin{bmatrix} m \\ b \end{bmatrix}_q = q^{ms}$.*

Proof. This is the q-analog of [Brundan et al. 2020, Lemma A.1]. See [Brundan and Kleshchev 2022, Lemma 3.1(3)] for its proof. □

Let $\mathcal{O}_q(n)$ be Manin's quantized coordinate algebra of $n \times n$ matrices [1988], which is the $\mathbb{Z}[q, q^{-1}]$-algebra on generators $\{x_{i,j} \mid 1 \leq i, j \leq n\}$ subject to the relations

$$x_{i,j}x_{k,l} = \begin{cases} x_{k,l}x_{i,j} & \text{if } i < k \text{ and } j > l, \\ x_{k,l}x_{i,j} - (q - q^{-1})x_{i,l}x_{k,j} & \text{if } i > k \text{ and } j > l, \\ q^{-1}x_{k,l}x_{i,j} & \text{if } i = k \text{ and } j > l, \\ qx_{k,l}x_{i,j} & \text{if } i < k \text{ and } j = l. \end{cases} \tag{3-5}$$

We view $\mathcal{O}_q(n)$ as a bialgebra with comultiplication $\Delta : \mathcal{O}_q(n) \to \mathcal{O}_q(n) \otimes \mathcal{O}_q(n)$ and counit $\varepsilon : \mathcal{O}_q(n) \to \mathbb{Z}[q, q^{-1}]$ defined by

$$\Delta(x_{i,k}) = \sum_{j=1}^{n} x_{i,j} \otimes x_{j,k}, \quad \varepsilon(x_{i,j}) = \delta_{i,j}. \tag{3-6}$$

Lemma 3.4. *In $\mathcal{O}_q(2)$, we have for $a, b \geq 0$ that*

$$x_{2,2}^a x_{1,1}^b = \sum_{s=0}^{\min(a,b)} q^{-s(s-1)/2}(q^{-1} - q)^s [s]_q^! \begin{bmatrix} a \\ s \end{bmatrix}_q \begin{bmatrix} b \\ s \end{bmatrix}_q x_{2,1}^s x_{1,1}^{b-s} x_{2,2}^{a-s} x_{1,2}^s.$$

Proof. Use induction on a to check that

$$x_{2,2}^a x_{1,1} = x_{1,1}x_{2,2}^a - (q - q^{-1})[a]x_{2,1}x_{2,2}^{a-1}x_{1,2}.$$

This treats the case $b = 1$. Then proceed by induction on b using (3-3). □

Lemma 3.5. *In $\mathcal{O}_q(2)$, we have for $a \geq 0$ and $i, j \in \{1, 2\}$ that*

$$\Delta(x_{i,j}^a) = \sum_{s=0}^{a} \begin{bmatrix} a \\ s \end{bmatrix}_q x_{i,1}^s x_{i,2}^{a-s} \otimes x_{2,j}^{a-s} x_{1,j}^s.$$

Proof. Exercise. □

The character group of the n-dimensional torus consisting of diagonal matrices in GL_n is naturally identified with the abelian group $\mathbb{Z}^n$, with standard coordinates $\varepsilon_1, \ldots, \varepsilon_n$. There is a scalar product on $\mathbb{Z}^n$ such that $\varepsilon_i \cdot \varepsilon_j = \delta_{i,j}$. We also have the *dominance order* on $\mathbb{Z}^n$ defined by $\lambda \leq \mu$ if the difference $\mu - \lambda$ is a sum of simple roots $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, n - 1$.

The algebra $\mathcal{O}_q(n)$ admits two different gradings. It is $\mathbb{Z}$-graded with $x_{i,j}$ in degree one, and it is bigraded by the character group $\mathbb{Z}^n$ with $x_{i,j}$ of bidegree $(\varepsilon_i, \varepsilon_j)$:

$$\mathcal{O}_q(n) = \bigoplus_{r \geq 0} \mathcal{O}_q(n, r) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} \mathcal{O}_q[\lambda, \mu]. \tag{3-7}$$

These two gradings are compatible with each other:

$$\mathcal{O}_q(n,r) = \bigoplus_{\lambda,\mu\in\Lambda(n,r)} \mathcal{O}_q[\lambda,\mu], \tag{3-8}$$

where $\Lambda(n,r) := \{\lambda \vDash r \mid \ell(\lambda) = n\}$ is the set of all $\lambda \in \mathbb{Z}^n$ such that $\lambda_1, \dots, \lambda_n \geq 0$ and $\lambda_1 + \dots + \lambda_n = r$. It is also important to observe that $\mathcal{O}_q(n,r)$ is a subcoalgebra of $\mathcal{O}_q(n)$.

Let

$$\mathrm{I}(n,r) := \{\boldsymbol{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r \mid 1 \leq i_1, \dots, i_r \leq n\} = \bigcup_{\lambda\in\Lambda(n,r)} \mathrm{I}_\lambda. \tag{3-9}$$

For $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{I}(n,r)$, we use the shorthand $x_{\boldsymbol{i},\boldsymbol{j}} := x_{i_1,j_1} \cdots x_{i_r,j_r}$. Then $\mathcal{O}_q(n,r)$ is free as a $\mathbb{Z}[q,q^{-1}]$-module with the following basis, which we call the *normally ordered monomial basis*:

$$\{x_{\boldsymbol{i},\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in \mathrm{I}(n,r),\ j_1 \leq \dots \leq j_r \text{ and } i_s \geq i_{s+1} \text{ when } j_s = j_{s+1}\}. \tag{3-10}$$

There are several different proofs of this, e.g., in [Brundan 2006, §6] it is derived from another realization of $\mathcal{O}_q(n)$ as a braided tensor product of quantum symmetric algebras; normally ordered here corresponds to the "terminal double indexes" in that article. Another relevant basis is

$$\{x_{\boldsymbol{i},\boldsymbol{j}} \mid \boldsymbol{i}, \boldsymbol{j} \in \mathrm{I}(n,r),\ i_1 \geq \dots \geq i_r \text{ and } j_s \leq j_{s+1} \text{ when } i_s = i_{s+1}\}. \tag{3-11}$$

This is the monomial basis in [Brundan 2006] indexed by "initial double indexes".

Following [Brundan 2006, Theorem 16], the *bar involution* on $\mathcal{O}_q(n)$ is the antilinear map

$$- : \mathcal{O}_q(n) \to \mathcal{O}_q(n), \tag{3-12}$$

which fixes all of the generators $x_{i,j}$ and satisfies

$$\overline{xy} = q^{\lambda\cdot\mu-\lambda'\cdot\mu'}\,\bar{y}\,\bar{x} \tag{3-13}$$

for x of bidegree (λ,λ') and y of bidegree (μ,μ'). It is indeed an involution.

Lemma 3.6. *The bar involution is an antilinear coalgebra automorphism.*

Proof. Let $\overline{\Delta}$ denote the composition $-\otimes - \circ \Delta$. We must show that $\overline{\Delta}(x) = \Delta(\bar{x})$ for any $x \in \mathcal{O}_q(n)$. This follows by induction on degree. □

For $\lambda, \mu \in \Lambda(n,r)$, recall the set $\mathrm{Mat}(\lambda,\mu)$ of matrices with these row and column sums from Section 2, which parametrizes the orbits Π_A of S_r on $I_\lambda \times I_\mu$.

For $A \in \mathrm{Mat}(\lambda, \mu)$, let

$$x_A := x_{\boldsymbol{i},\boldsymbol{j}} \quad \text{for } (\boldsymbol{i}, \boldsymbol{j}) \in \Pi_A \text{ such that } j_1 \leq \cdots \leq j_d \text{ and } i_k \geq i_{k+1} \text{ when } j_k = j_{k+1}. \tag{3-14}$$

In other words, if A corresponds to $P \in \mathrm{Row}(\lambda, \mu)$ under (2-2) then $\boldsymbol{i} = \swarrow(P)$ and $\boldsymbol{j} = \boldsymbol{i}^\mu$; the notation $\swarrow(P)$ means the sequence obtained by reading the entries of P in the order suggested by the arrow. Hence $\boldsymbol{i} = \boldsymbol{i}^\lambda \cdot d(A)w_0$ where w_0 is the longest element of S_r. The set $\{x_A \mid \lambda, \mu \in \Lambda(n, r),\ A \in \mathrm{Mat}(\lambda, \mu)\}$ is the normally ordered monomial basis of $\mathcal{O}_q(n, r)$ from (3-10), we have merely parametrized it in a more convenient way. By [Brundan 2006, Theorem 16] again, the image of the normally ordered monomial x_A under the bar involution is

$$\bar{x}_A := x_{\boldsymbol{i},\boldsymbol{j}} \quad \text{for } (\boldsymbol{i}, \boldsymbol{j}) \in \Pi_A \text{ such that } i_1 \geq \cdots \geq i_r \text{ and } j_k \leq j_{k+1} \text{ when } i_k = i_{k+1}. \tag{3-15}$$

In other words, if A^{T} corresponds to $Q \in \mathrm{Row}(\mu, \lambda)$ under (2-2) then $\boldsymbol{i} = \boldsymbol{i}^\lambda \cdot w_r$ and $\boldsymbol{j} = \nwarrow(Q)$. The set $\{\bar{x}_A \mid \lambda, \mu \in \Lambda(n, r),\ A \in \mathrm{Mat}(\lambda, \mu)\}$ is the basis for $\mathcal{O}_q(n, r)$ from (3-11).

Recall the partial order (2-6) on $\mathrm{Mat}(\lambda, \mu)$. The bar involution acts on the normally ordered monomial basis in a unitriangular fashion:

$$\bar{x}_A = x_A + (\text{a } \mathbb{Z}[q, q^{-1}]\text{-linear combination of } x_B\text{'s for } B > A).$$

This may be seen explicitly by using (3-5) to rewrite (3-15) in terms of normally ordered monomials. So one can apply Lusztig's Lemma to define another basis for $\mathcal{O}_q[\lambda, \mu]$, the *dual canonical basis* $\{b_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$. The dual canonical basis element b_A is the unique bar-invariant vector in $\mathcal{O}_q[\lambda, \mu]$ such that $b_A \equiv x_A \left(\mathrm{mod} \sum_{B \in \mathrm{Mat}(\lambda,\mu)} q\mathbb{Z}[q]x_B\right)$. The dual canonical basis is discussed further in [Brundan 2006] (and many other places). In particular, the polynomials $p_{A,B}(q) \in \mathbb{Z}[q]$ defined from

$$x_B = \sum_{A \in \mathrm{Mat}(\lambda,\mu)} p_{A,B}(q) b_A \tag{3-16}$$

are (renormalized) Kazhdan–Lusztig polynomials: writing $P_{x,y}(t) \in \mathbb{Z}[t]$ for the usual Kazhdan–Lusztig polynomial associated to $x, y \in S_r$, we have

$$p_{A,B}(q) = q^{\ell(d_A^+) - \ell(d_B^+)} P_{d_A^+, d_B^+}(q^{-2}). \tag{3-17}$$

This is explained in [Brundan 2006, Remark 10]. We have $p_{A,B}(q) = 0$ unless $A \geq B$, $p_{A,A}(q) = 1$, and $p_{A,B}(q) \in q\mathbb{N}[q]$ if $A > B$. The last assertion, which follows from positivity of Kazhdan–Lusztig polynomials, will not be needed here.

Lemma 3.7. *Suppose we are given $A', B' \in \mathrm{Mat}(\lambda', \mu')$ for $\lambda', \mu' \in \Lambda(n, r)$ and $1 \le i, j \le n$ such that $\lambda'_i = \mu'_j = 0$. Let A and B be the matrices obtained from A' and B' by removing the i-th row and j-th column. Then $p_{A,B}(q) = p_{A',B'}(q)$.*

Proof. This is clear from the nature of the defining relations (3-5) for $\mathcal{O}_q(n)$: they only depend on the relative positions of the indices in the total order on the set $\{1, \dots, n\}$, not on the actual values. □

Example 3.8. For $\lambda, \mu \in \Lambda(2, r)$ and $A \in \mathrm{Mat}(\lambda, \mu)$, we have

$$b_A = x_{2,1}^{a_{2,1}} x_{1,1}^{a_{1,1}-\min(a_{1,1},a_{2,2})} (x_{1,1}x_{2,2} - q x_{2,1}x_{1,2})^{\min(a_{1,1},a_{2,2})} x_{2,2}^{a_{2,2}-\min(a_{1,1},a_{2,2})} x_{1,2}^{a_{1,2}}.$$

This follows from a special case of [Brundan 2006, Theorem 20], which gives a closed formula for the dual canonical basis element b_A for all $A \in \mathrm{Mat}(\lambda, \mu)$ if either λ or μ has at most two nonzero parts. Expanding the binomial gives

$$b_A = x_A - q^M \begin{bmatrix} m \\ 1 \end{bmatrix}_q x_{A+B} + q^{2(M-1)} \begin{bmatrix} m \\ 2 \end{bmatrix}_q x_{A+2B} - \dots + (-1)^m q^{m(M+1-m)} \begin{bmatrix} m \\ m \end{bmatrix}_q x_{A+mB},$$

where $m := \min(a_{1,1}, a_{2,2})$, $M := \max(a_{1,1}, a_{2,2})$ and $B := \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

Lemma 3.9. *There is a surjective bialgebra homomorphism*

$$\mathtt{Y}^* : \mathcal{O}_q(m+n) \twoheadrightarrow \mathcal{O}_q(m) \otimes \mathcal{O}_q(n),$$
$$x_{i,j} \mapsto \begin{cases} x_{i,j} \otimes 1 & \text{if } 1 \le i, j \le m, \\ 1 \otimes x_{i-m,j-m} & \text{if } m+1 \le i, j \le m+n, \\ 0 & \text{otherwise.} \end{cases}$$

This intertwines the bar involution on $\mathcal{O}_q(m+n)$ with the bar involution $- \otimes -$ on $\mathcal{O}_q(m) \otimes \mathcal{O}_q(n)$.

Proof. The existence of this algebra homomorphism follows from the relations. Then one checks that it is a coalgebra homomorphism too. Finally, for the statement about the bar involution, note for an $m \times m$ matrix A and an $n \times n$ matrix B that $\mathtt{Y}^*$ sends $x_{\mathrm{diag}(A,B)}$ to $x_A \otimes x_B$ and $\bar{x}_{\mathrm{diag}(A,B)}$ to $\bar{x}_A \otimes \bar{x}_B$. □

There is also an antilinear algebra antiautomorphism

$$\overline{\mathtt{T}}^* : \mathcal{O}_q(n) \to \mathcal{O}_q(n), \quad x_{i,j} \mapsto x_{j,i}. \tag{3-18}$$

This is a coalgebra antiautomorphism, i.e., $\overline{\mathtt{T}}^* \otimes \overline{\mathtt{T}}^* \circ \Delta = \mathtt{P} \circ \Delta \circ \overline{\mathtt{T}}^*$ where $\mathtt{P}$ is the tensor flip. Comparing (3-14) and (3-15), we see that $\overline{\mathtt{T}}^*(x_A) = \bar{x}_{A^{\mathtt{T}}}$ where $A^{\mathtt{T}}$ is the transpose matrix. Since $\overline{\mathtt{T}}^*$ is an involution, it follows that it commutes with the bar involution. Let

$$\mathtt{T}^* := - \circ \overline{\mathtt{T}}^* = \overline{\mathtt{T}}^* \circ - : \mathcal{O}_q(n) \to \mathcal{O}_q(n). \tag{3-19}$$

This is a linear coalgebra antiautomorphism (but *not* an algebra antiautomorphism) which commutes with the bar involution and sends x_A to $x_{A^{\mathrm{T}}}$. It follows that

$$\mathrm{T}^*(b_A) = b_{A^{\mathrm{T}}}. \tag{3-20}$$

The dual canonical basis element b_A indexed by $A = I_n$, which is minimal in the Bruhat order, is the *quantum determinant*

$$\det_q := \sum_{w \in S_n} (-q)^{\ell(w)} x_{w(1),1} \cdots x_{w(n),n}. \tag{3-21}$$

This is central in $\mathcal{O}_q(n)$. It is also a group-like element, i.e., $\Delta(\det_q) = \det_q \otimes \det_q$ and $\varepsilon(\det_q) = 1$. The coordinate algebra of the *quantum general linear group* q-GL_n is the Ore localization of $\mathcal{O}_q(n)$ at the quantum determinant. The bialgebra structure on $\mathcal{O}_q(n)$ extends to make this into a Hopf algebra. We will not work explicitly with this Hopf algebra here, but its existence underpins all subsequent language and notation.

By a *polynomial representation of* q-GL_n we mean a right $\mathcal{O}_q(n)$-comodule. We use the notation $\mathrm{Hom}_{q\text{-}\mathrm{GL}_n}(-,-)$ to denote morphisms in the category of polynomial representations. Since $\mathcal{O}_q(n)$ is a bialgebra, this is a monoidal category. For example, we have the *natural representation* of q-GL_n, which is the free $\mathbb{Z}[q, q^{-1}]$-module V with basis $v_1, \ldots, v_n$ and comodule structure map $\eta : V \to V \otimes \mathcal{O}_q(n, 1)$ defined from

$$\eta(v_j) = \sum_{i=1}^{n} v_i \otimes x_{i,j}. \tag{3-22}$$

It is a polynomial representation of degree 1, hence, its r-th tensor power $V^{\otimes r}$ is a polynomial representation of degree r, meaning that it is a right $\mathcal{O}_q(n, r)$-comodule.

The category of polynomial representations of q-GL_n is also braided, with braiding c that is uniquely determined by requiring that $c_{V,V} \in \mathrm{End}_{q\text{-}\mathrm{GL}_n}(V \otimes V)$ is the $\mathbb{Z}[q, q^{-1}]$-linear map defined by (1-5). We have $(c_{V,V} + q)(c_{V,V} - q^{-1}) = 0$, hence, $c_{V,V}$ has eigenvalues $-q$ and q^{-1}. After localizing at $[2] = q + q^{-1}$, the tensor square $V \otimes V$ decomposes as the direct sum of the corresponding eigenspaces. The q^{-1}-eigenspace is spanned by

$$\{v_j \otimes v_i + q v_i \otimes v_j \mid 1 \le i < j \le n\} \cup \{v_k \otimes v_k \mid 1 \le k \le n\}. \tag{3-23}$$

The *quantum exterior algebra*

$$\textstyle\bigwedge(V) = \bigoplus_{r \ge 0} \bigwedge^r V \tag{3-24}$$

is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by the quadratic tensors from (3-23). This is studied in [Parshall and Wang 1991] (see also [Brundan 2006, §5]), where it is proved that $\bigwedge^r V$ is free as a $\mathbb{Z}[q, q^{-1}]$-module with basis

$$\{v_I := v_{i_1} \wedge \cdots \wedge v_{i_r} \mid I = \{i_1 < \cdots < i_r\} \subseteq \{1, \ldots, n\}\}.$$

The comodule structure map η for $\bigwedge^r V$ satisfies $\eta(v_J) = \sum_I v_I \otimes x_{I,J}$ where

$$x_{I,J} := \sum_{w \in S_r} (-q)^{\ell(w)} x_{i_{w(1)}, j_1} \cdots x_{i_{w(r)}, j_r} \tag{3-25}$$

for $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$. These so-called *quantum minors* include the quantum determinant (3-21) as a special case.

4. The q-Schur algebra

We continue to work over $\mathbb{Z}[q, q^{-1}]$ like in the previous section. The *q-Schur algebra* is the $\mathbb{Z}[q, q^{-1}]$-linear dual

$$S_q(n, r) := \mathcal{O}_q(n, r)^* = \bigoplus_{\lambda, \mu \in \Lambda(n,r)} \mathcal{O}_q[\lambda, \mu]^*. \tag{4-1}$$

It is an algebra with multiplication $S_q(n, r) \otimes S_q(n, r) \to S_q(n, r)$ defined by the dual map to the restriction $\mathcal{O}_q(n, r) \to \mathcal{O}_q(n, r) \otimes \mathcal{O}_q(n, r)$ of the comultiplication on $\mathcal{O}_q(n)$. For this, we are identifying $f \otimes g \in S_q(n, r) \otimes S_q(n, r)$ with an element of $(\mathcal{O}_q(n, r) \otimes \mathcal{O}_q(n, r))^*$ so that $\langle f \otimes g, x \otimes y\rangle := \langle f, x\rangle\langle g, y\rangle$ for $f, g \in S_q(n, r)$, $x, y \in \mathcal{O}_q(n, r)$.

The unit element $1 \in S_q(n, r)$ is the restriction of the counit ε to $\mathcal{O}_q(n, r)$. For $\lambda \in \Lambda(n, r)$, let 1_λ be the function which is equal to ε on $\mathcal{O}_q[\lambda, \lambda]$ and is zero on all other summands $\mathcal{O}_q[\lambda, \mu]$ in the decomposition (3-8). This defines mutually orthogonal idempotents $\{1_\lambda \mid \lambda \in \Lambda(n, r)\}$ in $S_q(n, r)$ whose sum is the identity. Moreover, $1_\lambda S_q(n, r) 1_\mu = \mathcal{O}_q[\lambda, \mu]^*$.

The dual map to the bar involution on $\mathcal{O}_q(n, r)$ defines a bar involution on $S_q(n, r)$ which we denote with the same notation, so $\langle \bar{f}, x\rangle = \overline{\langle f, \bar{x}\rangle}$ for $f \in S_q(n, r)$, $x \in \mathcal{O}_q(n, r)$. Lemma 3.6 implies that $- : S_q(n, r) \to S_q(n, r)$ is an antilinear algebra automorphism. The dual of the restriction $\mathcal{O}_q(m + n, r) \to \bigoplus_{a+b=r} \mathcal{O}_q(m, a) \otimes \mathcal{O}_q(n, b)$ of the homomorphism Υ^* from Lemma 3.9 defines an injective algebra homomorphism

$$\Upsilon_r : \bigoplus_{a+b=r} S_q(m, a) \otimes S_q(n, b) \hookrightarrow S_q(m + n, r), \qquad \xi_A \otimes \xi_B \mapsto \xi_{\mathrm{diag}(A,B)}. \tag{4-2}$$

This intertwines the bar involutions $-\otimes-$ on each $S_q(m,a)\otimes S_q(n,b)$ with the bar involution on $S_q(m+n,r)$. The dual of (3-19) gives us a transposition involution $\mathtt{T}: S_q(n,r)\to S_q(n,r)$. This is a linear algebra antiautomorphism.

The dual bases to $\{x_A \mid A\in \mathrm{Mat}(\lambda,\mu)\}$ and $\{b_A \mid A\in \mathrm{Mat}(\lambda,\mu)\}$ give bases for the free $\mathbb{Z}[q,q^{-1}]$-module $1_\lambda S_q(n,r)1_\mu$, which we denote by $\{\xi_A \mid A\in \mathrm{Mat}(\lambda,\mu)\}$, the *standard basis*, and $\{\theta_A \mid A\in \mathrm{Mat}(\lambda,\mu)\}$, the *canonical basis*. The canonical basis element $\theta_A\in 1_\lambda S_q(n,r)1_\mu$ is the unique bar-invariant element that satisfies $\theta_A\equiv \xi_A \ \big(\mathrm{mod}\ \sum_{B\in\mathrm{Mat}(\lambda,\mu)} q\mathbb{Z}[q]\xi_B\big)$. In particular, θ_A is the sum of ξ_A with a $q\mathbb{N}[q]$-linear combination of ξ_B for $B<A$, because by (3-16) we have

$$\theta_A=\sum_{B\in\mathrm{Mat}(\lambda,\mu)} p_{A,B}(q)\xi_B, \tag{4-3}$$

where $p_{A,B}(q)$ is the Kazhdan–Lusztig polynomial from (3-17). There is also a geometric construction of the canonical basis via intersection cohomology. This is explained in [Beilinson et al. 1990, §1.4], where the standard basis element ξ_A is denoted $[A]$ and θ_A is denoted $\{A\}$ (up to some renormalization).

The counit ε is zero on all of the normally ordered monomials in $\mathcal{O}_q[\lambda,\lambda]$ except for $x_{1,1}^{\lambda_1}\cdots x_{n,n}^{\lambda_n}$, proving the first equality in

$$1_\lambda=\xi_{\mathrm{diag}(\lambda_1,\dots,\lambda_n)}=\theta_{\mathrm{diag}(\lambda_1,\dots,\lambda_n)}. \tag{4-4}$$

The second equality follows because

$$\overline{\xi}_A=\xi_A+(\text{a }\mathbb{Z}[q,q^{-1}]\text{-linear combination of }\xi_B\text{'s for }B<A)$$

and $A=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$ is minimal in the Bruhat ordering, so $\xi_{\mathrm{diag}(\lambda_1,\dots,\lambda_n)}$ is bar invariant. More generally, since the homomorphism Υ_r is bar equivariant, we have

$$\Upsilon_r(\theta_A\otimes\theta_B)=\theta_{\mathrm{diag}(A,B)}. \tag{4-5}$$

Also, by (3-20), we have

$$\mathtt{T}(\xi_A)=\xi_{A^{\mathtt{T}}},\qquad \mathtt{T}(\theta_A)=\theta_{A^{\mathtt{T}}}. \tag{4-6}$$

Example 4.1. For $A\in\mathrm{Mat}(\lambda,\mu)$ with $\lambda,\mu\in\Lambda(2,r)$ we have

$$\theta_A=\sum_{s=0}^{\min(a_{1,2},a_{2,1})} q^{s(s+\max(a_{1,1},a_{2,2}))}\begin{bmatrix} s+\min(a_{1,1},a_{2,2})\\ s\end{bmatrix}_q \xi_{A-sB}, \tag{4-7}$$

where $B:=\left[\begin{smallmatrix}-1&1\\1&-1\end{smallmatrix}\right]$. This follows by inverting the transition matrix in Example 3.8.

For $n\times n$ matrices A, B, C with nonnegative integer entries, define

$$Z(A,B,C):=\langle \xi_A\otimes\xi_B,\Delta(x_C)\rangle\in\mathbb{Z}[q,q^{-1}]. \tag{4-8}$$

These are the *structure constants* for multiplication in the standard basis of the q-Schur algebra: we have

(4-9) $$\xi_A \circ \xi_B := \sum_C Z(A, B, C)\xi_C.$$

This formula can be viewed as a q-analog of Schur's product rule. For a completely different approach to the definition of these structure constants (counting points over a finite field), see [Beilinson et al. 1990, §1.1]. The structure constants have the following stabilization property, which will be relevant in the next section.

Lemma 4.2. *Suppose we are given $A' \in \mathrm{Mat}(\lambda', \mu')$, $B' \in \mathrm{Mat}(\mu', \nu')$ and $C' \in \mathrm{Mat}(\lambda', \nu')$ for $\lambda', \mu', \nu' \in \Lambda(n, r)$ and $1 \le i, j, k \le n$ such that $\lambda'_i = \mu'_j = \nu'_k = 0$. Let A, B, C be the matrices obtained by removing the i-th row and j-th column of A', the j-th row and k-th column of B', and the i-th row and k-th column of C', respectively. Then we have $Z(A, B, C) = Z(A', B', C')$.*

Proof. This follows for the same reason as Lemma 3.7. □

Let H_r be the Hecke algebra of the symmetric group, that is, the $\mathbb{Z}[q, q^{-1}]$-algebra on generators $\tau_1, \dots, \tau_{r-1}$ subject to the relations

$$(\tau_i + q)(\tau_i - q^{-1}) = 0, \qquad \tau_i\tau_j = \tau_j\tau_i \quad \text{if } |i - j| > 1, \qquad \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}.$$

For $w \in S_r$, we have the corresponding element $\tau_w \in H_r$ defined from a reduced expression for w, and the elements $\{\tau_w \mid w \in S_r\}$ give a basis for H_w as a free $\mathbb{Z}[q, q^{-1}]$-module. Recall also that the Hecke algebra has its own antilinear bar involution $- : H_w \to H_w$, $\tau_w \mapsto \tau_{w^{-1}}^{-1}$.

Lemma 4.3. *Suppose that $r \le n$ and let $\omega := (1^r\ 0^{n-r}) \in \Lambda(n, r)$. There is an algebra isomorphism $H_r \xrightarrow{\sim} 1_\omega S_q(n, r)1_\omega$ sending τ_w to the standard basis element ξ_A for the matrix $A \in \mathrm{Mat}(\omega, \omega)$ such that $a_{w(i),i} = 1$ for $i = 1, \dots, r$ and all other entries are zero. This map intertwines the bar involutions on H_r and $S_q(n, r)$.*

Proof. Check that the relation

$$\tau_w\tau_i = \begin{cases} \tau_{ws_i} & \text{if } w(i) < w(i+1), \\ \tau_{ws_i} - (q - q^{-1})\tau_w & \text{if } w(i) > w(i+1) \end{cases}$$

holds in $S_q(n, r)$ by explicitly calculating the corresponding structure constants. This is well known so we omit the details. □

Let V be the natural representation of q-GL_n. In addition to our definition of $S_q(n, r)$ by dualizing $\mathcal{O}_q(n, r)$, and the approach in [Beilinson et al. 1990] where the q-Schur algebra arises as the endomorphism algebra of a permutation representation of the finite general linear group, the q-Schur algebra can be realized as an endomorphism algebra for an action of the Hecke algebra H_r on the tensor space $V^{\otimes r}$. To explain this, note that $V^{\otimes r}$ has basis $v_{\boldsymbol{i}} := v_{i_1} \otimes \cdots \otimes v_{i_r}$ for $\boldsymbol{i} \in \mathrm{I}(n, r)$. There is a *right* action of H_r on $V^{\otimes r}$ such that τ_i acts as the braiding $1^{\otimes(i-1)} \otimes c_{V,V} \otimes 1^{r-i-1}$ from (1-5). Since $V^{\otimes r}$ is a polynomial representation of degree r, it is a left $S_q(n, r)$-module. The action of H_r commutes with the action of $S_q(n, r)$. Hence, there is a well-defined algebra homomorphism

$$S_q(n, r) \to \mathrm{End}_{H_r}(V^{\otimes r}). \tag{4-10}$$

This homomorphism is actually an algebra *isomorphism*. There are several ways to see this, e.g., it can be deduced from [Dipper and James 1986]. In fact, in [Dipper and James 1986], the authors work with a different realization of the right H_r-module $V^{\otimes r}$ as a direct sum of permutation modules. In this form, one obtains a basis for the endomorphism algebra on the right-hand side of (4-10) quite easily from the Mackey theorem, and then just needs to check that this basis is also the image of the standard basis for $S_q(n, r)$ under the homomorphism (4-10). Since this is quite important for us, we go through some details in the next paragraph.

For $\lambda \in \Lambda(n, r)$, let H_λ be the parabolic subalgebra of H_r associated to S_λ. Let X_λ be the free $\mathbb{Z}[q, q^{-1}]$-module of rank one with basis m_λ viewed as a right H_λ-module so that $m_\lambda \tau_i = q^{-1} m_\lambda$ for each $\tau_i \in H_\lambda$. The (right) *permutation module* is the induced module $M(\lambda) := X_\lambda \otimes_{H_\lambda} H_r$. There is a unique H_r-module homomorphism

$$f_\lambda : M(\lambda) \to 1_\lambda V^{\otimes r}, \quad m_\lambda \otimes 1 \mapsto v_{\boldsymbol{i}^\lambda}. \tag{4-11}$$

This is actually an *isomorphism* because the vectors $\{m_\lambda \otimes \tau_w \mid w \in (S_\lambda \backslash S_r)_{\min}\}$ give a basis for $M(\lambda)$, and f_λ maps them to the basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \in \mathrm{I}_\lambda\}$ for $1_\lambda V^{\otimes r}$. Summing over all $\lambda \in \Lambda(n, r)$, this gives us an H_r-module isomorphism

$$f : \bigoplus_{\lambda \in \Lambda(n,r)} M(\lambda) \xrightarrow{\sim} V^{\otimes r}. \tag{4-12}$$

The following lemma explains how to transport the natural action of $S_q(n, r)$ on $V^{\otimes r}$ through f to obtain an action on this direct sum of permutation modules.

Lemma 4.4. *Suppose that $\lambda, \mu \in \Lambda(n, r)$ and $A \in \mathrm{Mat}(\lambda, \mu)$. The diagram*

$$\begin{array}{ccc} 1_\mu V^{\otimes r} & \xrightarrow{\xi_A} & 1_\lambda V^{\otimes r} \\ {\scriptstyle f_\mu}\uparrow & & \uparrow{\scriptstyle f_\lambda} \\ M(\mu) & \longrightarrow & M(\lambda) \end{array}$$

commutes, where the top map is defined by acting on the left with ξ_A, and the bottom map is the H_r-module homomorphism sending

$$m_\mu \otimes 1 \mapsto \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} q^{\ell(w_0)-\ell(w)} m_\lambda \otimes \tau_{d_A} \tau_w, \tag{4-13}$$

where $\mu^+ \vDash r$ is as in Lemma 2.1 and w_0 is the longest element of $(S_{\mu^+} \backslash S_\mu)_{\min}$.

Proof. The comodule structure map η of $V^{\otimes r}$ satisfies

$$\eta(v_{\boldsymbol{j}}) = \sum_{\boldsymbol{i} \in \mathrm{I}(n,r)} v_{\boldsymbol{i}} \otimes x_{\boldsymbol{i},\boldsymbol{j}}.$$

Hence, for $\boldsymbol{j} \in \mathrm{I}_\mu$, we have

$$\xi_A v_{\boldsymbol{j}} = \sum_{\boldsymbol{i} \in \mathrm{I}_\lambda} \langle \xi_A, x_{\boldsymbol{i},\boldsymbol{j}} \rangle v_{\boldsymbol{i}}. \tag{4-14}$$

By (3-14), $x_A = x_{\boldsymbol{i}^\lambda \cdot d_A w_0, \boldsymbol{i}^\mu}$. The S_μ-orbit of $\boldsymbol{i}^\lambda \cdot d_A w_0$ is $\{\boldsymbol{i}^\lambda \cdot d_A w \mid w \in (S_{\mu^+} \backslash S_\mu)_{\min}\}$. Also for $w \in (S_{\mu^+} \backslash S_\mu)_{\min}$ we have $x_{\boldsymbol{i}^\lambda \cdot d_A w, \boldsymbol{i}^\mu} = q^{\ell(w_0)-\ell(w)} x_{\boldsymbol{i}^\lambda \cdot d_A w_0, \boldsymbol{i}^\mu}$ as one needs to use the last relation in (3-5) a total of $\ell(w_0) - \ell(w)$ times. Putting this together shows that

$$\xi_A v_{\boldsymbol{i}^\mu} = \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} q^{\ell(w_0)-\ell(w)} v_{\boldsymbol{i}^\lambda \cdot d_A w}.$$

The lemma now follows since f_λ sends $m_\lambda \otimes 1$ to $v_{\boldsymbol{i}^\lambda}$, f_μ sends $m_\mu \otimes 1$ to $v_{\boldsymbol{i}^\mu}$, and $v_{\boldsymbol{i}^\lambda \cdot d_A w} = v_{\boldsymbol{i}^\lambda} \tau_{d_A} \tau_w$ as $i_1^\lambda \leq \cdots \leq i_r^\lambda$. ☐

Let m be another natural number. For $\lambda \in \Lambda(m, r)$, let $Y(\lambda)$ be the free $\mathbb{Z}[q, q^{-1}]$-module of rank one with basis n_λ viewed as a left H_λ-module so that $\tau_i n_\lambda = -q n_\lambda$ for each $\tau_i \in H_\lambda$. The (left) signed permutation module is the induced module $N(\lambda) := H_r \otimes_{H_\lambda} Y(\lambda)$.

Lemma 4.5. *There is an algebra isomorphism*

$$S_q(m, r) \xrightarrow{\sim} \mathrm{End}_{H_r}\Bigl(\bigoplus_{\lambda \in \Lambda(m,r)} N(\lambda) \Bigr)$$

sending $\xi_A \in 1_\lambda S_q(m, r) 1_\mu$ to the unique H_r-module homomorphism such that

$$1 \otimes n_\mu \mapsto \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} (-1)^{\ell(w)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} \tau_w^{-1} \tau_{d_A}^{-1} \otimes n_\lambda,$$

where μ^+ is as in Lemma 2.1 and w_0 is the longest element of $(S_{\mu^+} \backslash S_\mu)_{\min}$, and $1 \otimes n_\nu \mapsto 0$ for $\nu \neq \mu$.

Proof. We start from the algebra isomorphism (4-10). Using (4-12) and Lemma 4.4, and replacing n by m, this gives us an algebra isomorphism

$$S_q(m,r) \xrightarrow{\sim} \mathrm{End}_{H_r}\Bigl(\bigoplus_{\lambda\in\Lambda(m,r)} M(\lambda)\Bigr)$$

such that $\xi_A \in 1_\lambda S_q(m,r)1_\mu$ acts on $m_\mu \otimes 1 \in M(\mu)$ according to (4-13), and it acts as zero on all other summands. Then we use the algebra antiautomorphism $H_r \to H_r$, $\tau_x \mapsto (-1)^{\ell(x)}\tau_x^{-1}$. The pull-back of the right H_r-module $M(\lambda)$ along this map is isomorphic to the left H_r-module $N(\lambda)$, there being a unique isomorphism such that $m_\lambda \otimes \tau_x \mapsto (-1)^{\ell(x)}\tau_x^{-1} \otimes n_\lambda$ for all $x \in S_r$. We deduce that $\mathrm{End}_{H_r}\bigl(\bigoplus_{\lambda\in\Lambda(m,r)} M(\lambda)\bigr) \cong \mathrm{End}_{H_r}\bigl(\bigoplus_{\lambda\in\Lambda(m,r)} N(\lambda)\bigr)$. It just remains to note that the action of $\xi_A \in 1_\lambda S(m,r)1_\mu$ on $\bigoplus_{\lambda\in\Lambda(m,r)} M(\lambda)$ translates into the action on $\bigoplus_{\lambda\in\Lambda(m,r)} N(\lambda)$ described explicitly in the statement of the lemma. □

The goal now is to replace H_r and the signed permutation modules $N(\lambda)$ in Lemma 4.5 with the quantum general linear group q-GL_n and its polynomial representations

$$\textstyle\bigwedge^\lambda V := \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_{\ell(\lambda)}} V. \tag{4-15}$$

Lemma 4.6. *Take $\lambda, \mu \in \Lambda(m,r)$ and $A \in \mathrm{Mat}(\lambda,\mu)$. There is a unique q-GL_n-module homomorphism $\phi_A : \bigwedge^\mu V \to \bigwedge^\lambda V$ such that the diagram*

$$\begin{array}{ccc} V^{\otimes r} & \longrightarrow & V^{\otimes r} \\ \big\downarrow & & \big\downarrow \\ \bigwedge^\mu V & \xrightarrow{\phi_A} & \bigwedge^\lambda V \end{array}$$

commutes, where the vertical maps are the quotient maps and the top map is right multiplication by $\sum_{w\in(S_{\mu^+}\backslash S_\mu)_{\min}} (-1)^{\ell(w)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} \tau_w^{-1}\tau_{d_A}^{-1}$ where μ^+ is defined as in Lemma 2.1 and w_0 is the longest element of $(S_{\mu^+}\backslash S_\mu)_{\min}$.

Proof. By the definition of quantum exterior powers, the kernel of the projection $V^{\otimes r} \twoheadrightarrow \bigwedge^\mu V$ is generated by the kernels of the endomorphisms $\tau_j - q^{-1} = \tau_j^{-1} - q$ for all j with $s_j \in S_\mu$. Hence, we need to show for such a j and $v \in V^{\otimes r}$ with $v\tau_j^{-1} = qv$ that the vector

$$v' := \sum_{w\in(S_{\mu^+}\backslash S_\mu)_{\min}} (-1)^{\ell(w)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} v\tau_w^{-1}\tau_{d_A}^{-1}$$

is in the sum of the kernels of the maps $\tau_i^{-1} - q$ for all i with $s_i \in S_\lambda$. We have

$$(S_{\mu^+}\backslash S_\mu)_{\min} = X \sqcup Xs_j \sqcup Y$$

such that $\ell(xs_j) = \ell(x) + 1$ for all $x \in X$, and $ys_jy^{-1} \in S_{\mu^+}$ for all $y \in Y$. This follows from [Dipper and James 1989, Lemma 1.1]. For $x \in X$, we have

$$(-1)^{\ell(x)+\ell(d_A)} q^{\ell(w_0)-\ell(x)} v\tau_x^{-1}\tau_{d_A}^{-1} + (-1)^{\ell(xs_j)+\ell(d_A)} q^{\ell(w_0)-\ell(xs_j)} v\tau_j^{-1}\tau_x^{-1}\tau_{d_A}^{-1} = 0$$

as $v\tau_j^{-1} = qv$. This implies that

$$v' = \sum_{y\in Y} (-1)^{\ell(y)+\ell(d_A)} q^{\ell(w_0)-\ell(y)} v\tau_y^{-1}\tau_{d_A}^{-1}.$$

It remains to show for $y \in Y$ that $v\tau_y^{-1}\tau_{d_A}^{-1}$ is in the kernel of $\tau_i^{-1} - q$ for some i with $s_i \in S_\lambda$. We have $d_A y s_j = t d_A y$ for $t := d_A(ys_jy^{-1})d_A^{-1}$. Since $ys_jy^{-1} \in S_{\mu^+}$, we deduce using Lemma 2.1 that $t \in S_\lambda$ (in fact, it is in $S_{\lambda^-} \le S_\lambda$ in the notation from the lemma), and that $\ell(td_Ay) = \ell(t) + \ell(d_A) + \ell(y)$. Since $\ell(d_Ays_j) \le \ell(d_A) + \ell(y) + 1$, we deduce that $\ell(t) = 1$. Hence, $t = s_i$ for some i such that $s_i \in S_\lambda$. Moreover $v\tau_y^{-1}\tau_{d_A}^{-1}\tau_i^{-1} = v\tau_j^{-1}\tau_y^{-1}\tau_{d_A}^{-1} = qv\tau_y^{-1}\tau_{d_A}^{-1}$. □

The following theorem is the quantum analog of [Donkin 1993, Proposition 3.11]. See [Donkin 1998, 4.2(19)] for a closely related result already in the quantum setting.

Theorem 4.7. *Fix $m, r \in \mathbb{N}$. For any $n \ge 0$, there is a surjective algebra homomorphism*

$$g_n : S_q(m,r) \twoheadrightarrow \mathrm{End}_{q\text{-}\mathrm{GL}_n}\Big(\bigoplus_{\lambda\in\Lambda(m,r)} {\textstyle\bigwedge}^\lambda V\Big) \tag{4-16}$$

sending $\xi_A \in 1_\lambda S_q(m,r)1_\mu$ to the endomorphism that is equal to the homomorphism ϕ_A from Lemma 4.6 on the summand $\bigwedge^\mu V$, and is zero on all other summands. Moreover, g_n is an isomorphism if $n \ge r$.

Proof. Using the base change functor $\Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} -$, it suffices to prove the analogous statement when $\mathbb{Z}[q,q^{-1}]$ is replaced by a field $\Bbbk$ and q is any nonzero element. In the remainder of the proof, we assume we are working over a field in this way, writing $q\text{-}\mathrm{GL}_n(\Bbbk)$ for the quantum general linear group over $\Bbbk$, whose coordinate algebra is $\Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{O}_q(n)$. The category of polynomial representations of $q\text{-}\mathrm{GL}_n(\Bbbk)$ is a highest weight category satisfying standard homological properties. This is justified, for instance, in [Parshall and Wang 1991] or [Donkin 1998].[2] In the next paragraph, we treat the case that $n \ge r$. Then the existence and surjectivity of g_n for $n < r$ follows from the existence and surjectivity of g_N for $N \ge r$ by an argument involving truncation to the subgroup $q\text{-}\mathrm{GL}_n < q\text{-}\mathrm{GL}_N$ using [Donkin 1998, 4.2(11)] (this requires the standard homological properties).

[2]It can also be deduced by using the results of Section 7 to show that $S_q(n,r)$ is a split quasihereditary algebra.

So now assume that $n \geq r$ and that we are working over a field. We must show that g_n is a well-defined algebra isomorphism. To see this, we use the *Schur functor*, that is, the idempotent truncation functor $\pi : S_q(n,r)\text{-mod} \to H_r\text{-mod}$ defined by the idempotent 1_ω, notation as in Lemma 4.3. This sends an $S_q(n,r)$-module to its ω-weight space viewed as an H_r-module via the isomorphism from that lemma. We have $\pi\big(\bigwedge^\lambda V\big) \cong N(\lambda)$, there being a unique such isomorphism sending the canonical image of $v_1 \otimes \cdots \otimes v_r$ in $\bigwedge^\lambda V$ to $1 \otimes n_\lambda$. Moreover, the Schur functor induces an isomorphism

$$\operatorname{Hom}_{S_q(n,r)}\big(\textstyle\bigwedge^\mu V, \bigwedge^\lambda V\big) \xrightarrow{\sim} \operatorname{Hom}_{H_r}(N(\mu), N(\lambda)).$$

This follows by general principles (e.g., see [Jantzen and Seitz 1992, Theorem 2.12]) because the head of $\bigwedge^\mu V$ and the socle of $\bigwedge^\lambda V$ are p-restricted, i.e., they only involve irreducible modules L which are not annihilated by π. Indeed, these modules are both submodules and quotient modules of the tensor space $V^{\otimes r}$, which has p-restricted head and socle because $V^{\otimes r} \cong S_q(n,r)1_\omega$ by the isomorphisms (4-10) and (4-11), hence,

$$\operatorname{Hom}_{S_q(n,r)}(L, V^{\otimes r}) \cong \operatorname{Hom}_{S_q(n,r)}(V^{\otimes r}, L) \cong \operatorname{Hom}_{S_q(n,r)}(S_q(n,r)1_\omega, L) \cong 1_\omega L$$

for any self-dual module L. Consequently, π induces an algebra isomorphism

$$\operatorname{End}_{q\text{-GL}_n}\Big(\bigoplus_{\lambda\in\Lambda(m,r)} \textstyle\bigwedge^\lambda V\Big) \cong \operatorname{End}_{H_r}\Big(\bigoplus_{\lambda\in\Lambda(m,r)} N(\lambda)\Big).$$

Composing this with the isomorphism from Lemma 4.5 gives the desired isomorphism g_n.

It just remains to identify the endomorphism $g_n(\xi_A)$ with ϕ_A. For this, it suffices to check for $\xi_A \in 1_\lambda S_q(m,r)1_\mu$ that the maps $g_n(\xi_A)$ and ϕ_A are equal on the canonical image of $v_1 \otimes \cdots \otimes v_r$ in $\bigwedge^\mu V$. By the definition from Lemma 4.6, ϕ_A sends this vector to the canonical image of

$$\sum_{w\in(S_{\mu^+}\backslash S_\mu)_{\min}} (-1)^{\ell(x)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} (v_1 \otimes \cdots \otimes v_r)\tau_w^{-1}\tau_{d_A}^{-1}$$

in $\bigwedge^\lambda V$. On the other hand, $g_n(\xi_A)$ takes this vector to the image of

$$\sum_{w\in(S_{\mu^+}\backslash S_\mu)_{\min}} (-1)^{\ell(x)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} \tau_w\tau_{d_A}^{-1}(v_1 \otimes \cdots \otimes v_r),$$

where the left action of H_r on $1_\omega V^{\otimes r}$ comes from the left action of $S_q(n,r)$ via the isomorphism of Lemma 4.3. Now observe for any $x \in S_r$ that $\tau_x(v_1 \otimes \cdots \otimes v_r) = (v_1 \otimes \cdots \otimes v_r)\tau_x$ as, by the definitions, both vectors are equal to $v_{x(1)} \otimes \cdots \otimes v_{x(r)}$. $\square$

5. The q-Schur category

It is easy to adapt (4-8) to define $Z(A, B, C) \in \mathbb{Z}[q, q^{-1}]$ for $A \in \mathrm{Mat}(\lambda, \mu)$, $B \in \mathrm{Mat}(\mu, \nu)$, $C \in \mathrm{Mat}(\lambda, \nu)$ and compositions $\lambda, \mu, \nu \vDash r$ that are not necessarily of the same length. To do so, we pick any $n \geq \ell(\lambda), \ell(\mu), \ell(\nu)$ and let λ', μ' and ν' be compositions of length n obtained from λ, μ and ν by adding some extra entries equal to zero. Let $A' \in \mathrm{Mat}(\lambda', \mu')$, $B' \in \mathrm{Mat}(\mu', \nu')$ and $C' \in \mathrm{Mat}(\lambda', \nu')$ be the matrices obtained by inserting corresponding rows and columns of zeros into A, B and C; see (2-3) and (2-4) for an example. Then we define $Z(A, B, C)$ to be the structure constant $Z(A', B', C')$ for the q-Schur algebra $S_q(n, r)$ exactly as defined earlier. The stability from Lemma 4.2 implies that this is well-defined independent of all choices.

The following theorem defines the *q-Schur category with* 0*-strings*. The version without 0-strings discussed in the introduction is the full subcategory with object set $\Lambda_s \subset \Lambda$.

Theorem 5.1. *There is a $\mathbb{Z}[q, q^{-1}]$-linear strict monoidal category q-**Schur** with*

- *objects that are all compositions $\lambda \in \Lambda$;*
- *for $\lambda \vDash r$ and $\mu \vDash s$, the morphism space $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ is $\{0\}$ unless $r = s$, and it is the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{\xi_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$ if $r = s$;*
- *tensor product of objects is defined by concatenation of compositions;*
- *tensor product of morphisms (horizontal composition) is defined by $\xi_A \star \xi_B := \xi_{\mathrm{diag}(A,B)}$;*
- *vertical composition of morphisms is defined as in (4-9).*

The strict identity object $\mathbb{1}$ is the composition of length zero, and the identity endomorphism 1_λ of an object $\lambda \in \Lambda$ is $\xi_{\mathrm{diag}(\lambda_1,\dots,\lambda_{\ell(\lambda)})}$.

Proof. Most of the axioms of strict monoidal category are straightforward. The fact that vertical composition is associative is a consequence of associativity of multiplication in the q-Schur algebra. To check the interchange law, we must show that

$$(\xi_A \star 1_\sigma) \circ (1_\mu \star \xi_B) = (1_\lambda \star \xi_B) \circ (\xi_A \star 1_\rho)$$

for $\lambda, \mu \vDash a$, $\sigma, \rho \vDash b$ and $A \in \mathrm{Mat}(\lambda, \mu)$, $B \in \mathrm{Mat}(\sigma, \rho)$, that is,

$$\xi_{\mathrm{diag}(A,\sigma_1,\dots,\sigma_{\ell(\sigma)})} \circ \xi_{\mathrm{diag}(\mu_1,\dots,\mu_{\ell(\mu)},B)} = \xi_{\mathrm{diag}(\lambda_1,\dots,\lambda_{\ell(\lambda)},B)} \circ \xi_{\mathrm{diag}(A,\rho_1,\dots,\rho_{\ell(\rho)})}.$$

Using the stability from Lemma 4.2, we may assume that $\ell(\lambda) = \ell(\mu) = m$ and $\ell(\sigma) = \ell(\rho) = n$. We have $(\xi_A \otimes 1_\sigma)(1_\mu \otimes \xi_B) = (1_\lambda \otimes \xi_B)(A \otimes 1_\rho)$ in the algebra $S_q(m, a) \otimes S_q(n, b)$. Now apply the algebra homomorphism Υ_{a+b} from (4-2). □

Remark 5.2. It is clear from the definition that the path algebra of the full subcategory of q-**Schur** generated by objects in $\Lambda(n, r)$ may be identified with the q-Schur algebra, that is,

$$S_q(n, r) = \bigoplus_{\lambda, \mu \in \Lambda(n,r)} \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda). \tag{5-1}$$

By (4-10) and Lemma 4.4, we have $1_\lambda S_q(n, r) 1_\mu \cong \mathrm{Hom}_{H_r}(M(\mu), M(\lambda))$ for $\lambda, \mu \in \Lambda(n, r)$. It follows that the full subcategory of q-**Schur** generated by objects in $\Lambda(r) := \bigcup_{n \geq 0} \Lambda(n, r)$ is isomorphic to the category q-**Schur**(r) with object set $\Lambda(r)$ and morphism spaces

$$\mathrm{Hom}_{q\text{-}\mathbf{Schur}(r)}(\mu, \lambda) := \mathrm{Hom}_{H_r}(M(\mu), M(\lambda)), \tag{5-2}$$

with the natural composition law. The categories q-**Schur**(r) for all r can then be assembled to obtain an alternative approach to the definition of q-**Schur**, with tensor product arising from the bifunctors q-**Schur**$(r) \times q$-**Schur**$(s) \to q$-**Schur**$(r+s)$ induced by the natural embeddings $H_r \otimes H_s \hookrightarrow H_{r+s}$. We have emphasized the based approach in Theorem 5.1 since it allows composition of standard basis elements to be computed effectively using the coalgebra structure on $\mathcal{O}_q(n)$. This will be used several times later on.

We have defined the morphism space $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ so that it comes equipped with the standard basis $\{\xi_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$. We can also transfer the canonical basis from the q-Schur algebra to q-**Schur**, as follows. Take any $\lambda, \mu \models r$ and $A, B \in \mathrm{Mat}(\lambda, \mu)$. There is a corresponding Kazhdan–Lusztig polynomial $p_{A,B}(q) \in \mathbb{Z}[q]$. To define this, we again pick any $n \geq \ell(\lambda), \ell(\mu)$, add extra zeros to λ and μ to make them into compositions of the same length n, and add corresponding rows and columns of zeros to A and B to obtain $A', B' \in \mathrm{Mat}(\lambda', \mu')$. Then we let $p_{A,B}(q) := p_{A',B'}(q)$, where the latter polynomial comes from (4-3). This is well defined independent of the choices thanks to Lemma 3.7. It is also clear from the proof of that lemma that the slightly more general polynomials $p_{A,B}(q)$ still satisfy (3-17). Let

$$\theta_A := \sum_{B \in \mathrm{Mat}(\lambda,\mu)} p_{A,B}(q) \xi_B, \tag{5-3}$$

thereby defining the canonical basis $\{\theta_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$ for $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$.

Lemma 5.3. *There is an antilinear strict monoidal functor* $- : q$-**Schur** $\to q$-**Schur** *which is the identity on objects and, on the morphism space* $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$, *is the unique antilinear map which fixes the canonical basis* $\{\theta_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$.

Proof. Since the bar involution for q-Schur algebras is an antilinear algebra automorphism, this prescription gives a well-defined antilinear functor. To see that it is strict monoidal, it suffices to observe that $\theta_A \star \theta_B = \theta_{\mathrm{diag}(A,B)}$. This follows from (4-5). □

In a similar way, we upgrade the involution $\mathtt{T}$ on $S_q(n,r)$ to a strict linear monoidal functor

$$\mathtt{T} : q\text{-}\mathbf{Schur} \to (q\text{-}\mathbf{Schur})^{\mathrm{op}}, \tag{5-4}$$

which is the identity on objects, commutes with the bar involution, and sends $\xi_A \mapsto \xi_{A^{\mathtt{T}}}$, $\theta_A \mapsto \theta_{A^{\mathtt{T}}}$. This follows by (4-6).

Theorem 5.4. *There is a full $\mathbb{Z}[q,q^{-1}]$-linear monoidal functor Σ_n from q-**Schur** to the category of polynomial representations of q-GL_n sending the object $\lambda \models d$ to the polynomial representation $\bigwedge^\lambda V$ of degree r from (4-15), and the morphism ξ_A for $\lambda, \mu \models r$ and $A \in \mathrm{Mat}(\lambda,\mu)$ to the homomorphism $\phi_A : \bigwedge^\mu V \to \bigwedge^\lambda V$ from Lemma 4.6.*

Proof. To see that Σ_n is a functor, we must show that $\Sigma_n(\xi_A \circ \xi_B) = \Sigma_n(\xi_A) \circ \Sigma_n(\xi_B)$ for $A \in \mathrm{Mat}(\lambda,\mu)$, $B \in \mathrm{Mat}(\mu,\nu)$, $\lambda,\mu,\nu \models r$ and $r \geq 0$. In view of the definition of vertical composition in q-**Schur**, this follows because $\phi_A \circ \phi_B = \sum_{C \in \mathrm{Mat}(\lambda,\nu)} Z(A,B,C)\phi_C$ by Theorem 4.7, taking $m \geq \ell(\lambda), \ell(\mu), \ell(\nu)$. The same theorem also shows that Σ_n is full. Finally, to see that Σ_n is a monoidal functor, we need to check that $\phi_A \otimes \phi_B = \phi_{\mathrm{diag}(A,B)}$. This is clear from the explicit description of these maps given by Lemma 4.6. □

Remark 5.5. (1) By the final statement from Theorem 4.7, the proof of Theorem 5.4 also shows that the functor Σ_n defines an *isomorphism* $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda) \xrightarrow{\sim} \mathrm{Hom}_{q\text{-}\mathrm{GL}_n}\big(\bigwedge^\mu V, \bigwedge^\lambda V\big)$ providing $n \geq |\lambda|, |\mu|$. So one could say that Σ_n is *asymptotically faithful* as $n \to \infty$. In Corollary 8.4 below, we will give an explicit description of the kernel of Σ_n, that is, the tensor ideal of q-**Schur** consisting of the morphisms that it annihilates.

(2) Let $\Bbbk$ be a field viewed as a $\mathbb{Z}[q,q^{-1}]$-algebra in some way, and consider the specialization $q\text{-}\mathbf{Schur}(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}$. The functor Σ_n in Theorem 5.4 induces a $\Bbbk$-linear monoidal functor from q-**Schur**$(\Bbbk)$ to the category of polynomial representations of q-$\mathrm{GL}_n(\Bbbk)$. By the proofs of Theorems 4.7 and 5.4, this induced functor is also full.

By *merges*, *splits*, and *positive crossings*, we mean the morphisms $\xi_{[a\ b]}$, $\xi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$, and $\xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ for $a, b \geq 0$. The images $\phi_{[a\ b]}$, $\phi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$, and $\phi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ of these special morphisms under the functor Σ_n from Theorem 5.4 are the natural projection

$$\bigwedge^a V \otimes \bigwedge^b V \twoheadrightarrow \bigwedge^{a+b} V, \quad v \otimes w \mapsto v \wedge w, \tag{5-5}$$

the inclusion

$$\bigwedge^{a+b} V \hookrightarrow \bigwedge^a V \otimes \bigwedge^b V,$$

$$v_{i_1} \wedge \cdots \wedge v_{i_{a+b}} \mapsto q^{-ab} \sum (-q)^{\ell(w)} v_{i_{w(1)}} \wedge \cdots \wedge v_{i_{w(a)}} \otimes v_{i_{w(a+1)}} \wedge \cdots \wedge v_{i_{w(a+b)}},$$

where the sum is over $w \in (S_{a+b}/S_a \times S_b)_{\min}$, and the isomorphism

$$(-1)^{ab} c^{-1}_{\bigwedge^b V, \bigwedge^a V} : \bigwedge^a V \otimes \bigwedge^b V \xrightarrow{\sim} \bigwedge^b V \otimes \bigwedge^a V, \tag{5-6}$$

where $c_{\bigwedge^b V, \bigwedge^a V} : \bigwedge^b V \otimes \bigwedge^a V \to \bigwedge^a V \otimes \bigwedge^b V$ is the braiding on the monoidal category of polynomial representations of q-GL_n. This follows from the explicit description of ϕ_A in Lemma 4.6. We refer to the morphisms $\bar{\xi}_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ for $a, b \geq 0$ as *negative crossings*. The following lemma implies that the image of $\bar{\xi}_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ under the functor Σ_n is the isomorphism

$$(-1)^{ab} c_{\bigwedge^a V, \bigwedge^b V} : \bigwedge^a V \otimes \bigwedge^b V \xrightarrow{\sim} \bigwedge^b V \otimes \bigwedge^a V. \tag{5-7}$$

Lemma 5.6. *For $a, b \geq 0$, we have*

$$\bar{\xi}_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} = \xi^{-1}_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]}.$$

Also the merge and split morphisms $\xi_{[a\ b]}$ and $\xi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$ are invariant under the bar involution, hence, they coincide with the canonical basis elements $\theta_{[a\ b]}$ and $\theta_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$.

Proof. The part about merge and split morphisms is trivial as these matrices are not comparable to any other in the Bruhat ordering. For the first part, we show that $\bar{\xi}_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} = 1_{(a,b)}$. This identity (with a and b switched) together with the image of this identity under the bar involution implies the result. Take any $A \in \mathrm{Mat}((a, b), (a, b))$ and consider the coefficient of ξ_A when the product $\bar{\xi}_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} \in S_q(2, a+b)$ is expanded in terms of the standard basis. Since multiplication in $S_q(2, a+b)$ is dual to comultiplication in $\mathcal{O}_q(2, a+b)$, this coefficient is equal to the $x_{2,1}^b x_{1,2}^a \otimes x_{2,1}^a x_{1,2}^b$-coefficient of $\Delta(x_{2,1}^{a_{2,1}} x_{1,1}^{a_{1,1}} x_{2,2}^{a_{2,2}} x_{1,2}^{a_{1,2}})$ when expanded in terms of the basis $\{\bar{x}_B \otimes x_C \mid B, C \in \mathrm{Mat}((a, b), (a, b))\}$. By Lemma 3.5,

$$\Delta(x_{2,1}^{a_{2,1}} x_{1,1}^{a_{1,1}} x_{2,2}^{a_{2,2}} x_{1,2}^{a_{1,2}}) = \sum_{a'_{2,1}=0}^{a_{2,1}} \sum_{a'_{1,1}=0}^{a_{1,1}} \sum_{a'_{2,2}=0}^{a_{2,2}} \sum_{a'_{1,2}=0}^{a_{1,2}} \begin{bmatrix} a_{2,1} \\ a'_{2,1} \end{bmatrix}_q \begin{bmatrix} a_{1,1} \\ a'_{1,1} \end{bmatrix}_q \begin{bmatrix} a_{2,2} \\ a'_{2,2} \end{bmatrix}_q \begin{bmatrix} a_{1,2} \\ a'_{1,2} \end{bmatrix}_q$$
$$\times x_{2,1}^{a'_{2,1}} x_{2,2}^{a_{2,1}-a'_{2,1}} x_{1,1}^{a'_{1,1}} x_{1,2}^{a_{1,1}-a'_{1,1}} x_{2,1}^{a'_{2,2}} x_{2,2}^{a_{2,2}-a'_{2,2}} x_{1,1}^{a'_{1,2}} x_{1,2}^{a_{1,2}-a'_{1,2}}$$
$$\otimes x_{2,1}^{a_{2,1}-a'_{2,1}} x_{1,1}^{a'_{2,1}} x_{2,1}^{a_{1,1}-a'_{1,1}} x_{1,1}^{a'_{1,1}} x_{2,2}^{a_{2,2}-a'_{2,2}} x_{1,2}^{a'_{2,2}} x_{2,2}^{a_{1,2}-a'_{1,2}} x_{1,2}^{a'_{1,2}}.$$

To get $x_{2,1}^a x_{1,2}^b$ on straightening using (3-5) into the normal order in the second tensor position, we must have $a'_{2,1} = a'_{1,1} = 0$, $a'_{1,2} = a_{1,2}$ and $a'_{2,2} = a_{2,2}$. This term is

(5-8) $$x_{2,2}^{a_{2,1}} x_{1,2}^{a_{1,1}} x_{2,1}^{a_{2,2}} x_{1,1}^{a_{1,2}} \otimes x_{2,1}^b x_{1,2}^a = q^{-a_{2,1}a_{2,2}-a_{1,1}a_{1,2}} x_{2,1}^{a_{2,2}} x_{2,2}^{a_{2,1}} x_{1,1}^{a_{1,2}} x_{1,2}^{a_{1,1}} \otimes x_{2,1}^b x_{1,2}^a.$$

Because we are using the ordering from (3-11) for monomials in the first tensor (rather than the normal ordering), we only get a nonzero coefficient when $A = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, when the coefficient is 1. This shows the product is $1_{(a,b)}$. □

Remark 5.7. By a similar argument to the proof of Lemma 5.6, one can also prove the "quadratic relation"

$$\xi_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} = \sum_{s=0}^{\min(a,b)} q^{-s(s-1)/2-s(a+b-2s)} (q^{-1}-q)^s [s]_q^! \, \xi_{\left[\begin{smallmatrix} a-s & s \\ s & b-s \end{smallmatrix}\right]}.$$

Indeed, from (5-8), the coefficient of $\xi_{\left[\begin{smallmatrix} a-s & s \\ s & b-s \end{smallmatrix}\right]}$ in $\xi_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ is $q^{-s(a-s)-s(b-s)}$ times the coefficient of $x_{2,1}^b x_{1,2}^a$ when

$$x_{2,1}^{b-s} x_{2,2}^s x_{1,1}^s x_{1,2}^{a-s}$$

is expanded in terms of the normally ordered monomial basis. The latter coefficient is $q^{-s(s-1)/2}(q^{-1}-q)^s [s]_q^!$ by Lemma 3.4.

More generally, a *merge of n strings* is a morphism of the form ξ_A for a $1\times n$ matrix A, a *split of n strings* is a morphism of the form ξ_A for an $n\times 1$ matrix A, and a *positive permutation of n strings* is a morphism of the form ξ_A for an $n\times n$ matrix A such that in each row and column there is at most one nonzero entry. Positive permutations of n strings can be parametrized instead by $w \in S_n$ and a composition μ of length n, setting

(5-9) $$\tau_{w;\mu} := \xi_A, \quad \text{where } A \in \mathrm{Mat}(\mu\cdot w^{-1}, \mu) \text{ has } a_{w(1),1} = \mu_1, \dots, a_{w(n),n} = \mu_n.$$

If $\mu \in \Lambda(n, r)$ then the same formula defines an element of $1_{\mu\cdot w^{-1}} S_q(n, r) 1_\mu$; for example, for $w \in S_r \leq S_n$, the image of $\tau_w \in H_r$ under the isomorphism of Lemma 4.3 is $\tau_{w;\omega}$. For $1 \leq i < n$, we have

(5-10) $$\tau_{s_i;\mu} = 1_{(\mu_1,\dots,\mu_{i-1})} \star \xi_{\left[\begin{smallmatrix} 0 & \mu_{i+1} \\ \mu_i & 0 \end{smallmatrix}\right]} \star 1_{(\mu_{i+2},\dots,\mu_n)}.$$

So $\tau_{s_i;\mu}$, which we call a *simple permutation of n strings*, is a positive crossing tensored on the left and right with the appropriate identity morphisms. The following lemma implies that any positive permutation of n strings can be obtained by composing simple permutations.

Lemma 5.8. *Suppose that* $\mu \in \Lambda(n,r)$ *and* $w \in S_n$ *is a permutation such that* $w(i) < w(i+1)$ *for some* $1 \le i < n$. *Then* $\tau_{ws_i;\mu} = \tau_{w;\mu\cdot s_i} \circ \tau_{s_i;\mu}$.

Proof. It suffices to prove the analogous statement in the q-Schur algebra $S_q(n,r)$. There is a left action of S_n on $\mathrm{I}(n,r)$ by its action on entries. This commutes with the right action of S_r. We claim that the left action of $S_q(n,r)$ on $V^{\otimes r}$ satisfies $\tau_{w;\mu} v_{\boldsymbol{i}^\mu} = v_{w\cdot \boldsymbol{i}^\mu}$. To see this, the normally ordered monomial in $\mathcal{O}_q(n,r)$ that is dual to the standard basis vector $\tau_{w;\mu}$ is $x_{w\cdot\boldsymbol{i}^\mu,\boldsymbol{i}^\mu}$. Moreover, $w\cdot\boldsymbol{i}^\mu$ is the only $\boldsymbol{i} \in \mathrm{I}(n,r)$ such that $x_{w\cdot\boldsymbol{i}^\mu,\boldsymbol{i}^\mu}$ appears in the normally ordered monomial basis expansion of $x_{\boldsymbol{i},\boldsymbol{i}^\mu}$. So the claim follows from (4-14).

To prove the lemma, it suffices to show that $\tau_{w;\mu\cdot s_i}\tau_{s_i;\mu}$ and $\tau_{ws_i;\mu}$ act in the same way on $v_{\boldsymbol{i}^\mu}$. The latter gives $v_{ws_i\cdot\boldsymbol{i}^\mu}$ by the claim. Also $\tau_{s_i;\mu}v_{\boldsymbol{i}^\mu} = v_{s_i\cdot\boldsymbol{i}^\mu}$. So we are reduced to checking that $\tau_{w;\mu\cdot s_i}v_{s_i\cdot\boldsymbol{i}^\mu} = v_{ws_i\cdot\boldsymbol{i}^\mu}$. Let $d \in (S_{\mu\cdot s_i}\backslash S_r)_{\min}$ be the unique Grassmann permutation such that $\boldsymbol{i}^{\mu\cdot s_i}\cdot d = s_i\cdot\boldsymbol{i}^\mu$. The action of H_r on $V^{\otimes r}$ was defined using (1-5), from which we see that $v_{\boldsymbol{i}^{\mu\cdot s_i}}\tau_d = v_{\boldsymbol{i}^{\mu\cdot s_i}\cdot d}$. Similarly, because $w(i) < w(i+1)$, we get that $v_{w\cdot\boldsymbol{i}^{\mu\cdot s_i})}\tau_d = v_{(w\cdot\boldsymbol{i}^{\mu\cdot s_i})\cdot d}$. So

$$\begin{aligned}\tau_{w;\mu\cdot s_i}v_{s_i\cdot\boldsymbol{i}^\mu} &= \tau_{w;\mu\cdot s_i}v_{\boldsymbol{i}^{\mu\cdot s_i}\cdot d} = \tau_{w;\mu\cdot s_i}v_{\boldsymbol{i}^{\mu\cdot s_i}}\tau_d = v_{w\cdot\boldsymbol{i}^{\mu\cdot s_i}}\tau_d \\ &= v_{(w\cdot\boldsymbol{i}^{\mu\cdot s_i})\cdot d} = v_{w\cdot(\boldsymbol{i}^{\mu\cdot s_i}\cdot d)} = v_{w\cdot(s_i\cdot\boldsymbol{i}^\mu)} = v_{ws_i\cdot\boldsymbol{i}^\mu}. \qquad \square\end{aligned}$$

A special case of the next lemma implies that

$$\xi_{[a_1+\cdots+a_s\ b_1+\cdots+b_t]} \circ (\xi_{[a_1\ \cdots\ a_s]} \star \xi_{[b_1\ \cdots\ b_t]}) = \xi_{[a_1\ \cdots\ a_s\ b_1\ \cdots\ b_t]}, \tag{5-11}$$

$$(\xi_{[a_1\ \cdots\ a_s]^{\mathrm{T}}} \star \xi_{[b_1\ \cdots\ b_t]^{\mathrm{T}}}) \circ \xi_{[a_1+\cdots+a_s\ b_1+\cdots+b_t]^{\mathrm{T}}} = \xi_{[a_1\ \cdots\ a_s\ b_1\ \cdots\ b_t]^{\mathrm{T}}}, \tag{5-12}$$

for $a_1,\dots,a_s,b_1,\dots,b_t \ge 0$. Hence, all merges/splits of n strings can be expressed as compositions of tensor products of merges/splits of 2 strings and appropriate identity morphisms.

Lemma 5.9. *Suppose that* $\lambda,\mu \vDash r$, $A \in \mathrm{Mat}(\lambda,\mu)$ *and* $1 \le i \le \ell(\lambda)$, $1 \le j \le \ell(\mu)$.

(a) *Let* B *be obtained from* A *by replacing its* i*-th row by two rows of length* $\ell(\mu)$, *the first of which has entries* $a_{i,1},\dots,a_{i,j},0,\dots,0$ *with sum* λ_i', *and the second has entries* $0,\dots,0,a_{i,j+1},\dots,a_{i,\ell(\mu)}$ *with sum* λ_i'' (*so* $\lambda_i'+\lambda_i''=\lambda_i$). *Then*

$$\xi_A = (1_{(\lambda_1,\dots,\lambda_{i-1})} \star \xi_{[\lambda_i'\ \lambda_i'']} \star 1_{(\lambda_{i+1},\dots,\lambda_{\ell(\lambda)})}) \circ \xi_B.$$

(b) *Let* B *be obtained from* A *by replacing its* j*-th column by two columns of length* $\ell(\lambda)$, *the first of which has entries* $a_{1,j},\dots,a_{i,j},0,\dots,0$ *with sum* μ_j', *and the second has entries* $0,\dots,0,a_{i+1,j},\dots,a_{\ell(\lambda),j}$ *with sum* μ_j'' (*so* $\mu_j'+\mu_j''=\mu_j$).

Then

$$\xi_A = \xi_B \circ (1_{(\mu_1,\dots,\mu_{j-1})} \star \xi_{\left[\begin{smallmatrix}\mu_j' \\ \mu_j''\end{smallmatrix}\right]} \star 1_{(\mu_{j+1},\dots,\mu_{\ell(\mu)})}).$$

Proof. We just prove (b). Then (a) follows on applying T. By the way that composition in q-**Schur** is defined, the statement we are trying to prove reduces to the following claim about multiplication in $S_q(n,r)$:

Claim. *Suppose that* $\lambda, \mu \in \Lambda(n,r)$, $A \in \mathrm{Mat}(\lambda,\mu)$ *and* $1 \le i \le n$, $1 \le j \le n-1$ *with* $\mu_{j+1} = 0$. *Let* $B \in \mathrm{Mat}(\lambda,\mu')$ *be obtained from* A *by replacing the* j*-th and* $(j+1)$*-th columns with* $[a_{i,1} \ \cdots \ a_{i,j} \ 0 \ \cdots \ 0]^{\mathrm{T}}$ *and* $[0 \ \cdots \ 0 \ a_{i+1,j} \ \cdots \ a_{n,j}]^{\mathrm{T}}$, *respectively, and* $C \in \mathrm{Mat}(\mu',\mu)$ *be* $\mathrm{diag}\big(\mu_1,\dots,\mu_{j-1}, \left[\begin{smallmatrix}\mu_j' & 0 \\ \mu_{j+1}' & 0\end{smallmatrix}\right], \mu_{j+2},\dots,\mu_n\big)$. *Then* $\xi_A = \xi_B \xi_C$ *in* $S_q(n,r)$.

To see this, it suffices to show for $D \in \mathrm{Mat}(\lambda,\mu)$ that g_D, the $x_B \otimes x_C$-coefficient of $\Delta(x_D)$ when expanded in terms of normally ordered monomials, is equal to $\delta_{A,D}$. We have that

$$\begin{aligned} x_B &= x_{\boldsymbol{i}_1,(1^{\mu_1})} \cdots x_{\boldsymbol{i}_{j-1},((j-1)^{\mu_{j-1}})} \\ &\qquad \cdot (x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}}) x_{\boldsymbol{i}_{j+2},((j+2)^{\mu_{j+2}})} \cdots x_{\boldsymbol{i}_n,(n^{\mu_n})}, \\ x_C &= x_{1,1}^{\mu_1} \cdots x_{j-1,j-1}^{\mu_{j-1}} (x_{j+1,j}^{\mu_{j+1}'} x_{j,j}^{\mu_j'}) x_{j+2,j+2}^{\mu_{j+2}} \cdots x_{n,n}^{\mu_n}, \\ x_D &= x_{\boldsymbol{j}_1,(1^{\mu_1})} \cdots x_{\boldsymbol{j}_{j-1},((j-1)^{\mu_{j-1}})} (x_{n,j}^{d_{n,j}} \cdots x_{1,j}^{d_{1,j}}) x_{\boldsymbol{j}_{j+2},((j+2)^{\mu_{j+2}})} \cdots x_{\boldsymbol{j}_n,(n^{\mu_n})}, \end{aligned}$$

where $\boldsymbol{i}_k := (n^{a_{n,k}} \ \cdots \ 1^{a_{1,k}})$ and $\boldsymbol{j}_k = (n^{d_{n,k}} \ \cdots \ 1^{d_{1,k}})$. It is easy to see that the $x_{\boldsymbol{i}_k,(k^{\mu_k})} \otimes x_{k,k}^{\mu_k}$-coefficient of $\Delta(x_{\boldsymbol{j}_k,(k^{\mu_k})})$ is 0 unless $\boldsymbol{j}_k = \boldsymbol{i}_k$, when it is 1. This implies that $g_D = 0$ unless $\boldsymbol{j}_k = \boldsymbol{i}_k$ for each $k = 1,\dots,j-1, j+2,\dots,n$ in which case, by weight considerations, we have $d_{1,j} = a_{1,j},\dots,d_{n,j} = a_{n,j}$, hence, $D = A$. Thus, we are reduced to showing that $g_A = 1$.

Our argument shows that g_A is the coefficient of

$$x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}} \otimes x_{j+1,j}^{\mu_{j+1}'} x_{j,j}^{\mu_j'}$$

in the normally ordered expansion of

$$\Delta(x_{n,j}^{a_{n,j}} \cdots x_{1,j}^{a_{1,j}}) = \sum_{\boldsymbol{k} \in \mathrm{I}(n,r)} x_{(n^{a_{n,j}} \ \dots \ 1^{a_{1,j}}),\boldsymbol{k}} \otimes x_{\boldsymbol{k},(j^{\mu_j})}.$$

To complete the proof, we claim for $\boldsymbol{k} \in \mathrm{I}(n,r)$ that $x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}}$ appears with nonzero coefficient in the normally ordered expansion of $x_{(n^{a_{n,j}} \ \dots \ 1^{a_{1,j}}),\boldsymbol{k}}$ if and only if $\boldsymbol{k} = ((j+1)^{\mu_{j+1}'} \ j^{\mu_j'})$, in which case the coefficient is 1. Certainly, $\boldsymbol{k}$ must be a permutation of $((j+1)^{\mu_{j+1}'} \ j^{\mu_j'})$. For any such $\boldsymbol{k}$ and any $\boldsymbol{h}$ that is a permutation of $(n^{a_{n,j}} \cdots 1^{a_{1,j}})$, we define the *height* of the monomial $x_{\boldsymbol{h},\boldsymbol{k}}$ to be $\sum_s h_s$ where the sum is over all $1 \le s \le \mu_j$ such that $k_s = j$. The monomial $x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}}$ is of height $\sum_{k=1}^{i} k a_{k,j}$. Also $x_{(n^{a_{n,j}} \cdots 1^{a_{1,j}}),\boldsymbol{k}}$ is of the same height if $\boldsymbol{k} = ((j+1)^{\mu'_{j+1}}\ j^{\mu'_j})$, and otherwise its height is strictly bigger. In order to straighten $x_{(n^{a_{n,j}} \cdots 1^{a_{1,j}}),\boldsymbol{k}}$, we need to use the commutation relations $x_{p,j+1}x_{p,j} = q^{-1}x_{p,j}x_{p,j+1}$ and $x_{p,j+1}x_{q,j} = x_{q,j}x_{p,j+1} - (q-q^{-1})x_{p,j}x_{q,j+1}$ for $p > q$. Monomials arising from the "error term" $x_{p,j}x_{q,j+1}$ are of strictly greater height, so do not contribute to the coefficient, and the others have the same height. The claim follows. ☐

Now take any $A \in \operatorname{Mat}(\lambda, \mu)$ and define λ^-, μ^+ as in Lemma 2.1. Note that $n := \ell(\lambda^-) = \ell(\mu^+) = \ell(\lambda)\ell(\mu)$. We can convert A into a matrix $A^\circ \in \operatorname{Mat}(\lambda^-, \mu^+)$ with at most one nonzero entry in each row and column by applying a sequence of the operations $A \mapsto B$ described in Lemma 5.9(a)–(b). For example,

$$\begin{bmatrix}1&0&3\\2&2&1\end{bmatrix} \mapsto \begin{bmatrix}1&0&0\\0&0&3\\2&2&1\end{bmatrix} \mapsto \begin{bmatrix}1&0&0\\0&0&0\\0&0&3\\2&2&1\end{bmatrix} \mapsto \begin{bmatrix}1&0&0\\0&0&0\\0&0&3\\2&0&0\\0&2&1\end{bmatrix}$$

$$\mapsto \begin{bmatrix}1&0&0\\0&0&0\\0&0&3\\2&0&0\\0&2&0\\0&0&1\end{bmatrix} \mapsto \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&3\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix} \mapsto \begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&3\\0&2&0&0&0\\0&0&0&2&0\\0&0&0&0&1\end{bmatrix} \mapsto \begin{bmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&3&0\\0&2&0&0&0&0\\0&0&0&2&0&0\\0&0&0&0&0&1\end{bmatrix}.$$

The $n \times n$ matrix A° obtained in this way is uniquely determined. It corresponds to the permutation of n strings arising from the middle part of the double coset diagram of A. Lemma 5.9 plus (5-11) and (5-12) gives us an explicit algorithm to express the standard basis element ξ_A as a composition

$$\xi_A = \xi_{A^-} \circ \xi_{A^\circ} \circ \xi_{A^+}, \tag{5-13}$$

where ξ_{A^-} is a tensor product of $\ell(\lambda)$ merges of $\ell(\mu)$ strings and ξ_{A^+} is a tensor product of $\ell(\mu)$ splits of $\ell(\lambda)$ strings. The double coset diagrams of $A^- \in \operatorname{Mat}(\lambda, \lambda^-)$ and $A^+ \in \operatorname{Mat}(\mu^+, \mu)$ are given explicitly by the top part or the bottom part of the diagram of A, respectively.

Lemma 5.10. *For $a, b \ge 0$, we have* $\xi_{[a\ b]} \circ \xi_{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]} = \begin{bmatrix}a+b\\a\end{bmatrix}_q \xi_{[a+b]}$.

Proof. The $x_{1,1}^a x_{1,2}^b \otimes x_{2,1}^b x_{1,1}^a$-coefficient of $\Delta(x_{1,1}^{a+b})$ is $\begin{bmatrix}a+b\\a\end{bmatrix}_q$ by Lemma 3.5. ☐

Lemma 5.11. *For $a, b, c, d \geq 0$ with $a + b = c + d$, we have*

$$\theta_{\left[\begin{smallmatrix} 0 & c \\ a & d-a \end{smallmatrix}\right]} = \xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]} \circ \xi_{[a\ b]} = \sum_{s=0}^{\min(a,c)} q^{s(s+d-a)} \xi_{\left[\begin{smallmatrix} s & c-s \\ a-s & s+d-a \end{smallmatrix}\right]} \quad \textit{if } a \leq d \textit{ and } b \geq c,$$

$$\theta_{\left[\begin{smallmatrix} c-b & b \\ d & 0 \end{smallmatrix}\right]} = \xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]} \circ \xi_{[a\ b]} = \sum_{t=0}^{\min(b,d)} q^{t(t+c-b)} \xi_{\left[\begin{smallmatrix} t+c-b & b-t \\ d-t & t \end{smallmatrix}\right]} \quad \textit{if } a \geq d \textit{ and } b \leq c.$$

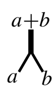
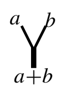
Proof. We just prove this when $a \leq d$, the other case is similar. Since the merge $\xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]}$ and the split $\xi_{[a\ b]}$ are bar invariant, and the canonical basis element θ_A is the unique bar invariant element equal to ξ_A plus a $q\mathbb{Z}[q]$-linear combination of other ξ_B, the first equality follows from the second one. To prove the second equality, we must show that the coefficient of $\xi_{\left[\begin{smallmatrix} s & c-s \\ a-s & s+d-a \end{smallmatrix}\right]}$ of $\xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]} \circ \xi_{[a\ c+d-a]}$ is equal to $q^{s(s+d-a)}$. This is the coefficient of $x_{2,1}^d x_{1,1}^c \otimes x_{1,1}^a x_{1,2}^{c+d-a}$ in $\Delta(x_{2,1}^{c-s} x_{1,1}^s x_{2,2}^{s+d-a} x_{1,2}^{c-s})$, which may be computed by the same argument as was used in the proof of Lemma 5.6. $\square$

6. Presentations

We start now to represent morphisms in q-**Schur** by string diagrams. Let $\mathbb{1}$ be the strict identity object, that is, the composition () of length zero. For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \Lambda$, the identity endomorphism 1_λ in q-**Schur** will be represented by a sequence of strings labeled from left to right by $\lambda_1, \ldots, \lambda_\ell$, which we think of as indicating the *thicknesses* of the strings. We are including strings of zero thickness. For $a, b \geq 0$, we use the string diagrams

(6-1)

 $: (0) \to \mathbb{1}$,  $: \mathbb{1} \to (0)$,

 $: (a, b) \to (a+b)$,  $: (a+b) \to (a, b)$

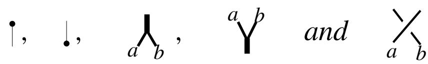
to denote the standard basis vectors ξ_A where A is the 0×1 matrix, the 1×0 matrix, the matrix $[a\ \ b]$ or the matrix $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, respectively. Henceforth, in string diagrams for morphisms in q-**Schur**, we will omit thickness labels on strings when they are implicitly determined by the other labels. We represent the positive crossing $\xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ by the string diagram

(6-2)  $: (a, b) \to (b, a)$.

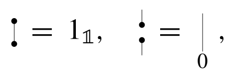
This morphism is invertible by Lemma 5.6, so it makes sense to define

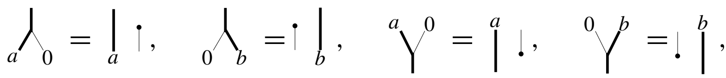
 $:= \Big($  $\Big)^{-1}$.

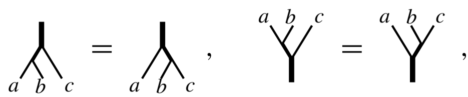
Theorem 6.1. *The $\mathbb{Z}[q, q^{-1}]$-linear monoidal category q-**Schur** is generated by the objects (r) for $r \geq 0$ and the morphisms*

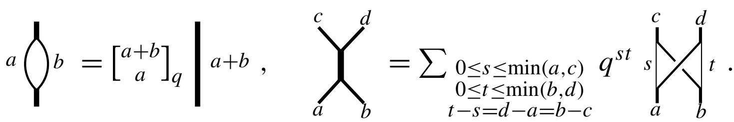


for $a, b \geq 0$, subject only to the following relations for admissible $a, b, c, d \geq 0$:

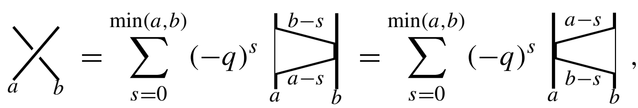
(6-3) 

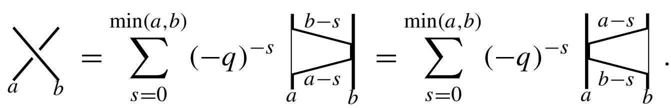
(6-4) 

(6-5) 

(6-6) 

Positive and negative crossings can be written in terms of other generating morphisms since

(6-7) 

(6-8) 

The following hold:

(a) *There is a unique braiding $c : -\star- \xrightarrow{\sim} -\star^{\mathrm{rev}}-$ making q-**Schur** into a braided monoidal category such that $c_{(a),(b)} =$*  .

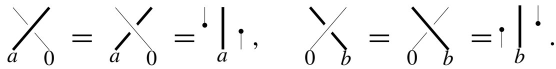
(b) *For any $A \in \mathrm{Mat}(\lambda, \mu)$, the standard basis element ξ_A is represented as a string diagram by the double coset diagram for A with all crossings drawn as positive crossings.*

(c) *The antilinear involution $-: q$-**Schur** $\to q$-**Schur** is defined on string diagrams by interchanging positive and negative crossings.*

(d) *The linear isomorphism $\mathtt{T} : q$-**Schur** $\to q$-**Schur**$^{\mathrm{op}}$ maps a string diagram to its rotation through 180° around a horizontal axis.*

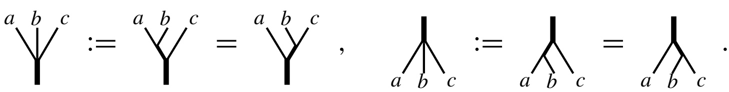
Before we prove this, we give some comments. The relations (6-3) imply that $(0) \cong \mathbb{1}$. The relations (6-4) mean that splits and merges with a string of thickness zero can be expressed in terms of the other generating morphisms, hence, can be eliminated from any string diagram. Using (6-3), (6-4) and the definition of negative crossings, the second relation in (6-6) implies that

(6-9)



This means that crossings involving a string of thickness zero can also be expressed in terms of other morphisms, so these can be eliminated from string diagrams too. Then all remaining strings of thickness zero can be contracted to dots on the top and bottom boundaries. In this way, any string diagram is equivalent to one without strings of thickness zero. The relations (6-5) mean that we can introduce further diagrams as shorthand for more general splits and merges of n strings. For example, splits and merges of 3 strings are

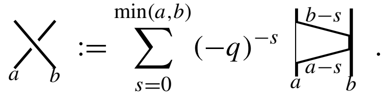
(6-10)



By (5-11) and (5-12), these are the standard basis vectors $\xi_{[a\ b\ c]}$ and $\xi_{\left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]}$, respectively.

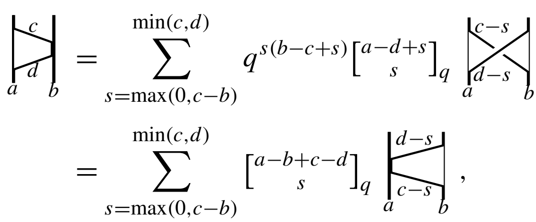
Proof of Theorem 6.1. Let q-**Schur**′ be the strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category defined by the generators and relations in the statement of the theorem. We also *define* the negative crossings in q-**Schur**′ by setting

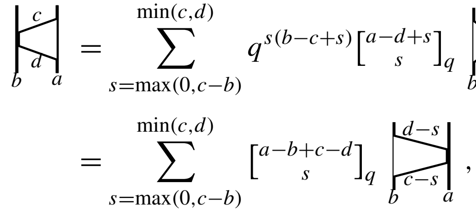
(6-11)

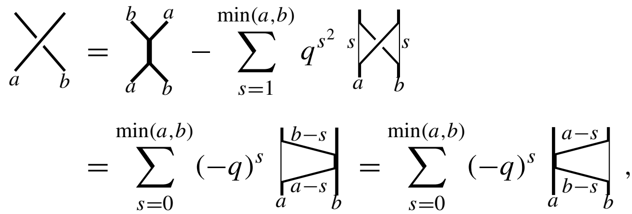


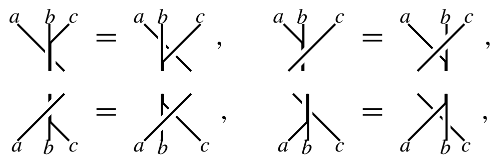
At this point, some calculations are needed to deduce the following additional relations from the defining relations in q-**Schur**′ (for all $a, b, c, d \geq 0$ that make sense):

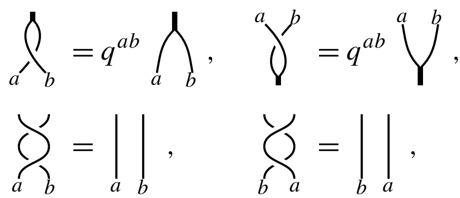
(6-12)

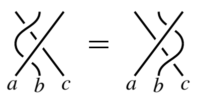


(6-13) 

(6-14) 

(6-15) 

(6-16) 

(6-17) 

The derivations of these relations are similar to those in the appendix of [Brundan et al. 2020] (which treats the $q = 1$ case); see the appendix to the version of this article available on the arxiv.

Now we prove (a) but for the presented category q-**Schur**$'$ rather than q-**Schur** itself; then (a) for q-**Schur** follows at the end when we have established that

$$q\text{-}\mathbf{Schur}' \cong q\text{-}\mathbf{Schur}.$$

We need natural isomorphisms $c_{\lambda,\mu} : \lambda\star\mu \xrightarrow{\sim} \mu\star\lambda$ for all compositions λ, μ. Given that $c_{(a),(b)}$ is the positive crossing, there is no choice for the definition of more general $c_{\lambda,\mu}$ in order for the hexagon axioms for a braided monoidal category to hold: it must be defined by composing positive crossings according to a reduced expression for the Grassmann permutation taking $1, \dots, \ell(\lambda)$ to $\ell(\mu)+1, \dots, \ell(\mu)+\ell(\lambda)$ and $\ell(\lambda)+1, \dots, \ell(\lambda)+\ell(\mu)$ to $1, \dots, \ell(\mu)$. As any two reduced expressions for a Grassmann permutation are equivalent by commuting braid relations, the resulting morphism is well defined by the interchange law. The morphism $c_{\lambda,\mu}$ is an isomorphism since the positive crossing  is invertible; its two-sided inverse is  according to the last two relations in (6-16). Naturality follows from (6-15).

Next, we show that there is a strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal functor F : q-**Schur**$'\to q$-**Schur** taking $(r)\mapsto(r)$ and the generating morphisms of q-**Schur**$'$ to the morphisms in q-**Schur** represented by the same diagrams. To prove this, we just need to check relations: (6-3) and (6-4) are trivial to check in q-**Schur**, (6-5) follows from (5-11) and (5-12), and (6-6) follows from Lemmas 5.10 and 5.11. By definition, the functor F takes the positive crossing in q-**Schur**$'$ to the positive crossing in q-**Schur**, so the identity (6-7) in q-**Schur** follows by applying F to (6-14). We have observed already that the negative crossing in q-**Schur**$'$ is the two-sided inverse of the positive crossing in q-**Schur**$'$, hence,

$F\Big($  $\Big) =$ 

since the negative crossing in q-**Schur** is also the inverse of the positive crossing by the original definition. To prove that (6-8) holds in q-**Schur**, the first equality follows by applying F to (6-11). The second equality follows by applying the bar involution to the second equality of (6-7), remembering that this fixes splits and merges in q-**Schur** thanks to Lemma 5.6.

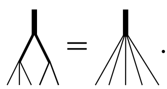
For any $A\in\mathrm{Mat}(\lambda,\mu)$, let ξ'_A be the morphism in q-**Schur**$'$ obtained by taking the (reduced) double coset diagram for A, replacing all crossings by positive crossings, and interpreting the result as a morphism by composing generators as the diagram suggests. The resulting morphism is well defined independent of the choices made when doing this. For the split of $\ell(\mu)$ strings at the bottom and the merge of $\ell(\lambda)$ strings at the top, this depends on (6-5) as explained in the comments after the statement of the theorem. For the permutation of $\ell(\lambda)\ell(\mu)$ strings in the middle, one needs to draw the diagram according to a choice of a reduced expression, but the resulting morphism is independent of this by (6-17). We are ready to prove (b) by showing that $F(\xi'_A)=\xi_A$. This follows from the factorization of ξ_A explained in (5-13), together with Lemma 5.8, (5-11) and (5-12), since these results show ξ_A can be obtained from merges, splits and positive crossings in exactly the same way as ξ'_A is obtained from the corresponding generating morphisms for q-**Schur**$'$.

Now we can prove that F is an isomorphism. It is clear that it defines a bijection between the object sets of q-**Schur**$'$ and q-**Schur** (both are identified with Λ). Since the morphisms ξ_A $(A\in\mathrm{Mat}(\lambda,\mu))$ form a basis for $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda)$ by the definition of q-**Schur**, we deduce using the previous paragraph that F is full. It just remains to show that it is faithful, which we do by proving that the morphisms ξ'_A ($A \in \mathrm{Mat}(\lambda, \mu)$) span $\mathrm{Hom}_{q\text{-}\mathbf{Schur}'}(\mu, \lambda)$ as a $\mathbb{Z}[q, q^{-1}]$-module. This follows from our next claim, since the merge and split morphisms f described in the claim for all λ, λ' generate q-**Schur**$'$ as a $\mathbb{Z}[q, q^{-1}]$-linear category by (6-7).

Claim. *For any $\lambda, \lambda', \mu \in \Lambda$, $A \in \mathrm{Mat}(\lambda, \mu)$ and $f : \lambda \to \lambda'$ that consists of a merge or split of 2 strings tensored on the left and/or right by some identity morphisms, the composition $f \circ \xi'_A$ is a $\mathbb{Z}[q, q^{-1}]$-linear combination of the morphisms ξ'_A ($A \in \mathrm{Mat}(\lambda, \mu)$).*

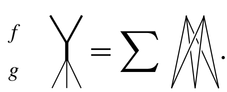
To prove the claim, there are two cases:

- Suppose first that f has a merge of two strings connecting to the i-th and $(i+1)$-th thick strings at the top of ξ'_A. The double coset diagram of A has a merge of r strings at its i-th vertex and merge of s strings at its $(i+1)$-th vertex. We use (6-5) to convert $f \circ \xi'_A$ into a diagram which has a merge of $r + s$ strings at its i-th vertex. For example,

(6-18) 

The permutation arising in the middle section of the resulting diagram is not necessarily reduced, but it can be converted to a scalar multiple of some ξ'_B using (6-5), (6-6), (6-16) and (6-17).

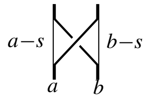
- Now suppose that f has a split connecting to the i-th vertex at the top of the double coset diagram of A. Say this vertex in the double coset diagram is part of an n-fold merge. Using (6-5), (6-6) and (6-15), we rewrite the composition of the split in f and this merge in ξ'_A as a sum of other ξ'_B. For example,

(6-19) 

Then compose these diagrams with the remainder of the diagram, using (6-15) then (6-5) again to commute the splits at the bottom of this part of the resulting diagrams downwards past the positive crossings in ξ'_A.

All that is left is to prove (c) and (d). Part (c) follows because the bar involution on q-**Schur** fixes merges and splits and interchanges positive and negative crossings by Lemma 5.6; it obviously fixes the other two generating morphisms ↑ and ↓. Part (d) follows using (b) because $\mathtt{T}$ takes ξ_A to $\xi_{A^{\mathtt{T}}}$. □

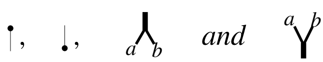
Corollary 6.2. *In q-**Schur**, we have*

 $= \sum_{s=0}^{\min(a,b)} q^{-s(s-1)/2+r(a+b-2s)}(q^{-1}-q)^s [s]_q^!$  .

Proof. This is a translation of Remark 5.7 into the graphical description of q-**Schur** provided by the theorem. □

We also have the following theorem, which gives an alternative presentation for q-**Schur** with fewer generators and relations.

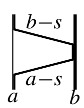
Theorem 6.3. *The strict $\mathbb{Z}[q,q^{-1}]$-monoidal category q-**Schur** is generated by the objects (r) for $r \geq 0$ and the morphisms*



subject only to the relations (6-3) and (6-4) for $a, b \geq 0$, (6-5) for $a, b, c > 0$, and one of the two square-switch relations from (1-3) for all $a, b, c, d \geq 0$ with $d \leq a$ and $c \leq b + d$.

Proof. Let q-**Schur**$'$ be the strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal category defined by the new presentation in the statement of the theorem, assuming for clarity that the *second* relation in (1-3) is the chosen one. All of the relations of q-**Schur**$'$ hold in q-**Schur** thanks to (6-13). So there is a strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal functor F : q-**Schur**$' \to q$-**Schur** taking $(r) \mapsto (r)$ and the generating morphisms for q-**Schur**$'$ to the morphisms represented by the same diagrams in q-**Schur**. In the next paragraph, we show that F is an isomorphism, proving the theorem for this choice of square-switch. The proof of the theorem if one instead chooses the first square-switch relation from (1-3), i.e., the one that is known to hold in q-**Schur** by (6-12), is very similar — one simply needs to rotate all calculations in a vertical axis.

To prove that F is an isomorphism, we use the presentation from Theorem 6.1 to construct a two-sided inverse $G : q$-**Schur** $\to q$-**Schur**$'$. This is defined on objects so that $(r) \mapsto (r)$ and, on generating morphisms, it maps the positive crossing to

(6-20)  $:= \sum_{s=0}^{\min(a,b)} (-q)^s$  $\in \mathrm{Hom}_{q\text{-}\mathbf{Schur}'}((a) \star (b) \to (b) \star (a))$

and the other generating morphisms for q-**Schur** to the morphisms represented by the same diagrams in q-**Schur**$'$. That G is indeed a two-sided inverse of F follows using (6-7). It remains to show that G is well defined, which is another relations check. The relations (6-3) and (6-4) hold in q-**Schur**$'$ by its definition. If one or more of a, b, c is zero, the relations (6-5) follow easily from (6-3) and (6-4), so the relations (6-5) also hold in q-**Schur**$'$ for all $a, b, c \geq 0$. The first relation from (6-6) follows from the chosen square-switch relation taking $b = 0$ and $c = d$. It remains to show that the second relation from (6-6) holds in q-**Schur**$'$ using only (6-3), (6-4) and (6-5) and square-switch. This is explained in the appendix to the `arxiv` version of this paper; see (a) of the corollary there. □

7. A straightening formula for codeterminants

Definition 7.1. Let $\mathcal{O}$ be a commutative Noetherian ring and $K = \bigoplus_{\lambda,\mu \in \Lambda} 1_\lambda K 1_\mu$ be a locally unital $\mathcal{O}$-algebra with (mutually orthogonal) distinguished idempotents 1_λ ($\lambda \in \Lambda$) for some index set Λ. We say that K is a *based quasihereditary algebra* with *weight poset* Λ^+ if we are given a subset $\Lambda^+ \subseteq \Lambda$, an upper finite partial order $\leq$ on Λ^+, and finite sets $X(\lambda, \kappa) \subset 1_\lambda K 1_\kappa$ and $Y(\kappa, \lambda) \subset 1_\kappa K 1_\lambda$ for $\lambda \in \Lambda, \kappa \in \Lambda^+$, such that the following axioms hold:

- The products xy for $(x, y) \in \bigcup_{\lambda,\mu\in\Lambda} \bigcup_{\kappa\in\Lambda^+} X(\lambda, \kappa) \times Y(\kappa, \mu)$ give a basis for K as a free $\mathcal{O}$-module. We refer to this as the *triangular basis*.
- For $\lambda, \mu \in \Lambda^+$, we have $X(\lambda, \mu) \neq \varnothing \Rightarrow \lambda \leq \mu$, $Y(\lambda, \mu) \neq \varnothing \Rightarrow \lambda \geq \mu$, and $X(\lambda, \lambda) = Y(\lambda, \lambda) = \{1_\lambda\}$.

We say that it is a *symmetrically based quasihereditary algebra* if in addition there is an algebra anti-involution $\mathtt{T} : K \to K$ such that $Y(\kappa, \lambda) = \mathtt{T}(X(\lambda, \kappa))$ for all $\lambda \in \Lambda$ and $\kappa \in \Lambda^+$ (in this case, there is no need to specify $Y(\kappa, \lambda)$ in the first place).

Remark 7.2. When $\mathcal{O}$ is a field, Definition 7.1 is [Brundan and Stroppel 2024, Definition 5.1]. When the set Λ is finite, it is a simplified version of the definition of based quasihereditary algebra given in [Kleshchev and Muth 2020]. In that case, as explained in detail in [Kleshchev and Muth 2020], K is also a standardly full-based algebra in the sense of [Du and Rui 1998], and a split quasihereditary algebra in the sense of [Cline et al. 1990]. In the symmetrically based case, K is a cellular algebra in the sense of [Graham and Lehrer 1996], and when K is the path algebra of an $\mathcal{O}$-linear category $\mathbf{C}$ with object set Λ, Definition 7.1 is equivalent to $\mathbf{C}$ being a strictly object-adapted cellular category in the sense of [Elias and Lauda 2016, Definition 2.1] (the opposite partial order is used there). The far-reaching consequences for the representation theory of K are well known, and are discussed in these references.

For the remainder of the section, K is the path algebra

$$K := \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda) \tag{7-1}$$

of the q-Schur category with 0-strings. This is a locally unital $\mathbb{Z}[q, q^{-1}]$-algebra with the distinguished system $\{1_\lambda \mid \lambda \in \Lambda\}$ of mutually orthogonal idempotents coming from the identity endomorphisms of the objects of q-**Schur**. Recall the set $\mathrm{Row}(\lambda, \mu)$ of *row tableaux* of shape μ and content λ from Section 2, and the bijection $A : \mathrm{Row}(\lambda, \mu) \xrightarrow{\sim} \mathrm{Mat}(\lambda, \mu)$ from (2-2). We start now to index the standard and canonical bases by the sets $\mathrm{Row}(\lambda, \mu)$ instead of $\mathrm{Mat}(\lambda, \mu)$, introducing the shorthand

$$\varphi_P := \xi_{A(P)}, \quad \beta_P := \theta_{A(P)} \tag{7-2}$$

for $P \in \mathrm{Row}(\lambda, \mu)$. For a partition κ, let $\mathrm{Std}(\lambda, \kappa)$ be the usual set of *semistandard tableau of shape* κ *and content* λ, that is, the subset of $\mathrm{Row}(\lambda, \kappa)$ consisting of the row tableaux of shape κ and content λ whose entries are also strictly increasing down columns.

Lemma 7.3. *For $\lambda, \mu \vDash r$, the $\mathbb{Z}[q, q^{-1}]$-module $1_\lambda K 1_\mu$ is spanned by the products $\varphi_P \mathtt{T}(\varphi_Q)$ for $P \in \mathrm{Row}(\lambda, \kappa)$, $Q \in \mathrm{Row}(\mu, \kappa)$, where κ is the dominant conjugate of μ.*

Proof. The dominant conjugate κ of μ is the unique partition whose parts are a permutation of the nonzero parts of μ. Using a morphism of the form $\tau_{w;\mu}$ from (5-9), we deduce $\mu \cong \kappa$ in q-**Schur**. Consequently, any element of $1_\lambda K 1_\mu = \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ is a morphism which factors through κ. Since the morphisms φ_P for $P \in \mathrm{Row}(\lambda, \kappa)$ give the standard basis for $1_\lambda K 1_\kappa = \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\kappa, \lambda)$ and the morphisms $\mathtt{T}(\varphi_Q)$ for $Q \in \mathrm{Row}(\mu, \kappa)$ give the standard basis for $1_\kappa K 1_\mu = \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \kappa)$, we deduce that the products $\varphi_P \mathtt{T}(\varphi_Q)$ span $1_\lambda K 1_\mu$. □

Now we come to the main combinatorial lemma. To formulate it, we use certain lexicographic total orders on tableaux and partitions. On partitions, $\geq_{\mathrm{lex}}$ is just the usual lexicographical ordering; it is a refinement of the dominance ordering on partitions into a total order. To define the required ordering $\leq_{\mathrm{lex}}$ on tableaux of the same shape, given any tableau T, we let $\overleftarrow{\Sigma}(T)$ be the sequence obtained by reading its entries in order from right to left along rows, starting with the top row. Then we declare that $S \leq_{\mathrm{lex}} T$ if and only if $\overleftarrow{\Sigma}(S) \leq_{\mathrm{lex}} \overleftarrow{\Sigma}(T)$ in the lexicographic ordering on sequences.

Lemma 7.4. *For $\lambda \vDash r$, $\kappa \vdash r$ and $P \in \mathrm{Row}(\lambda, \kappa)$ which is* ***not*** *semistandard, φ_P can be written as a $\mathbb{Z}[q, q^{-1}]$-linear combination of the elements*

- *φ_S for $S \in \mathrm{Row}(\lambda, \kappa)$ with $S <_{\mathrm{lex}} P$;*
- *$\varphi_{P'} \mathtt{T}(\varphi_{Q'})$ for $P' \in \mathrm{Row}(\lambda, \kappa')$ and $Q' \in \mathrm{Row}(\kappa, \kappa')$ of shape $\kappa' \vdash r$ with $\kappa' >_{\mathrm{lex}} \kappa$.*

Proof. Take P as in the statement. Since P is not semistandard, we may choose $a \geq 1$ and $0 \leq m < n \leq \kappa_{a+1}$ so that the entries of P in rows a and $a+1$ look like

$$\begin{array}{c} i_1 \leq \cdots \leq i_m < i_{m+1} \leq \cdots \leq i_n \leq i_{n+1} \leq \cdots \leq i_{\kappa_a} \\ \veebar \\ j_1 \leq \cdots \leq j_m \leq j_{m+1} = \cdots = j_n < j_{n+1} \leq \cdots \leq j_{\kappa_{a+1}}. \end{array}$$

Let U be the row tableau which is identical to P everywhere except in rows a and $a+1$, which are replaced by *three* (possibly empty) rows as in the diagram:

$$\begin{array}{l} i_1 \leq \cdots \leq i_m \\ j_1 \leq \cdots \leq j_n \leq i_{m+1} \leq \cdots \leq i_{\kappa_a} \\ j_{n+1} \leq \cdots \leq j_{\kappa_{a+1}}. \end{array}$$

Let μ be the shape of the tableau U. Let V be the row tableau of shape κ and content μ with all entries on row b equal to b for $b < a$, entries $a^m\ (a+1)^{\kappa_a - m}$ on row a, entries $(a+1)^n\ (a+2)^{\kappa_{a+1}-n}$ on row $a+1$, and all entries on row b equal to $b+1$ for $b > a+1$. Expanding in terms of the standard basis, we have

$$\varphi_U \varphi_V = \sum_{S \in \mathrm{Row}(\lambda,\kappa)} g_S \varphi_S \tag{7-3}$$

for coefficients $g_S \in \mathbb{Z}[q, q^{-1}]$. We claim that $g_S = 0$ unless $S \leq_{\mathrm{lex}} P$ and that $g_P = 1$. This suffices to prove the lemma. Indeed, assuming the claim, we rearrange (7-3) to obtain

$$\varphi_P = \varphi_U \varphi_V - \sum_{S <_{\mathrm{lex}} P} g_S \varphi_S.$$

The second term on the right-hand side is already of the desired form. To understand the first term, note that the first $a-1$ rows of U are of lengths $\kappa_1, \ldots, \kappa_{a-1}$, and it also has a row of length $\kappa_a + n - m > \kappa_a$. Consequently, the dominant conjugate of the shape μ of U is greater than κ in the ordering $>_{\mathrm{lex}}$. So, by Lemma 7.3, the first term can be rewritten as a sum $\varphi_{P'} \mathtt{T}(\varphi_{Q'})$ for row tableaux P', Q' of dominant shape $\kappa' >_{\mathrm{lex}} \kappa$. This is also of the desired form.

It just remains to prove the claim. Take $S \in \mathrm{Row}(\lambda, \kappa)$. Recalling that $x_S = x_{\leftarrow\!\!\!\Sigma(S), \boldsymbol{i}^\kappa}$ and $x_U = x_{\leftarrow\!\!\!\Sigma(U), \boldsymbol{i}^\mu}$, the definition of multiplication in K gives that g_S is the $x_{\leftarrow\!\!\!\Sigma(U), \boldsymbol{i}^\mu} \otimes x_{\leftarrow\!\!\!\Sigma(V), \boldsymbol{i}^\kappa}$-coefficient of

$$\Delta(x_{\leftarrow\!\!\!\Sigma(S), \boldsymbol{i}^\kappa}) = \sum_{\boldsymbol{k} \in \mathrm{I}_\mu} x_{\leftarrow\!\!\!\Sigma(S), \boldsymbol{k}} \otimes x_{\boldsymbol{k}, \boldsymbol{i}^\kappa}$$

when expanded in terms of the normally ordered monomial basis. To straighten $x_{\boldsymbol{k}, \boldsymbol{i}^\kappa}$ into normal order, we only need the fourth relation from (3-5), and see that this coefficient is nonzero if and only if $\boldsymbol{k} = \underleftarrow{\Sigma}(R)$ for a tableau R of shape κ (not necessarily a row tableau) that is obtained from V by shuffling entries within rows a and $a+1$. Moreover, the coefficient is 1 in the case that $R = V$. To complete the proof, we show for such a tableau R that the $x_{\underleftarrow{\Sigma}(U),\boldsymbol{i}^\mu}$-coefficient of $x_{\underleftarrow{\Sigma}(S),\underleftarrow{\Sigma}(R)}$ is zero unless $S \leq_{\text{lex}} P$, it is 1 if $S = P$ and $R = V$, and it is zero if $S = P$ and $R \neq V$. Suppose the entries in rows a and $a+1$ of S are

$$\begin{gathered} i'_1 \leq \cdots \leq i'_{\kappa_a} \\ j'_1 \leq \cdots \leq j'_{\kappa_{a+1}}. \end{gathered}$$

In order to convert the monomial $x_{\underleftarrow{\Sigma}(S),\underleftarrow{\Sigma}(R)}$ into normal order, we must apply the relations to commute products of the form $x_{i'_c,a+1}x_{i'_b,a}$ for $1 \leq b < c \leq \kappa_a$ or $x_{j'_c,a+2}x_{j'_b,a+1}$ for $1 \leq b < c \leq \kappa_{a+1}$. This can be done using the second and third relations from (3-5). We deduce that

$$x_{\underleftarrow{\Sigma}(S),\underleftarrow{\Sigma}(R)} = \sum_{\substack{v \in (S_{\kappa_r}/S_m \times S_{\kappa_a - m})_{\min} \\ w \in (S_{\kappa_{a+1}}/S_n \times S_{\kappa_{a+1}-n})_{\min}}} g_{v,w} x_{\underleftarrow{\Sigma}(T_{v,w}),\boldsymbol{i}^\mu}$$

for some scalars $g_{v,w} \in \mathbb{Z}[q, q^{-1}]$ with $g_{1,1} = \delta_{R,V}$, where $T_{v,w}$ is the tableau of shape μ obtained from S by replacing its rows a and $a+1$ by three rows according to the diagram:

$$\begin{array}{ll} i'_{v(1)} \leq \cdots \leq i'_{v(m)} & \\ j'_{w(1)} \leq \cdots \leq j'_{w(n)} & i'_{v(m+1)} \leq \cdots \leq i'_{v(\kappa_a)} \\ j'_{w(n+1)} \leq \cdots \leq j'_{w(\kappa_{a+1})}. & \end{array}$$

In particular, if $S = P$ then $T_{1,1} = U$. Using the fourth relation, the $x_{\underleftarrow{\Sigma}(U),\boldsymbol{i}^\mu}$-coefficient of $x_{\underleftarrow{\Sigma}(T_{v,w}),\boldsymbol{i}^\mu}$ is nonzero if and only if $T_{v,w} \sim_{\text{row}} U$, i.e., they have the same entries in each row counted with multiplicity, and the coefficient is 1 if $T_{v,w} = U$. Now it remains to check that

- $T_{v,w} \sim_{\text{row}} U \Rightarrow S \leq_{\text{lex}} P$;
- $T_{v,w} \sim_{\text{row}} U$ and $S = P \Rightarrow (v, w) = (1, 1)$.

To see this, suppose that $T_{v,w} \sim_{\text{row}} U$. All rows of S are clearly equal to the corresponding rows of P except perhaps for rows a and $a+1$. Also the sequences $i'_{v(1)} \leq \cdots \leq i'_{v(m)}$ and $j'_{w(n+1)} \leq \cdots \leq j'_{w(\kappa_{a+1})}$ are equal to $i_1 \leq \cdots \leq i_m$ and $j_{n+1} \leq \cdots \leq j_{\kappa_{a+1}}$, respectively. So the a-th row of S is obtained by taking all of the entries in the a-th row of U together with $\kappa_a - m$ entries from row $a+1$, and row $a+1$ of S is obtained by taking all of the remaining entries from row $a+1$ of

U plus all of the entries in row $a+2$. It follows that $S \leq_{\text{lex}} P$. Moreover, if $S = P$, then $v = w = 1$ due to the assumptions that $i_m < i_{m+1}$ and $j_n < j_{n+1}$. □

Theorem 7.5. *The path algebra* $K = \bigoplus_{\lambda,\mu\in\Lambda} 1_\lambda K 1_\mu$ *of* q-**Schur** *is a symmetrically based quasihereditary algebra. The required data from Definition 7.1 is as follows*:

- *The weight poset is the set* $\Lambda^+ \subset \Lambda$ *of partitions ordered by the dominance ordering.*
- *The anti-involution* $\mathtt{T} : K \to K$ *is the transposition map arising from (5-4).*
- $X(\lambda, \kappa) = \{\varphi_P \mid P \in \mathrm{Std}(\lambda, \kappa)\}$.

In particular, for $\lambda, \mu \models r$, *the* ***codeterminants***

(7-4) $$\{\varphi_P \mathtt{T}(\varphi_Q) \mid \kappa \vdash r,\ P \in \mathrm{Std}(\lambda, \kappa),\ Q \in \mathrm{Std}(\mu, \kappa)\}$$

give a basis for $1_\lambda K 1_\mu$ *as a free* $\mathbb{Z}[q, q^{-1}]$*-module.*

Proof. The second axiom follows because there is a unique semistandard tableau of shape and content κ, and there only exist semistandard tableaux of shape κ' and content κ if $\kappa \leq \kappa'$. It just remains to check for $\lambda, \mu \models r$ that the set (7-4) is a basis for $1_\lambda K 1_\mu$ as a free $\mathbb{Z}[q, q^{-1}]$-module. By the original definition, $1_\lambda K 1_\mu$ is a free $\mathbb{Z}[q, q^{-1}]$-module with basis labeled by $\mathrm{Mat}(\lambda, \mu)$. It is well known that $|\mathrm{Mat}(\lambda, \mu)| = \sum_{\kappa\vdash r} |\mathrm{Std}(\lambda, \kappa) \times \mathrm{Std}(\mu, \kappa)|$, e.g., this follows from the Robinson–Schensted–Knuth-type correspondence in (7-5) below. So the set (7-4) is of size $\leq \operatorname{rank} 1_\lambda K 1_\mu$. It remains to show that the set (7-4) spans $1_\lambda K 1_\mu$ as a $\mathbb{Z}[q, q^{-1}]$-module.

By Lemma 7.3, the elements $\varphi_P \mathtt{T}(\varphi_Q)$ for $P \in \mathrm{Row}(\lambda, \kappa)$, $Q \in \mathrm{Row}(\mu, \kappa)$ and $\kappa \vdash r$ span $1_\lambda K 1_\mu$. To complete the proof, we show by induction on the lexicographic orderings that any such $\varphi_P \mathtt{T}(\varphi_Q)$ can be written as a $\mathbb{Z}[q, q^{-1}]$-linear combination of $\varphi_{P'} \mathtt{T}(\varphi_{Q'})$ such that either $P' \in \mathrm{Std}(\lambda, \kappa)$, $Q' \in \mathrm{Std}(\mu, \kappa)$ with $P' \leq_{\text{lex}} P$, $Q' \leq_{\text{lex}} Q$, or $P' \in \mathrm{Std}(\lambda, \kappa')$, $Q' \in \mathrm{Std}(\mu, \kappa')$ for $\kappa' >_{\text{lex}} \kappa$. Applying $\mathtt{T}$ if necessary, we may assume that P is not semistandard. Applying Lemma 7.4, we see that $\varphi_P \mathtt{T}(\varphi_Q)$ is a linear combination of elements $\varphi_S \mathtt{T}(\varphi_Q)$ for $S \in \mathrm{Row}(\lambda, \kappa)$ with $S <_{\text{lex}} P$, and $\varphi_{P'} \mathtt{T}(\varphi_{Q'}) \mathtt{T}(\varphi_Q) = \varphi_{P'} \mathtt{T}(\varphi_Q \varphi_{Q'})$ with P' of shape $\kappa' >_{\text{lex}} \kappa$. Both types of elements can then be expanded into the required form by induction; for the second type, one first expands $\varphi_Q \varphi_{Q'}$ as a sum of terms φ_R for $R \in \mathrm{Row}(\mu, \kappa')$, then applies $\mathtt{T}$ to obtain a linear combination of $\varphi_{P'} \mathtt{T}(\varphi_R)$'s, before invoking the induction hypothesis. □

Remark 7.6. Let us explain how the canonical basis fits into this picture. In [Du and Rui 1998, §5.3], one finds a Robinson–Schensted–Knuth-type correspondence giving a bijection

$$\text{(7-5)} \qquad \mathrm{Mat}(\lambda,\mu) \xrightarrow{\sim} \bigcup_{\kappa\in\Lambda^+} \mathrm{Std}(\lambda,\kappa)\times\mathrm{Std}(\mu,\kappa), \qquad A \mapsto (P(A), Q(A)),$$

which we explain more fully shortly. Also let $\kappa(A)$ be the common shape of the tableaux $P(A)$ and $Q(A)$ and recall (7-2). Then [Du and Rui 1998, Theorem 5.3.3] can be reformulated as follows:

Theorem. *The path algebra K of q-**Schur** has another triangular basis*

$$\text{(7-6)} \qquad \Big\{ \beta_P \mathtt{T}(\beta_Q) \;\Big|\; (P,Q) \in \bigcup_{\substack{\lambda,\mu\in\Lambda \\ \kappa\in\Lambda^+}} \mathrm{Std}(\lambda,\kappa)\times\mathrm{Std}(\mu,\kappa) \Big\}$$

making it a symmetrically based quasihereditary algebra with $X(\lambda,\kappa)$ equal to $\{\beta_P \mid P \in \mathrm{Std}(\lambda,\kappa)\}$ and all other data as in Theorem 7.5. For $A \in \mathrm{Mat}(\lambda,\mu)$ we have

$$\text{(7-7)} \qquad \theta_A \equiv \beta_{P(A)}\mathtt{T}(\beta_{Q(A)}) \quad \Big(\mathrm{mod} \sum_{B\in\mathrm{Mat}(\lambda,\mu) \text{ with } \kappa(B)>\kappa(A)} \mathbb{Z}[q,q^{-1}]\theta_B\Big).$$

So the canonical basis is a cellular basis which is equivalent to the triangular basis (7-6), that is, it defines the same two-sided cell ideals and induces the same basis in each two-sided cell.

To define the map (7-5) explicitly, take $A \in \mathrm{Mat}(\lambda,\mu)$ corresponding to $R \in \mathrm{Row}(\lambda,\mu)$ under the bijection (2-2). Let $\boldsymbol{i} = (i_1,\dots,i_r) \in I_\lambda$ be the sequence $\overleftarrow{\Sigma}(R)$. Then we use *column insertion*[3] to insert $i_1,\dots,i_r$ in order into the empty tableau, to end up with a semistandard tableau $P(A) \in \mathrm{Std}(\lambda,\kappa)$ for some $\kappa \vdash r$. We also obtain another semistandard tableau $Q(A) \in \mathrm{Std}(\mu,\kappa)$, namely, the *recording tableau* defined so that the entry of the box that gets added at the r-th step of the algorithm is i_r^μ. This concise description of the map (7-5) is equivalent to the more complicated description in [Du and Rui 1998, §5.3]. It takes some combinatorial work (omitted here) to establish the equivalence. For example, suppose that $A = \left[\begin{smallmatrix}1&0&2\\0&1&0\end{smallmatrix}\right]$, $\lambda = (3,1)$ and $\mu = (1,1,2)$. Then $\boldsymbol{i} = (1,2,1,1)$ and $\boldsymbol{i}^\mu = (1,2,3,3)$. Column insertion of the sequence $\boldsymbol{i}$ gives $\varnothing \xrightarrow{1} \begin{array}{|c|}\hline 1\\\hline\end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \xrightarrow{1} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}$. So we get that

$$P(A) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}, \qquad Q(A) = \begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1}\end{array}, \qquad \kappa(A) = (3,1).$$

[3] We mean the following algorithm to insert i into a semistandard tableau: start with the first column; if i is bigger than all entries in the column then we add i to the bottom of that column and stop; otherwise, we find the smallest entry j in the column that is greater than or equal to i, replace that entry by i, then repeat to insert j into the next column to the right.

8. Tilting modules

For $n \geq 0$, let $\mathbf{I}_n$ be the two-sided tensor ideal of q-**Schur** generated by the identity morphisms $1_{(r)}$ for all $r > n$, then set

$$q\text{-}\mathbf{Schur}_n := q\text{-}\mathbf{Schur}/\mathbf{I}_n.$$

This is a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category.

Theorem 8.1. *The path algebra K_n of q-**Schur**$_n$ is a symmetrically based quasi-hereditary algebra, with one possible triangular basis arising from the images of the codeterminants from (7-4) for all $\kappa \in \Lambda^+$ satisfying $\kappa_1 \leq n$, and another one given by the images of the canonical basis products from (7-6) for the same κ. Also the images of the canonical basis elements θ_A for $A \in \bigcup_{\lambda,\mu\in\Lambda} \mathrm{Mat}(\lambda, \mu)$ such that $\kappa(A)_1 \leq n$ give a cellular basis for K_n.*

Proof. The two-sided tensor ideal $\mathbf{I}_n$ is equal to the ordinary two-sided ideal of q-**Schur** generated by the morphisms 1_κ for all partitions $\kappa \in \Lambda^+$ with $\kappa_1 > n$. This follows because every object $\lambda \in \Lambda$ which has some part $r > n$ is isomorphic to such a partition κ. Hence, $\mathbf{I}_n$ corresponds to the two-sided ideal $I_n \lhd K$ of the path algebra K of q-**Schur** generated by the idempotents 1_κ for all $\kappa \in \Lambda^+$ with $\kappa_1 > n$, and $K_n = K/I_n$. The set $\{\kappa \in \Lambda^+ \mid \kappa_1 > n\}$ is an upper set in the poset Λ^+, hence, I_n is a *cell ideal* in the based quasihereditary algebra K. Consequently, by [Brundan and Stroppel 2024, Corollary 5.6], the quotient algebra K_n is also a symmetrically based quasihereditary algebra with bases as described in the statement of the theorem. □

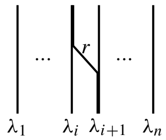
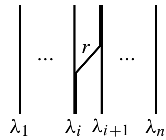
Now let $\Bbbk$ be a field viewed as a $\mathbb{Z}[q, q^{-1}]$-algebra in some way, and consider the $\Bbbk$-linear monoidal categories

$$q\text{-}\mathbf{Schur}(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur} \quad \text{and} \quad q\text{-}\mathbf{Schur}_n(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}_n.$$

From the bases as free $\mathbb{Z}[q, q^{-1}]$-modules discussed in the proof of Theorem 8.1, it follows that q-**Schur**$_n(\Bbbk)$ may be identified with the quotient of q-**Schur**$(\Bbbk)$ by the two-sided tensor ideal $\mathbf{I}_n(\Bbbk)$ generated by the morphisms $1_{(r)}$ for $r > n$.

Let q-**Tilt**$_n^+(\Bbbk)$ be the monoidal category of polynomial tilting modules for q-$\mathrm{GL}_n(\Bbbk)$, that is, the full additive Karoubian monoidal subcategory of the category of polynomial representations of q-$\mathrm{GL}_n(\Bbbk)$ generated by the exterior powers $\bigwedge^r V$ for $1 \leq r \leq n$. Here, to avoid too much more notation, we are reusing $\bigwedge^r V$ to denote the specializations of the $\mathbb{Z}[q, q^{-1}]$-modules from before. Note also that we defined the braided monoidal category q-**Tilt**$_n^+(\Bbbk)$ in the introduction in a different way in terms modules over the algebra $U_n(\Bbbk)$, but the two definitions are equivalent.

This identification requires the specific choice of comultiplication Δ described in the introduction in order for the induced homomorphism $\dot{U}_n \to K$ to map

(8-1) $E_i^{(r)} 1_\lambda \mapsto$  , $F_i^{(r)} 1_\lambda \mapsto$ 

for $1 \leq i < n$, $r \geq 0$ and $\lambda \in \mathbb{N}^n$ with $\lambda_{i+1} \geq r$ or $\lambda_i \geq r$, respectively (they map to zero for all other λ). To see that the defining relations of $\dot{U}_n$ hold in K, most of them are easy: this is the origin of the square-switch relation. The Serre relation is deduced from the other relations in [Cautis et al. 2014, Lemma 2.2.1].

Remark 8.2. When $0 \leq a - d \leq b - c$, the expressions in (6-12) and (6-13) are the canonical basis elements $\theta_{\left[\begin{smallmatrix} a-d & c \\ d & b-c \end{smallmatrix}\right]}$ and $\theta_{\left[\begin{smallmatrix} b-c & d \\ c & a-d \end{smallmatrix}\right]}$ from Example 4.1. They are also the images under the homomorphism (8-1) of the canonical basis elements $E^{(c)} F^{(d)} 1_{(a,b)}$ and $F^{(c)} E^{(d)} 1_{(b,a)}$ of $\dot{U}_2$.

The monoidal functor Σ_n from Theorem 5.4 extends to define a $\Bbbk$-linear monoidal functor q-$\mathbf{Schur}(\Bbbk) \to q$-$\mathbf{Tilt}_n^+(\Bbbk)$. Since $\bigwedge^r V = \{0\}$ for $r > n$, this factors through the quotient q-$\mathbf{Schur}_n(\Bbbk)$ to induce a $\Bbbk$-linear monoidal functor $\overline{\Sigma}_n : q$-$\mathbf{Schur}_n(\Bbbk) \to q$-$\mathbf{Tilt}_n^+(\Bbbk)$.

Theorem 8.3. *For any field $\Bbbk$, the functor $\overline{\Sigma}_n : q$-$\mathbf{Schur}(\Bbbk) \to q$-$\mathbf{Tilt}_n^+(\Bbbk)$ induces a $\Bbbk$-linear monoidal equivalence between the additive Karoubi envelope of q-$\mathbf{Schur}_n(\Bbbk)$ and q-$\mathbf{Tilt}_n^+(\Bbbk)$.*

Proof. We saw already in Remark 5.5(2) that $\overline{\Sigma}_n$ is full. It is dense by the definition of q-$\mathbf{Tilt}_n^+(\Bbbk)$. It just remains to show that it is faithful. Thus, we must show that the surjective $\Bbbk$-linear map $\mathrm{Hom}_{q\text{-}\mathbf{Schur}_n(\Bbbk)}(\mu, \lambda) \twoheadrightarrow \mathrm{Hom}_{q\text{-}\mathrm{GL}_n(\Bbbk)}\big(\bigwedge^\mu V, \bigwedge^\lambda V\big)$ induced by the functor is also injective for any $\lambda, \mu \vDash r$. By Theorem 8.1, we know that the morphism space on the left is of dimension $\sum_{\kappa \vdash r} |\mathrm{Std}(\lambda, \kappa) \times \mathrm{Std}(\mu, \kappa)|$. This is also the dimension of $\mathrm{Hom}_{q\text{-}\mathrm{GL}_n(\Bbbk)}\big(\bigwedge^\mu V, \bigwedge^\lambda V\big)$. Indeed, in the highest weight category of polynomial representations of q-$\mathrm{GL}_n(\Bbbk)$, the tilting module $\bigwedge^\mu V$ has a filtration with sections that are standard modules $\Delta(\kappa')$ for partitions κ with $\kappa_1 \leq n$, and $\bigwedge^\lambda V$ has a filtration with sections that are costandard modules $\nabla(\kappa')$ for the same κ. By the Littlewood–Richardson rule, the multiplicities $\big(\bigwedge^\mu V : \Delta(\kappa')\big)$ and $\big(\bigwedge^\mu V : \nabla(\kappa')\big)$ are $|\mathrm{Row}(\mu, \kappa)|$ and $|\mathrm{Row}(\lambda, \kappa)|$. Since

$$\dim \mathrm{Ext}^i_{q\text{-}\mathrm{GL}_n(\Bbbk)}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}\delta_{i,0},$$

this is enough to prove that $\mathrm{Hom}_{q\text{-}\mathrm{GL}_n(\Bbbk)}\big(\bigwedge^\mu V, \bigwedge^\lambda V\big)$ has the same dimension as $\mathrm{Hom}_{q\text{-}\mathbf{Schur}_n(\Bbbk)}(\mu, \lambda)$. □

Corollary 8.4. *The kernel of* Σ_n *from Theorem 5.4 is equal to* $\mathbf{I}_n$.

Proof. Let $\mathbf{J}_n$ be the kernel of Σ_n. Since $\bigwedge^r V = \{0\}$ for $r > n$, $\mathbf{I}_n \subseteq \mathbf{J}_n$. Hence, Σ_n factors through the quotient to induce a full $\mathbb{Z}[q, q^{-1}]$-linear monoidal functor from q-$\mathbf{Schur}_n$ to the category of polynomial representations of q-GL_n. It is sufficient to show that this induced functor is also faithful. This follows because it remains an isomorphism on base change to $\mathbb{Q}(q)$ by a special case of Theorem 8.3. □

Proofs of results in the introduction. Recall that in the introduction we were discussing the q-Schur category without 0-strings. This is the full subcategory of the q-Schur category with 0-strings generated by the objects Λ_s. The path algebra H of the category without 0-strings from (1-4) is the idempotent truncation $H = \bigoplus_{\lambda,\mu\in\Lambda_s} 1_\lambda K 1_\mu$ of the path algebra K of the category with 0-strings from (7-1). The set Λ^+ indexing the special idempotents is a subset of $\Lambda_s \subset \Lambda$. In view of this, Theorem 3 follows immediately from Theorem 7.5. Every object of the q-Schur category with 0-strings is isomorphic to an object of the q-Schur category without 0-strings. So the two path algebras K and H are Morita equivalent, and the restriction of the equivalence from Theorem 8.3 remains an equivalence. Theorem 4 follows. Finally, we explain how to establish the presentations in Theorems 1 and 2. These are similar to the ones in Theorems 6.1 and 6.3, respectively, but we have omitted the relations involving the generators ⫯ and ⫰. Instead, (1-2) and (1-3) need to be interpreted in a different way when strings labeled by 0 are present — simply omit those strings so that the splits and merges become identity morphisms. That these relations hold follows from the ones in Theorems 6.1 and 6.3 by contracting 0-strings. To complete the proof of Theorem 1, one needs to show that we have a full set of relations. This follows by a straightening argument which is the same as the one used in the proof of Theorem 6.1. Then Theorem 2 follows from Theorem 1 by the same argument that was used to deduce Theorem 6.3 from Theorem 6.1.

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THE BINARY ACTIONS OF SIMPLE GROUPS OF LIE TYPE OF CHARACTERISTIC 2

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In memory of Gary Seitz

Let $\mathcal{C}$ be a conjugacy class of involutions in a group G. We study the graph $\Gamma(\mathcal{C})$ whose vertices are elements of $\mathcal{C}$ with $g, h \in \mathcal{C}$ connected by an edge if and only if $gh \in \mathcal{C}$. For $t \in \mathcal{C}$, we define the *component group* of t to be the subgroup of G generated by all vertices in $\Gamma(\mathcal{C})$ that lie in the connected component of the graph that contains t.

We classify the component groups of all involutions in simple groups of Lie type over a field of characteristic 2. We use this classification to partially classify the transitive binary actions of the simple groups of Lie type over a field of characteristic 2 for which a point stabiliser has even order. The classification is complete unless the simple group in question is a symplectic or unitary group.

1. Introduction

Let G be a finite group acting on a finite set Ω. Let $I, J \in \Omega^n$ be n-tuples of elements of Ω, for some $n \geq 2$, written $I = (I_1, \ldots, I_n)$ and $J = (J_1, \ldots, J_n)$. For $r \leq n$, we say that I and J are r*-related*, and we write $I \underset{r}{\sim} J$, when for each choice of indices $1 \leq k_1 < k_2 < \cdots < k_r \leq n$, there exists $g \in G$ such that $I_{k_i}^g = J_{k_i}$ for all i. We say that the action of G on Ω has *relational complexity* k if, for all $n \geq 2$ and all $I, J \in \Omega^n$, $I \underset{k}{\sim} J$ implies that $I \underset{n}{\sim} J$, and k is the minimal such positive integer. This definition originated with Cherlin [3], motivated by considerations in model theory. In particular Cherlin observed that the relational complexity of a finite permutation group is the least k for which the group can be viewed as an automorphism group acting naturally on a homogeneous relational structure whose relations are k-ary.

There has been particular interest in the foundational case of permutation groups of relational complexity 2, which were coined *binary groups* by Cherlin. In this case, the relational structures in question are homogeneous edge-coloured directed

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graphs. Many problems and conjectures on binary actions were posed in [3], and in recent years there has been substantial progress on some of these. Following [4; 15], we now have a full classification of the primitive binary permutation groups in [8]. More recently, a general study of the binary actions of finite simple groups has been initiated in [6; 7]; in particular, we now have a full classification of the binary actions of the alternating groups.

The first aim of this paper is to contribute to the classification of the binary actions of the simple groups by proving a result for simple groups of Lie type over a field of characteristic 2. Our main result in this direction is the following.

Theorem 1. *Let $G = G(q)$ be a simple group of Lie type over $\mathbb{F}_q$, where $q = 2^a$. Suppose that G has a binary action on a set Ω and that there exists $\omega \in \Omega$ such that G_ω, the stabiliser in G of ω, is a proper subgroup of G of even order. Then one of the following holds*:

(i) *$G = {}^2B_2(2^{2a+1})$ with $a \geq 1$ and G_ω is the centre of a Sylow 2-subgroup of G.*

(ii) *$G = \mathrm{Sp}_{2n}(2^a)$ or $\mathrm{PSU}_n(2^a)$, with $n \geq 2$, and G_ω contains the centre of a long root subgroup of G.*

(iii) *$G = \mathrm{Sp}_4(2^a)$ with $a \geq 2$ and G_ω contains a short root subgroup of G.*

For comments on the hypotheses of the theorem, see the remark after Theorem 2 below. Note that the theorem applies with G equal to $\mathrm{Sp}_4(2)'$, $G_2(2)'$ and ${}^2F_4(2)'$ (and also to $\mathrm{SL}_2(2^a)$, via the isomorphism $\mathrm{PSU}_2(q) \cong \mathrm{SL}_2(q)$ in (ii)). Note too that the theorem applies to all actions of G — there is no assumption of transitivity, for instance. Let us briefly discuss the three listed items in the theorem.

In (i), G is a Suzuki group and the transitive binary actions of G are completely classified [7]. In this case we know that the action of G on the set of right cosets of a nontrivial proper subgroup H is binary if and only if H is the centre of a Sylow 2-subgroup of G.

For (ii) and (iii), our expectation is that the result would remain true were the word "contains" to be replaced, in each case, with the word "is". This would amount to a classification of the nontrivial transitive binary actions of the simple groups of Lie type over a field of characteristic 2 for which a point stabiliser has even order.

To emphasise this point we present a result — Proposition 6.3 — which implies that, were we to make the aforementioned replacement and restrict our attention to transitive actions, then the resulting list of actions would all be binary.

Our proof of Theorem 1 makes use of methods introduced in [6], in particular the notion of a *component group*. Our second main result pertains to a special case of this notion which we now define (more detail is given in Section 2).

G	t	$\Delta(t)$
$\mathrm{SL}_6(2)$	(J_2^3)	2^9
$\mathrm{PSU}_6(2)$	(J_2^3)	2^9
$\mathrm{PSU}_7(2)$	(J_2^3, J_1)	2^9
$\mathrm{Sp}_6(2)$	$W(2)+V(2)$	2^6
$\Omega_{12}^+(2)$	$W(2)^3$	2^{15}
$\mathrm{Sp}_{12}(2)$	$W(2)^3$	2^{15}

Table 1. Exceptional involution classes.

Let G be a finite group, let t be an involution in G and let $\mathcal{C} = t^G$ be the conjugacy class of t. We define a graph $\Gamma(\mathcal{C})$, whose vertices are elements of $\mathcal{C}$; two vertices $g, h \in \mathcal{C}$ are connected by an edge in $\Gamma(\mathcal{C})$ if $gh \in \mathcal{C}$. We write $\Delta(t)$ for the group generated by all vertices in $\Gamma(\mathcal{C})$ that lie in the connected component of the graph that contains t. We call $\Delta(t)$ the *component group* of the element t.

We are ready to state our second main result. For this result we do not include $\mathrm{Sp}_4(2)'$ or $G_2(2)'$ in the hypothesis (but we do include ${}^2F_4(2)'$). Note also that the group $G = \mathrm{SL}_2(q)$ is included under item (ii) of the theorem, since $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q)$.

Theorem 2. *Let $G = G(q)$ be a simple group of Lie type over $\mathbb{F}_q$, where $q = 2^a$, and let $t \in G$ be an involution. Then one of the following holds*:

(i) $\Delta(t) = G$.

(ii) *t is a long root involution, $G = \mathrm{Sp}_{2n}(q)$, $\mathrm{PSU}_n(q)$ or ${}^2B_2(q)$, and $\Delta(t)$ is the centre of a long root subgroup* (*the centre of a Sylow 2-subgroup when $G = {}^2B_2(q)$*).

(iii) *t is a short root involution, $G = \mathrm{Sp}_4(q)$ $(q > 2)$, and $\Delta(t)$ is a short root subgroup.*

(iv) *G, t and $\Delta(t)$ are as in Table 1.*

The notation for the classes of $\mathrm{Sp}_6(2)$ and $\Omega_{12}^+(2)$ in Table 1 is described in Section 4.4. Each line of the table corresponds to a single conjugacy class in G, except for the line for $G = \Omega_{12}^+(2)$, where $W(2)^3$ corresponds to two G-classes, interchanged by an outer automorphism. For the classes in Table 1, $\Delta(t)$ is elementary abelian, and in all cases except $G = \mathrm{PSU}_7(2)$, we have $\Delta(t) = O_2(C_G(t))$.

Remark. Let us remark on the hypotheses of Theorem 1 — specifically, the characteristic 2 assumption for $G = G(q)$, and the assumption that a point stabiliser has even order. Other cases are the subject of future projects, and will require additional ideas. For example, if $G = G(q)$ with q odd and $t \in G$ is an involution, then there are many more possibilities for the component group $\Delta(t)$ than those in the conclusion of Theorem 2; in some cases it is even possible that the graph $\Gamma(\mathcal{C})$ has no edges at all. Also the even order assumption on the point stabiliser is necessary for our methods, through the application of Lemma 2.2.

Structure of the paper. In Section 2 we discuss the graph $\Gamma(\mathcal{C})$ in a more general context than that given above and we state a number of results that connect this graph to the binary actions.

In Section 3 we describe some computational methods that pertain to the graph $\Gamma(\mathcal{C})$, concluding with several lemmas that will be important for the proofs of Theorems 1 and 2. In Section 4 we give the proof of Theorem 2 and in Section 5 we give the (very short) proof of Theorem 1. In the final section we prove a proposition which implies a partial converse to Theorem 1.

2. Binary actions and component groups

All groups mentioned in this paper are finite, and all group actions are on finite sets.

2.1. ***Component groups.*** We mentioned (a special case of) the graph $\Gamma(\mathcal{C})$ in the introduction. This graph was first defined in [6] and we give the definition here.

Definition 2.1. Given a conjugacy class $\mathcal{C}$ in a group G we define a graph, $\Gamma(\mathcal{C})$, whose vertices are elements of $\mathcal{C}$; two vertices $g, h \in \mathcal{C}$ are connected in $\Gamma(\mathcal{C})$ if g and h commute and either gh^{-1} or hg^{-1} is in $\mathcal{C}$.

The next lemma connects the graph $\Gamma(\mathcal{C})$ to the notion of a binary action; this lemma first appeared as [6, Corollary 2.16]. The *fixity* of an element in a group acting on a set Ω is the number of points of Ω fixed by the element.

Lemma 2.2. *Let G act transitively on Ω, let H be the stabiliser of a point in Ω, let p be a prime dividing $|H|$, let $\mathcal{C}$ be a conjugacy class of elements of order p of maximal fixity and let g be in $H \cap \mathcal{C}$. If the action is binary, then H contains all vertices in the connected component of $\Gamma(\mathcal{C})$ that contains g.*

In what follows we will write "element of maximal p-fixity" as shorthand for "element of prime order p of maximal fixity".

In the notation of Lemma 2.2, we say that the *component group* of g in G is the group generated by the connected component of $\Gamma(\mathcal{C})$ that contains g; we write this as $\Delta(g)$. So the conclusion of Lemma 2.2 could be "H contains $\Delta(g)$".

It turns out that Definition 2.1 and Lemma 2.2 can be stated slightly more generally. To do this we need to revisit an example from [6], on which the proof of

Lemma 2.2 is based. Here and below, we write $\mathrm{Fix}(g)$ to denote the fixed set of a permutation g.

Example 2.3. Let p be a prime and let g_1, g_2 be distinct commuting elements of order p in a group G acting on a set Λ. Set $g_3 = g_1 g_2^{-1}$ and write F_i for the fixed set of g_i for $i = 1, 2, 3$. Assume that

$$|F_1| = |F_2| = |F_3| \geq 1$$

and assume that F_1 and F_2 are distinct (which, in turn, means that F_3 is distinct from both F_1 and F_2).

Then $\langle g_1, g_2 \rangle$ acts on the set $F = F_1 \cup F_2 \cup F_3$ and we write τ_0 for the (nontrivial) permutation of F_3 induced by g_1 (and g_2) on the set F_3. Finally let τ be the permutation of F which equals τ_0 on F_3 and fixes all points in $F \setminus F_3$.

Write $I = (f_1, \dots, f_k)$, where $F = \{f_1, \dots, f_k\}$ and $k = |F|$, and write $J = (f_1^\tau, \dots, f_k^\tau)$. It is easy to verify directly that

(1) $I \underset{2}{\sim} J$;

(2) $I \underset{k}{\sim} J$ if and only if there exists a permutation $h \in G$ that fixes F setwise and that induces the permutation τ on F.

In particular, if the action of G on Λ is binary, then there exists a permutation $h \in G$ that fixes F setwise and that induces the permutation τ on F. Note that $\mathrm{Fix}(h)$ properly contains $\mathrm{Fix}(g_i)$ for $i = 1, 2$.

Definition 2.4. Given a union of conjugacy classes $\mathcal{D}$ in a group G we define a graph, $\Gamma(\mathcal{D})$, whose vertices are elements of $\mathcal{D}$; two vertices $g, h \in \mathcal{D}$ are connected in $\Gamma(\mathcal{D})$ if g and h commute and either gh^{-1} or hg^{-1} is in $\mathcal{D}$.

For $g \in \mathcal{D}$, we write $\Delta(g, \mathcal{D})$ for the group generated by the vertices in the connected component of $\Gamma(\mathcal{D})$ that contains g. In particular, if $\mathcal{C}$ is the conjugacy class containing g, then $\Delta(g, \mathcal{C}) = \Delta(g)$, the component group of g.

Lemma 2.5. *Let G act transitively on Ω, let H be the stabiliser of a point in Ω, let p be a prime dividing $|H|$, let $\mathcal{D}$ be a union of conjugacy classes of elements of maximal p-fixity and let g be in $H \cap \mathcal{D}$. If the action is binary, then $\Delta(g, \mathcal{D}) \leq H$.*

Proof. Assume, for a contradiction, that $\Delta(g, \mathcal{D}) \not\leq H$. Then there exist $g_1, g_2 \in \mathcal{D}$ such that g_1 and g_2 are adjacent in $\Gamma(\mathcal{D})$, $g_1 \in H$ and $g_2 \notin H$. This immediately implies that g_1 and g_2 commute and that $\mathrm{Fix}(g_1)$ is distinct from $\mathrm{Fix}(g_2)$. Now the setup of Example 2.3 applies. Since the action of G on Ω is assumed to be binary, there must exist an element h whose fixed set properly contains $\mathrm{Fix}(g_1)$. This contradicts the fact that g_1 is an element of maximal p-fixity and we are done. □

2.2. ***Terminal component groups.*** The next lemma inspires the definitions that follow.

Lemma 2.6. *Let G act transitively on Ω, let H be the stabiliser of a point in Ω, let p be a prime dividing $|H|$, let $\mathcal{D}$ be a union of conjugacy classes of elements of maximal p-fixity and let g be in $H \cap \mathcal{D}$. If the action is binary, then every element of $\Delta(g, \mathcal{D})$ that has order p is also an element of maximal p-fixity.*

Proof. By assumption, the action of G is binary and so Lemma 2.5 implies that the stabiliser of any point fixed by g contains $\Delta(g, \mathcal{D})$. Thus every element of $\Delta(g, \mathcal{D})$ fixes at least as many elements of Ω as g. □

Now consider a finite group G and an element g contained in $\mathcal{D}$, a union of conjugacy classes of elements of order p. Let $\mathcal{D}_1 = \mathcal{D}$ and let $\Gamma_1(\mathcal{D}) = \Gamma(\mathcal{D})$ and $\Delta_1(g) = \Delta(g, \mathcal{D})$.

For a positive integer i, define $\mathcal{D}_i$ to be the union of conjugacy classes in G which satisfy two criteria:

(1) elements of these conjugacy classes have order p;

(2) for any such conjugacy class, C, we have $C \cap \Delta(g, \mathcal{D}_{i-1}) \neq \varnothing$.

Now define $\Delta_i(g, \mathcal{D}) = \Delta(g, \mathcal{D}_i)$. We have an ascending chain of subgroups:

$$\Delta_1(g, \mathcal{D}) \leq \Delta_2(g, \mathcal{D}) \leq \Delta_3(g, \mathcal{D}) \leq \cdots .$$

We define $\Delta_\infty(g, \mathcal{D})$ to be the union of all of the subgroups in this chain. In the case where $\mathcal{D} = \mathcal{C}$, the conjugacy class containing g, we write $\Delta_\infty(g) = \Delta_\infty(g, \mathcal{D})$ and call this group the *terminal component group of g in G*. This definition allows us to strengthen Lemma 2.6:

Lemma 2.7. *Let G act transitively on Ω, let H be the stabiliser of a point in Ω, let p be a prime dividing $|H|$, let $\mathcal{D}$ be a union of conjugacy classes of elements of maximal p-fixity and let g be in $H \cap \mathcal{D}$. If the action is binary, then H contains $\Delta_\infty(g, \mathcal{D})$.*

In particular if g is any element of maximal p-fixity, then H contains $\Delta_\infty(g)$, the terminal component group of g.

Proof. We suppose that $\mathcal{D}_i$ is a union of conjugacy classes of elements of maximal p-fixity. We will use induction to prove that, for all positive integers i,

(S1) $\Delta_i(g, \mathcal{D}) \leq H$; and

(S2) $\mathcal{D}_{i+1}$ is a union of conjugacy classes of elements of maximal p-fixity.

For $i = 1$, (S1) follows from Lemma 2.5 and (S2) follows from Lemma 2.6. Thus the base case of our inductive argument is proved.

Let us assume that (S1) and (S2) are true for i. Since (S2) is true for i, Lemma 2.5 implies that (S1) is true for $i+1$. Now Lemma 2.6 implies that (S2) is true for $i+1$. The result follows by induction. □

Remark 2.8. We will be interested primarily in the groups $\Delta(g)$ and $\Delta_\infty(g)$ for various elements g of prime order in G.

The most obvious scenario where the more general notion of $\Delta(g, \mathcal{D})$ and $\Delta_\infty(g, \mathcal{D})$ is of interest would be setting $\mathcal{D}$ to be the *rational conjugacy class* containing g (i.e., the union of conjugacy classes that contain an i-th power of g, where i is coprime to the order of g). Clearly, in any action of G, if g is an element of maximal p-fixity, then the same will be true of all of the rational conjugates of g.

3. Computing with component groups

The proof of Theorem 2 involves the calculation of several component groups in certain exceptional cases, using a computer. We have relied on GAP and Sagemath, but it is not possible to use a naive approach for the task at hand. We describe how the computation was made possible. Besides, the "transport group", to be defined shortly, is probably a useful object to think of, even in theoretical situations.

In what follows, G is a finite group, $s \in G$ is an involution and $\mathcal{C}$ is its conjugacy class. We wish to compute the connected component X of $\Gamma(\mathcal{C})$ containing s; the subgroup generated by this connected component is then determined readily by standard algorithms.

The techniques we describe would work equally well with s an element of prime order p, although we stick to $p = 2$ for simplicity of language (and because this is what the paper is about).

3.1. ***Introducing the transport group.*** The naive approach to our problem would consist in finding all the neighbours of s in the graph $\Gamma(\mathcal{C})$, by scanning all the elements of $\mathcal{C}$ one by one; then finding the neighbours of these neighbours; and then iterating until no new vertex can be appended to the connected component. This is way too slow in practice.

We will mostly reduce things to the computation of the original neighbours of s, with no need to dive deeper in the graph. An essential ingredient is this lemma:

Lemma 3.1. *Let Y be a finite, connected graph, and let T be a subgroup of* $\mathrm{Aut}(Y)$ *with the following property: there is a vertex v such that all its neighbours are of the form v^t for $t \in T$. Then the action of T is vertex transitive.*

Proof. Let $w \in Y$. To show that $w \in v^T$, we proceed by induction on the length n of a minimal path from v to w in the graph, the case $n = 1$ being given. We see immediately that w has a neighbour u such that $u \in v^T$, by induction. If $u = v^t$ for $t \in T$, then $w^{t^{-1}}$ is a neighbour of v, so that $w^{t^{-1}} = v^{t'}$ for some $t' \in T$, and $w = v^{t't}$ as desired. □

We will apply this with $Y = X$, the connected component of s, viewed as an induced subgraph of $\Gamma(\mathcal{C})$, and of course with $v = s$. By definition, all the vertices of $\Gamma(\mathcal{C})$ are conjugates of s within G, so that we may find elements $t_i \in G$ such that the neighbours of s can be computed to be of the form $s^{t_1}, \ldots, s^{t_d}$. Each t_i acts on $\Gamma(\mathcal{C})$ by graph automorphisms, and takes s to an element of X, and so it preserves the connected subgraph X. We can then apply the lemma to the subgroup $T = \langle t_1, \ldots, t_d \rangle$, and we find that $X = s^T$.

This provides already a comfortable speedup, but we also need to select the group T more carefully, as finding the neighbours of v is in itself a costly operation. Since these neighbours must commute with s by definition, we elect to:

(1) Compute first the centraliser $Z = C_G(s)$.

(2) Determine the conjugacy classes of involutions in Z.

(3) For each such class, pick one representative x, and check whether there is $t \in G$ such that $s^t = x$ (so that $x \in \mathcal{C}$), and then check whether $sx \in \mathcal{C}$; if not, discard the conjugacy class.

(4) Finally, define the *transport group* T to be generated by Z and all the elements t obtained during the previous step.

The lemma then applies with this T, as is checked readily (simply note that in step (3), we only need one representative, as there is an element of Z (which is thus in T) which fixes s and takes x to any other involution in the conjugacy class).

Having gone through these steps, it is very easy to compute the order of T, and $|T|/|Z|$ is the size of $s^T = X$. Very often, we find that $X = \Gamma(\mathcal{C})$ so that, when G is simple, we can conclude that $\Delta(s) = G$. Otherwise, standard algorithms allow us to iterate over the elements of X and compute the group they generate.

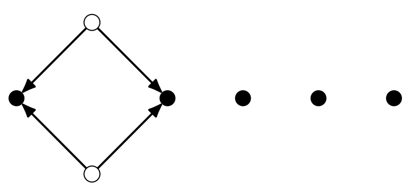
Remark 3.2. To give a bit of perspective, let us mention the sort of performance we obtained with $G = \Omega_{12}^{+}(2)$ for some randomly chosen involution s. The naive algorithm took about 8 minutes to compute the neighbours of s, and there are around 800 of these, so we expected to have to wait 4 days to get the list of the vertices in X which are at distance 2 from s. By contrast, the algorithm above concludes that $\Delta(s) = G$ in less than 20 seconds.

3.2. ***Randomised calculations.*** The above algorithm can still take a lot of time in certain cases (for example, when the centraliser Z has a complicated structure). However, another advantage of the approach we have just described is that it can be turned easily into a randomised algorithm. The idea is simply to draw elements of $\mathcal{C}$ at random until one finds a neighbour of s in $\Gamma(\mathcal{C})$; then, compute $t \in G$ such that the vertex just found is s^t, and keep a list of the elements t thus obtained. At any moment, we can compute the group T generated by $C_G(s)$ and the t's that have been collected. If at any point $T = G$ (which happens frequently with our examples), of course we may stop and conclude that $\Delta(s) = G$ (when G is simple). Otherwise, we can stop after a certain number of tries, and what we have is a subgroup T of the "real" transport group, and we can compute the group $\Delta'(s)$ generated by s^T, which is a subgroup of the "real" component group $\Delta(s)$.

In this situation, we can try to compute the terminal component group $\Delta_\infty(s)$ instead, as in Section 2.1. Concretely, we check whether $\Delta'(s)$ (and so also $\Delta(s)$) contains an involution s_1 such that $s_1 \notin \mathcal{C}$; if there is one, we can run the algorithm with s_1, obtaining $\Delta'(s_1)$, a subgroup of $\Delta(s_1)$, and crucially, $\Delta'(s_1)$ is also a subgroup of $\Delta_\infty(s)$, by definition. We can continue and find s_2 in either $\Delta'(s)$ or $\Delta'(s_1)$, compute $\Delta'(s_2)$, which is yet another subgroup of $\Delta_\infty(s)$, and so on. We stop when there are no more elements to try (because all of their conjugacy classes have been attempted), or when there is s_i such that $\Delta'(s_i) = G$, in which case we conclude that $\Delta_\infty(s) = G$ also.

This search is best organised, and more easily summarised, as follows. Our program constructs a graph whose vertices are the conjugacy classes of involutions in G. When we have found that $\Delta(s) = G$ for an involution s, we paint the corresponding vertex black. Otherwise, the vertex is left in white, and we add a directed arrow from the class of s to that of any s_1 such that $s_1 \in \Delta'(s)$, our computed approximation of $\Delta(s)$. A sufficient condition for $\Delta_\infty(s) = G$ is thus that from the corresponding vertex, there is a directed path leading to a black vertex.

As an example, here is a graph produced when working with $G = \Omega^+_{12}(2)$. It shows that $\Delta_\infty(s) = G$ for any involution s:



In general, the graphs thus produced only reflect partial information, as obtained at a given point in time by the randomised calculation. Typically, we then focus our efforts on the white vertices, and start the full, nonrandom algorithm on them, to see if $\Delta(s)$ is really smaller than G. Of course, this may take a little longer.

3.3. ***Results.*** To conclude this section we record three results that will be important for the proof of Theorem 2. These were all proved using the computational methods described above.

Lemma 3.3. *Let G be one of the groups $\mathrm{SL}_6(2)$, $\mathrm{PSU}_n(2)$ $(4 \le n \le 8)$, $\mathrm{Sp}_{2n}(2)$ $(3 \le n \le 6)$, $\Omega^{\pm}_{2n}(2)$ $(4 \le n \le 6)$, and let $t \in G$ be an involution. Then one of the following holds*:

(i) $\Delta(t) = G$.

(ii) $G = \mathrm{PSU}_n(2)$ *or* $\mathrm{Sp}_{2n}(2)$ *and t is a long root involution.*

(iii) *G, t are as in Table 1 of Theorem 2.*

Lemma 3.4. *Let G be equal to ${}^2F_4(2)'$ or ${}^3D_4(2)$ and let $t \in G$ be an involution. Then $\Delta(t) = G$.*

Lemma 3.5. *Let G be a group occurring in Table 1 and let t be the associated involution. Then $\Delta_\infty(t) = G$.*

Notice that Lemma 3.5 implies that if $G = G(q)$ is a simple group of Lie type over $\mathbb{F}_q$ where $q = 2^a$ and $t \in G$ is an involution, then either $\Delta_\infty(t) = G$ or else G and t are as described at points (ii) and (iii) of Theorem 2.

4. Proof of Theorem 2

4.1. ***Long root elements.*** Here we prove a lemma identifying component groups of long root elements for arbitrary characteristic. In the statement we do not include $G = \mathrm{Sp}_4(2)'$, $G_2(2)'$ or ${}^2G_2(3)'$, but we do include ${}^2F_4(2)'$.

Lemma 4.1. *Let $G = G(q)$ be a simple group of Lie type over $\mathbb{F}_q$, where $q = p^a$, and let t be a long root element. Then one of the following holds*:

(i) $\Delta(t) = G$.

(ii) $G = \mathrm{PSp}_{2n}(q)$ $(n \ge 1)$ *or* $\mathrm{PSU}_n(q)$ $(n \ge 3)$, *and $\Delta(t)$ is the centre of a long root subgroup.*

(iii) $G = {}^2G_2(q)$ *or* ${}^2B_2(q)$, *and $\Delta(t)$ is the centre of a Sylow p-subgroup of G.*

Proposition 6.3 asserts that if G is one of the groups listed in items (ii) or (iii), t is a long root element and $H = \Delta(t)$, then either $G = \mathrm{PSp}_2(q)$ with q odd or the action of G on the set of right cosets of H in G is binary.

Proof. First assume that G is not one of the groups under (ii) or (iii), and is of untwisted type. Let Φ be the root system of G, and Φ_L the set of long roots in Φ (so $\Phi_L = \Phi$ if there is only one root length). We can take $t = u_\alpha(1)$ with $\alpha \in \Phi_L$. As $G \neq C_n(q)$, there exists $\beta \in \Phi_L$ such that $\alpha + \beta \in \Phi_L$, and $\langle U_{\pm\alpha}, U_{\pm\beta}\rangle = A_2(q)$. Working in $\mathrm{SL}_3(q)$ with the usual representation

$$u_\alpha(c) = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad u_\beta(c) = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}, \quad u_{\alpha+\beta}(c) = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix},$$

we see that the graph $\Gamma(\mathcal{C})$ has, for any $c_i \in \mathbb{F}_q \setminus 0$, the edges

$$u_{-\beta}(c_1) \text{ --- } u_\alpha(c_2) \text{ --- } u_{\alpha+\beta}(c_3) \text{ --- } u_\beta(c_4) \text{ --- } u_{-\alpha}(c_5).$$

Hence $\Delta(t)$ contains $\langle U_{\pm\alpha}, U_{\pm\beta}\rangle = A_2(q)$.

Now take $\alpha = -\alpha_0$, where α_0 is the highest root. Then for any long root ρ such that $\beta = \alpha_0 - \rho \in \Phi_L$, we see from the above that $\Delta(t)$ contains $U_{\pm\beta}$. If there is only one root length, these root groups $U_{\pm\beta}$, together with $U_{\pm\alpha}$, generate G; so $\Delta(t) = G$ in this case. Finally, suppose there are two root lengths. Then Φ is of type B_n ($n \geq 3$), G_2 or F_4, and the root groups $U_{\pm\beta}$ for $\beta \in \Phi_L$ generate a subsystem subgroup $D_{n-1}(q)$, $A_2(q)$ or $D_4(q)$, respectively. So $\Delta(t)$ contains this subsystem subgroup, which is maximal in G. Also $\Delta(t)$ is invariant under $C_G(t)$, which is the derived group of a parabolic subgroup of G, and is not contained in the subsystem subgroup. It follows that $\Delta(t) = G$ in this case also.

Now assume that G is of twisted type, and is not one of the groups under (ii) or (iii). For $G = {}^2D_n(q)$ ($n \geq 4$), ${}^3D_4(q)$ or ${}^2E_6(q)$, the above argument gives the result, as there is a subsystem A_2 spanned by long roots. And for $G = {}^2F_4(q)'$, we argue as follows. The involution t lies in a subgroup ${}^2F_4(2)'$ of G, and Lemma 3.4 implies that $\Delta(t) \geq {}^2F_4(2)'$. As $\Delta(t)$ is $C_G(t)$-invariant, it follows that $\Delta(t) = G$.

Now we let G be one of the groups under (ii) and (iii) and we must describe $\Delta(t)$. We start by assuming that $G = \mathrm{PSp}_{2n}(q)$. With respect to a suitable standard basis $e_1, \ldots, e_n, f_n, \ldots, f_1$, modulo the scalars $Z = \langle -I\rangle$, we can take

$$t = \begin{pmatrix} 1 & & \lambda \\ & I_{2n-2} & \\ & & 1 \end{pmatrix}, \tag{1}$$

for some scalar $\lambda \neq 0$, and then

$$C_G(t) = \left\{ \begin{pmatrix} \epsilon & x & c \\ & A & AJx^T \\ & & \epsilon \end{pmatrix} : A \in \mathrm{Sp}_{2n-2}(q),\ x \in \mathbb{F}_q^{2n-2},\ c \in \mathbb{F}_q,\ \epsilon = \pm 1 \right\} \Big/ Z,$$

where J is the matrix of the form restricted to $e_2, \dots, e_n, f_n, \dots, f_2$. The only elements $u \in C_G(t)$ for which both u and tu^{-1} are conjugate to t are those with $A = I_{2n-2}$ and $x = 0$. Hence

$$u \in \left\{ \begin{pmatrix} 1 & & c \\ & I_{2n-2} & \\ & & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\} = U, \tag{2}$$

a long root subgroup. Thus $\Delta(t) = U$ in this case.

The proof of (ii) for $G = \mathrm{PSU}_n(q)$ is very similar. Take V to be an n-dimensional unitary space over $\mathbb{F}_{q^2}$, with unitary form having Gram matrix with 1's on the reverse diagonal and 0's elsewhere. Then we can take t to be as in (1) with $\lambda + \lambda^q = 0$, and we compute as above that the only elements $u \in C_G(t)$ for which both u and tu^{-1} are conjugate to t lie in a subgroup U defined as in (2), but with the condition on the scalar $c \in \mathbb{F}_{q^2}$ being $c + c^q = 0$. Then U is the centre of a long root subgroup of G, and $\Delta(t) = U$, proving (ii) in this case.

Now let $G = {}^2G_2(q)$ or ${}^2B_2(q)$ as in (iii), with $q = p^{2a+1}$, where $p = 2$ or 3, and $a \geq 1$. Then $C_G(t) = P$, a Sylow p-subgroup of G. The structure and fusion of P is described in [13; 14]: $Z(P)$ is elementary abelian of order q and has all its nonidentity elements conjugate to t; also there is no element $u \in P \setminus Z(P)$ such that u and tu^{-1} are conjugate to t. Hence $\Delta(t) = Z(P)$ in these cases, completing the proof. □

We now prove Theorem 2 separately for the various Lie types.

4.2. *Linear groups.*

Lemma 4.2. *Theorem 2 holds for $G = \mathrm{PSL}_n(q)$, $q = 2^a$.*

Proof. Let $G = \mathrm{PSL}_n(q)$ with $q = 2^a$. If $n = 2$, the only involution class in G contains a root element $t = J_2 \in \mathrm{SL}_2(q) = \mathrm{Sp}_2(q)$ (where J_i denotes an $i \times i$ unipotent Jordan block), and this class is covered by Lemma 4.1.

So assume $n \geq 3$. We begin by establishing the result for $n \leq 4$. For $n = 3$, the group $\mathrm{PSL}_3(q)$ has only one class of involutions, with representative $t = (J_2, J_1)$, a long root element, so this case is covered by Lemma 4.1. Now let $n = 4$, $G = \mathrm{PSL}_4(q)$. Again, root elements (J_2, J_1^2) are covered by Lemma 4.1, so let $t = (J_2^2) \in G$. We can take $t = J_2 \otimes I \in \mathrm{SL}_2(q) \otimes \mathrm{SL}_2(q) < G$. Then $C_G(t)$ contains $I \otimes \mathrm{SL}_2(q)$, and each involution u in this group is conjugate to t; also $tu = J_2 \otimes J_2$ is conjugate to t. Hence t is joined in $\Gamma(\mathcal{C})$ to all such involutions u. It follows that $\Delta(t)$ contains $I \otimes \mathrm{SL}_2(q)$. Similarly, arguing with neighbours of u, we see that $\mathrm{SL}_2(q) \otimes I \leq \Delta(t)$. Thus

$$\Delta(t) \geq \mathrm{SL}_2(q) \otimes \mathrm{SL}_2(q).$$

Also, if we write t as $\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$, then as in the proof of Lemma 3.3, we see that $\Delta(t)$ contains $N := O_2(C_G(t)) \cong q^4$. Using [2, Tables 8.8 and 8.9], we see that for $q \geq 4$, the only overgroups in G of $\mathrm{SL}_2(q) \otimes \mathrm{SL}_2(q) = \Omega_4^+(q)$ are $O_4^+(q)$ and $\mathrm{Sp}_4(q)$, and hence $\langle \mathrm{SL}_2(q) \otimes \mathrm{SL}_2(q), N \rangle = G$. Finally, for $q = 2$ we argue in $A_8 \cong L_4(2)$ that $\Delta(t) = G$ here too. This completes the proof for $n = 4$. Now suppose $n \geq 5$. Adopt, for $\lambda \in \mathbb{F}_q$, the notation

$$J_2(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad J_2 = J_2(1), \quad t = (J_2^a, J_1^b),$$

where $2a + b = n$. By Lemma 4.1 we can assume that $a \geq 2$.

Assume next that $q > 2$, and let $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. Suppose $b \geq 1$. Write $t = t_1 \oplus t_2$, where $t_1 = (J_2, J_1)$, $t_2 = (J_2^{a-1}, J_1^{b-1})$. Let $u_2 = (J_2(\alpha)^{a-1}, J_1^{b-1})$. If $u_1 \in \mathrm{SL}_3(q)$ is an involution joined to t_1 in the graph on $t_1^{\mathrm{SL}_3(q)}$, then t is joined to $u = u_1 \oplus u_2$ in the graph on t^G. Hence by the result for $\mathrm{SL}_3(q)$, we have $\Delta(t) \geq \mathrm{SL}_3(q)$. This holds for any choice of blocks (J_2, J_1) in t, and these $\mathrm{SL}_3(q)$ subgroups generate G. So $\Delta(t) = G$ in the case where $b \geq 1$.

Now suppose $b = 0$ (still assuming that $q > 2$). For this case, write $t = t_1 \oplus t_2$, where $t_1 = (J_2^2)$, $t_2 = (J_2^{a-2})$. Let $u_2 = (J_2(\alpha)^{a-2})$, and argue as above using the result for $\mathrm{SL}_4(q)$ that $\Delta(t)$ contains the subgroup $\mathrm{SL}_4(q)$ corresponding to t_1. This holds for any choice of blocks J_2^2 in t, so again we see that $\Delta(t) = G$.

It remains to deal with the case where $q = 2$. It is convenient first to deal with $n = 5, 6$. For these cases we have $t = (J_2^2, J_1^{n-4})$ (excluding the exceptional case $(J_2^3) \in \mathrm{SL}_6(2)$ as it is conclusion (iii) of Theorem 2). For $n = 5$, we write $t = t_1 \oplus t_2$, where $t_1 = (J_2^2)$, $t_2 = J_1$. Arguing as above using the result for $\mathrm{SL}_4(q)$, we see that $\Delta(t)$ contains $S := SL_4(q)$. Also $\Delta(t)$ is $C_G(t)$-invariant, and we can compute a $C_G(t)$-conjugate S^c of S such that $\langle S, S^c \rangle = G$. Thus $\Delta(t) = G$ in the case $n = 5$. For $n = 6$ we argue similarly, taking $t = t_1 \oplus t_2$, where $t_1 = (J_2, J_1)$, $t_2 = (J_2, J_1)$ and using the result for $\mathrm{SL}_3(q)$.

Finally, assume that $n \geq 7$ (with $q = 2$). Write $t = t_1 \oplus t_2$, where

$$t_1 = \begin{cases} (J_2, J_1) & \text{if } a = 2, \\ (J_2^2) & \text{if } a \geq 3. \end{cases}$$

Then t_1, t_2 are involutions in $\mathrm{SL}_{n_1}(2)$, $\mathrm{SL}_{n_2}(2)$, where $n_1 = 3$ or 4, $n_2 = n - n_1$. If $u_i \in \mathrm{SL}_{n_i}(2)$ $(i = 1, 2)$ are involutions joined to t_i in the graphs on $t_i^{\mathrm{SL}_{n_i}(2)}$, then $u = u_1 \oplus u_2$ is joined to t in $\Gamma(\mathcal{C})$. Hence inductively we have $\Delta(t) \geq \mathrm{SL}_{n_1}(2) \times \mathrm{SL}_{n_2}(2)$ (the second factor possibly replaced by 2^9 in the case where $n = 10$, $t = (J_2^5)$). By reordering the blocks in t, we obtain several different such subgroups in $\Delta(t)$, and these generate G. This completes the proof. □

4.3. *Unitary groups.*

Lemma 4.3. *Theorem 2 holds for $G = \mathrm{PSU}_n(q)$ with $n \geq 3$, $q = 2^a$.*

Proof. Let $G = \mathrm{PSU}_n(q)$ with $q = 2^a$. If $n = 3$ then there is only one class of involutions, namely the long root elements, and these are covered by Lemma 4.1. So we may suppose that $n \geq 4$, and that $t \in G$ is an involution that is not a long root element.

If $n = 4$ then $t = (J_2^2)$ and we argue as in the proof of Lemma 4.2 (second paragraph) that $\Delta(t) = G$, noting at the end that the case $q = 2$ is covered by Lemma 3.3.

Now assume that $n \geq 5$ and $q > 2$. Let $t = (J_2^a, J_1^b)$ (with $a \geq 2$, as t is not a long root element). We argue by induction on n. Write $t = t_1 \oplus t_2$, where $t_1 = (J_2^a, J_1^{b-1})$ if $b \geq 1$, and $t_1 = (J_2^{a-1})$ if $b = 0$. Then inductively, $\Delta(t_1) = \mathrm{SU}_{n-1}(q)$ or $\mathrm{SU}_{n-2}(q)$ in the respective cases. Moreover $\Delta(t)$ contains this subgroup: in the first case this is clear, and in the second we can use neighbours of t of the form $u_1 \oplus u_2$, where $u_2 = J_2(\alpha)$, as in the proof of Lemma 4.2. As $\Delta(t)$ is $C_G(t)$-invariant, it follows that $\Delta(t) = G$.

Finally, consider the case $n \geq 5$, $q = 2$. By Lemma 3.3, we may assume that $n \geq 9$. Let $t = (J_2^a, J_1^b)$ with $a \geq 2$. If $b \geq 1$, write $t = t_1 \oplus t_2$, where $t_1 = (J_2^a, J_1^{b-1})$, and argue inductively as above (using Lemma 3.3 for the base case where $t_1 \in \mathrm{SU}_8(2)$). Now suppose $b = 0$, and write $t = t_1 \oplus t_2$, where $t_1 = (J_2^2)$, $t_2 = (J_2^{a-2})$. If $a > 5$, then inductively we have $\Delta(t) \geq \mathrm{SU}_4(2) \times \mathrm{SU}_{n-4}(2)$; also $\Delta(t)$ contains several subgroups of this form corresponding to different pairs of J_2-blocks, and these generate G. And if $a = 5$, then according to the entry for $\mathrm{PSU}_6(2)$ in Table 1, $\Delta(t_2) = 2^9$, and so we see that $\Delta(t)$ contains $\Delta(t_1) \times \Delta(t_2) = \mathrm{SU}_4(2) \times 2^9$. The collection of such subgroups $\mathrm{SU}_4(2)$, one for each pair of J_2-blocks in t, generates G, and hence $\Delta(t) = G$ in this case, completing the proof. □

4.4. ***Symplectic and orthogonal groups.*** We consider the symplectic and orthogonal groups $G = \mathrm{Sp}_{2n}(q)$, $\Omega_{2n}^{\epsilon}(q)$ with $q = 2^a$ and $\epsilon = \pm$. Note that $O_{2n}^{\epsilon}(q) < \mathrm{Sp}_{2n}(q)$, and let $V = V_{2n}(q)$ be the natural module. We begin by describing the involution classes in G, with notation and results taken from [12, Chapters 4-6]. There is an involution in $O_2^{\epsilon}(q) \setminus \Omega_2^{\epsilon}(q)$ (hence also in $\mathrm{Sp}_2(q)$), which we denote by $V(2)$. Also the group $\Omega_4^{+}(q)$ has a subgroup $\mathrm{SL}_2(q)$ stabilising a pair of totally singular 2-spaces, and we denote an involution in this SL_2 subgroup by $W(2)$; it acts as J_2^2 on the 4-dimensional space. Denote by $W(1)$ the identity element of $\Omega_2^{\epsilon}(q)$. Then, with one exceptional case, every involution $t \in G$ is uniquely determined up to conjugacy by an orthogonal decomposition

$$t = W(1)^a + W(2)^b + V(2)^c, \tag{3}$$

where $c \le 2$ (and $n = a + 2b + c$). The exceptional case is $t = W(2)^b \in \Omega_{4b}^+(q)$, in which case there are two G-classes of involutions which are interchanged by elements of $O_{4b}^+(q) \setminus \Omega_{4b}^+(q)$.

Note that in (3), c can only be 1 if $G = \mathrm{Sp}_{2n}(q)$. Also, when G is orthogonal and $c = 2$, there are actually two possibilities for the blocks $V(2)^2$ — in the notation of [9], they could be $V(2)+V(2) \in \Omega_4^+(q)$ or $V(2)+V_\alpha(2) \in \Omega_4^-(q)$ (where $\alpha \in \mathbb{F}_q$ is such that the quadratic $x^2+x+\alpha$ is irreducible over $\mathbb{F}_q$). For our proof below it is not necessary to distinguish between these cases, so we use the notation $V(2)^2$ for both.

Lemma 4.4. *Theorem 2 holds for $G = \mathrm{Sp}_{2n}(q)$ with $n \ge 2$, $q = 2^a$.*

Proof. Let $G = \mathrm{Sp}_{2n}(q)$ with $q = 2^a$. Note that the class of long root elements of G is represented by $t = V(2) + W(1)^{n-1}$, and the conclusion of Theorem 2 for this class follows from Lemma 4.1. Henceforth we do not consider this class.

Suppose first that $n = 2$ and $q > 2$. The classes of long and short root elements of G are interchanged by a graph automorphism, so the conclusion of the theorem for short root elements follows from Lemma 4.1. The remaining involution class in G contains $t = V(2)^2$. This element lies in a subgroup $\mathrm{Sp}_4(2)' \cong A_6$ of G, and arguing in A_6 we see that $\Delta(t) \ge A_6$. As $\Delta(t)$ is $C_G(t)$-invariant, $\Delta(t) = G$ for this class.

Next consider $n = 3$. If $q = 2$ then $G = \mathrm{Sp}_6(2)$, which is covered by Lemma 3.3, so assume $q > 2$. The classes to consider are those containing $t = W(2) + W(1)$, $V(2)^2 + W(1)$ and $W(2) + V(2)$. In the first two cases t lies in a subgroup $\mathrm{Sp}_6(2)$, so Lemma 3.3 shows that $\Delta(t) \ge \mathrm{Sp}_6(2)$, and now the $C_G(t)$-invariance of $\Delta(t)$ shows that it is equal to G. In the last case, t is conjugate to $V(2)^3$, and writing $t = t_1 \oplus t_2$ with $t_1 = V(2)^2 \in \mathrm{Sp}_4(q)$, we see that $\Delta(t)$ contains $\Delta(t_1) = \mathrm{Sp}_4(q)$ and hence that $\Delta(t) = G$.

If $n = 4$ or 5 then since any representative t as in (3) lies in a subgroup $\mathrm{Sp}_{2n}(2)$ of G, Lemma 3.3 shows that $\Delta(t)$ contains $\mathrm{Sp}_{2n}(2)$, and then $C_G(t)$-invariance gives $\Delta(t) = G$. The same argument applies for $n = 6$, except for the class $t = W(2)^3$ (the exceptional class in $\mathrm{Sp}_{12}(2)$ in Table 1). In this case, Table 1 gives the conclusion if $q = 2$, and for $q > 2$ we write $t = t_1 \oplus t_2$ with $t_1 = W(2)^2$, $t_2 = W(2)$ and see that $\Delta(t)$ contains $\Delta(t_1) = \mathrm{Sp}_8(q)$, hence that $\Delta(t) = G$.

Suppose finally that $n \ge 7$. Let $t \in G$ be as in (3). We argue by induction on n, having established the base cases $n = 3, 4, 5, 6$.

Assume $a \ge 1$, and write $t = t_1 \oplus t_2$ with $t_1 = W(1)^{a-1} + W(2)^b + V(2)^c$, $t_2 = W(1)$. Then $\Delta(t)$ contains $\Delta(t_1)$, which inductively is $\mathrm{Sp}_{2n-2}(q)$, unless $t_1 = W(2)^3 \in \mathrm{Sp}_{12}(2)$ and $G = \mathrm{Sp}_{14}(2)$. In the former case, $\Delta(t) = G$ in the usual way; in the latter, rewrite t as $(W(2)^2) \oplus (W(2) + W(1)) \in \mathrm{Sp}_8(2) \times \mathrm{Sp}_6(2)$ - then Lemma 3.3 gives $\Delta(t) \ge \mathrm{Sp}_8(2) \times \mathrm{Sp}_6(2)$, and now we see that $\Delta(t) = G$ as usual.

So we may assume $a = 0$. Then $b \geq 3$, and we can write $t = t_1 \oplus t_2$ with $t_1 = W(2)^2$, $t_2 = W(2)^{b-2} + V(2)^c$. Then $\Delta(t)$ contains $\Delta(t_1) = \mathrm{Sp}_8(q)$, and this is the case for any pair of $W(2)$-blocks making up t_1. Hence $\Delta(t) \geq \mathrm{Sp}_{2n-2c}(q)$, and then $\Delta(t) = G$ in the usual way. □

Lemma 4.5. *Theorem 2 holds for $G = \Omega^{\epsilon}_{2n}(q)$ with $q = 2^a$, $n \geq 4$ and $\epsilon = \pm$.*

Proof. Let $t \in G$ be as in (3). The proof goes by induction on n, and runs along entirely similar lines to that of Lemma 4.4. We first establish the bases cases $n = 4, 5, 6$ in exactly the same way, using Lemma 3.3 to see that $\Delta(t)$ contains a subgroup $\Omega^{\delta}_{2n}(2)$ and hence that $\Delta(t) = G$, in all cases except $t = W(2)^3 \in \Omega^{+}_{12}(q)$. For the latter class, the case $q = 2$ is covered by Lemma 3.3, and for $q > 2$ we write $t = t_1 \oplus t_2$ with $t_1 = W(2)^2$, $t_2 = W(2)$; hence $\Delta(t) \geq \Delta(t_1) = \mathrm{Sp}_8(q)$, from which we deduce that $\Delta(t) = G$.

Finally, for $n \geq 7$ we argue by induction exactly as in the proof of Lemma 4.4. □

4.5. *Exceptional groups of Lie type.*

Lemma 4.6. *Theorem 2 holds for $G = G(q)$, a simple group of exceptional Lie type, where $q = 2^a$.*

Note that we exclude $G_2(2)'$ in the hypothesis.

Proof. First we consider the simple groups $G(q)$ of untwisted Lie type. A convenient list of conjugacy classes of involutions in these groups $G(q)$ can be found in the tables in [12, Chapter 22]. Class representatives are products of involutions in Levi subgroups A_1^k, as listed in the tables.

Case $G = E_8(q)$. There are four involution classes, corresponding to Levi subgroups A_1^k for $k = 1, 2, 3, 4$. For $k = 1$ this is a root involution, covered by Lemma 4.1.

For $k = 2$, we can take as representative $t = u_{\alpha_4}(1)u_{\alpha_8}(1)$ with the usual labelling of the Dynkin diagram. Then t lies in a subsystem subgroup $A_7(q)$ generated by $U_{\pm\alpha_i}$ for $i = 1, 3, 4, 5, 6, 7, 8$; it also lies in a subsystem $A_6(q)$ generated by $U_{\pm\alpha_i}$ for $i = 2, 4, 5, 6, 7, 8$. By Lemma 4.2, $\Delta(t)$ contains both of these subsystem subgroups. Together they generate G, so $\Delta(t) = G$.

Now consider $k = 3$, so $t \in A_1^3$. Here we can take $t = u_{\alpha_4}(1)u_{\alpha_6}(1)u_{\alpha_8}(1)$, and we see that $\Delta(t)$ contains subsystem subgroups $A_7(q)$ and $A_6(q)$ as in the previous case.

Finally consider $k = 4$. A Levi subgroup A_1^4 lies in a subsystem subgroup A_8, so Lemma 4.2 gives $\Delta(t) \geq A_8(q)$, which is maximal. Also $\Delta(t)$ is $C_G(t)$-invariant, and $C_G(t) = [q^{84}].C_4(q)$ by [12]. Hence $\Delta(t) = G$.

Case $G = E_7(q)$. Apart from root elements, there are four involution classes, labelled A_1^2, $(A_1^3)^{(1)}$, $(A_1^3)^{(2)}$, A_1^4, with representatives

$$\begin{aligned} A_1^2 &: u_{\alpha_4}(1)u_{\alpha_7}(1), \\ (A_1^3)^{(1)} &: u_{\alpha_2}(1)u_{\alpha_5}(1)u_{\alpha_7}(1), \\ (A_1^3)^{(2)} &: u_{\alpha_3}(1)u_{\alpha_5}(1)u_{\alpha_7}(1), \\ A_1^4 &: u_{\alpha_2}(1)u_{\alpha_3}(1)u_{\alpha_5}(1)u_{\alpha_7}(1). \end{aligned}$$

The A_1^2 class is dealt with as for $E_8(q)$. For t in one of the A_1^3 classes, t lies in a subsystem subgroup $D_6(q)$, so by Lemma 4.5 we have $\Delta(t) \geq D_6(q)$; one can adjoin roots to those defining the A_1^3 Levis to obtain several different D_6 subsystem subgroups, all of which lie in $\Delta(t)$ and generate G. Finally, for t in the A_1^4 class, t again lies in a subsystem $D_6(q)$, so $\Delta(t)$ contains $D_6(q)$ and is $C_G(t)$-invariant (here $C_G(t) = [q^{42}] \cdot C_3(q)$), which is enough to force $\Delta(t) = G$.

Cases $G = E_6(q)$, $F_4(q)$. These are dealt with in similar fashion — the class representatives t lie in subsystem subgroups $D_5(q)$ or $D_4(q)$, respectively, so $\Delta(t)$ contains this subgroup and is $C_G(t)$-invariant, which forces $\Delta(t) = G$.

Case $G = G_2(q)$. Here $q \geq 4$ (as we excluded $G_2(2)'$ from the hypothesis). Apart from long root involutions, there is one involution class $\tilde{A}_1$, with representative $t = u_\alpha(1)$, where α is a short root. Take $\alpha = \alpha_1 + 2\alpha_2$, where α_1, α_2 are simple roots with α_2 short, and let Φ be the root system. Now $\tilde{A}_1 = \langle U_{\pm\alpha} \rangle$ is centralised by $A_1 = \langle U_{\pm\alpha_1} \rangle$, and the action of $A_1\tilde{A}_1$ on the 6-dimensional module $V_6 = V_G(\lambda_1)$ is $1 \otimes 1 \oplus 0 \otimes 1^{(2)}$ (where 1 denotes the natural 2-dimensional module — see [12, Chapter 11]). In particular the long and short root involutions act on V_6 as (J_2^2, J_1^2) and (J_2^3), respectively. For $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$, and for any involution $x \in A_1$, the element $u = xu_\alpha(\lambda)$ is conjugate to both t and tu; so u is joined to t in the graph $\Gamma(\mathcal{C})$. Hence $\Delta(t)$ contains $A_1 \times U_\alpha$. It is also $C_G(t)$-invariant, and $C_G(t) = [q^3].A_1(q)$, from which we see that $\Delta(t) \geq C_G(t)$. Since u is joined to t, also $\Delta(t) \geq C_G(u)$. We need to finally check that $\langle C_G(t), C_G(u) \rangle = G$. We shall prove this with $x = u_{\alpha_1}(1)$ (so $u = u_{\alpha_1}(1)u_\alpha(\lambda)$).

Suppose that $\langle C_G(t), C_G(u) \rangle \neq G$, and let M be a maximal subgroup of G containing $\langle C_G(t), C_G(u) \rangle$. From the maximal subgroups of G (see, for example, [2]), we see that M must be the parabolic subgroup $P = \langle U_{\pm\alpha_1}, U_\beta : \beta \in \Phi^+ \rangle$.

Write $U = \langle U_\beta : \beta \in \Phi^+ \rangle$. Since $C_G(u) \leq P$, there exists $v \in U$ such that $A_1^{v^{-1}} \leq C_G(u)$, and so $u^v \in C_G(A_1) = \tilde{A}_1$. Since $u^v \in U$, it follows that $u^v \in \tilde{A}_1 \cap U = U_\alpha$. However we can see from the commutator relations in U that this is not possible.

Therefore $\langle C_G(t), C_G(u) \rangle = G$, completing the proof for $G_2(q)$.

We now move on to the twisted exceptional groups $G(q)$. For these, there are convenient lists of involution class representatives given in [1].

Case $G = {}^2E_6(q)$. The involution class representatives t lie in a subsystem subgroup ${}^2D_5(q)$, so by Lemma 4.5, $\Delta(t)$ contains this subgroup and is $C_G(t)$-invariant, which forces $\Delta(t) = G$.

Case $G = {}^2F_4(q)'$. For $q = 2$, Lemma 3.4 implies that for both the involution class representatives $t \in {}^2F_4(2)'$ we have $\Delta(t) \geq {}^2F_4(2)'$. Now assume $q \geq 4$. Note that G has two classes of involutions. From [5, p. 75] we see that there are no involutions in ${}^2F_4(2) \setminus {}^2F_4(2)'$. Hence both involution classes are represented by elements t in a subgroup ${}^2F_4(2)'$ of G, and so by the $q = 2$ case we have $\Delta(t) \geq {}^2F_4(2)'$. Also $\Delta(t)$ is $C_G(t)$-invariant, and it follows that $\Delta(t) = G$.

Case $G = {}^3D_4(q)$. Again there are two involution classes, and both are represented by involutions t in a subgroup $G_2(q)$ of G. Hence by the $G_2(q)$ case, for $q \geq 4$ we have $\Delta(t) \geq G_2(q)$, and so $\Delta(t) = G$ by the $C_G(t)$-invariance. Finally, for $q = 2$ we have $\Delta(t) = G$ by Lemma 3.4.

Case $G = {}^2B_2(q)$. Here $q \geq 8$ and there is just one involution class t^G, and by Lemma 4.1, $\Delta(t)$ is the centre of a Sylow 2-subgroup.

This completes the proof for exceptional groups. □

5. Proof of Theorem 1

For the first part of this section $G = G(q)$ is a simple group of Lie type over $\mathbb{F}_q$, where $q = 2^a$. We consider the binary action of G on a set Ω. We assume that H is the stabiliser in G of a point $\omega \in \Omega$ and that H is a proper subgroup of G of even order.

By [6, Lemma 2.5] we know that the action of G on $(G : H)$, the set of cosets of H in G, is binary (or, put another way, we can assume that the action of G is transitive). Since H has even order, there exist involutions in G that fix elements of $(G : H)$. We take g to be an involution in H of maximal 2-fixity for the action of G on $(G : H)$. Now the following lemma immediately yields Theorem 1:

Lemma 5.1. *One of the following holds*:

(i) *The element g is a long root involution, $G = \mathrm{Sp}_{2n}(q)$, $\mathrm{PSU}_n(q)$ or ${}^2B_2(q)$, and $\Delta(g)$ is the centre of a long root subgroup that is contained in H (the centre of a Sylow 2-subgroup when $G = {}^2B_2(q)$).*

(ii) *The element g is a short root involution, $G = \mathrm{Sp}_4(q)$ with $q > 2$, and $\Delta(g)$ is a short root subgroup that is contained in H.*

If $G = {}^2B_2(q)$, then $H = \Delta(g)$.

Proof. The four possibilities for $\Delta(g)$ are given by Theorem 2. We examine these in turn, noting that Lemma 2.2 implies that $\Delta(g) \leq H$.

If item (i) of Theorem 2 holds, then we have $H = G$ which is a contradiction. Items (ii) and (iii) are listed in the statement of the lemma.

Finally, if item (iv) of Theorem 2 holds, then G, g and $\Delta(g)$ are shown in Table 1. But, in this case, Lemma 3.5 implies that $\Delta_\infty(g) = G$ and Lemma 2.7 implies that $\Delta_\infty(g) \leq H$. Once again we have a contradiction.

To complete the proof of the lemma we use the fact that if $G = {}^2B_2(q)$, then G has a single conjugacy class of involutions and so the fact that $H = \Delta(g)$ follows from [7, Theorem 1.1]. □

6. A partial converse to Theorem 1

The main result of this section, Proposition 6.3, pertains to the groups listed at items (ii) and (iii) of Lemma 4.1. We show that particular actions of these groups are binary. These are the first nonregular transitive binary actions to have been described for the families $\mathrm{PSp}_n(q)$, $\mathrm{PSU}_n(q)$ and ${}^2G_2(q)$.

As we mention in the introduction, we conjecture that the actions considered in Proposition 6.3 include all of the transitive binary actions of a simple group of Lie type over a field of characteristic 2, for which a point stabiliser has even order.

Recall that a group H is a *TI-subgroup* of a group G if H is a subgroup of G with the property that $H \cap H^g$ is either H or $\{1\}$ for $g \in G$. Now we need the following result.

Lemma 6.1 [7, Lemma 2.3]. *Suppose that G acts on Ω, the set of cosets of a TI-subgroup H. The action is binary if and only if, given any three distinct conjugates H_1, H_2 and H_3 of H, we have $H_1 \cap (H_2 H_3) = \{1\}$.*

Lemma 6.2. *Let $S = \mathrm{SL}_2(q)$ where $q = p^a$ for some prime p and positive integer a. Let H be a Sylow p-subgroup of G. For all H_1, H_2 and H_3, distinct conjugates of H, we have $H_1 \cap (H_2 H_3) = \{1\}$.*

Proof. Since S acts 2-transitively on the set of Sylow p-subgroups of S, we can assume that H_2 is the set of strictly upper-triangular matrices and H_3 is the set of strictly lower-triangular matrices. Now consider $h_2 \in H_2$ and $h_3 \in H_3$ and observe that

$$h_2 h_3 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1{+}ab & a \\ b & 1 \end{pmatrix}.$$

We wish to determine if $h_2 h_3$ can lie in a third conjugate, H_1, of H. If this were the case, then $h_2 h_3$ would be an element of order p. This requires that $h_2 h_3$ has two eigenvalues equal to 1 and so the characteristic polynomial is $\lambda^2 - 2\lambda + 1$. But this would imply that a or b is equal to 0. Thus if we take h_2 and h_3 to be nontrivial, then $h_2 h_3$ is not an element of order p. The result follows. □

Lemmas 6.1 and 6.2 show that the action of $\mathrm{SL}_2(q)$ on the cosets of a Sylow p-subgroup is binary. But in contrast, for q odd the action of $\mathrm{PSL}_2(q)$ is not binary, as shown in [7].

Proposition 6.3. *Let G and H be one of the following*:

(1) $G = \mathrm{PSp}_{2n}(q)$ $(n \geq 1)$ *or* $\mathrm{PSU}_n(q)$ $(n \geq 3)$ *and H is the centre of a long root subgroup of G.*

(2) $G = {}^2G_2(q)$ *or* ${}^2B_2(q)$, *and H is the centre of a Sylow p-subgroup of G with* $p = 3$ *or* 2, *respectively.*

Then the action of G on the set of right cosets of H is binary if and only if G is not equal to $\mathrm{PSp}_2(q)$ *with q odd.*

Note that q is an arbitrary prime power. When q is even, this lemma gives a partial converse to Theorem 1.

Proof. It is well known, and easy to confirm directly, that in every case H is a TI-subgroup of G. We will, therefore, prove Proposition 6.3 using Lemma 6.1.

If G is as in item (1), then we suppose, first, that $G = \mathrm{PSp}_2(q) = \mathrm{PSL}_2(q)$. In this case the action of G on the set of right cosets of H is binary if and only if q is even, by [7, Theorem 1.2].

Suppose next that $G = \mathrm{PSU}_3(q)$. Then the action of G on the set of conjugates of H is 2-transitive and any distinct pair of conjugates of H generate a subgroup isomorphic to $\mathrm{SL}_2(q)$. If (H_2, H_3) is such a pair, then H_2, H_3 are Sylow p-subgroups of $\langle H_2, H_3\rangle$ and so, if H_1 is a third conjugate of H satisfying $H_1 \cap H_2 H_3 \neq \{1\}$, then H_1 must be a subgroup of $\langle H_2, H_3\rangle$ (since H is a TI-subgroup). But now Lemma 6.2 implies that in this case $H_1 \cap H_2 H_3 = \{1\}$. Thus the criterion of Lemma 6.1 is satisfied and we conclude that the action is binary.

Suppose next that $G = \mathrm{PSp}_{2n}(q)$ with $n \geq 2$ or $\mathrm{PSU}_n(q)$ with $n \geq 4$. Then the action of G on the set of conjugates of H has rank 3 (see [11]) and so G has 3 orbits in its action by conjugation on the set of pairs of conjugates of H, one of which includes the pair (H, H), which we can ignore.

A second orbit includes the pair (H, H^{opp}) where H^{opp} is the centre of an ‘opposite’ root subgroup of H; in particular $\langle H, H^{\mathrm{opp}}\rangle \cong \mathrm{SL}_2(q)$. Now the argument proceeds just as for $\mathrm{PSU}_3(q)$, via Lemma 6.2, to lead us to conclude that the criterion of Lemma 6.1 is satisfied when (H_2, H_3) is in this orbit.

The third orbit contains all pairs of conjugates of H that commute with each other. We take $G = \mathrm{PSp}_{2n}(q)$ first and display two such conjugates explicitly as follows: we let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a hyperbolic basis for $V = \mathbb{F}_q^n$ with respect to an alternating form φ and we set $H_2^\dagger$ to be the long root subgroup consisting of linear maps g that fix every basis vector except for e_1 and for which there exists $\alpha_g \in \mathbb{F}_q$ such that

$$e_1^g = e_1 + \alpha_g f_1.$$

Similarly $H_3^\dagger$ is the long root subgroup consisting of linear maps g that fix every basis vector except for e_2 and for which there exists $\alpha_g \in \mathbb{F}_q$ such that

$$e_2^g = e_2 + \alpha_g f_2.$$

It is clear that nontrivial elements of $H_2^\dagger$ and $H_3^\dagger$ have a single Jordan block of size 2, whereas an element of $H_2^\dagger H_3^\dagger \setminus (H_2^\dagger \cup H_3^\dagger)$ has two Jordan blocks of size 2.

Writing H_2 (resp. H_3) for the projective image of $H_2^\dagger$ (resp. $H_3^\dagger$) we conclude that the only long root elements in $\langle H_2, H_3\rangle$ lie in $H_2 \cup H_3$. Since H is a TI-subgroup, we conclude that if H_1 is a third conjugate of H, then $H_1 \cap H_2H_3 = \{1\}$ and the criterion of Lemma 6.1 is verified in this case too. We conclude, therefore, that the action of G on the set of right cosets of H is binary.

The proof for $G = \mathrm{PSU}_n(q)$ is identical except that our basis is written with respect to a Hermitian form and we require that $\alpha_g \in \mathbb{F}_{q^2}$ satisfies $\alpha_g + \alpha_g^q = 0$.

If G is as in item (2), then the result for $G = {}^2B_2(q)$ follows from [7, Theorem 1.1]. We are left with the case $G = {}^2G_2(q)$ for which we use facts from [14]. The group H is the centre of a Sylow 3-subgroup of G, and all of its nonidentity elements lie in the same conjugacy class C; a representative is denoted by X in [14]. Moreover $H \cong (\mathbb{F}_q, +)$, and we can parametrise the elements of H as $\{h(c) : c \in \mathbb{F}_q\}$, where $h(c)h(d) = h(c+d)$.

Let H_2, H_3 be distinct conjugates of H. We claim that

$$H_2H_3 \cap C = (H_2 \cup H_3) \setminus \{1\}, \tag{4}$$

from which the result will follow in the usual way from Lemma 6.1. To prove (4), we use character theory to count the number of triples in the set

$$S := \{(x_1, x_2, x_3) \in C^3 : x_1x_2x_3 = 1\}.$$

By the well-known Frobenius formula (see, for example, [10, 30.4]), we have

$$|S| = \frac{|C|^3}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(X)^3}{\chi(1)}.$$

All the values $\chi(X)$ for $\chi \in \mathrm{Irr}(G)$ are listed in [14, p. 87], and we compute from these that

$$|S| = (q^3 + 1)(q - 1)(q - 2).$$

The number of triples in S of the form $(h(c), h(d), h(-c-d))$ is $(q-1)(q-2)$, and if we multiply this by the number of conjugates of H, we obtain $|S|$. It follows that every triple in S is a conjugate of such a triple $(h(c), h(d), h(-c-d))$, and (4) follows. □

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FINITE SIMPLE GROUPS HAVE MANY CLASSES OF p-ELEMENTS

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Dedicated with admiration and thanks to the memory of our colleague Gary M. Seitz

For an element x of a finite group T, the $\mathrm{Aut}(T)$-class of x is $\{x^\sigma \mid \sigma \in \mathrm{Aut}(T)\}$. We prove that the order $|T|$ of a finite nonabelian simple group T is bounded above by a function of the parameter $m(T)$, where $m(T)$ is the maximum, over all primes p, of the number of $\mathrm{Aut}(T)$-classes of elements of T of p-power order. This bound is a substantial generalisation of the results of Pyber (1992) and of Héthelyi and Külshammer (2005), and it has implications for relative Brauer groups of finite extensions of global fields.

1. Introduction

In 1992 Pyber [33] showed that a group of order n contains at least $O\left(\frac{\log n}{(\log\log n)^8}\right)$ conjugacy classes of elements. This solved a problem of Brauer from 1963 [3], who had asked for a significant improvement on his lower bound of $\log\log n$. In Section 1.1 we briefly discuss the interesting story around these bounds, which date back to work of Landau in 1903 and extend to recent work in 2017. The special case of Pyber's bound for a nonabelian simple group T could be turned around to state that $|T| < c^{f(m)}$ where m is the number of $\mathrm{Aut}(T)$-classes in T, c is a constant, and f is the particular function $f(m) = (\log m)^2 \cdot \log\log m$. This alternative statement of Pyber's result was used in [32, Theorem 4.4] to prove a conjecture about maximal subgroups of a finite group which are "covering subgroups", and in turn, this application had consequences for Kronecker classes of algebraic number fields (see [32, Section 4]).

Many classical results concerning conjugacy classes of elements in groups have analogues in the case where the conjugacy classes are restricted to those consisting of elements of prime power order. For example, given a core-free subgroup H of a group G, not only is a there a conjugacy class of elements of G that does not meet H, but Fein, Kantor and Schacher [9] showed that there is a conjugacy class of elements of prime power order that avoids H. Similarly, not only does every

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nonlinear irreducible character of a finite group vanish on some conjugacy class of elements, but Malle, Navarro and Olsson [27] showed that each such character must vanish on some conjugacy class of elements of prime power order. Such analogues usually have interesting applications: for example, the Fein–Kantor–Schacher result is equivalent (see [9, Section 3]) to the fact that the relative Brauer group of a nontrivial finite extension of global fields is infinite.

In this paper we prove a new bound (Theorem 1.1) on the order of a finite simple group related to its p-elements, that is, elements of p-power order for various primes p. The bound is a substantial generalisation of the results of Pyber and others, in that the parameter m above is replaced by

$$m(T) = \max_{p \text{ prime}} m_p(T), \tag{1}$$

where $m_p(T) = \#\{\mathrm{Aut}(T)\text{-classes of elements of } p\text{-elements in } T\}$.

Theorem 1.1. *There exists an increasing function f on the natural numbers such that, for a finite nonabelian simple group T, the order of T is at most $f(m(T))$.*

In the proof of Theorem 1.1 for exceptional groups of Lie type, we are indebted to the work of Gary Seitz, to whose memory this paper is dedicated, and his coauthors Martin Liebeck and Jan Saxl, for their classification of the subgroups of maximal rank of these groups [25]. Their results gave us the detailed information about certain tori and their normalisers on which our proof is based. In addition, in the proof of Theorem 1.1 for classical groups, we use the description by Aschbacher and Seitz [1] of conjugacy classes of involutions in Chevalley groups of even characteristic to bound the dimension (Lemma 4.1).

The function $f(n)$ we obtain in the proof of Theorem 1.1 involves an $n!$ term. It is possible that a better function might be obtained, see Remark 4.3 for further comments. Since the function $f(n)$ in Theorem 1.1 is increasing, the bound can be turned around to give a lower bound in terms of $|T|$ for the number of $\mathrm{Aut}(T)$-classes of p-elements in T.

Corollary 1.2. *There exists an increasing function g on the natural numbers such that, for a finite nonabelian simple group T, there is a prime p dividing $|T|$ such that the number of* $\mathrm{Aut}(T)$*-classes of elements of p-power order in T is at least $g(|T|)$.*

There are numerous bounds in the literature that relate $|T|$ with various parameters concerning numbers of conjugacy classes or $\mathrm{Aut}(T)$-classes. For example, by [28, Theorems 1.2 and 1.4], for a given prime p dividing $|T|$, the order $|T|$ is bounded above in terms of the number of its p-regular conjugacy classes (elements of order coprime to p) and also, apart from certain rank 1 Lie type simple groups, $|T|$ is bounded above in terms of the number of its p-singular conjugacy classes (elements of order a multiple of p). One motivation for proving Theorem 1.1 is a conjecture concerning finite groups G with a proper subgroup that meets all $\mathrm{Aut}(G)$-classes of elements of G of prime power order [32, Conjecture 4.3′]. Thus, rather than considering p-singular or p-regular elements for some fixed prime p, we must work with all elements of prime power order. In future work [12] we apply Theorem 1.1 to prove an important case of [32, Conjecture 4.3′], which has consequences for relative Brauer groups of field extensions as discussed in [9; 12; 14].

Finally we note that many bounds of this type in the literature are available for general finite groups, and it would be interesting to know if the bounds in Theorem 1.1 and Corollary 1.2 can be used to obtain similar bounds for larger families of finite groups.

This paper is organised as follows. In Section 2 we prove some preliminary numerical results and a result relating the normalisers of cyclic subgroups S of a group G with our parameter $m(G)$ (1). We treat the alternating groups in Section 3 and the bulk of the work takes place in Sections 4 and 5 where we consider the classical groups and the exceptional groups of Lie type, respectively. Finally in Section 6 we complete the proof of Theorem 1.1.

1.1. ***Commentary on Landau's and Pyber's theorems.*** Landau's theorem [23] from 1903 states that, for a given positive integer k, there are only finitely many finite groups having exactly k conjugacy classes of elements, and so such a group must have order bounded in terms of k. Brauer [3] made this bound explicit in 1963, showing that a group of order n has at least $\log\log n$ conjugacy classes, and asked for a substantially better bound. Providing an improvement was the main focus of Pyber's 1992 paper [33], where he proved that a group of order $n \geqslant 4$ must have at least $c \log n/(\log\log n)^8$ conjugacy classes for some "computable constant" c. The relevance for this paper is Pyber's bound for nonabelian simple groups [33, Lemma 4.4]: if T is a finite nonabelian simple group and $a = |\mathrm{Aut}(T)|$, then the number of $\mathrm{Aut}(T)$-classes in T is at least

$$2^{c(\log a/\log\log a)^{1/2}} \quad \text{for some constant } c.$$

Since 1992 there have been numerous contributions that strengthened these bounds (see [2] for an overview). Currently the best lower bound for the number $k(G)$ of conjugacy classes of an arbitrary finite group G is given by Baumeister, Maróti and Tong-Viet in 2018 [2, Theorem 1.1]:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall \text{ finite groups } G \text{ with } |G| \geqslant 3,\ k(G) \geqslant \frac{\delta \log |G|}{(\log\log |G|)^{3+\epsilon}}.$$

Many better lower bounds are available for restricted classes of groups, for example, for G soluble then Keller [19] proved that $k(G) \geqslant \frac{c \log |G|}{\log\log |G|}$ for some constant c; a purely logarithmic lower bound $k(G) > \log_3 |G|$ was given in [2, Theorem 1.2] for groups with trivial soluble radical; and a better than logarithmic lower bound was obtained for nilpotent groups by Jaikin-Zapirain [18]. These bounds have been exploited to obtain related bounds concerning irreducible complex representations, for example, bounding the number of irreducible characters of odd degree [11; 16] in connection with the McKay conjecture. Also, as mentioned above, there are various results in [28] that give lower bounds for the number of conjugacy classes of p-regular elements, or p-singular elements, or the total number of classes of elements of prime power order (adding over the prime divisors) in [15], but to our knowledge our bound in terms of classes of p-elements, for a certain single prime p, is new.

Remark 1.3. Our proof of Theorem 1.1 for simple classical groups hints towards a possible bound for the order of a simple group in terms of another property. For a group G and a prime s we define

$$m_{s\text{-exp}}(G) = \#\{\mathrm{Aut}(G)\text{-classes of elements of order } \exp(G)_s\},$$

where $\exp(G)_s$ is the s-part of the exponent $\exp(G)$, and set $m_{\exp}(G)$ to be the maximum of $m_{s\text{-exp}}(G)$ over all primes s dividing $|G|$. In addition, for a set $\mathcal{S}$ of primes we set $m_{\mathcal{S}\text{-exp}}(G)$ to be the maximum of $m_s(G)$ over all primes $s \in \mathcal{S}$.

The above concepts are motivated by choices made in the proof of Proposition 4.2. It turns out that for simple classical groups T, apart from the characteristic p, the set $\mathcal{S}(T)$ of primes s we consider all have the property that the Sylow s-subgroups are cyclic and hence have order $\exp(T)_s$, and the s-elements we consider are those of maximal order $\exp(T)_s$. Thus for a simple classical group T of characteristic p, we prove that $|T|$ is bounded above by a function of

$$m'(T) = \max\{m_p(T), m_{\mathcal{S}(T)\text{-exp}}(T)\},$$

where $m_p(T)$ is as in (1). See Remark 4.4 for further details. This motivated us to consider whether a similar bound holds for other simple groups. In Remark 3.2 we show, using the prime number theorem, that for large enough m, $|\mathrm{Alt}(m)|$ is bounded above by a function of $m_{\exp}(\mathrm{Alt}(m))$. We believe this style of bound has not previously been studied. It would be interesting to know if such a bound holds also for the simple exceptional groups of Lie type.

2. Preparatory lemmas

For a prime s and integer n, let n_s denote the s-part of n, that is, the highest power of s dividing n, and let $n_{s'} = n/n_s$ denote the part of n prime to s. For an integer m define $\mathcal{S}(m)$ to be the set of all prime divisors s of $q^m - 1$ such that $(q^m - 1)_s$ does not divide $q^t - 1$ for any $t < m$. In our analysis we need that

$$\mathrm{Prod}(m, q) := \prod_{s \in \mathcal{S}(m)} (q^m - 1)_s$$

is large enough, for example, larger than q or some constant multiple of q. We use the following lemma to treat the classical groups.

Lemma 2.1. *Let q be a prime power and m a positive integer.*

(i) *Suppose that s is a prime divisor of q^m-1. If t is the smallest positive integer such that $(q^m-1)_s$ divides q^t-1, then t divides m.*

(ii) $q+1$ *divides* $\mathrm{Prod}(2,q)$, *and* q^2+1 *divides* $\mathrm{Prod}(4,q)$.

(iii) *If m is odd and $m>1$, then $2\notin\mathcal{S}(2m)$ and* $\mathrm{Prod}(2m,q)$ *divides* q^m+1.

(iv) *Suppose that m is an odd prime. Then*:

(1) *If $s\in\mathcal{S}(m)$ and $s\,|\,q-1$, then $s=m$; if $s\in\mathcal{S}(2m)$ and $s\,|\,q^2-1$, then $s=m$ and $m\,|\,q+1$.*

(2) $(q^m-1)/(q-1)$ *divides* $\mathrm{Prod}(m,q)$.

(3) $(q^m+1)/(q+1)$ *divides* $\mathrm{Prod}(2m,q)$.

Proof. (i) Let $s^b=(q^m-1)_s$ and let t be the least positive integer such that s^b divides q^t-1. Then $t\leqslant m$. Write $m=kt+r$, where $1\leqslant r\leqslant t$. Then s^b divides $\gcd(q^m-1,q^t-1)=q^{\gcd(m,t)}-1$, and $\gcd(m,t)=\gcd(r,t)\leqslant r\leqslant t$. By the minimality of t we have $r=t$, and hence t divides m.

(ii) If a prime $s\,|\,q^2-1$ then either $s\in\mathcal{S}(2)$ or $(q^2-1)_s$ divides $q-1$, and hence q^2-1 divides $(q-1)\cdot\mathrm{Prod}(2,q)$, so $\mathrm{Prod}(2,q)$ is divisible by $q+1$. Similarly, if a prime $s\,|\,q^4-1$ then, by part (i), either $s\in\mathcal{S}(4)$ or $(q^4-1)_s$ divides q^2-1, and hence q^2+1 divides $\mathrm{Prod}(4,q)$.

(iii) Next assume that m is odd and $m>1$. If q is even then $2\notin\mathcal{S}(2m)$ by the definition of $\mathcal{S}(2m)$. It turns out that this also holds if q is odd: by [13, Lemma 2.5],

$$(q^{2m}-1)_2=((q^2)^m-1)_2=(q^2-1)_2$$

and as $m>1$, again $2\notin\mathcal{S}(2m)$ by the definition of $\mathcal{S}(2m)$. Thus, for any q, $\mathcal{S}(2m)$ consists of odd primes. Since $\gcd(q^m-1,q^m+1)=(2,q-1)$, it follows that for any odd prime s, $(q^{2m}-1)_s$ divides exactly one of q^m+1 and q^m-1. Moreover, if $s\in\mathcal{S}(2m)$ then $(q^{2m}-1)_s$ does not divide q^m-1 by definition, and hence, for each $s\in\mathcal{S}(2m)$, we have that $(q^{2m}-1)_s$ divides q^m+1. Hence $\mathrm{Prod}(2m,q)$ divides q^m+1, proving (iii).

(iv) Now assume that m is an odd prime and let s be a prime dividing q^m-1. Then either $s\in\mathcal{S}(m)$ or $(q^m-1)_s$ divides $q-1$, and hence q^m-1 divides $(q-1)\cdot\mathrm{Prod}(m,q)$, so $\mathrm{Prod}(m,q)$ is divisible by $(q^m-1)/(q-1)$. Furthermore, if $s\in\mathcal{S}(m)$ and $s\,|\,q-1$, then s must divide $(q^m-1)/(q-1)$ by the definition of $\mathcal{S}(m)$, and hence s divides

$$\gcd\left(\frac{q^m-1}{q-1},q-1\right)=\gcd(m,q-1).$$

Since m is prime this implies $s = m$. Similarly, if $s \in \mathcal{S}(2m)$ and s divides $q^2 - 1$, then s must divide $(q^{2m} - 1)/(q^2 - 1)$ by the definition of $\mathcal{S}(2m)$, and hence s divides

$$\gcd\left(\frac{q^{2m}-1}{q^2-1}, q^2 - 1\right) = \gcd(m, q^2 - 1).$$

Since m is prime this implies that $s = m$ divides $q^2 - 1$. Further, by part (iii), $s \mid q^m + 1$ and hence s divides $\gcd(q^m + 1, q^2 - 1) = q + 1$. Thus parts (1) and (2) are proved.

Finally, if a prime s divides $q^{2m} - 1$ and t is minimal such that $(q^{2m} - 1)_s$ divides $q^t - 1$, then either $s \in \mathcal{S}(2m)$, or $t \in \{m, 2, 1\}$. Thus $q^{2m} - 1$ divides

$$\mathrm{Prod}(2m, q) \cdot \frac{(q^m - 1)(q^2 - 1)}{\gcd(q^m - 1, q^2 - 1)} = \mathrm{Prod}(2m, q) \cdot (q^m - 1)(q + 1).$$

It follows that $(q^m + 1)/(q + 1)$ divides $\mathrm{Prod}(2m, q)$. □

Definition 2.2. We say that a group G is *prime power bounded by* n, or simply, *pp-bounded by* n, if $m(G) \leqslant n$, where $m(G)$ is as in (1). In other words, for each prime p dividing $|G|$, the number of $\mathrm{Aut}(G)$-classes of elements of p-power order in G is at most n.

In the following lemma ϕ denotes the Euler ϕ-function, namely for a positive integer m, $\phi(m)$ is the number of positive integers at most m and coprime to m.

Lemma 2.3. *Suppose that G is a group that is pp-bounded by n. Let $S \leqslant G$ be a nontrivial cyclic s-subgroup of order s^b, where s is prime, and let $N = N_{\mathrm{Aut}(G)}(S)$. Then there is a bijection $\mathcal{C} \to \mathcal{D}$, where*

$$\mathcal{C} := \{x^{\mathrm{Aut}(G)} \mid x \in S, o(x) = s^b\}, \quad \mathcal{D} := \{x^N \mid x \in S, o(x) = s^b\}$$

and $|\mathcal{C}| = \phi(|S|)/r$ with $r = |N_{\mathrm{Aut}(G)}(S) : C_{\mathrm{Aut}(G)}(S)|$. Furthermore,

$$\phi(|S|) = s^{b-1}(s-1) \leqslant rn \quad \textit{and} \quad \phi(|S|) \textit{ divides } r(n!).$$

Proof. Note that if $x_1, x_2 \in S$ have order s^b and are such that $x_1^g = x_2$ for some $g \in \mathrm{Aut}(G)$, then $S^g = \langle x_1 \rangle^g = \langle x_1^g \rangle = \langle x_2 \rangle = S$, so that $g \in N$. Conversely, if $x_1, x_2 \in S$ are conjugate by an element of N, then they are also conjugate under the action of $\mathrm{Aut}(G)$. This gives a bijection $\mathcal{C} \to \mathcal{D}$ as claimed.

For $x^N \in \mathcal{D}$, we know $\langle x \rangle = S$, so $|x^N| = |N : C_N(x)| = |N : C_N(S)|$. Note that $C_{\mathrm{Aut}(G)}(S) \leqslant N_{\mathrm{Aut}(G)}(S) = N$ so that $C_{\mathrm{Aut}(G)}(S) = C_N(S)$. Hence $|N : C_N(S)| = r$. Thus each N-class in $\mathcal{D}$ has length exactly r. Since S contains $\phi(|S|)$ elements of order $|S|$ (where ϕ is Euler's function), r divides $\phi(|S|)$ and $|\mathcal{D}| = \phi(|S|)/r$. Since G is pp-bounded by n, we have $\phi(|S|)/r = |\mathcal{D}| \leqslant n$, and hence $\phi(|S|)$ divides $r(n!)$. □

Notation 2.4. Throughout the paper $\log x$ denotes the natural logarithm of x.

Lemma 2.5. *For all $a > 1$ we have*

$$\frac{a^{1/4}}{4} < \frac{\sqrt{a}}{\log a}.$$

Proof. The inequality in the statement is equivalent to $\log a < 4a^{1/4}$. Exponentiating both sides, we see the latter inequality is equivalent to $a < e^{4a^{1/4}}$, that is,

$$f(a) := e^{4a^{1/4}} - a > 0.$$

Now

$$f'(a) = \frac{e^{4a^{1/4}}}{a^{3/4}} - 1 \quad \text{and} \quad f''(a) = \frac{e^{4a^{1/4}}}{a^{3/2}}\Big(1 - \frac{3}{4a^{1/4}}\Big).$$

For $a \geqslant 1$ we have $f''(a) > 0$, so that $f'(a)$ is increasing for $a \geqslant 1$. Further, $f'(1) = e^4 - 1 > 0$, so $f'(a) > 0$ for all $a \geqslant 1$. Thus $f(a)$ is increasing for $a \geqslant 1$, and since $f(1) = e^4 - 1 > 0$, we have $f(a) > 0$ for all $a \geqslant 1$, as required. □

Lemma 2.6. *Suppose that $q = p^a$, $p \geqslant 2$ and $a \geqslant 1$. Then*

$$\frac{\sqrt{q}\log 2}{2} \leqslant \frac{q}{a}.$$

Proof. Note that $a = \log q/\log p \leqslant \log q/\log 2$. Now since $\log x \leqslant 2\sqrt{x}$ for all $x \in \mathbb{R}$ with $x > 0$, we have

$$a \leqslant \frac{2\sqrt{q}}{\log 2}$$

and thus $a\sqrt{q} \leqslant 2q/\log 2$, which yields

$$\frac{\sqrt{q}\log 2}{2} \leqslant \frac{q}{a}.$$ □

3. The alternating groups

We first consider the alternating groups.

Lemma 3.1. *If $T \cong \mathrm{Alt}(m)$ is pp-bounded by n, then $|T| \leqslant \frac{1}{2}(3n+2)!$.*

Proof. Suppose that $T \cong \mathrm{Alt}(m)$ is pp-bounded by n. If $m \neq 6$, we note that T has $\lfloor m/3 \rfloor$ $\mathrm{Aut}(T)$-classes of elements of elements order 3. This gives $m \leqslant 3n+2$, and hence $|T| = \frac{1}{2}\cdot m! \leqslant \frac{1}{2}\cdot(3n+2)!$. If $m = 6$, then T has two classes of 2-elements, so $n \geqslant 2$. Then certainly $|T| = 360 \leqslant \frac{1}{2}\cdot(3n+2)!$ holds. □

Remark 3.2. As discussed in Remark 1.3, another type of bound on $|T|$ for a simple group T is in terms of the number of $\mathrm{Aut}(T)$-classes of elements with order equal to $\exp(T)_p$, the p-part of the exponent of T. We explain briefly that there is such a bound when $T \cong \mathrm{Alt}(m)$, for sufficiently large m. Recall that $\exp(\mathrm{Alt}(m)) = \mathrm{lcm}\{m' : 1 \leqslant m' \leqslant m\}$.

Let $\pi(x)$ denote the number of integers less than or equal to x that are prime. Based on the prime number theorem, we know (for sufficiently large x) that

$$c\frac{x}{\log x} < \pi(x) < d\frac{x}{\log x} \tag{2}$$

for explicit constants c and d. For example, if $x \geqslant 11$, then we can take $c = 1$, and for x "sufficiently large" we can take $d = 1.04423$; see [34, pp. 176–177]. These bounds allow us to prove existence of primes in certain intervals, somewhat analogously to Bertrand's postulate which asserts that there is a prime between m and $2m$ for all positive integers m. We are interested here in the existence of a prime p such that $\sqrt{m/3} < p \leqslant \sqrt{m/2}$; and we claim that such a prime exists for sufficiently large m. Indeed, if $\sqrt{m/3}$ is large enough so that (2) holds with $c = 1$, then we have

$$\pi(\sqrt{m/2}) > \frac{\sqrt{m/2}}{\log\sqrt{m/2}} \quad \text{and} \quad \pi(\sqrt{m/3}) < d\frac{\sqrt{m/3}}{\log\sqrt{m/3}},$$

and a sufficient condition for p to exist is that

$$\frac{\sqrt{m/2}}{\log\sqrt{m/2}} > d\frac{\sqrt{m/3}}{\log\sqrt{m/3}},$$

which is equivalent to

$$\log m > \frac{\frac{1}{\sqrt{2}}\log 3 - \frac{1}{\sqrt{3}}d\log 2}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}d}$$

and since d is a constant, this holds for sufficiently large m.

Thus, if m is large enough we may choose a prime p such that $\sqrt{m/3} < p \leqslant \sqrt{m/2}$, or equivalently, $2p^2 \leqslant m < 3p^2$. We may assume that $p > 3$ (by taking $m > 27$) and then we see that the p-part $\exp(T)_p$ of the exponent is exactly p^2. Meanwhile, the inequality $2p^2 \leqslant m$ implies that T has at least p $\mathrm{Aut}(T)$-classes of elements of order p^2, namely, for $1 \leqslant i \leqslant p$, elements that have a single p^2-cycle and exactly i cycles of length p. Such elements with different values of i cannot be conjugate in $\mathrm{Aut}(T) \cong \mathrm{Sym}(m)$ because they have different cycle types, so there are at least p $\mathrm{Aut}(T)$-classes of elements with exponent $\exp(T)_p$, that is, $m_{\exp}(T) \geqslant p$. Thus $m < 3p^2 \leqslant 3(m_{\exp}(T))^2$, and hence $|T|$ is bounded above by $\frac{1}{2}(3(m_{\exp}(T))^2)!$.

4. Finite simple classical groups

Let T be a finite simple classical group defined over a field of order $q = p^a$, as in one of the lines of Table 1. The table records also the natural module V for T and its dimension, and the covering group G of T in $\mathrm{GL}(V)$.

Lemma 4.1. *Let T and d be as in* Table 1. *Then the number of unipotent conjugacy classes is at least d.*

T	d	V	G	conditions
$\mathrm{PSL}(d,q)$	$d \geqslant 2$	$\mathbb{F}_q^d$	$\mathrm{SL}_d(q)$	$(d,q) \neq (2,2)$ or $(2,3)$
$\mathrm{PSU}(d,q)$	$d \geqslant 3$	$\mathbb{F}_{q^2}^d$	$\mathrm{SU}_d(q)$	$(d,q) \neq (3,2)$
$\mathrm{PSp}(2d,q)$	$d \geqslant 2$	$\mathbb{F}_q^{2d}$	$\mathrm{Sp}_{2d}(q)$	$(d,q) \neq (2,2)$
$\mathrm{P}\Omega^\circ(2d{+}1,q)$	$d \geqslant 3$	$\mathbb{F}_q^{2d+1}$	$\Omega_{2d+1}^\circ(q)$	q odd
$\mathrm{P}\Omega^\epsilon(2d,q)$	$d \geqslant 4$	$\mathbb{F}_q^{2d}$	$\Omega_{2d}^\epsilon(q)$	$\epsilon \in \{+,-\}$

Table 1. Simple classical groups, their covering groups and natural modules.

Proof. Note that the identity element forms a unipotent conjugacy class, and since there is always a nonidentity unipotent conjugacy class, the assertion holds if $d = 2$. Thus we may assume that $d \geqslant 3$. Furthermore, making use of the identity, we need to show the existence of $d-1$ nonidentity unipotent classes. The unipotent conjugacy classes in classical groups are well known; see, e.g., [5; 10; 24; 36] or [1] for unipotent elements of order 2.

Let J_i denote an $i \times i$ Jordan block with all 1's down the diagonal. Then for each i such that $2 \leqslant i \leqslant d$, the group $T = \mathrm{PSL}(d,q)$ contains an element corresponding to the matrix that is the direct sum of J_i and I_{d-i}. These are clearly not conjugate under any element of $\mathrm{P\Gamma L}(d,q)$ or under the inverse transpose map, so the number of $\mathrm{Aut}(T)$-conjugacy classes of nonidentity unipotent elements is at least $d-1$. Each of these conjugacy classes meets $\mathrm{PSU}(d,q)$ nontrivially and so for $T = \mathrm{PSU}(d,q)$ the number of $\mathrm{Aut}(T)$-conjugacy classes of nonidentity unipotent elements is also at least $d-1$.

Suppose now that $T = \mathrm{PSp}(2d,q)$ (with $d \geqslant 3$). For each $i \leqslant d$, the group T contains an element whose corresponding matrix has Jordan canonical form that is the direct sum of i copies of J_2 and $I_{2(d-i)}$. This gives us d different conjugacy classes in T that are clearly not fused in $\mathrm{P\Gamma Sp}(2d,q)$. Since $2d > 4$, we have $\mathrm{Aut}(T) = \mathrm{P\Gamma Sp}(2d,q)$, and the result is proved.

Next suppose that $T = \mathrm{P}\Omega^\circ(2d+1,q)$ and q is odd. For each even i with $1 \leqslant i \leqslant 2d+1$, there are unipotent elements whose corresponding matrix has Jordan canonical form being the direct sum of J_i and I_{2d+1-i}. This gives d distinct classes that are clearly not fused in $\mathrm{P\Gamma O}(2d+1,q) = \mathrm{Aut}(T)$, as required.

Finally, we have $T = \mathrm{P}\Omega^\epsilon(2d,q)$ with $\epsilon \in \{\pm\}$. First suppose that q is odd. For each i with $1 \leqslant i \leqslant d$, there are unipotent elements in T whose corresponding matrix has Jordan canonical form being the direct sum of J_{2i} and $I_{2(d-i)}$. This gives d distinct classes that are clearly not fused under inner, diagonal or field automorphisms of T. Moreover, when $2d = 8$ and $\epsilon = +$, from [4, Proposition 3.55] we see that none of these classes are fused under a triality automorphism. Hence T has at least d $\mathrm{Aut}(T)$-classes of unipotent elements. Suppose now that q is even. We again follow the notation of [1, Section 8] for involutions in T and use the geometric description in [5, Section 3.5.4]. For each even $i < d$ we get two T-classes of involutions with Jordan canonical form consisting of i copies of J_2. When $i = d$ and d is even, we get one such T-class when $\epsilon = -$ and three when $\epsilon = +$. Fusing of these classes in $\mathrm{P\Gamma O}^{\pm}(2d, q)$ occurs only when $\epsilon = +$ and $i = d$ is even, in which case two of the three classes fuse. When d is odd we have $\mathrm{Aut}(T) = \mathrm{P\Gamma O}^{\pm}(2d, q)$ and so we see that T has at least $d - 1$ $\mathrm{Aut}(T)$-classes of involutions as required. Suppose now that d is even. Then the number of T-classes of involutions is $2(\lfloor (d-1)/2 \rfloor) + 3 = d + 1$ when $\epsilon = +$ and $2(\lfloor (d-1)/2 \rfloor) + 1 = d$ when $\epsilon = -$. Thus if $\epsilon = -$, or if $\epsilon = +$ and $2d \neq 8$, we see that T has at least d $\mathrm{Aut}(T)$-classes of involutions. Finally, if $2d = 8$ and $\epsilon = +$, then two of the classes with $i = 4$ (denoted a_4 and a_4' in [5]) are fused with one of the classes with $i = 2$ (denoted c_2 in [5]) under triality [4, Proposition 3.55]. Hence in this final case we also get at least $d - 1$ $\mathrm{Aut}(T)$-classes of involutions, as required. □

Proposition 4.2. *There is an increasing integer function g such that, if T is a finite simple classical group as in Table 1, and T is pp-bounded by n, then $|T| < g(n)$.*

Proof. The group T has a natural module V as in Table 1, and it is convenient to work with the preimage $G \leqslant \mathrm{GL}(V)$ acting linearly on V. We set $q = p^a$ with p prime. Let φ denote the natural map $\varphi : G \to T$, and for a subgroup H of T let $\hat{H}$ denote the full preimage of H under φ. We first observe that, by Lemma 4.1, the number of $\mathrm{Aut}(T)$-classes of unipotent elements of T is at least d, and hence $d \leqslant c_1(n)$ with $c_1(n) = n$.

There is a prime m satisfying $\frac{d}{2} < m \leqslant d$, and we choose the smallest possible value for m, except that we choose $m = 3$ if $d = 3$. Then one of the following holds:

(i) $m < d$.

(ii) $d = m = 2$ with $T = \mathrm{PSL}(2, q)$ or $\mathrm{PSp}(4, q)$.

(iii) $d = m = 3$ with $T = \mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, $\mathrm{PSp}(6, q)$, or $\mathrm{P\Omega}^{\circ}(7, q)$.

Choice of decomposition. If $m < d$ then we choose a decomposition $V = U \oplus W$, with $\dim(U) = m$ in the linear and unitary cases, and $\dim(U) = 2m$ in the symplectic and orthogonal cases (and possibly $\dim(W) = 0$). Additionally, if G preserves a form, we choose U to be nondegenerate and $W = U^{\perp}$. Finally, in the orthogonal case, we choose U of minus-type.

Choice of cyclic subgroup. In our arguments we will work with a cyclic subgroup of G that stabilises the decomposition $V = U \oplus W$ and acts trivially on W (see [30, Section 3] for a description of the linear action of these tori). More specifically:

(L) If $T = \mathrm{PSL}(d, q)$ we consider a cyclic subgroup H of order $\frac{q^m - 1}{(q-1)\cdot(m, q-1)}$ such that $\hat{H}^U$ is a Singer cycle of $\mathrm{SL}(U)$ and $\hat{H}$ fixes W pointwise.

case	G	$\lvert\hat{H}\rvert$	$\hat{H}^U$	$\hat{H}^W$	notes
(L)	$\mathrm{SL}(d,q)$	$(q^m-1)/(q-1)$	$\leqslant \mathrm{GL}(1,q^m)$	1	$m=d$ if $d=2$ or 3
(U)	$\mathrm{SU}(d,q)$	$(q^m+1)/(q+1)$	$\leqslant \mathrm{GU}(1,q^m)$	1	$m=d$ if $d=3$
(Sp)	$\mathrm{Sp}(2d,q)$	q^m+1	$\leqslant \mathrm{Sp}(2,q^m)$	1	$m=d$ if $d=2$ or 3
(O)	$\Omega(2d+1,q)$	$(q^m+1)_{2'}$	$\leqslant \mathrm{GO}^-(2,q^m)$	1	$m=d$ if $d=3$
(O)	$\Omega^{\pm}(2d,q)$	$(q^m+1)_{2'}$	$\leqslant \mathrm{GO}^-(2,q^m)$	1	

Table 2. Choices for a cyclic subgroup $\hat{H}$ of G.

(U) If $T=\mathrm{PSU}(d,q)$ then m is an odd prime and we consider a cyclic subgroup H of order $\frac{q^m+1}{(q+1)\cdot(m,q+1)}$ such that $\hat{H}^U$ is a Singer cycle of $\mathrm{SU}(U)$ and $\hat{H}$ fixes W pointwise.

(Sp) If $T=\mathrm{PSp}(2d,q)$ we consider a cyclic subgroup $\hat{H}$ of order q^m+1 such that $\hat{H}^U$ is a maximal torus of $\mathrm{Sp}(2m,q)$ and $\hat{H}$ fixes W pointwise.

(O) If $T=\mathrm{P}\Omega^{\epsilon}(2d,q)$ or $\mathrm{P}\Omega^{\circ}(2d+1,q)$ with $m<d$, and also in the exceptional case where $T=\mathrm{P}\Omega^{\circ}(7,q)$ with q odd and $d=m=3$, then the parameter m is an odd prime and U is of minus-type and dimension $2m$. By [22, Lemma 4.1.1(ii)], the stabiliser in G of the decomposition $V=U\oplus W$ contains the subgroup

$$\Omega(U)\times 1\cong\Omega^-(2m,q),$$

which fixes W pointwise. We consider a cyclic subgroup $\hat{H}$ of $\Omega(U)\times 1$ of order $\frac{q^m+1}{(q+1)_2}$ such that $\hat{H}^U$ is contained in a Singer cycle of $\Omega(U)$. Note that

$$|\mathrm{O}(U):\Omega(U)|=2\cdot(2,q-1)$$

(see [22, Table 2.1.C]) and that $(q^m+1)/(q+1)$ is odd since m is odd, so the 2-part $(q^m+1)_2=(q+1)_2$ and $|\hat{H}|=(q^m+1)_{2'}$.

Thus the cyclic subgroup $\hat{H}$ has the properties given in the appropriate row of Table 2. First we consider the linear case.

Case 1. $T=\mathrm{PSL}(d,q)$.

Here $|\hat{H}|=(q^m-1)/(q-1)$ and we set $H=\varphi(\hat{H})$. Also $H\cong\hat{H}$ if $m<d$, or if $m=d=3$ with $\gcd(3,q-1)=1$, or if $m=d=2$ with q even, since in these cases $\hat{H}$ contains no nontrivial scalar matrix. We assume first that $|H|=(q^m-1)/(q-1)$, and comment at the end on how to adjust our argument to deal with the exceptional cases when $m=d=2$ or 3 and m divides $q-1$.

The centraliser of $\hat{H}^U$ in $\mathrm{GL}(U)$ is the full Singer cycle $C=\mathrm{GL}(1,q^m)$ and $N_{\mathrm{GL}(U)}(\hat{H})=C.m$ (see [29, Lemma 2.1] or [17, Satz II.7.3, p. 187]). Setting $q=p^a$ with p prime, we have

$$N_{\Gamma\mathrm{L}(d,q)}(\hat{H})=\big(\mathrm{GL}(1,q^m)\cdot m\times\mathrm{GL}(d-m,q)\big)\cdot a$$

(or $\mathrm{GL}(1, q^m)\cdot ma$ if $m = d$). It follows that $r := |N_{\mathrm{Aut}(T)}(H)/C_{\mathrm{Aut}(T)}(H)|$ divides $2am$ (noting that in general an outer automorphism corresponding to the "inverse transpose map" will act nontrivially on H if $d > 2$).

Let $s \in \mathcal{S}(m)$ as defined in Section 2, and let $\hat{S}$ be the Sylow s-subgroup of $\hat{H}$ and $S = \varphi(\hat{S})$. Since $|\hat{S}|$ does not divide $q-1$ by the definition of $\mathcal{S}(m)$, we have $|S| = s^b$ for some $b \geqslant 1$ and $\phi(|S|) = s^{b-1}(s-1)$. Also $\hat{S}$ is irreducible on U, and S and H have the same centraliser and normaliser in $\mathrm{Aut}(T)$ (see [29, Lemma 2.1]). Hence $|N_{\mathrm{Aut}(T)}(S)/C_{\mathrm{Aut}(T)}(S)| = |N_{\mathrm{Aut}(T)}(H)/C_{\mathrm{Aut}(T)}(H)| = r$ divides $2am$. Further, by Lemma 2.3, the number of $\mathrm{Aut}(T)$-classes of elements of T of order $|S|$ is $\phi(|S|)/r$ and divides $n!$. Thus $\phi(|S|)$ divides $r \cdot n!$, which divides $2am \cdot n!$. Since $m \leqslant d$ and $d \leqslant c_1(n) = n$, it follows that $2m \leqslant 2n$ and hence $2m$ divides $(2n)!$. Thus

$$\phi(|S|) = s^{b-1}(s-1) \quad \text{divides } \ c_2(n)\, a, \tag{3}$$

where $c_2(n) = (2n)! \cdot n!$. Note that the bound $c_2(n)\, a$ is independent of the prime $s \in \mathcal{S}(m)$. The condition (3) implies that $s - 1$ is a divisor of $c_2(n)\, a$, and since the number of divisors of $c_2(n)\, a$ is less than $2\sqrt{c_2(n)\, a}$ (see [31, Section 8.3]), it follows that $|\mathcal{S}(m)| < 2\sqrt{c_2(n)\, a}$.

Now $|H| = |\varphi(\hat{H})| = (q^m - 1)/(q-1)$ and recall that m is prime. If $s \in \mathcal{S}(m)$ and s does not divide $q - 1$, then $(q^m - 1)_s = |H|_s$ and in this case by (3) we have

$$(q^m - 1)_s = |S| = s^b = \frac{s\phi(|S|)}{(s-1)} \leqslant 2c_2(n)\, a.$$

On the other hand if $s \in \mathcal{S}(m)$ and s divides $q - 1$, then $s = m$ by Lemma 2.1(iv). In this case,

$$\frac{(q^m-1)_m}{(q-1)_m} = |S| = \frac{m\phi(|S|)}{(m-1)} \leqslant 2c_2(n)\, a.$$

Putting this together we see that

$$\mathrm{Prod}(m, q) = \prod_{s \in \mathcal{S}(m)} (q^m - 1)_s \leqslant (q-1)_m \cdot (2c_2(n)\, a)^{2\sqrt{c_2(n)\, a}}.$$

Thus we have

$$\frac{\mathrm{Prod}(m, q)}{(q-1)_m} \leqslant (2c_2(n)\, a)^{2\sqrt{c_2(n)\, a}}.$$

By Lemma 2.1(iv), we obtain a lower bound:

$$\frac{\mathrm{Prod}(m, q)}{(q-1)_m} \geqslant \frac{1}{(q-1)_m} \cdot \frac{q^m - 1}{q-1} \geqslant \frac{q^m - 1}{(q-1)^2}.$$

Suppose first that $m \geqslant 3$. Then $(q^m - 1)/(q-1)^2 > q = p^a$. If $a = 1$, this gives the upper bound $q < (2c_2(n))^{2\sqrt{c_2(n)}}$ and we are done. Suppose that $a > 1$. Now

$$(2c_2(n)\, a)^{2\sqrt{c_2(n)\, a}} = \left((2c_2(n))^{2\sqrt{c_2(n)}}\right)^{\sqrt{a}} (e^{2\sqrt{c_2(n)}})^{\log(a)\sqrt{a}} \leqslant c_3(n)^{\sqrt{a}\log a}$$

for some function $c_3(n)$. Hence $a \log p < \sqrt{a} \log a \log(c_3(n))$ which yields

$$\frac{\sqrt{a} \log p}{\log a} < \log(c_3(n)).$$

Since $a > 1$, we have $\sqrt{a}/\log a \geqslant 1$, so $\log p \leqslant \log c_3(n)$ and hence $p \leqslant c_3(n)$. Also, for all $a > 1$, Lemma 2.5 gives

$$\frac{a^{1/4} \log 2}{4} \leqslant \frac{\sqrt{a} \log 2}{\log a} \leqslant \frac{\sqrt{a} \log p}{\log a},$$

and hence $a \leqslant \left(\frac{4}{\log 2} \log(c_3(n))\right)^4$. Thus, when $m \geqslant 3$, we have shown that all of a, p and d are bounded above by functions of n, and hence $|T|$ is also bounded above by some function of n. If $m = d = 2$ with q even, then $(q-1)_m = 1$ and, by Lemma 2.1(ii), $p^a < q+1 \leqslant \mathrm{Prod}(m, q)$. Our arguments therefore give

$$p^a < \mathrm{Prod}(m, q) \leqslant c_3(n)^{\sqrt{a} \log a},$$

and the same argument yields the required bound.

Now we treat the two exceptional cases: if $m = d = 3$ with 3 dividing $q - 1$, then $\hat{H}$ has order $(q^3 - 1)/(q - 1) = q^2 + q + 1$ and its image $H = \varphi(\hat{H})$ has order $\frac{1}{3}(q^2 + q + 1)$. Also, by Lemma 2.1(iv), we have $(q^3 - 1)/(q - 1) \leqslant \mathrm{Prod}(m, q)$. Thus the argument above yields

$$p^a = q < \frac{(q^3 - 1)}{(q - 1)^2} \leqslant \frac{\mathrm{Prod}(3, q)}{(q - 1)_3} = 3 \cdot |H| \leqslant c_3'(n)^{\sqrt{a} \log a},$$

and the required bound.

Finally assume that $m = d = 2$ with q odd. Here T has unique conjugacy classes of cyclic subgroups of orders $\frac{1}{2}(q+1)$ and $\frac{1}{2}(q-1)$, and one of these orders is odd, say $\frac{1}{2}(q+\delta)$. We choose $H < T$ with H cyclic of odd order $\frac{1}{2}(q+\delta)$, and for each prime s dividing $\frac{1}{2}(q+\delta)$, we consider the Sylow s-subgroup S of H. The index $|N_{\mathrm{Aut}(T)}(S) : C_{\mathrm{Aut}(T)}(S)|$ is $2a$, and hence by Lemma 2.3, the number of $\mathrm{Aut}(T)$-classes of elements of T of order $|S|$ divides $2a \cdot n!$. Now a very similar argument to that above shows that the product over the primes s of the orders of these Sylow subgroups of H (which equals $|H|$) is bounded above by $c_3(n)^{\sqrt{a} \log a}$ for some function $c_3(n)$, and since $|H| \geqslant \frac{1}{2}(q-1) \geqslant p^a/3$, we conclude that p and a, and hence also q are bounded by some function of n. Thus $|T|$ is bounded by a function of n, which completes the proof in the linear case.

Case 2. *The remaining cases with $m = d$ in* (ii) *and* (iii).

Here T and m are as in Table 3; in each case T contains a cyclic subgroup H with order as in the respective row of Table 3. For each prime s dividing $|H|$, we consider the Sylow s-subgroup S of H. Then H and S have the same centralisers and normalisers in $\mathrm{Aut}(T)$ and the index $r = |N_{\mathrm{Aut}(T)}(S) : C_{\mathrm{Aut}(T)}(S)|$ is as in Table 3.

T	m	$\lvert H\rvert$	r
$\mathrm{PSp}(4,q)$	2	$(q^2+1)/(2,q-1)$	$4a\cdot(2,q)$
$\mathrm{PSU}(3,q)$	3	$(q^2-q+1)/(3,q+1)$	$6a$
$\mathrm{PSp}(6,q)$	3	$(q^3+1)/(2,q+1)$	$6a$
$\mathrm{P}\Omega^\circ(7,q)$	3	$(q^3+1)_{2'}$	$6a$

Table 3. Orders of cyclic subgroups of T in the remaining cases for $m=d$.

By Lemma 2.3, the number of $\mathrm{Aut}(T)$-classes of elements of T of order $|S|$ is equal to $\phi(|S|)/r$ and hence $\phi(|S|)$ divides $r\cdot n!$, which divides $24a\cdot n!$ in each of these four cases, and this bound is independent of s. Thus in each case

$$|S| = s\cdot\frac{\phi(|S|)}{s-1} < 2\phi(|S|) \leqslant 48a\cdot n!\,.$$

The number of divisors of $24a\cdot n!$ is less than $2\sqrt{24a\cdot n!}$ by [31, Section 8.3], so $|H|$, which is the product of $|S|$ over all s, satisfies

$$|H| \leqslant (48a\cdot n!)^{2\sqrt{24a\cdot n!}}.$$

On the other hand, in all cases $|H|>q=p^a$. If $a=1$, then this bounds q by a function of n and we are done. Suppose that $a>1$. Then

$$|H| \leqslant (48a\cdot n!)^{2\sqrt{24a\cdot n!}} = \left((48\cdot n!)^{2\sqrt{24\cdot n!}}\right)^{\sqrt{a}}(e^{2\sqrt{24\cdot n!}})^{\log(a)\sqrt{a}} \leqslant c(n)^{\sqrt{a}\log a}$$

for some function $c(n)$. Thus $p^a=q<|H|\leqslant c(n)^{\sqrt{a}\log a}$, and so taking logs we obtain $(\sqrt{a}\log p)/\log a \leqslant \log c(n)$. Arguing as in the previous case, this bounds both a and p, and hence also q and $|T|$, by some function of n.

Case 3. *The other classical groups with $m<d$.*

Here m is an odd prime, $d\geqslant 4$, and the preimage G of T has a cyclic subgroup $\hat{H}$ as in case (U), (Sp), or (O) of Table 2. Since $m<d$, $\hat{H}$ has nontrivial fixed point space in V and hence contains no nontrivial scalars, so $H:=\varphi(\hat{H})\cong\hat{H}$. In all cases $|H|$ divides q^m+1.

By (iii) and (iv) of Lemma 2.1, we have $2\notin\mathcal{S}(2m)$, and $\mathrm{Prod}(2m,q)$ divides q^m+1 and is divisible by $(q^m+1)/(q+1)$. Moreover, by Lemma 2.1(iv)(1), if $s\in\mathcal{S}(2m)$ then s divides q^m+1, and either $(q^m+1)_s=\mathrm{Prod}(2m,q)_s$ divides $(q^m+1)/(q+1)$, or $s=m$ divides $q+1$. In the former case we have

$$(q^m+1)_s = |H|_s = \mathrm{Prod}(2m,q)_s,$$

while if $s=m$ divides $q+1$ then, since m is odd, either we again have

$$(q^m+1)_s = |H|_s = \mathrm{Prod}(2m,q)_s,$$

or $T=\mathrm{PSU}(m,q)$ and $|H|_m=(q^m+1)_m/(q+1)_m$.

Now we argue as in the linear case for the primes in $\mathcal{S}(2m)$ (rather than $\mathcal{S}(m)$). Let $s \in \mathcal{S}(2m)$ and let $\hat{S}$ be the Sylow s-subgroup of $\hat{H}$ and $S = \varphi(\hat{S})$ the Sylow s-subgroup of H. An identical argument to that in the linear case shows that the number of $\mathrm{Aut}(T)$-classes of elements of T of order $|S|$ is $\phi(|S|)/r$ and divides $n!$, where $r = |N_{\mathrm{Aut}(T)}(H)/C_{\mathrm{Aut}(T)}(H)|$, and for each of the groups in this case r divides $12ma$. The argument thus shows that $\phi(|S|)$ divides $c_2(n)\,a$ for some function $c_2(n)$, and that there are at most $2\sqrt{c_2(n)\,a}$ primes in $\mathcal{S}(2m)$. For each prime s where $(q^m+1)_s = |H|_s = \mathrm{Prod}(2m,q)_s$, we therefore have

$$(q^m+1)_s = |S| = \frac{s\phi(|S|)}{(s-1)} < 2c_2(n)\,a.$$

There is at most one prime in $\mathcal{S}(2m)$ for which this condition fails, namely the prime $s = m$ if $m \,|\, q+1$ and $T = \mathrm{PSU}(d,q)$. In this exceptional case we have instead that

$$\frac{(q^m+1)_m}{(q+1)_m} = |S| = \frac{m\phi(|S|)}{(m-1)} < 2c_2(n)\,a.$$

Since s is odd, s divides exactly one of q^m+1 and q^m-1. Since $s \in \mathcal{S}(2m)$, we must therefore have that $(q^m+1)_s = (q^{2m}-1)_s$. Thus in the case where m does not arise as an exceptional prime in $\mathcal{S}(2m)$, we have

$$\begin{aligned}\mathrm{Prod}(2m,q) &= \prod_{s\in\mathcal{S}(2m)} (q^{2m}-1)_s \\ &= \prod_{s\in\mathcal{S}(2m)} (q^m+1)_s \leqslant (2c_2(n)\,a)^{2\sqrt{c_2(n)\,a}} \leqslant c_3(n)^{\sqrt{a}\log a}\end{aligned}$$

for some $c_3(n)$, and as $\mathrm{Prod}(2m,q) \geqslant (q^m+1)/(q+1) > q = p^a$, we obtain the required bound on q and $|T|$ as before. On the other hand, if m does arise as an exception, so in particular $T = \mathrm{PSU}(d,q)$, then we obtain

$$\frac{\mathrm{Prod}(2m,q)}{(q+1)_m} = \frac{1}{(q+1)_m} \prod_{s\in\mathcal{S}(2m)} (q^m+1)_s \leqslant (2c_2(n)\,a)^{2\sqrt{c_2(n)\,a}} \leqslant c_3'(n)^{\sqrt{a}\log a}$$

for some $c_3'(n)$. Now

$$\frac{\mathrm{Prod}(2m,q)}{(q+1)_m} \geqslant \frac{(q^m+1)}{(q+1)^2} \geqslant \frac{(q^3+1)}{(q+1)^2} > q-2$$

and the usual argument yields the required bound. □

Remark 4.3. We comment on the function $f(n)$ which appears in Theorem 1.1. This function is the maximum over the functions for different families of simple groups. From the linear case when $d > 2$ and $q = p$, we have $c_1(n) = n$ so that $c_2(n) = (2n)! \cdot n!$. Then

$$c_3(n) = 2((2n)! \cdot n!)^{2\sqrt{(2n)!\cdot n!}}$$

and so we obtain

$$\log p \leqslant \log c_3(n) = (2\sqrt{(2n)! \cdot n!}) \log(2 \cdot (2n)! \cdot n!).$$

Since we have $|T| \leqslant p^{d^2}$, we have shown $\log |T| \leqslant d^2 \log p$, which gives

$$\log |T| \leqslant n^2 (2\sqrt{(2n)! \cdot n!}) \log(2 \cdot (2n)! \cdot n!).$$

The functions arising from the other cases of classical simple groups are similar.

Remark 4.4. In the course of the proof of Proposition 4.2 for simple classical groups T, we make use of s-elements, for carefully chosen primes s. One of these primes is the characteristic $s = p$, and for all the other primes s, we bound the number of $\mathrm{Aut}(T)$-classes of elements of maximal order $\exp(T)_s = |T|_s$, as discussed in Remark 1.3. Thus if $\mathcal{S}(T)$ is the set of these primes s, then in our proof of Proposition 4.2 we have bounded $|T|$ by a function of the parameter

$$m'(T) = \max\{m_p(T), m_{\mathcal{S}(T)\text{-exp}}(T)\}$$

defined in Remark 1.3. We showed in Remark 3.2 that a similar bound holds for alternating groups. However, our approach to the exceptional groups of Lie type is different from the classical case (see Section 5 below), and it is an open question if such a bound holds for the simple exceptional groups of Lie type.

5. The exceptional groups

In this section we complete the proof of Theorem 1.1 for the simple groups of Lie type that are exceptional. Our treatment is divided into two cases, depending on whether or not there is a torus subgroup whose normaliser is maximal. Remarkably, most families have this property.

Lemma 5.1. *For each simple group T in Table 4, there exists a cyclic subgroup H of the claimed order. Let s be prime dividing $|H|$, H_s be the Sylow s-subgroup of H and z be the corresponding value from Table 4. Then one of the following holds*:

(1) *H_s is characteristic in $N_T(H)$, $N_{\mathrm{Aut}(T)}(H) = N_{\mathrm{Aut}(T)}(H_s)$ and*

$$|N_{\mathrm{Aut}(T)}(H) : C_{\mathrm{Aut}(T)}(H_s)|$$

divides $z|\mathrm{Out}(T)|$.

(2) *$T = E_7(q)$, q is odd, $s = 2$ and there is a nontrivial characteristic cyclic 2-subgroup Y_2 of $N_T(H)$ of order at least $|H|_2/2$ such that*

$$N_{\mathrm{Aut}(T)}(H) = N_{\mathrm{Aut}(T)}(Y_2) \quad \text{and} \quad |N_{\mathrm{Aut}(T)}(H) : C_{\mathrm{Aut}(T)}(Y_2)|$$

divides $z|\mathrm{Out}(T)|$.

(3) *$T = E_7(q)$, q is odd, $s = 2$ and $|H|_2 \leqslant 2$.*

T	$\lvert H\rvert$	$N_T(H)$	z	$\lvert\mathrm{Out}(T)\rvert$	notes
${}^2B_2(q)$	$q \pm \sqrt{2q}+1$	$H.4$	4	a	$q=2^{2t+1}$ [35, Theorem 9]
${}^2G_2(q)$	$q \pm \sqrt{3q}+1$	$H.6$	6	a	$q=3^{2t+1}$ [20, Theorem C]
${}^2F_4(q)$	$q^2+q+1\pm\sqrt{2q}(q+1)$	$H.12$	12	a	$q=2^{2t+1}$ [26, Main Theorem]
${}^3D_4(q)$	q^4-q^2+1	$H.4$	4	a	[21, Theorem]
$F_4(q)$	q^4-q^2+1	$H.12$	12	$2a$	$q>2$ even [8, Table 8]
$E_6(q)$	$(q^2+q+1)/e$	$({}^3D_4(q)\times H).3$	3	$2ae$	[8, Table 9]
${}^2E_6(q)$	$(q^2-q+1)/e'$	$({}^3D_4(q)\times H).3$	3	ae'	$q>2$ [8, Table 10]
$E_7(q)$	$(q-1)/d$	$(E_6(q)\times H).2$	2	ad	$q\equiv 2 \pmod 3$ [25, Table 5.1]
$E_7(q)$	$(q+1)/d$	$({}^2E_6(q)\times H).2$	2	ad	$q\not\equiv 2 \pmod 3$ [25, Table 5.1]
$E_8(q)$	$(q^4-1)(q^4-q^3+q)+1$	$H.30$	30	a	[25, Table 5.2]

Table 4. Choices for a cyclic subgroup H for exceptional groups T defined over a field of size $q=p^a$, p a prime, with $N_T(H)$ maximal, $d=(2,q-1)$, $e=(3,q-1)$, $e'=(3,q+1)$.

Proof. The existence of a cyclic subgroup H of T of the prescribed order is given by the reference in the notes column of Table 4. (For $T=E_7(q)$ see also [7, Table 4.1].) Further, each reference proves that $N_{\mathrm{Aut}(T)}(H)$ is a maximal subgroup of $\mathrm{Aut}(T)$.

Let s be a prime dividing $|H|$ and let H_s be the Sylow s-subgroup of H. We claim that s is coprime to z, except for the case $T=E_7(q)$, q is odd and $s=2$. This is easy to verify, for example, when $T=E_6(q)$ we have $|H|$ is coprime to 3 unless $q\equiv 1 \pmod 3$, in which case $q\equiv 1,4,7 \pmod 9$ and so $q^2+q+1\equiv 3 \pmod 9$, which implies that $|H|=\frac{1}{3}(q^2+q+1)$ is coprime to 3; or, for example, when $T=E_8(q)$ then $|H|=(q^4-1)(q^4-q^3+q)+1$ is coprime to 30.

Suppose first that $T=E_7(q)$, q is odd and $s=2$. We may assume that $|H|_2>2$ otherwise statement (3) of Lemma 5.1 holds. If $O_2(N_T(H))\leqslant H$, then $O_2(N_T(H))=H_2$ and we set $Y_2=H_2$. Otherwise, $O_2(N_T(H))=(H_2).2$. Write $H_2=\langle y\rangle$. Then since $\langle y^2\rangle$ is a normal subgroup of $O_2(N_T(H))=\langle y\rangle.2$ of index 4, we have that $\Phi\big(O_2(N_T(H))\big)=\langle y\rangle$ or $\langle y^2\rangle$ (depending on whether $O_2(N_T(H))/\langle y^2\rangle$ is cyclic or elementary abelian, respectively). Hence in this case we set $Y_2=\Phi(O_2(N_T(H)))$. In both cases, Y_2 is a characteristic cyclic subgroup of $N_T(H)$ of order at least $|H|_2/2$ (in particular, $Y_2\neq 1$). It follows that $N_{\mathrm{Aut}(T)}(H)\leqslant N_{\mathrm{Aut}(T)}(Y_2)$ and the maximality of $N_{\mathrm{Aut}(T)}(H)$ in $\mathrm{Aut}(T)$ gives equality. Finally, $C_T(H)$ contains $(N_T(H))''\cong E_6(q)$ or ${}^2E_6(q)$, and $H\leqslant C_T(H)$, so we have that $|N_{\mathrm{Aut}(T)}(H):C_{\mathrm{Aut}(T)}(H)|$ divides $z|\mathrm{Out}(T)|$. This proves that statement (2) of Lemma 5.1 holds.

Assume now that T is any of the groups in Table 4, but that if $T=E_7(q)$ and q is odd, then $s\neq 2$. Let H_s be the Sylow s-subgroup of H. Now $N_T(H)=(D\times H).Z$, where D is either trivial or nonabelian simple and Z is a cyclic group of order z, as in Table 4. Since we have excluded the case $s=2$ when $T=E_7(q)$ and q is

T	$\|H\|$	M	$N_T(H)$	z	$\|\mathrm{Out}(T)\|$	notes
$G_2(q)$	q^2+q+1	$\mathrm{SL}(3,q):2$	$H:6$	6	$\leqslant 2a$	[6, Theorem 2.3; 20, Theorem A]
$F_4(q)$	q^4+1	$2.\Omega_9(q)$	$H.4$	4	a	q odd [8, Table 7; 25, Table 5.1]

Table 5. Choices for a cyclic subgroup H of exceptional groups T when $N_T(H)$ is not maximal, but $N_T(H) < M$ for a unique class of maximal M, $q = p^a$ with p prime.

odd, we have that s is coprime to z. It follows that $H_s = O_s(N_T(H))$, and so H_s is characteristic in $N_T(H)$. Thus

$$N_{\mathrm{Aut}(T)}(H) \leqslant N_{\mathrm{Aut}(T)}(H_s)$$

and the maximality of $N_{\mathrm{Aut}(T)}(H)$ gives equality. Finally, since $C_T(H)$ contains $D \times H$, we conclude that $|N_{\mathrm{Aut}(T)}(H) : C_{\mathrm{Aut}(T)}(H)|$ divides $z|\mathrm{Out}(T)|$. This proves that statement (1) of Lemma 5.1 holds. □

Lemma 5.2. *For each simple group T in Table 5, there exists a cyclic subgroup H of the claimed order. Let s be a prime dividing $|H|$, H_s be the Sylow s-subgroup of H and z be the corresponding value in Table 5. Then one of the following holds*:

(1) *H_s is characteristic in $N_T(H)$, $N_{\mathrm{Aut}(T)}(H) = N_{\mathrm{Aut}(T)}(H_s)$ and*

$$|N_{\mathrm{Aut}(T)}(H) : C_{\mathrm{Aut}(T)}(H_s)|$$

divides $z|\mathrm{Out}(T)|$.

(2) $T = G_2(q)$, $q \equiv 1 \pmod 3$, $s = 3$ *and* $|H_3| = 3$.

(3) $T = F_4(q)$, $s = 2$ *and* $|H_2| = 2$.

Proof. The proof is similar to that of Lemma 5.1, with adjustments since $N_T(H)$ is not maximal in T. Let $T = G_2(q)$ or $T = F_4(q)$ with q odd. For each row of Table 5, take H to be a cyclic subgroup of the specified order inside the specified maximal subgroup M of T (the maximality of M in T is justified by the references in the notes column of Table 5).

Claim. *$N_T(H) = N_M(H)$, and the structure of the normaliser is as displayed in the respective column of Table 5.*

Proof of Claim. Suppose first that $T = G_2(q)$ and note that $|H|$ has order divisible by a prime t that is a primitive prime divisor of $q^3 - 1$ (and therefore $t \geqslant 5$). Choose $q_0 = p^b$ with b a minimal divisor of a such that $N_T(H) \leqslant G_2(q_0) =: T_0$. Now, since T_0 contains no normal cyclic subgroups, we have $N_T(H) \leqslant L$ for some maximal subgroup L of T_0 (and note that L cannot be a subfield subgroup by the minimality of q_0). From the references in the notes column, we see that L and t are as in Table 6.

L	q_0	t	notes	$q_0^2+q_0+1 \geqslant$
$2^3.\mathrm{PSL}(3,2)$	p	7	p odd	13
$\mathrm{PSL}(2,13)$	4	7		21
J_2	4	7		21
J_1	11	7, 19		133
$\mathrm{PSL}(2,13)$	$q_0=p$	7, 13	$p \equiv 1,3,4,9,10,12 \pmod{13}$	13
$\mathrm{PSL}(2,13)$	$q_0=p^2$	7, 13	$p \equiv 2,5,6,7,8,11 \pmod{13}$	21
$\mathrm{PSL}(2,8)$	$q_0=p$	7	$p \equiv 1,8 \pmod 9$	307
$\mathrm{PSL}(2,8)$	$q_0=p^3$	7	$p \equiv 2,4,5,7 \pmod 9$	73
$\mathrm{PSU}(3,3):2$	$q_0=p$	7	$p \geqslant 5$	31
$\mathrm{SL}(3,q_0):2$		t		

Table 6. Possibilities for L with $N_T(H) \leqslant L$ for $T = G_2(q)$.

L	t
$\mathrm{PGL}(2,13)$	13
$\mathrm{PSL}(2,17)$	17
$\mathrm{PSL}(2,25).2$	13
$\mathrm{PSL}(2,27).(3)$	13
$3^3 \rtimes \mathrm{SL}(3,3)$	13
$2.\Omega_9(q_0)$	

Table 7. Possibilities for L with $N_T(H) \leqslant L$ for $T = F_4(q)$.

Now H is cyclic of order $q^2+q+1 \geqslant q_0^2+q_0+1$, and on comparing the entry for L in Table 6 with the bound in the last column of that table, we see that L contains a cyclic subgroup of order $|H|$ only if either $L = \mathrm{PSL}(2,13)$, $t = 13$ and $q = q_0 = p = 3$, or $L = \mathrm{SL}(3,q_0):2$. In the first case, we have $|H| = 13 = q^2+q+1$ and $N_{\mathrm{PSL}(2,13)}(H) = H:6$, as in Table 5. In the latter case, we have

$$|N_{\mathrm{SL}(3,q_0):2}(H)| = (q_0^2+q_0+1).6 \leqslant |N_M(H)|.$$

Hence $q = q_0$ and $N_T(H) = N_M(H) = H.6$, again as in Table 5.

Suppose now that $T = F_4(q)$, where q is odd, and note that there is a primitive prime divisor t of q^8-1 that divides $|H|$ (from which it follows that $t \geqslant 11$). As above, choose $q_0 = p^b$ with b a minimal divisor of a such that $N_T(H) \leqslant F_4(q_0) =: T_0$, and let L be a maximal subgroup of T_0 containing $N_T(H)$. Then t divides $|L|$, and so from [8, Tables 1 and 8] we see that L and t are as in Table 7.

For the groups in the first 5 rows of Table 7, we have that a subgroup of L of order t is self-centralising, and thus $|H| = t = q^4+1$. Now, since q is odd, we have $|H| = t = q^4+1 \geqslant q_0^4+1 \geqslant 82$, a contradiction. Hence the only possibility is that $L \cong 2.\Omega_9(q_0)$, and comparing the orders of $N_{2.\Omega_9(q_0)}(H)$ and $N_M(H)$, we have $q = q_0$ and $N_T(H) = N_M(H) = H.4$ as in Table 5. □

We now turn to the statements of the lemma. First consider the case $T = G_2(q)$. Then s is coprime to z, unless $s = 3$ and $q \equiv 1 \pmod 3$. If $s = 3$ and $q \equiv 1 \pmod 3$, then $q^2 + q + 1 \equiv 3 \pmod 9$, and so $|H|_3 = (q^2 + q + 1)_3 = 3$, and statement (2) of the lemma holds. Suppose now that if $s = 3$ holds, then $q \not\equiv 1 \pmod 3$. Then $|H| = q^2 + q + 1$ is coprime to 3 and odd, so $O_s(N_T(H)) = H_s$ is a characteristic subgroup of $N_T(H)$, and thus $N_{\mathrm{Aut}(T)}(H) = N_{\mathrm{Aut}(T)}(H_s)$. Arguing similarly to Lemma 5.1, we see that statement (1) of the lemma holds.

Consider now the case $T = F_4(q)$. Since q is odd, we have that $q \equiv \pm 1 \pmod 4$, and hence $q^4 + 1 \equiv 2 \pmod 4$. Thus $|H|_2 = 2$ and part (3) of this lemma holds if $s = 2$. If s is odd, then s is coprime to z and hence $O_s(N_T(H)) = H_s$ satisfies part (1) of the lemma, similarly to the previous cases. □

Proposition 5.3. *There is an increasing integer function h such that, if T is a simple exceptional group of Lie type, and T is pp-bounded by n, then* $|T| < h(n)$.

Proof. Let T be a simple exceptional group of Lie type defined over the field with q elements. Note that since the rank of T is bounded by a constant, we simply need to prove that q is bounded by a function of n. In particular, we may assume that $q > 2$ so that we may use all rows of Table 4.

Case 1. *T is one of the groups appearing in* Table 4.

We apply Lemma 5.1 and consider the outcomes in turn. Let H be the cyclic subgroup of T defined in Lemma 5.1 and let $\pi(H)$ be the set of prime divisors of $|H|$. For $s \in \pi(H)$ let H_s be the Sylow s-subgroup of H. Then, in all cases,

$$\frac{q-1}{2} \leqslant |H| \quad \text{and} \quad |H| = \prod_{s \in \pi(H)} |H_s|. \tag{4}$$

Case 1(i). *If $T = E_7(q)$, we assume that q is even.*

Now H_s is cyclic and $|H_s| = s^b$ for some $b \geqslant 1$. Let $\mathcal{C}(s)$ be the set of $\mathrm{Aut}(T)$-classes of elements $g \in T$ such that $g^{\mathrm{Aut}(T)} \cap H \neq \varnothing$ and $|g| = s^b$. Since T is pp-bounded by n, it follows from Lemma 2.3 that $|\mathcal{C}(s)|$ divides $z|\mathrm{Out}(T)|n!$, which implies that $(s-1)$ divides $360a(n!)$ (using the lowest common multiple of the entries in the z column of Table 4). Since the number of divisors of $360a(n!)$ is at most $2\sqrt{360an!}$, and different values of s give distinct divisors, we have

$$|\pi(H)| \leqslant 2\sqrt{360a(n!)}. \tag{5}$$

Further, for each $s \in \pi(H)$, Lemma 2.3 combined with the value of the index $|N_{\mathrm{Aut}(T)}(H) : C_{\mathrm{Aut}(T)}(H)|$ from Lemma 5.1 yields

$$|H_s| = s^b = s^{b-1}(s-1)\frac{s}{s-1} = \phi(|H_s|)\frac{s}{s-1} \leqslant 2z|\mathrm{Out}(T)|n \leqslant 60an. \tag{6}$$

Thus putting (4), (5) and (6) together, we have

$$\frac{q}{4} \leqslant \frac{q-1}{2} \leqslant |H| \leqslant |\pi(H)| \max_{s\in\pi(H)} (|H_s|) \leqslant 120an\sqrt{360a(n!)}.$$

This gives

$$\frac{q}{a^{3/2}} \leqslant f(n),$$

with $f(n) = 480n\sqrt{360(n!)}$. By Lemma 2.6,

$$\frac{q}{a^{3/2}} = \left(\frac{q}{a}\right)^{3/2} q^{-1/2} \geqslant \left(\frac{\log 2}{2} q^{1/2}\right)^{3/2} q^{-1/2} = Cq^{1/4}$$

for a constant C. Thus we obtain $q^{1/4} \leqslant f(n)/C$, which shows that q is bounded by a function of n, and hence $|T| \leqslant h_1(n)$, for a computable function h_1.

Case 1(ii). *$T = E_7(q)$ and q is odd.*

From parts (2) and (3) of Lemma 5.1 we have

$$|H| = |H_2|\left(\prod_{s\in\pi(H),\, s \text{ odd}} |H_s|\right) \leqslant \begin{cases} 2\left(\prod_{s\in\pi(H),\, s \text{ odd}} |H_s|\right) & \text{if } |H_2| = 2, \\ 2|Y_2|\left(\prod_{s\in\pi(H),\, s \text{ odd}} |H_s|\right) & \text{if } |H_2| > 2. \end{cases}$$

Since T is pp-bounded by n, an identical argument to that given above shows that

$$|\pi(H)| \leqslant 2\sqrt{z|\mathrm{Out}(T)|(n!)} \leqslant 2\sqrt{4a(n!)}.$$

Further, for $s \in \pi(H)$ with s odd, since T is pp-bounded by n, Lemmas 2.3 and 5.1 give, exactly as above, $|H_s| \leqslant z|\mathrm{Out}(T)|n \leqslant 4an$. For $s = 2$, suppose that $|H_2| > 2$, so that Lemma 5.1(2) holds. Then Lemma 2.3 gives

$$\frac{|Y_2|}{2} = \phi(|Y_2|) \leqslant z|\mathrm{Out}(T)|n = 4an,$$

so that $|Y_2| \leqslant 8an$. Thus

$$|H| \leqslant 2(8an)(2\sqrt{4a(n!)})$$

and since $|H| \geqslant \frac{q-1}{2}$, we obtain

$$\frac{q-1}{a^{3/2}} \leqslant 64n\sqrt{4(n!)}.$$

Similarly to the above, q is bounded by a function of n, and hence $|T| \leqslant h_2(n)$, for a computable function h_2.

Case 2. *T appears in* Table 5.

Here $T = G_2(q)$ or $T = F_4(q)$ with q odd. We apply Lemma 5.2 and consider the outcomes in turn. Let H be the cyclic subgroup of T defined in Lemma 5.1 and let H_s be the Sylow s-subgroup of H and let $\pi(H)$ be the set of prime divisors of $|H|$.

From parts (2) and (3) of Lemma 5.2, we may write

$$|H| = \begin{cases} (3, q-1)\left(\prod_{s\in\pi(H),\, s\neq 3} |H_s|\right) & \text{if } T = G_2(q), \\ 2\left(\prod_{s\in\pi(H),\, s\neq 2} |H_s|\right) & \text{if } T = F_4(q). \end{cases}$$

Now for any $s \in \pi(H)$ such that $(T, s) \neq (G_2(q), 3)$ or $(F_4(q), 2)$, Lemma 5.2(1) applies, and we argue similarly to the previous cases to obtain $|\pi(H)| \leqslant 2\sqrt{12an!}$. Further, we have $|H_s| \leqslant 2z|\mathrm{Out}(T)|n \leqslant 24an$. This gives

$$q \leqslant \frac{|H|}{3} \leqslant 2\sqrt{12an!}\,(24an)$$

and therefore $|T|$ is bounded by a computable function $h_3(n)$.

Finally, we define $h(n)$ to be the maximum over the functions h_1, h_2 and h_3, and thus in any of the cases above, we have $|T| \leqslant h(n)$, as required. □

6. Proof of Theorem 1.1

We set

$$f(n) = \max\left\{\tfrac{1}{2}(3n+2)!,\ g(n),\ h(n)\right\} + |M|,$$

where $g(n)$ is the function in Proposition 4.2, $h(n)$ is the function in Proposition 5.3 and M is the Monster sporadic simple group. It is clear from the functions g and h that f is an increasing function. Let T be a nonabelian finite simple group. If T is sporadic, then $|T| \leqslant |M| \leqslant f(n)$. If T is alternating, classical or an exceptional group of Lie type, then Lemma 3.1, Propositions 4.2 and 5.3, respectively, shows that $|T| \leqslant f(n)$. This completes the proof of Theorem 1.1.

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MONOGAMOUS SUBVARIETIES OF THE NILPOTENT CONE

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In memory of Gary, who influenced us greatly

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of prime characteristic not 2, whose Lie algebra is denoted $\mathfrak{g}$. We call a subvariety $\mathfrak{X}$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ *monogamous* if for every $e \in \mathfrak{X}$, the $\mathfrak{sl}_2$-triples (e, h, f) with $f \in \mathfrak{X}$ are conjugate under the centraliser $C_G(e)$. Building on work by the first two authors, we show there is a unique maximal closed G-stable monogamous subvariety $\mathcal{V} \subset \mathcal{N}$ and that it is an orbit closure, hence irreducible. We show that $\mathcal{V}$ can also be characterised in terms of Serre's G-complete reducibility.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic $p \neq 2$, and G a simple algebraic $\Bbbk$-group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Three elements $e, h, f \in \mathfrak{g}$ form an $\mathfrak{sl}_2$-triple if the subalgebra $\langle e, h, f\rangle$ is a homomorphic image of $\mathfrak{sl}_2(\Bbbk)$. That is, (e, h, f) satisfy the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

When the characteristic is 2 these relations degenerate leading to a qualitatively different theory; see [Stewart and Thomas 2024] for more details. This justifies our underlying assumption of $p \neq 2$. Theorems of Jacobson [1951], Morozov [1942] and Kostant [1959] say that if $\Bbbk$ is of characteristic 0, then for any nilpotent $e \in \mathfrak{g}$ there exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ which is unique up to conjugacy by the centraliser of e in G.

Over fields of positive odd characteristic, for any nilpotent $e \in \mathfrak{g}$ there exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ except in the case G is of type G_2, $p = 3$, and e is in the $\tilde{A}_{1(3)}$ class [Stewart and Thomas 2018, Theorem 1.7]. We continue the investigation into generalising Kostant's uniqueness theorem to fields of small characteristic. Let $\mathfrak{X}$ be a subset of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. We say that $\mathfrak{X}$ is *monogamous* if the

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following property holds:

> Let (e, h, f) and (e, h', f') be $\mathfrak{sl}_2$-triples with $e, f, f' \in \mathfrak{X}$. Then (e, h, f) is $C_G(e)$-conjugate to (e, h', f').

The main theorem of [Stewart and Thomas 2018] proves that $\mathcal{N}$ is monogamous if and only if $p > h(G)$, where $h(G)$ is the Coxeter number for G. When G is of classical type, Goodwin and Pengelly [2024] showed that there exists a unique maximal G-stable closed subvariety of $\mathcal{N}$ that is monogamous, and gave an explicit description of these. This paper completes the story by treating the exceptional types. Define the following subset of $\mathcal{N}$:

$$\mathcal{V} := \left\{ x \in \mathcal{N} \;\middle|\; \begin{array}{l} x^{[p]} = 0, \\ x \text{ is not distinguished in a Levi subalgebra with a factor of type } A_{p-1}, \\ x \text{ is not subregular if } G \text{ is of type } G_2 \text{ and } p = 3. \end{array} \right\}.$$

Theorem 1.1. *Let G be a simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 2$. Then $\mathcal{V}$ is the unique maximal G-stable closed monogamous subvariety of $\mathcal{N}$. Furthermore, $\mathcal{V}$ is irreducible, being the closure of a single orbit as specified in Tables 1 and 2 below.*

In [Stewart and Thomas 2018], a close relationship was found between uniqueness of $\mathfrak{sl}_2$-subalgebras and the existence of so-called non-G-cr $\mathfrak{sl}_2$-subalgebras. The notion of G-complete reducibility for subgroups of G is due to Serre [2005], and the natural generalisation to subalgebras of $\mathfrak{g}$ was introduced by McNinch [2007].

G	m	λ	
A_{m-1}	$a(p-1)+r$	$((p-1)^a, r)$	
$B_{\frac{m-1}{2}}$	$p+a(p-1)+r$ $(r>0)$	$(p, (p-1)^a, r-1, 1)$	a even
		$(p, (p-1)^{a-1}, p-2, r+1)$	a odd
	$p+a(p-1)$	$(p, (p-1)^a)$	a even
		$(p, (p-1)^{a-1}, p-2, 1)$	a odd
	$\leq p$	(m)	
$C_{\frac{m}{2}}$	$a(p-1)+r$	$((p-1)^a, r)$	
$D_{\frac{m}{2}}$	$p+a(p-1)+r$	$(p, (p-1)^a, r)$	a even
		$(p, (p-1)^{a-1}, p-2, r, 1)$	a odd
	$\leq p$	$(m-1, 1)$	

Table 1. Partition λ corresponding to the orbit O_λ such that $\mathcal{V} = \overline{O}_\lambda$ in the classical types, where $a \geq 0$ and $0 \leq r < p-1$.

G	p	O	G	p	O
G_2	3	$\tilde{A}_1^{(3)}$	E_6	3	A_1^3
	5	$G_2(a_1)$		5	$D_4(a_1)$
	≥ 7	G_2		7	$E_6(a_3)$
F_4	3	$A_1\tilde{A}_1$		11	$E_6(a_1)$
	5	$F_4(a_3)$		≥ 13	E_6
	7	$F_4(a_2)$			
	11	$F_4(a_1)$			
	≥ 13	F_4			

G	p	O	G	p	O
E_7	3	A_1^4	E_8	3	A_1^4
	5	$A_3A_2A_1$		5	A_3^2
	7	$E_7(a_5)$		7	$E_8(a_7)$
	11	$E_7(a_3)$		11	$E_8(a_6)$
	13	$E_7(a_2)$		13	$E_8(a_5)$
	17	$E_7(a_1)$		17	$E_8(a_4)$
	≥ 19	E_7		19	$E_8(a_3)$
				23	$E_8(a_2)$
				29	$E_8(a_1)$
				≥ 31	E_8

Table 2. Orbits O such that $\mathcal{V} = \overline{O}$ in the exceptional types.

Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we say that $\mathfrak{h}$ is *G-completely reducible* (G-cr for short) if for every parabolic subalgebra $\mathfrak{p}$ such that $\mathfrak{h} \subseteq \mathfrak{p}$ there exists some Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ with $\mathfrak{h} \subseteq \mathfrak{l}$.

We say $\mathfrak{X} \subseteq \mathcal{N}$ is *A_1-G-cr* if every subalgebra generated by an $\mathfrak{sl}_2$-triple (e, h, f) with $e, f \in \mathfrak{X}$ is G-cr.

Theorem 1.2. *Let G be a simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 2$. Then $\mathcal{V}$ is the unique maximal G-stable closed A_1-G-cr subvariety of $\mathcal{N}$.*

The proof follows very quickly from Theorem 1.1; see Section 4.

Remark 1.3. It would be interesting to know more about the geometry of the nilpotent variety $\mathcal{V}$. In type A, Donkin [1990] showed that the closure of each orbit is normal. Orbit closures in the remaining classical types are considered by Xiao and Shu [2015]. For exceptional types $G_2, F_4, \ldots, E_8$, results of Thomsen [2000] show that our varieties $\mathcal{V}$ are in fact Gorenstein normal varieties with rational singularities as long as $p \geq 5, 11, 7, 11, 13$, respectively.

2. Preliminaries

Throughout, $\Bbbk$ is an algebraically closed field of characteristic $p > 2$ and G is a simple $\Bbbk$-group with $\mathfrak{g} = \mathrm{Lie}(G)$. There is an inherited $[p]$-map on $\mathfrak{g}$ and we use $x^{[p]}$ to denote the image of $x \in \mathfrak{g}$ under this map. The variety of all nilpotent elements in $\mathfrak{g}$, often called the nilpotent cone, is denoted by $\mathcal{N}$. The restricted nullcone is the subvariety of $\mathcal{N}$ consisting of elements x such that $x^{[p]} = 0$ and we denote it by $\mathcal{N}_p$. The distribution of nilpotent elements among $\mathfrak{sl}_2$-subalgebras of $\mathfrak{g}$ is insensitive to central isogeny, and so we assume that whenever G is classical, it is one of $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ and write $G = \mathrm{Cl}(V)$ for brevity; if G is exceptional, we take it to be simply connected.

Recall that a prime p is bad for G if $p = 2$ and G is of type B, C or D; if $p \le 3$ and G is exceptional; or if $p \le 5$ and G is of type E_8; otherwise it is good. In some examples we require a choice of base for the root system associated to $\mathfrak{g}$; we use Bourbaki notation [2005]. Finally, we fix a maximal torus T of G.

2.1. ***Nilpotent orbits and Hasse diagrams.*** The orbits for the action of G on $\mathcal{N}$ are called nilpotent orbits. There are finitely many such and they are classified. In case G is of exceptional type, we describe an orbit $O = G \cdot x$ by a label indicating a Levi subalgebra in which e is distinguished; for these labels we refer to [Liebeck and Seitz 2012].

When $G = \mathrm{Cl}(V)$, the classification of orbits in terms of the action on V is well-known and can be found in [Jantzen 2004, Section 1], but we recap it here for ease of reference. Set $m = \dim V$. If $G = \mathrm{SL}(V)$, orbits are parametrised by partitions of m according to the Jordan decomposition of their elements' actions on V; we write $x \sim (\lambda_1, \ldots, \lambda_r)$ where $\lambda_1 \ge \cdots \ge \lambda_r$ is the partition of m corresponding to x. In types B and C orbits are parametrised by partitions of m with an even number of even parts and an even number of odd parts, respectively. In type D it is slightly more complicated. A partition is called very even if it only has even parts and they all occur with even multiplicity. There is one orbit for each partition of m with an even number of even parts that is not very even; and two orbits for each very even partition of m.

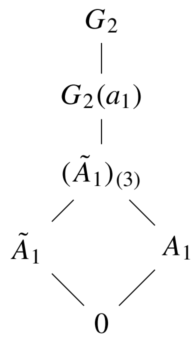
To check that $\mathcal{V}$ is a closed subvariety of $\mathcal{N}$ we require information about the Hasse diagrams for the closure relation on nilpotent orbits. For classical types, apart from type D, the closure order on orbits is precisely the dominance order on partitions. In type D we start with the Hasse diagram for the dominance order on partitions with an even number of even parts. Then we replace each very even partition λ with two nodes λ_1, λ_2 and replace each edge from λ to μ with two edges from λ_i to μ. For exceptional types the picture is actually incomplete in general. But if p is good for G, the existence of Springer morphisms implies that the Hasse diagrams remain the same as those in characteristic 0 [Spaltenstein 1982, Thèoréme III 5.2]. These can be found in [Spaltenstein 1982, pp. 247–250] and are reproduced in [Carter 1993, Section 13.4] with labels closer to those in [Liebeck and Seitz 2012]. However, those in [Carter 1993] are missing edges in the E_6, E_7 and E_8 diagrams. Specifically, there should be an edge between the following pairs of labels:

E_6: $(D_4(a_1), A_3)$.

E_7: $(D_6(a_2), D_5(a_1){+}A_1)$, $(D_5(a_1), D_4)$, $(D_4(a_1), 2A_2{+}A_1)$, $(D_4(a_1), A_2{+}3A_1)$.

E_8: $(E_6+A_1, E_8(b_6))$, $(E_8(a_7), D_6(a_2))$, (A_3+A_1, A_3).

In bad characteristic, there are not even the same number of nilpotent and unipotent orbits; for certain bad primes there are more nilpotent orbits than in characteristic 0. The full Hasse diagram for G_2 when $p = 3$ can be deduced from [Stuhler 1971]:



For the remaining types we will have to work harder to obtain partial information about the closure relations.

We can now prove part of Theorem 1.1.

Lemma 2.1. *The subset $\mathcal{V} \subseteq \mathcal{N}$ is a closed G-stable subvariety; moreover, it is the closure of a single orbit in each case, as specified in Tables 1 and 2.*

Proof. Suppose $G = \mathrm{Cl}(V)$ with $\dim V = m$. An orbit corresponding to a partition λ of m is contained in the restricted nullcone if and only if the largest part of λ is at most p. Let $G = \mathrm{SL}(V)$ or $\mathrm{Sp}(V)$ (resp. $\mathrm{SO}(V)$), and let $x \in \mathcal{N}$ with partition represented by λ. Then x is not distinguished in a Levi subalgebra with a factor of type A_{p-1} precisely when λ contains no parts of size p (resp. at most one part of size p). Now every orbit represented in Table 1 represents a single orbit in $\mathcal{V}$: for G of type D, each λ given in Table 1 is not very even. Observe that any other orbit in $\mathcal{V}$ is represented by a partition lower than λ in the dominance ordering, and hence is contained in $\overline{O}_\lambda$; and vice-versa, by definition of $\mathcal{V}$.

Now suppose G is of exceptional type. We use the tables in the corrected arxiv version of [Stewart 2016] to determine the orbits in the restricted nullcone. A nilpotent element x is distinguished in a Levi subalgebra with a factor of type A_{p-1} exactly when the labelling of its orbit contains an A_{p-1} part. Thus in good characteristic, as well as for G of type G_2, the result then follows by inspecting the Hasse diagrams.

$\mathcal{O}$	A_3^2	$D_4(a_1)A_2$	$A_3A_2A_1$	A_3A_2
λ	$(5, 4^2, 1^3)$	$(5, 3^3, 1^2)$	$(5, 3^2, 2^2, 1)$	$(5, 3^2, 1^5)$
$\mathcal{O}$	$D_4(a_1)A_1$	$D_4(a_1)$	$A_3A_1^2$	$A_2^2A_1^2$
λ	$(5, 3, 2^2, 1^4)$	$(5, 3, 1^8)$	$(5, 2^4, 1^3)$	$(3^5, 1)$
$\mathcal{O}$	A_3A_1	$A_2^2A_1$	A_3	A_2^2
λ	$(5, 2^2, 1^7)$	$(3^4, 2^2)$	$(5, 1^{11})$	$(3^4, 1^4)$
$\mathcal{O}$	$A_2A_1^3$	$A_2A_1^2$	A_2A_1	A_2
λ	$(3^3, 2^2, 1^3)$	$(3^3, 1^7)$	$(3^2, 2^2, 1^6)$	$(3^2, 1^{10})$
$\mathcal{O}$	A_1^4	A_1^3	A_1^2	A_1
λ	$(3, 2^4, 1^5)$	$(2^6, 1^4)$	$(2^4, 1^8)$	$(2^2, 1^{12})$

Table 3. D_8 partitions for nilpotent orbits in $\mathcal{V}$ for E_8, $p = 5$.

In the remaining cases we use case-by-case analysis. First let G be of type E_8 and $p = 5$. Note that every class is distinguished in $\mathrm{Lie}(L)$ for L some Levi subgroup of G. The Levi subgroups in question are all conjugate to subgroups of M, a maximal subgroup of G of type D_8. Let V be the 16-dimensional standard module for M. For each nontrivial class in $\mathcal{V}$ we choose a representative e in $\mathrm{Lie}(M)$ and calculate the Jordan block sizes for the action of e on V; these are in Table 3. Note that for some classes there are many non-M-conjugate choices for e. For example, there are three non-M-conjugate Levi subgroups of M of type A_3^2; these correspond to the subsets of simple roots $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 5, 6, 8\}$ and $\{1, 2, 3, 6, 7, 8\}$. A regular nilpotent element of the corresponding Levi subalgebras will act on V with Jordan blocks of sizes (4^4), (4^4) and $(5, 4^2, 1^3)$, respectively.

Note that the final partition is higher in the dominance order than all other partitions in Table 3. Therefore, the closure of the M-orbit of a representative of the class A_3^2 contains a representative of every class in $\mathcal{V}$. It remains to prove that there are no more G-classes in the closure of the A_3^2 class. By [Stewart 2016, Table 10], the Jordan block sizes for the adjoint action of nilpotent elements in the A_3^2-class are $(5^{38}, 4^{12}, 1^{10})$. By embedding G into SL_{248}, it follows that the Jordan block sizes for the adjoint action of every nilpotent element in the closure of the A_3^2-class will be lower than $(5^{38}, 4^{12}, 1^{10})$ in the dominance order. Using [loc. cit.], we check that every nonrestricted class has a Jordan block of size greater than 5 and all remaining classes (which have labels with an A_4 part) have at least 45 blocks of size 5.

Now let $p = 3$. When G is of type F_4, the subset $\mathcal{V}$ consists of the zero element and the union of the three classes with labels A_1, $\tilde{A}_1$ and $A_1\tilde{A}_1$. All three nontrivial classes have representatives contained in $\mathrm{Lie}(M)$ where M is a subgroup of type B_3. We may choose these representatives so that the corresponding partitions of 7 are $(2^2, 1^3)$, $(3, 1^4)$ and $(3, 2^2)$, respectively. Therefore, all three classes are contained in the closure of the $A_1\tilde{A}_1$-class. By [Liebeck and Seitz 2012, Table 22.1.4], the three classes in $\mathcal{V}$ for G of type E_6 (which are A_1, A_1^2 and A_1^3) are all contained in an F_4-subalgebra. Therefore the closure of the A_1^3-class contains all three classes.

When G is of type E_7, the nonzero elements of $\mathcal{V}$ consist of the union of the five classes with labels A_1, A_1^2, $(A_1^3)^{(1)}$, $(A_1^3)^{(2)}$ and A_1^4. All such classes have representatives contained in $\mathrm{Lie}(M)$ where M is a subgroup of type D_6. We may choose these representatives so that the corresponding partitions of 12 are $(2^2, 1^8)$, $(3, 1^9)$, (2^6), $(3, 2^2, 1^5)$ and $(3, 2^4, 1)$, respectively. Thus, all the classes in $\mathcal{V}$ are contained in the closure of the A_1^4-class. The discussion in [Liebeck and Seitz 2012, Section 16.1.2] shows that the four nontrivial classes in $\mathcal{V}$ for G of type E_8 (which are A_1, A_1^2, A_1^3 and A_1^4) are contained in an E_7-subalgebra. Thus the closure of the A_1^4-class contains all classes in $\mathcal{V}$.

A final routine use of the tables in [Stewart 2016] allows us to complete the proof. For example, when G is of type E_7 the Jordan block sizes for the adjoint action of a nilpotent element in the A_1^4-class are $(3^{28}, 2^{14}, 1^{21})$. Every nonrestricted class has a block of size greater than 3 and all other remaining classes have at least 33 blocks of size 3. □

2.2. *G-cr subalgebras.*

Proposition 2.2. *Suppose $e \in \mathcal{N}_p$. If e is contained in an $\mathfrak{sl}_2$-triple then there exists a G-cr subgroup $X \le G$ of type A_1 such that* $\mathrm{Lie}(X)$ *contains e.*

Proof. If $G = \mathrm{SL}(V)$ then $e^{[p]} = 0$ implies e has Jordan blocks of size at most p, which means e is regular in a Levi subalgebra of type $A_{r_1} \times \cdots \times A_{r_i}$ where $r_j \le p - 1$ for each j. The image of $X = \mathrm{SL}_2$ under the completely reducible representation given by $L(r_1) \oplus \cdots \oplus L(r_i)$ satisfies the demands of the theorem, where r_j now represents a (restricted) high weight. So assume G is not of type A. Then if p is good for G, it is very good, and the result follows from [McNinch 2005, Proposition 33, Theorem 52].

So we may assume p is bad, and therefore that G is exceptional. As before, the orbits of $\mathcal{N}_p$ can be worked out from the tables in [Stewart 2016] and there are not very many. By inspection, it follows that the label of every restricted nilpotent class is denoted by sums of A_r for $r < p$ and $D_4(a_1)$ if $G = E_8$, $p = 5$ or is $G_2(a_1)$ when $G = G_2$, $p = 3$; the class $(\tilde{A}_1)_{(3)}$ is excluded since it is not contained in an $\mathfrak{sl}_2$-triple.

We first deal with the final case. The subsystem subgroup $A_2 < G_2$ contains an A_2-irreducible subgroup X of type A_1. By [Stewart 2010, Theorem 1], all simple subgroups of G_2 are G_2-cr when $p = 3$. The restriction of the nontrivial 7-dimensional G_2-module to X is $L(2)^2 + L(0)$. It follows that the nilpotent elements contained in $\mathrm{Lie}(X)$ have Jordan blocks of size $(3^2, 1)$ and thus are in the $G_2(a_1)$ class by [Stewart 2016, Table 4].

In the remaining cases, every class is a distinguished element in $\mathfrak{l} = \mathrm{Lie}(L)$ for some Levi subgroup L with simple factors only of type A_r with $r < p$ or D_4. By [Serre 2005, Proposition 3.2], a subgroup X of L is G-cr if and only if it is L-cr. Furthermore a subgroup X of a central product $L = L_1 L_2$ is L-cr if and only if the projection of X to both L_1 and L_2 is L-cr. Therefore, it suffices to deal with the cases where L is simple and simply connected of type A_r ($r < p$) or D_4 — but these cases have already been tackled. □

If X is G-cr then so is $\mathrm{Lie}(X)$ by [McNinch 2007, Theorem 1]; so we get the following.

Corollary 2.3. *Suppose $e \in \mathcal{N}_p$. If e is contained in an $\mathfrak{sl}_2$-triple then there exists a G-cr subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2$ of $\mathfrak{g}$ containing e.*

The following is used a couple of times, and is [McNinch 2007, Lemma 4].

Lemma 2.4. *Let L be a Levi factor of a parabolic subgroup of G. Suppose that we have a Lie subalgebra $\mathfrak{s} \subset \mathfrak{l} = \mathrm{Lie}(L)$. Then $\mathfrak{s}$ is G-cr if and only if $\mathfrak{s}$ is L-cr.*

Proposition 2.5. *Suppose $e \in \mathcal{N}$ is distinguished in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with a factor of type A_{p-1}. Then there is an $\mathfrak{sl}_2$-triple (e, h, f) such that $\mathfrak{s} := \langle e, h, f\rangle$ is non-G-cr and $f \in \overline{L \cdot e}$.*

Proof. By Lemma 2.4 it suffices to treat the case that $L = \mathrm{SL}(V)$ with $\dim V = p$. In that case, let $\mathfrak{s} = \langle e, h, f\rangle$ be the image of $\mathfrak{sl}_2$ under the representation given by the p-dimensional baby Verma module $Z_0(0)$; see [Jantzen 1998, Section 5.4]. As $V \downarrow X = Z_0(0)$ is a nontrivial extension of the irreducible module $L(p-2)$ by the trivial module we have that $\mathfrak{s}$ is not L-cr. It is easy to see that one of e or f has a full Jordan block on V and is therefore regular. But the whole of $\mathcal{N}(L)$ is the closure of a regular nilpotent element so we are done. □

Lemma 2.6. *Let p be a good prime for G and (e, h, f) be an $\mathfrak{sl}_2$-triple with $e, f \in \mathcal{N}$. Suppose that e and f are distinguished in Levi subalgebras of $\mathfrak{g}$ with no factors of type A_{p-1}. If $\mathfrak{s} := \langle e, f\rangle$ is G-cr then $\mathfrak{s}$ is a p-subalgebra.*

Proof. Suppose $\mathfrak{s}$ is not a p-subalgebra. Then by [Stewart and Thomas 2018, Lemma 4.3], $\mathfrak{s}$ is L-irreducible in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with $L = L_1 L_2 \cdots L_r$ and L_1 of type A_{rp-1}, say, for some $r \in \mathbb{N}$. Therefore, the projection $\bar{\mathfrak{s}}$ of $\mathfrak{s}$ to $\mathfrak{l}_1 = \mathrm{Lie}(L_1)$ is also L_1-irreducible, so that $\bar{\mathfrak{s}}$ acts irreducibly on the rp-dimensional natural L_1-module. All irreducible representations of $\mathfrak{sl}_2$ have dimension at most p by [Block 1962, Lemma 5.1], thus $r = 1$. Moreover, the classification of p-dimensional irreducible $\mathfrak{sl}_2$-modules in [Jantzen 1998, Section 5.4] shows that the image of e or f in $\bar{\mathfrak{s}}$ is regular in L_1, a contradiction. □

3. Monogamy of $\mathcal{V}$

We start with an observation that $\mathcal{V}$ can be characterised using the following partial order on $\mathcal{N}$.

Definition 3.1. Let $x, y \in \mathcal{N}$. We say $x \preceq y$ (resp. $x \prec y$) if $\operatorname{rank}(\operatorname{ad}(x)^{p-1}) \leq \operatorname{rank}(\operatorname{ad}(y)^{p-1})$ (resp. $\operatorname{rank}(\operatorname{ad}(x)^{p-1}) < \operatorname{rank}(\operatorname{ad}(y)^{p-1})$).

Note that $\operatorname{rank}(\operatorname{ad}(x)^{p-1})$ can be calculated from the adjoint Jordan blocks of x of size at least p, and if G is exceptional, this can be done by reference to [Stewart 2016, Section 3.1]. The next lemma follows from a simple case-by-case check, using Tables 1 and 2, the Hasse diagrams for nilpotent orbit closures and [Stewart 2016, Section 3.1].

Lemma 3.2. *Let $x, y \in \mathcal{N}$ such that $x \in \mathcal{V}$ and $y \notin \mathcal{V}$. Then $x \prec y$.*

Remark 3.3. Comparing ranks of $(p-1)$-th powers is necessary for the partial order to differentiate nilpotent orbits contained in $\mathcal{V}$. For example, let G be of type E_6, $p=5$, and take $x, y \in \mathcal{N}$ to be representatives of the $D_4(a_1)$ and A_4 classes, respectively. Then we have $x \in \mathcal{V}$ and $y \notin \mathcal{V}$. Using [Stewart 2016, Table 16] we see that $\operatorname{rank}(\operatorname{ad}(x)) = \operatorname{rank}(\operatorname{ad}(y)) = 78$, however $\operatorname{rank}(\operatorname{ad}(x)^{p-1}) = 11 < 15 = \operatorname{rank}(\operatorname{ad}(y)^{p-1})$.

Let $\mathfrak{X} \subseteq \mathcal{N}$. We say that $\mathfrak{X}$ is *partially monogamous* if the following holds:

> If (e, h, f) and (e, h', f') are two $\mathfrak{sl}_2$-triples with $e, f, f' \in \mathfrak{X}$ and $f, f' \preceq e$, then f and f' are conjugate under the action of $C_G(e)$.

Lemma 3.4. *Let $\mathfrak{X}$ be a subvariety of $\mathcal{N}_p$. Then $\mathfrak{X}$ is monogamous if and only if it is partially monogamous.*

Proof. One direction is trivial. Suppose $\mathfrak{X}$ is partially monogamous but not monogamous. Then there exist $\mathfrak{sl}_2$-triples (e, h, f) and (e, h', f') with $e, f, f' \in \mathfrak{X}$ such that (e, h, f) is not $C_G(e)$-conjugate to (e, h', f'). Since $\mathfrak{X}$ is partially monogamous it follows that either $f \npreceq e$ or $f' \npreceq e$; without loss of generality we assume the former. Thus $\operatorname{rank}(\operatorname{ad}(e)^{p-1}) < \operatorname{rank}(\operatorname{ad}(f)^{p-1})$, and in particular, e and f are not conjugate.

Let $(f, \tilde{h}, \tilde{e})$ be an $\mathfrak{sl}_2$-triple with f conjugate to $\tilde{e}$, which exists by Proposition 2.2. Then the two $\mathfrak{sl}_2$-triples $(f, -h, e)$ and $(f, \tilde{h}, \tilde{e})$ satisfy $f, e, \tilde{e} \in \mathfrak{X}$ and $e, \tilde{e} \preceq f$. But as $\mathfrak{X}$ is partially monogamous, we have that f is conjugate to $\tilde{e}$, which is in turn conjugate to e, a contradiction. $\square$

Theorem 1.1 for classical types follows from Lemma 2.1 and the main theorem of [Goodwin and Pengelly 2024]. For the remainder of this section we suppose G is of exceptional type.

3.1. ***Bad characteristic.*** We first treat the case when p is bad for G. Fix $0 \neq e \in \mathcal{V}$ for the remainder of this section. We use the representatives as in [Liebeck and Seitz 2012], presented in [Stewart 2016]. If G is of type G_2 and $p = 3$, then the element e with label $(\tilde{A}_1)_{(3)}$ cannot be extended to an $\mathfrak{sl}_2$-triple by [Stewart and Thomas 2018, Theorem 1.7]. So we exclude that case from now on.

Lemma 3.5. *The normaliser $N_G(\langle e\rangle)$ (and centraliser $C_G(e)$) is smooth if and only if the class of e does not occur in the following table*:

G	p	class of e
G_2	3	$G_2(a_1)$
F_4	3	F_4, $\tilde{A}_2A_1$
E_6	3	E_6, $E_6(a_1)$, $E_6(a_3)$, A_5, $A_2^2A_1$, A_2^2
E_8	3	E_8, $E_8(a_1)$, $E_8(a_3)$, E_7, E_6A_1, $E_8(b_6)$, A_7, E_6, $E_6(a_3)A_1$, A_5A_1, $A_2^2A_1^2$, $A_2^2A_1$
	5	E_8, A_4A_3

Proof. Every element e has a cocharacter τ for which $\operatorname{im}(\tau)$ is contained in $N_G(\langle e\rangle)$ but not $C_G(e)$. Thus, the dimension of $N_G(\langle e\rangle)$ is precisely $\dim C_G(e)+1$. Similarly, $\dim \mathfrak{n}_{\mathfrak{g}}(\langle e\rangle) = \dim \mathfrak{c}_g(\langle e\rangle) + 1$ thanks to the existence of $\mathfrak{sl}_2$-triples. Therefore $N_G(\langle e\rangle)$ is smooth precisely when $C_G(e)$ is smooth.

It is straightforward to use Magma to calculate the dimension of $\mathfrak{c}_{\mathfrak{g}}(e)$. Comparing these dimensions with the dimension of $C_G(e)$ presented in [Liebeck and Seitz 2012, Tables 22.1.1–22.1.5] completes the proof. □

Observe that the set of classes in Lemma 3.5 does not intersect $\mathcal{V}$, so we may now deduce an important reduction.

Proposition 3.6. *There exists an $\mathfrak{sl}_2$-triple $(e, \bar{h}, \bar{f})$ with $\bar{f}$ conjugate to e and $\bar{h} \in \mathfrak{t} = \operatorname{Lie}(T)$. Moreover, if (e, h, f) is also an $\mathfrak{sl}_2$-triple then h is $C_G(e)$-conjugate to $\bar{h}$.*

Proof. We know from Proposition 2.2 that there is an $\mathfrak{sl}_2$-triple $(e, \bar{h}, \bar{f})$ with $\bar{f}$ in the same nilpotent class as e. By Lemma 3.5, the group $N_G(\langle e\rangle)$ is smooth. Therefore, all maximal tori in $\mathfrak{n}_{\mathfrak{g}}(\langle e\rangle)$ are $N_G(\langle e\rangle)$-conjugate. A computation in Magma shows that $\mathfrak{n}_{\mathfrak{g}}(\langle e\rangle) \cap \mathfrak{t}$ is a maximal torus of $\mathfrak{n}_{\mathfrak{g}}(\langle e\rangle)$. So we may assume that $\bar{h}$ is contained in $\mathfrak{t}$ (noting that if $(\lambda e, \bar{h}^g, \bar{f}^g)$ is an $\mathfrak{sl}_2$-triple then so is $(e, \bar{h}^g, \lambda \bar{f}^g)$).

For the final part, first note that since $[h, e] = 2e$ we have $[h^{[p]}, e] = \operatorname{ad}(h)^p e = 2e$ thanks to Fermat's little theorem. Therefore $\mathfrak{h} = \langle h^{[p]^r} \mid r = 0, 1, \ldots\rangle$ is an abelian p-closed subalgebra of $\mathfrak{n}_{\mathfrak{g}}(\langle e\rangle)$. It follows from [Strade and Farnsteiner 1988, Chapter 2, Corollary 4.2] that $\mathfrak{h} = \mathfrak{t}' \oplus \mathfrak{n}'$ where $\mathfrak{t}'$ is the set of semisimple elements of $\mathfrak{h}$. Since $\mathfrak{t}'$ is a torus, the above argument shows that up to $N_G(\langle e\rangle)$-conjugacy we may assume that $\mathfrak{t}'$ is contained in $\mathfrak{t}$. In particular, $\bar{h} \in \mathfrak{t}'$.

Because $\mathfrak{c}_{\mathfrak{g}}(\langle e\rangle)$ has codimension 1 in $\mathfrak{n}_{\mathfrak{g}}(\langle e\rangle)$ and $\bar{h} \notin \mathfrak{c}_{\mathfrak{g}}(\langle e\rangle)$ we see that the torus $\mathfrak{t}'$ decomposes as $\mathfrak{t}' = \mathfrak{c}_{\mathfrak{t}'}(e) \oplus \langle \bar{h}\rangle$. Furthermore, $\mathfrak{n}' \subset \mathfrak{c}_{\mathfrak{g}}(\langle e\rangle)$. It follows that $h = \bar{h} + h'$ for some $h' \in \mathfrak{c}_{\mathfrak{g}}(e) \cap \mathfrak{c}_{\mathfrak{g}}(\bar{h})$.

Since $h = [e, f]$ and $\bar{h} = [e, \bar{f}]$ we also have $h' \in \operatorname{im}(\operatorname{ad}(e))$. Thus

$$h' \in W = \mathfrak{c}_{\mathfrak{g}}(\langle e, h\rangle) \cap \operatorname{im}(\operatorname{ad}(e)).$$

Another Magma check shows that every element in W is p-nilpotent.

In particular, all eigenvalues of h' are 0. Since $h = \bar{h} + h'$ and $[h, f] = -2f$ we must have $[\bar{h}, f] = -2f$. Therefore, $f \in F = \ker(\operatorname{ad}(\bar{h}) + 2I_{\dim \mathfrak{g}})$ and so $h = [e, f] \in \operatorname{im}(\operatorname{ad}(e))(F)$. Note that $\bar{f} \in F$ also, so $\bar{h} \in \operatorname{im}(\operatorname{ad}(e))(F)$ and hence $h' \in \operatorname{im}(\operatorname{ad}(e))(F)$.

Thus $h' \in W \cap \operatorname{im}(\operatorname{ad}(e))(F)$. A final straightforward check in Magma shows that $W \cap \operatorname{im}(\operatorname{ad}(e))(F) = 0$, as required. □

We now describe an ad-hoc method to prove that if (e, h, f') is an $\mathfrak{sl}_2$-triple with $f' \in \mathcal{V}$ and $f' \preceq e$ then f' is uniquely determined up to $C := (C_G(e) \cap C_G(h))$-conjugacy. In principle, this can be implemented by hand, but for speed and accuracy we have used Magma. Applying Proposition 3.6 and Lemma 3.4 then completes the proof that $\mathcal{V}$ is monogamous.

Setup: By Proposition 3.6, there exists an $\mathfrak{sl}_2$-triple (e, h, f) with $h \in \mathfrak{t} = \operatorname{Lie}(T)$ and $f \in \mathcal{V}$ in the same nilpotent class as e. Let (e, h, f') be an $\mathfrak{sl}_2$-triple with $f' \in \mathcal{V}$ and $f' \preceq e$. Since

$$[h, f'] = -2f' \tag{1}$$

we have $f' \in F := \ker(\operatorname{ad}(h) + 2I_{\dim(\mathfrak{g})})$. We set up a generic element of the subspace F, namely $\tilde{f} = \sum x_i v_i \in \mathfrak{g}$ where the x_i are variables and $v_1, \dots, v_{\dim(F)}$ is a basis for F. One can view the set of all $\tilde{f}$ as describing a subvariety $\mathcal{F}$ of $\mathfrak{g}$. In Steps 1 to 3 below, we add in additional equations and thus replace $\mathcal{F}$ with successively smaller sets (still called $\mathcal{F}$ by abuse of notation).

Step 1: The equation

$$[e, \tilde{f}] = h \tag{2}$$

yields a set of linear equations among the x_i. We use these to constrain $\tilde{f}$ and thus reduce the dimension of $\mathcal{F}$. Now every element of $\mathcal{F}$ forms an $\mathfrak{sl}_2$-triple with e.

Example 3.7. We give an example where Step 1 is sufficient. Let G be of type E_7, $p = 3$ and $e = e_{\alpha_2} + e_{\alpha_5} + e_{\alpha_7}$. Then e is a representative of the $(A_1^3)^{(1)}$ orbit and $e \in \mathcal{V}$ by Lemma 2.1. On this occasion it is obvious that (e, h, f) is an $\mathfrak{sl}_2$-triple with $h = h_2 + h_5 + h_7 \in \mathfrak{t}$ and $f = e_{-\alpha_2} + e_{-\alpha_5} + e_{-\alpha_7}$.

Let $F := \ker(\mathrm{ad}(h) + 2I_{\dim(\mathfrak{g})})$. A straightforward calculation shows that the space F is 27-dimensional with a basis of root vectors $v_1 = e_{r_1}, \ldots, v_{27} = e_{r_{27}}$ for some set of roots $r_1, \ldots, r_{27}$; in particular $r_{12} = -\alpha_2$, $r_{13} = -\alpha_5$ and $r_{14} = -\alpha_7$.

We let $\tilde{f} = \sum_i x_i v_i$ as above. We then compute $[e, \tilde{f}] = h$. For $i \neq 12, 13, 14$ we find that the left-hand side has a coordinate of the form λx_i for $\lambda = 1$ or 2. Thus $x_i = 0$ for $i \neq 12, 13, 14$. On the other hand the coordinate of h_2 is seen to be equal to $x_{14} + 2$. Thus x_{14} is -1. Similarly, the coordinates of h_5 and h_7 are $x_{13} + 2$ and $x_{12} + 2$, respectively. We have therefore determined all the variables in $\tilde{f}$ and in fact $\tilde{f} = f$, which is sufficient.

Step 2: The adjoint action of C preserves $\mathcal{F}$. Find a set of variables $\{x_i \mid i \in Z\}$ such that every C-orbit in $\mathcal{F}$ contains a representative with $x_i = 0$ for $i \in Z$. Thus we may assume that these variables are zero in $\tilde{f}$, further reducing $\mathcal{F}$.

Example 3.8. We give an example where Steps 1 and 2 are sufficient. Let G be of type G_2 and $p = 3$. Consider $e = e_{10}$ which is a representative of the $\tilde{A}_1$ orbit, thus contained in $\mathcal{V}$ by Lemma 2.1.

Clearly, if $h = h_{10}$, $f = e_{-10}$, then (e, h, f) is an $\mathfrak{sl}_2$-triple with $f \in \mathcal{V}$. Define $F := \ker(\mathrm{ad}(h) + 2I_{\dim(\mathfrak{g})})$. This is 3-dimensional and we build $\tilde{f}$ as above:

$$\tilde{f} = x_1 e_{-11} + x_2 e_{-10} + x_3 e_{21}.$$

After Step 1 we find

$$\tilde{f} = x_1 e_{-11} + e_{-10} + x_3 e_{21}.$$

Now we apply elements of $C = C_G(e) \cap C_G(h)$ to $\tilde{f}$. First consider $x_{-01}(t) \in C$. We calculate that

$$x_{-01}(t) \cdot \tilde{f} = (t + x_1) e_{-11} + e_{-10} + x_3 e_{21}.$$

Therefore, by setting $t = -x_1$, we see that every C-orbit in $\mathcal{F}$ contains a representative with $x_1 = 0$. We're down to

$$\tilde{f} = e_{-10} + x_3 e_{21}.$$

Finally, conjugation by $x_{31}(t) \in C$ sends $\tilde{f}$ to $e_{-10} + (t + x_3) e_{21}$. Thus we conclude that $\tilde{f} = f$, as required.

Step 3: Finally, we impose the condition that $\tilde{f}$ should represent an element $f' \in \mathcal{V}$ with $f' \preceq e$. Since every element in $\mathcal{V}$ is p-nilpotent, the equation

$$\mathrm{ad}(\tilde{f})^p = 0 \tag{3}$$

yields further polynomial equations we want the x_i to satisfy.

Forcing $\mathcal{F}$ to only contain elements f' with $f' \preceq e$ is slightly more subtle since we cannot simply calculate the 'rank' of $M = \mathrm{ad}(\tilde{f})^{p-1}$. Let $R = \mathrm{rank}(\mathrm{ad}(e)^{p-1})$

and ϵ be a map evaluating the remaining variables to choices in $\Bbbk$ (so each $f' \in \mathcal{F}$ is simply some $\epsilon(\tilde{f})$). We find a subset $r_1, \dots, r_R$ of rows and subset $c_1, \dots, c_R$ of columns such that, up to the reordering of rows and columns, the corresponding submatrix S of M is upper triangular and all diagonal entries are elements of $\mathbb{F}_p^*$. Then any element $f' \in \mathcal{F}$ will satisfy $\operatorname{rank}(\operatorname{ad}(f')^{p-1}) \geq R$. We only want those elements $f' \preceq e$ which means $\operatorname{rank}(\operatorname{ad}(f')^{p-1}) \leq R$. Thus, given any row r of M the element $\epsilon(r)$ is in the span of $\epsilon(r_1), \dots, \epsilon(r_R)$. In particular, a row r' of M with zeroes at all columns $c_1, \dots, c_R$ evaluates to zero. This final set of conditions is enough to force all remaining variables to be 0.

Example 3.9. Here we require Step 3. Let G be of type G_2 and $p = 3$. Consider $e = e_{01}$ which is a representative of the A_1 orbit, thus contained in $\mathcal{V}$ by Lemma 2.1.

Take $h = h_{01}$, $f = e_{-01}$. Then (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with $f \in \mathcal{V}$. Define $F := \ker(\operatorname{ad}(h) + 2I_{\dim(\mathfrak{g})})$. This is 5-dimensional and we build $\tilde{f}$ as above:

$$\tilde{f} = x_1 e_{-32} + x_2 e_{-01} + x_3 e_{-10} + x_4 e_{11} + x_5 e_{32}.$$

After Step 1 we find

$$\tilde{f} = x_1 e_{-32} + e_{-01} + x_3 e_{-10} + x_4 e_{11} + x_5 e_{32}.$$

There are no elements of $C = C_G(e) \cap C_G(h)$ which we can use to reduce $\tilde{f}$, so we move onto Step 3.

The equation $\operatorname{ad}(\tilde{f})^p = 0$ gives many relations amongst the remaining variables but none that allow us to conveniently reduce $\tilde{f}$. Consider the matrix $M = \operatorname{ad}(\tilde{f})^{p-1}$. The first, eighth, tenth and thirteenth column of M consist only of zeroes, so we remove them, leaving the matrix M' as follows:

$$\begin{pmatrix}
x_1x_5 & 0 & 0 & x_5 & 2x_4^2 & 0 & 0 & x_4x_5 & 0 & x_5^2 \\
0 & 2x_4 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 2x_4 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2x_1x_5+x_3x_4 & 0 & 0 & x_3x_5+x_4^2 & 0 & 0 \\
0 & 2x_1x_4+2x_3^2 & 0 & 0 & 0 & x_1x_5+2x_3x_4 & 0 & 0 & x_3x_5+x_4^2 & 0 \\
0 & x_3 & 0 & 0 & 0 & 2x_4 & 0 & 0 & x_5 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_1x_4+x_3^2 & 0 & 0 & 2x_1x_5+x_3x_4 & 0 & 0 \\
x_1 & 0 & 0 & 1 & x_3 & 0 & 0 & x_4 & 0 & x_5 \\
0 & 2x_1 & 0 & 0 & 0 & 2x_3 & 0 & 0 & 2x_4 & 0 \\
0 & 0 & 2x_1 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 \\
x_1^2 & 0 & 0 & x_1 & x_1x_3 & 0 & 0 & 2x_3^2 & 0 & x_1x_5 \\
0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_3 & 0
\end{pmatrix}$$

We calculate that

$$R = \operatorname{rank}(\operatorname{ad}(e)^{p-1}) = 1.$$

Therefore, if $\epsilon(\tilde{f}) = f' \preceq e$ for some evaluation map ϵ, the rank of $\epsilon(M')$ is at most one. Observe that $M'_{10,4} = 1$ and so the rank of $\epsilon(M')$ is at least one. It follows that every row of $\epsilon(M')$ is a multiple of the tenth row of $\epsilon(M')$.

Consider the sixth row of M'. This only has nonzero entries in columns 2, 6 and 9, namely x_3, $2x_4$ and x_5. Since the tenth row is zero in columns 2, 6 and 9, the sixth row of $\epsilon(M')$ is zero. Hence $x_3 = x_4 = x_5 = 0$.

Similarly, row 11 of $\epsilon(M')$ is zero. Thus $x_1 = 0$, and we conclude that $\tilde{f} = f$.

3.2. ***Good characteristic.*** Suppose p is a good prime for G. As in the bad characteristic case, we describe an algorithm to deduce that $\mathcal{V}$ is monogamous. In good characteristic there is a considerable amount of theory at our disposal. In particular, every $e \in \mathcal{N}$ has an associated cocharacter: that is, a homomorphism $\tau : \mathbb{G}_m \to G$ such that under the adjoint action, we have $\tau(t) \cdot e = t^2 e$ and τ evaluates in the derived subgroup of the Levi subgroup in which e is distinguished.

Lemma 3.10. *Suppose p is good for G, and let (e, h_1, f_1) be an $\mathfrak{sl}_2$-triple with $e, f_1 \in \mathcal{V}$. Then there is a cocharacter τ associated to e such that* $\operatorname{Lie}(\tau(\mathbb{G}_m)) = \langle h_1 \rangle$. *Thus if (e, h_2, f_2) is also an $\mathfrak{sl}_2$-triple with $f_2 \in \mathcal{V}$, then h_2 is $C_G(e)$-conjugate to h_1. Moreover, if $h_1 = h_2$ and $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ is the grading of $\mathfrak{g}$ with respect to τ we have*

$$f_1 - f_2 \in \bigoplus_{r>0} \mathfrak{g}_e(-2 + rp),$$

where $\mathfrak{g}_e(i) := \mathfrak{c}_{\mathfrak{g}}(e) \cap \mathfrak{g}(i)$.

Proof. We start by proving that h_i is toral. By Lemma 2.6, the subalgebra $\mathfrak{s}_i = \langle e, h_i, f_i \rangle$ is either a p-subalgebra or non-G-cr. In the former case, we are done. In the latter case, the argument in the proof of [Stewart and Thomas 2018, Lemma 6.1] applies, showing h_i is toral.

Now we apply [Stewart and Thomas 2018, Proposition 2.8]. This yields cocharacters τ_i associated to e such that $\operatorname{Lie}(\tau_i(\mathbb{G}_m)) = \langle h_i \rangle$. By [Jantzen 2004, Lemma 5.3], any two cocharacters associated to e are $C_G(e)$-conjugate. Therefore, h_1 and h_2 are $C_G(e)$-conjugate and so up to $C_G(e)$-conjugacy we may assume they are equal. Set $h = h_1 = h_2$.

Since $[e, f_1 - f_2] = h - h = 0$ we know $f_1 - f_2 \in \mathfrak{c}_{\mathfrak{g}}(e)$. Furthermore, $[h, f_1 - f_2] = -2(f_1 - f_2)$ and hence

$$f_1 - f_2 \in \bigoplus_{r} \mathfrak{g}(-2 + rp).$$

The conclusion follows by noting that $\mathfrak{c}_{\mathfrak{g}}(e)$ is contained in the nonnegative graded part of $\mathfrak{g}$. □

Fix $0 \neq e \in \mathcal{V}$ for the remainder of this section. Choose a cocharacter τ associated to e such that $h \in \operatorname{Lie}(\tau(\mathbb{G}_m)) \subset \mathfrak{t}$ with $[h, e] = 2e$. In practice, we use the representatives and associated cocharacters given in [Lawther and Testerman 2011]. We know from Pommerening [1977; 1980] and Lemma 3.10 that there exists a unique $\bar{f} \in \mathfrak{g}(-2)$ such that $(e, h, \bar{f})$ is an $\mathfrak{sl}_2$-triple. Furthermore, if (e, h, f) is another $\mathfrak{sl}_2$-triple then $f = \bar{f} + f'$ with $f' \in \bigoplus_{r>0} \mathfrak{g}_e(-2+rp)$. Therefore, we need to prove that if $f \in \mathcal{V}$ then up to $C = C_G(e) \cap C_G(h)$-conjugacy we have $f = \bar{f}$, i.e., that $f' = 0$.

To do this we use the ad-hoc method from Section 3.1. Indeed, by Lemma 3.4 it suffices to prove that $f = \bar{f}$ when $f \preceq e$. We now apply Steps 1–3 starting with the space $F = f + \bigoplus_{r>0} \mathfrak{g}_e(-2+rp)$.

Example 3.11. We give a final example, this time in good characteristic. Let G be of type E_7 and $p = 7$. Consider

$$e = e_{\substack{100000\\0}} + e_{\substack{010000\\0}} + e_{\substack{001000\\0}} + e_{\substack{000100\\0}} + e_{\substack{000010\\0}},$$

which is a representative of the $(A_5)^{(2)}$ orbit; thus $e \in \mathcal{V}$ by Lemma 2.1. By [Lawther and Testerman 2011, p. 109], e has an associated cocharacter with the following τ-weights on simple roots $\tau = \substack{2\,2\ \ 2\,2\,2\,{-5}\\ \ \ \ {-9}}$. One uses the inverse of the Cartan matrix to convert this into a sum of coroots, yielding

$$h = 2h_1 + 6h_3 + 5h_4 + 6h_5 + 2h_6 \in \operatorname{Lie}(\tau(\mathbb{G}_m))$$

(this process is how one gets from the diagram of the distinguished cocharacters in Section 11 to the cocharacters given in Table 3 of [ibid.]). The unique $\bar{f} \in \mathfrak{g}(-2)$ such that $(e, h, \bar{f})$ is an $\mathfrak{sl}_2$-triple is then given by

$$\bar{f} = 2e_{-\substack{100000\\0}} + 6e_{-\substack{010000\\0}} + 5e_{-\substack{001000\\0}} + 6e_{-\substack{000100\\0}} + 2e_{-\substack{000010\\0}}.$$

Let $F = f + \bigoplus_{r>0} \mathfrak{g}_e(-2+rp)$, which is 6-dimensional. We build a generic element $\tilde{f}$ of F as in Section 3.1 with six variables. Following Step 1 by enforcing the linear equations from $[e, \tilde{f}] = h$ yields

$$\tilde{f} = \bar{f} + x_1 e_{-\substack{123211\\2}} + x_2 e_{-\substack{001100\\1}} + x_2 e_{-\substack{011000\\1}} + x_3 e_{-\substack{000001\\0}} + x_4 e_{\substack{111111\\0}} - x_5 e_{\substack{122110\\1}} + x_5 e_{\substack{112210\\1}} + x_6 e_{\substack{234321\\2}}.$$

On this occasion $C := C_G(e) \cap C_G(h)$ is finite and we move on to Step 3.

Let $M = \operatorname{ad}(\tilde{f})^{p-1}$. We calculate that

$$R = \operatorname{rank}(\operatorname{ad}(e)^{p-1}) = 13.$$

So if $\epsilon(\tilde{f}) = f' \preceq e$ for some evaluation map ϵ, we have that the rank of $\epsilon(M)$ is at most 13.

Ordering the basis of $\mathfrak{g}$ as in Magma, we use the 13×13 submatrix S of M corresponding to the rows r and columns c where

$$r = \{75, 125, 62, 94, 87, 129, 120, 97, 42, 82, 23, 34, 108\},$$
$$c = \{37, 100, 24, 52, 50, 109, 92, 60, 14, 40, 5, 9, 72\}.$$

The submatrix S is upper triangular and all diagonal entries are elements of $\mathbb{F}_p^*$. The only other nonzero entries in S can be found in row one, which is

$$(1\ \ 0\ \ 4x_2\ \ 0\ \ 0\ \ 0\ \ 5x_5\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0).$$

We find that 42 rows of M have zero entries in every column in c, so each of these rows is zero. An example of such a row is the eighth row of M. In row 8 we find x_4, $3x_5$ and $-x_6$ in columns 11, 15 and 70, respectively. It follows that $x_4 = x_5 = x_6 = 0$. Similarly the 133-rd row of M then allows us to deduce that $x_1 = x_2 = x_3 = 0$. Thus $\tilde{f} = f$ as required.

4. Proof of Theorems 1.1 and 1.2

Proposition 2.2 shows that for each $e \in \mathcal{V}$ there exists an $\mathfrak{sl}_2$-triple (e, h, f) with $\mathfrak{s} = \langle e, h, f\rangle = \mathrm{Lie}(X)$ for a G-cr subgroup $X < G$ of type A_1. Thus f is G-conjugate to e and hence $f \in \mathcal{V}$. We have demonstrated in Section 3 that any other $\mathfrak{sl}_2$-triple (e, h', f') with $f' \in \mathcal{V}$ is $C_G(e)$-conjugate to (e, h, f). Therefore $\mathfrak{s}' = \langle e, h', f'\rangle$ is G-conjugate to $\mathfrak{s}$ and hence G-cr.

It remains to prove that $\mathcal{V}$ is the unique maximal closed G-stable subvariety of $\mathcal{N}$ satisfying both the monogamy and A_1-G-cr conditions.

For G of classical type, it follows from [Goodwin and Pengelly 2024, Theorem 1.1] that $\mathcal{V}$ is maximal with respect to being monogamous and the unique subvariety with this property. For the A_1-G-cr property, the ingredients are in [ibid.] but let us spell out the details, as these essentially make up the strategy for the groups of exceptional type used below.

Proposition 4.1. *Let G be a simple algebraic group of classical type. Then $\mathcal{V}$ is the unique maximal closed G-stable A_1-G-cr subvariety of $\mathcal{N}$.*

Proof. Suppose $\mathfrak{X}$ is a G-stable closed subvariety of $\mathcal{N}$ satisfying the A_1-G-cr condition and $\mathfrak{X} \not\subseteq \mathcal{V}$. Let $e \in \mathfrak{X} \setminus \mathcal{V}$.

First suppose e is distinguished in a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ with L having a factor of type A_{p-1}. Proposition 2.5 shows that e is contained in an $\mathfrak{sl}_2$-triple generating a non-G-cr subalgebra, a contradiction (these non-G-cr subalgebras are also exhibited in [Goodwin and Pengelly 2024, Section 2.4]).

By definition of $\mathcal{V}$, we may now assume that $e^{[p]} \neq 0$. The discussion before Proposition 2.2 in [ibid.] exhibits an $\mathfrak{sl}_2$-triple (e, h, f) with $f^{[p]} = 0$ and f in $\overline{G\cdot e}$, thus $f \in \mathfrak{X}$. The argument in the first paragraph shows that neither e nor f are distinguished in a Levi subalgebra with a factor of type A_{p-1}. By Lemma 2.6, the $\mathfrak{sl}_2$-subalgebra $\langle e, f \rangle$ is non-G-cr, a final contradiction. □

Proposition 4.2. *Let G be a simple algebraic group of exceptional type. The variety $\mathcal{V}$ is the unique maximal closed G-stable subvariety of $\mathcal{N}$ satisfying both the monogamy and A_1-G-cr conditions.*

Proof. Suppose $\mathfrak{X}$ is a G-stable closed subvariety of $\mathcal{N}$ satisfying either the monogamy or A_1-G-cr condition and $\mathfrak{X} \not\subseteq \mathcal{V}$.

First suppose there exists $e \in \mathfrak{X}$ which is distinguished in a Levi subalgebra $\mathfrak{l} = \operatorname{Lie}(L)$ with a factor of type A_{p-1}. Then Propositions 2.2 and 2.5 furnish us with two $\mathfrak{sl}_2$-triples (e, h, f) and (e, h', f') such that the first generates a G-cr subalgebra and the second generates a non-G-cr subalgebra. Moreover, f is in the same G-class as e and f' is in the closure of the G-class of e. Hence $\mathfrak{X}$ does not satisfy either condition, a contradiction.

Thus, we now assume every element of $\mathfrak{X}$ is distinguished in a Levi subalgebra with no factors of type A_{p-1}. Since $\mathfrak{X} \not\subseteq \mathcal{V}$, there exists a nilpotent class in $\mathfrak{X}$ with representative e distinguished in a Levi subalgebra $\mathfrak{l} = \operatorname{Lie}(L)$ of $\mathfrak{g}$ such that $e^{[p]} \neq 0$.

Suppose p is good for L. From [Premet and Stewart 2019, Section 2.4] we find an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{l}$ with $f^{[p]} = 0$. Since p is good for L, we may simply inspect the Hasse diagrams of each factor of L to deduce that every restricted nilpotent class is contained in the closure of each nonrestricted distinguished class. Thus, $f \in X$. Furthermore, $\mathfrak{s} = \langle e, f \rangle \cong \mathfrak{sl}_2$ is a non-L-cr subalgebra by Lemma 2.6. Hence by Lemma 2.4, $\mathfrak{X}$ does not satisfy the A_1-G-cr condition. Proposition 2.2 yields an $\mathfrak{sl}_2$-triple (f, h', e') which generates a G-cr $\mathfrak{sl}_2$-subalgebra, and moreover e' is in the same G-class as f. Therefore, f is contained in two nonconjugate $\mathfrak{sl}_2$-triples. Thus $\mathfrak{X}$ does not satisfy the monogamy condition either.

In the remaining cases p is bad for L (and hence for G) so L has an exceptional factor (including the cases $L = G$). For each class, we choose e to be the representative as in [Liebeck and Seitz 2012]. Then [Liebeck and Seitz 2012, Theorem 1(iii)(b)] provides a parabolic subgroup $P = QL$ of G and a 1-dimensional torus $T_1 < Z(L)$ with the following properties: $e \in \mathfrak{q}_{\geq 2} := \operatorname{Lie}(Q_{\geq 2})$ and moreover, the closure of the P-orbit of e is equal to $\mathfrak{q}_{\geq 2}$, where here $Q_{\geq 2}$ is the product of all root groups for which the T_1-weight is at least 2. Thus, $\mathfrak{q}_{\geq 2} \subseteq \mathfrak{X}$. Unless G is of type G_2 (this case is dealt with momentarily), a straightforward calculation shows that $\mathfrak{q}_{\geq 2}$ contains a representative of the A_{p-1}-class. Thus, so does $\mathfrak{X}$, which is a contradiction.

Finally, let G be of type G_2 and $p = 3$. The only two classes not contained in $\mathcal{V}$ are the regular and the subregular. Since the closure of the regular class contains the subregular class it suffices to assume $\mathfrak{X}$ contains the subregular class.

A representative for this orbit is $e = e_{\alpha_2} + e_{-3\alpha_1-\alpha_2}$. This is a regular nilpotent element in $\mathfrak{m} = \mathrm{Lie}(M)$ where M is the standard subsystem subgroup of type A_2 corresponding to the simple roots α_2 and $-3\alpha_1 - 2\alpha_2$.

As in the proof of Proposition 2.5, there exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{m}$ such that $\mathfrak{s} = \langle e, f \rangle$ is non-M-cr. Furthermore, f is in the orbit labelled A_1 (both as an A_2-orbit and G_2-orbit). We claim that $\mathfrak{s}$ is non-G-cr. By Proposition 2.2, the element f is contained in an $\mathfrak{sl}_2$-triple generating a G-cr subalgebra and by the claim, the $\mathfrak{sl}_2$-triple $(f, -h, e)$ generates a non-G-cr subalgebra. Hence $\mathfrak{X}$ does not satisfy either condition.

For the claim, note that $\mathfrak{s}$ is certainly G-reducible since it is non-M-cr. All G-cr $\mathfrak{sl}_2$-subalgebras which are G-reducible are contained in a Levi subalgebra. In this low-rank case, it immediately follows that all such $\mathfrak{sl}_2$-subalgebras are G-conjugate to either $\mathfrak{l}_1 = \langle e_{\pm\alpha_1} \rangle$ or $\mathfrak{l}_2 = \langle e_{\pm\alpha_2} \rangle$. Therefore a G-cr $\mathfrak{sl}_2$-subalgebra only contains nilpotent elements in the A_1 or $\tilde{A}_1$ classes. The claim follows since $\mathfrak{s}$ contains e which is in the subregular class. □

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AN EXTENSION OF GOW'S THEOREM

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In memory of our good friend and esteemed colleague Gary Seitz

We extend Gow's theorem to finite groups G whose generalized Fitting subgroup is $Z(G)S$ for a quasisimple Lie-type group S of simply connected type in characteristic p, and whose center $Z(G)$ has p'-order.

A result of Rod Gow [2000, Theorem 2] asserts that the product $a^G b^G$ of any two regular semisimple classes in a finite simple group of Lie type G contains every nontrivial semisimple element $x \in G$. This result has been used in many applications. It has also been extended to any quasisimple Lie-type group of simply connected type: the product $a^G b^G$ of any two regular semisimple classes in G contains every noncentral semisimple element $x \in G$; see [Guralnick and Tiep 2015, Lemma 5.1].

We will further extend Gow's theorem. Let p be a prime and let $\underline{\boldsymbol{G}}$ be a simple, simply connected algebraic group defined over $\overline{\mathbb{F}}_p$. Let $F : \underline{\boldsymbol{G}} \to \underline{\boldsymbol{G}}$ be a Steinberg endomorphism, so that

$$S := \underline{\boldsymbol{G}}^F$$

is quasisimple. (In particular, we do not view $\mathrm{PSL}_2(9)$ as $\mathrm{Sp}_4(2)'$, $\mathrm{SU}_3(3)$ as $G_2(2)'$, or $\mathrm{SL}_2(8)$ as ${}^2G_2(3)'$.)

We will consider finite groups G with

$$F^*(G) = \boldsymbol{Z}(G)S \quad \text{and} \quad p \nmid |\boldsymbol{Z}(G)|, \tag{1}$$

(so $\boldsymbol{C}_G(S) = \boldsymbol{C}_G(F^*(G)) = \boldsymbol{Z}(G)\boldsymbol{Z}(S)$ is a p'-group), and aim to show that the product $a^G b^G$ of two particular conjugacy classes in G will cover all elements $g \in G$ of a certain kind. Before going on we state a special case of a consequence of our main result which is less technical. Versions of this result have already been used in [Acciarri et al. 2023; Guralnick et al. 2025].

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Corollary 1. *Let G be as above. Assume that $a \in G$ has order prime to p, $|\boldsymbol{C}_S(a)|$ has order prime to p and that $s \in S \setminus \boldsymbol{Z}(S)$ is semisimple. Then $s^t = [a, u]$ for some $t, u \in S$.*

In fact, one need not assume that g has order prime to p. See our main result Theorem 6.

Let a and b be elements of G, and set $G_1 := \langle S, a, b\rangle$. Then $S \lhd G_1$, and hence $S \lhd F^*(G_1)$. It follows that $F(G_1)$, as well as any quasisimple subnormal subgroup $T \neq S$ of G_1, centralizes S. Hence by (1), $F^*(G_1) = \boldsymbol{Z}(G_1)S$. Furthermore,

$$\boldsymbol{Z}(G_1) \leq \boldsymbol{C}_G(S) = \boldsymbol{Z}(G)\boldsymbol{Z}(S)$$

is also a p'-subgroup. Hence, for our purposes, we may assume

$$G = \langle S, a, b\rangle. \tag{2}$$

Let St denote the Steinberg character of S, and for $x \in G$, write x_p for the p-part of x. By [Feit 1993, Corollary D], St extends to a rational-valued character St_G of G (called the *basic p-Steinberg character* of G). By [Feit 1995, Theorem C], there is a Sylow p-subgroup P and a p-subgroup D of G, of order

$$|D| = p^d = |G/S|_p,$$

such that $P = Q \rtimes D$ for a Sylow p-subgroup Q of S and the following statement holds. For any element $x \in G$, $\mathsf{St}_G(x) \neq 0$ if and only if $x_p \in D$ (up to conjugation), in which case

$$\mathsf{St}_G(x) = \pm|\boldsymbol{C}_S(x)|_p.$$

In view of these results, the proper generalization to G of regular semisimple classes in S will be that $a, b \in G$ satisfy

$$a_p \in D \text{ up to conjugation in } G \quad \text{and} \quad p \nmid |\boldsymbol{C}_S(a)|, \tag{3}$$

and

$$b_p \in D \text{ up to conjugation in } G \quad \text{and} \quad p \nmid |\boldsymbol{C}_S(b)|, \tag{4}$$

Certainly, $g \in G$ can belong to $a^G b^G$ only when it does so in the solvable group G/S, so we will assume

$$gS \in (aS)^{G/S}(bS)^{G/S} \quad \text{and} \quad g_p \in D \text{ up to conjugation in } G. \tag{5}$$

For instance, if G/S is abelian, then the first condition in (5) is equivalent to $g \in abS$.

Proposition 2. (i) *If $p > 2$ and $Q \in \mathrm{Syl}_p(S)$, then $\boldsymbol{C}_G(Q) = \boldsymbol{Z}(G)\boldsymbol{Z}(S)\boldsymbol{Z}(Q)$.*

(ii) *If $g \in G \setminus \boldsymbol{Z}(G)\boldsymbol{Z}(S)$ and $g_p \in D$, then p divides $[S : \boldsymbol{C}_S(g)]$.*

Proof. Since $Z := \boldsymbol{Z}(G)\boldsymbol{Z}(S)$ is a p'-group centralizing Q and normal in G, we may work in $\overline{G} := G/Z$ and identify Q with $\overline{Q} := QZ/Z$. Note that $Z \cap S = \boldsymbol{Z}(S)$. Moreover, as S is perfect, we have $\boldsymbol{C}_G(S/\boldsymbol{Z}(S)) = \boldsymbol{C}_G(S) = Z$. It follows that

$$\overline{S} := S/\boldsymbol{Z}(S) \lhd \overline{G} \leq \mathrm{Aut}(\overline{S}),$$

i.e., $\overline{G}$ is almost simple.

Consider any element $x \in \overline{C} := \boldsymbol{C}_{\overline{G}}(\overline{Q})$. Then $H := \langle \overline{S}, x\rangle \leq \overline{G}$ is also almost simple, whence $\boldsymbol{O}_{p'}(H) = 1$, and $R := \langle \overline{Q}, x_p\rangle$ is a Sylow p-subgroup of H centralized by x. It follows that $R = \boldsymbol{O}_p(\boldsymbol{N}_H(R))$. By [Gorenstein et al. 1998, Corollary 3.1.4], $\boldsymbol{O}_p(\boldsymbol{N}_H(R)) = F^*(\boldsymbol{N}_H(R))$, whence x belongs to $\boldsymbol{C}_{\boldsymbol{N}_H(R)}(R) \leq R$ and so $x = x_p$ is a p-element. Thus $\overline{C}$ is a p-group.

Similarly, $\overline{Q} = \boldsymbol{O}_p(\boldsymbol{N}_{\overline{S}}(\overline{Q})) = F^*(\boldsymbol{N}_{\overline{S}}(\overline{Q}))$, and so

$$\overline{C} \cap \overline{S} = \boldsymbol{C}_{\overline{S}}(\overline{Q}) = \boldsymbol{C}_{\boldsymbol{N}_{\overline{S}}(\overline{Q})}(\overline{Q}) \leq \overline{Q}.$$

It follows that $\overline{C} \cap \overline{S} = \boldsymbol{Z}(\overline{Q})$.

(i) Now we assume $p > 2$ and show that $\overline{C} \leq \overline{S}$, which implies that $\boldsymbol{C}_G(Q) = \boldsymbol{Z}(Q)Z$. Assume the contrary: $\overline{C} \not\leq \overline{S}$. Since $\overline{C}$ is a p-group, we can find a p-element $x \in \overline{C} \setminus \overline{S}$; in particular, $[x, \overline{Q}] = 1$. Now $R := \langle x, \overline{Q}\rangle$ is Sylow p-subgroup of $H := \langle \overline{S}, x\rangle$ and $x \in \boldsymbol{Z}(R)$. As $\overline{S} \lhd H \leq \overline{G} \leq \mathrm{Aut}(\overline{S})$, we still have $\boldsymbol{O}_{p'}(H) = 1$. Now, if $p > 2$ then $\boldsymbol{Z}(R) \leq F^*(H) = \overline{S}$ by [Glauberman et al. 2020, Corollary 1.2], and hence $x \in \overline{S}$, contrary to the choice of x.

(ii) Assume the contrary that $p \nmid [S : \boldsymbol{C}_S(g)]$. Conjugating g suitably, we may assume that $g \in \boldsymbol{C}_G(Q)$ with $Q \in \mathrm{Syl}_p(S)$ as before.

Suppose first that $p > 2$. Then $g \in \boldsymbol{Z}(G)\boldsymbol{Z}(S)\boldsymbol{Z}(Q)$ by (i), and so $g_p \in S$. But g_p is conjugate to an element in D by assumption and $D \cap S = 1$, so $g_p = 1$. It follows that $g \in \boldsymbol{Z}(G)\boldsymbol{Z}(S)$, a contradiction.

Thus we have $p = 2$. Then

$$\mathsf{St}_G(1) = |Q| = |\boldsymbol{C}_S(g)|_p = \pm\mathsf{St}_G(g).$$

On the other hand, St is trivial at $\boldsymbol{Z}(S)$, so the generalized center of St_G contains $Z = \boldsymbol{Z}(G)\boldsymbol{Z}(S)$ and hence equals G as G/Z is almost simple with socle $\overline{S}$. As the generalized center of St_G contains g, we conclude that $g \in Z$, again a contradiction. □

Fix any element $g \in G$ satisfying (5). Then $S\boldsymbol{C}_G(g) \leq G$, so

$$\mathbb{Z} \ni [G : S\boldsymbol{C}_G(g)] = \frac{|G| \cdot |\boldsymbol{C}_S(g)|}{|S| \cdot |\boldsymbol{C}_G(g)|} = \frac{[G : \boldsymbol{C}_G(g)]}{[S : \boldsymbol{C}_S(g)]}. \tag{6}$$

Write

$$\frac{[G : \boldsymbol{C}_G(g)]_p}{[S : \boldsymbol{C}_S(g)]_p} = p^e. \tag{7}$$

Lemma 3. *Let X be a finite group, which is abelian-by-cyclic, that is, X has a normal abelian subgroup $A \lhd X$ such that X/A is cyclic. Suppose $x, y, z \in X$ are such that*

$$X = \langle x, y \rangle \quad \textit{and} \quad z \equiv xy \pmod{[X, X]}.$$

Then

$$\sum_{\alpha \in \mathrm{Irr}(X)} \frac{\alpha(x)\alpha(y)\overline{\alpha(z)}}{\alpha(1)} = |X/[X, X]|.$$

Proof. The condition $z \equiv xy \pmod{[X, X]}$ implies that

$$\sum_{\alpha \in \mathrm{Irr}(X),\, \alpha(1)=1} \frac{\alpha(x)\alpha(y)\overline{\alpha(z)}}{\alpha(1)} = |X/[X, X]|.$$

Hence it suffices to show that the contribution of any nonlinear $\alpha \in \mathrm{Irr}(X)$ to the sum in the statement is 0. Consider any irreducible constituent λ of $\alpha|_A$. Suppose λ is not X-invariant. As $X = \langle x, y \rangle$, we may assume that λ is not x-invariant, in which case $\alpha(x) = 0$ by Clifford's theorem and the contribution is 0 as claimed.

Suppose now that λ is X-invariant. Then for any $a \in A$ and $t \in X$, as $\lambda(1) = 1$ we have

$$\lambda(tat^{-1}a^{-1}) = \lambda(tat^{-1})/\lambda(a) = 1,$$

whence $[t, a] \in \mathrm{Ker}(\lambda)$ and $\mathrm{Ker}(\lambda) \lhd X$. It follows that $A/\mathrm{Ker}(\lambda) \leq \mathbf{Z}(X/\mathrm{Ker}(\lambda))$. But X/A is cyclic, so $X/\mathrm{Ker}(\lambda)$ is abelian. Now λ is the unique irreducible constituent of α_A, so $\mathrm{Ker}(\lambda) \leq \mathrm{Ker}(\alpha)$, and hence α, viewed as an irreducible character of $X/\mathrm{Ker}(\lambda)$, must be linear, contrary to the assumption $\alpha(1) > 1$. □

Proposition 4. *Under the assumptions (1)–(5), assume in addition that G/S is abelian-by-cyclic. Then*

$$\Sigma_1 := \sum_{\chi \in \mathrm{Irr}(G|\mathsf{St})} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \cdot |g^G|$$

is a rational integer whose p-part is at most p^{d+e}.

Proof. As mentioned above, St extends to St_G. Hence, by Gallagher's theorem [Isaacs 2006, (6.17)], any $\chi \in \mathrm{Irr}(G|\mathsf{St})$ is of the form

$$\chi = \mathsf{St}_G \alpha$$

with $\alpha \in \mathrm{Irr}(G/S)$. Using (2), (5) and Lemma 3, we see that

$$\sum_{\alpha \in \mathrm{Irr}(G/S)} \frac{\alpha(a)\alpha(b)\overline{\alpha(g)}}{\alpha(1)}$$

is a rational integer whose p-part is at most p^d.

On the other hand, by (3) and (4) we see that

$$\frac{\mathsf{St}_G(a)\mathsf{St}_G(b)\overline{\mathsf{St}_G(g)}}{\mathsf{St}_G(1)}\cdot|g^G| = \pm\frac{|\boldsymbol{C}_S(g)|_p\cdot|G|_p\cdot|G|_{p'}}{|S|_p\cdot|\boldsymbol{C}_G(g)|_p\cdot|\boldsymbol{C}_G(g)|_{p'}} = \pm\frac{[G:\boldsymbol{C}_G(g)]_p}{[S:\boldsymbol{C}_S(g)]_p}\cdot[G:\boldsymbol{C}_G(g)]_{p'}$$

is p^e times a p'-integer. Hence the statement follows. □

Recall that St is the only p-defect zero character of S. By the main result of [Humphreys 1971], all the remaining characters of S belong to p-blocks of maximal defect. The next result deals with these characters.

Proposition 5. *Under the assumptions (1)–(5), assume in addition that G/S has a cyclic Sylow p-subgroup and a normal p-complement, and that $g \notin \boldsymbol{Z}(G)\boldsymbol{Z}(S)$. Then*

$$\Sigma_2 := \sum_{\chi\in\mathrm{Irr}(G)\setminus\mathrm{Irr}(G|\mathrm{St})} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)}\cdot|g^G|$$

is p^{d+e+1} times an algebraic integer.

Proof. By the hypothesis we can write $G/S = (H/S)\rtimes D$ for some normal subgroup $H \geq S$ of G. Note that any $\chi\in\mathrm{Irr}(G)\setminus\mathrm{Irr}(G|\mathrm{St})$ lies above some $\theta\in\mathrm{Irr}(H)$ which does not lie above St. Suppose θ is not G-invariant. As $G=\langle H,a,b\rangle$, we may assume that θ is not a-invariant, in which case $\chi(a)=0$ by Clifford's theorem, and the contribution of χ to Σ_2 is 0.

Hence we need to count the total contribution to Σ_2 of the characters $\chi\in\mathrm{Irr}(G|\theta)$, where $\theta\notin\mathrm{Irr}(H|\mathrm{St})$ is G-invariant. Since G/H is cyclic, any such θ extends to a character χ_1 of G, and we may write

$$\mathrm{Irr}(G|\theta) = \{\chi_1\mu \mid \mu\in\mathrm{Irr}(G/H)\}.$$

By the assumption $\theta\notin\mathrm{Irr}(H|\mathrm{St})$, every irreducible constituent of $\theta|_S$ belongs to an S-block B_S of maximal p-defect.

Conjugating g suitably, we may assume that $g_p\in D$. Note that $g_{p'}\in H$, so $g=g_pg_{p'}$ belongs to

$$K := \langle H, g_p\rangle \lhd G.$$

Set

$$\chi_2 := (\chi_1)|_K \in \mathrm{Irr}(K|\theta).$$

Now the p-block B of H that contains θ covers B_S, and $p\nmid|H/S|$, so B has maximal defect; see, e.g., [Navarro 1998, Theorem 9.26]. But $K/H \hookrightarrow D$ is a p-group, so by [Navarro 1998, Corollary 9.6] there is a unique p-block B_2 of K that covers B. In particular,

$$\mathrm{Irr}(K|\theta)\subseteq\mathrm{Irr}(B_2).$$

Moreover, B is K-invariant as θ is K-invariant, whence B_2 is of maximal defect by [Navarro 1998, Theorem 9.17]. It follows that B_2 contains a character χ_0 of height zero, and so of p'-degree.

As χ_0 and χ_2 belong to the same block, we know that the two algebraic integers

$$\omega_{\chi_i}(g) = \frac{\chi_i(g)}{\chi_i(1)} \cdot |g^K|$$

for $i \in \{0, 2\}$ are congruent modulo p. By Proposition 2, $|g^S|$ is divisible by p, so $|g^K|$ is divisible by p as well; see the computation in (6). But $p \nmid \chi_0(1)$, so $p \mid \omega_{\chi_0}(g)$. It follows that

$$p \text{ divides } \omega_{\chi_2}(g) = \frac{\chi_2(g)}{\chi_2(1)} \cdot |g^K| = \frac{\chi_1(g)}{\chi_1(1)} \cdot |g^K|. \tag{8}$$

Next, (6) applied to $S \lhd H$ with $p \nmid |H/S|$ shows that

$$|g^S|_p = |g^H|_p.$$

On the other hand, g_p centralizes g, and $K = \langle H, g_p \rangle$, so $H\mathbf{C}_K(g) = K$, showing that $g^K = g^H$. Hence $|g^S|_p = |g^K|_p$, and (7) becomes

$$p^e = \frac{|g^G|_p}{|g^K|_p}.$$

Together with (8), we now obtain

$$p^{e+1} \text{ divides } \omega_{\chi_1}(g) = \frac{\chi_1(g)}{\chi_1(1)} \cdot |g^G|.$$

Now, (5) implies that $g \equiv ab \pmod{G/H}$, and so

$$\sum_{\mu \in \mathrm{Irr}(G/H)} \frac{\mu(a)\mu(b)\overline{\mu(g)}}{\mu(1)} = |G/H| = p^d.$$

It follows that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \cdot |g^G| = \omega_{\chi_1}(g) \sum_{\mu \in \mathrm{Irr}(G/H)} \frac{\mu(a)\mu(b)\overline{\mu(g)}}{\mu(1)} = p^d \omega_{\chi_1}(g)$$

is p^{d+e+1} times an algebraic integer. □

Theorem 6. *Under the assumptions (1)–(5), assume in addition that all the following conditions hold*:

(a) *G/S is abelian-by-cyclic.*

(b) *G/S has cyclic Sylow p-subgroups and a normal p-complement.*

(c) $g \notin \mathbf{Z}(G)\mathbf{Z}(S)$.

Then $g \in a^G b^G$.

Proof. By Propositions 4 and 5,

$$|g^G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} = \Sigma_1 + \Sigma_2$$

is $p^s(u + p^t v)$, where $u \in \mathbb{Z} \setminus p\mathbb{Z}$, v is an algebraic integer, $0 \le s \le d+e$, and $t \ge 1$. Now if $u + p^t v = 0$, then $v = -u/p^t$ is rational and an algebraic integer, so $v \in \mathbb{Z}$ and $u \in p\mathbb{Z}$, a contradiction. Thus

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(a)\chi(b)\overline{\chi(g)}}{\chi(1)} \neq 0,$$

and so $g \in a^G b^G$ by Frobenius' character formula. □

Proof of Corollary 1. To prove this, we may replace G by $\langle S, a, b\rangle$ with $b := a^{-1}$. Then (3)–(4) hold with $g := s$. Now G/S is cyclic, so $s \in a^G b^G$ by Theorem 6. But $a^G = a^S$ and $b^G = b^S$ since $G = \langle S, a\rangle = \langle S, b\rangle$, so the statement follows. □

In what follows, q is always a power of the prime p. We will use the structure of $\mathrm{Aut}(\bar{S})$ as described in [Gorenstein et al. 1998, Theorem 2.5.12], in particular the notation $\mathrm{Inndiag}(\bar{S})$ and $\mathrm{Outdiag}(\bar{S})$.

Theorem 7. *Under the assumptions (1)–(5), assume in addition that all the following conditions hold for g, $\bar{S} = S/\boldsymbol{Z}(S)$, and $\bar{G} = G/\boldsymbol{Z}(G)\boldsymbol{Z}(S)$:*

(a) *$g \notin \boldsymbol{Z}(G)\boldsymbol{Z}(S)$.*

(b) *If $\bar{S} = \mathrm{PSL}_n(q)$ with $n \ge 3$, or $\bar{S} = P\Omega^+_{2n}(q)$ with $n \ge 4$, or $S = E_6(q)$, then the quotient $\bar{G}/(\bar{G} \cap \mathrm{Inndiag}(\bar{S}))$ is cyclic.*

Then $g \in a^G b^G$.

Proof. Recall that $G/S \cong \bar{G}/\bar{S}$ is a subgroup of $O := \mathrm{Out}(\bar{S})$. By Theorem 6, we need to show that $A = G/S$ satisfies both of the conditions (a) and (b) listed therein. Note that both (a) and (b) in Theorem 6 follow from the condition

(9) A admits a normal abelian p'-subgroup B with A/B being cyclic.

In turn, (9) is a consequence of the condition

(10) $O := \mathrm{Out}(\bar{S})$ admits a normal abelian p'-subgroup J with O/J being cyclic.

(Indeed, taking $B := A \cap J$ we have $A/B \hookrightarrow O/J$.)

Set $J := \mathrm{Outdiag}(\bar{S}) := \mathrm{Inndiag}(\bar{S})/\bar{S}$. Now, if $\bar{S}$ is a *twisted* group, i.e., the parameter d for $\bar{S} \cong {}^d\Sigma(q)$ in [Gorenstein et al. 1998, Theorem 2.5.12] is greater than one, then (10) holds (for this choice of J). It remains to consider the untwisted groups, that is, the ones with $d = 1$.

If $\bar{S} = \mathrm{PSL}_2(q)$ then (10) holds. Suppose that $\bar{S} = \mathrm{PSL}_n(q)$ with $n \geq 3$, or $\bar{S} = P\Omega^+_{2n}(q)$ with $n \geq 4$, or $S = E_6(q)$. Then taking $B := (\bar{G} \cap \mathrm{Inndiag}(\bar{S}))/\bar{S}$, we see that A/B is cyclic by assumption (b) in Theorem 7, hence (9) holds.

In the remaining cases, $\bar{S}$ is of type B_n, C_n, G_2, F_4, E_7, or E_8, hence O/J is cyclic, and so (10) holds. □

Next we deduce another consequence of Theorem 6. For the definition of the *reduced Clifford group* $\Gamma^+(\mathbb{F}_q^n) = \mathrm{CSpin}^\epsilon_n(q)$, see, for example, [Tiep and Zalesski 2005, §6]; in particular, it contains $\mathrm{Spin}^\epsilon_n(q)$ as a normal subgroup with factor C_{q-1}.

Theorem 8. *Let q be a prime power, and let (G, S) be any of the following pairs of groups*:

(a) $G = \mathrm{GL}_n(q)$ *with* $n \geq 2$, $(n, q) \neq (2, 2), (2, 3)$, *and* $S = \mathrm{SL}_n(q)$.

(b) $G = \mathrm{GU}_n(q)$ *with* $n \geq 2$, $(n, q) \neq (2, 2), (2, 3), (3, 2)$, *and* $S = \mathrm{SU}_n(q)$.

(c) $G = \mathrm{CSp}_{2n}(q)$ *with* $n \geq 2$, $(n, q) \neq (2, 2)$, *and* $S = \mathrm{Sp}_{2n}(q)$.

(d) $G = \mathrm{CSpin}^\epsilon_n(q)$ *with* $n \geq 5$, $2 \nmid q$, *and* $\epsilon = \pm$, *and* $S = \mathrm{Spin}^\epsilon_n(q)$.

(e) $G = \mathrm{GO}^\epsilon_n(q)$ *or* $\mathrm{SO}^\epsilon_n(q)$ *with* $n \geq 5$, $2 \nmid q$, *and* $\epsilon = \pm$, *and* $S = \Omega^\epsilon_n(q)$.

Suppose that $a, b \in G$ are such that $p \nmid |\mathbf{C}_S(a)|$ and $p \nmid |\mathbf{C}_S(b)|$. If $g \in G$ is any noncentral p'-element such that $g \in abS$, then $g \in a^G b^G$.

Proof. For all of the above pairs but (e), we have that S is a quasisimple Lie-type group of simply connected type, $S \lhd G$, $F^*(G) = \mathbf{Z}(G)S$, and $\mathbf{Z}(S) \leq \mathbf{Z}(G)$. Furthermore, G/S is abelian of p'-order, and (3), (4), and (5) are all fulfilled. Hence the statement follows from Theorem 6. In the case of (e), the same proof of Theorem 6 applies. □

Note that Theorem 6 also applies to $\mathrm{GO}^\epsilon_{2n}(q)$ with $2 \mid q$ and $n \geq 3$. But we do not include them in Theorem 8 since the subgroup D is now of order 2 and so conditions (3)–(5) are more complicated than those formulated in Theorem 8.

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ON DIMENSIONS OF ROCK BLOCKS OF CYCLOTOMIC QUIVER HECKE SUPERALGEBRAS

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To the memory of Gary Seitz

We explicitly compute the dimensions of certain idempotent truncations of RoCK blocks of cyclotomic quiver Hecke superalgebras. Equivalently, this amounts to a computation of the value of the Shapovalov form on certain explicit vectors in the basic representations of twisted affine Kac–Moody Lie algebras of type A.

1. Introduction

Our goal is to obtain a technical (but neat) result computing the dimensions of certain idempotent truncations of RoCK blocks of cyclotomic quiver Hecke superalgebras. Equivalently, this amounts to a computation of the value of the Shapovalov form on certain explicit vectors in the basic representation $V(\Lambda_0)$ of the twisted affine Kac–Moody Lie algebra $\mathfrak{g}$ of type $A_{2\ell}^{(2)}$.

To state the result, let $\mathfrak{g}$ be the Kac–Moody Lie algebra of type $A_{2\ell}^{(2)}$, with a normalized invariant form $(\cdot\,|\,\cdot)$ on the corresponding weight lattice P, and the Weyl group W. Let

$$I = \{0, 1, \dots, \ell\}, \quad J := I \setminus \{\ell\},$$

$\{\alpha_i \mid i \in I\}$ be the simple roots, $\{\Lambda_i \mid i \in I\}$ be the fundamental dominant weights, and $Q_+ \subset P$ be the set of $\mathbb{Z}_{\geq 0}$-linear combinations of the simple roots. For the negative Chevalley generators $\{f_i \mid i \in I\}$ of $\mathfrak{g}$, we have the divided powers $f_i^{(k)} := f_i^k/k! \in U(\mathfrak{g})$ for $k \in \mathbb{Z}_{\geq 0}$. Fix a nonzero highest weight vector v_+ of the irreducible $\mathfrak{g}$-module $V(\Lambda_0)$ with highest weight Λ_0. Let $(\cdot\,,\cdot)$ be the Shapovalov form on $V(\Lambda_0)$ such that $(v_+, v_+) = 1$.

For every $w \in W$ with reduced decomposition $w = r_{i_t} \cdots r_{i_1}$, setting

$$a_k := (r_{i_{k-1}} \cdots r_{i_1} \Lambda_0 \mid \alpha_{i_k}^\vee) \quad (k = 1, \dots, t), \tag{1.1}$$

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it is easy to see that $a_1, \dots, a_t \in \mathbb{Z}_{\geq 0}$ and

$$v_w := f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$$

is a nonzero vector of the weight space $V(\Lambda_0)_{w\Lambda_0}$, which does not depend on the choice of reduced decomposition and satisfies $(v_w, v_w) = 1$.

For every $j \in J$ and $m \in \mathbb{Z}_{\geq 0}$, we define the divided power monomial

$$f(m, j) := f_j^{(m)} \cdots f_1^{(m)} f_0^{(2m)} f_1^{(m)} \cdots f_j^{(m)} f_{j+1}^{(2m)} \cdots f_{\ell-1}^{(2m)} f_\ell^{(m)} \in U(\mathfrak{g}) \tag{1.2}$$

(with $f(0, j)$ interpreted as 1). Recalling the null-root $\delta := 2\alpha_0 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$, note that the monomial $f(m, j)$ has total weight $-m\delta$. More generally, given $d, n \in \mathbb{Z}_{>0}$, a composition $\mu = (\mu_1, \dots, \mu_n)$ of d into n nonnegative parts and a tuple $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$ we define

$$f(\mu, \boldsymbol{j}) := f(\mu_n, j_n) \cdots f(\mu_1, j_1) \in U(\mathfrak{g}). \tag{1.3}$$

For a composition $\omega_d := (1, \dots, 1)$ of d, we also define

$$f(\omega_d) := \sum_{\boldsymbol{j} \in J^d} f(\omega_d, \boldsymbol{j}) = \sum_{j_1, \dots, j_d \in J} f(1, j_d) \cdots f(1, j_1). \tag{1.4}$$

If the weight space $V(\Lambda_0)_{\Lambda_0 - \theta}$ for $\theta \in Q_+$ is nonzero, then we can write $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some w in the Weyl group W and a unique $d \in \mathbb{Z}_{\geq 0}$. We then say that θ is *RoCK* if $(\theta \mid \alpha_0^\vee) \geq 2d$ and $(\theta \mid \alpha_i^\vee) \geq d - 1$ for $i = 1, \dots, \ell$. This is equivalent to the cyclotomic quiver Hecke algebra $R_\theta^{\Lambda_0}$ being a *RoCK block*, as defined in [Kleshchev and Livesey 2022, Section 4.1]. To each composition $\mu = (\mu_1, \dots, \mu_n)$ and tuple $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$, we can define the corresponding divided power idempotents $e(\mu, \boldsymbol{j}) \in R_\theta^{\Lambda_0}$. The idempotents $e(\omega_d, \boldsymbol{j})$ are then distinct and orthogonal to each other for distinct $\boldsymbol{j} \in J^d$, so we also have the idempotent $e(\omega_d) := \sum_{\boldsymbol{j} \in J^d} e(\omega_d, \boldsymbol{j})$.

Main Theorem. *Let weight $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ be RoCK, $n \in \mathbb{Z}_{>0}$, $\mu = (\mu_1, \dots, \mu_n)$ be a composition of d with n parts, and $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$. Set*

$$|\mu, \boldsymbol{j}|_{\ell-1} := \sum_{\substack{1 \leq r \leq n \\ j_r = \ell-1}} \mu_r. \tag{1.5}$$

Then

$$\begin{aligned} \dim e(\mu, \boldsymbol{j}) R_\theta^{\Lambda_0} e(\omega_d) &= (f(\mu, \boldsymbol{j}) v_w, f(\omega_d) v_w) \\ &= \binom{d}{\mu_1 \cdots \mu_n} 4^{d - |\mu, \boldsymbol{j}|_{\ell-1}} 3^{|\mu, \boldsymbol{j}|_{\ell-1}}. \end{aligned}$$

The first equality in the theorem is a known consequence of the Kang–Kashiwara–Oh categorification, so the main work is to prove of the second equality. This is proved in Theorem 5.6.

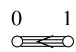
The dimension formula for $e(\mu, \boldsymbol{j})R_\theta^{\Lambda_0}e(\omega_d)$ obtained in the Main Theorem is a shadow of the fact that the Gelfand–Graev idempotent truncation of a RoCK block $R_\theta^{\Lambda_0}$ is isomorphic to a generalized Schur algebra corresponding to a certain Brauer tree algebra. In fact, our Main Theorem is a key step needed for the proof of this isomorphism in [Kleshchev 2024].

2. Shapovalov forms

2.1. ***Lie theoretic notation.*** Let $\mathfrak{g}$ be the Kac–Moody Lie algebra of type $A_{2\ell}^{(2)}$ (over $\mathbb{C}$); see [Kac 1990, Chapter 4]. We set

$$p := 2\ell + 1.$$

The Dynkin diagram of $\mathfrak{g}$ has vertices labeled by $I = \{0, 1, \dots, \ell\}$:

if $\ell \geq 2$ and


if $\ell = 1$.

We have the standard Chevalley generators $\{e_i, f_i, h_i \mid i \in I\}$ of $\mathfrak{g}$ and the Chevalley anti-involution

$$\sigma : \mathfrak{g} \to \mathfrak{g}, \quad e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad h_i \mapsto h_i.$$

We have the *weight lattice* P of $\mathfrak{g}$, the subset of all *dominant integral weights* $P_+ \subset P$, and the set $\{\alpha_i \mid i \in I\} \subset P$ of the *simple roots* of $\mathfrak{g}$. We denote by Q the sublattice of P generated by the simple roots and set

$$Q_+ := \left\{ \sum_{i \in I} m_i \alpha_i \;\middle|\; m_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I \right\} \subset Q.$$

For $\theta = \sum_{i \in I} m_i \alpha_i \in Q_+$, its *height* is

$$\mathrm{ht}(\theta) := \sum_{i \in I} m_i.$$

For $\theta \in Q_+$ of height n, we set

$$I^\theta := \{(i_1, \cdots, i_n) \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \theta\}.$$

We have the null-root

$$\delta = \sum_{i=0}^{\ell-1} 2\alpha_i + \alpha_\ell$$

with $\mathrm{ht}(\delta) = 2\ell + 1 = p$.

We denote by $(\cdot\,|\,\cdot)$ a normalized invariant form on P whose Gram matrix with respect to the linearly independent set $\alpha_0, \alpha_1, \dots, \alpha_\ell$ is

$$\begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 4 & -2 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & 4 & -2 & 0 \\ 0 & 0 & 0 & \dots & -2 & 4 & -4 \\ 0 & 0 & 0 & \dots & 0 & -4 & 8 \end{pmatrix} \quad \text{if } \ell \geq 2 \qquad \text{and} \qquad \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \quad \text{if } \ell = 1.$$

Recall from [Kac 1990, §§3.7, 3.13] the (affine) *Weyl group* W generated by the fundamental reflections $\{r_i \mid i \in I\}$ as a Coxeter group. The form $(\cdot\,|\,\cdot)$ is W-invariant; see [Kac 1990, Proposition 3.9].

For $\Lambda \in P_+$, we denote by $V(\Lambda)$ the *irreducible integrable* $\mathfrak{g}$*-module* of highest weight Λ. Fix a nonzero highest weight vector $v_+ \in V(\Lambda)_\Lambda$. The *Shapovalov form* is the unique symmetric bilinear form $(\cdot\,,\cdot)$ on $V(\Lambda)$ such that $(v_+, v_+) = 1$ and

$$(2.1) \qquad (xv, w) = (v, \sigma(x)w) \quad (x \in \mathfrak{g},\ v, w \in V(\Lambda)).$$

Lemma 2.2. *Let* $\Lambda \in P_+$ *and* $w \in W$ *with a reduced decomposition* $w = r_{i_t} \cdots r_{i_1}$. *Then*

(i) $a_k := (r_{i_{k-1}} \cdots r_{i_1}\Lambda \mid \alpha_{i_k}^\vee) \in \mathbb{Z}_{\geq 0}$ *for all* $k = 1, \dots, t$;

(ii) $v_w := f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$ *is a nonzero vector of the weight space* $V(\Lambda)_{w\Lambda}$, *which does not depend on the choice of a reduced decomposition of* w;

(iii) $(v_w, v_w) = 1$.

Proof. This is well-known. For example, for all the claims, except the independence of a reduced decomposition, one can consult [Kleshchev and Livesey 2022, Lemma 2.4.11]. One way to see the independence of a reduced decomposition is to first note that (iii) determines v_w uniquely up to a sign, and if $f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$ is such vector corresponding to another reduced decomposition, we cannot have $v_w = -f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$, since, by the Kang–Kashiwara–Oh categorification [Kang et al. 2013], the vectors $f_{i_t}^{(a_t)} \cdots f_{i_1}^{(a_1)} v_+$ and $f_{j_t}^{(b_t)} \cdots f_{j_1}^{(b_1)} v_+$ correspond to modules over a certain algebra when $V(\Lambda)$ is identified with a Grothendieck group. □

2.2. ***Quantized enveloping algebra.*** Let q be an indeterminate and consider the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials and the field $\mathbb{C}(q)$ of rational functions. For $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we have the following elements of $\mathbb{Z}[q, q^{-1}]$:

$$(2.3) \qquad q_i := q^{(\alpha_i|\alpha_i)/2}, \quad [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i^! := [1]_i[2]_i \cdots [n]_i.$$

Let $U_q(\mathfrak{g})$ be the *quantized enveloping algebra of type* $A_{2\ell}^{(2)}$, i.e., the associative unital $\mathbb{C}(q)$-algebra with generators $\{E_i, F_i, K_i^{\pm 1} \mid i \in I\}$ subject only to the quantum Serre relations

$$\begin{aligned} T_i E_j T_i^{-1} &= q^{(\alpha_i|\alpha_j)} E_j, \\ T_i F_j T_i^{-1} &= q^{-(\alpha_i|\alpha_j)} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{T_i - T_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{i,j}} (-1)^r E_i^{(r)} E_j E_i^{(1-a_{i,j}-r)} &= 0 \quad (i \neq j), \\ \sum_{r=0}^{1-a_{i,j}} (-1)^r F_i^{(r)} F_j F_i^{(1-a_{i,j}-r)} &= 0 \quad (i \neq j), \end{aligned}$$

where we have set $T_i := K_i^{(\alpha_i|\alpha_i)/2}$, $E_i^{(r)} := E_i/[r]_i^!$, $F_i^{(r)} := F_i/[r]_i^!$.

There is a $\mathbb{C}(q)$-linear anti-involution $\sigma_q : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ with

$$\sigma_q : E_i \mapsto q_i F_i T_i^{-1} = q_i^{-1} T_i^{-1} F_i, \quad F_i \mapsto q_i^{-1} T_i E_i = q_i E_i T_i, \quad T_i \mapsto T_i.$$

For $\Lambda \in P_+$, we denote by $V_q(\Lambda)$ the *irreducible integrable module* for $U_q(\mathfrak{g})$ of highest weight Λ. We fix a nonzero highest weight vector $v_{+,q} \in V_q(\Lambda)_\Lambda$, so $E_i v_{+,q} = 0$ and $T_i v_{+,q} = q^{(\alpha_i|\Lambda)} v_{+,q}$ for all $i \in I$. There is a unique symmetric bilinear form $(\cdot\,,\cdot)_q$ on $V_q(\Lambda)$ such that $(v_{+,q}, v_{+,q})_q = 1$ and

$$(xv, w)_q = (v, \sigma_q(x)w)_q \quad (x \in U_q(\mathfrak{g}),\ v, w \in V_q(\Lambda)); \tag{2.4}$$

see [Kashiwara et al. 1996, Appendix D]. We refer to $(\cdot\,,\cdot)_q$ as the (quantum) Shapovalov form.

2.3. ***Putting q to* 1.** Let V_q be a $U_q(\mathfrak{g})$-module which decomposes as a direct sum of finite-dimensional integral weight spaces $V_q = \bigoplus_{\mu \in P} V_{q,\mu}$. Suppose also that there exists a full rank $\mathbb{Z}[q, q^{-1}]$-sublattice $V_{q,\mu,\mathbb{Z}[q,q^{-1}]} \subset V_{q,\mu}$ for every μ such that $V_{q,\mathbb{Z}[q,q^{-1}]} := \bigoplus_{\mu \in P} V_{q,\mu,\mathbb{Z}[q,q^{-1}]}$ is stable under all E_i and F_i. Considering $\mathbb{C}$ as a $\mathbb{Z}[q, q^{-1}]$-module with q acting as 1, we change scalars to get the complex vector space $V_q|_{q=1} := \mathbb{C} \otimes_{\mathbb{Z}[q,q^{-1}]} V_{q,\mathbb{Z}[q,q^{-1}]}$ with linear operators $e_i := 1 \otimes E_i$, $f_i := 1 \otimes F_i$ and $h_i := 1 \otimes ((T_i - T_i^{-1})/(q_i - q_i^{-1}))$ for all i. These linear operators are easily checked to satisfy the Serre relations for $\mathfrak{g}$; see [Jantzen 1996, 5.13; Lusztig 1988, Theorem 4.12 and §4.14]. Thus $V_q|_{q=1}$ becomes a $\mathfrak{g}$-module. (This module depends on the choice of the $\mathbb{Z}[q, q^{-1}]$-sublattice).

Given a symmetric bilinear form $(\cdot\,,\cdot)_q$ on V_q which satisfies (2.4) and is $\mathbb{Z}[q, q^{-1}]$-valued on $V_{q,\mathbb{Z}[q,q^{-1}]}$, we obtain, extending scalars, a symmetric bilinear form $(\cdot\,,\cdot)_q|_{q=1}$ on $V_q|_{q=1}$ satisfying (2.1).

The above constructions can be applied to the irreducible modules $V_q(\Lambda)$ with highest weight $\Lambda \in P_+$ by considering the $\mathbb{Z}[q, q^{-1}]$-sublattice spanned by all vectors of the form $F_{i_1} \cdots F_{i_r} v_+$. Taking into account that the formal characters of $V_q(\Lambda)$ and $V(\Lambda)$ agree by [Lusztig 1988, Theorem 4.12 and §4.14], this shows that there is a unique isomorphism $V(\Lambda) \xrightarrow{\sim} V_q(\Lambda)|_{q=1}$ mapping v_+ onto $1 \otimes v_{+,q}$. Identifying $V(\Lambda)$ and $V_q(\Lambda)|_{q=1}$ under this isomorphism, the quantum Shapovalov form $(\cdot\,,\cdot)_q$ yields the usual Shapovalov form $(\cdot\,,\cdot)$, that is, $(\cdot\,,\cdot)_q|_{q=1} = (\cdot\,,\cdot)$. In particular:

Lemma 2.5. *Let $(\cdot\,,\cdot)_q$ be the Shapovalov form on $V_q(\Lambda)$ and $(\cdot\,,\cdot)$ be the Shapovalov form on $V(\Lambda)$. Then for all $i_1, \dots, i_n, j_1, \dots, j_n \in I$, we have*

$$(F_{i_1} \cdots F_{i_n} v_{+,q}, F_{j_1} \cdots F_{j_n} v_{+,q})_q \in \mathbb{Z}[q, q^{-1}]$$

and

$$(f_{i_1} \cdots f_{i_n} v_+, f_{j_1} \cdots f_{j_n} v_+) = (F_{i_1} \cdots F_{i_n} v_{+,q}, F_{j_1} \cdots F_{j_n} v_{+,q})_q|_{q=1}.$$

Another example of passing from V_q to $V_q|_{q=1}$ will be considered in Section 4.

3. Combinatorics

Recall that we have set $p = 2\ell + 1$.

3.1. ***Partitions, multipartitions, tableaux.*** We denote by $\mathscr{P}$ the set of all partitions and by $\mathscr{P}(n)$ the set of all partitions of $n \in \mathbb{Z}_{\geq 0}$. For $\lambda \in \mathscr{P}(n)$ we write $|\lambda| = n$. Collecting equal parts of $\lambda \in \mathscr{P}$, we can write it in the form

$$\lambda = (l_1^{m_1}, \dots, l_k^{m_k}) \quad \text{with } l_1 > \dots > l_k > 0 \text{ and } m_1, \dots, m_k \geq 1. \tag{3.1}$$

We then define

$$\|\lambda\|_q := \prod_{r \text{ with } p|l_r} \prod_{s=1}^{m_r} (1 - (-q^2)^s), \tag{3.2}$$

and

$$h(\lambda) := \sum_{r=1}^{k} m_r, \quad h_p(\lambda) := \sum_{r \text{ with } p|l_r} m_r. \tag{3.3}$$

In other words, $h(\lambda)$ is the number of (positive) parts of λ and $h_p(\lambda)$ is the number of (positive) parts of λ divisible by p. If $m_r > 1$ implies $p \mid l_r$ for all $1 \leq r \leq k$ then λ is called *p-strict*. Note that 0-strict also makes sense and means simply *strict*, i.e., all parts are distinct. We denote by $\mathscr{P}_p(n)$ the set of all p-strict partitions of n, and let $\mathscr{P}_p := \bigsqcup_{n \geq 0} \mathscr{P}_p(n)$. We use the similar notation $\mathscr{P}_0(n)$ and $\mathscr{P}_0$ for strict partitions.

For $\lambda \in \mathscr{P}_0$ we define its *parity*:

$$p_\lambda := \begin{cases} 1 & \text{if } \lambda \text{ has odd number of positive even parts,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let λ be a p-strict partition. As usual, we identify λ with its *Young diagram*

$$\lambda = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \le \lambda_r\}.$$

We refer to the element $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ as the *node* in row r and column s. We define a preorder '$\le$' on the nodes via $(r, s) \le (r', s')$ if and only if $s \le s'$. For partitions (equivalently Young diagrams) $\alpha \subseteq \lambda$, we define

$$q(\lambda/\alpha) := |\{r \in \mathbb{Z}_{>0} \mid \lambda\backslash\alpha \text{ has a node in column } r \text{ but not in column } r+1\}|. \tag{3.5}$$

We label the nodes with the elements of the set I as follows: the labeling follows the repeating pattern $0, 1, \dots, \ell-1, \ell, \ell-1, \dots, 1, 0$, starting from the first column and going to the right; see Example 3.8 below. If a node $\mathtt{A} \in \lambda$ is labeled with i, we say that $\mathtt{A}$ has *residue* i and write $\operatorname{Res} \mathtt{A} = i$. Recalling α_i's and Q_+ from Section 2.1, define the *residue content* of λ to be

$$\operatorname{cont}(\lambda) := \sum_{\mathtt{A} \in \lambda} \alpha_{\operatorname{Res} \mathtt{A}} \in Q_+.$$

We always write $\operatorname{cont}(\lambda) = \sum_{i \in I} c_i^\lambda \alpha_i$, and

$$c_{\neq 0}^\lambda := c_1^\lambda + \cdots + c_\ell^\lambda = |\lambda| - c_0^\lambda. \tag{3.6}$$

Following [Morris 1965; Leclerc and Thibon 1997], we can associate to every $\lambda \in \mathscr{P}_p$ its *$\bar{p}$-core*

$$\operatorname{core}(\lambda) \in \mathscr{P}_p$$

obtained from λ by removing certain nodes. Clearly, from the definition, the number of nodes removed to go from λ to $\operatorname{core}(\lambda)$ is divisible by p, so the *$\bar{p}$-weight* of λ

$$\operatorname{wt}(\lambda) := \frac{|\lambda| - |\operatorname{core}(\lambda)|}{p}$$

is a nonnegative integer. A partition $\rho \in \mathscr{P}_p$ is called a *$\bar{p}$-core* if $\operatorname{core}(\rho) = \rho$. By [Kleshchev and Livesey 2022, Lemma 3.1.39], we have:

Lemma 3.7. *A p-strict partition λ is a $\bar{p}$-core if and only if*

$$\operatorname{cont}(\lambda) = \Lambda_0 - w\Lambda_0 \quad \textit{for some } w \in W.$$

Example 3.8. Let $\ell = 2$, so $p = 5$. The partition $\lambda = (16, 11, 10, 10, 9, 4, 1)$ is 5-strict. The residues of the nodes are

0	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0
0	1	2	1	0	0	1	2	1	0	0					
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1							
0	1	2	1												
0															

The $\bar{5}$-core of λ is (1).

The partition $\lambda \in \mathscr{P}_p$ is determined by its $\bar{p}$-core $\mathrm{core}(\lambda)$ and $\bar{p}$-quotient

$$\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)}), \tag{3.9}$$

which is an *I-multipartition* of d, in other words, $\lambda^{(0)}, \dots, \lambda^{(\ell)}$ are partitions and $|\lambda^{(0)}| + |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = d$. We denote the set of all such multipartitions by $\mathscr{P}^I(d)$, and set

$$\mathscr{P}^I := \bigsqcup_{d \geq 0} \mathscr{P}^I(d).$$

We refer the reader to [Morris and Yaseen 1986, p. 27] and [Kleshchev and Livesey 2022, §2.3b] for details on this. For a $\bar{p}$-core partition ρ, we define

$$\mathscr{P}_p(\rho, d) := \{\lambda \in \mathscr{P}_p \mid \mathrm{core}(\lambda) = \rho \text{ and } \mathrm{wt}(\lambda) = d\}.$$

The map

$$\mathscr{P}_p(\rho, d) \to \mathscr{P}^I(d), \quad \lambda \mapsto \mathrm{cont}(\lambda), \tag{3.10}$$

is a bijection; see [Morris and Yaseen 1986, Theorem 2]. The condition $\lambda \in \mathscr{P}_p(\rho, d)$ is equivalent to $\mathrm{cont}(\lambda) = \mathrm{cont}(\rho) + d\delta$; see [Kleshchev and Livesey 2022, Lemma 2.3.9].

A multipartition $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^I$ is *strict* if its 0-th component $\lambda^{(0)}$ is a strict partition (and $\lambda^{(1)}, \dots, \lambda^{(\ell)}$ are arbitrary partitions). For $d \in \mathbb{Z}_{\geq 0}$, we denote by $\mathscr{P}^I_0(d)$ the set of all strict multipartitions of d. Note that $\mathrm{quot}(\lambda) \in \mathscr{P}^I_0(d)$ if and only if $\lambda \in \mathscr{P}_0(\rho, d)$, so the bijection (3.10) restricts to the bijection

$$\mathscr{P}_0(\rho, d) \to \mathscr{P}^I_0(d), \quad \lambda \mapsto \mathrm{cont}(\lambda). \tag{3.11}$$

We identify a multipartition $\underline{\lambda}$ with its Young diagram

$$\underline{\lambda} = \{(i, r, s) \in I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda^{(i)}_r\}.$$

We refer to the element $(i, r, s) \in I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ as the *node* in row r and column s of component i. For each i, we consider (the Young diagram of) the partition $\lambda^{(i)}$ as a subset $\lambda^{(i)} \subseteq \underline{\lambda}$ consisting of the nodes of $\underline{\lambda}$ in its i-th component.

Let $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$. A *composition* of d with n parts is a tuple $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_1, \dots, \mu_n \in \mathbb{Z}_{\geq 0}$ such that $\mu_1 + \cdots + \mu_n = d$. We denote by $\Lambda(n, d)$ the set of all compositions of d with n parts. We will need a special composition of d with d parts:

$$\omega_d := (1^d) = (1, \dots, 1). \tag{3.12}$$

Recall that J denotes $I \setminus \{\ell\}$. A *colored composition of d with n parts* is a pair $(\mu, \boldsymbol{j})$ where $\mu = (\mu_1, \dots, \mu_n)$ is a composition of d with n parts and $\boldsymbol{j} = (j_1, \dots, j_n) \in J^n$. We denote by $\Lambda^{\mathrm{col}}(n, d)$ the set of all colored compositions of d with n parts.

Let $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^I(d)$ and $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. A *colored tableau of shape $\underline{\lambda}$ and type* $(\mu, \boldsymbol{j})$ is a function $\mathsf{T} : \underline{\lambda} \to \mathbb{Z}_{>0}$ such that

(1) $\mathsf{T}(i, r, s) \leq \mathsf{T}(i, r, s+1)$ and $\mathsf{T}(i, r, s) \leq \mathsf{T}(i, r+1, s)$ whenever these make sense;

(2) for all $k = 1, \dots, n$, we have $|\mathsf{T}^{-1}(\{k\})| = \mu_k$ and $\mathsf{T}^{-1}(\{k\}) \subseteq \lambda^{(j_k)} \sqcup \lambda^{(j_k+1)}$;

(3) for all $k = 1, \dots, n$, no two nodes of $\mathsf{T}^{-1}(\{k\}) \cap \lambda^{(j_k)}$ are in the same column, and no two nodes of $\mathsf{T}^{-1}(\{k\}) \cap \lambda^{(j_k+1)}$ are in the same row.

Denote by $\mathsf{CT}(\underline{\lambda}; \mu, \boldsymbol{j})$ the set of all colored tableaux of shape $\underline{\lambda}$ and type $(\mu, \boldsymbol{j})$. For $\mathsf{T} \in \mathsf{CT}(\underline{\lambda}; \mu, \boldsymbol{j})$ and $1 \leq k \leq n$ we denote by $q_k(\mathsf{T})$ the number of positive integers r such that $\mathsf{T}^{-1}(k) \cap \lambda^{(0)}$ contains a node in column r but not in column $r+1$. We then set $q(\mathsf{T}) = q_1(\mathsf{T}) + \cdots + q_n(\mathsf{T})$ and define

$$K(\underline{\lambda}; \mu, \boldsymbol{j}) := \sum_{\mathsf{T} \in \mathsf{CT}(\underline{\lambda}; \mu, \boldsymbol{j})} 2^{q(\mathsf{T})}. \tag{3.13}$$

3.2. ***Addable and removable nodes.*** Let λ be a p-strict partition and $i \in I$. A node $\mathtt{A} \in \lambda$ is called *i-removable* (for λ) if one of the following holds:

(R1) $\operatorname{Res} \mathtt{A} = i$ and $\lambda_{\mathtt{A}} := \lambda \setminus \{\mathtt{A}\}$ is again a p-strict partition; such $\mathtt{A}$'s are also called *properly i-removable*.

(R2) The node $\mathtt{B}$ immediately to the right of $\mathtt{A}$ belongs to λ, $\operatorname{Res} \mathtt{A} = \operatorname{Res} \mathtt{B} = i$, and both $\lambda_{\mathtt{B}} = \lambda \setminus \{\mathtt{B}\}$ and $\lambda_{\mathtt{A},\mathtt{B}} := \lambda \setminus \{\mathtt{A}, \mathtt{B}\}$ are p-strict partitions.

A node $\mathtt{B} \notin \lambda$ is called *i-addable* (for λ) if one of the following holds:

(A1) $\operatorname{Res} \mathtt{B} = i$ and $\lambda^{\mathtt{B}} := \lambda \cup \{\mathtt{B}\}$ is again a p-strict partition; such $\mathtt{B}$'s are also called *properly i-addable*.

(A2) The node $\mathtt{A}$ immediately to the left of $\mathtt{B}$ does not belong to λ, $\operatorname{Res} \mathtt{A} = \operatorname{Res} \mathtt{B} = i$, and both $\lambda^{\mathtt{A}} = \lambda \cup \{\mathtt{A}\}$ and $\lambda^{\mathtt{A},\mathtt{B}} := \lambda \cup \{\mathtt{A}, \mathtt{B}\}$ are p-strict partitions.

We note that (R2) and (A2) above are only possible if $i = 0$. For $i \in I$, we denote by $\mathrm{Ad}_i(\lambda)$ (resp. $\mathrm{Re}_i(\lambda)$) the set of all i-removable (resp. i-addable) nodes for λ. We also denote by $\mathrm{PAd}_i(\lambda)$ (resp. $\mathrm{PRe}_i(\lambda)$) the set of all properly i-removable (resp. properly i-addable) nodes for λ.

Let $\lambda \in \mathscr{P}_p$ be written in the form (3.1). Suppose $\mathtt{A} \in \mathrm{PRe}_i(\lambda)$. Then there is $1 \le r \le k$ such that

$$\mathtt{A} = (m_1 + \cdots + m_r, l_r).$$

Recalling the preorder '$\le$' on the nodes defined above, we set

$$\eta_{\mathtt{A}}(\lambda) := \sharp\{\mathtt{C} \in \mathrm{Re}_i(\lambda) \mid \mathtt{C} > \mathtt{A}\} - \sharp\{\mathtt{C} \in \mathrm{Ad}_i(\lambda) \mid \mathtt{C} > \mathtt{A}\},$$

$$\zeta_{\mathtt{A}}(\lambda) := \begin{cases} (1 - (-q^2)^{m_r}) & \text{if } p \mid l_r, \\ 1 & \text{otherwise,} \end{cases}$$

$$d_{\mathtt{A}}(\lambda) := q_i^{\eta_{\mathtt{A}}(\lambda)} \zeta_{\mathtt{A}}(\lambda).$$

Note that

$$d_{\mathtt{A}}(\lambda)|_{q=1} = \begin{cases} 0 & \text{if } p \mid l_r \text{ and } m_r \text{ is even,} \\ 2 & \text{if } p \mid l_r \text{ and } m_r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.14}$$

Suppose $\mathtt{B} \in \mathrm{PAd}_i(\lambda)$. Then there is r such that $1 \le r \le k+1$ and

$$\mathtt{B} = (m_1 + \cdots + m_{r-1} + 1, l_r + 1),$$

where we interpret l_{k+1} as 0. We define

$$\eta^{\mathtt{B}}(\lambda) := \sharp\{\mathtt{C} \in \mathrm{Ad}_i(\lambda) \mid \mathtt{C} < \mathtt{B}\} - \sharp\{\mathtt{C} \in \mathrm{Re}_i(\lambda) \mid \mathtt{C} < \mathtt{B}\}, \tag{3.15}$$

$$\zeta^{\mathtt{B}}(\lambda) := \begin{cases} (1 - (-q^2)^{m_r}) & \text{if } r \le k \text{ and } p \mid l_r, \\ 1 & \text{otherwise,} \end{cases} \tag{3.16}$$

$$d^{\mathtt{B}}(\lambda) := q_i^{\eta^{\mathtt{B}}(\lambda)} \zeta^{\mathtt{B}}(\lambda). \tag{3.17}$$

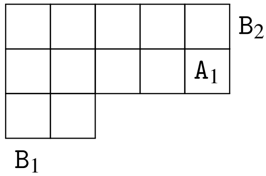
Note that

$$d^{\mathtt{B}}(\lambda)|_{q=1} = \begin{cases} 0 & \text{if } r \le k,\ p \mid l_r \text{ and } m_r \text{ is even,} \\ 2 & \text{if } r \le k,\ p \mid l_r \text{ and } m_r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.18}$$

Example 3.19. Let $\ell = 2$ so $p = 5$. The partition $\lambda = (5, 5, 2)$ is 5-strict, and the residues of its boxes are labeled on the diagram

0	1	2	1	0
0	1	2	1	0
0	1			

The only 0-removable node is marked as $\mathtt{A}_1$, and the 0-addable nodes are marked as $\mathtt{B}_1, \mathtt{B}_2$:



We have $d^{\mathtt{B}_1}(\lambda) = 1$ and $d^{\mathtt{B}_2}(\lambda) = (1-q^4)$. On the other hand for the partition $\mu = (5)$ and the node $\mathtt{B} = (1, 6)$, we have $d^{\mathtt{B}}(\mu) = (1+q^2)$.

3.3. *Symmetric functions.* We denote by Λ the algebra of symmetric functions in the variables $x_1, x_2, \ldots$ over $\mathbb{C}$ with the basis

$$\{\mathsf{s}_\lambda \mid \lambda \in \mathscr{P}\}$$

of *Schur functions* and the inner product $\langle \cdot, \cdot \rangle$ such that $\langle \mathsf{s}_\lambda, \mathsf{s}_\mu \rangle = \delta_{\lambda,\mu}$; see [Macdonald 1995]. We also have the *monomial symmetric functions* m_λ, the *elementary symmetric functions* $\mathsf{e}_r = \mathsf{s}_{(1^r)}$ and the *complete symmetric functions* $\mathsf{h}_r = \mathsf{s}_{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$. *Pieri's rules* [Macdonald 1995, (5.16), (5.17)] say that

$$\mathsf{s}_\lambda \mathsf{h}_r = \sum_\mu \mathsf{s}_\mu \quad \text{and} \quad \mathsf{s}_\lambda \mathsf{e}_r = \sum_\nu \mathsf{s}_\nu, \tag{3.20}$$

where the first sum is over all partitions μ obtained by adding r nodes to λ with no two nodes added in the same column, and the second sum is over all partitions ν obtained by adding r nodes to λ with no two nodes added in the same row.

Suppose that for $s_1, \ldots, s_t \in \mathbb{Z}_{\geq 0}$, we have that $\mathsf{f}_{s_u} = \mathsf{e}_{s_u}$ or h_{s_u}. Under the characteristic map [Macdonald 1995, I.7], the symmetric function $\mathsf{f}_{s_1} \cdots \mathsf{f}_{s_t}$ corresponds to an induced representation of the symmetric group $\mathfrak{S}_{s_1+\cdots+s_t}$ of dimension given by the multinomial coefficient $\binom{s_1+\cdots+s_t}{s_1\cdots s_t}$, while the symmetric function $\mathsf{s}_{(1)}^r$ with $r \in \mathbb{Z}_{\geq 0}$ corresponds to the regular representation of the symmetric group $\mathfrak{S}_r$. Hence,

$$(\mathsf{f}_{s_1} \cdots \mathsf{f}_{s_t}, \mathsf{s}_{(1)}^r) = \begin{cases} \binom{r}{s_1\cdots s_t} & \text{if } s_1 + \cdots + s_t = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Denoting by $\mathsf{p}_r \in \Lambda$ the r-th power sum symmetric function, let Ω be the (unital) subalgebra of Λ generated by $\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \ldots$. Then Ω has bases

$$\{\mathsf{P}_\lambda \mid \lambda \in \mathscr{P}_0\} \quad \text{and} \quad \{\mathsf{Q}_\lambda \mid \lambda \in \mathscr{P}_0\},$$

where the elements P_λ and Q_λ are *Schur's P- and Q-symmetric functions*; see [Stembridge 1989, §6]. We have $\mathsf{P}_\lambda = 2^{-h(\lambda)}\mathsf{Q}_\lambda$ for all $\lambda \in \mathscr{P}_0$. Let $[\cdot, \cdot]$ be an inner product on Ω such that $[\mathsf{P}_\lambda, \mathsf{Q}_\lambda] = \delta_{\lambda,\mu}$ for all $\lambda, \mu \in \mathscr{P}_0$; see [Stembridge 1989, §§5, 6].

We also have the symmetric functions

$$\mathsf{q}_r = 2\mathsf{P}_{(r)} \in \Omega \quad (r \in \mathbb{Z}_{>0})$$

(and $\mathsf{q}_0 := 1$); see [Stembridge 1989, (5.3), (6.6)]. We have the analogue of the Pieri's rule (which goes back to [Morris 1964] but can be most easily seen from [Stembridge 1989, Theorem 8.3]):

$$\mathsf{P}_\lambda \mathsf{q}_r = \sum_\mu 2^{q(\mu/\lambda)} \mathsf{P}_\mu, \tag{3.22}$$

where the sum is over all strict partitions μ obtained by adding r nodes to λ with no two nodes added in the same column and $q(\mu/\lambda)$ is as in (3.5).

By [Stembridge 1989, Proposition 5.6(b)], the inner product $[\mathsf{q}_{s_1} \dots \mathsf{q}_{s_t}, \mathsf{q}_1^r]$ is the coefficient of $\mathsf{m}_{(s_1,\dots,s_r)}$ in $\mathsf{q}_1^r = (2x_1+2x_2+\dots)^r$ (we may assume that $s_1 \geq \dots \geq s_r$ so $(s_1, \dots, s_r)$ is a partition), whence

$$[\mathsf{q}_{s_1} \dots \mathsf{q}_{s_t}, \mathsf{q}_1^r] = \begin{cases} 2^r \binom{r}{s_1 \cdots s_t} & \text{if } s_1 + \dots + s_t = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

We consider the algebra

$$\mathsf{Sym}^I := \Omega \otimes \Lambda^{(1)} \otimes \dots \otimes \Lambda^{(\ell)},$$

where each algebra $\Lambda^{(i)}$ is just a copy of Λ. This has bases

$$\{\pi_{\underline{\lambda}} := \mathsf{P}_{\lambda^{(0)}} \otimes \mathsf{s}_{\lambda^{(1)}} \otimes \dots \otimes \mathsf{s}_{\lambda^{(\ell)}} \mid \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_0^I\} \tag{3.24}$$

and

$$\{\kappa_{\underline{\lambda}} := \mathsf{Q}_{\lambda^{(0)}} \otimes \mathsf{s}_{\lambda^{(1)}} \otimes \dots \otimes \mathsf{s}_{\lambda^{(\ell)}} \mid \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_0^I\}, \tag{3.25}$$

which are dual with respect to the inner product $(\cdot\,,\cdot)_{\mathsf{Sym}}$ defined as

$$(f_0 \otimes f_1 \otimes \dots \otimes f_\ell, g_0 \otimes g_1 \otimes \dots \otimes g_\ell)_{\mathsf{Sym}} := [f_0, g_0]\langle f_1, g_1\rangle \cdots \langle f_\ell, g_\ell\rangle. \tag{3.26}$$

Let $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. Recalling (3.13), set

$$\Pi_{\mu,\boldsymbol{j}} := \sum_{\underline{\lambda} \in \mathscr{P}_0^I(d)} K(\underline{\lambda}; \mu, \boldsymbol{j}) \pi_{\underline{\lambda}}. \tag{3.27}$$

Example 3.28. Let $n = 1$ so $(\mu, \boldsymbol{j})$ is of the form $((d), j) \in \Lambda^{\mathrm{col}}(1, d)$. Then

$$\Pi_{(d),j} := \begin{cases} 1 \otimes \mathsf{e}_d \otimes 1^{\otimes \ell-1} + 2\sum_{k=1}^d \mathsf{P}_{(k)} \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1} & \text{if } j = 0, \\ \sum_{k=0}^d 1^{\otimes j} \otimes \mathsf{h}_k \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1-j} & \text{if } 1 \leq j < \ell. \end{cases} \tag{3.29}$$

Lemma 3.30. *Let $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. Suppose $n \geq 2$ and set $\nu := (\mu_1, \dots, \mu_{n-1})$, $\boldsymbol{k} := (j_1, \dots, j_{n-1})$, $m := \mu_n$, $j := j_n$. Then*

$$\Pi(\mu, \boldsymbol{j}) = \Pi(\nu, \boldsymbol{k})\Pi((m), j).$$

Proof. Suppose $j \neq 0$. Using (3.29), we see that $\Pi(\nu, \boldsymbol{k})\Pi((m), j)$ equals

$$\Big(\sum_{\underline{\alpha}\in\mathscr{P}_0^I(d-m)} K(\underline{\alpha};\nu,\boldsymbol{k})\pi_{\underline{\alpha}}\Big)\Big(\sum_{k=0}^{m} 1^{\otimes j}\otimes \mathsf{h}_k\otimes \mathsf{e}_{m-k}\otimes 1^{\otimes \ell-1-j}\Big)$$
$$= \sum_{\substack{\underline{\alpha}\in\mathscr{P}_0^I(d-m)\\ 0\le k\le m}} K(\underline{\alpha};\nu,\boldsymbol{k})\mathsf{P}_{\alpha^{(0)}}\otimes \mathsf{s}_{\alpha^{(1)}}\otimes\cdots\otimes \mathsf{s}_{\alpha^{(j)}}\mathsf{h}_k\otimes \mathsf{s}_{\alpha^{(j+1)}}\mathsf{e}_{m-k}\otimes\cdots\otimes \mathsf{s}_{\alpha^{(\ell)}}$$
$$= \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\mu,\boldsymbol{j})\mathsf{P}_{\lambda^{(0)}}\otimes \mathsf{s}_{\lambda^{(1)}}\otimes\cdots\otimes \mathsf{s}_{\lambda^{(j)}}\otimes \mathsf{s}_{\lambda^{(j+1)}}\otimes\cdots\otimes \mathsf{s}_{\lambda^{(\ell)}},$$

where the last equality follows from Pieri's rules (3.20).

Suppose now that $j = 0$. Using (3.29), we see that $\Pi(\nu, \boldsymbol{k})\Pi((m), 0)$ equals

$$\Big(\sum_{\underline{\alpha}\in\mathscr{P}_0^I(d-m)} K(\underline{\alpha};\nu,\boldsymbol{k})\pi_{\underline{\alpha}}\Big)\Big(\sum_{k=0}^{d} \mathsf{q}_k\otimes \mathsf{e}_{d-k}\otimes 1^{\otimes \ell-1}\Big)$$
$$= \sum_{\substack{\underline{\alpha}\in\mathscr{P}_0^I(d-m)\\ 0\le k\le m}} K(\underline{\alpha};\nu,\boldsymbol{k})\mathsf{P}_{\alpha^{(0)}}\mathsf{q}_k\otimes \mathsf{s}_{\alpha^{(1)}}\mathsf{e}_{d-k}\otimes \mathsf{s}_{\alpha^{(2)}}\otimes\cdots\otimes \mathsf{s}_{\alpha^{(\ell)}}$$
$$= \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\mu,\boldsymbol{j})\mathsf{P}_{\lambda^{(0)}}\otimes \mathsf{s}_{\lambda^{(1)}}\otimes\cdots\otimes \mathsf{s}_{\lambda^{(\ell)}},$$

where the last equality follows from Pieri's rules (3.22) and (3.20). □

Recalling (3.12), we let

$$\Pi_{\omega_d} := \sum_{\boldsymbol{j}\in J^d} \Pi_{\omega_d,\boldsymbol{j}}. \tag{3.31}$$

Corollary 3.32. *We have*

$$\Pi_{\omega_d} = \sum_{k_0+k_1+\cdots+k_\ell=d} 2^{k_1+\cdots+k_{\ell-1}}\binom{d}{k_0\,k_1\cdots k_\ell}\mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell}.$$

Proof. We apply induction on d. In the base case $d = 1$, by (3.29) and using $\mathsf{q}_1 = 2\mathsf{P}_{(1)}$ we have

$$\begin{aligned}\Pi_{\omega_1} &= \Pi_{(1),0}+\Pi_{(1),1}+\cdots+\Pi_{(1),\ell-1}\\ &= (1\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes\ell-1}+2\mathsf{P}_{(1)}\otimes 1^{\otimes \ell})+(1\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes\ell-1}+1\otimes 1\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes \ell-2})\\ &\qquad +\cdots+(1^{\otimes\ell-1}\otimes \mathsf{s}_{(1)}\otimes 1+1^{\otimes \ell-1}\otimes 1\otimes \mathsf{s}_{(1)})\\ &= \mathsf{q}_1\otimes 1^{\otimes\ell}+1^{\otimes \ell}\otimes \mathsf{s}_{(1)}+2\sum_{i=1}^{\ell-1}1^{\otimes i}\otimes \mathsf{s}_{(1)}\otimes 1^{\otimes \ell-i},\end{aligned}$$

as required.

For the inductive step, for $d > 1$, it follows from Lemma 3.30 that $\Pi_{\omega_d} = \Pi_{\omega_{d-1}}\Pi_{\omega_1}$. So, by the inductive assumption, we get

$$\begin{aligned}\Pi_{\omega_d} &= \Pi_{\omega_{d-1}}\Pi_{\omega_1}\\ &= \Bigg(\sum_{m_0+m_1+\cdots+m_\ell=d-1} 2^{m_1+\cdots+m_{\ell-1}}\binom{d-1}{m_0\, m_1\cdots m_\ell}\mathsf{q}_1^{m_0}\otimes \mathsf{s}_{(1)}^{m_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{m_\ell}\Bigg)\\ &\qquad\qquad\times\Bigg(\sum_{n_0+n_1+\cdots+n_\ell=1} 2^{n_1+\cdots+n_{\ell-1}}\mathsf{q}_1^{n_0}\otimes \mathsf{s}_{(1)}^{n_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{n_\ell}\Bigg)\\ &= \sum_{k_0+k_1+\cdots+k_\ell=d} 2^{k_1+\cdots+k_{\ell-1}}\binom{d}{k_0\, k_1\cdots k_\ell}\mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell}\end{aligned}$$

thanks to the identity $\binom{d}{k_0\, k_1\cdots k_\ell} = \sum_{r \text{ with } k_r>0}\binom{d-1}{k_0\cdots k_{r-1}\cdots k_r-1\, k_{r+1}\cdots k_\ell}$. ☐

3.4. ***Another description of*** $\boldsymbol{\Pi_{\mu,j}}$. Let $M_{n,I}$ denote the set of $n\times I$-matrices with nonnegative integer entries,

$$M_{n,I} = \{(a_{r,i})_{1\le r\le n,\, i\in I} \mid a_{r,i}\in\mathbb{Z}_{\ge 0}\}.$$

For $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$, we define the sets of matrices

$$\begin{aligned}M_{n,I}(\mu) &:= \Big\{(a_{r,i})\in M_{n,I}\;\Big|\;\sum_{i\in I} a_{r,i} = \mu_r \text{ for } r=1,\dots,n\Big\},\\ M_{n,I}(\boldsymbol{j}) &:= \{(a_{r,i})\in M_{n,I}\mid a_{r,i}=0 \text{ if } i\neq j_r,\, j_r+1 \text{ for } r=1,\dots,n\},\\ M(\mu,\boldsymbol{j}) &:= M_{n,I}(\mu)\cap M_{n,I}(\boldsymbol{j}).\end{aligned}$$

Let $A=(a_{r,i})\in M(\mu,\boldsymbol{j})$. For $1\le r\le n$ and $i\in I\setminus\{0\}$, we define

$$\psi_A(r,i) := \begin{cases}\mathsf{h}_{a_{r,i}} & \text{if } i=j_r,\\ \mathsf{e}_{a_{r,i}} & \text{if } i=j_r+1,\\ 1 & \text{otherwise.}\end{cases}$$

We now set

$$\psi_A^{(0)} := \mathsf{q}_{a_{1,0}}\cdots\mathsf{q}_{a_{n,0}},\qquad \psi_A^{(i)} := \psi_A(1,i)\cdots\psi_A(n,i)\quad (\text{for } i\in I\setminus\{0\}),$$

and

$$\psi_A := \psi_A^{(0)}\otimes\psi_A^{(1)}\otimes\cdots\otimes\psi_A^{(\ell)}\in \mathsf{Sym}^I.$$

Example 3.33. Suppose $n=1$ and so $(\mu,\boldsymbol{j})\in\Lambda^{\mathrm{col}}(1,d)$ is of the form $((d), j)$. The set $M((d), j)$ consists of all matrices of the form

$$\{A(j,k) := (0\ \cdots\ 0\ k\ \ d-k\ \ 0\ \cdots\ 0)\mid 0\le k\le d\}$$

with k in position j. Note that by definition we have

$$\begin{aligned} \psi_{A_{0,0}} &= \mathsf{q}_0 \otimes \mathsf{e}_d \otimes 1^{\otimes \ell-1} = 1 \otimes \mathsf{e}_d \otimes 1^{\otimes \ell-1}, \\ \psi_{A_{0,k}} &= \mathsf{q}_k \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1} = 2\mathsf{P}_{(k)} \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1} \quad (1 \le k \le d), \\ \psi_{A_{j,k}} &= 1^{\otimes j} \otimes \mathsf{h}_k \otimes \mathsf{e}_{d-k} \otimes 1^{\otimes \ell-1-j} \quad (1 \le j < \ell). \end{aligned}$$

In particular, comparing with (3.29), we deduce that $\Pi_{(d),j} = \sum_{A \in M((d),j)} \psi_A$.

Proposition 3.34. *Let* $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. *Then*

$$\Pi_{\mu,\boldsymbol{j}} = \sum_{A \in M(\mu,\boldsymbol{j})} \psi_A.$$

Proof. We apply induction on n. For the base $n = 1$, see Example 3.33. Suppose $n \ge 2$ and set

$$\nu := (\mu_1, \dots, \mu_{n-1}), \quad \boldsymbol{k} := (j_1, \dots, j_{n-1}), \quad m := \mu_n, \quad j := j_n.$$

Then $\Pi(\mu, \boldsymbol{j}) = \Pi(\nu, \boldsymbol{k})\Pi((m), j)$ by Lemma 3.30. By the inductive assumption,

$$\Pi(\nu, \boldsymbol{k}) = \sum_{B \in M(\nu,\boldsymbol{k})} \psi_B \quad \text{and} \quad \Pi((m), j) = \sum_{C \in M((m),j)} \psi_C,$$

so it suffices to observe that

$$\Big(\sum_{B \in M(\nu,\boldsymbol{k})} \psi_B\Big)\Big(\sum_{C \in M((m),j)} \psi_C\Big) = \sum_{A \in M(\mu,\boldsymbol{j})} \psi_A,$$

which comes from the definitions. □

3.5. *Computing the inner product* $(\Pi_{\mu,\boldsymbol{j}}, \Pi_{\omega_d})_{\mathsf{Sym}}$. Recall the inner product $(\cdot\,,\cdot)_{\mathsf{Sym}}$ from (3.26). Throughout this subsection, we fix $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. For $A = (a_{r,i}) \in M(\mu, \boldsymbol{j})$ and $i \in I$, we define

$$|a_{*,i}| := \sum_{r=1}^{n} a_{r,i}.$$

Then we have compositions

$$a_{*,i} := (a_{1,i}, \dots, a_{n,i}) \in \Lambda(n, |a_{*,i}|) \quad (i \in I)$$

and multinomial coefficients

$$\binom{|a_{*,i}|}{a_{*,i}} := \binom{|a_{*,i}|}{a_{1,i} \cdots a_{n,i}} = \frac{|a_{*,i}|!}{a_{1,i}! \cdots a_{n,i}!}.$$

Lemma 3.35. *Let* $A \in M(\mu, \boldsymbol{j})$, *and* $k_0, k_1, \dots, k_\ell \in \mathbb{Z}_{\ge 0}$. *Then*

$$(\psi_A, \mathsf{q}_1^{k_0} \otimes \mathsf{s}_{(1)}^{k_1} \otimes \cdots \otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}} = \begin{cases} 2^{|a_{*,0}|} \prod_{i \in I} \binom{|a_{*,i}|}{a_{*,i}} & \text{if } |a_{*,i}| = k_i \text{ for all } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$(\psi_A, \mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}} = (\psi_A^{(0)}\otimes\psi_A^{(1)}\otimes\cdots\otimes\psi_A^{(\ell)}, \mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}}$$
$$= [\psi_A^{(0)}, \mathsf{q}_1^{k_0}]\,(\psi_A^{(1)}, \mathsf{s}_{(1)}^{k_1})\cdots(\psi_A^{(\ell)}, \mathsf{s}_{(1)}^{k_\ell}).$$

Now, by (3.23), we have

$$[\psi_A^{(0)}, \mathsf{q}_1^{k_0}] = [\mathsf{q}_{a_{1,0}}\cdots \mathsf{q}_{a_{n,0}}, \mathsf{q}_1^{k_0}] = \begin{cases} 2^{k_0}\binom{k_0}{a_{1,0}\cdots a_{n,0}} & \text{if } k_0 = a_{1,0}+\cdots+a_{n,0},\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, by (3.21), we have, for $i = 1, \dots, \ell$,

$$(\psi_A^{(i)}, \mathsf{s}_{(1)}^{k_i})) = (\psi_A(1,i)\cdots\psi_A(n,i), \mathsf{s}_{(1)}^{k_i})) \begin{cases} \binom{k_i}{a_{1,i}\cdots a_{n,i}} & \text{if } k_i = a_{1,i}+\cdots+a_{n,i},\\ 0 & \text{otherwise.}\end{cases}$$

This implies the required equality. □

Recall the notation $|\mu, \boldsymbol{j}|_{\ell-1}$ from (1.5).

Theorem 3.36. *Let* $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. *Then*

$$(\Pi_{\mu,\boldsymbol{j}}, \Pi_{\omega_d})_{\mathsf{Sym}} = \binom{d}{\mu_1\cdots\mu_n} 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}.$$

Proof. By Proposition 3.34 and Corollary 3.32, we have that $(\Pi_{\mu,\boldsymbol{j}}, \Pi_{\omega_d})_{\mathsf{Sym}}$ equals

$$\sum_{A\in M(\mu,\boldsymbol{j})}\ \sum_{k_0+k_1+\cdots+k_\ell=d} 2^{k_1+\cdots+k_{\ell-1}}\binom{d}{k_0\,k_1\cdots k_\ell}(\psi_A, \mathsf{q}_1^{k_0}\otimes \mathsf{s}_{(1)}^{k_1}\otimes\cdots\otimes \mathsf{s}_{(1)}^{k_\ell})_{\mathsf{Sym}}.$$

By Lemma 3.35, this equals

$$\sum_{A\in M(\mu,\boldsymbol{j})} 2^{|a_{*,1}|+\cdots+|a_{*,\ell-1}|}\binom{d}{|a_{*,0}|\,|a_{*,1}|\cdots|a_{*,\ell}|}2^{|a_{*,0}|}\prod_{i\in I}\binom{|a_{*,i}|}{a_{*,i}}$$
$$= \sum_{A\in M(\mu,\boldsymbol{j})} 2^{|a_{*,0}|+|a_{*,1}|+\cdots+|a_{*,\ell-1}|}\frac{d!}{\prod_{i\in I}\prod_{r=1}^n a_{r,i}!}$$
$$= \sum_{A\in M(\mu,\boldsymbol{j})} 2^{d-|a_{*,\ell}|}\binom{d}{\mu_1\cdots\mu_n}\prod_{r=1}^n\binom{\mu_r}{a_{r,0}\cdots a_{r,\ell}},$$

and it remains to prove that

$$\sum_{A\in M(\mu,\boldsymbol{j})} 2^{d-|a_{*,\ell}|}\prod_{r=1}^n\binom{\mu_r}{a_{r,0}\cdots a_{r,\ell}} = 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\,3^{|\mu,\boldsymbol{j}|_{\ell-1}}. \tag{3.37}$$

Define

$$d_j := \sum_{\substack{1\le r\le n\\ j_r=j}} \mu_r \qquad (j\in J).$$

In particular, $d_{\ell-1} = |\mu, \boldsymbol{j}|_{\ell-1}$ and $d_0 + d_1 + \cdots + d_{\ell-1} = d$. Note that permuting the parts of $(\mu_1, \ldots, \mu_n)$ and $(j_1, \ldots, j_n)$ by the same permutation in $\mathfrak{S}_n$ does not change the left-hand side of (3.37), so we may assume without loss of generality that $\boldsymbol{j} = (0^{n_0}, 1^{n_1}, \ldots, (\ell-1)^{n_{\ell-1}})$ with $n_0 + n_1 + \cdots + n_{\ell-1} = n$ and

$$\mu = (\lambda_1^{(0)}, \ldots, \lambda_{n_0}^{(0)}, \ldots, \lambda_1^{(\ell-1)}, \ldots, \lambda_{n_{\ell-1}}^{(\ell-1)})$$

with $(\lambda_1^{(j)}, \ldots, \lambda_{n_j}^{(j)}) \in \Lambda(n_j, d_j)$ for all $j \in J$. Then, the matrices $A \in M(\mu, \boldsymbol{j})$ look like

$$A = \begin{pmatrix} B^{(0)} \\ \vdots \\ B^{(\ell-1)} \end{pmatrix},$$

where, for each j, the matrix $B^{(j)} = (b_{r,i}^{(j)})_{1 \le r \le n_j,\, i \in I}$ is an arbitrary matrix with non-negative integer values such that $b_{r,i}^{(j)} = 0$ unless $i \in \{j, j+1\}$ and $b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}$ for all $r = 1, \ldots, n_j$. So, the left-hand side of (3.37) equals XY where

$$X := \prod_{j=0}^{\ell-2} \prod_{r=1}^{n_j} \sum_{b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}} 2^{\lambda_r^{(j)}} \binom{\lambda_r^{(j)}}{b_{r,j}^{(j)}\ b_{r,j+1}^{(j)}}$$

and

$$Y := \prod_{r=1}^{n_{\ell-1}} \sum_{b_{r,\ell-1}^{(\ell-1)} + b_{r,\ell}^{(\ell-1)} = \lambda_r^{(\ell-1)}} 2^{b_{r,\ell-1}^{(\ell-1)}} \binom{\lambda_r^{(\ell-1)}}{b_{r,\ell-1}^{(\ell-1)}\ b_{r,\ell}^{(\ell-1)}}.$$

Now, using the formula $\sum_{a+b=c} \binom{c}{a\ b} = 2^c$, we get

$$X = 2^{d_0 + \cdots + d_{\ell-2}} \prod_{j=0}^{\ell-2} \prod_{r=1}^{n_j} \sum_{b_{r,j}^{(j)} + b_{r,j+1}^{(j)} = \lambda_r^{(j)}} 2^{\lambda_r^{(j)}} = 4^{d_0 + \cdots + d_{\ell-2}} = 4^{d - d_{\ell-1}} = 4^{d - |\mu, \boldsymbol{j}|_{\ell-1}},$$

and, using the formula $\sum_{a+b=c} 2^a \binom{c}{a\ b} = 3^c$, we get

$$Y = \prod_{r=1}^{n_{\ell-1}} 3^{\lambda_r^{(\ell-1)}} = 3^{d_{\ell-1}} = 3^{|\mu, \boldsymbol{j}|_{\ell-1}},$$

completing the proof. □

4. Fock space

4.1. ***Fock spaces $\mathscr{F}_q$ and $\mathscr{F}$.*** The (*q-deformed*) *level-*1 *Fock space* $\mathscr{F}_q$, as defined in [Kashiwara et al. 1996] (see also [Leclerc and Thibon 1997]), is the $\mathbb{Q}(q)$-vector space with basis $\{u_\lambda \mid \lambda \in \mathscr{P}_p\}$ labeled by the p-strict partitions

$$\mathscr{F}_q := \bigoplus_{\lambda \in \mathscr{P}_p} \mathbb{Q}(q) \cdot u_\lambda.$$

There is a structure of a $U_q(\mathfrak{g})$-module on $\mathscr{F}_q$ such that

$$E_i u_\lambda = \sum_{\mathtt{A} \in \mathrm{PRe}_i(\lambda)} d_{\mathtt{A}}(\lambda) u_{\lambda_{\mathtt{A}}}, \tag{4.1}$$

$$F_i u_\lambda = \sum_{\mathtt{B} \in \mathrm{PAd}_i(\lambda)} d^{\mathtt{B}}(\lambda) u_{\lambda^{\mathtt{B}}}, \tag{4.2}$$

$$T_i u_\lambda = q^{(\alpha_i \mid \Lambda_0 - \mathrm{cont}(\lambda))} u_\lambda. \tag{4.3}$$

Example 4.4. In the set up of Example 3.19 we have

$$F_0 u_{(5,5,2)} = (1 - q^4) u_{(6,5,2)} + u_{(5,5,2,1)}.$$

As established in [Kashiwara et al. 1996, Appendix D], there is a bilinear form $(\cdot\,,\cdot)$ on $\mathscr{F}_q$ which satisfies

$$(u_\lambda, u_\mu)_q = \delta_{\lambda,\mu} \|\lambda\|_q \tag{4.5}$$

and

$$(xv, w)_q = (v, \sigma_q(x) w)_q$$

for all $x \in U_q(\mathfrak{g})$ and $v, w \in \mathscr{F}_q$. The following well-known result allows us to identify $V_q(\Lambda_0)$ with the submodule of $\mathscr{F}_q$ generated by $u_\varnothing$, where $\varnothing$ stands for the partition (0) of 0; cf. [Kleshchev and Livesey 2022, Lemma 2.4.20].

Lemma 4.6. *There exists a unique isomorphism of $U_q(\mathfrak{g})$-modules $V_q(\Lambda_0) \xrightarrow{\sim} U_q(\mathfrak{g}) \cdot u_\varnothing$ mapping $v_{+,q}$ onto $u_\varnothing$. Moreover, identifying $V_q(\Lambda_0)$ with the submodule $U_q(\mathfrak{g}) \cdot u_\varnothing \subseteq \mathscr{F}_q$ via this isomorphism, the Shapovalov form $(\cdot\,,\cdot)_q$ on $V_q(\Lambda_0)$ is the restriction of the form $(\cdot\,,\cdot)_q$ on $\mathscr{F}_q$ to $V_q(\Lambda_0)$.*

We now apply the construction of Section 2.3 to go from the $U_q(\mathfrak{g})$-module $\mathscr{F}_q$ to the $\mathfrak{g}$-module $\mathscr{F}_q|_{q=1} = \mathbb{C} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{F}_{q,\mathbb{Z}[q,q^{-1}]}$ with the lattice $\mathscr{F}_{q,\mathbb{Z}[q,q^{-1}]} := \bigoplus_{\lambda \in \mathscr{P}_p} \mathbb{Z}[q, q^{-1}] \cdot u_\lambda$. We will denote $1 \otimes u_\lambda \in \mathbb{C} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{F}_{q,\mathbb{Z}[q,q^{-1}]}$ again by u_λ. So we have a $\mathfrak{g}$-module

$$\mathscr{F}_q|_{q=1} = \bigoplus_{\lambda \in \mathscr{P}_p} \mathbb{C} \cdot u_\lambda$$

with the action

$$e_i u_\lambda = \sum_{\mathtt{A} \in \mathrm{PRe}_i(\lambda)} (d_{\mathtt{A}}(\lambda)|_{q=1}) u_{\lambda_{\mathtt{A}}}, \quad f_i u_\lambda = \sum_{\mathtt{B} \in \mathrm{PAd}_i(\lambda)} (d^{\mathtt{B}}(\lambda)|_{q=1}) u_{\lambda^{\mathtt{B}}}, \tag{4.7}$$

and the form $(\cdot,\cdot) := (\cdot,\cdot)_q|_{q=1}$ which satisfies

$$(u_\lambda, u_\mu) = \delta_{\lambda,\mu}\|\lambda\|_q|_{q=1}$$

and $(xv, w) = (v, \sigma(x)w)$ for all $x \in \mathfrak{g}$ and $v, w \in \mathscr{F}_q|_{q=1}$.

Recalling (3.14) and (3.18), it is easy to see that $\mathscr{R} := \mathrm{span}_{\mathbb{C}}(u_\lambda \mid \lambda \in \mathscr{P}_p \setminus \mathscr{P}_0)$ is a $\mathfrak{g}$-submodule of $\mathscr{F}_q|_{q=1}$. Consider the *reduced Fock space*

$$\mathscr{F} := (\mathscr{F}_q|_{q=1})/\mathscr{R}.$$

Denoting $u_\lambda + \mathscr{R} \in (\mathscr{F}_q|_{q=1})/\mathscr{R}$ by u_λ again, we have that

$$\mathscr{F} = \bigoplus_{\lambda\in\mathscr{P}_0} \mathbb{C}\cdot u_\lambda$$

with the action of the Chevalley generators f_i given by

$$f_i u_\lambda = \sum_{\mathtt{B}\in A_i(\lambda)} c(\lambda, \mathtt{B}) u_{\lambda^{\mathtt{B}}}, \tag{4.8}$$

where

$$A_i(\lambda) := \{\mathtt{B} \in \mathrm{PAd}_i(\lambda) \mid \lambda^{\mathtt{B}} \in \mathscr{P}_0\}$$

and, recalling (3.3),

$$c(\lambda, \mathtt{B}) := \begin{cases} 2 & \text{if } h_p(\lambda^{\mathtt{B}}) = h_p(\lambda) - 1, \\ 1 & \text{otherwise.} \end{cases} \tag{4.9}$$

(We are not going to need the action of the Chevalley generators e_i.) Moreover, $\mathscr{F}$ inherits the form $(\cdot,\cdot)$ which satisfies

$$(u_\lambda, u_\mu) = \delta_{\lambda,\mu} 2^{h_p(\lambda)}, \tag{4.10}$$

and $(xv, w) = (v, \sigma(x)w)$ for all $x \in \mathfrak{g}$ and $v, w \in \mathscr{F}$. We now have from Lemmas 4.6 and 2.5:

Lemma 4.11. *There is a unique isomorphism of $\mathfrak{g}$-modules from $V(\Lambda_0)$ to the submodule $U(\mathfrak{g})\cdot u_\varnothing \subseteq \mathscr{F}$ generated by $u_\varnothing$, mapping v_+ onto $u_\varnothing$. Moreover, identifying $V(\Lambda_0)$ with $U(\mathfrak{g})\cdot u_\varnothing \subseteq \mathscr{F}$ via this isomorphism, the Shapovalov form $(\cdot,\cdot)$ on $V(\Lambda_0)$ is the restriction of the form $(\cdot,\cdot)$ on $\mathscr{F}$ to $V(\Lambda_0)$.*

Let ρ be a $\bar{p}$-core. By definition, we have $h_p(\rho) = 0$ and $\rho \in \mathscr{P}_0$. Note that by Lemma 3.7, there is $w \in W$ such that $\mathrm{cont}(\rho) = \Lambda_0 - w\Lambda_0$. We have the element $v_w \in V(\Lambda_0)$ defined in Lemma 2.2, and the element $u_\rho \in \mathscr{F}$. The following lemma shows that these agree:

Lemma 4.12. *Let $\iota : V(\Lambda_0) \xrightarrow{\sim} U(\mathfrak{g})\cdot u_\varnothing$, $v_+ \mapsto u_\varnothing$, be the isomorphism of Lemma 4.11. If ρ is a $\bar{p}$-core and $w \in W$ is such that* $\mathrm{cont}(\rho) = \Lambda_0 - w\Lambda_0$ *then $\iota(v_w) = u_\rho$.*

Proof. By Lemma 2.2, for certain $a_1, \dots, a_l$, we have

$$\iota(v_w) = \iota(F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} v_+) = F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} \iota(v_+) = F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing.$$

It follows from Lemma 3.7 that $\mathscr{F}_{w\Lambda_0} = \mathscr{F}_{\Lambda_0 - \mathrm{cont}(\rho)}$ is 1-dimensional and hence spanned by u_ρ. It follows from the formulas (4.8) and (4.9) that $F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing = k u_\rho$ for $k \in \mathbb{Z}_{>0}$. Now, $(u_\rho, u_\rho) = 2^{h_p(\rho)} = 1$. On the other hand,

$$(F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing, F_{i_l}^{(a_l)} \cdots F_{i_1}^{(a_1)} u_\varnothing) = (\iota(v_w), \iota(v_w)) = (v_w, v_w) = 1$$

by Lemmas 4.11 and 2.2(iii). So $k = 1$. □

4.2. ***The elements χ_λ.*** In this subsection we introduce a new basis of $\mathscr{F}$. Recalling (3.3), (3.4) and (3.6), we consider the following rescalings of the basis vectors u_λ:

$$\chi_\lambda := 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} u_\lambda \quad (\lambda \in \mathscr{P}_0). \tag{4.13}$$

These elements correspond to the irreducible supercharacters of the double covers of symmetric groups, see [Fayers et al. 2024, §5], and the formula (4.13) was communicated to us by M. Fayers.

Set

$$a(\lambda, \mathtt{B}) := \begin{cases} 2 & \text{if } p_\lambda = 1 \text{ and } p_{\lambda^\mathtt{B}} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.14. *Let $i \in I$ and $\lambda \in \mathscr{P}_0$. Then $f_i \chi_\lambda = \sum_{\mathtt{B} \in A_i(\lambda)} a(\lambda, \mathtt{B}) u_{\lambda^\mathtt{B}}$.*

Proof. We have

$$\begin{aligned} f_i \chi_\lambda &= 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} f_i u_\lambda \\ &= 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} \sum_{\mathtt{B} \in A_i(\lambda)} c(\lambda, \mathtt{B}) u_{\lambda^\mathtt{B}} \\ &= 2^{(p_\lambda - h_p(\lambda) - c^\lambda_{\neq 0})/2} \sum_{\mathtt{B} \in A_i(\lambda)} c(\lambda, \mathtt{B}) 2^{-(p_{\lambda^\mathtt{B}} - h_p(\lambda^\mathtt{B}) - c^{\lambda^\mathtt{B}}_{\neq 0})/2} \chi_{\lambda^\mathtt{B}}, \end{aligned}$$

so we need to prove that

$$a(\lambda, \mathtt{B}) = 2^{(p_\lambda - p_{\lambda^\mathtt{B}} - h_p(\lambda) + h_p(\lambda^\mathtt{B}) - c^\lambda_{\neq 0} + c^{\lambda^\mathtt{B}}_{\neq 0})/2} c(\lambda, \mathtt{B}). \tag{4.15}$$

If $i \neq 0$ then either $p_\lambda = 1$ and $p_{\lambda^\mathtt{B}} = 0$, or $p_\lambda = 0$ and $p_{\lambda^\mathtt{B}} = 1$. Moreover, $c^{\lambda^\mathtt{B}}_{\neq 0} = c^\lambda_{\neq 0} + 1$. Hence

$$\begin{aligned} 2^{(p_\lambda - p_{\lambda^\mathtt{B}} - h_p(\lambda) + h_p(\lambda^\mathtt{B}) - c^\lambda_{\neq 0} + c^{\lambda^\mathtt{B}}_{\neq 0})/2} c(\lambda, \mathtt{B}) &= 2^{(p_\lambda - p_{\lambda^\mathtt{B}} + 1)/2} \\ &= \begin{cases} 2 & \text{if } p_\lambda = 1 \text{ and } p_{\lambda^\mathtt{B}} = 0, \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

which is $a(\lambda, B)$ as required.

On the other hand, if $i = 0$ then $c^{\lambda}_{\neq 0} = c^{\lambda^{\mathrm{B}}}_{\neq 0}$. Let λ be written in the form (3.1), and consider the following three cases:

(1) $B = (m_1 + \cdots + m_k + 1, 1)$. In this case, we have $c(\lambda, B) = 1$, $p_\lambda = p_{\lambda^{\mathrm{B}}}$ and $h_p(\lambda) = h_p(\lambda^{\mathrm{B}})$, which immediately gives (4.15).

(2) $B = (m_1 + \cdots + m_{r-1} + 1, 1)$ for some $1 \le r \le k$ and $p \mid l_r$. In this case, we have $h_p(\lambda^{\mathrm{B}}) = h_p(\lambda) - 1$, $c(\lambda, B) = 2$ and $p_{\lambda^{\mathrm{B}}} \neq p_\lambda$. If $p_{\lambda^{\mathrm{B}}} = 1$ and $p_\lambda = 0$ then both sides of (4.15) equal 1. If $p_{\lambda^{\mathrm{B}}} = 0$ and $p_\lambda = 1$ then both sides of (4.15) equal 2.

(3) $B = (m_1 + \cdots + m_{r-1} + 1, 1)$ for some $1 \le r \le k$ and $p \nmid l_r$. In this case, we have $h_p(\lambda^{\mathrm{B}}) = h_p(\lambda) + 1$, $c(\lambda, B) = 1$ and $p_{\lambda^{\mathrm{B}}} \neq p_\lambda$. If $p_{\lambda^{\mathrm{B}}} = 1$ and $p_\lambda = 0$ then both sides of (4.15) equal 1. If $p_{\lambda^{\mathrm{B}}} = 0$ and $p_\lambda = 1$ then both sides of (4.15) equal 2. □

5. Shapovalov form for RoCK weights

Suppose that $\theta \in Q_+$ satisfies $V(\Lambda_0)_{\Lambda_0-\theta} \neq 0$. Then $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some w in the Weyl group W, and unique $d \in \mathbb{Z}_{\ge 0}$. We say that θ is *RoCK* if $(\theta \mid \alpha_0^\vee) \ge 2d$ and $(\theta \mid \alpha_i^\vee) \ge d - 1$ for $i = 1, \dots, \ell$. This is equivalent to the cyclotomic quiver Hecke superalgebra $R^{\Lambda_0}_\theta$ being a *RoCK block*, as defined in [Kleshchev and Livesey 2022, Section 4.1].

Throughout Section 5, we fix a RoCK weight $\theta \in Q_+$, so that $\theta = \Lambda_0 - w\Lambda_0 + d\delta$ for some $w \in W$ and $d \in \mathbb{Z}_{\ge 0}$, and $(\theta \mid \alpha_0^\vee) \ge 2d$, $(\theta \mid \alpha_i^\vee) \ge d - 1$ for $i = 1, \dots, \ell$.

We have the element $v_w \in V(\Lambda_0)_{w\Lambda_0}$ defined in Lemma 2.2.

5.1. ***Computation of $f(\mu, j)u_\rho$.*** For each $m \in \mathbb{Z}_{\ge 0}$, $j \in J$, $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$, recall the divided power monomials $f(m, j)$ and $f(\mu, \boldsymbol{j})$ defined in (1.2) and (1.3). We also have a sum $f(\omega_d)$ of monomials defined in (1.4).

Let $\mu \in \mathscr{P}_0(\rho, c)$ with $c \le d$. Recall the notion of a p-quotient $\mathrm{quot}(\mu) = (\mu^{(0)}, \dots, \mu^{(\ell)})$ of μ from (3.9) and the notation (3.5). Recall that $\mathrm{quot}(\mu) \in \mathscr{P}_0^I(c)$ since μ is strict. The following result follows immediately from [Fayers et al. 2024, Proposition 6.6 and (5.1)] and Lemma 4.14.

Lemma 5.1. *Let $j \in J$ and $c, k \in \mathbb{Z}_{\ge 0}$ satisfy $c + k \le d$. For $\alpha \in \mathscr{P}_0(\rho, c)$, in the reduced Fock space $\mathscr{F}$, we have*

$$f(k, j)\chi_\alpha = \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)}) + (k(2\ell-1) + h(\alpha^{(0)}) - h(\lambda^{(0)}) + p_\alpha - p_\lambda)/2} \chi_\lambda,$$

where the sum is over all $\lambda \in \mathscr{P}_0(\rho, c+k)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ *is obtained from* $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ *by adding k nodes to the components $\alpha^{(j)}$ and $\alpha^{(j+1)}$, with no two nodes added in the same column of $\alpha^{(j)}$ or in the same row of $\alpha^{(j+1)}$.*

Corollary 5.2. *Let $j \in J$ and $c, k \in \mathbb{Z}_{\geq 0}$ satisfy $c+k \leq d$. For $\alpha \in \mathscr{P}_0(\rho, c)$, in the reduced Fock space $\mathscr{F}$, we have*

$$f(k, j)u_\alpha = \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})-h(\lambda^{(0)})} u_\lambda,$$

where the sum is over all $\lambda \in \mathscr{P}_0(\rho, c+k)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ *is obtained from* $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ *by adding k nodes to the components $\alpha^{(j)}$ and $\alpha^{(j+1)}$, with no two nodes added in the same column of $\alpha^{(j)}$ or in the same row of $\alpha^{(j+1)}$.*

Proof. Note that $h_p(\alpha) = h(\alpha^{(0)})$. Moreover, for λ's appearing in the sum, we have $c^\lambda_{\neq 0} - c^\alpha_{\neq 0} = k(2\ell-1)$ and $h_p(\lambda) = h(\lambda^{(0)})$. So we have by (4.13) and Lemma 5.1,

$$\begin{aligned}
f(k, j)u_\alpha &= 2^{(-p_\alpha+h_p(\alpha)+c^\alpha_{\neq 0})/2} f(k, j)\chi_\alpha \\
&= 2^{(-p_\alpha+h_p(\alpha)+c^\alpha_{\neq 0})/2} \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)})+(k(2\ell-1)+h(\alpha^{(0)})-h(\lambda^{(0)})+p_\alpha-p_\lambda)/2} \chi_\lambda \\
&= 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})+(k(2\ell-1)-h(\lambda^{(0)})-p_\lambda+c^\alpha_{\neq 0})/2} \sum_\lambda 2^{(p_\lambda-h_p(\lambda)-c^\lambda_{\neq 0})/2} u_\lambda \\
&= \sum_\lambda 2^{q(\lambda^{(0)}/\alpha^{(0)})+h(\alpha^{(0)})-h(\lambda^{(0)})} u_\lambda,
\end{aligned}$$

as required. □

Lemma 5.3. *Let $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(n, d)$. Then*

$$f(\mu, \boldsymbol{j})u_\rho = \sum_{\lambda \in \mathscr{P}_0(\rho, d)} K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) 2^{-h(\lambda^{(0)})} u_\lambda.$$

Proof. We apply induction on n. For the induction base case $n = 1$ we apply Corollary 5.2 to see that

$$f(\mu_1, j_1)u_\rho = \sum_\lambda 2^{q(\lambda^{(0)}/\rho^{(0)})+h(\rho^{(0)})-h(\lambda^{(0)})} u_\lambda = \sum_\lambda 2^{q(\lambda^{(0)}/\rho^{(0)})} 2^{-h(\lambda^{(0)})} u_\lambda,$$

where the sums are over all $\lambda \in \mathscr{P}_0(\rho, d)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\mathrm{quot}(\rho) = (\varnothing, \dots, \varnothing)$ by adding k nodes to the components $\rho^{(j)} = \varnothing$ and $\rho^{(j+1)} = \varnothing$, with no two nodes added in the same column of $\rho^{(j)}$ or in the same row of $\rho^{(j+1)}$, and

$$q(\lambda^{(0)}/\rho^{(0)}) = |\{r \in \mathbb{Z}_{>0} \mid \lambda^{(0)} \text{ contains a node in column } r \text{ but not in column } r+1\}|.$$

It follows from the definitions that the λ's appearing in the latter sum are exactly the λ's with $K(\mathrm{quot}(\lambda); (\mu_1), j_1) \neq 0$, and for those λ we have $K(\mathrm{quot}(\lambda); (\mu_1), j_1) = 2^{q(\lambda^{(0)}/\rho^{(0)})}$. This establishes the induction base.

For the inductive step, suppose $n > 1$. Let

$$\nu := (\mu_1, \dots, \mu_{n-1}), \quad \boldsymbol{k} := (j_1, \dots, j_{n-1}), \quad c = \mu_1 + \dots + \mu_{n-1}.$$

In particular,

$$f(\mu, \boldsymbol{j}) = f(\mu_n, j_n) f(\nu, \boldsymbol{k}).$$

By the inductive assumption, we have

$$\begin{aligned} f(\mu, \boldsymbol{j}) u_\rho &= f(\mu_n, j_n) f(\nu, \boldsymbol{k}) u_\rho \\ &= \sum_{\alpha \in \mathscr{P}_0(\rho, c)} K(\mathrm{quot}(\alpha); \nu, \boldsymbol{k}) 2^{-h(\alpha^{(0)})} f(\mu_n, j_n) u_\alpha \\ &= \sum_{\alpha \in \mathscr{P}_0(\rho, c)} K(\mathrm{quot}(\alpha); \nu, \boldsymbol{k}) 2^{-h(\alpha^{(0)})} \sum_{\lambda} 2^{q(\lambda^{(0)}/\alpha^{(0)}) + h(\alpha^{(0)}) - h(\lambda^{(0)})} u_\lambda, \end{aligned}$$

where the second sum is over all $\lambda \in \mathscr{P}_0(\rho, d)$ such that $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ is obtained from $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ by adding μ_n nodes to the components $\alpha^{(j_n)}$ and $\alpha^{(j_n+1)}$, with no two nodes added in the same column of $\alpha^{(j_n)}$ or in the same row of $\alpha^{(j_n+1)}$. It remains to note that λ's appearing in the expression above are exactly those with $K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) \neq 0$, and for such λ we have

$$K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) = \sum_{\alpha} 2^{q(\lambda^{(0)}/\alpha^{(0)})} K(\mathrm{quot}(\alpha); \nu, \boldsymbol{k}),$$

where the sum is over all $\alpha \in \mathscr{P}_0(\rho, c)$ such that $\mathrm{quot}(\alpha) = (\alpha^{(0)}, \dots, \alpha^{(\ell)})$ is obtained from $\mathrm{quot}(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ by removing μ_n nodes from the components $\lambda^{(j_n)}$ and $\lambda^{(j_n+1)}$, with no two nodes removed in the same column of $\lambda^{(j_n)}$ or in the same row of $\lambda^{(j_n+1)}$. □

Recall the definition of $\Pi_{\mu,\boldsymbol{j}}$ from (3.27).

Corollary 5.4. *Let* $(\mu, \boldsymbol{j}) \in \Lambda^{\mathrm{col}}(m, d)$ *and* $(\nu, \boldsymbol{i}) \in \Lambda^{\mathrm{col}}(n, d)$. *Then*

$$(f(\mu, \boldsymbol{j}) u_\rho, f(\nu, \boldsymbol{i}) u_\rho) = (\Pi_{\mu,\boldsymbol{j}}, \Pi_{\nu,\boldsymbol{i}})_{\mathsf{Sym}}.$$

Proof. For $\lambda \in \mathscr{P}_0(\rho, d)$, we have $h_p(\lambda) = h(\lambda^{(0)})$. So, by Lemma 5.3 and (4.10), taking into account the bijection (3.11), we have

$$\begin{aligned} &(f(\mu, \boldsymbol{j}) u_\rho, f(\nu, \boldsymbol{i}) u_\rho) \\ &\qquad = \sum_{\lambda \in \mathscr{P}_0(\rho, d)} K(\mathrm{quot}(\lambda); \mu, \boldsymbol{j}) K(\mathrm{quot}(\lambda); \nu, \boldsymbol{i}) 2^{-2h(\lambda^{(0)})} (u_\lambda, u_\lambda) \\ &\qquad = \sum_{\underline{\lambda} \in \mathscr{P}_0^I(d)} K(\underline{\lambda}; \mu, \boldsymbol{j}) K(\underline{\lambda}; \nu, \boldsymbol{i}) 2^{-h(\lambda^{(0)})}. \end{aligned}$$

Since the bases $\{\pi_{\underline{\lambda}} \mid \underline{\lambda} \in \mathscr{P}_0^I\}$ from (3.24) and $\{\kappa_{\underline{\lambda}} \mid \underline{\lambda} \in \mathscr{P}_0^I\}$ from (3.25) are dual to each other with respect to the inner product $(\,\cdot\,,\,\cdot\,)_{\mathsf{Sym}}$, we have

$$\begin{aligned}\sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} & K(\underline{\lambda};\mu,\boldsymbol{j})K(\underline{\lambda};\nu,\boldsymbol{i})2^{-h(\lambda^{(0)})}\\ &= \Big(\sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\mu,\boldsymbol{j})2^{-h(\lambda^{(0)})}\kappa_{\underline{\lambda}},\ \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\nu,\boldsymbol{i})\pi_{\underline{\lambda}}\Big)_{\mathsf{Sym}}\\ &= \Big(\sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\mu,\boldsymbol{j})\pi_{\underline{\lambda}},\ \sum_{\underline{\lambda}\in\mathscr{P}_0^I(d)} K(\underline{\lambda};\nu,\boldsymbol{i})\pi_{\underline{\lambda}}\Big)_{\mathsf{Sym}}\\ &= (\Pi_{\mu,\boldsymbol{j}},\Pi_{\nu,\boldsymbol{i}})_{\mathsf{Sym}},\end{aligned}$$

as required. □

Theorem 5.5. *Let* $(\mu,\boldsymbol{j}) \in \Lambda^{\mathrm{col}}(m,d)$. *Then*

$$(f(\mu,\boldsymbol{j})u_\rho, f(\omega_d)u_\rho) = \binom{d}{\mu_1\cdots\mu_n} 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}.$$

Proof. By Corollary 5.4, we have $(f(\mu,\boldsymbol{j})u_\rho, f(\omega_d)u_\rho) = (\Pi_{\mu,\boldsymbol{j}},\Pi_{\omega_d})_{\mathsf{Sym}}$, and the theorem follows from Theorem 3.36. □

Recall the vector $v_w \in V(\Lambda_0)_{w\Lambda_0}$ defined in Lemma 2.2.

Theorem 5.6. *Let* $(\mu,\boldsymbol{j}) \in \Lambda^{\mathrm{col}}(m,d)$. *Then*

$$(f(\mu,\boldsymbol{j})v_w, f(\omega_d)v_w) = \binom{d}{\mu_1\cdots\mu_n} 4^{d-|\mu,\boldsymbol{j}|_{\ell-1}}\, 3^{|\mu,\boldsymbol{j}|_{\ell-1}}.$$

Proof. In view of Lemmas 4.11 and 4.12, this follows from Theorem 5.5. □

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REPRESENTATION GROWTH OF FUCHSIAN GROUPS AND MODULAR FORMS

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To the memory of Gary Seitz

Let Γ be a cocompact, oriented Fuchsian group which is not on an explicit finite list of possible exceptions and q a sufficiently large prime power not divisible by the order of any nontrivial torsion element of Γ. Then $|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))| \sim c_{q,n} q^{(1-\chi(\Gamma))n^2}$, where $c_{q,n}$ is periodic in n. Within a fixed congruence class for q and for n, $c_{q,n}$ can be expressed as a Puiseux series in $1/q$. Moreover, this series is essentially the q-expansion of a meromorphic modular form of half-integral weight.

1. Introduction

Let Γ be a cocompact and oriented Fuchsian group (which, in what follows, we shall call simply a Fuchsian group). Concretely, this means that Γ has a presentation

$$\langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_r | z_1^{a_1}, \ldots, z_r^{a_r}, [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_r \rangle,$$

where $a_1, a_2, \ldots, a_r$ is a fixed (possibly empty) nondecreasing sequence of integers $a_i \geq 2$ such that the Euler characteristic

$$\chi(\Gamma) := 2 - 2g - \sum_{i=1}^{r} \Big(1 - \frac{1}{a_i}\Big)$$

is negative. Let $\mathbb{F}_q$ be a finite field. We investigate the asymptotic growth in n of

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the number of homomorphisms from Γ to the group $\mathrm{GL}_n(q)$, which we denote G_n when the value of q is understood.

There are two complementary points of view. On the one hand we can fix n and consider the homomorphism scheme $\mathrm{Hom}(\Gamma, \mathrm{GL}_n) := \mathrm{Hom}(\Gamma, \mathrm{GL}_{n,\mathbb{Z}})$, which is defined over $\mathbb{Z}$. For a fixed characteristic $p > 0$, we can think of the fiber of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ over Spec $\mathbb{F}_p$ as the variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_{n,\mathbb{F}_p})$ of homomorphisms from Γ to GL_n over $\mathbb{F}_p$. (Note that in this paper, a *variety* will be just an affine scheme of finite type over a field; in particular, it need not be either irreducible or reduced.) Applying the Lang–Weil theorem to this variety as q ranges over powers of p, we see that the number of homomorphisms $\rho\colon \Gamma \to \mathrm{GL}_n(q)$ determines its *dimension*, by which we mean the maximum dimension of any of its irreducible components.

On the other hand, for fixed q, we can partition homomorphisms ρ according to the r-tuple of G_n-conjugacy classes

$$(C_1, \dots, C_r) = (\rho(z_1)^{G_n}, \dots, \rho(z_r)^{G_n}).$$

Each C_i must consist of elements of order dividing a_i. For given $(C_1, \dots, C_r)$ satisfying this divisibility condition, the number of homomorphisms $\Gamma \to G_n$ with $\rho(z_i) \in C_i$ for all i is given by a theorem of Hurwitz [Liebeck and Shalev 2004, Proposition 3.2]:

$$|G_n|^{2g-1}|C_1|\cdots|C_r| \sum_{\chi\in\mathrm{Irr}(G_n)} \frac{\chi(C_1)\cdots\chi(C_r)}{\chi(1)^{2g+r-2}}. \tag{1-1}$$

Summing (1-1) over all possible r-tuples $(C_1, \dots, C_r)$ we obtain a formula which can potentially be used for fixed q to understand the asymptotic behavior of $|\mathrm{Hom}(\Gamma, G_n)|$ as $n \to \infty$.

These two ways of counting $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))$ are in some sense complementary. For instance, just as we can use character methods to determine the dimension of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$, we can use the dimension of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ to get an upper bound on $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))$ for all large q. For the large n limit, the first way of counting seems to be the more appropriate, and an analysis of (1-1) has led us to this conjecture:

Conjecture 1. *Let A denote the least common multiple of $a_1, \dots, a_r$, which we take to be 1 if $r = 0$. Let q be a prime power relatively prime to A.*

(a) *There exists a $2A$-periodic sequence $c_{q,1}, c_{q,2}, \dots$ of positive numbers such that*

$$|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))| \sim c_{q,n} q^{(1-\chi(\Gamma))n^2}$$

uniformly in q and Γ, that is,

$$\frac{|\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))|}{c_{q,n} q^{(1-\chi(\Gamma))n^2}}$$

approaches 1 in the limit as $n \to \infty$, uniformly in q and Γ.

(b) *There exist a 2-dimensional array $e_{\Gamma,q,n}$ of rational numbers and a 2-dimensional array $f_{\Gamma,q,n}$ of half-integral weight meromorphic modular forms, periodic in both q and n, such that*

$$c_{q,n} = (q-1)q^{e_{\Gamma,q,n}} f_{\Gamma,q,n}\left(\frac{i\log q}{2\pi}\right).$$

Moreover, $f_{\Gamma,q,n}$ is holomorphic on the upper half plane and has integer Fourier coefficients at $i\infty$.

The problem of estimating the number of representations of a given Fuchsian group over a finite field seems to have been first considered by Liebeck and Shalev [2005b]. There are a number of significant differences in emphasis between that paper and ours; Γ need not be oriented in their paper, and the target of homomorphisms from Γ could be a quasisimple group $G(q)$ instead of $\mathrm{GL}_n(q)$. They were primarily interested in the "geometric" direction, that is, n fixed and $q\to\infty$. A key limitation of their paper is that their method requires $g\geq 2$.

Under this hypothesis, they showed that the contribution in (1-1) from nonlinear characters is negligible, which reduces the problem of estimating $|\mathrm{Hom}(\Gamma,\mathrm{GL}_n(q))|$ to that of estimating the numbers $j_{q,n}(a_i)$ of elements $x\in\mathrm{GL}_n(q)$ satisfying $x^{a_i}=1$. They gave an asymptotic formula for $j_{q,n}(a)$ when n is fixed and $q\to\infty$, using work of Lawther [2005]. In the case $r=0$ (the surface group case) they proved that $|\mathrm{Hom}(\Gamma,\mathrm{GL}_n(q))|$ is asymptotic to $(q-1)|\mathrm{GL}_n(q)|^{2g-1}$. If $\eta(z)$ is the Dedekind function, then

$$\frac{|\mathrm{GL}_n(q)|}{q^{1/24}\eta\left(\frac{i\log q}{2\pi}\right)} = q^{n^2}\prod_{i=n+1}^{\infty}(1-q^{-i})^{-1}\sim q^{n^2}.$$

Setting $e_{\Gamma,q,n}=\frac{1}{24}(2g-1)$ and $f_{\Gamma,q,n}=\eta^{2g-1}$ for all q,n, we deduce Conjecture 1 for surface groups. In general, their analysis depends crucially on the fact that for $g\geq 2$, the trivial upper bound on $|\chi(C_i)|$ is good enough to allow us to ignore nonlinear characters of G_n. This is certainly not the case when $g=1$, let alone when $g=0$. However, the new character bounds developed by Bezrukavnikov, Liebeck, Shalev, and Tiep [Bezrukavnikov et al. 2018] give us hope of making progress even for $g=0$.

Using these bounds, Liebeck, Shalev, and Tiep proved [Liebeck et al. 2020, Theorem 1.1] that for every Γ satisfying $\chi(\Gamma)<-2$, if $q\equiv 1 \pmod{A}$, then

$$|\mathrm{Hom}(\Gamma,\mathrm{GL}_n(q))|\leq f(n)q^{(1-\chi(\Gamma))n^2+1}$$

where $f(n)$ does not depend on q. This immediately gives an upper bound for the dimension of the representation variety $\mathrm{Hom}(\Gamma,\mathrm{GL}_{n,K})$ where K is any field in which $A\neq 0$, provided that n is sufficiently large:

$$\dim\mathrm{Hom}(\Gamma,\mathrm{GL}_{n,K})\leq(1-\chi(\Gamma))n^2+1.$$

They also proved a lower bound on dimension:

$$\dim \operatorname{Hom}(\Gamma, \mathrm{GL}_{n,K}) \geq (1-\chi(\Gamma))(n^2-1) - \sum_i a_i.$$

Using ideas from [Bezrukavnikov et al. 2018; Taylor and Tiep 2020], we prove a new exponential character bound Theorem 2.9, which applies to all semisimple elements and which plays an essential role in the proof of the main theorems of this paper. The reason we can prove Conjecture 1 only when q is sufficiently large is that the exponent in our bound only approaches its optimal value as $q \to \infty$.

Proposition 3.2 gives a list, consisting of thirty-one triangle groups and one quadrilateral group, where even for large q, our bounds are not strong enough to prove the conjecture. When Γ is not on this list and q is sufficiently large and relatively prime to the a_i, $|\operatorname{Hom}(\Gamma, G_n)|$ behaves as predicted.

Theorem A. *There exists an absolute constant q_0 such that if Γ is a Fuchsian group which is not on the finite list of groups excluded by Proposition 3.2, then Conjecture 1 holds for Γ for all prime powers $q > q_0$ which are prime to A.*

In particular, the theorem holds for all Fuchsian groups Γ with Euler characteristic less than $-\frac{1}{6}$.

We deduce Theorem A from an analogue of [Liebeck and Shalev 2005b, Theorem 1.2 (i)]. Let $J_{q,n}(a_1, \ldots, a_r)$ denote the cardinality of the set

$$\Big\{(t_1, \ldots, t_r) \in \mathrm{GL}_n(q) \mid t_i^{a_i} = 1 \ \forall i, \ \prod_i \det(t_i) = 1\Big\}. \tag{1-2}$$

Theorem B. *If Γ is not on the excluded list of Proposition 3.2, and $q > q_0$ is prime to A then*

$$|\operatorname{Hom}(\Gamma, \mathrm{GL}_n(q))| = (1+o(1))(q-1)J_{q,n}(a_1, \ldots, a_r)|\mathrm{GL}_n(q)|^{2g-1},$$

where the term $o(1)$ does not depend on q.

Theorem A allows us to compute the exact dimension of $\operatorname{Hom}(\Gamma, \mathrm{GL}_n)$ when n is sufficiently large. Given Γ and n, we define $\sigma_{\Gamma,n}$ to be either 1 or -1 according to the rule that it is -1 if and only if $a_i \in 2\mathbb{Z}$ implies $n/a_i \in \mathbb{Z}$, and

$$\sum_{\{i \mid a_i \in 2\mathbb{Z}\}} \frac{n}{a_i} \in 1 + 2\mathbb{Z}.$$

Let $\{x\}$ denote the fractional part of x. We have:

Theorem C. *There exists an absolute constant N such that if Γ is not on the excluded list of Proposition 3.2, $n > N$, and K is any field of characteristic $p \geq 0$, such that $p \nmid a_i$ for any i, we have*

$$\dim \operatorname{Hom}(\Gamma, \mathrm{GL}_{n,K}) = \sigma_{\Gamma,n} + (1-\chi(\Gamma))n^2 - \sum_{i=1}^{r} a_i \Big\{\frac{n}{a_i}\Big\}\Big\{-\frac{n}{a_i}\Big\}.$$

In particular,

$$\dim \operatorname{Hom}(\Gamma, \mathrm{GL}_{n,K}) \geq -\tfrac{1}{2} + (1-\chi(\Gamma))n^2 - \sum_{i=1}^{r} \tfrac{1}{4} a_i.$$

Unfortunately, the letter q has a standard meaning both for finite fields and for modular forms. We use it only in the former sense, but we evaluate modular forms f at $i \log q/(2\pi)$, which amounts to plugging $1/q$ into the q-expansion for f.

2. An asymptotic character bound

The goal of this section is to prove an asymptotic version of the character bounds in [Bezrukavnikov et al. 2018, Theorem 1.1] and [Taylor and Tiep 2020, Theorem 1.9] when G is a finite group of Lie-type A. We will achieve this by combining the approach of [Bezrukavnikov et al. 2018] with the *character level* approach developed in [Guralnick et al. 2020] to bound $|\chi(g)|$.

To this end, let us recall the approach of [Bezrukavnikov et al. 2018]. Throughout this section, q is a prime power and $\mathcal{G} = \mathcal{G}(\overline{\mathbb{F}}_q)$ is the group of $\overline{\mathbb{F}}_q$-points of a connected reductive $\mathbb{F}_q$-group scheme. We assume $G = \mathcal{G}(\mathbb{F}_q) = \mathcal{G}^F$ is the finite group of $\mathbb{F}_q$-points, where $F : \mathcal{G} \to \mathcal{G}$ is the Frobenius endomorphism determined by its structure as a scheme over $\mathbb{F}_q$.

The main case of interest to us will be when the underlying group scheme is GL_n^ϵ, where $\epsilon \in \{+,-\}$ and we set $\mathrm{GL}_n^+ := \mathrm{GL}_n$, the general linear group, and $\mathrm{GL}_n^- := \mathrm{GU}_n$, the general unitary group. In this setting,

$$\mathcal{G} = \mathcal{G}_n := \mathrm{GL}_n(\overline{\mathbb{F}}_q)$$

is the general linear group of dimension $n > 0$ and F is either F_q or σF_q, where $F_q : \mathcal{G} \to \mathcal{G}$ is the standard Frobenius endomorphism and $\sigma : \mathcal{G} \to \mathcal{G}$ is the inverse transpose automorphism.

Suppose $L = \mathcal{L}^F$, where $\mathcal{L} < \mathcal{G}$ is a proper F-stable Levi subgroup of $\mathcal{G}$. Assume $g \in \mathcal{G}^F$ is an element such that $\boldsymbol{C}_{\mathcal{G}}(g) \leqslant \mathcal{L}$. By [Taylor and Tiep 2020, Lemma 13.3], for every irreducible character χ of G, we have

$$\chi(g) = {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)(g) = \sum_{\eta \in \mathrm{Irr}(L)} \langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle \eta(g), \tag{2-1}$$

where ${}^*R_{\mathcal{L}}^{\mathcal{G}}$ denotes Deligne–Lusztig restriction. We also write $R_{\mathcal{L}}^{\mathcal{G}}$ for Deligne–Lusztig induction.

Following [Bezrukavnikov et al. 2018, Theorem 1.1], we define the constant $\alpha(L)$ to be the maximum over nontrivial unipotent elements $u \in L$ of

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}};$$

if L contains no such elements we take $\alpha(L) = 0$. Here $u^{\mathcal{G}}$ denotes the $\mathcal{G}$-conjugacy class of u, and similarly $u^{\mathcal{L}}$ denotes the $\mathcal{L}$-conjugacy class. From the proof of [Bezrukavnikov et al. 2018, Theorem 1.1], see also [Taylor and Tiep 2020, §2], we get

$$|\eta(g)| \leqslant \eta(1) \leqslant B_1 \left(\frac{q+1}{q-1}\right)^{D/2} \chi(1)^{\alpha(L)}, \tag{2-2}$$

for any $\eta \in \mathrm{Irr}(L)$ with $\langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle \neq 0$, where $B_1 > 0$ is a constant that depends on $\mathcal{G}^F$. Furthermore, $D = \dim v^{\mathcal{G}}$, where $v^{\mathcal{G}} = \mathcal{O}_{\chi}^*$ is the *wave front set* of χ, defined by work of Kawanaka [1986], Lusztig [1984], and Taylor [2016].

If $f_\eta \in \mathbb{Q}[X]$ is the degree polynomial of η, so that $\eta(1) = f_\eta(q)$, then the constant B_1 is chosen such that $B_1 f_\eta \in \mathbb{Z}[X]$. When the underlying group scheme is GL_n^ϵ we have that $\mathcal{L}^F$ is a direct product of groups $\mathrm{GL}_{n_i}^{\epsilon_i}(q)$. Therefore, in this case, the constant B_1 can be taken to be 1 because the degree polynomial of any irreducible character of $\mathcal{L}^F$ is already contained in $\mathbb{Z}[X]$.

Recall that ${}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)$ is a virtual character but is a true character if $\mathcal{L}$ is split. To bound $|\chi(g)|$ it suffices, by the triangle inequality, (2-1), and (2-2), to bound

$$\sum_{\eta \in \mathrm{Irr}(L)} |\langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle| \leqslant \sum_{\eta \in \mathrm{Irr}(L)} \langle \eta, {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle^2 = \langle {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi), {}^*R_{\mathcal{L}}^{\mathcal{G}}(\chi)\rangle, \tag{2-3}$$

where $\langle -, - \rangle$ is the usual inner product on class functions.

We know from [Bezrukavnikov et al. 2018, Proposition 2.2] and its proof, as well as the arguments in [Taylor and Tiep 2020, §13], that (2-3) is always bounded above by

$$(n!)^2. \tag{2-4}$$

However, we can do significantly better if n is large compared to both q and the *true level*

$$\mathfrak{l}^*(\chi) = j$$

of χ, as defined in [Guralnick et al. 2020, Definition 1(i)]. When $G = \mathrm{GL}_n(q)$ then j is the smallest integer for which χ is a constituent of τ^j, where $\tau(g)$ is the number of fixed points of g acting on the natural module $V = \mathbb{F}_q^n$ of G.

In the next subsections we give upper bounds for (2-3) that incorporate the true level of χ.

2.1. ***Elements with split centralizer in*** $\mathbf{GL}_n(q)$**.** In this subsection we consider the group scheme GL_n so that

$$G = G_n := \mathrm{GL}_n(q).$$

Fix a proper split Levi subgroup $\mathcal{L}$, and let

$$L = \mathcal{L}^F = \mathrm{GL}_{m_1}(q) \times \mathrm{GL}_{m_2}(q) \times \cdots \times \mathrm{GL}_{m_t}(q) \subset G, \tag{2-5}$$

where $m_i \in \mathbb{Z}_{\geq 1}$ and $\sum_{i=1}^{t} m_i = n$. In this case ${}^*R^{\mathcal{G}}_{\mathcal{L}}$ is just Harish-Chandra restriction. With $1 \leq j < \frac{1}{2}n$ fixed, consider a split Levi subgroup $\mathcal{M}$ and set

$$M = \mathcal{M}^F \cong \mathrm{GL}_j(q) \times \mathrm{GL}_{n-j}(q) \subset G.$$

By [Guralnick et al. 2020, Theorem 3.9(i)], $\mathfrak{l}^*(\chi) = j$ implies that χ is an irreducible constituent of the Harish-Chandra induction

$$R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}})$$

for a unique irreducible character α of G_j. Conjugating $\mathcal{M}$ by a suitable element $g \in G$, we may assume that L and M are block-diagonal subgroups in the same basis $(e_1, e_2, \ldots, e_n)$ of V.

To bound (2-3), it therefore suffices to bound

$$\langle {}^*R^{\mathcal{G}}_{\mathcal{L}} R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}}), {}^*R^{\mathcal{G}}_{\mathcal{L}} R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}}) \rangle.$$

By the Mackey formula for Harish-Chandra restriction and induction [Dipper and Fleischmann 1992, Theorem 1.14],

$$ {}^*R^{\mathcal{G}}_{\mathcal{L}} R^{\mathcal{G}}_{\mathcal{M}}(\alpha \boxtimes 1_{G_{n-j}}) = \sum_{x \in L \backslash \mathcal{S}(\mathcal{L},\mathcal{M})/M} R^{\mathcal{L}}_{\mathcal{L}\cap {}^x\mathcal{M}} {}^*R^{{}^x\mathcal{M}}_{\mathcal{L}\cap {}^x\mathcal{M}}((\alpha \boxtimes 1_{G_{n-j}})^x), \tag{2-6}$$

where $\mathcal{S}(\mathcal{L}, \mathcal{M})$ is the set of elements $y \in G$ such that $\mathcal{L} \cap {}^y\mathcal{M}$ contains a maximal torus of $\mathcal{G}$, and the summation runs through the (L, M) double cosets of this set.

For our pair of split Levi subgroups $(\mathcal{L}, \mathcal{M})$, there is an explicit description of $L \backslash \mathcal{S}(\mathcal{L}, \mathcal{M})/M$, as described in [Brundan et al. 2001, §2.2c]. Embed the symmetric group S_n in G_n via permutation matrices, and consider the Young subgroups

$$\mathsf{S}_\lambda = \mathsf{S}_{m_1} \times \mathsf{S}_{m_2} \times \cdots \times \mathsf{S}_{m_t}, \quad \mathsf{S}_\mu = \mathsf{S}_j \times \mathsf{S}_{n-j}$$

of the embedded S_n. Then in (2-6) we can just choose x as representatives of the set $\mathsf{S}_\lambda \backslash \mathsf{S}_n / \mathsf{S}_\mu$, one for each double coset. The set of double cosets $\mathsf{S}_\lambda \backslash \mathsf{S}_n / \mathsf{S}_\mu$ is in bijection with S_λ-orbits on the set of $\mathsf{S}_n/\mathsf{S}_\mu$, which may be identified with the set of j-subsets of $\{1, 2, \ldots, n\}$. Hence each such double coset can be labeled uniquely by a t-tuple

$$\kappa = (k_1, k_2, \ldots, k_t), \quad 0 \leq k_i \leq m_i, \ \sum_{i=1}^{t} k_i = j. \tag{2-7}$$

Correspondingly, we can choose $x = x_\kappa$ to be the element of G that sends the first j basis vectors $e_1, \ldots, e_j$ of V to

$$e_1, \ldots, e_{k_1}, e_{m_1+1}, e_{m_1+2}, \ldots, e_{m_1+k_2}, \ldots, e_{m_1+\cdots+m_{t-1}+1}, \ldots, e_{m_1+m_2+\cdots+m_{t-1}+k_t}$$

in the increasing order of the subscripts, and sends the last $n - j$ basis vectors $e_{j+1}, \ldots, e_n$ to the remaining $n - j$ basis vectors, again in the increasing order of the subscripts. We will say that $x_\kappa(e_i) = e_{x_\kappa(i)}$, $1 \leq i \leq n$.

For the reader's convenience, let us give a justification for this statement in the case $q \geq 3$. Suppose $y \in G$ is such that $\mathcal{L} \cap {}^y\mathcal{M}$ contains a maximal torus $\mathcal{T}$ of $\mathcal{G}$. Then $\mathcal{T}$ is a maximal torus of $(\mathcal{L} \cap {}^y\mathcal{M})^\circ$ which is F-stable and connected. By the Lang–Steinberg theorem, conjugating $\mathcal{T}$ suitably, we may assume that it is F-stable. Then

$$T := \mathcal{T}^F \cong C_{(q^{a_1})-1} \times C_{(q^{a_2})-1} \times \cdots \times C_{(q^{a_s})-1}$$

for some integers $a_1, a_2, \dots, a_s \geq 1$. Since $q \geq 3$, all cyclic direct factors in this decomposition are nontrivial, and hence $V = \mathbb{F}_q^n$ is a direct sum of s simple $\mathbb{F}_q T$-modules $W_1, \dots, W_s$ of dimension $a_1, a_2, \dots, a_s$, which are pairwise non-isomorphic (indeed, they have pairwise distinct kernels). On the other hand, the $\mathbb{F}_q L$-module V decomposes as the sum $\bigoplus_{i=1}^t V_i$ of $\mathbb{F}_q L$-modules, where

$$V_1 := \langle e_1, \dots, e_{m_1} \rangle_{\mathbb{F}_q},\ V_2 := \langle e_{m_1+1}, \dots, e_{m_1+m_2} \rangle_{\mathbb{F}_q}, \dots, V_t := \langle e_{m_1+\cdots+m_{t-1}+1}, \dots, e_n \rangle_{\mathbb{F}_q}.$$

Since $T \leq L$, each V_i is a direct sum of some of these W_l, $1 \leq l \leq s$. Similarly, since $V = \mathbb{F}_q^j \oplus \mathbb{F}_q^{n-j}$ as an $\mathbb{F}_q {}^yM$-module and $T \leq {}^yM$, each of $\mathbb{F}_q^j$ and $\mathbb{F}_q^{n-j}$ is a direct sum of some of these W_l. Using the left multiplication by L and right multiplication by M if needed, we may assume that $\mathbb{F}_q^j$ is spanned by

$$e_1, \dots, e_{k_1}, e_{m_1+1}, e_{m_1+2}, \dots, e_{m_1+k_2}, \dots, e_{m_1+\cdots+m_{t-1}+1}, \dots, e_{m_1+m_2+\cdots+m_{t-1}+k_t}$$

(k_i first vectors in the indicated basis of V_i for each $1 \leq i \leq t$), and $\mathbb{F}_q^{n-j}$ is spanned by the remaining $n-j$ basis vectors.

It is well known (and can be proved by an easy induction on $t \geq 1$) that the total number N of t-tuples κ in (2-7) is

$$N = \binom{j+t-1}{j} = t \cdot \frac{t+1}{2} \cdots \frac{t+j-1}{j} \leq t^j \leq n^j \tag{2-8}$$

since $t \leq n$. For each such κ, $x = x_\kappa$ sends e_i to $e_{x(i)}$, and we can write

$$\begin{aligned} {}^xM = xMx^{-1} = \mathrm{GL}(\langle e_{x(1)}, \dots, e_{x(j)} \rangle_{\mathbb{F}_q}) \times \mathrm{GL}(\langle e_{x(j+1)}, \dots, e_{x(n)} \rangle_{\mathbb{F}_q}) \\ \cong \mathrm{GL}_j(q) \times \mathrm{GL}_{n-j}(q) \end{aligned}$$

Now, $L \cap {}^xM$ fixes each of the subspaces

$$\langle e_1, \dots, e_{m_1} \rangle_{\mathbb{F}_q} \cap \langle e_{x(1)}, \dots, e_{x(j)} \rangle_{\mathbb{F}_q} = \langle e_1, \dots, e_{k_1} \rangle_{\mathbb{F}_q}$$

and

$$\langle e_1, \dots, e_{m_1} \rangle_{\mathbb{F}_q} \cap \langle e_{x(j+1)}, \dots, e_{x(n)} \rangle_{\mathbb{F}_q} = \langle e_{k_1+1}, \dots, e_{m_1} \rangle_{\mathbb{F}_q}$$

of V_1, and similarly for V_i with $2 \leq i \leq t$. It follows that

$$L \cap {}^xM = \prod_{i=1}^t (K_i \times M_i),$$

where for each $1 \le i \le t$, $K_i \cong \mathrm{GL}_{k_i}(q)$ is contained in the $\mathrm{GL}_j(q)$-factor of ${}^xM$, and $M_i \cong \mathrm{GL}_{m_i-k_i}(q)$ is contained in the $\mathrm{GL}_{n-j}(q)$-factor of ${}^xM$. Moreover, $\prod_{i=1}^t K_i$ is a split Levi subgroup of $\mathrm{GL}_j(q)$, and $\prod_{i=1}^t M_i$ is a split Levi subgroup of $\mathrm{GL}_{n-j}(q)$. Now, applying [Giannelli et al. 2017, Lemma 2.7(i)] twice, we obtain

$$ {}^*R^{{}^x\mathcal{M}}_{\mathcal{L}\cap{}^x\mathcal{M}}((\alpha \boxtimes 1_{G_{n-j}})^x) = {}^*R^{\mathcal{G}_j}_{K_1\times\cdots\times K_t}(\alpha^x) \boxtimes 1_{M_1\times\cdots\times M_t}. $$

Recall that α^x is an irreducible character of $\mathrm{GL}_j(q)$. So, by (2-4), the total sum of multiplicities of irreducible constituents β in ${}^*R^{{}^x\mathcal{M}}_{\mathcal{L}\cap{}^x\mathcal{M}}((\alpha \boxtimes 1_{G_{n-j}})^x)$ is at most

$$ (j!)^2. \tag{2-9} $$

Consider any such irreducible constituent

$$ \beta = \alpha_1 \boxtimes \alpha_2 \boxtimes \cdots \boxtimes \alpha_t \boxtimes 1_{M_1\times\cdots\times M_t}. $$

By [Guralnick et al. 2020, Lemma 2.5(ii)],

$$ R^{\mathcal{L}}_{\mathcal{L}\cap{}^x\mathcal{M}}(\beta) = \boxtimes_{i=1}^{t} R^{\mathcal{G}_{m_i}}_{\mathcal{G}_{k_i}\times\mathcal{G}_{m_i-k_i}}(\alpha_i \boxtimes 1_{G_{m_i-k_i}}). $$

Let $\tau_{q,n}$ denote the permutation character of G_n on $\mathbb{F}_q^n$, see [Guralnick et al. 2020, Equation 3.1]. Then the character

$$ \gamma_i := R^{\mathcal{G}_{m_i}}_{\mathcal{G}_{k_i}\times\mathcal{G}_{m_i-k_i}}(\alpha_i \boxtimes 1_{G_{m_i-k_i}}) $$

is contained in $(\tau_{m_i,q})^{k_i}$ by [Guralnick et al. 2020, Proposition 3.2]. If $k_i = 0$, then the total number $N(\gamma_i)$ of multiplicities of irreducible constituents of γ_i is 1. If $1 \le k_i \le \frac{1}{2}m_i$, then

$$ N(\gamma_i) \le \langle \gamma_i, \gamma_i \rangle \le \langle \tau^{k_i}_{m_i,q}, \tau^{k_i}_{m_i,q} \rangle = \langle \tau^{2k_i}_{m_i,q}, 1_{G_{m_i}} \rangle, $$

which is the number of G_{m_i}-orbits on ordered $2k_i$-tuples of vectors in $\mathbb{F}_q^{m_i}$, and hence is at most $8q^{k_i^2} \le q^{4k_i^2}$ by [Guralnick et al. 2020, Lemma 2.4]. Suppose $\frac{1}{2}m_i < k_i \le m_i$. Then γ_i is a character of degree at most $q^{m_ik_i} < q^{2k_i^2}$, and hence $N(\gamma_i) < q^{2k_i^2}$. Thus in all cases we have

$$ N(\gamma_i) \le q^{4k_i^2}. $$

It follows that the total number $N(\beta)$ of multiplicities of irreducible constituents of

$$ R^{\mathcal{L}}_{\mathcal{L}\cap{}^x\mathcal{M}}(\beta) = \boxtimes_{i=1}^{t} \beta_i $$

is at most

$$ q^{4\sum_{i=1}^t k_i^2} \le q^{4(\sum_{i=1}^t k_i)^2} = q^{4j^2}. $$

Combining this with (2-8) and (2-9), we have proved:

Proposition 2.1. *Let $G = \mathcal{G}^F = \mathrm{GL}_n(q)$ and let χ be any irreducible character G of true level $j \le \frac{1}{2}n$. If $L = \mathcal{L}^F$ is a proper split Levi subgroup of G, then the total number A of irreducible constituents (counting multiplicities) of the Harish-Chandra restriction ${}^*R^{\mathcal{G}}_{\mathcal{L}}(\chi)$ is at most $n^j(j!)^2q^{4j^2}$.*

Corollary 2.2. *Let $G = \mathrm{GL}_n(q)$ and let $g \in G$ be any element such that $\boldsymbol{C}_G(g)$ is contained in a split Levi subgroup $\mathcal{L}$ of G. Let $\chi \in \mathrm{Irr}(G)$ be of true level $j \le \frac{1}{2}n$, and let $D = \dim \mathfrak{v}^{\mathcal{G}}$, with $\mathfrak{v}^{\mathcal{G}} = \mathcal{O}^*_\chi$ being the wave front set of χ. Then*

$$|\chi(g)| \le n^j(j!)^2q^{4j^2}\left(\frac{q+1}{q-1}\right)^{D/2}\chi(1)^{\alpha(L)}.$$

Proof. As mentioned above, in our case the constant B_1 in (2-2) can be taken to be 1. We now combine (2-1), (2-2), and (2-3) with Proposition 2.1. □

2.2. ***The general case.*** For semisimple elements whose centralizer is a nonsplit Levi subgroup, the bound in Corollary 2.2 can be very poor; for instance, it says nothing at all about character values for elements in anisotropic tori. However, the following result is almost as good for all semisimple elements as Corollary 2.2 is in the split case, and moreover works for both GL_n and GU_n:

Theorem 2.3. *Let $G = \mathrm{GL}^\epsilon_n(q)$ and let $g \in G$ be any element such that $\boldsymbol{C}_{\mathcal{G}}(g)$ is contained in a proper F-stable Levi subgroup $\mathcal{L}_1$. Define $L_1 := \mathcal{L}_1^F$. Let $\chi \in \mathrm{Irr}(G)$ be of true level j, $0 \le j \le n$, and let $D = \dim \mathfrak{v}^{\mathcal{G}}$, with $\mathfrak{v}^{\mathcal{G}} = \mathcal{O}^*_\chi$ being the wave front set of χ. Then*

$$|\chi(g)| \le n^{3j}\left(\frac{q+1}{q-1}\right)^{D/2}\chi(1)^{\alpha(L_1)}.$$

To prove this result we will use Deligne–Lusztig theory. However, before developing the necessary results about Deligne–Lusztig characters we recall a few facts about cosets. Assume $\mathfrak{G}$ is a group. The set of conjugacy classes of $\mathfrak{G}$ will be denoted by $\mathrm{Cl}(\mathfrak{G})$ and if $x \in \mathfrak{G}$ then $x^{\mathfrak{G}} \in \mathrm{Cl}(\mathfrak{G})$ denotes the conjugacy class containing x. A *subcoset* of $\mathfrak{G}$ is a coset $Hw \subseteq \boldsymbol{N}_{\mathfrak{G}}(H)$ of a subgroup $H \leqslant \mathfrak{G}$. Given any subsets $X, Y \subseteq \mathfrak{G}$ we define

$$\boldsymbol{N}_X(Y) := X \cap \boldsymbol{N}_{\mathfrak{G}}(Y), \quad \boldsymbol{C}_X(Y) := X \cap \boldsymbol{C}_{\mathfrak{G}}(Y),$$

where $\boldsymbol{N}_{\mathfrak{G}}(Y)$ and $\boldsymbol{C}_{\mathfrak{G}}(Y)$ are the usual normalizer and centralizer of Y. As usual

$$XY := \{xy \mid x \in X \text{ and } y \in Y\}.$$

Now assume that $W\gamma \subseteq \mathfrak{G}$ is a finite subcoset. We denote by $\mathrm{cf}(W\gamma)$ the space of W-invariant functions $f : W\gamma \to \mathbb{C}$, which we call class functions. This space has an inner product $\langle -, - \rangle$ and if $Hw \subseteq W\gamma$ is a subcoset then we have induction $\mathrm{Ind}^{W\gamma}_{Hw} : \mathrm{cf}(Hw) \to \mathrm{cf}(W\gamma)$ and restriction maps $\mathrm{Res}^{W\gamma}_{Hw} : \mathrm{cf}(W\gamma) \to \mathrm{cf}(Hw)$ which satisfy Frobenius reciprocity with respect to $\langle -, -\rangle$, see [Bonnafé 2006, § 1.C] or [Taylor and Tiep 2020, § 4].

The function $\pi_w = \pi_w^{W\gamma}$ taking the value $|\boldsymbol{C}_W(w)|$ at any W-conjugate of $w \in W\gamma$ and the value 0 otherwise is clearly contained in $\mathrm{cf}(W\gamma)$. We will need the following elementary calculation.

Lemma 2.4. *For any subcoset $Hw \subseteq W\gamma$ and $x \in W\gamma$ we have*

$$\mathrm{Res}_{Hw}^{W\gamma}(\pi_x^{W\gamma}) = \sum_{\substack{z\in H\backslash W/\boldsymbol{C}_W(x)\\ {}^z x\in Hw}} \frac{|\boldsymbol{C}_W(x)|}{|\boldsymbol{C}_H({}^z x)|}\pi_{{}^z x}^{Hw} = \sum_{\substack{z\in H\backslash W\\ {}^z x\in Hw}} \pi_{{}^z x}^{Hw}$$

Proof. The first equality is easy and the second follows because

$$Hz\boldsymbol{C}_W(x) = \bigsqcup_{c\in \boldsymbol{C}_H({}^z x)\backslash \boldsymbol{C}_W({}^z x)} Hcz. \qquad \square$$

We can also produce class functions in the following way. Consider the subgroup $W\langle\gamma\rangle \leqslant \boldsymbol{N}_{\mathfrak{G}}(W)$ and let $\rho \in \mathrm{Irr}(W)$ be a γ-invariant irreducible character. The representation affording ρ can be extended to a representation of $W\langle\gamma\rangle$ containing γ^n in its kernel, for some $n > 0$. The trace function $\tilde{\rho} : W\langle\gamma\rangle \to \mathbb{C}$ of such a representation is what we call an extension of ρ. Note that the group $W\langle\gamma\rangle$ may be infinite but, by design, $\tilde{\rho}$ factors through a finite quotient.

The restriction $\mathrm{Res}_{W\gamma}^{W\langle\gamma\rangle}(\tilde{\rho})$ of such an extension, which we usually again denote by $\tilde{\rho}$, is called an irreducible character of $W\gamma$. The set of irreducible characters is denoted by $\mathrm{Irr}(W\gamma)$. We say $\mathcal{B} \subseteq \mathrm{Irr}(W\gamma)$ is a basis if it is a basis of $\mathrm{cf}(W\gamma)$. Every basis is orthonormal and is obtained by choosing for each γ-stable $\rho \in \mathrm{Irr}(W)$ exactly one extension to $W\langle\gamma\rangle$, see [Digne and Michel 2020, Proposition 11.6.3]. We need the following analogue of [Taylor and Tiep 2020, Corollary 4.11]:

Lemma 2.5. *Assume $Hw \subseteq W\gamma$ is a subcoset and $\rho_i \in \mathrm{Irr}(W)$, with $i \in \{1, 2\}$, is γ-invariant. If $\tilde{\rho}_i$ is an extension of ρ_i to $W\langle\gamma\rangle$ then*

$$|\langle \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_1), \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_2)\rangle| \leqslant \langle \mathrm{Res}_H^W(\rho_1), \mathrm{Res}_H^W(\rho_2)\rangle$$

Proof. Expanding out in a basis $\mathcal{B} \subseteq \mathrm{Irr}(Hw)$ and using the triangle inequality,

$$\begin{aligned}|\langle \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_1), \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_2)\rangle| &\leqslant \sum_{\tilde{\eta}\in\mathcal{B}} |\langle \tilde{\eta}, \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_1)\rangle \overline{\langle \tilde{\eta}, \mathrm{Res}_{Hw}^{W\gamma}(\tilde{\rho}_2)\rangle}| \\ &= \sum_{\tilde{\eta}\in\mathcal{B}} |\langle \mathrm{Ind}_{Hw}^{W\gamma}(\tilde{\eta}), \tilde{\rho}_1\rangle||\langle \mathrm{Ind}_{Hw}^{W\gamma}(\tilde{\eta}), \tilde{\rho}_2\rangle|.\end{aligned}$$

Therefore, using [Taylor and Tiep 2020, Lemma 4.10] we obtain

$$\begin{aligned}|\langle \operatorname{Res}^{W\gamma}_{Hw}(\tilde{\rho}_1), \operatorname{Res}^{W\gamma}_{Hw}(\tilde{\rho}_2)\rangle| &\leqslant \sum_{\eta\in\operatorname{Irr}(H)} \langle \operatorname{Ind}^W_H(\eta), \rho_1\rangle\langle \operatorname{Ind}^W_H(\eta), \rho_2\rangle \\ &= \sum_{\eta\in\operatorname{Irr}(H)} \langle \eta, \operatorname{Res}^W_H(\rho_1)\rangle\langle \eta, \operatorname{Res}^W_H(\rho_2)\rangle \\ &= \langle \operatorname{Res}^W_H(\rho_1), \operatorname{Res}^W_H(\rho_2)\rangle. \qquad \square\end{aligned}$$

Recall that $\mathcal{G} = \mathcal{G}(\overline{\mathbb{F}}_q)$ is the group of $\overline{\mathbb{F}}_q$-points of a connected reductive $\mathbb{F}_q$-group, with Frobenius F. We form the semidirect product $\mathcal{G}\langle F\rangle$ with the infinite cyclic group generated by F, defined so that $FgF^{-1} = F(g)$ for all $g \in \mathcal{G}$. If $\mathcal{H}n \subseteq \mathcal{G}F$ is a subcoset then the centralizer $\boldsymbol{C}_{\mathcal{H}}(n) \leqslant \boldsymbol{C}_{\mathcal{G}}(n)$ is a finite group. Moreover, if $\mathcal{H} \leqslant \mathcal{G}$ is closed and connected, then by the Lang–Steinberg theorem, $\mathcal{H}$ acts transitively by conjugation on $\mathcal{H}n$. If $\mathcal{H}$ is a Levi subgroup of $\mathcal{G}$, resp., maximal torus of $\mathcal{G}$, then we call $\mathcal{H}n$ a *Levi subcoset*, resp., a *toral subcoset*.

We define

$$\mathscr{C}(\mathcal{G}F) := \{(g, n) \in \mathcal{G} \times \mathcal{G}F \mid gn = ng\}$$

to be the set of commuting pairs. The group $\mathcal{G}$ acts by simultaneous conjugation on $\mathscr{C}(\mathcal{G}F)$. We write $[g, n]$ for the orbit of $(g, n) \in \mathscr{C}(\mathcal{G}F)$ and $\mathscr{C}(\mathcal{G}F)/\mathcal{G}$ for the set of orbits.

Lemma 2.6. *The map $g^{\boldsymbol{C}_{\mathcal{G}}(F)} \mapsto [g, F]$ is a well-defined bijection*

$$\operatorname{Cl}(\boldsymbol{C}_{\mathcal{G}}(F)) \to \mathscr{C}(\mathcal{G}F)/\mathcal{G}.$$

Proof. Clearly this is injective. If $(g, n) \in \mathscr{C}(\mathcal{G}F)$ then by the Lang–Steinberg theorem $n = F^h$ for some $h \in \mathcal{G}$ so $[g, n] = [{}^h g, F]$. □

Let $\operatorname{cf}(\mathscr{C}(\mathcal{G}F))$ be the set of $\mathcal{G}$-invariant functions $f : \mathscr{C}(\mathcal{G}F) \to \mathbb{C}$. Via Lemma 2.6 we can identify $\operatorname{cf}(\mathscr{C}(\mathcal{G}F))$ with the space $\operatorname{cf}(\boldsymbol{C}_{\mathcal{G}}(F))$ of $\mathbb{C}$-valued class functions on the finite group $\boldsymbol{C}_{\mathcal{G}}(F)$. We define $\operatorname{Irr}(\mathscr{C}(\mathcal{G}F))$ to be those functions corresponding to $\operatorname{Irr}(\boldsymbol{C}_{\mathcal{G}}(F))$. The advantage of working with $\mathscr{C}(\mathcal{G}F)$ is that we can work with the different (inner) forms $\boldsymbol{C}_{\mathcal{G}}(gF)$ of $\mathcal{G}$ simultaneously.

If $\mathcal{L}w \subseteq \mathcal{G}F$ is a Levi subcoset then we can define Deligne–Lusztig induction and restriction maps

$$R^{\mathcal{G}F}_{\mathcal{L}w} : \operatorname{cf}(\mathscr{C}(\mathcal{L}w)) \to \operatorname{cf}(\mathscr{C}(\mathcal{G}F)) \quad \text{and} \quad {}^*R^{\mathcal{G}F}_{\mathcal{L}w} : \operatorname{cf}(\mathscr{C}(\mathcal{G}F)) \to \operatorname{cf}(\mathscr{C}(\mathcal{L}w)).$$

For our purposes this can be done as follows. We start first with the case of a coset $\mathcal{L}F$ where $\mathcal{L} \leqslant \mathcal{G}$ is an F-stable Levi subgroup. Making the identifications $\operatorname{cf}(\boldsymbol{C}_{\mathcal{G}}(F)) \to \operatorname{cf}(\mathscr{C}(\mathcal{G}F))$ and $\operatorname{cf}(\boldsymbol{C}_{\mathcal{L}}(F)) \to \operatorname{cf}(\mathscr{C}(\mathcal{L}F))$ we define

$$R^{\mathcal{G}F}_{\mathcal{L}F} := R^{\mathcal{G}}_{\mathcal{L}}, \quad {}^*R^{\mathcal{G}F}_{\mathcal{L}F} := {}^*R^{\mathcal{G}}_{\mathcal{L}}.$$

Now consider a general Levi subcoset $\mathcal{L}w \subseteq \mathcal{G}F$. We pick an element $g \in \mathcal{G}$ such that $F^g \in \mathcal{L}w$, so that ${}^g(\mathcal{L}w) = \mathcal{L}_1 F$ where $\mathcal{L}_1 := {}^g\mathcal{L}$ is an F-stable Levi subgroup of $\mathcal{G}$. If $\iota_g : \mathcal{G}\langle F\rangle \to \mathcal{G}\langle F\rangle$ is the inner automorphism defined by $\iota_g(x) = {}^g x$ then $\iota_g(\mathcal{L}w) = \mathcal{L}_1 F$ and we define

$$R^{\mathcal{G}F}_{\mathcal{L}w} := R^{\mathcal{G}F}_{\mathcal{L}_1 F} \circ (\iota_g^{-1})^* \quad \text{and} \quad {}^*R^{\mathcal{G}F}_{\mathcal{L}w} := (\iota_g)^* \circ {}^*R^{\mathcal{G}F}_{\mathcal{L}_1 F},$$

where $(\iota_g^{-1})^*$ is the map $f \mapsto f \circ \iota_g^{-1}$ and likewise for $(\iota_g)^*$.

We note that the maps $R^{\mathcal{G}F}_{\mathcal{L}w}$ and ${}^*R^{\mathcal{G}F}_{\mathcal{L}w}$ are defined only up to composition with $(\iota_n)^*$ for some $n \in \boldsymbol{N}_{\mathcal{G}}(\mathcal{L}w) = \boldsymbol{N}_{\boldsymbol{C}_{\mathcal{G}}(w)}(\mathcal{L})\mathcal{L}$. We need the following interpretation of the Mackey formula.

Lemma 2.7. *If $\mathcal{L}w \subseteq \mathcal{G}F$ is a Levi subcoset and $\mathcal{T}x \subseteq \mathcal{G}F$ is a toral subcoset then*

$$ {}^*R^{\mathcal{G}F}_{\mathcal{L}w} \circ R^{\mathcal{G}F}_{\mathcal{T}x} = \sum_{\substack{z \in \mathcal{L}\backslash\mathcal{G} \\ {}^z(\mathcal{T}x) \subseteq \mathcal{L}w}} R^{\mathcal{L}w}_{{}^z(\mathcal{T}x)} \circ (\iota_z^{-1})^* $$

Proof. Fix elements $h_w, h_x \in \mathcal{G}$ such that $w = F^{h_w}$ and $x = F^{h_x}$ and let $\mathcal{L}_1 := {}^{h_w}\mathcal{L}$ and $\mathcal{T}_1 := {}^{h_x}\mathcal{T}$ be corresponding F-stable subgroups of $\mathcal{G}$. According to the Mackey formula, see [Digne and Michel 2020, Theorem 9.2.6], we have

$$ {}^*R^{\mathcal{G}}_{\mathcal{L}_1} \circ R^{\mathcal{G}}_{\mathcal{T}_1} = \sum_{\substack{u \in L_1\backslash G/T_1 \\ {}^u\mathcal{T}_1 \leqslant \mathcal{L}_1}} T^{\mathcal{L}_1}_{{}^u\mathcal{T}_1} \circ (\iota_u^{-1})^* = \sum_{\substack{u \in L_1\backslash G \\ {}^u\mathcal{T}_1 \leqslant \mathcal{L}_1}} R^{\mathcal{L}_1}_{{}^u\mathcal{T}_1} \circ (\iota_u^{-1})^*, $$

where $L_1 = \boldsymbol{C}_{\mathcal{L}_1}(F)$, $G = \boldsymbol{C}_{\mathcal{G}}(F)$, and $T_1 = \boldsymbol{C}_{\mathcal{T}_1}(F)$. The second equality follows because if ${}^u\mathcal{T}_1 \leqslant \mathcal{L}_1$ then ${}^uT_1 \leqslant L_1$, so

$$ L_1 u T_1 = L_1({}^uT_1)u = L_1 u. $$

Consider the isomorphism of varieties $\psi : \mathcal{G} \to \mathcal{G}$ given by $\psi(v) = h_w v h_x^{-1}$. If $F' : \mathcal{G} \to \mathcal{G}$ is the morphism defined by $F'(v) = wvx^{-1}$ then we have $F\psi = \psi F'$. From this it follows that we have bijections

$$ (\mathcal{L}\backslash\mathcal{G})^{F'} \xrightarrow{\psi} \boldsymbol{C}_{\mathcal{L}_1\backslash\mathcal{G}}(F) \longrightarrow L_1\backslash G, $$

where $\boldsymbol{C}_{\mathcal{L}_1\backslash\mathcal{G}}(F) = \{\mathcal{L}_1 u \in \mathcal{L}_1\backslash\mathcal{G} \mid \mathcal{L}_1 u F = \mathcal{L}_1 F u\}$, and $(\mathcal{L}\backslash\mathcal{G})^{F'}$ denotes the cosets fixed by F'. The second bijection is a simple consequence of the Lang–Steinberg theorem. If $z \in \mathcal{G}$ then ${}^{\psi(z)}\mathcal{T}_1 \leqslant \mathcal{L}_1$ if and only if ${}^z\mathcal{T} \leqslant \mathcal{L}$ and $F'(\mathcal{L}z) = \mathcal{L}z$ if and only if ${}^z x \in \mathcal{L}w$. It is clear that the combination of these two conditions is equivalent to the condition ${}^z(\mathcal{T}x) \subseteq \mathcal{L}w$.

Finally, conjugating the expression above we get

$$ {}^*R^{\mathcal{G}F}_{\mathcal{L}w} \circ R^{\mathcal{G}}_{\mathcal{T}x} = \sum_{\substack{z \in \mathcal{L}\backslash\mathcal{G} \\ {}^z(\mathcal{T}x) \leqslant \mathcal{L}w}} (\iota_{h_w})^* \circ R^{\mathcal{L}_1}_{{}^{\psi(z)}\mathcal{T}_1} \circ (\iota_{\psi(z)}^{-1})^* \circ (\iota_{h_x}^{-1})^*. $$

It suffices to show that $(\iota_{h_w})^* \circ R^{\mathcal{L}_1}_{\psi(z)\mathcal{T}_1} = R^{\mathcal{L}}_{{}^z\mathcal{T}} \circ (\iota_{h_w})^*$ when $F'(z) = z$, where $R^{\mathcal{L}}_{{}^z\mathcal{T}} = R^{\mathcal{L}w}_{({}^z\mathcal{T})w}$ is defined with respect to the Frobenius w on $\mathcal{L}$. However, the arguments to prove this are identical to those used to prove [Digne and Michel 2020, Proposition 11.3.10]; see also the arguments by Bonnafé in [Navarro et al. 2008]. We omit the details. □

From now on we assume the underlying group scheme is GL^{ϵ}_n so that

$$\mathcal{G} = \mathcal{G}_n = \mathrm{GL}_n(\overline{\mathbb{F}}_q).$$

We will assume $\mathcal{T} \leqslant \mathcal{G}$ is the diagonal maximal torus and we denote by $W := \boldsymbol{N}_{\mathcal{G}}(\mathcal{T})/\mathcal{T} \cong \mathsf{S}_n$ the corresponding Weyl group. The quotient $\boldsymbol{N}_{\mathcal{G}\langle F\rangle}(\mathcal{T})/\mathcal{T} = W\langle F\rangle$ is isomorphic to the semidirect product $W \rtimes \langle F\rangle$, where we identify F with its natural image $\mathcal{T}F$. The coset $WF \subseteq W\langle F\rangle$ is, by definition, the set of toral subcosets $\mathcal{T}n$ where $n \in \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T})$.

We wish to reinterpret Lemma 2.7 in the language of Lusztig's almost characters. Recall that GL^{ϵ}_n is self-dual. If $(w, s) \in WF \times \mathcal{T}$ is a pair such that $ws = sw$ then we set

$$R^{\mathcal{G}F}_w(s) := R^{\mathcal{G}F}_x(\theta),$$

where $x \in WF$ and $\theta \in \mathrm{Irr}(\mathscr{C}(w))$ correspond to (w, s) under a bijection obtained as in [Digne and Michel 2020, Proposition 11.1.16] from duality.

If $s \in \mathcal{T}$ and $\boldsymbol{C}_{WF}(s) \neq \varnothing$ then, following Lusztig [1984, §8.4], we define

$$\mathcal{R}^{\mathcal{G}F}_s : \mathrm{cf}(\boldsymbol{C}_{WF}(s)) \to \mathrm{cf}(\mathscr{C}(\mathcal{G}F)),\ \ f \mapsto \frac{1}{|\boldsymbol{C}_W(s)|} \sum_{w \in \boldsymbol{C}_{WF}(s)} f(w) R^{\mathcal{G}F}_w(s) \in \mathrm{cf}(\mathscr{C}(\mathcal{G}F)),$$

where, as defined above, $\boldsymbol{C}_X(s) = \{w \in X \mid ws = sw\}$ for any subset $X \subseteq W\langle F\rangle$.

Corollary 2.8. *Assume $\mathcal{T} \leqslant \mathcal{L} \leqslant \mathcal{G}$ is a Levi subgroup and $w \in \boldsymbol{N}_{WF}(\mathcal{L})$. Then for any $s \in \mathcal{T}$ with $\boldsymbol{C}_{WF}(s) \neq \varnothing$ we have*

$${}^*R^{\mathcal{G}F}_{\mathcal{L}w} \circ \mathcal{R}^{\mathcal{G}F}_s = \sum_{\substack{z \in H\backslash W/\boldsymbol{C}_W(s) \\ \boldsymbol{C}_{Hw}({}^z s) \neq \varnothing}} \mathcal{R}^{\mathcal{L}w}_{{}^z s} \circ \mathrm{Res}^{\boldsymbol{C}_{WF}({}^z s)}_{\boldsymbol{C}_{Hw}({}^z s)} \circ (\iota_z^{-1})^*$$

where $H = \boldsymbol{N}_{\mathcal{L}}(\mathcal{T})/\mathcal{T}$ is the Weyl group of $\mathcal{L}$.

Proof. By linearity it is enough to check both sides agree when evaluated at $\pi_x^{\boldsymbol{C}_{WF}(s)}$ for some $x \in \boldsymbol{C}_{WF}(s)$. But in that case $\mathcal{R}^{\mathcal{G}F}_s(\pi_x^{\boldsymbol{C}_{WF}(s)}) = R^{\mathcal{G}F}_x(s)$. Assume (x, s) corresponds to (y, θ). Evaluating at θ we have by Lemma 2.4 that

$${}^*R^{\mathcal{G}F}_{\mathcal{L}w}(R^{\mathcal{G}F}_y(\theta)) = \sum_{\substack{z \in \mathcal{L}\backslash\mathcal{G} \\ {}^z y \subseteq \mathcal{L}w}} R^{\mathcal{L}w}_{{}^z y}({}^z\theta).$$

If ${}^z y \subseteq \mathcal{L}w$ then ${}^z\mathcal{T} \leqslant \mathcal{L}$ so ${}^{lz}\mathcal{T} = \mathcal{T}$ for some $l \in \mathcal{L}$. Therefore, we can take the sum over cosets $\boldsymbol{N}_{\mathcal{L}}(\mathcal{T})\backslash\boldsymbol{N}_{\mathcal{G}}(\mathcal{T})$ or similarly $H\backslash W$. As ${}^z y \subseteq \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T})$ the condition

${}^{z}y \subseteq \mathcal{L}w$ is equivalent to

$$ {}^{z}y \subseteq \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T}) \cap \mathcal{L}w = \boldsymbol{N}_{\mathcal{L}}(\mathcal{T})w $$

which in turn is equivalent to ${}^{z}y \in Hw$. Breaking the sum in Lemma 2.7 along double cosets, as in the proof of Lemma 2.4, gives

$$ {}^{*}R^{\mathcal{G}F}_{\mathcal{L}w}(R^{\mathcal{G}F}_{y}(\theta)) = \sum_{z \in H\backslash W/\boldsymbol{C}_W(s)} \sum_{\substack{c \in \boldsymbol{C}_H({}^{z}s)/\boldsymbol{C}_W({}^{z}s) \\ {}^{cz}y \in Hw}} R^{\mathcal{L}w}_{{}^{cz}y}({}^{cz}\theta). $$

Picking a different double coset representative we can assume that ${}^{z}y \in Hw$. We claim that $\boldsymbol{C}_{Hw}({}^{z}s) = \boldsymbol{C}_H({}^{z}s){}^{z}y$. Certainly ${}^{z}x \in \boldsymbol{C}_{Hw}({}^{z}s)$ by assumption. Now if $d \in \boldsymbol{C}_W({}^{z}s)$ then $d({}^{z}x) \in Hw$ if and only if $d \in \boldsymbol{C}_H({}^{z}s) = H \cap \boldsymbol{C}_W({}^{z}s)$. The statement now follows from Lemma 2.4. $\square$

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. By assumption, χ has true level j. Embed the maximal diagonal torus $\mathcal{T}$ in the natural F-stable Levi subgroup $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_1 \cong \mathcal{G}_j$ and $\mathcal{M}_2 \cong \mathcal{G}_{n-j}$. Note that F stabilizes $\mathcal{M}_1$ and $\mathcal{M}_2$. We will identify $\mathcal{M}\langle F\rangle$ with a subgroup of $\mathcal{M}_1\langle F\rangle \times \mathcal{M}_2\langle F\rangle$, where we again denote by F its restriction to $\mathcal{M}_i$.

By [Guralnick et al. 2020, Theorem 3.9] our assumption on the level of χ implies that χ is a constituent of

$$ R^{\mathcal{G}F}_{\mathcal{M}F}(\alpha \boxtimes 1) $$

for some $\alpha \in \operatorname{Irr}(\mathscr{C}(\mathcal{M}_1 F))$. Note that

$$ \boldsymbol{C}_{\mathcal{M}}(F) = \boldsymbol{C}_{\mathcal{M}_1}(F)\boldsymbol{C}_{\mathcal{M}_2}(F) \cong \mathrm{GL}^{\epsilon}_{j}(q) \times \mathrm{GL}^{\epsilon}_{n-j}(q). $$

If $W_i \leqslant W$ is the subgroup $\boldsymbol{N}_{\mathcal{M}_i}(\mathcal{T})/\mathcal{T}$ then the subgroup $W_1W_2 \leqslant W$ is a direct product with $W_1 \cong \mathsf{S}_j$ and $W_2 \cong \mathsf{S}_{n-j}$.

By [Digne and Michel 2020, Theorem 11.7.3] we have $\alpha = \pm\mathcal{R}^{\mathcal{M}_1 F}_{s}(\tilde{\phi})$ for some $s \in \mathcal{M}_1 \cap \mathcal{T}$, with $\boldsymbol{C}_{W_1 F}(s) \neq \varnothing$, and some irreducible character $\tilde{\phi} \in \operatorname{Irr}(\boldsymbol{C}_{W_1 F}(s))$ afforded by a representation over $\mathbb{Q}$, see [Lusztig 1984, Proposition 3.2]. Note that $\boldsymbol{C}_{W_1W_2}(s) = \boldsymbol{C}_{W_1}(s)W_2$ is a reflection group and we have

$$ \boldsymbol{C}_{W_1W_2F}(s) = \boldsymbol{C}_{W_1F}(s)W_2. $$

It is known, see [Digne and Michel 2020, Proposition 11.6.6], that

$$ R^{\mathcal{G}F}_{\mathcal{M}F}(\alpha \boxtimes 1_{\mathcal{M}_2 F}) = \pm\mathcal{R}^{\mathcal{G}F}_{s}(\operatorname{Ind}^{\boldsymbol{C}_{WF}(s)}_{\boldsymbol{C}_{W_1W_2F}(s)}(\tilde{\phi} \boxtimes 1)) $$

from which it follows that $\chi = \pm\mathcal{R}^{\mathcal{G}F}_{s}(\tilde{\psi})$ for some irreducible constituent $\tilde{\psi} \in \operatorname{Irr}(\boldsymbol{C}_{WF}(s))$ of the induced function $\operatorname{Ind}^{\boldsymbol{C}_{WF}(s)}_{\boldsymbol{C}_{W_1W_2F}(s)}(\tilde{\phi} \boxtimes 1)$.

We denote again by $\tilde{\phi}$ and $\tilde{\psi}$ irreducible characters of $\boldsymbol{C}_{W_1}(s)\langle F\rangle$ and $\boldsymbol{C}_W(s)\langle F\rangle$ respectively, yielding $\tilde{\phi}$ and $\tilde{\psi}$ upon restriction to the respective cosets. By [Taylor and Tiep 2020, Lemma 4.10],

$$|\langle \mathrm{Ind}_{\boldsymbol{C}_{W_1W_2F}(s)}^{\boldsymbol{C}_{WF}(s)}(\tilde{\phi}\boxtimes 1),\, \tilde{\psi}\rangle| \leqslant \langle \mathrm{Ind}_{\boldsymbol{C}_{W_1W_2}(s)}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1),\, \psi\rangle.$$

In particular, ψ is a constituent of $\mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)$.

Following the proof of Corollary 2.2 it suffices to show that

$$\sum_{\eta\in\mathrm{Irr}(L_1)} |\langle \eta, {}^*R_{\mathcal{L}_1F}^{\mathcal{G}F}(\chi)\rangle| \leqslant \langle {}^*R_{\mathcal{L}_1F}^{\mathcal{G}F}(\chi), {}^*R_{\mathcal{L}_1F}^{\mathcal{G}F}(\chi)\rangle \leqslant n^{3j}. \tag{2-10}$$

A straightforward argument shows that we may find a Levi subgroup $\mathcal{T}\leqslant\mathcal{L}\leqslant\mathcal{G}$, an element $w\in \boldsymbol{N}_{\mathcal{G}F}(\mathcal{T})$, and an element $h\in\mathcal{G}$ such that ${}^h(\mathcal{L}w)=\mathcal{L}_1F$, see [Digne and Michel 2020, Proposition 11.4.1]. With this we need only bound

$$\langle {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\chi), {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\chi)\rangle = \langle {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\mathcal{R}_s^{\mathcal{G}}(\tilde{\psi})), {}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\mathcal{R}_s^{\mathcal{G}}(\tilde{\psi}))\rangle.$$

By Corollary 2.8,

$${}^*R_{\mathcal{L}w}^{\mathcal{G}F}(\mathcal{R}_s^{\mathcal{G}}(\tilde{\psi})) = \sum_{\substack{z\in H\backslash W/\boldsymbol{C}_W(s)\\ \boldsymbol{C}_{Hw}({}^zs)\neq\varnothing}} \mathcal{R}_{{}^zs}^{\mathcal{L}w}(\mathrm{Res}_{\boldsymbol{C}_{Hw}({}^zs)}^{\boldsymbol{C}_{WF}({}^zs)}({}^z\tilde{\psi})),$$

where $H=\boldsymbol{N}_{\mathcal{L}}(\mathcal{T})/\mathcal{T}$ is the Weyl group of $\mathcal{L}$. The number of double cosets appearing in this sum is bounded above by

$$|H\backslash W/\boldsymbol{C}_W(s)| \leqslant |W/\boldsymbol{C}_W(s)| \leqslant |W/W_2| = |\mathsf{S}_n/\mathsf{S}_{n-j}| \leqslant n^j$$

because $W_2\leqslant \boldsymbol{C}_W(s)$.

Now, by the disjointness of Deligne–Lusztig characters, see [Digne and Michel 2020, Proposition 11.3.2], the summands are pairwise orthogonal because each ${}^zs$ lies in a distinct $\mathcal{L}$-conjugacy class. So, using Lemma 2.5 it suffices to bound

$$|\langle \mathrm{Res}_{\boldsymbol{C}_{Hw}({}^zs)}^{\boldsymbol{C}_{WF}({}^zs)}(\tilde{\psi}), \mathrm{Res}_{\boldsymbol{C}_{Hw}({}^zs)}^{\boldsymbol{C}_{WF}({}^zs)}(\tilde{\psi})\rangle| \leqslant \langle \mathrm{Res}_{\boldsymbol{C}_H({}^zs)}^{\boldsymbol{C}_W({}^zs)}(\psi), \mathrm{Res}_{\boldsymbol{C}_H({}^zs)}^{\boldsymbol{C}_W({}^zs)}(\psi)\rangle \leqslant \psi(1)^2.$$

But we know ψ is a constituent of $\mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)$ so

$$\psi(1)^2 \leqslant \mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)(1)^2 \leqslant \mathrm{Ind}_{\boldsymbol{C}_{W_1}(s)W_2}^{\boldsymbol{C}_W(s)}(\phi\boxtimes 1)(1)^2 \leqslant |W/W_2|^2 \leqslant n^{2j}. \qquad \square$$

2.3. ***An asymptotic version of*** [Bezrukavnikov et al. 2018, Theorem 1.1] ***and*** [Taylor and Tiep 2020, Theorem 1.9]**.** Now we can prove the main result of the section:

Theorem 2.9. *For any $\epsilon>0$, there are some explicit positive constants $N_0=N_0(\epsilon)$ and $q_0=q_0(\epsilon)$ such that the following statement holds for all integers $n\geq N_0$ and all prime powers $q\geq q_0$. Let $\mathcal{G}=\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and let $F:\mathcal{G}\to\mathcal{G}$ be a Frobenius* *endomorphism so that* $\mathcal{G}^F \in \{\mathrm{GL}_n(q), \mathrm{GU}_n(q)\}$. *Suppose we are in one of the following two cases.*

(i) $G := \mathcal{G}^F$ *and* $g \in G$ *is any element such that* $\boldsymbol{C}_{\mathcal{G}}(g)$ *is contained in a proper* F*-stable Levi subgroup* $\mathcal{L}$ *of* $\mathcal{G}$ *and* $L := \mathcal{L}^F$.

(ii) $G := [\mathcal{G}, \mathcal{G}]^F \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q)\}$, $g \in G$, *and either*

  (a) $\boldsymbol{C}_{[\mathcal{G},\mathcal{G}]}(g)$ *is contained in a proper split Levi subgroup* $\mathcal{L}$ *of* $[\mathcal{G}, \mathcal{G}]$ *and* $L := \mathcal{L}^F$, *or*

  (b) g *is noncentral semisimple with* $L := \boldsymbol{C}_G(g)$.

Then, for all $\chi \in \mathrm{Irr}(G)$,

$$|\chi(g)| \le \chi(1)^{\alpha(L)+\epsilon} \tag{2-11}$$

Proof. Note that (2-11) is obvious if $\alpha := \alpha(L) \ge 1 - \epsilon$. So in what follows we will assume

$$\alpha + \epsilon < 1,$$

in particular $0 < \epsilon < 1$.

(A) First we prove (2-11) in the cases of (i) and (ii)(a). Note that the upper bound on $|\chi(g)|$ in Theorem 2.3 is obtained by combining (2-1), (2-2), and (2-10). If $\mathcal{L}$ is split, $\phi := {}^*R^{\mathcal{G}^F}_{\mathcal{L}_1{}^F}(\chi)$ is a true character of $\mathcal{L}_1^F$ (in the notation of the proof of Theorem 2.3), hence this upper bound is actually an upper bound on the degree of ϕ. Arguing as in part (ii) of the proof of [Bezrukavnikov et al. 2018, Theorem 1.1], it therefore suffices to prove (2-11) for

$$G \in \{\mathrm{GL}_n(q), \mathrm{GU}_n(q)\}.$$

Let $j = \mathfrak{l}(\chi)$ be the *level* of χ in the sense of [Guralnick et al. 2020, Definition 1(ii)]. This means that multiplying χ by a suitable linear character of G, we may assume that $\mathfrak{l}^*(\chi) = j$. Applying Theorem 2.3, it suffices to prove

$$n^{3j}\left(\frac{q+1}{q-1}\right)^{D/2} \le \chi(1)^{\epsilon}. \tag{2-12}$$

Note that the degree of any irreducible character of G is a monic polynomial in variable q with integer coefficients, in fact a product of a power of q and cyclotomic polynomials in q. Writing $D = \dim \upsilon^{\mathcal{G}}$, with $\upsilon^{\mathcal{G}} = \mathcal{O}^*_\chi$ being the wave front set of χ, the degree of this polynomial is $\frac{1}{2}D$ [Bezrukavnikov et al. 2018, Equation 2.1]. Since $\Phi_k(q) \ge (q-1)^{\deg \Phi_k}$ for any cyclotomic polynomial Φ_k, we therefore have

$$\chi(1) \ge (q-1)^{D/2}.$$

Choosing $q_0 = q_0(\epsilon) \ge 3$ such that

$$\frac{q_0+1}{q_0-1} \le (q_0-1)^{\epsilon/3},$$

it remains to prove

$$n^{3j} \le \chi(1)^{2\epsilon/3} \tag{2-13}$$

for $q \ge q_0$ and $n \ge N_0$.

Choosing $N_0 \ge 4$, we have $\frac{1}{4}n^2 - 2 > \frac{1}{16}n^2$ for $n \ge N_0$, and so, when $j > \frac{1}{2}n$, we have

$$\chi(1) > q^{n^2/16} > q^{n^2\epsilon/16}$$

by [Guralnick et al. 2020, Theorem 1.2(ii)]. Next, if $\frac{1}{4}n < j \le \frac{1}{2}n$, then $j(n-j) > \frac{1}{8}n^2$, and so

$$\chi(1) > q^{n^2/8}$$

by [Guralnick et al. 2020, Theorem 1.2(i)]. If $0 \le j \le \frac{1}{4}n$, then $j(n-j) \ge \frac{3}{4}nj$, and so

$$\chi(1) \ge q^{3nj/4} \tag{2-14}$$

again by [Guralnick et al. 2020, Theorem 1.2(i)].

First we work in the setting

$$\tfrac{1}{12}n\epsilon \le j \le n.$$

Then (2-14) and the above arguments show that

$$\chi(1) \ge q^{n^2\epsilon/16}.$$

Choose $N_0 \ge 4$ such that

$$n \le q_0^{n\epsilon^2/72} \tag{2-15}$$

for all $n \ge N_0$. Then for $q \ge q_0$ we now have

$$n^{3j} \le n^{3n} \le q_0^{n^2\epsilon^2/24} \le \chi(1)^{2\epsilon/3},$$

yielding (2-13) in this case.

Assume now that $j \le \frac{1}{12}n\epsilon \le \frac{1}{12}n$. Then for $n \ge N_0$ and $q \ge q_0$ we now have by (2-14) and (2-15) that

$$n^{3j} \le q_0^{n\epsilon^2 j/24} < q^{n\epsilon j/2} \le \chi(1)^{2\epsilon/3},$$

proving (2-13) in this case as well.

(B) Now we handle the case (ii)(b), embedding G in $\tilde{G} := \mathcal{G}^F$. Letting $\tilde{L} := \boldsymbol{C}_{\tilde{G}}(g)$, note that $\tilde{G} = G\tilde{L}$ and $g \in \tilde{L}$. Letting $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$ lie above χ, by Clifford's theorem we have

$$\tilde{\chi}|_G = \sum_{i=1}^{t} \chi^{x_i},$$

where $x_1, \dots, x_t$ can be chosen from $\tilde{L}$. Since every x_i centralizes g, we have

$$\tilde{\chi}(1) = t\chi(1), \quad \tilde{\chi}(g) = t\chi(g).$$

Let $\mathcal{L} = \boldsymbol{C}_{[\mathcal{G},\mathcal{G}]}(g)$ and $\tilde{\mathcal{L}} = \boldsymbol{C}_{\mathcal{G}}(g)$, so that $L = \mathcal{L}^F$ and $\tilde{L} = \tilde{\mathcal{L}}^F$. We have $\tilde{\mathcal{L}} = \mathcal{L}\boldsymbol{Z}(\mathcal{G})$ because $\mathcal{G} = [\mathcal{G}, \mathcal{G}]\boldsymbol{Z}(\mathcal{G})$, and so every unipotent element of $\tilde{L}$ is contained in L. Moreover, if $u \in \tilde{L}$ is unipotent then $u^{\tilde{\mathcal{L}}} = u^{\mathcal{L}}$ and $u^{\mathcal{G}} = u^{[\mathcal{G},\mathcal{G}]}$, whence $\alpha(\tilde{L}) = \alpha(L)$. By the case (i) proved in (A), $|\tilde{\chi}(g)| \le \tilde{\chi}(1)^{\alpha+\epsilon}$. As $\alpha + \epsilon < 1$, it follows that $|\chi(g)| \le \chi(1)^{\alpha+\epsilon}$. □

3. Some numerical estimates

This section is devoted to numerical estimates which allow us to determine a finite list of possible exceptions to Conjecture 1.

Let

$$f_{a,x}(\delta) := \min\Bigl(\Bigl(\frac{1}{a} + \delta\Bigr)x,\ \frac{1}{2}\Bigl(\frac{1}{a} + \frac{a}{a-1}\delta^2\Bigr)\Bigr).$$

The main goal of this section is to give an explicit finite list of possible exceptions to the rule that if $2 \le a_1 \le a_2 \le \cdots \le a_r$ are integers such that $\sum_{i=1}^{r} \frac{1}{a_i} < r - 2$, then

$$\sum_{i=1}^{r} f_{a_i,x}(\delta_i) < x + \sum_{i=1}^{r} \frac{a_i\delta_i^2}{a_i - 1}$$

when $x \in \bigl[0, \frac{1}{2}\bigr]$ and $\delta_i \in \bigl[0, \frac{a_i-1}{a_i}\bigr]$ for all i. We will see that when the rule holds, Conjecture 1 holds for the corresponding genus 0 Fuchsian group. There is only one exception with $r \ge 4$, and there are thirty-one with $r = 3$.

Proposition 3.1. *For all $a \ge 2$, $x \in \bigl(0, \frac{1}{2}\bigr]$, and $\delta \in \bigl[0, 1 - \frac{1}{a}\bigr]$, the function*

$$\frac{f_{a,x}(\delta) - \frac{a\delta^2}{a-1}}{x}$$

is bounded above by $\max\bigl(\frac{x}{a}, G_{a,x}, H_{a,x}\bigr)$, *where*

$$G_{a,x} = \begin{cases} \dfrac{(a-1)x+4}{4a} & \text{if } x \le \dfrac{2\sqrt{3a+1}-4}{3(a-1)}, \\ -\infty & \text{if } x > \dfrac{2\sqrt{3a+1}-4}{3(a-1)}, \end{cases}$$

and

$$H_{a,x} = \begin{cases} \dfrac{\sqrt{(a-1)^2x^2+2(a-1)x-(a-1)}}{a} - \dfrac{(a-1)x}{a} + \dfrac{1-x}{ax} & \text{if } (a-1)x^2+ax \ge 1, \\ -\infty & \text{if } (a-1)x^2+ax < 1. \end{cases}$$

In particular,

$$f_{a,x}(\delta) < \frac{2x}{\sqrt{a}} + \frac{a\delta^2}{a-1}. \tag{3-1}$$

Proof. Let

$$g_{a,x}(\delta) := \Big(\frac{1}{a}+\delta\Big)x - \frac{a\delta^2}{a-1}, \quad h_a(\delta) := \frac{1}{2a} - \frac{a\delta^2}{2(a-1)},$$

so

$$f_{a,x}(\delta) - \frac{a\delta^2}{a-1} = \min(g_{a,x}(\delta), h_a(\delta)).$$

For each fixed integer $a \geq 2$, we wish to determine as a function of $x \in \big(0, \frac{1}{2}\big]$, the (unique) element $\delta_0(x) \in \big[0, \frac{a-1}{a}\big]$ for which $\min(g_{a,x}(\delta), h_a(\delta))$ achieves its maximum as a function of δ.

We note first that as functions of δ, $g_{a,x}(\delta)$ and $h_a(\delta)$ are strictly concave, and $h_a(\delta)$ is decreasing on $[0, \infty)$. Therefore, $\delta_0(x)$ must either be the unique critical point $\frac{(a-1)x}{2a}$ of $g_{a,x}(\delta)$, the minimum solution of $g_{a,x}(\delta) = h_a(\delta)$ in the interval $\big[0, \frac{a-1}{a}\big]$, or one of the endpoints 0 and $\frac{a-1}{a}$ of the interval. For the endpoints, we have

$$\min(g_{a,x}(0), h_a(0)) \leq g_{a,x}(0) = \frac{x}{a},$$

and

$$\min\Big(g_{a,x}\Big(\frac{a-1}{a}\Big), h_a\Big(\frac{a-1}{a}\Big)\Big) \leq h_a\Big(\frac{a-1}{a}\Big) = \frac{1}{a} - \frac{1}{2} \leq 0.$$

If the maximum occurs at $\frac{(a-1)x}{2a}$, it must be

$$g_{a,x}\Big(\frac{(a-1)x}{2a}\Big) = \frac{(a-1)x^2+4x}{4a},$$

and this quantity must be less than or equal to

$$h_a\Big(\frac{(a-1)x}{2a}\Big) = \frac{4-(a-1)x^2}{8a},$$

so

$$x \leq \frac{2\sqrt{3a+1}-4}{3(a-1)}.$$

Thus, for all x,

$$G_{a,x} \leq \frac{1}{a} + \frac{\sqrt{3a+1}-2}{6a} \leq \frac{1}{a} + \frac{1}{\sqrt{2a}} < \frac{2}{\sqrt{a}}. \tag{3-2}$$

The graphs of $g_{a,x}(\delta)$ and $h_a(\delta)$ intersect only if

$$(a-1)x^2 + 2x \geq 1, \tag{3-3}$$

in which case the smaller δ-value satisfying $g_{a,x}(\delta) = h_a(\delta)$ is

$$\delta = \frac{(a-1)x - \sqrt{(a-1)^2x^2 + 2(a-1)x - (a-1)}}{a}.$$

If this is $\delta_0(x)$, we have

$$H_{a,x} = \frac{\sqrt{(a-1)^2x^2+2(a-1)x-(a-1)}\,x}{a} - \frac{(a-1)x^2}{a} + \frac{1-x}{a} \leq \frac{1-x}{a} \leq \frac{1}{a}. \tag{3-4}$$

By (3-3), $x > \frac{1}{2\sqrt{a}}$, so (3-4) implies

$$\frac{g_{a,x}(\delta_0(x))}{x} < \frac{2}{\sqrt{a}}.$$

Together with (3-2), this implies the proposition. □

Proposition 3.2. *Let $r \geq 3$ be an integer, $a_1 \leq a_2 \leq \cdots \leq a_r$ be integers greater or equal to 2, and δ_i be nonnegative numbers with $\frac{1}{a_i} + \delta_i \leq 1$ for all i. We assume that the tuple $a_1 \cdots a_r$ is not in the following list*:

(i) $23c,\ 7 \leq c \leq 24$.

(ii) $24c,\ 5 \leq c \leq 9$.

(iii) $25c,\ 5 \leq c \leq 7$.

(iv) 266.

(v) $33c,\ 4 \leq c \leq 6$.

(vi) 344.

(vii) 2223.

Then for all $x \in \left(0, \frac{1}{2}\right]$,

$$\sum_{i=1}^{r} f_{a_i,x}(\delta_i) < -2rx\epsilon + (r-2)x + \sum_{i=1}^{r} \frac{a_i\delta_i^2}{a_i - 1},$$

where ϵ is a positive constant which does not depend on x, r, the a_i, or the δ_i.

Proof. By (3-1) for $a \geq 100$ and machine computation for $2 \leq a < 100$,

$$\text{(3-5)}\qquad \frac{f_{a,x}(\delta) - \frac{a\delta^2}{a-1}}{x} \leq \begin{cases} 0.555 & \text{if } a = 2, \\ 0.399 & \text{if } a = 3, \\ 0.318 & \text{if } 4 \leq a < 100, \\ 0.2 & \text{if } 100 \leq a < 10000, \\ 0.02 & \text{if } 10000 \leq a. \end{cases}$$

Therefore, if $r \geq 5$,

$$r - 2 - \sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \frac{a_i\delta_i^2}{a_i-1}}{x} > r - 2 - 0.56r \geq (2r)0.02.$$

If $r = 4$ and $a_4 \geq 4$, then

$$r - 2 - \sum_{i=1}^{r} \frac{f_{a_i,x}(\delta_i) - \frac{a_i\delta_i^2}{a_i-1}}{x} > 2 - 0.56 \cdot 3 - 0.318 = 0.002,$$

while if $r=4$ and $a_3\ge 3$, then

$$r-2-\sum_{i=1}^{r}\frac{f_{a_i,x}(\delta_i)-\dfrac{a_i\delta_i^2}{a_i-1}}{x}>2-0.56\cdot 2-0.4\cdot 2=0.08.$$

The only remaining possibility for $r=4$ is 2223.

For $r=3$, we may assume $a_2\ge 3$, so if $a_3\ge 10000$,

$$r-2-\sum_{i=1}^{r}\frac{f_{a_i,x}(\delta_i)-\dfrac{a_i\delta_i^2}{a_i-1}}{x}>1-0.56-0.4-0.02=0.02.$$

The triples with $a_3<10000$ can be handled exhaustively by machine, by partitioning the x-interval $\left[0,\frac{1}{2}\right]$ into subintervals on which $G_{a,x}$ and $H_{a,x}$ are bounded above. □

Lemma 3.3. *Let g be a positive integer and r a nonnegative integer such that if $g=1$, then $r>0$. Let $a_1\le a_2\le\cdots\le a_r$ be a (possibly empty) sequence of integers at least 2. Then for all $x\in\left[0,\frac{1}{2}\right]$,*

$$\sum_{i=1}^{r} f_{a_i,x}(\delta_i)<-0.22(r+1)x+(2g+r-2)x+\sum_{i=1}^{r}\frac{a_i\delta_i^2}{a_i-1}.$$

Proof. If $g=1$ and $r\ge 1$, (3-5) implies

$$\sum_{i=1}^{r}\frac{f_{a_i,x}(\delta_i)-\dfrac{a_i\delta_i^2}{a_i-1}}{x}<0.56r\le -0.44r+r\le -0.22(r+1)+(2g+r-2).$$

If $g\ge 2$ and $r\ge 0$,

$$\begin{aligned}\sum_{i=1}^{r}\frac{f_{a_i,x}(\delta_i)-\dfrac{a_i\delta_i^2}{a_i-1}}{x}&\le 0.56r\\ &<-0.22(r+1)+(2+r)\le -0.22(r+1)+(2g+r-2).\end{aligned}$$ □

4. Asymptotics of $j_{q,n}(a)$

Let t be an element of $G_n=\mathrm{GL}_n(q)$ of order a. We assume q is prime to a. Let $\zeta=\zeta_a$ be a primitive a-th root of unity in $\overline{\mathbb{F}}_q$, so $z^a-1=0$ has distinct roots $\zeta,\zeta^2,\ldots,\zeta^a=1$ in $\overline{\mathbb{F}}_q$. Let m_i denote the multiplicity of ζ^i as an eigenvalue of t. We write $i\sim j$ if ζ^i and ζ^j have the same Frobenius orbit. Then:

(1) $m_i\in\mathbb{Z}$ for all i.

(2) $m_1+\cdots+m_a=n$.

(3) $m_i=m_j$ whenever $i\sim j$.

(4) $m_i\ge 0$ for all i.

The element t is determined up to conjugacy in $\mathrm{GL}_n(q)$ by the vector $(m_1, \dots, m_a)$. For given n, the vector is determined by $m_1, \dots, m_{a-1}$, so the number of possibilities is $O(n^{a-1})$.

Let S denote the subset of $\{1, \dots, a\}$ consisting of the smallest element in each Frobenius orbit, and let l_s be the size of the orbit of s. The centralizer of t in $\mathrm{GL}_n(q)$ can be written $\prod_{s\in S} \mathrm{GL}_{m_s}(q^{l_s})$, so the conjugacy class $C = t^{G_n}$ satisfies

$$|C| = \frac{q^{n^2}\prod_{j=1}^{n}(1-q^{-j})}{\prod_{s\in S}\bigl(q^{l_s m_s^2}\prod_{j=1}^{m_s}(1-q^{-l_s j})\bigr)} \tag{4-1}$$
$$= \frac{q^{n^2-\sum_{i=1}^{a} m_i^2}\, q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{i l_s \log q}{2\pi}\bigr)} \, \frac{\prod_{s\in S}\prod_{j=m_s+1}^{\infty}(1-q^{-l_s j})}{\prod_{j=n+1}^{\infty}(1-q^{-j})},$$

where $\eta(z)$ is the Dedekind eta-function. The second multiplicand on the right-hand side can be bounded above in terms of a, and it approaches 1 as $\inf_i m_i$ goes to ∞.

Writing

$$n^2 - \sum_{i=1}^{a} m_i^2 = n^2\Bigl(1-\frac{1}{a}\Bigr) - \sum_{i=1}^{a}\Bigl(\frac{n}{a}-m_i\Bigr)^2, \tag{4-2}$$

we see that if $m_i \le n/2a$, then

$$|C| = O(q^{(1-1/a-1/(4a^2))n^2}),$$

so the sum of $|C|$ over all conjugacy classes with $\inf_i m_i \le n/2a$ is $o(q^{(1-1/a)n^2})$.

We define $j_{q,n,k}(a)$ to be the number of elements $t \in \mathrm{GL}_n(q)$ with $t^a = 1$ and $\det(a) = \zeta^k$. Consider the subset of $\mathbb{Z}^a$ satisfying conditions (1)–(3) and the congruence condition

$$\sum_{i=1}^{r} i m_i \equiv k \pmod{a}, \tag{4-3}$$

which is equivalent to the condition $\det(t) = \zeta^k$. This is a coset $\lambda_n + \Lambda$, where Λ is a subgroup of $\mathbb{Z}^a$ which does not depend on n and $\lambda_n \in \mathbb{Z}^a$ has coordinate sum n. Moreover, adding 2 to each m_i and $2a$ to n preserves the sets satisfying conditions (1)–(3) and (4-3), so

$$\lambda_{n+2a} = \lambda_n + (2, 2, \dots, 2).$$

Thus,

$$\lambda'_n := \lambda_n - \Bigl(\frac{n}{a}, \frac{n}{a}, \dots, \frac{n}{a}\Bigr)$$

is periodic in n with period $2A$ and has coordinate sum 0.

By (4-1) and (4-2),

$$\begin{aligned}
j_{q,n,k}(a) &= \sum_{(m_1,\ldots,m_a)\in(\lambda_n+\Lambda)\cap\mathbb{N}^a} \frac{q^{n^2}\prod_{j=1}^{n}(1-q^{-j})}{\prod_{s\in S}\bigl(q^{l_s m_s^2}\prod_{j=1}^{m_s}(1-q^{-l_s j})\bigr)} \\
&= \sum_{(m_1,\ldots,m_a)\in(\lambda_n+\Lambda)\cap\mathbb{N}^a} \frac{q^{n^2-\sum_{i=1}^a m_i^2} q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} + o(q^{(1-1/a)n^2}) \\
&= \frac{q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} \sum_{(m_1,\ldots,m_a)\in\lambda_n+\Lambda} q^{n^2-\sum_{i=1}^a m_i^2} + o(q^{(1-1/a)n^2}) \\
&= \frac{q^{\frac{1}{24}(1-a)}\eta\bigl(\frac{i\log q}{2\pi}\bigr)}{\prod_{s\in S}\eta\bigl(\frac{il_s\log q}{2\pi}\bigr)} q^{(1-1/a)n^2} \sum_{\lambda'\in\lambda'_n+\Lambda} q^{-\lambda'\cdot\lambda'} + o(q^{(1-1/a)n^2}),
\end{aligned}$$

where the implicit constant on the right-hand side does not depend on q. Defining

$$\theta_v(z) := \sum_{\lambda\in v+\Lambda} e^{2\pi i(\lambda\cdot\lambda)z} \quad \text{and} \quad f_n(z) := \frac{\eta(z)\theta_{\lambda'_n}(z)}{\prod_{s\in S}\eta(l_s z)},$$

we have proved the following:

Proposition 4.1. *The periodic sequence $f_1, f_2, f_3, \ldots$ of half-integral weight modular forms with integral q-expansions satisfies*

$$j_{q,n,k}(a) = \left(f_n\left(\frac{i\log q}{2\pi}\right) + o(1)\right) q^{\frac{1}{24}(1-a)} q^{(1-1/a)n^2},$$

where the $o(1)$ term does not depend on q.

From this, it is easy to deduce:

Corollary 4.2. *Let $a_1, \ldots, a_r$ denote positive integers with least common multiple A. Then there exists a $2A$-periodic sequence of meromorphic modular forms $f_1, f_2, f_3, \ldots$ with integral Fourier coefficients, holomorphic except possibly at $i\infty$, such that*

$$J_{q,n}(a_1,\ldots,a_r) = \left(f_n\left(\frac{i\log q}{2\pi}\right) + o(1)\right) q^{\frac{1}{24}(r-a_1-\cdots-a_r)} q^{(r-1/a_1-\cdots-1/a_r)n^2}.$$

Proof. Let

$$\Sigma(a_1,\ldots,a_r) := \bigl\{(k_1,\ldots,k_r)\in(\mathbb{Z}/a_1\mathbb{Z})\times\cdots\times(\mathbb{Z}/a_r\mathbb{Z}) \mid \prod\zeta_{a_i}^{k_i} = 1\bigr\}.$$

If $t_i^{a_i} = 1$, then $\det(t_i) = \zeta_{a_i}^{k_i}$ for a well-defined $k_i \in \mathbb{Z}/a_i\mathbb{Z}$. Every element $(t_1, \ldots, t_r)$ of (1-2) determines $(k_1, \ldots, k_r) \in \Sigma(a_1, \ldots, a_r)$ such that $\det(t_i) = \zeta_{a_i}^{k_1}$. Therefore,

$$J_{q,n}(a_1,\ldots,a_r) = \sum_{(k_1,\ldots,k_r)\in\Sigma(a_1,\ldots,a_r)} \prod_{i=1}^r j_{q,n,k_i}(a_i),$$

and the corollary follows immediately from Proposition 4.1. □

5. Counting Fuchsian group representations

We now prove the main results of the paper. We continue with the notation of the previous section.

If $t^a = 1$, and $m_1, \dots, m_a$ are the eigenvalue multiplicities of t, define $\delta := -1/a + \sup_i m_i/n$. Thus, $\delta \ge 0$. Let j be chosen so $m_j = n/a + \delta n$.

Proposition 5.1. *For all $\epsilon > 0$, there exist N and q_0 depending only on ϵ such that if $n > N$, $q > q_0$, $x \in \left(0, \tfrac{1}{2}\right]$ and χ is an irreducible character of $G_n = \mathrm{GL}_n(q)$ of degree $q^{xn^2} > 1$, then*

$$\frac{\log |t^{G_n}||\chi(t)|}{n^2 \log q} \le \left(1 - \frac{1}{a} - \frac{a\delta^2}{a-1}\right) + \frac{1}{n} + \epsilon x + f_{a,x}(\delta).$$

Proof. Since

$$\sum_{i \ne j} \left(\frac{n}{a} - m_i\right) \ge \delta n,$$

by the Cauchy–Schwartz inequality,

$$\sum_{i \ne j} \left(\frac{n}{a} - m_i\right)^2 \ge \frac{\delta^2 n^2}{a-1}.$$

By (4-2), the dimension of the centralizer of t in the algebraic group GL_n is

$$\frac{n^2}{a} + \sum_i \left(\frac{n}{a} - m_i\right)^2 \ge \frac{n^2}{a} + \delta^2 n^2 + (a-1)\frac{\delta^2 n^2}{(a-1)^2} = \frac{n^2}{a} + \frac{a\delta^2 n^2}{a-1}. \tag{5-1}$$

The order of the centralizer L of t in G_n is less than or equal to q to the power of the centralizer dimension, so the centralizer bound $|\chi(t)| \le |L|^{1/2}$ for irreducible characters $\chi \in \mathrm{Irr}(G_n)$ implies

$$\begin{aligned} |t^{G_n}||\chi(t)| \le \frac{|G_n|}{|L|^{\frac{1}{2}}} &\le q^{n^2 - \frac{1}{2}\left(\frac{n^2}{a} + \frac{a\delta^2 n^2}{a-1}\right)} (1 - q^{-1})^{-\frac{1}{2}n} \\ &\le q^{n^2\left(1 - \frac{1}{2a} - \frac{a\delta^2}{2(a-1)}\right) + \frac{1}{2}n}. \end{aligned} \tag{5-2}$$

On the other hand, by [Bezrukavnikov et al. 2018, Theorem 1.10],

$$\alpha(L) \le \frac{\sup_i m_i}{n} = \frac{1}{a} + \delta.$$

The character bound Theorem 2.9 therefore implies

$$|\chi(t)| \le \chi(1)^{\frac{1}{a} + \delta + \epsilon} = q^{n^2 x\left(\frac{1}{a} + \delta + \epsilon\right)},$$

so

$$|t^{G_n}||\chi(t)| \le q^{n^2\left(1 - \frac{1}{a} - \frac{a\delta^2}{a-1}\right) + n} q^{n^2 x\left(\frac{1}{a} + \delta + \epsilon\right)}.$$

Combining this with (5-2), we get

$$\begin{aligned}\frac{\log|t^{G_n}||\chi(t)|}{n^2\log q}&\\ &\le \min\Bigl(\Bigl(1-\frac{1}{a}-\frac{a\delta^2}{a-1}\Bigr)+\frac{1}{n}+x\Bigl(\frac{1}{a}+\delta+\epsilon\Bigr),\Bigl(1-\frac{1}{2a}-\frac{a\delta^2}{2(a-1)}\Bigr)+\frac{1}{2n}\Bigr)\\ &\le \Bigl(1-\frac{1}{a}-\frac{a\delta^2}{a-1}\Bigr)+\frac{1}{n}+\epsilon x+\min\Bigl(x\Bigl(\frac{1}{a}+\delta\Bigr),\frac{1}{2a}+\frac{a\delta^2}{2(a-1)}\Bigr)\\ &= \Bigl(1-\frac{1}{a}-\frac{a\delta^2}{a-1}\Bigr)+\frac{1}{n}+\epsilon x+f_{a,x}(\delta).\end{aligned}$$

□

Proposition 5.2. *There exist absolute constants $\epsilon > 0$, q_0, and N with the following property. If $2 \le a_1 \le a_2 \le \cdots \le a_r$ is not excluded in Proposition 3.2 above, $q > q_0$ is a prime power relatively prime to all a_i, $n > N$, elements $t_i \in G_n = \mathrm{GL}_n(q)$ satisfy $t_i^{a_i} = 1$, and χ is a nonlinear irreducible character of G_n, then*

$$\frac{\prod_i |t_i^{G_n}|}{|G_n|}\frac{\prod_i |\chi(t_i)|}{\chi(1)^{r-2}} \le 4q^{n^2(-1+\sum_i (1-\frac{1}{a_i}))}\chi(1)^{-\epsilon}.$$

Proof. Let $x := \log_{q^{n^2}} \chi(1)$, which by [Landazuri and Seitz 1974] is at least $1/(2n)$. We fix q_0 as in Proposition 5.1 and ϵ as in Proposition 3.2 and choose $N > 2/\epsilon$. Let $\delta_i := \mu_i - 1/a_i$, where $\mu_i n$ is the highest multiplicity of any eigenvalue of t_i. By Proposition 5.1 and Proposition 3.2,

$$\begin{aligned}\frac{\log\bigl(\prod_i |t_i^{G_n}|\prod_i |\chi(t_i)|\bigr)}{n^2\log q} &\le \sum_{i=1}^r\Bigl(1-\frac{1}{a_i}-\frac{a_i\delta_i^2}{a_i-1}+\frac{1}{n}+\epsilon x+f_{a_i,x}(\delta_i)\Bigr)\\ &= \frac{r}{n}+rx\epsilon+\sum_i\Bigl(1-\frac{1}{a_i}\Bigr)+\sum_i\Bigl(f_{a_i,x}(\delta_i)-\frac{a_i\delta_i^2}{a_i-1}\Bigr)\\ &< \frac{r}{n}-rx\epsilon+\sum_i\Bigl(1-\frac{1}{a_i}\Bigr)+(r-2)x\\ &< -x\epsilon+\sum_i\Bigl(1-\frac{1}{a_i}\Bigr)+(r-2)x.\end{aligned}$$

Since $|G_n| \ge \frac{1}{4}q^{n^2}$, the proposition follows. □

The positive genus variant of this result is as follows:

Lemma 5.3. *There exist absolute constants q_0 and N such that for every $g \ge 1$, every sequence $2 \le a_1 \le \cdots \le a_r$ (assumed nonempty if $g = 1$), every prime power $q > q_0$ relatively prime to a_i for all $i \le r$, every $n > N$, every tuple $(t_1, \dots, t_r) \in G_n^r$ satisfying $t_i^{a_i} = 1$ for all i, and every nonlinear $\chi \in \mathrm{Irr}(G_n)$, we have*

$$\prod_i |t_i^{G_n}||G_n|^{2g-1}\frac{\prod_i |\chi(t_i)|}{\chi(1)^{2g+r-2}} \le q^{n^2(2g-1+\sum_i (1-\frac{1}{a_i}))}\chi(1)^{-0.22}.$$

Proof. The proof is exactly the same as that of Proposition 5.2 except that we use Lemma 3.3 instead of Proposition 3.2. □

Lemma 5.4. *Let* $H_i = \mathrm{GL}_{n_i}(q_i)$, *where* $\lim_{i\to\infty} n_i = \infty$. *Then for all* $\epsilon > 0$,

$$\sum_{\chi\in\mathrm{Irr}(H_i)} \chi(1)^{-\epsilon} = q_i - 1 + o(q_i^{-\frac{1}{3}\epsilon n_i})$$

Proof. By [Liebeck and Shalev 2005a, Theorem 1.2],

$$\sum_{\chi\in\mathrm{Irr}(\mathrm{SL}_{n_i}(q_i))} \chi(1)^{-\frac{1}{2}\epsilon} = 1 + o(1),$$

so if D_i denotes the minimum degree of a nontrivial character of $\mathrm{SL}_{n_i}(q_i)$,

$$\sum_{\chi\in\mathrm{Irr}(\mathrm{SL}_{n_i}(q_i))} \chi(1)^{-\epsilon} = 1 + o(D_i^{-\frac{1}{2}\epsilon}) = 1 + o(q_i^{-\frac{1}{3}\epsilon n_i})$$

by [Landazuri and Seitz 1974]. The relation which assigns to each element of $\mathrm{Irr}(H_i)$ all the elements in $\mathrm{Irr}(\mathrm{SL}_{n_i}(q_i))$ which are constituents of its restriction is at most $q-1$ to 1 and nonincreasing in degree. There are $q_i - 1$ linear characters for H_i, all mapping to the trivial character of $\mathrm{SL}_{n_i}(q_i))$. Therefore,

$$1 - q_i + \sum_{\chi\in\mathrm{Irr}(H_i)} \chi(1)^{-\epsilon} = o((q_i-1)q_i^{-\frac{1}{3}\epsilon n_i}),$$

and the lemma follows. □

We can now prove Theorems A and B.

Proof. We assume first that $g=0$, so Γ is determined by $2 \le a_1 \le a_2 \le \cdots \le a_r$. A homomorphism $\Gamma \to G_n$ is determined by the images $t_1, \ldots, t_r \in G_n$ of $z_1, \ldots, z_r \in \Gamma$, which satisfy $t_i^{a_i} = 1$ and $t_1 \cdots t_r = 1$. We can partition the set of homomorphisms according to the conjugacy classes $C_1, \ldots, C_r$ to which the t_i belong. By the Frobenius formula, the total number of homomorphisms is

$$\sum_{(C_1,\ldots,C_r)} \frac{|C_1\times\cdots\times C_r|}{|G_n|} \sum_{\chi\in\mathrm{Irr}(G_n)} \frac{\chi(C_1)\cdots\chi(C_r)}{\chi(1)^{r-2}}. \tag{5-3}$$

The determinant condition implies that each linear character in the inner sum contributes 1, and there are a total of $q-1$ such characters. Their total contribution is therefore

$$(q-1)\sum \frac{|C_1\times\cdots\times C_r|}{|G_n|} = (q-1)J_{q,n}(a_1,\ldots,a_r)|G_n|^{-1}. \tag{5-4}$$

By Proposition 5.2 and Lemma 5.4, the contribution of all nonlinear characters χ to (5-3) is $o(q^{-\frac{1}{3}\epsilon n} q^{(1-\chi(\Gamma))n^2})$, where ϵ is the positive absolute constant defined in Proposition 5.2. By Corollary 4.2,

$$|\mathrm{Hom}(\Gamma, G)| = (q-1)q^{\frac{1}{24}(r-a_1-\cdots-a_r)}\left(f_n\left(\frac{i\log q}{2\pi}\right) + o(1)\right)q^{(1-\chi(\Gamma))n^2},$$

which implies Theorem A and Theorem B in the genus 0 case. The proof in the higher genus case is the same except that we use Lemma 5.3 instead of Proposition 5.2. □

Proposition 5.5. *Let a and n be positive integers, $a \geq 2$, and let q be a prime power which is $1 \pmod a$. The minimum dimension of the centralizer in GL_n of a semisimple element $t \in \mathrm{GL}_n(q)$ of order dividing a is $n^2/a + a\{n/a\}\{-n/a\}$. If a is odd, t can be chosen to have determinant 1. If a is even and $n/a \notin \mathbb{Z}$, then t can be chosen to have determinant 1 or -1. If a is even and $n/a \in \mathbb{Z}$, then t must have determinant $(-1)^{n/a}$; if this is -1, there is no element in $\mathrm{GL}_n(q)$ whose centralizer has dimension $n^2/a + 1$, but there is an element $t' \in \mathrm{SL}_n(q)$ with centralizer dimension $n^2/a + 2$.*

Proof. If the multiplicities $m_1, \dots, m_a$ of the eigenvalues $\zeta_a, \dots, \zeta_a^a$ of a semisimple $t \in \mathrm{GL}_n(q)$ satisfying $t^a = 1$ are written $n/a + \epsilon_i$, then the centralizer of t has dimension

$$\sum_i m_i^2 = \frac{n^2}{a} + \sum_i \epsilon_i^2.$$

As $\sum_i \epsilon_i = 0$, either all are zero (which can only happen in the case that a divides n), or at least one is positive and at least one is negative. In the latter case, if any $\epsilon_i \geq 1$, then by reducing this by 1 and increasing some negative ϵ_j by 1, we decrease $\sum_i \epsilon_i^2$, and likewise if some $\epsilon_i \leq -1$. As all ϵ_i are $\{n/a\} \pmod 1$, each must be $\{n/a\}$ or $\{n/a\} - 1$, and since they sum to zero, there must be $a - a\{n/a\}$ of the former and $a\{n/a\}$ of the latter, implying $\sum_i \epsilon_i^2 = a\{n/a\}\{-n/a\}$.

Next, we claim that as long as $\{n/a\} \neq 0$, there exists some sequence $m_1, \dots, m_a$ consisting of $a\{n/a\}$ copies of $\lceil n/a \rceil$ and $a - a\{n/a\}$ copies of $\lfloor n/a \rfloor$ such that $\prod_i \zeta_a^{im_i}$ is any desired power of ζ_a. To prove this, it suffices to show that if $0 < k < a$, the sums of k-element subsets S of $\{0, 1, \dots, a-1\}$ represent all residue classes $\pmod a$. Indeed, if $S \neq \{a-1, a-2, \dots, a-k\}$, there exists $s \in S$ such that $s + 1 \in \{0, 1, \dots, a-1\} \setminus S$. Thus, the set of sums of k-element subsets S includes all integers from $\binom{k}{2}$ to $\binom{a}{2} - \binom{a-k}{a}$, a total of $k(a-k) + 1 \geq a$ consecutive integers, which therefore represent all congruence classes $\pmod a$.

Finally, assume a divides n, so $m_1 = \cdots = m_a = n/a$ gives the minimum value n^2/a of $\sum_i m_i^2$. Any other choice of $(m_1, \dots, m_a)$ must have all ϵ_i integral and at least two nonzero, so $\sum_i m_i^2 \geq n^2/a + 2$. If $m_1 = \cdots = m_a$ and n/a is even or a is odd, then $\sum_i im_i$ is divisible by a, so $\det(t) = 1$. If a is even and n/a is odd, then $m_1 = \cdots = m_a$ gives $\det(t) = -1$. In this last case, setting $\epsilon_1 = 1$ and $\epsilon_{a/2+1} = -1$ and all other $\epsilon_i = 0$, we get $\sum_i im_i$ is divisible by a and $\sum_i m_i^2 = n^2/a + 2$. □

Let E_Γ denote the set of i such that a_i is even. As in the statement of Theorem C, for each positive integer n, we define $\sigma_{\Gamma,n} := -1$ if $n/a_i \in \mathbb{Z}$ for all $i \in E_\Gamma$, and $\sum_{i \in E_\Gamma} n/a_i$ is odd; otherwise $\sigma_{\Gamma,n} := 1$.

Proposition 5.6. *Suppose* $q \equiv 1 \pmod{a_i}$ *for all* i*. If* $(t_1, \dots, t_r)$ *is an* r*-tuple of semisimple elements in* $\mathrm{GL}_n(q)$ *such that* $t_i^{a_i} = 1$ *and* $\prod_i \det(t_i) = 1$*, the minimum possible sum of the dimensions of the centralizers of the* t_i *in* GL_n *is*

$$1 - \sigma_{\Gamma,n} + \sum_{i=1}^{r} \Bigl(\frac{n^2}{a_i} + a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\} \Bigr).$$

Proof. If there is at least one a_i which is even and such that $n/a_i \notin \mathbb{Z}$, then we can choose t_i to have either determinant 1 or -1 and centralizer dimension

$$\frac{n^2}{a_i} + a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\}. \tag{5-5}$$

For $j \neq i$, we can choose t_j to have determinant in $\{\pm 1\}$ and centralizer $n^2/a_j + a_j\{n/a_j\}\{-n/a_j\}$. Therefore, we can choose minimal centralizer dimension for all t_i while imposing the condition $\prod_i \det(t_i) = 1$.

If $n/a_i \in \mathbb{Z}$ for all a_i even, and the set of i such that a_i is even and n/a_i is odd has even cardinality, then we may choose t_i whose centralizer has dimension (5-5) for all i and such that $\det(t_i) = 1$ except when a_i is even and n/a_i is odd. In these cases, of which there are an odd number, $\det(t_i) = 1$, so again $\prod_i \det(t_i) = 1$.

What remains is the case $\sigma_{\Gamma,n} = -1$, and here if t_i has centralizer dimension (5-5) for all i, then the product $\prod_i \det(t_i)$ is -1 times a product of elements of odd order, so it cannot be 1. On the other hand, if we choose one t_i with a_i even and n/a_i odd, and choose it to have determinant 1 and centralizer dimension $n^2/a_i + 2$, and all other t_j have minimal centralizer dimension and $\det(t_j) = \pm 1$, then the product $\prod_i \det(t_i)$ equals 1. □

Proof of Theorem C. By Theorem B, there exist q_0 and N such that if $q > q_0$ is relatively prime to A and $n > N$, then

$$\tfrac{1}{2} < \frac{q^{(1-2g)n^2} |\mathrm{Hom}(\Gamma, \mathrm{GL}_n(q))|}{q J_{q,n}(a_1, \dots, a_r)} < \tfrac{3}{2}.$$

For any fixed such q and n, let $X_{q,n}$ denote the variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ over the field $\mathbb{F}_q$. Then, for all positive integers m,

$$\tfrac{1}{2} < \frac{q^{(1-2g)mn^2} |X_{q,n}(\mathbb{F}_{q^m})|}{q^m J_{q^m,n}(a_1, \dots, a_r)} < \tfrac{3}{2}.$$

By Proposition 5.6,

$$\begin{aligned}
\dim X_{q,n} &= \limsup_m \log_{q^m} |X_{q,n}(\mathbb{F}_{q^m})| \\
&= 1 + (2g-1)n^2 + \limsup_m \log_{q^m} J_{q^m,n}(a_1, \dots, a_r) \\
&= 1 + (2g-1)n^2 + rn^2 - 1 + \sigma_{\Gamma,n} - \sum_{i=1}^{r} \Bigl(\frac{n^2}{a_i} + a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\} \Bigr) \\
&= \sigma_{\Gamma,n} + (1 - \chi(\Gamma))n^2 - \sum_{i=1}^{r} a_i \Bigl\{ \frac{n}{a_i} \Bigr\} \Bigl\{ -\frac{n}{a_i} \Bigr\}.
\end{aligned}$$

The first claim of the theorem follows in the positive characteristic case.

For characteristic zero, we consider the scheme $\mathrm{Hom}(\Gamma, \mathrm{GL}_{n,\mathbb{Z}})$ over $\mathrm{Spec}\,\mathbb{Z}$ whose $\mathbb{F}_p$ fiber is the n-dimensional representation variety of Γ over $\mathbb{F}_p$. By the constructibility of the set of dimensions of irreducible fiber components [EGA IV$_3$ 1966, proposition 9.5.5], the dimension of the generic fiber must be the same as the common dimension of any infinite set of fibers over closed points.

Finally, for the second claim of the theorem, we observe that $\{t\}\{-t\} \le \frac{1}{4}$ for all real t, so for all i,

$$a_i\left\{\frac{n}{a_i}\right\}\left\{-\frac{n}{a_i}\right\} \le \tfrac{1}{4}a_i.$$

We have $\sigma_{\Gamma,n} > -\frac{1}{2}$ unless there is at least one value of i for which a_i is even and n/a_i is integral. For this value of i, $a_i \ge 2$, so

$$\sigma_{\Gamma,n} - \sum_{i=1}^{r} a_i\left\{\frac{n}{a_i}\right\}\left\{-\frac{n}{a_i}\right\} \ge -\tfrac{1}{2} - \sum_{i=1}^{r} \tfrac{1}{4}a_i. \qquad \square$$

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D_4-TYPE SUBGROUPS OF $F_4(q)$

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We treat the action of the simple group $F_4(q)$ on the cosets of subgroups $D_4(q)$, ${}^2D_4(q)$ and ${}^3D_4(q)$ and their extensions by graph automorphisms. We obtain the ranks and decompose the corresponding permutation characters; we show that, even allowing for the application of field automorphisms, the only two primitive multiplicity-free actions arising are those of $F_4(2)$ on cosets of $D_4(2).S_3$ and ${}^3D_4(2).3$. For these two actions, we calculate the subdegrees; we find that all suborbits are self-paired, but that the action gives rise to no distance-transitive graph.

1. Introduction

This paper represents a further contribution to the programs described in [15], namely the classification of finite primitive permutation groups with the property that the action is on the vertices of a distance-transitive graph, or (more generally) all suborbits are self-paired, or (more generally still) the permutation character is multiplicity-free. Results of [1; 23] essentially reduce these classifications to the consideration of almost simple groups; many cases have already been studied and resolved. In [15] the case of the action of $F_4(q)$ on cosets of $B_4(q)$ was treated; in this paper, which in many ways may be seen as a continuation of [15], we consider two further actions of $F_4(q)$, on cosets of $D_4(q).S_3$ and of ${}^3D_4(q).3$. In conjunction with [17] (about which we shall say more later), we shall show that each of these actions is only multiplicity-free if $q = 2$ (and, moreover, that if $q > 2$ the application of field automorphisms never makes the action multiplicity-free); in these two cases we shall calculate the subdegrees, and show that all suborbits are self-paired, but that the action gives rise to no distance-transitive graph.

In fact, we shall consider other subgroups beside the two just mentioned. We shall say that a subgroup of $F_4(q)$ is a *D_4-type subgroup* if it is of the form ${}^mD_4(q).\Gamma$, where $m \in \{1, 2, 3\}$ (interpreting ${}^1D_4(q)$ to mean simply $D_4(q)$), and Γ is a group of graph automorphisms of ${}^mD_4(q)$. For completeness, we shall decompose the permutation characters of the actions of $F_4(q)$ on cosets of all D_4-type subgroups.

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We let $\boldsymbol{G}$ be a simple algebraic group of type F_4 over k, the algebraic closure of $\mathbb{F}_p$, where p is a prime. We let $\boldsymbol{T}_0$ be a maximal torus of $\boldsymbol{G}$, and Φ be the set of roots of $\boldsymbol{G}$ relative to $\boldsymbol{T}_0$; we choose a Borel subgroup $\boldsymbol{B}$ of $\boldsymbol{G}$ containing $\boldsymbol{T}_0$, and let Φ^+ and Σ be the sets of positive and simple roots determined by $\boldsymbol{B}$. As in [24; 25], we let

$$\Sigma = \left\{\epsilon_2-\epsilon_3, \epsilon_3-\epsilon_4, \epsilon_4, \tfrac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)\right\},$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ form an orthonormal basis of a 4-dimensional Euclidean space; then

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j : 1 \le i < j \le 4\} \cup \{\epsilon_i : 1 \le i \le 4\} \cup \left\{\tfrac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\},$$

and $\Phi = \Phi^+ \cup -\Phi^+$. For convenience, where there is no danger of confusion, we shall write $\pm i \pm j$ for $\pm\epsilon_i \pm \epsilon_j$, $\pm i$ for $\pm\epsilon_i$, and $+{-}{-}{-}$ for $\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)$, etc. in what follows. We write $\boldsymbol{N} = N_{\boldsymbol{G}}(\boldsymbol{T}_0)$, and $W = \boldsymbol{N}/\boldsymbol{T}_0$ for the Weyl group of $\boldsymbol{G}$.

Given a root $\alpha \in \Phi$, we let $\boldsymbol{U}_\alpha$ be the corresponding root subgroup of $\boldsymbol{G}$; we write $\boldsymbol{U} = \prod_{\alpha\in\Phi^+} \boldsymbol{U}_\alpha$. For each α there is an isomorphism $x_\alpha : k \to \boldsymbol{U}_\alpha$ such that for all $\lambda \in k$ and $t \in \boldsymbol{T}_0$ we have ${}^t x_\alpha(\lambda) = x_\alpha(\alpha(t)\lambda)$ (where as usual a group element as superscript denotes conjugation on the appropriate side). We assume that the isomorphisms x_α are chosen as in [4, Section 1.9] such that if for all $\lambda \in k^*$ we set

$$n_\alpha(\lambda) = x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda), \quad h_\alpha(\lambda) = n_\alpha(\lambda)n_\alpha(-1),$$

then as in [3, Lemma 6.4.4] for all $\lambda \in k^*$ we have $n_\alpha(\lambda) \in \boldsymbol{N}$ and $h_\alpha(\lambda) \in \boldsymbol{T}_0$; the maps $h_\alpha : k^* \to \boldsymbol{T}_0$ are the coroots. For all $\lambda, \mu \in k$ we also have the Chevalley commutator relations

$$[x_\alpha(\lambda), x_\beta(\mu)] = \begin{cases} 1 & \text{if } \alpha+\beta \notin \Phi, \\ x_{\alpha+\beta}(N_{\alpha,\beta}\lambda\mu) & \text{if } \alpha+\beta \in \Phi,\ 2\alpha+\beta, \alpha+2\beta \notin \Phi, \\ x_{\alpha+\beta}(N_{\alpha,\beta}\lambda\mu)x_{2\alpha+\beta}\left(-\tfrac{1}{2}N_{\alpha,\beta}N_{\alpha,\alpha+\beta}\lambda^2\mu\right) & \text{if } \alpha+\beta, 2\alpha+\beta \in \Phi, \\ x_{\alpha+\beta}(N_{\alpha,\beta}\lambda\mu)x_{\alpha+2\beta}\left(\tfrac{1}{2}N_{\alpha,\beta}N_{\alpha+\beta,\beta}\lambda\mu^2\right) & \text{if } \alpha+\beta, \alpha+2\beta \in \Phi, \end{cases}$$

where we assume that the structure constants $N_{\alpha,\beta}$ are as given in [25]. For each α we write $n_\alpha = n_\alpha(1)$, and $w_\alpha = n_\alpha \boldsymbol{T}_0 \in W$.

Much as in [24], we shall write elements of $\boldsymbol{T}_0$ in the form $(\mu_1, \mu_2, \mu_3, \mu_4; \nu)$ with $\mu_1, \mu_2, \mu_3, \mu_4, \nu \in k^*$ and $\nu^2 = \mu_1\mu_2\mu_3\mu_4$, where for $\lambda \in k^*$ we set

$$\begin{aligned} h_{2-3}(\lambda) &= (1, \lambda, \lambda^{-1}, 1; 1), \\ h_{3-4}(\lambda) &= (1, 1, \lambda, \lambda^{-1}; 1), \\ h_4(\lambda) &= (1, 1, 1, \lambda^2; \lambda), \\ h_{+---}(\lambda) &= (\lambda, \lambda^{-1}, \lambda^{-1}, \lambda^{-1}; \lambda^{-1}). \end{aligned}$$

The action of W on $\boldsymbol{T}_0$ is then determined by

$$
\begin{aligned}
{}^{w_{2-3}}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\mu_1, \mu_3, \mu_2, \mu_4; \nu),\\
{}^{w_{3-4}}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\mu_1, \mu_2, \mu_4, \mu_3; \nu),\\
{}^{w_4}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\mu_1, \mu_2, \mu_3, {\mu_4}^{-1}; \nu{\mu_4}^{-1}),\\
{}^{w_{+---}}(\mu_1, \mu_2, \mu_3, \mu_4; \nu) &= (\nu, \nu{\mu_3}^{-1}{\mu_4}^{-1}, \nu{\mu_2}^{-1}{\mu_4}^{-1}, \nu{\mu_2}^{-1}{\mu_3}^{-1}; \mu_1).
\end{aligned}
$$

We let

$$
\begin{aligned}
\boldsymbol{H} &= \langle \boldsymbol{U}_\alpha : \alpha \in \Phi(\boldsymbol{H})\rangle,\\
\boldsymbol{C} &= \langle \boldsymbol{U}_\alpha : \alpha \in \Phi(\boldsymbol{C})\rangle,\\
\boldsymbol{A} &= \langle \boldsymbol{U}_\alpha : \alpha \in \Phi(\boldsymbol{A})\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi(\boldsymbol{H}) &= \{\pm\epsilon_i \pm \epsilon_j : 1 \le i < j \le 4\},\\
\Phi(\boldsymbol{C}) &= \left\{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_3 \pm \epsilon_4), \pm\epsilon_3, \pm\epsilon_4, \pm\tfrac{1}{2}(\epsilon_1 - \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\},\\
\Phi(\boldsymbol{A}) &= \left\{\pm\epsilon_4, \pm\tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 \pm \epsilon_4)\right\};
\end{aligned}
$$

then $\boldsymbol{H}$, $\boldsymbol{C}$ and $\boldsymbol{A}$ are simple algebraic groups over k of types D_4, C_3 and $\tilde{A}_2$, respectively, where as usual a tilde denotes a root system comprising short roots, and each is simply connected, as it is easy to see that the $\mathbb{Z}$-linear span of the relevant coroots equals the group of cocharacters of the relevant torus. We let

$$
\begin{aligned}
W_{\boldsymbol{H}} &= \langle w_{1-2}, w_{2-3}, w_{3-4}, w_{3+4}\rangle,\\
W_{\boldsymbol{C}} &= \langle w_{3-4}, w_4, w_{+---}\rangle,\\
W_{\boldsymbol{A}} &= \langle w_4, w_{+---}\rangle,
\end{aligned}
$$

so that $W_{\boldsymbol{H}}$, $W_{\boldsymbol{C}}$ and $W_{\boldsymbol{A}}$ are the Weyl groups of $\boldsymbol{H}$, $\boldsymbol{C}$ and $\boldsymbol{A}$, respectively.

We write

$$
\tau_1 = 1, \qquad \tau_2 = h_{+---}(-1)n_4, \qquad \tau_3 = n_4 n_{+---}.
$$

For $m \in \{1, 2, 3\}$ the element τ_m is of order m and acts on $\boldsymbol{H}$ as a graph automorphism; moreover $\tau_m \in \boldsymbol{A}$, and $C_{\boldsymbol{A}}(\tau_m)$ is connected (if $p \neq m$ then τ_m is semisimple, so the connectedness of $C_{\boldsymbol{A}}(\tau_m)$ follows from the simple connectedness of $\boldsymbol{A}$ and [4, Theorem 3.5.6]; if instead $p = m$ then τ_m is unipotent, and an easy calculation shows the connectedness of $C_{\boldsymbol{A}}(\tau_m)$). If $p \neq 2$, as in [13] we may set $y_2 = x_4(1)n_4x_4\left(\tfrac{1}{2}\right)$; then ${\tau_2}^{y_2} = h_{+---}(-1) = (-1, -1, -1, -1; -1)$. If instead $p = 2$ we may set $y_2 = x_{-4}(1)$; then ${\tau_2}^{y_2} = x_4(1)$. Likewise, if $p \neq 3$ we may take $\omega \in k^*$ with $\omega^3 = 1 \neq \omega$, and set $y_3 = x_{+--+}(\omega^2)x_{+---}(-\omega)x_4(1)n_4n_{+---}n_4x_4\left(\tfrac{1}{3}\right)x_{+---}\left(\tfrac{1-\omega}{3}\right)x_{+--+}\left(\tfrac{1+2\omega}{3}\right)$; then ${\tau_3}^{y_3} = h_4(\omega) = (1, 1, 1, \omega^2; \omega)$. If instead $p = 3$ we may set $y_3 = x_{-++-}(1)$ $x_{-4}(1)x_{-+++}(-1)$; then ${\tau_3}^{y_3} = x_4(1)x_{+---}(1)$.

We let q be a power of p, and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be the Frobenius map determined by $x_\alpha(\lambda)^F = x_\alpha(\lambda^q)$ for all $\lambda \in k$ and $\alpha \in \Phi$. As in [15; 25] we set $x = (4, q-1)$, $y = (3, q-1)$ and $z = (3, q+1)$. We take $d \in \{0, 1\}$ and $e \in \{0, \pm 1\}$ such that $q \equiv d \pmod 2$ and $q \equiv e \pmod 3$. For $m \in \{1, 2, 3\}$ we set

$$f = \begin{cases} 3 & \text{if } m = 1, \\ 1 & \text{if } m = 2, \\ 0 & \text{if } m = 3; \end{cases}$$

the element τ_m commutes with F, and as $\tau_m \in C_{\boldsymbol{A}}(\tau_m)$, by the Lang–Steinberg theorem we may take $g_m \in C_{\boldsymbol{A}}(\tau_m)$ with ${g_m}^F.{g_m}^{-1} = \tau_m$ (choosing $g_1 = 1$), and let $\boldsymbol{H}_m = \boldsymbol{H}^{g_m}$.

For any F-stable subset $\boldsymbol{X}$ of $\boldsymbol{G}$ we write $\boldsymbol{X}^F$ for the set of points of $\boldsymbol{X}$ fixed by F. We set $G = \boldsymbol{G}^F$, $T_0 = {\boldsymbol{T}_0}^F$, $B = \boldsymbol{B}^F$, $U_\alpha = {\boldsymbol{U}_\alpha}^F$ for $\alpha \in \Phi$, $U = \boldsymbol{U}^F$, $N = \boldsymbol{N}^F$, $A = \boldsymbol{A}^F$, $C = \boldsymbol{C}^F$ and $H_m = (\boldsymbol{H}_m)^F$ for $m \in \{1, 2, 3\}$; then $G = F_4(q)$ and $H_m = {}^m D_4(q)$ with $H_m < G$, and we have

$$\begin{aligned} |G| &= q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1), \\ |H_1| &= q^{12}(q^2-1)(q^6-1)(q^4-1)^2, \\ |H_2| &= q^{12}(q^2-1)(q^6-1)(q^8-1), \\ |H_3| &= q^{12}(q^2-1)(q^6-1)(q^8+q^4+1). \end{aligned}$$

For $m \in \{1, 2, 3\}$ we define the map $F_m = F\tau_m : \boldsymbol{G} \to \boldsymbol{G}$ by $g^{F_m} = (g^F)^{\tau_m}$ for $g \in \boldsymbol{G}$, and set $\hat{H}_m = {}^{g_m}H_m = \boldsymbol{H}^{F_m}$ (where we extend the superscript notation to mean the set of points fixed by F_m).

Now as $g_1 = \tau_1 = 1$, and for $m \in \{2, 3\}$ we have chosen g_m to commute with τ_m, given $m, r \in \{1, 2, 3\}$ with $\{m, r\} \neq \{2, 3\}$ we have ${\tau_r}^{g_m} = \tau_r$. It follows that, up to conjugacy, the D_4-type subgroups of G are H_1, H_2 and H_3; $H_1.2$ and $H_2.2$, where $H_m.2 = H_m\langle\tau_2\rangle$; $H_1.3$ and $H_3.3$, where $H_m.3 = H_m\langle\tau_3\rangle$; and $H_1.S_3 = H_1\langle\tau_2, \tau_3\rangle$. Of these, only $H_1.S_3$ and $H_3.3$ are maximal subgroups of G, since $H_2.2$ lies inside $B_4(q) = \langle U_{\pm(1-2)}, U_{\pm(2-3)}, U_{\pm(3-4)}, U_{\pm 4}\rangle$.

The structure of the rest of the paper is as follows. In Section 2 we develop a method for decomposing the permutation characters which occur here. In Section 3 we apply this to the cases where the D_4-type subgroup is H_m for $m \in \{1, 2, 3\}$; in Section 4 we do the same for the other D_4-type subgroups. Finally in Section 5 we consider the contributions of the actions treated here to the classification programs mentioned above.

2. Decomposing permutation characters

In this section we begin by giving information about conjugacy classes, and then explain a method for calculating scalar products of characters which uses it; this will allow us to decompose the permutation characters of interest here. After some preliminary results we prove the claim on which the method is based.

In what follows we shall make several statements to the effect that a certain expression is a 'polynomial in q'. By this we shall mean that, for a stated set of zero or more parameters which may take only finitely many possible values, if we fix their values there is a polynomial with rational coefficients such that, for any appropriate q, the expression equals the evaluation of the polynomial at q.

2.1. ***Conjugacy classes in D_4-type subgroups.*** We begin by providing information concerning conjugacy classes in D_4-type subgroups of G, which will be needed in much of the rest of the paper. We shall see that, for $m, r \in \{1, 2, 3\}$ with $\{m, r\} \neq \{2, 3\}$, it suffices to consider the H_m-classes in the coset $H_m\tau_r$; equivalently we may consider the $\hat{H}_m$-classes in the coset $\hat{H}_m\tau_r$ and then conjugate by g_m (note that $\hat{H}_1 = H_1$ as $g_1 = 1$, and that $\hat{H}_m\tau_r = {}^{g_m}(H_m\tau_r)$ as g_m commutes with τ_r).

Of fundamental importance is Jordan decomposition, but the way in which this is applied will depend on whether or not r equals p. We define

$$\bar{r} = \frac{r}{(r, p)} = \begin{cases} r & \text{if } r \neq p, \\ 1 & \text{if } r = p; \end{cases}$$

given an element of the coset $\hat{H}_m\tau_r$, its semisimple part lies in the coset $\hat{H}_m\tau_{\bar{r}}$, and indeed all semisimple elements of $\boldsymbol{H}\langle\tau_r\rangle$ lie in $\boldsymbol{H}\langle\tau_{\bar{r}}\rangle$. If $\bar{r} = r$ we begin with semisimple classes in $\hat{H}_m\tau_r$ and then take unipotent classes in $\hat{H}_m$ lying in centralizers of semisimple elements; if however $\bar{r} \neq r$ we instead begin with unipotent classes in $\hat{H}_m\tau_r$ and then take semisimple classes in $\hat{H}_m$ lying in centralizers of unipotent elements. In the former case, for each unipotent class in the $\hat{H}_m$-centralizer we shall need information concerning which unipotent class in the $\boldsymbol{G}$-centralizer contains it (sometimes partial information is sufficient for our purposes); frequently inspection allows us to determine the class precisely, but if necessary we can always calculate Jordan structure on an appropriate module (the natural module for groups of classical type, the 26-dimensional module for F_4 itself) and use results from either [18] or [14].

First take $r = 1$, so that our concern is with the $\hat{H}_m$-classes in $\hat{H}_m$ itself. We start with semisimple $\hat{H}_m$-classes in $\hat{H}_m$. Recall that any semisimple element of $\hat{H}_m = \boldsymbol{H}^{F_m}$ lies in an F_m-stable maximal torus of $\boldsymbol{H}$, and that there is a bijection between the $\boldsymbol{H}^{F_m}$-classes of F_m-stable maximal tori of $\boldsymbol{H}$ and the F_m-conjugacy classes of $W_{\boldsymbol{H}}$; as $F_m = F\tau_m$, and F acts trivially on W while τ_m corresponds to the Weyl group element 1, w_4 or w_4w_{+---} according as $m = 1$, 2 or 3, we see that the F_m-conjugacy classes of $W_{\boldsymbol{H}}$ correspond to the $W_{\boldsymbol{H}}$-classes in the coset $W_{\boldsymbol{H}}$, $W_{\boldsymbol{H}}w_4$ or $W_{\boldsymbol{H}}w_{+---}w_4$, respectively. In fact the semisimple classes of $B_4(q)$ are listed in the Appendix of [15], from which it is straightforward to deduce the semisimple

$\hat{H}_m$-classes in $\hat{H}_m$ for $m = 1$ and $m = 2$, while those for $m = 3$ are given in [5, Table 2.1]. The $\boldsymbol{G}$-centralizer of a semisimple element is determined by the roots with respect to a maximal torus containing it which take the value 1 at it, and so may easily be determined by inspection (in fact for $m = 1$ and $m = 2$ this information is given in the Appendix of [15]).

Next consider $\hat{H}_m$-classes in $\hat{H}_m$ which are not semisimple. Given a semisimple element of $\hat{H}_m$, the unipotent classes lying in its centralizer in either $\hat{H}_m$ or $\boldsymbol{G}$ may usually be obtained from the various results and tables in [18] (see Theorems 3.1, 7.1 and Tables 8.1a, 8.2a, 8.4a, 8.5a, 22.2.4); the exceptions are that no table is given for $B_4(q)$ (which here appears as the centralizer of a semisimple element only in odd characteristic) but it may be constructed using the results given in Chapters 3–7 therein (alternatively a list is given in [25], which of course also gives the unipotent classes of $F_4(q)$), and that ${}^3D_4(q)$ is not treated but results are given in [5]. In this way we obtain a complete set of representatives of the $\hat{H}_m$-classes in $\hat{H}_m$; conjugation by g_m then gives a complete set of representatives of the H_m-classes in H_m.

Now take $r \in \{2, 3\}$, so that $m \in \{1, r\}$. We note that for given r the $\hat{H}_m$-classes in $\hat{H}_m \tau_r$ for the two values of m are closely related by Shintani descent (see [6, I.7.2; 22, Proposition 5]): there is a bijection between the sets of $\hat{H}_1$-classes in $\hat{H}_1 \tau_r$ and $\hat{H}_r$-classes in $\hat{H}_r \tau_r$ which preserves centralizer orders. Indeed, we may proceed as follows: given an H_1-class in $H_1 \tau_r$ (i.e., an $\hat{H}_1$-class in $\hat{H}_1 \tau_r$), we may take a class representative $h\tau_r$ and use the Lang–Steinberg theorem to write h in the form $\bar{x}^{F_r}.\bar{x}^{-1}$ for some $\bar{x} \in \boldsymbol{H}$; Shintani descent gives the corresponding $\hat{H}_r$-class in $\hat{H}_r \tau_r$ as that containing $h^\dagger \tau_r$ where $h^\dagger = \bar{x}^{-1}.\bar{x}^{F_1}$; conjugating by g_r then gives the corresponding H_r-class in $H_r \tau_r$ as that containing $h' \tau_r$, where $h' = (h^\dagger)^{g_r}$.

First assume $r \neq p$; then $\tau_r = \tau_{\bar{r}}$ is a semisimple element of G. Here we begin by observing that $\tau_{\bar{r}}$ is a quasisemisimple automorphism of $\boldsymbol{H}$ (recall from [28, Section 9] that this means that it stabilizes both a Borel subgroup and a maximal torus therein). We make use of [20, 1.14]: we write $\boldsymbol{T}_{\bar{r}} = \boldsymbol{T}_0 \cap C_{\boldsymbol{H}}(\tau_{\bar{r}})$, so that $\boldsymbol{T}_{\bar{r}}$ is a maximal torus of $C_{\boldsymbol{H}}(\tau_{\bar{r}})$; we let $\boldsymbol{N}_{\bar{r}} = \{n \in \boldsymbol{H} : {}^n(\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}) = \boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}\}$, and then its identity component $\boldsymbol{N}_{\bar{r}}{}^\circ$ is equal to $\boldsymbol{T}_{\bar{r}}$, so that $\boldsymbol{N}_{\bar{r}}/\boldsymbol{T}_{\bar{r}}$ is finite; any quasisemisimple element lying in $\boldsymbol{H}\tau_{\bar{r}}$ is $\boldsymbol{H}$-conjugate to an element of $\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}$; any two elements of $\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}$ which are $\boldsymbol{H}$-conjugate are in fact conjugate by an element of $\boldsymbol{N}_{\bar{r}}$, and thus lie in the same orbit under the action of the finite group $\boldsymbol{N}_{\bar{r}}/\boldsymbol{T}_{\bar{r}}$. Using this we may prove the following.

Lemma 2.1. *Any semisimple element of $\hat{H}_m \tau_{\bar{r}}$ is $\hat{H}_m$-conjugate to an element $(s\tau_{\bar{r}})^{x'}$ where $s \in \boldsymbol{T}_{\bar{r}}$ and $x' \in \boldsymbol{H}$ with ${x'}^{F_m}.{x'}^{-1} \in \boldsymbol{N}_{\bar{r}}$.*

Proof. Suppose $\check{s} \in \hat{H}_m$ with $\check{s}\tau_{\bar{r}}$ semisimple; then $\check{s}$ is quasisemisimple, so there exist $h \in \boldsymbol{H}$ and $s \in \boldsymbol{T}_{\bar{r}}$ such that $\check{s}\tau_{\bar{r}} = (s\tau_{\bar{r}})^h$. As both $\check{s}$ and $\tau_{\bar{r}}$ are F_m-stable we have $(s\tau_{\bar{r}})^h = (s^{F_m}.\tau_{\bar{r}})^{h^{F_m}}$, so $s\tau_{\bar{r}} = (s^{F_m}.\tau_{\bar{r}})^{h^{F_m}.h^{-1}}$; as $\boldsymbol{T}_{\bar{r}}$ is also F_m-stable, both

$s\tau_{\bar r}$ and $s^{F_m}.\tau_{\bar r}$ lie in $\boldsymbol{T}_{\bar r}\tau_{\bar r}$, so there exists $n \in \boldsymbol{N}_{\bar r}$ such that $s\tau_{\bar r} = (s^{F_m}.\tau_{\bar r})^n$. Using the Lang–Steinberg theorem we may write $n = {x'}^{F_m}.{x'}^{-1}$ for some $x' \in \boldsymbol{H}$; then

$$((s\tau_{\bar r})^{x'})^{F_m} = (s^{F_m}.\tau_{\bar r})^{{x'}^{F_m}} = (s^{F_m}.\tau_{\bar r})^{nx'} = (s\tau_{\bar r})^{x'},$$

so that $(s\tau_{\bar r})^{x'}$ is F_m-stable, and

$$(\check{s}\tau_{\bar r})^{h^{-1}n^{-1}} = (s\tau_{\bar r})^{n^{-1}} = s^{F_m}.\tau_{\bar r} = (\check{s}\tau_{\bar r})^{(h^{F_m})^{-1}},$$

so that $h^{-1}n^{-1}h^{F_m} \in C_{\boldsymbol{H}}(\check{s}\tau_{\bar r})$. As $\boldsymbol{H}$ is simply connected and $\check{s}\tau_{\bar r}$ is a quasisemisimple automorphism of $\boldsymbol{H}$, by [28, Theorem 8.2] $C_{\boldsymbol{H}}(\check{s}\tau_{\bar r})$ is connected, so we may again use the Lang–Steinberg theorem to write $h^{-1}n^{-1}h^{F_m} = c^{-1}c^{F_m}$ for some $c \in C_{\boldsymbol{H}}(\check{s}\tau_{\bar r})$; then as $h^{-1}x'({x'}^{F_m})^{-1}h^{F_m} = c^{-1}c^{F_m}$ we see that $ch^{-1}x' \in \boldsymbol{H}$ is F_m-stable and thus lies in $\hat{H}_m$, and $(\check{s}\tau_{\bar r})^{ch^{-1}x'} = (\check{s}\tau_{\bar r})^{h^{-1}x'} = (s\tau_{\bar r})^{x'}$, so that $\check{s}\tau_{\bar r}$ is $\hat{H}_m$-conjugate to $(s\tau_{\bar r})^{x'}$ as required. □

Thus $\boldsymbol{T}_{\bar r}$ and $\boldsymbol{N}_{\bar r}/\boldsymbol{T}_{\bar r}$ play the roles of 'maximal torus' and 'Weyl group' for the coset $\hat{H}_m\tau_{\bar r}$. We find that they are as follows: for $\bar r = 2$ we have

$$\boldsymbol{T}_2 = \left\{\left(\lambda, \mu, \frac{\nu^2}{\lambda\mu}, 1; \nu\right) : \lambda, \mu, \nu \in k^*\right\},$$
$$\boldsymbol{N}_2/\boldsymbol{T}_2 = \langle n_{1-2}\boldsymbol{T}_2, n_{2-3}\boldsymbol{T}_2, n_{3-4}n_{3+4}\boldsymbol{T}_2, (-1,-1,-1,-1;-1)\boldsymbol{T}_2\rangle;$$

for $\bar r = 3$ we have

$$\boldsymbol{T}_3 = \left\{\left(\lambda, \mu, \frac{\lambda}{\mu}, 1; \lambda\right) : \lambda, \mu \in k^*\right\},$$
$$\boldsymbol{N}_3/\boldsymbol{T}_3 = \langle n_{2-3}\boldsymbol{T}_3, n_{1-2}n_{3-4}n_{3+4}\boldsymbol{T}_3, (\omega^2, \omega, \omega, \omega^2; 1)\boldsymbol{T}_3\rangle.$$

In each case we see that $|\boldsymbol{N}_{\bar r}/\boldsymbol{T}_{\bar r}| = |C_W(\tau_{\bar r}\boldsymbol{T}_0)|$. For $\bar r = 2$ we in fact have $\boldsymbol{N}_2/\boldsymbol{T}_2 \cong C_W(\tau_2\boldsymbol{T}_0) = C_W(w_4) = \langle w_{1-2}, w_{2-3}, w_{3-4}w_{3+4}\rangle \times \langle w_4\rangle = W(B_3) \times 2$; however for $\bar r = 3$ we have $\boldsymbol{N}_3/\boldsymbol{T}_3 \cong S_3 \times S_3$ but $C_W(\tau_3\boldsymbol{T}_0) = C_W(w_4w_{+---}) = \langle w_{2-3}, w_{1-2}w_{3-4}w_{3+4}\rangle \times \langle w_4w_{+---}\rangle = W(G_2) \times 3$. Write

$$n_0 = n_{1-2}n_{1+2}n_{3-4}n_{3+4}, \qquad h_0 = \begin{cases} (-1,-1,-1,-1;-1) & \text{if } r = 2,\\ (\omega^2, \omega, \omega, \omega^2; 1) & \text{if } r = 3.\end{cases}$$

In $\boldsymbol{N}_{\bar r}/\boldsymbol{T}_{\bar r}$ we then have conjugacy class representatives $n\boldsymbol{T}_{\bar r}$ as follows: if $\bar r = 2$ we may take $n = n'n''$, where

$$n' \in \{1, n_0, h_0, h_0n_0\}, \quad n'' \in \{1, n_{3-4}n_{3+4}, n_{1-2}, n_{1-2}n_{2-4}n_{2+4}, n_{1-2}n_{2-3}\};$$

if $\bar r = 3$ we may take $n = n'n''$, where

$$n' \in \{1, n_0, h_0\}, \quad n'' \in \{1, n_{2-3}, n_{1+3}n_{2-3}\}.$$

For each such element n, using the Lang–Steinberg theorem we may write $n = {x'}^{F_m}.{x'}^{-1}$ for some $x' \in \boldsymbol{H}$; by Lemma 2.1 the various F_m-stable elements $(s\tau_{\bar r})^{x'}$

with $s \in \boldsymbol{T}_{\bar{r}}$ between them represent all semisimple $\hat{H}_m$-classes in $\hat{H}_m\tau_{\bar{r}}$. Note that the condition for F_m-stability of $(s\tau_{\bar{r}})^{x'}$ is $s\tau_{\bar{r}} = (s^{F_m}.\tau_{\bar{r}})^n$; as $s \in \boldsymbol{T}_{\bar{r}}$ we have $s^{F_{\bar{r}}} = s^{F_1}$, so that there is a natural correspondence between the semisimple $\hat{H}_1$-classes in $\hat{H}_1\tau_{\bar{r}}$ and the semisimple $\hat{H}_{\bar{r}}$-classes in $\hat{H}_{\bar{r}}\tau_{\bar{r}}$, as we expect from Shintani descent.

Observe that $\boldsymbol{T}_{\bar{r}}$ commutes with the element $y_{\bar{r}}$ defined above, and we have seen that ${\tau_{\bar{r}}}^{y_{\bar{r}}} = h_{+---}(-1)$ or $h_4(\omega)$ according as $\bar{r} = 2$ or 3, so for all $s \in \boldsymbol{T}_{\bar{r}}$ we have $(s\tau_{\bar{r}})^{y_{\bar{r}}} = s{\tau_{\bar{r}}}^{y_{\bar{r}}} \in \boldsymbol{T}_0$; moreover a straightforward calculation shows that $\boldsymbol{T}_0 \cap \boldsymbol{H}^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}$, so that $\boldsymbol{T}_0 \cap (\boldsymbol{H}\langle\tau_{\bar{r}}\rangle)^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}\langle{\tau_{\bar{r}}}^{y_{\bar{r}}}\rangle$. Write

$$\Upsilon_2 = \langle\epsilon_4\rangle, \qquad \Upsilon_3 = \left\langle\epsilon_4, \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\right\rangle;$$

then $\Upsilon_{\bar{r}}$ is a root subsystem of Φ of type $\tilde{A}_{\bar{r}-1}$ such that $\boldsymbol{T}_{\bar{r}} = \bigcap_{\beta \in \Upsilon_{\bar{r}}} \ker\beta$. Indeed, given a root subsystem Υ of Φ of type $\tilde{A}_{\bar{r}-1}$, we define

$$\ker_{\bar{r}} \Upsilon = \bigcap_{\beta \in \Upsilon} \ker\beta \cup \bigcap_{\beta \in \Upsilon} (\ker \bar{r}\beta \setminus \ker\beta)$$

(if $\bar{r} = 2$ and $\beta \in \Upsilon$ then $\ker_{\bar{r}} \Upsilon = \ker 2\beta$; if however $\bar{r} = 3$ and $\{\beta_1, \beta_2\}$ is a simple system of Υ then $\ker_{\bar{r}} \Upsilon = \{s' \in \boldsymbol{T}_0 : \beta_1(s') = \beta_2(s') \in \{1, \omega^{\pm 1}\}\}$ because if $s' \in \boldsymbol{T}_0$ and $\beta_1(s'), \beta_2(s') \in \{1, \omega^{\pm 1}\}$ with $\beta_1(s') \neq \beta_2(s')$ then exactly one of β_1, β_2 and $\beta_1 + \beta_2$ takes the value 1 at s'); from the expression above for ${\tau_{\bar{r}}}^{y_{\bar{r}}}$ we see that $\boldsymbol{T}_{\bar{r}}\langle{\tau_{\bar{r}}}^{y_{\bar{r}}}\rangle = \ker_{\bar{r}} \Upsilon_{\bar{r}}$.

Note that if we set $\boldsymbol{T}_1 = \boldsymbol{T}_0$, $\boldsymbol{N}_1 = \boldsymbol{N} \cap \boldsymbol{H}$, $y_1 = 1$ and $\Upsilon_1 = \varnothing$, then the above (apart from the comment about Shintani descent) also holds for $\bar{r} = 1$.

For each semisimple $\hat{H}_m$-class representative $(s\tau_{\bar{r}})^{x'}$ we may then take a set of representatives of the unipotent conjugacy classes in the $\hat{H}_m$-centralizer, and form products in the usual way, to obtain a complete set of representatives of the $\hat{H}_m$-classes in $\hat{H}_m\tau_{\bar{r}}$; finally, conjugation by g_m gives a complete set of representatives of the H_m-classes in $H_m\tau_{\bar{r}} = H_m\tau_r$. Indeed, as a check we find that the sum of the sizes of the classes so obtained is $|H_m|$.

Now assume $r = p$; then τ_r is a unipotent element of G. Here we refer to [21, Tables X and VIII], which list representatives and centralizer orders for the unipotent H_1-classes lying in $H_1\tau_r$ for $r = 2$ and $r = 3$; the elements are given as $u_i\tau_r$ for $1 \leq i \leq 10$ and $1 \leq i \leq 7$, respectively. Using Shintani descent as described above it is straightforward to obtain representatives ${u_i}^{\dagger}\tau_r$ for the unipotent $\hat{H}_r$-classes lying in $\hat{H}_r\tau_r$. Using the expression for ${\tau_r}^{y_r}$ given above, we may also identify the unipotent $\boldsymbol{G}$-classes containing the elements $u_i\tau_r$ and ${u_i}^{\dagger}\tau_r$, either by inspection or by computing Jordan blocks on the 26-dimensional module for $\boldsymbol{G}$ and referring to [14] (whose notation we use), as described above; we find that for each i the elements $u_i\tau_r$ and ${u_i}^{\dagger}\tau_r$ lie in the same $\boldsymbol{G}$-class, which we give in Table 1.

i	$(u_i\tau_2)^{\boldsymbol{G}} = (u_i{}^{\dagger}\tau_2)^{\boldsymbol{G}}$
1	$\tilde{A}_1$
2	$A_1\tilde{A}_1$
3, 4	B_2
5	$C_3(a_1)$
6	$A_2\tilde{A}_1$
7	$C_3(a_1)^{(2)}$
8, 9	B_3
10	$F_4(a_1)$

i	$(u_i\tau_3)^{\boldsymbol{G}} = (u_i{}^{\dagger}\tau_3)^{\boldsymbol{G}}$
1	$\tilde{A}_2$
2	$\tilde{A}_2A_1$
3	C_3
4, 5, 6	$F_4(a_2)$
7	F_4

Table 1. Unipotent $\boldsymbol{G}$-classes meeting $\hat{H}_m\tau_r$ for $r = 2$ and $r = 3$.

For each unipotent H_1-class representative $u_i\tau_r$, we may then identify its centralizer $C_{H_1}(u_i\tau_r)$; we take representatives s of the semisimple classes therein and form products $su_i\tau_r$. Combining the results for the various values of i gives a complete set of representatives of the H_1-classes in $H_1\tau_r$; again, as a check we find that the sum of the sizes of the classes so obtained is $|H_1|$. In fact in almost all cases the elements s commute with both $u_i{}^{\dagger}$ and τ_r, so as they are stable under F_1 they are also stable under $F_r = F_1\tau_r$ and hence lie in $C_{\hat{H}_r}(u_i{}^{\dagger}\tau_r)$; in the exceptional cases we must replace the elements s by conjugates of them. In this way we also obtain a complete set of representatives of the $\hat{H}_r$-classes in $\hat{H}_r\tau_r$; conjugation by g_r then gives a complete set of representatives of the H_r-classes in $H_r\tau_r$.

Before concluding this section we note that the semisimple G-classes in G are listed in [25] for $p \neq 2$ and in [24] for $p = 2$, each of which groups together classes containing elements with equal centralizers. These groupings, which we shall call *types*, may be described combinatorially, in terms of Weyl group elements by which maximal tori are twisted and root systems of centralizers; the finitely many possibilities for the type are the same for all q (and p), although for a given q not all need occur, either because p is a bad prime for G or because q is small. For each type, both [25] and [24] use a single notation for varying q (although the notations used by [25] and [24] are different).

Here we shall similarly say that a *type* of H_m-class in $H_m\tau_r$ is a collection of such H_m-classes all of which contain elements with equal G-centralizers whose semisimple parts also have equal G-centralizers. The finitely many possibilities for the type of such a semisimple H_m-class are again the same for all q (although once more not all occur for all q); for each type we may likewise use a single notation for varying q. Classes which are not semisimple are however more complicated, because the behavior of unipotent classes in bad characteristic differs from that in good characteristic; nevertheless we may do the following.

Note that the triple (d, e, x) must be one of

$$(0, 1, 1),\ (0, -1, 1),\ (1, 0, 2),\ (1, 0, 4), (1, 1, 2),\ (1, 1, 4),\ (1, -1, 2),\ (1, -1, 4)$$

(the first two cover the cases where $p = 2$, the second two those where $p = 3$, and the last four those where $p > 3$). Fix one such triple (d, e, x) and restrict attention to the prime powers q associated to it, of which there are infinitely many. We then find that there are finitely many possibilities for the type of H_m-class in $H_m\tau_r$, that they are the same for all such q, and that for a given type of H_m-class in $H_m\tau_r$ the order of the H_m-centralizer is a polynomial in q: if $p = r$ then for each unipotent class appearing in Table 1 we have obtained a collection of semisimple elements lying in the centralizer, and after forming products to obtain class representatives the statements follow by inspection; if instead $p \neq r$ then for a given type of semisimple H_m-class in $H_m\tau_r$ inspection of tables of unipotent classes in the various simple factors appearing in the centralizer (see, for example, [18] as mentioned above) shows that the parametrization of such unipotent classes is the same for all q, and that for a given unipotent class the centralizer order is a polynomial in q, from which the statements follow.

Indeed from the above we see that more is true: for a fixed triple (d, e, x) and type of H_m-class in $H_m\tau_r$, if we take one such H_m-class and let s be the semisimple part of an element thereof, then the number of H_m-classes of the type concerned which contain elements with semisimple part s is the same for all q.

Thus for a fixed triple (d, e, x) we may extend the notion of 'type of H_m-class in $H_m\tau_r$' to cover all prime powers q associated to it, and H_m-centralizer orders are polynomials in q. This will be crucial to the approach to calculating character scalar products which we now describe.

2.2. ***Character scalar products.*** Recall that, for any subgroup H of G, when G acts on the cosets of H the value at $g \in G$ of the permutation character ${1_H}^G$ is the number of cosets $g'H$ for $g' \in G$ with $gg'H = g'H$. For any generalized character χ of G we have by Frobenius reciprocity

$$({1_H}^G, \chi)_G = (1_H, \chi|_H)_H = \frac{1}{|H|} \sum_{g \in H} \chi(g) = \sum_{[g] \subset H} \frac{\chi(g)}{|C_H(g)|},$$

where the final sum is over all conjugacy classes $[g]$ in H.

We shall be interested in the cases where $H = H_m\langle\tau_r\rangle$, for $m, r \in \{1, 2, 3\}$ with $\{m, r\} \neq \{2, 3\}$; by taking first $r = 1$, and then $r \in \{2, 3\}$ (and considering the inverses of classes if $r = 3$), in the final sum it will suffice to treat the $H_m\langle\tau_r\rangle$-classes which lie in $H_m\tau_r$, which are of course simply the H_m-classes lying in $H_m\tau_r$. We may then see the scalar product as a sum of contributions from the different types of H_m-class in $H_m\tau_r$, with each contribution being the fraction with numerator given by the sum of the character values concerned and denominator equal to the common order of the $H_m\langle\tau_r\rangle$-centralizer (which is r times the order of the H_m-centralizer).

Now take χ to be a generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$, where $\boldsymbol{T}$ is an F-stable maximal torus of $\boldsymbol{G}$ and θ is a linear character of $\boldsymbol{T}^F$, and consider the contribution to the scalar product $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})_G$ from a given type of H_m-class in $H_m\tau_r$. At the end of Section 2.1 we observed that, for a fixed triple (d, e, x), the order of the H_m-centralizer appearing in the denominator is a polynomial in q, and we shall see that the same is true of the number of H_m-classes of the type concerned. We wish to show that the same is also true of the numerator; in order to make such a statement, we first need to say a little more about the characters $R_{\boldsymbol{T},\theta}$.

To begin with, an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$ may be written as ${}^g\boldsymbol{T}_0$ for some $g \in \boldsymbol{G}$, and then we have $g^{-1}g^F \in \boldsymbol{N}$; if we write $w = g^{-1}g^F\boldsymbol{T}_0 \in W$, we say that $\boldsymbol{T}$ is obtained from $\boldsymbol{T}_0$ by twisting with w — although different choices for g may give different elements w of W, the F-conjugacy class $[w]$ of w in W is uniquely determined by $\boldsymbol{T}$, and indeed there is a bijection between F-conjugacy classes in W and G-classes of F-stable maximal tori of $\boldsymbol{G}$. (All of this is well known; see, for example, [4, Section 3.3].) In the present case F acts trivially on W, so that F-conjugacy in W is simply conjugacy.

Next, if $\boldsymbol{T} = {}^g\boldsymbol{T}_0$ as above the finite group $\boldsymbol{T}^F$ is equal to ${}^g({\boldsymbol{T}_0}^{(Fw^{-1})})$, where we write ${\boldsymbol{T}_0}^{(Fw^{-1})} = \{s_0 \in \boldsymbol{T}_0 : s_0 = {}^w({s_0}^F)\}$. Thus given a linear character θ of $\boldsymbol{T}^F$, we may write $\theta = {}^g\theta_0$, where θ_0 is the linear character of ${\boldsymbol{T}_0}^{(Fw^{-1})}$ defined by $\theta_0(s_0) = {}^g\theta_0({}^gs_0)$. We may then define $\Phi_{\theta_0} = \{\alpha \in \Phi : \ker\alpha \geq \ker\theta_0\}$ to be the root subsystem of Φ comprising those roots whose kernel contains $\ker\theta_0$; for fixed $\boldsymbol{T}$ and θ, different choices for g may give different root subsystems Φ_{θ_0}, but it is straightforward to see that the set of possible Φ_{θ_0} forms a single orbit under the action of W. Moreover, in the present case it is easy to see that two root subsystems of Φ lie in the same W-orbit if and only if they are isomorphic (where we require an isomorphism to preserve root lengths).

We may therefore associate to each generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$ a pair $([w], [\Phi'])$ consisting of an F-conjugacy class $[w]$ in W and an isomorphism class $[\Phi']$ of root subsystems of Φ. There are finitely many such pairs, and they are the same for all q, although for a given q not all pairs may be associated to a character $R_{\boldsymbol{T},\theta}$. We are now in a position to state our claim concerning the numerator in the contribution to the scalar product $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})_G$ from a given type of H_m-class in $H_m\tau_r$.

Claim 1. *For a fixed triple (d, e, x), pair $([w], [\Phi'])$ and type of H_m-class in $H_m\tau_r$, the sum over the corresponding H_m-classes $[su]$, where s and u are commuting semisimple and unipotent elements, respectively, of the character values $R_{\boldsymbol{T},\theta}(su)$ is a polynomial in q. Moreover the degree of this polynomial is at most $d_1 + d_2$, where $d_1 = \dim\big(Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle\big)$ is the degree of the polynomial giving the number of H_m-classes of the type concerned, and $d_2 = \frac{1}{2}(\dim C_{\boldsymbol{G}}(su) - \dim \boldsymbol{T})$.*

We shall prove Claim 1 in Section 2.4, following some preliminary results in Section 2.3. Once this has been done it will follow that, for a fixed triple (d, e, x), pair $([w], [\Phi'])$ and type of H_m-class, the contribution to the scalar product $({1_{H_m\langle\tau_r\rangle}}^G, R_{\boldsymbol{T},\theta})_G$ from the type of H_m-class is a ratio of two polynomials in q. Since this scalar product is an integer, it will suffice to determine the *nonnegligible part* of the contribution, where given two polynomials p_1 and p_2 with $p_2 \neq 0$ we let p_3 be the unique polynomial such that $p_1 - p_2 p_3$ is of strictly smaller degree than p_2, and say that the nonnegligible part of $p_1(q)/p_2(q)$ is $p_3(q)$. The scalar product will then be the sum of the nonnegligible parts of the contributions from the finitely many different types of H_m-class, and will therefore be a polynomial in q; by taking linear combinations of Deligne–Lusztig characters it will in fact follow that the same is true of all multiplicities of irreducible characters in the permutation character ${1_{H_m\langle\tau_r\rangle}}^G$.

Note that the degree bound in Claim 1 depends only on the type of H_m-class; we call the type of H_m-class *relevant* if this bound is greater than or equal to the degree in q of the order of the H_m-centralizer, and *irrelevant* otherwise. Nonnegligible parts of contributions from irrelevant types of H_m-class are then zero for all $R_{\boldsymbol{T},\theta}$; it therefore suffices to consider relevant types of H_m-class, and our calculations in subsequent sections will begin by identifying these. We shall see that this significantly reduces the number of classes requiring consideration.

We conclude this section by observing that the decompositions we shall obtain will of necessity be more complicated than was the case in the action of G on cosets of $B_4(q)$ treated in [15]. On one level this is simply because the subgroups H_m are considerably smaller than $B_4(q)$ (the ratio of orders is approximately q^8); however, there is a more fundamental difference between the actions here and that of [15]. In [16] a criterion was established for certain subgroups $\boldsymbol{H}$ of algebraic groups $\boldsymbol{G}$ to be spherical (i.e., to have finitely many orbits on the flag variety $\boldsymbol{G}/\boldsymbol{B}$); the proof proceeded by taking fixed points under Frobenius morphisms F and considering scalar products of permutation characters ${1_{\boldsymbol{H}^F}}^{\boldsymbol{G}^F}$ and principal series characters ${1_{\boldsymbol{B}^F}}^{\boldsymbol{G}^F}$. It was shown that if $\boldsymbol{H}$ is of the form $\langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha : \alpha \in \Psi\rangle$ for some subsystem Ψ of Φ, then $\boldsymbol{H}$ is spherical if Ψ satisfies the following condition: there do not exist $\alpha, \beta \in \Phi \setminus \Psi$ with $\alpha + \beta \in \Phi \setminus \Psi$. (Although it is not stated in [16] that this condition is also necessary for sphericality, it is nevertheless clear from the proof that if it is not met then the scalar product $({1_{\boldsymbol{H}^F}}^{\boldsymbol{G}^F}, {1_{\boldsymbol{B}^F}}^{\boldsymbol{G}^F})$ grows with q.) In the case of B_4, the condition is satisfied; correspondingly, the multiplicities in ${1_{B_4(q)}}^{F_4(q)}$ of unipotent characters lying in the principal series are forced to be constants rather than polynomials in q of positive degree (and indeed they are all 0 or 1). In the case of D_4, however, the condition is not satisfied (as we may take α and β to be the two short simple roots of G); thus some multiplicities in ${1_{H_m\langle\tau_r\rangle}}^G$ must turn out to be polynomials in q of positive degree.

$\Phi(\boldsymbol{T})_s$	p	$Z(C_{\boldsymbol{G}}(s))/Z(C_{\boldsymbol{G}}(s))^\circ$
$A_3\tilde{A}_1$	$\neq 2$	$\mathbb{Z}_4$
$A_2\tilde{A}_2$	$\neq 3$	$\mathbb{Z}_3$
B_4, A_1C_3, A_3, A_1B_2, ${A_1}^2\tilde{A}_1$, ${A_1}^2$	$\neq 2$	$\mathbb{Z}_2$
F_4, B_3, C_3, $A_2\tilde{A}_1$, $A_1\tilde{A}_2$, B_2, A_2, $\tilde{A}_2$, $A_1\tilde{A}_1$, A_1, $\tilde{A}_1$, $\varnothing$	any	1

Table 2. Root subsystems $\Phi(\boldsymbol{T})_s$ of $\Phi(\boldsymbol{T})$.

2.3. *Preliminary results.* We begin with some results concerning root subsystems. Let $\boldsymbol{T}$ be any maximal torus of $\boldsymbol{G}$. Write $\Phi(\boldsymbol{T}) \subset \operatorname{Hom}(\boldsymbol{T}, k^*)$ for the root system of $\boldsymbol{G}$ with respect to the maximal torus $\boldsymbol{T}$; for a root subsystem Ψ of $\Phi(\boldsymbol{T})$ define

$$\ker \Psi = \bigcap_{\alpha \in \Psi} \ker \alpha.$$

For $s \in \boldsymbol{T}$ define $\Phi(\boldsymbol{T})_s$ to be the root subsystem $\{\alpha \in \Phi(\boldsymbol{T}) : \alpha(s) = 1\}$ of $\Phi(\boldsymbol{T})$; by [4, Theorems 3.5.6, 3.5.3] $C_{\boldsymbol{G}}(s)$ is generated by $\boldsymbol{T}$ and the root groups relative to $\boldsymbol{T}$ corresponding to the roots in $\Phi(\boldsymbol{T})_s$, so $Z(C_{\boldsymbol{G}}(s)) = \ker \Phi(\boldsymbol{T})_s$.

Lemma 2.2. *Up to the action of the Weyl group of $\boldsymbol{T}$, the possibilities for the root subsystem $\Phi(\boldsymbol{T})_s$ and the component group $Z(C_{\boldsymbol{G}}(s))/Z(C_{\boldsymbol{G}}(s))^\circ$ are listed in Table 2.*

Proof. The root subsystems $\Phi(\boldsymbol{T})_s$ may be obtained using [9, Construction 4.1], which also gives the restrictions on the characteristic p: for example, the subsystem ${A_1}^2\tilde{A}_1$ is formed from the extended Dynkin diagram by removing the nodes corresponding to the first and third simple roots, whose coefficients in the highest root are 2 and 4, respectively, so the order of any element s of $\boldsymbol{T}$ having $\Phi(\boldsymbol{T})_s = {A_1}^2\tilde{A}_1$ must be a positive integer linear combination of 2 and 4 and thus must be even, whence p cannot be 2. A simple calculation in each case then gives the component group $Z(C_{\boldsymbol{G}}(s))/Z(C_{\boldsymbol{G}}(s))^\circ$. □

Given a root subsystem Ψ of $\Phi(\boldsymbol{T})$, we define

$$\Psi^{(p)} = \{\beta \in \Phi(\boldsymbol{T}) : p^i\beta \in \mathbb{Z}\Psi \text{ for some } i \in \mathbb{N}\};$$

clearly $\Psi^{(p)}$ is a root subsystem of $\Phi(\boldsymbol{T})$ containing Ψ, and we call it the p-closure of Ψ. We say that Ψ is p-closed if $\Psi^{(p)} = \Psi$; certainly $(\Psi^{(p)})^{(p)} = \Psi^{(p)}$, i.e., $\Psi^{(p)}$ is p-closed. Observe that the eight instances of pairs (Ψ, p) missing from Table 2 are not p-closed; indeed a simple calculation in each case shows that these p-closures are as given in Table 3.

Lemma 2.3. *Given a root subsystem Ψ of $\Phi(\boldsymbol{T})$, the following are true*:

(i) $\ker \Psi = \ker \Psi^{(p)}$;

Ψ	p	$\Psi^{(p)}$
$A_2\tilde{A}_2$	3	F_4
B_4, $A_3\tilde{A}_1$, A_1C_3	2	F_4
A_3, ${A_1}^2\tilde{A}_1$	2	B_3
A_1B_2	2	C_3
${A_1}^2$	2	B_2

Table 3. p-closures of root subsystems Ψ of $\Phi(\boldsymbol{T})$.

(ii) *there exists* $s \in \ker\Psi$ *such that* $\Psi^{(p)} = \Phi(\boldsymbol{T})_s$;

(iii) $\Psi^{(p)} = \bigcap\{\Phi(\boldsymbol{T})_s : s \in \boldsymbol{T},\ \Phi(\boldsymbol{T})_s \supseteq \Psi\}$; *and*

(iv) Ψ *is not p-closed if and only if* Ψ *is listed in Table 3.*

Proof. As the only p-th root of unity in k is 1, for $i \in \mathbb{N}$ we have $\ker p^i\beta = \ker\beta$. Thus if $\beta \in \Psi^{(p)}$ so that $p^i\beta \in \mathbb{Z}\Psi$, we have $\ker\Psi \subseteq \ker p^i\beta = \ker\beta$, proving (i). Since the only root subsystems of $\Phi(\boldsymbol{T})$ which are not of the form $\Phi(\boldsymbol{T})_s$ for some $s \in \boldsymbol{T}$ are the eight listed in Table 3, which are not p-closed, they cannot be $\Psi^{(p)}$; thus $\Psi^{(p)} = \Phi(\boldsymbol{T})_s$ for some $s \in \boldsymbol{T}$, and then for all $\alpha \in \Psi$ we have $\alpha(s) = 1$, proving (ii). Given $s \in \boldsymbol{T}$ with $\Phi(\boldsymbol{T})_s \supseteq \Psi$, for all $\alpha \in \Phi(\boldsymbol{T})_s$ we have $\alpha(s) = 1$; if $\beta \in \Psi^{(p)}$ then for some $i \in \mathbb{N}$ we have $p^i\beta \in \mathbb{Z}\Psi$, so $s \in \ker p^i\beta = \ker\beta$, whence $\Psi^{(p)} \subseteq \Phi(\boldsymbol{T})_s$. Thus $\Psi^{(p)}$ is contained in all the root subsystems of $\Phi(\boldsymbol{T})$ of the form $\Phi(\boldsymbol{T})_s$ which contain Ψ, and by (ii) it is one such root subsystem, proving (iii). Finally (iii) implies that every root subsystem of $\Phi(\boldsymbol{T})$ of the form $\Phi(\boldsymbol{T})_s$ is p-closed, proving (iv). □

Write $W(\boldsymbol{T}) = N_{\boldsymbol{G}}(\boldsymbol{T})/\boldsymbol{T}$ for the Weyl group of $\boldsymbol{T}$; given a root subsystem Ψ of $\Phi(\boldsymbol{T})$, let $\Psi^\perp$ be the root subsystem of $\Phi(\boldsymbol{T})$ consisting of the roots orthogonal to those in Ψ, and write $W(\Psi)$ and $W(\Psi^\perp)$ for the subgroups of $W(\boldsymbol{T})$ generated by reflections in the roots lying in Ψ and $\Psi^\perp$, respectively.

Lemma 2.4. *Given a root subsystem* Ψ *of* $\Phi(\boldsymbol{T})$, *the index of* $W(\Psi)W(\Psi^\perp)$ *in* $N_{W(\boldsymbol{T})}(W(\Psi))$ *is* 1 *or* 2 *according as the group of graph automorphisms of* Ψ *afforded by* $W(\boldsymbol{T})$ *is trivial or not*; *if it is nontrivial, then the coset* $N_{W(\boldsymbol{T})}(W(\Psi)) \setminus W(\Psi)W(\Psi^\perp)$ *contains the long word of* $W(\boldsymbol{T})$, *unless* $\Psi = {A_1}^2$ *or* ${A_1}^2\tilde{A}_1$, *in which case it contains the reflection in a short root in* $\Phi(\boldsymbol{T})$ *which is half the sum or difference of two long roots in* Ψ.

Proof. This is an easy calculation; note that Ψ has a nontrivial graph automorphism if and only if $\Psi = A_3\tilde{A}_1$, A_3, $A_2\tilde{A}_2$, $A_2\tilde{A}_1$, $A_1\tilde{A}_2$, A_2, $\tilde{A}_2$, ${A_1}^2\tilde{A}_1$ or ${A_1}^2$, and all graph automorphisms are afforded by elements of $W(\boldsymbol{T})$, unless $\Psi = A_2\tilde{A}_2$ in which case the only such graph automorphism acts nontrivially on each simple factor. □

For each root subsystem Ψ of $\Phi(\boldsymbol{T})$ set

$$\boldsymbol{Z}_\Psi = \{s \in \boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle : \Phi(\boldsymbol{T})_s \supseteq \Psi\}, \quad \tilde{\boldsymbol{Z}}_\Psi = \{s \in \boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle : \Phi(\boldsymbol{T})_s = \Psi\}.$$

Then $\boldsymbol{Z}_\Psi = \ker \Psi \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$, so for all $s \in \boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$ we have

$$\boldsymbol{Z}_{\Phi(\boldsymbol{T})_s} = Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle.$$

For a subset $\boldsymbol{X}$ of $\boldsymbol{T} \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$, let $\mathbb{1}_{\boldsymbol{X}}$ denote the indicator function of $\boldsymbol{X}$.

Lemma 2.5. *There exist integers $n_{\Psi,\Psi'}$ for root subsystems Ψ, Ψ' of $\Phi(\boldsymbol{T})$, with $n_{\Psi,\Psi} = 1$, and $n_{\Psi,\Psi'} = 0$ if $\Psi' \not\supseteq \Psi$, such that $\mathbb{1}_{\tilde{\boldsymbol{Z}}_\Psi} = \sum_{\Psi'} n_{\Psi,\Psi'} \mathbb{1}_{\boldsymbol{Z}_{\Psi'}}$.*

Proof. For each root subsystem Ψ of $\Phi(\boldsymbol{T})$, the set $\boldsymbol{Z}_\Psi$ is the disjoint union of the sets $\tilde{\boldsymbol{Z}}_{\Psi'}$ as Ψ' runs over the root subsystems of $\Phi(\boldsymbol{T})$ which contain Ψ, so $\mathbb{1}_{\boldsymbol{Z}_\Psi} = \sum_{\Psi' \supseteq \Psi} \mathbb{1}_{\tilde{\boldsymbol{Z}}_{\Psi'}}$. We may now use downward induction on subsystem size: for $\Psi = \Phi(\boldsymbol{T})$ we have $\mathbb{1}_{\tilde{\boldsymbol{Z}}_\Psi} = \mathbb{1}_{\boldsymbol{Z}_\Psi}$, while for a proper subsystem Ψ of $\Phi(\boldsymbol{T})$ we have $\mathbb{1}_{\tilde{\boldsymbol{Z}}_\Psi} = \mathbb{1}_{\boldsymbol{Z}_\Psi} - \sum_{\Psi' \supset \Psi} \mathbb{1}_{\tilde{\boldsymbol{Z}}_{\Psi'}}$, and by induction we may assume each term $\mathbb{1}_{\tilde{\boldsymbol{Z}}_{\Psi'}}$ with $\Psi' \supset \Psi$ is an integer linear combination of terms $\mathbb{1}_{\boldsymbol{Z}_{\Psi''}}$ with $\Psi'' \supseteq \Psi' \supset \Psi$. The result follows. □

The next result concerns commuting elements of $\boldsymbol{N}$. For convenience we confine our attention to elements of the group $\langle n_\alpha : \alpha \in \Phi\rangle$, which we call N'; note that elements of N' are F-stable, and N' has the normal subgroup $\{s \in \boldsymbol{T}_0 : s^2 = 1\} = \langle h_\alpha(-1) : \alpha \in \Phi\rangle \cong {\mathbb{Z}_{d+1}}^4$, with the quotient being naturally isomorphic to W. As in Section 2.1, write $n_0 = n_{1-2}n_{1+2}n_{3-4}n_{3+4} \in N'$, so that $n_0\boldsymbol{T}_0$ is the long word w_0 of W, which is central in W.

Lemma 2.6. *For each $w \in W$ there exists $n \in N'$ with $n\boldsymbol{T}_0 = w$ such that for all $w' \in C_W(w)$ there exists $n' \in C_{N'}(n)$ with $n'\boldsymbol{T}_0 = w'$.*

Proof. It suffices to treat representatives w of the 25 conjugacy classes in W; in Table 4 we list 25 corresponding elements n lying in N', and for each we give elements n' of N' such that the elements $w' = n'\boldsymbol{T}_0$ of W generate $C_W(w)$. It is in each case a straightforward calculation to check that n commutes with each element n' listed. □

Now suppose $\boldsymbol{T}$ is F-stable, and θ is a linear character of $\boldsymbol{T}^F$. We recall that the generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$ takes values as follows: given $s, u \in G$ commuting semisimple and unipotent elements, respectively, by [4, Theorem 7.2.8] we have

$$R_{\boldsymbol{T},\theta}(su) = \frac{1}{|C_G(s)|} \sum_{\substack{x' \in G \\ s^{x'} \in \boldsymbol{T}}} \theta(s^{x'}) Q^{C_{\boldsymbol{G}}(s)}_{{}^{x'}\boldsymbol{T}}(u)$$

(as already observed, because $\boldsymbol{G}$ is simply connected the centralizer $C_{\boldsymbol{G}}(s)$ is connected). Here $Q^{C_{\boldsymbol{G}}(s)}_{{}^{x'}\boldsymbol{T}}$ is the appropriate Green function; it is defined to be the

n	$n' \in C_{N'}(n)$
1	$n_{2-3}, n_{3-4}, n_4, n_{+---}$
n_{3-4}	$n_{3-4}, n_2, n_{+---}, n_{3+4}$
n_4	$n_4, n_{1-2}, n_{2-3}, n_3h_{3-4}(-1)$
$n_{3-4}n_{3+4}$	$n_{3-4}, n_4, n_{1-2}, n_2$
$n_{1-2}n_4$	$n_{1-2}, n_4, n_{1+2}, n_3h_{3-4}(-1)$
$n_{2-3}n_{3-4}$	$n_{2-3}n_{3-4}, n_1, n_{+---}, n_0$
$n_{+---}n_4$	$n_{+---}n_4, n_{2-3}, n_{1+2}, n_0$
$n_{3-4}n_4$	$n_{3-4}n_4, n_{1-2}, n_2h_{3-4}(-1)$
$n_{3-4}n_{1-2}n_{1+2}$	$n_{3+4}, n_2, n_{+--+}, n_{3-4}$
$n_{3-4}n_{3+4}n_2h_{1-2}(-1)$	$n_1, n_{2-3}, n_{3-4}, n_4h_{1-2}(-1)$
$n_{1-2}n_{2-3}n_{3-4}$	$n_{1-2}n_{2-3}n_{3-4}, n_{++++}, n_0$
$n_{1-2}n_{3-4}n_4$	$n_{3-4}n_4, n_{1-2}, n_{1+2}$
$n_{+--+}n_{3-4}n_4$	$n_{+--+}n_{3-4}n_4, n_{1+2}$
$n_{2-3}n_{3-4}n_4$	$n_{2-3}n_{3-4}n_4, n_1h_{+--+}(-1)$
$n_{++--}n_4n_{1-2}$	$n_{++--}n_4n_{1-2}, n_0$
$n_{2-3}n_{3-4}n_1$	$n_{2-3}n_{3-4}n_1, n_0$
n_0	$n_{2-3}, n_{3-4}, n_4, n_{+---}$
$n_{++++}n_1n_{2-3}n_{3-4}$	$n_{++++}n_1, n_{2-3}n_{3-4}, n_{1-4}n_{3+4}, n_0$
$n_{1-2}n_{1+2}n_{3-4}n_4$	$n_{3-4}n_4, n_{1-2}, n_2h_{3-4}(-1)$
$n_{+-+-}n_1n_2n_4$	$n_{+-+-}n_3h_{2-3}(-1), n_{2-4}, n_{1+2}, n_0$
$n_{1-2}n_{2-3}n_3n_4h_{+---}(-1)$	$n_{1-2}n_{2-3}n_3, n_4, n_{+-+-}$
$n_{1-2}n_2n_{3-4}n_4h_{3-4}(-1)$	$n_{1-2}n_2h_{3-4}(-1), n_{3-4}n_4, n_{1-3}n_{2-4}, n_{2+3}n_{+---}$
$n_{1-2}n_{2-3}n_{3-4}n_4$	$n_{1-2}n_{2-3}n_{3-4}n_4$
$n_{+---}n_{3-4}n_{2-3}n_2$	$n_{+---}n_{3-4}n_{2-3}n_2$
$n_{+---}n_{3+4}n_{2-3}n_3h_{1-2}(-1)$	$n_{++++}n_1, n_{2-3}n_{3-4}, n_{1-4}n_{3+4}, n_0$

Table 4. Commuting elements of N'.

restriction of the generalized Deligne–Lusztig character $R_{x'\boldsymbol{T},1}$ for the group $C_G(s)$ to the set of unipotent elements therein.

Some of the values taken by Green functions are given in [4]: if the unipotent element is the identity then by [4, Theorem 7.5.1] the value is, up to sign, the p-part of the index of the maximal torus concerned (and in particular, if the torus is maximally split the sign is '$+$', as may be seen by comparing the statement of [4, Theorem 7.5.1] with the definitions on [4, pp. 197, 199]); at the other extreme, if the unipotent element is regular then by [4, Proposition 8.4.1] the value is 1. Using [4, Proposition 3.3.5] for the first of these we see that both are polynomials in q (for all q, not just those for a fixed triple (d, e, x)).

For the purposes of this paper we are interested in Green functions for groups which occur as centralizers of semisimple elements of $\boldsymbol{G}$; an easy application of [4, Property 7.1.9] reduces to the consideration of simple groups appearing as factors in these centralizers. For these groups Green functions have been known for some time: for groups of type A, see Green's original paper [8]; for other groups of classical type, see [11] for $p > 2$ and [27] for $p = 2$; for the group F_4, see [26] for $p > 3$, [7] for $p = 3$ and [22] for $p = 2$. It follows that for a fixed triple (d, e, x) all Green function values are polynomials in q.

(It is in fact known that Green function values in general are polynomials in q, provided one restricts to those q lying in a given residue class modulo an appropriate modulus; the term 'polynomial on residue classes' has been used to describe this phenomenon. Frank Lübeck has in fact recently completed the determination of all (ordinary) Green functions for simple groups by computing those for $E_8(q)$ in bad characteristic in [19], the introduction to which contains a useful summary of the position; the author, who is far from being an expert on these matters, is grateful to him for his helpful comments.)

We end this discussion of Green functions with two lemmas. The first uses the orthogonality of Green functions to give a bound on degrees.

Lemma 2.7. *Given an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$, $s \in \boldsymbol{T}^F$ and $u \in C_G(s)$ unipotent, for a fixed triple (d, e, x) the degree as a polynomial in q of the Green function value $Q_{\boldsymbol{T}}^{C_G(s)}(u)$ is at most $\frac{1}{2}(\dim C_{\boldsymbol{G}}(su) - \dim \boldsymbol{T})$.*

Proof. By [4, Proposition 7.6.2] we have

$$\sum_{u' \in C_G(s) \text{ unipotent}} Q_{\boldsymbol{T}}^{C_G(s)}(u')^2 = \frac{|C_G(s)| \cdot |N_G(\boldsymbol{T})|}{|\boldsymbol{T}^F|^2};$$

restricting to the class $u^{C_G(s)}$ we have

$$|u^{C_G(s)}| \cdot Q_{\boldsymbol{T}}^{C_G(s)}(u)^2 \leq \frac{|C_G(s)| \cdot |N_G(\boldsymbol{T})|}{|\boldsymbol{T}^F|^2},$$

whence

$$Q_{\boldsymbol{T}}^{C_G(s)}(u) \leq \frac{|C_{C_G(s)}(u)|^{1/2} \cdot |N_G(\boldsymbol{T})|^{1/2}}{|\boldsymbol{T}^F|} = \left(\frac{|C_G(su)|}{|\boldsymbol{T}^F|}\right)^{1/2} \cdot |N_G(\boldsymbol{T}) : \boldsymbol{T}^F|^{1/2}.$$

As this is true for all q concerned, the result follows. □

The second concerns the effect on a Green function $Q_{\boldsymbol{T}}^{C_G(s)}$ of conjugating by an element of G which normalizes $C_{\boldsymbol{G}}(s)$.

Lemma 2.8. *Given an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$, $s \in \boldsymbol{T}^F$ and $u \in C_G(s)$ unipotent, if $x' \in G$ normalizes $C_{\boldsymbol{G}}(s)$ then $Q_{\boldsymbol{T}}^{C_G(s)}(u) = Q_{{}^{x'}\boldsymbol{T}}^{C_G(s)}({}^{x'}u)$.*

Proof. As in [4, Section 7.2] we have Lang's map $L : C_{\boldsymbol{G}}(s) \to C_{\boldsymbol{G}}(s)$ defined by $L(g) = g^{-1}g^F$. Take a Borel subgroup $\boldsymbol{B}'$ of $C_{\boldsymbol{G}}(s)$ containing $\boldsymbol{T}$, let $\boldsymbol{U}'$ be its unipotent radical and write $\tilde{\boldsymbol{X}} = L^{-1}(\boldsymbol{U}')$. It is then shown in the proof of

[4, Theorem 7.2.8] that $Q_{\boldsymbol{T}}^{C_{\boldsymbol{G}}(s)}(u) = (1/|\boldsymbol{T}^F|)\mathscr{L}(u, \tilde{\boldsymbol{X}})$, where $\mathscr{L}(u, \tilde{\boldsymbol{X}})$ is the Lefschetz number of u on $\tilde{\boldsymbol{X}}$. Conjugating everything by x', and noting that ${}^{x'}\tilde{\boldsymbol{X}} = L^{-1}({}^{x'}\boldsymbol{U}')$, we have

$$Q_{{}^{x'}\boldsymbol{T}}^{C_{\boldsymbol{G}}(s)}({}^{x'}u) = \frac{1}{|({}^{x'}\boldsymbol{T})^F|}\mathscr{L}({}^{x'}u, {}^{x'}\tilde{\boldsymbol{X}}).$$

As x' is F-stable we have $({}^{x'}\boldsymbol{T})^F = {}^{x'}\boldsymbol{T}^F$; and if we take the map $f : \tilde{\boldsymbol{X}} \to {}^{x'}\tilde{\boldsymbol{X}}$ given by conjugation by x' and apply [4, Property 7.1.5] we obtain $\mathscr{L}(u, \tilde{\boldsymbol{X}}) = \mathscr{L}({}^{x'}u, {}^{x'}\tilde{\boldsymbol{X}})$. Thus $Q_{\boldsymbol{T}}^{C_{\boldsymbol{G}}(s)}(u) = Q_{{}^{x'}\boldsymbol{T}}^{C_{\boldsymbol{G}}(s)}({}^{x'}u)$ as required. □

2.4. ***Proof of Claim 1.*** Take a triple (d, e, x), a pair $([w], [\Phi'])$ and a type of H_m-class in $H_m\tau_r$; take a representative w of the $(F\text{-})$conjugacy class $[w]$ in W, and take $n \in N$ with $n\boldsymbol{T}_0 = w$ as in Lemma 2.6. All of these will be fixed throughout this section; the statement that an expression is ‘a polynomial in q’ or that a set or number is ‘independent of q’ will always make this assumption.

Take a prime power q associated to (d, e, x), and let F be the corresponding Frobenius map. Take $g \in \boldsymbol{G}$ satisfying $g^{-1}g^F = n$; write $\boldsymbol{T} = {}^g\boldsymbol{T}_0$, so that $\boldsymbol{T}$ is an F-stable maximal torus of $\boldsymbol{G}$ twisted by w, and $T = \boldsymbol{T}^F$. Take a linear character θ of T such that, if as in Section 2.2 we write $\theta = {}^g\theta_0$ and $\Phi_{\theta_0} = \{\alpha \in \Phi : \ker\alpha \geq \ker\theta_0\}$, the root subsystem Φ_{θ_0} of Φ lies in the isomorphism class $[\Phi']$. Take an H_m-class $[su]$ in $H_m\tau_r$ of the type concerned, where s and u are commuting semisimple and unipotent elements, respectively, such that $s \in T \cap H_m\tau_{\bar{r}}$ and $u \in H_m\tau_{r/\bar{r}}$; write $s = {}^g s_0$ with $s_0 \in \boldsymbol{T}_0$, and $\Phi_{s_0} = \Phi(\boldsymbol{T}_0)_{s_0}$.

Our first result links w and s_0.

Lemma 2.9. *We have ${s_0}^F = {s_0}^w$, and the element w normalizes $C_W(s_0)$.*

Proof. As ${}^g s_0 = ({}^g s_0)^F = {}^{g^F}{s_0}^F$ we have ${s_0}^F = {s_0}^{g^{-1}g^F} = {s_0}^w$. If $c \in C_W(s_0)$, as F acts trivially on W we have ${s_0}^{wcw^{-1}} = ({s_0}^F)^{cw^{-1}} = ({s_0}^F)^{(c^F)w^{-1}} = (({s_0}^c)^F)^{w^{-1}} = ({s_0}^F)^{w^{-1}} = s_0$, whence $wcw^{-1} \in C_W(s_0)$ as required. □

Note that Lemma 2.9 implies that w acts by conjugation on the set of right cosets of $C_W(s_0)$; let the number of fixed points in this action be l, and choose $w_{(1)} = 1, \ldots, w_{(l)} \in W$ such that the fixed points are $C_W(s_0)w_{(i)}$ for $i \leq l$.

Lemma 2.10. *The distinct $\boldsymbol{G}$-conjugates of s lying in T are ${}^g({s_0}^{w_{(i)}})$ for $i \leq l$. Moreover, for each $i \leq l$, if we take $n_{(i)} \in \boldsymbol{N}$ with $n_{(i)}\boldsymbol{T}_0 = w_{(i)}$, there exists $y_{(i)} \in C_{\boldsymbol{G}}(s)$ such that if we set $x_i = y_{(i)}.{}^g n_{(i)}$ then $x_i \in G$ and $s^{x_i} = {}^g({s_0}^{w_{(i)}})$.*

Proof. Any $\boldsymbol{G}$-conjugate of $s = {}^g s_0$ lying in $\boldsymbol{T} = {}^g\boldsymbol{T}_0$ is of the form $s^{{}^g n'} = {}^g({s_0}^{n'})$ for some $n' \in \boldsymbol{N}$; if it is also F-stable and we write $w' = n'\boldsymbol{T}_0$ then ${}^g({s_0}^{n'}) = {}^{g^F}(({s_0}^F)^{(n')^F}) = {}^{g^F}({s_0}^{n(n')^F})$, so $n(n')^F(g^F)^{-1}gn'^{-1} \in C_{\boldsymbol{G}}(s_0)$, whence $ww'w^{-1}w'^{-1} \in C_W(s_0)$, i.e., $ww'w^{-1} \in C_W(s_0)w'$. Thus for some $i \leq l$ the conjugate is ${}^g(s_0{}^{w_{(i)}})$; as distinct values i give distinct conjugates, the first statement follows. Given $n_{(i)} \in \boldsymbol{N}$ with $n_{(i)}\boldsymbol{T}_0 = w_{(i)}$, we have $s^{{}^g n_{(i)}} = (s^F)^{({}^g n_{(i)})^F} = s^{({}^g n_{(i)})^F}$, so ${}^g n_{(i)}(({}^g n_{(i)})^F)^{-1} \in C_{\boldsymbol{G}}(s)$; by [4, Theorem 3.5.6] $C_{\boldsymbol{G}}(s)$ is connected, so by the Lang–Steinberg theorem there exists $y_{(i)} \in C_{\boldsymbol{G}}(s)$ with $y_{(i)}{}^{-1} y_{(i)}{}^F = {}^g n_{(i)}(({}^g n_{(i)})^F)^{-1}$. Set $x_i = y_{(i)}.{}^g n_{(i)}$; then $x_i \in G$ and $s^{x_i} = {}^g(s_0{}^{w_{(i)}})$ as required. □

In Lemma 2.10 we shall assume that $n_{(1)} = y_{(1)} = x_1 = 1$. Now set

$$\boldsymbol{Z} = \boldsymbol{Z}_{\Phi(\boldsymbol{T})_s} = Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle \tau_{\bar{r}}\rangle, \quad \tilde{\boldsymbol{Z}} = \tilde{\boldsymbol{Z}}_{\Phi(\boldsymbol{T})_s} = \{\tilde{s} \in \boldsymbol{Z} : C_{\boldsymbol{G}}(\tilde{s}) = C_{\boldsymbol{G}}(s)\}.$$

Note that as $s \in \boldsymbol{T}$ we have $\boldsymbol{T} \leq C_{\boldsymbol{G}}(s)$, so $Z(C_{\boldsymbol{G}}(s)) \leq C_{\boldsymbol{G}}(\boldsymbol{T}) = \boldsymbol{T}$ and hence $\boldsymbol{Z} \leq \boldsymbol{T}$. Write $Z = \boldsymbol{Z}^F$, so that $Z \leq T \cap H_m\langle \tau_{\bar{r}}\rangle$, and $\tilde{Z} = \tilde{\boldsymbol{Z}}^F$; for each root subsystem Ψ of $\Phi(\boldsymbol{T})$ write $Z_\Psi = \boldsymbol{Z}_\Psi{}^F$ and $\tilde{Z}_\Psi = \tilde{\boldsymbol{Z}}_\Psi{}^F$.

Our next result indicates that the elements of $\tilde{Z}$ all behave similarly when it comes to taking conjugates in T.

Lemma 2.11. *If $\tilde{s} \in \tilde{Z}$, the distinct $\boldsymbol{G}$-conjugates of $\tilde{s}$ lying in T are $\tilde{s}^{x_i}$ for $i \leq l$.*

Proof. Take $\tilde{s} \in \tilde{Z}$. For $i \leq l$ we have $\tilde{s}^{x_i} \in G$, and as $y_{(i)} \in C_{\boldsymbol{G}}(s) = C_{\boldsymbol{G}}(\tilde{s})$ and ${}^g n_{(i)} \in {}^g\boldsymbol{N} = N_{\boldsymbol{G}}(\boldsymbol{T})$ we have $\tilde{s}^{x_i} = \tilde{s}^{y_{(i)}.{}^g n_{(i)}} = \tilde{s}^{{}^g n_{(i)}} \in \boldsymbol{T}$, so $\tilde{s}^{x_i} \in T$. For $i, i' \leq l$ with $i \neq i'$, the elements $\tilde{s}^{x_i}$ and $\tilde{s}^{x_{i'}}$ are distinct since x_i and $x_{i'}$ lie in distinct right cosets of $C_G(s) = C_G(\tilde{s})$. Interchanging the roles of s and $\tilde{s}$ we see that the distinct $\boldsymbol{G}$-conjugates of $\tilde{s}$ lying in T are $\tilde{s}^{x_i}$ for $i \leq l$ as required. □

It is useful to characterize the set $\boldsymbol{Z}$ in terms of root subsystems of Φ.

Lemma 2.12. *The following are true*:

(i) *there is a root subsystem Υ of Φ of type $\tilde{A}_{\bar{r}-1}$ with $s_0 \in \ker_{\bar{r}} \Upsilon \setminus \ker \Upsilon$*;

(ii) *if Υ' is another such root subsystem then $\langle \Phi_{s_0}, \Upsilon\rangle = \langle \Phi_{s_0}, \Upsilon'\rangle$*;

(iii) *for any such root subsystem Υ we have $\boldsymbol{Z} = {}^g(\ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon)$.*

Proof. As $s \in \boldsymbol{H}_m\tau_{\bar{r}}$ we have ${}^{g_m}s \in \boldsymbol{H}\tau_{\bar{r}}$; because ${}^{g_m}s$ is a quasisemisimple element of $\boldsymbol{H}\tau_{\bar{r}}$, as in Section 2.1 there exists $h \in \boldsymbol{H}$ such that ${}^{hg_m}s \in \boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}$. Set $g' = y_{\bar{r}}{}^{-1}hg_m$; then ${}^{g'}s \in (\boldsymbol{T}_{\bar{r}}\tau_{\bar{r}})^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}\tau_{\bar{r}}{}^{y_{\bar{r}}} \subset \boldsymbol{T}_0$, so $Z(C_{\boldsymbol{G}}({}^{g'}s)) \leq \boldsymbol{T}_0$. As ${}^{g'g}s_0 = {}^{g'}s$ is a conjugate of s_0 lying in $\boldsymbol{T}_0$, there exists $n' \in \boldsymbol{N}$ such that ${}^{g'g}s_0 = s_0{}^{n'}$, and then if we write $c = n'g'g$ we have $c \in C_{\boldsymbol{G}}(s_0)$; set $w' = n'\boldsymbol{T}_0 \in W$ and $\Upsilon = {}^{w'}\Upsilon_{\bar{r}}$, then we have $s_0{}^{w'} = {}^{g'g}s_0 \in \ker_{\bar{r}} \Upsilon_{\bar{r}} \setminus \ker \Upsilon_{\bar{r}}$ so that $s_0 \in \ker_{\bar{r}} \Upsilon \setminus \ker \Upsilon$, proving (i).

To prove (ii), suppose Υ' is also a root subsystem of Φ of type $\tilde{A}_{\bar{r}-1}$ with $s_0 \in \ker_{\bar{r}} \Upsilon' \setminus \ker \Upsilon'$. The result is immediate if $\bar{r} = 1$; we shall treat separately the cases $\bar{r} = 2$ and $\bar{r} = 3$.

First suppose $\bar{r} = 2$; write $\Upsilon = \langle \beta\rangle$ and $\Upsilon' = \langle \beta'\rangle$, so that $\beta(s_0) = \beta'(s_0) = -1$. There are 12 possibilities for a root subsystem of Φ of type $\tilde{A}_1$. If $\Upsilon' = \Upsilon$ the result is immediate. If not, then according as β' is or is not orthogonal to β, either both $\beta + \beta'$ and $\beta - \beta'$ are long roots, or exactly one of $\beta + \beta'$ and $\beta - \beta'$ is a short root;

thus we may take a root $\gamma \in \Phi \cap \{\beta \pm \beta'\}$, and then as $\beta(s_0) = \beta'(s_0) = -1$ we have $\gamma(s_0) = 1$, so $\gamma \in \Phi_{s_0}$, and hence $\beta' \in \langle \Phi_{s_0}, \Upsilon \rangle$ and $\beta \in \langle \Phi_{s_0}, \Upsilon' \rangle$, proving (ii) in the case where $\bar{r} = 2$.

Now suppose instead $\bar{r} = 3$; write $\Upsilon = \langle \beta_1, \beta_2 \rangle$ and $\Upsilon' = \langle {\beta_1}', {\beta_2}' \rangle$, so that $\beta_1(s_0), \beta_2(s_0), {\beta_1}'(s_0), {\beta_2}'(s_0) \in \{\omega^{\pm 1}\}$. There are 16 possibilities for a root subsystem of Φ of type $\tilde{A}_2$ (the 12 positive short roots fall into three sets of 4 mutually orthogonal roots, namely $\{1, 2, 3, 4\}$, $\{+---, +-++, ++-+, +++-\}$ and $\{+--+, +-+-, ++--, ++++\}$, and such a subsystem must contain exactly one root from each set, with the choices in any two sets determining that in the third). If $\Upsilon' = \Upsilon$ the result is immediate. If not, then in at least two of the three sets of 4 mutually orthogonal short roots the root β' which lies in Υ' is different from the root β which lies in Υ, and then both $\beta + \beta'$ and $\beta - \beta'$ are long roots. Since in each case $\beta(s_0), \beta'(s_0) \in \{\omega^{\pm 1}\}$, for one of the two long roots (say γ) we have $\gamma(s_0) = 1$, so $\gamma \in \Phi_{s_0}$, and hence $\beta' \in \langle \Phi_{s_0}, \Upsilon \rangle$ and $\beta \in \langle \Phi_{s_0}, \Upsilon' \rangle$, proving (ii) in the case where $\bar{r} = 3$.

Finally, given $\check{s} \in Z(C_{\boldsymbol{G}}(s))$, write $\check{s} = {}^{g}\check{s}_0$ with $\check{s}_0 \in Z(C_{\boldsymbol{G}}(s_0))$; then as $c \in C_{\boldsymbol{G}}(s_0)$ we have ${}^{c}\check{s}_0 = \check{s}_0$, so

$$ {}^{g'}\check{s} = {}^{g'g}\check{s}_0 = {}^{n'^{-1}c}\check{s}_0 = {\check{s}_0}^{n'} = {\check{s}_0}^{w'}. $$

Since ${}^{g'}\check{s} \in Z(C_{\boldsymbol{G}}({}^{g'}s)) \leq \boldsymbol{T}_0$, we have

$$ \begin{aligned} \check{s} \in \boldsymbol{H}_m\langle \tau_{\bar{r}} \rangle &\iff {}^{g_m}\check{s} \in \boldsymbol{H}\langle \tau_{\bar{r}} \rangle \\ &\iff {}^{hg_m}\check{s} \in \boldsymbol{H}\langle \tau_{\bar{r}} \rangle \\ &\iff {}^{g'}\check{s} \in (\boldsymbol{H}\langle \tau_{\bar{r}} \rangle)^{y_{\bar{r}}} \\ &\iff {\check{s}_0}^{w'} = {}^{g'}\check{s} \in \boldsymbol{T}_0 \cap (\boldsymbol{H}\langle \tau_{\bar{r}} \rangle)^{y_{\bar{r}}} = \boldsymbol{T}_{\bar{r}}\langle {\tau_{\bar{r}}}^{y_{\bar{r}}} \rangle = \ker_{\bar{r}} \Upsilon_{\bar{r}} \\ &\iff \check{s}^{g} = \check{s}_0 \in \ker_{\bar{r}} {}^{w'}\Upsilon_{\bar{r}} = \ker_{\bar{r}} \Upsilon. \end{aligned} $$

Thus

$$ \boldsymbol{Z} = Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle \tau_{\bar{r}} \rangle = {}^{g}\big(Z(C_{\boldsymbol{G}}(s_0)) \cap \ker_{\bar{r}} \Upsilon\big) = {}^{g}(\ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon), $$

proving (iii). □

Define

$$ I = \{i \leq l : s^{x_i} \in \boldsymbol{Z}\}; $$

this set will be of some importance. Using Lemma 2.12 we may characterize I in terms of the Weyl group elements $w_{(i)}$.

Lemma 2.13. *We have*

$$ I = \{i \leq l : w_{(i)} \text{ preserves both } \Phi_{s_0} \text{ and } \langle \Phi_{s_0}, \Upsilon \rangle\}, $$

where the root subsystem Υ is as in Lemma 2.12.

Proof. Take $i \le l$. As $s^{x_i} = s^{{}^g n_{(i)}} = {}^g(s_0{}^{n_{(i)}}) = {}^g(s_0{}^{w_{(i)}})$, and $|\Phi_{s_0{}^{w_{(i)}}}| = |\Phi_{s_0}|$, using the characterization of $\boldsymbol{Z}$ given in Lemma 2.12(iii) we have

$$\begin{aligned} s^{x_i} \in \boldsymbol{Z} &\iff s_0{}^{w_{(i)}} \in \ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon \\ &\iff s_0{}^{w_{(i)}} \in \ker \Phi_{s_0} \cap (\ker_{\bar{r}} \Upsilon \setminus \ker \Upsilon) \\ &\iff s_0 \in \ker {}^{w_{(i)}}\Phi_{s_0} \cap (\ker_{\bar{r}} {}^{w_{(i)}}\Upsilon \setminus \ker {}^{w_{(i)}}\Upsilon) \\ &\iff {}^{w_{(i)}}\Phi_{s_0} = \Phi_{s_0} \text{ and } {}^{w_{(i)}}\langle \Phi_{s_0}, \Upsilon\rangle = \langle \Phi_{s_0}, {}^{w_{(i)}}\Upsilon\rangle = \langle \Phi_{s_0}, \Upsilon\rangle \end{aligned}$$

by Lemma 2.12(ii). □

Thus the set I is independent of q. Our next result shows that membership of I has consequences.

Lemma 2.14. *If $i \in I$, conjugation by x_i preserves the algebraic groups $C_{\boldsymbol{G}}(s)$ and $\boldsymbol{Z}$, the finite groups $C_G(s)$ and Z, and the set $\tilde{Z}$; in particular $s^{x_i} \in \tilde{Z}$.*

Proof. Take $i \in I$, so $s^{x_i} \in \boldsymbol{Z}$. As $\dim C_{\boldsymbol{G}}(s) = \dim C_{\boldsymbol{G}}(s^{x_i})$, both centralizers are connected, and $\boldsymbol{G}$-centralizers of elements of $\boldsymbol{Z}$ contain $C_{\boldsymbol{G}}(s)$, we have $C_{\boldsymbol{G}}(s) = C_{\boldsymbol{G}}(s^{x_i}) = C_{\boldsymbol{G}}(s)^{x_i}$. In addition, by Lemma 2.13 $w_{(i)}$ preserves Φ_{s_0} and $\langle \Phi_{s_0}, \Upsilon\rangle$, so $n_{(i)}$ normalizes $\ker \Phi_{s_0} \cap \ker_{\bar{r}} \Upsilon$; thus by Lemma 2.12(iii) ${}^g n_{(i)}$ normalizes $\boldsymbol{Z}$, and as $y_{(i)} \in C_{\boldsymbol{G}}(s)$ commutes with $\boldsymbol{Z}$ we see that x_i normalizes $\boldsymbol{Z}$. As x_i lies in G it also normalizes $C_G(s)$ and Z; thus given $\tilde{s} \in \tilde{Z}$ we have $\tilde{s}^{x_i} \in Z$, and $C_{\boldsymbol{G}}(\tilde{s}^{x_i}) = C_{\boldsymbol{G}}(\tilde{s})^{x_i} = C_{\boldsymbol{G}}(s)^{x_i} = C_{\boldsymbol{G}}(s)$, so $\tilde{s}^{x_i} \in \tilde{Z}$ as required. □

We now show that the elements of I give rise to permutations.

Lemma 2.15. *Given $i \in I$, there exists a permutation π_i of $\{1, \dots, l\}$ such that for all $i' \le l$ and $\tilde{s} \in \tilde{Z}$ we have $x_{\pi_i(i')} \in C_G(s)x_i x_{i'}$ and $\tilde{s}^{x_{\pi_i(i')}} = (\tilde{s}^{x_i})^{x_{i'}}$.*

Proof. Take $i \in I$. By Lemma 2.14 we have $s^{x_i} \in \tilde{Z}$, so by Lemma 2.11

$$\{(s^{x_i})^{x_1}, \dots, (s^{x_i})^{x_l}\} = \{s^{x_1}, \dots, s^{x_l}\};$$

thus there exists a permutation π_i of $\{1, \dots, l\}$ such that for all $i' \le l$ we have $s^{x_{\pi_i(i')}} = (s^{x_i})^{x_{i'}}$, whence $x_{\pi_i(i')} \in C_G(s)x_i x_{i'}$. For all $\tilde{s} \in \tilde{Z}$ we have $C_G(\tilde{s}) = C_G(s)$, so $\tilde{s}^{x_{\pi_i(i')}} = (\tilde{s}^{x_i})^{x_{i'}}$ as required. □

Define $\Pi = \{\pi_i : i \in I\}$; as may be expected, this is a group.

Lemma 2.16. *The set Π is a group which preserves I and acts without fixed points on $\{1, \dots, l\}$.*

Proof. Take $i, i' \in I$. Lemmas 2.14 and 2.15 give $s^{x_{\pi_i(i')}} = (s^{x_i})^{x_{i'}} \in \boldsymbol{Z}$, so $\pi_i(i') \in I$; thus π_i preserves I. For all $i'' \le l$ we have $s^{x_{\pi_i(\pi_{i'}(i''))}} = (s^{x_i})^{x_{\pi_{i'}(i'')}} = ((s^{x_i})^{x_{i'}})^{x_{i''}} = (s^{x_{\pi_i(i')}})^{x_{i''}} = s^{x_{\pi_{\pi_i(i')}(i'')}}$, so $\pi_i(\pi_{i'}(i'')) = \pi_{\pi_i(i')}(i'')$; therefore $\pi_i \circ \pi_{i'} = \pi_{\pi_i(i')}$. Since $\pi_1 = 1$ because $x_1 = 1$, and $\pi_i \circ \pi_{\pi_i{}^{-1}(1)} = \pi_1$, we see that Π is a group, with $\pi_i{}^{-1} = \pi_{\pi_i{}^{-1}(1)}$. Also if $i \ne i'$ then for all $i'' \le l$ we have $s^{x_{\pi_i(i'')}} = (s^{x_i})^{x_{i''}} \ne (s^{x_{i'}})^{x_{i''}} = s^{x_{\pi_{i'}(i'')}}$, so $\pi_i(i'') \ne \pi_{i'}(i'')$ as required. □

Our next result concerns root subsystems Ψ of $\Phi(\boldsymbol{T})$ which contain $\Phi(\boldsymbol{T})_s$.

Lemma 2.17. *Given a root subsystem Ψ of $\Phi(\boldsymbol{T})$ containing $\Phi(\boldsymbol{T})_s$, the order $|Z_\Psi|$ is a polynomial in q.*

Proof. Write $\Psi = {}^g\Psi_0$, so that Ψ_0 is a root subsystem of Φ containing Φ_{s_0}. As $\boldsymbol{Z}_\Psi = \ker\Psi \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$ and $\boldsymbol{Z} = \boldsymbol{Z}_{\Phi(\boldsymbol{T})_s} = \ker\Phi(\boldsymbol{T})_s \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle$, if we take a root subsystem Υ as in Lemma 2.12 we have

$$\boldsymbol{Z}_\Psi = \boldsymbol{Z} \cap \ker\Psi = {}^g(\ker\Phi_{s_0} \cap \ker_{\bar{r}}\Upsilon) \cap {}^g\ker\Psi_0 = {}^g(\ker\Psi_0 \cap \ker_{\bar{r}}\Upsilon)$$

by Lemma 2.12(iii). Now

$$\begin{aligned}|{}^g(\ker\Psi_0 \cap \ker_{\bar{r}}\Upsilon) : {}^g\ker\langle\Psi_0,\Upsilon\rangle| &= |(\ker\Psi_0 \cap \ker_{\bar{r}}\Upsilon) : \ker\langle\Psi_0,\Upsilon\rangle| \\ &= \begin{cases} 1 & \text{if } \Psi_0 \supseteq \langle\Phi_{s_0},\Upsilon\rangle, \\ \bar{r} & \text{if } \Psi_0 \not\supseteq \langle\Phi_{s_0},\Upsilon\rangle. \end{cases}\end{aligned}$$

Set $\overline{\Psi}_0 = \langle\Psi_0,\Upsilon\rangle$ and $\overline{\Psi} = {}^g\overline{\Psi}_0$; then we have ${}^g\ker\overline{\Psi}_0 = \ker\overline{\Psi} = \ker\overline{\Psi}^{(p)}$ by Lemma 2.3(i), and it suffices to prove that the order $|Z_{\overline{\Psi}}|$ is a polynomial in q.

By Lemma 2.3(ii) there exists $\check{s} \in \ker\overline{\Psi}$ such that $\overline{\Psi}^{(p)} = \Phi(\boldsymbol{T})_{\check{s}}$, so $\boldsymbol{Z}_{\overline{\Psi}} = Z(C_{\boldsymbol{G}}(\check{s}))$. The connected reductive group $C_{\boldsymbol{G}}(\check{s})$ has derived group $C_{\boldsymbol{G}}(\check{s})'$ and F-stable maximal torus $\boldsymbol{T}$; thus as in [4, Section 3.3] $\boldsymbol{T} \cap C_{\boldsymbol{G}}(\check{s})'$ is an F-stable maximal torus of $C_{\boldsymbol{G}}(\check{s})'$. Applying [4, Proposition 3.3.5] to $\boldsymbol{G}$ and $C_{\boldsymbol{G}}(\check{s})$ shows that the orders $|\boldsymbol{T}^F|$ and $\left|(\boldsymbol{T} \cap C_{\boldsymbol{G}}(\check{s})')^F\right|$ are both polynomials in q; by [4, Proposition 3.3.7] the order $\left|\left(Z(C_{\boldsymbol{G}}(\check{s}))^\circ\right)^F\right|$ is their ratio, so as it is an integer for all q it must itself be a polynomial in q. The possibilities for the component group $Z(C_{\boldsymbol{G}}(\check{s}))/Z(C_{\boldsymbol{G}}(\check{s}))^\circ$ are given in Table 2: in all but the first two cases F must act trivially on $Z(C_{\boldsymbol{G}}(\check{s}))/Z(C_{\boldsymbol{G}}(\check{s}))^\circ$; consideration of these two cases shows that the action of F on $Z(C_{\boldsymbol{G}}(\check{s}))/Z(C_{\boldsymbol{G}}(\check{s}))^\circ$ is independent of q (as the triple (d,e,x) is fixed), so the order $|Z_{\overline{\Psi}}| = \left|Z(C_{\boldsymbol{G}}(\check{s}))^F\right|$ is also a polynomial in q as required. □

As a consequence, certain sums of values taken by θ are polynomials in q.

Lemma 2.18. *Given $i' \le l$ and $s' \in Z$, the sums*

$$\sum_{\check{s}\in Z_{\Phi(\boldsymbol{T})_{s'}}} \theta(\check{s}^{x_{i'}}) \quad \textit{and} \quad \sum_{\check{s}\in \tilde{Z}_{\Phi(\boldsymbol{T})_{s'}}} \theta(\check{s}^{x_{i'}})$$

are polynomials in q.

Proof. Given $s' \in Z$, as $\Phi(\boldsymbol{T})_{s'} \supseteq \Phi(\boldsymbol{T})_s$ Lemma 2.17 shows that the order $|Z_{\Phi(\boldsymbol{T})_{s'}}|$ is a polynomial in q. The sum over a finite group of the values taken by a linear character is either 0 or the group order (as can be seen by considering the scalar product of the character with the trivial character). Applying this for $i' \le l$ to the group $Z_{\Phi(\boldsymbol{T})_{s'}}$ and its linear character whose value at the element $\check{s} = {}^g\check{s}_0$ is $\theta(\check{s}^{x_{i'}}) = {}^g\theta_0(({}^g\check{s}_0)^{{}^gn_{(i')}}) = \theta_0(\check{s}_0{}^{w_{(i')}})$, where $\theta = {}^g\theta_0$, shows that the first sum is a polynomial in q; Lemmas 2.5 and 2.17 then show that the same is true of the second sum. □

So far we have considered only semisimple elements. Recall that we have the H_m-class $[su]$; write $u_1 = u$ and let $[su_1], \dots, [su_a]$ be the distinct H_m-classes in $H_m\tau_r$ of the type concerned containing elements with semisimple part s, where for $j \le a$ the element u_j is unipotent, commutes with s and lies in $H_m\tau_{r/\bar{r}}$. For any $\tilde{s} \in \tilde{Z}$, likewise $[\tilde{s}u_1], \dots, [\tilde{s}u_a]$ are then the distinct H_m-classes in $H_m\tau_r$ of the type concerned containing elements with semisimple part $\tilde{s}$. As explained at the end of Section 2.1, the number a of these classes is independent of q.

Take $j \le a$. If $i \in I$, then as $u_j \in C_{\boldsymbol{G}}(s) = C_{\boldsymbol{G}}(s^{x_i})$ by Lemma 2.14, the element $s^{x_i}u_j$ has semisimple part s^{x_i} and unipotent part u_j. For $s^{x_i}u_j$ to lie in the G-class $(su_j)^G$ there must exist $g' \in G$ with $s^{g'} = s^{x_i}$ and $u_j{}^{g'} = u_j$; the first condition gives $g' \in C_G(s)x_i$, and then the second forces x_i to preserve the $C_G(s)$-class $u_j{}^{C_G(s)}$. We therefore set

$$I_j = \{i \in I : \ x_i \text{ preserves } u_j{}^{C_G(s)}\}.$$

Lemma 2.19. *For $j \le a$ the set I_j is independent of q.*

Proof. Take $i \in I$; then x_i normalizes $C_G(s)$ by Lemma 2.14. We seek to show that the action of x_i on the unipotent classes in $C_G(s)$ is the same for all q associated to the triple (d, e, x); the result will then follow.

We have $s = {}^g s_0$. Since we know by [4, Theorem 3.5.6] that $C_{\boldsymbol{G}}(s_0)$ is connected, by [4, Theorem 3.5.3] we have $C_{\boldsymbol{G}}(s_0) = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha : \alpha(s_0) = 1\rangle = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha : \alpha \in \Phi_{s_0}\rangle$; thus unipotent classes in $C_G(s)$ lie in $\langle {}^g\boldsymbol{U}_\alpha : \alpha \in \Phi_{s_0}\rangle$, and $C_W(s_0) = \langle w_\alpha : \alpha \in \Phi_{s_0}\rangle = W(\Phi_{s_0})$. By Lemma 2.9 we know that w normalizes $C_W(s_0)$; by Lemma 2.13 we know that $w_{(i)}$ preserves Φ_{s_0} and so normalizes $W(\Phi_{s_0})$. Thus both w and $w_{(i)}$ lie in $N_W(C_W(s_0)) = N_W(W(\Phi_{s_0}))$.

By Lemma 2.4 we see that the index of $W(\Phi_{s_0})W(\Phi_{s_0}{}^\perp)$ in $N_W(W(\Phi_{s_0}))$ is 1 or 2. We shall say that we are in case (i) if the index is 1, case (ii) if the index is 2 and $\Phi_{s_0} \notin \{A_1{}^2, A_1{}^2\tilde{A}_1\}$, and case (iii) if the index is 2 and $\Phi_{s_0} \in \{A_1{}^2, A_1{}^2\tilde{A}_1\}$. In case (ii) we then have $N_W(W(\Phi_{s_0})) = W(\Phi_{s_0})W(\Phi_{s_0}{}^\perp)\langle w_0\rangle$, where $w_0 = w_{1-2}w_{1+2}w_{3-4}w_{3+4}$ is the long word of W; in case (iii) we may assume the long roots in Φ_{s_0} are $\pm\epsilon_3 \pm \epsilon_4$ (and if there are short roots in Φ_{s_0} they are $\pm\epsilon_2$), and then we have $N_W(W(\Phi_{s_0})) = W(\Phi_{s_0})W(\Phi_{s_0}{}^\perp)\langle w_4\rangle$. Set

$$W^* = \begin{cases} W(\Phi_{s_0}) & \text{in case (i)},\\ W(\Phi_{s_0}) & \text{in case (ii)},\\ W(\Phi_{s_0})\langle w_4\rangle & \text{in case (iii)}, \end{cases} \qquad W^\dagger = \begin{cases} W(\Phi_{s_0}{}^\perp) & \text{in case (i)},\\ W(\Phi_{s_0}{}^\perp)\langle w_0\rangle & \text{in case (ii)},\\ W(\Phi_{s_0}{}^\perp) & \text{in case (iii)}; \end{cases}$$

then $N_W(W(\Phi_{s_0})) = W^* \times W^\dagger$. As $w, w_{(i)} \in N_W(W(\Phi_{s_0}))$ we may write $w = w^*w^\dagger$ and $w_{(i)} = w_{(i)}{}^*w_{(i)}{}^\dagger$ with $w^*, w_{(i)}{}^* \in W^*$ and $w^\dagger, w_{(i)}{}^\dagger \in W^\dagger$. Moreover the right coset $W(\Phi_{s_0})w_{(i)}$ is fixed under conjugation by w, so we have ${}^w w_{(i)} \in W(\Phi_{s_0})w_{(i)}$ and therefore $[w, w_{(i)}] \in W(\Phi_{s_0}) \le W^*$; as $[w, w_{(i)}] = [w^*, w_{(i)}{}^*].[w^\dagger, w_{(i)}{}^\dagger]$ we must have $[w^\dagger, w_{(i)}{}^\dagger] = 1$, whence $[w, w_{(i)}] = [w^*, w_{(i)}{}^*]$.

Note that we may choose $w_{(i)}$ to be any representative of the appropriate right coset of $W(\Phi_{s_0})$. In cases (i) and (ii) we may choose ${w_{(i)}}^* = 1$, and then $[w^*, {w_{(i)}}^*] = 1$. In case (iii) the fact that $W(\Phi_{s_0})$ is a proper subgroup of W^* makes the situation rather more complicated. Write $w^* = w^{**}{w_4}^b$ and ${w_{(i)}}^* = {w_{(i)}}^{**}{w_4}^c$ with $w^{**}, {w_{(i)}}^{**} \in W(\Phi_{s_0})$ and $b, c \in \{0, 1\}$. If $c = 0$ we may choose ${w_{(i)}}^{**} = 1$; if $c = b = 1$ we may choose ${w_{(i)}}^{**} = w^{**}$; if $c = 1$, $b = 0$ and w^{**} involves an even number of reflections in long roots we may choose ${w_{(i)}}^{**} = 1$ — in each of these instances we then have $[w^*, {w_{(i)}}^*] = 1$. If however $c = 1$, $b = 0$ and w^{**} involves an odd number of reflections in long roots then for any choice of ${w_{(i)}}^{**}$ we have $[w^*, {w_{(i)}}^*] = w_{3-4}w_{3+4}$. Thus overall the commutator $[w, w_{(i)}]$ is 1, unless we are in the particular situation in case (iii) where it is $w_{3-4}w_{3+4}$.

Suppose $[w, w_{(i)}] = 1$; then w commutes with $w_{(i)}$, and as the element n was chosen as in Lemma 2.6, we may assume that in Lemma 2.10 the element $n_{(i)}$ with $n_{(i)}\boldsymbol{T}_0 = w_{(i)}$ was chosen to lie in $C_{N'}(n)$. It follows that $({}^g n_{(i)})^F = {}^{g.g^{-1}g^F}{n_{(i)}}^F = {}^g({}^n n_{(i)}) = {}^g n_{(i)}$, so that ${}^g n_{(i)} \in G$; we may therefore assume that in Lemma 2.10 the element $y_{(i)}$ was chosen to be 1, whence $x_i = {}^g n_{(i)}$. Thus x_i is an F-stable element of $N_{\boldsymbol{G}}(\boldsymbol{T})$ corresponding to a fixed element of $W(\boldsymbol{T}) = N_{\boldsymbol{G}}(\boldsymbol{T})/\boldsymbol{T}$; any two such elements act identically on the set of unipotent classes in $C_G(s)$, because they differ by an F-stable element of $\boldsymbol{T}$, which thus lies in $C_G(s)$. As the element of $W(\boldsymbol{T})$ is fixed, we see that the action of x_i on the set of unipotent classes in $C_G(s)$ is the same for all q associated to the triple (d, e, x).

Now suppose instead that we are in the particular situation in case (iii) where $[w, w_{(i)}] = w_{3-4}w_{3+4}$. Observe that, if $w_{(i)}$ and $w_{(i')}$ represent two different right cosets of $W(\Phi_{s_0})$ and both involve w_4, the action of x_i on unipotent classes in $C_G(s)$ determines that of $x_{i'}$, since the quotient corresponds to a right coset representative $w_{(i'')}$ which commutes with w and therefore by the above its action is known; we may thus choose the right coset representative $w_{(i)}$ to be w_4. By conjugating w by w_4 if necessary we may assume that $w = w_{3-4}w^\dagger$ for some $w^\dagger \in W^\dagger = W({\Phi_{s_0}}^\perp) \le \langle w_{1-2}, w_2\rangle$. Again we have the element n. Write $n = n_{3-4}n^\dagger$, where inspection of Table 4 shows that we may take $n^\dagger \in \langle n_{1-2}, n_2\rangle$; by taking $n_{(i)} = n_4h_{1-2}(-1)$ we may ensure that $n_{(i)}$ commutes with $n^\dagger$. We then have

$$\begin{aligned}
n_{(i)}({}^n n_{(i)})^{-1} &= n_{(i)}n_{3-4}n^\dagger {n_{(i)}}^{-1}(n^\dagger)^{-1}{n_{3-4}}^{-1}\\
&= n_{(i)}n_{3-4}{n_{(i)}}^{-1}{n_{3-4}}^{-1}\\
&= n_4h_{1-2}(-1)n_{3-4}h_{1-2}(-1){n_4}^{-1}{n_{3-4}}^{-1}\\
&= n_4n_{3-4}{n_4}^{-1}{n_{3-4}}^{-1}\\
&= n_{3+4}{n_{3-4}}^{-1}.
\end{aligned}$$

Take $\lambda, \mu \in k^*$ satisfying $\lambda^{q^2-1} = -1$ and $\mu^q - \mu = \lambda^{q+1}$, and set

$$y_{(i)} = {}^g(x_{3+4}(\mu)h_{3+4}(\lambda)n_{3+4}x_{3+4}(-\lambda^{1-q})x_{3-4}(\lambda^{1-q})h_{3-4}(\lambda)n_{3-4}x_{3-4}(-\mu));$$

then $y_{(i)} \in C_{\boldsymbol{G}}(s)$ and calculation shows that ${y_{(i)}}^{-1}{y_{(i)}}^F = {}^g(n_{3+4}{n_{3-4}}^{-1}) = {}^g(n_{(i)}({}^n n_{(i)})^{-1}) = {}^g n_{(i)}.{}^{gn}{n_{(i)}}^{-1} = {}^g n_{(i)}(({}^g n_{(i)})^F)^{-1}$ as required in Lemma 2.10, while the square of the F-stable element $x_i = y_{(i)}.{}^g n_{(i)}$ fixes ${}^g\langle \boldsymbol{U}_{\pm 3\pm 4}\rangle$ pointwise.

Assume $\Phi_{s_0} = {A_1}^2$. If $p > 2$ and we write the nontrivial unipotent classes of $({}^g\langle \boldsymbol{U}_{\pm(3-4)}\rangle)^F$ as $\mathcal{C}_1$ and $\mathcal{C}_2$, then those of $({}^g\langle \boldsymbol{U}_{\pm(3+4)}\rangle)^F$ are ${\mathcal{C}_1}^{x_i}$ and ${\mathcal{C}_2}^{x_i}$; the element x_i then acts on the unipotent classes in $C_G(s)$ by fixing $\{1\}$, $\mathcal{C}_1{\mathcal{C}_1}^{x_i}$ and $\mathcal{C}_2{\mathcal{C}_2}^{x_i}$, and interchanging $\mathcal{C}_1$ with ${\mathcal{C}_1}^{x_i}$, $\mathcal{C}_2$ with ${\mathcal{C}_2}^{x_i}$, and $\mathcal{C}_1{\mathcal{C}_2}^{x_i}$ with $\mathcal{C}_2{\mathcal{C}_1}^{x_i}$. If instead $p = 2$ and we write the nontrivial unipotent class of $({}^g\langle \boldsymbol{U}_{\pm(3-4)}\rangle)^F$ as $\mathcal{C}$, then that of $({}^g\langle \boldsymbol{U}_{\pm(3+4)}\rangle)^F$ is $\mathcal{C}^{x_i}$; the element x_i then acts on the unipotent classes in $C_G(s)$ by fixing $\{1\}$ and $\mathcal{C}\mathcal{C}^{x_i}$, and interchanging $\mathcal{C}$ with $\mathcal{C}^{x_i}$. Thus in all characteristics the action of x_i on the unipotent classes in $C_G(s)$ is the same for all q associated to the triple (d, e, x).

Now assume instead $\Phi_{s_0} = {A_1}^2\tilde{A}_1$. The action of x_i on the unipotent classes in $({}^g\langle \boldsymbol{U}_{\pm 2}\rangle)^F$ is determined as in the cases above where w commutes with $w_{(i)}$; combined with the previous paragraph this now determines the action of x_i on the unipotent classes in $C_G(s)$, which again therefore is the same for all q associated to the triple (d, e, x). □

Take $j \le a$. Define $\Pi_j = \{\pi_i : i \in I_j\}$; clearly Π_j is a subgroup of Π which preserves I_j. Let ${I_j}'$ be a set of orbit representatives for the action of Π_j on $\{1, \dots, l\}$; by Lemma 2.19 we may choose ${I_j}'$ to be independent of q, and as by Lemma 2.16 each orbit has size $|I_j|$ we have $|I_j| \cdot |{I_j}'| = l$. Our final lemma in this section gives, for $\tilde{s} \in \tilde{Z}$, the value taken by the generalized Deligne–Lusztig character $R_{\boldsymbol{T},\theta}$ at $\tilde{s}u_j$.

Lemma 2.20. *Given $j \le a$ and $\tilde{s} \in \tilde{Z}$, we have*

$$R_{\boldsymbol{T},\theta}(\tilde{s}u_j) = \sum_{i' \in {I_j}'} Q^{C_{\boldsymbol{G}}(s)}_{x_{i'}\boldsymbol{T}}(u_j) \sum_{i \in I_j} \theta((\tilde{s}^{x_i})^{x_{i'}}).$$

Proof. Take $j \le a$ and $\tilde{s} \in \tilde{Z}$. Since $C_{\boldsymbol{G}}(\tilde{s}) = C_{\boldsymbol{G}}(s)$, by Lemma 2.11 the set $\{x' \in G : \tilde{s}^{x'} \in \boldsymbol{T}\}$ is the disjoint union of the right cosets $C_G(s)x_1, \dots, C_G(s)x_l$; each element of this set is thus of the form $cx_{\pi_i(i')}$ for unique $c \in C_G(s)$, $i \in I_j$ and $i' \in {I_j}'$. By Lemma 2.15 we have $x_{\pi_i(i')} = c'x_i x_{i'}$ for some $c' \in C_G(s)$, and $\tilde{s}^{cx_{\pi_i(i')}} = \tilde{s}^{x_{\pi_i(i')}} = (\tilde{s}^{x_i})^{x_{i'}}$; then

$$\begin{aligned} Q^{C_{\boldsymbol{G}}(s)}_{cx_{\pi_i(i')}\boldsymbol{T}}(u_j) &= Q^{C_{\boldsymbol{G}}(s)}_{cc'x_i x_{i'}\boldsymbol{T}}({}^{cc'}u_j) \\ &= Q^{C_{\boldsymbol{G}}(s)}_{x_{i'}\boldsymbol{T}}({u_j}^{x_i}) \\ &= Q^{C_{\boldsymbol{G}}(s)}_{x_{i'}\boldsymbol{T}}(u_j), \end{aligned}$$

where we obtain the first equality because $Q^{C_{\boldsymbol{G}}(s)}_{cc'x_i x_{i'}\boldsymbol{T}}$ is a $C_G(s)$-class function, the second by Lemma 2.8 and the third because x_i preserves ${u_j}^{C_G(s)}$.

The formula of [4, Theorem 7.2.8] quoted before Lemma 2.7 now gives

$$R_{\boldsymbol{T},\theta}(\tilde{s}u_j) = \frac{1}{|C_G(s)|} \sum_{\substack{c\in C_G(s)\\ i\in I_j\\ i'\in {I_j}'}} \theta(\tilde{s}^{cx_{\pi_i(i')}}) Q^{C_{\boldsymbol{G}}(s)}_{{}^{cx_{\pi_i(i')}}\boldsymbol{T}}(u_j)$$

$$= \sum_{\substack{i\in I_j\\ i'\in {I_j}'}} \theta((\tilde{s}^{x_i})^{x_{i'}}) Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) = \sum_{i'\in {I_j}'} Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) \sum_{i\in I_j} \theta((\tilde{s}^{x_i})^{x_{i'}})$$

as required. □

We may now prove our claim.

Proposition 2.21. *Claim 1 is true.*

Proof. Take $j \le a$. By Lemma 2.14, for all $i \in I_j$ and $\tilde{s} \in \tilde{Z}$ the element $\tilde{s}^{x_i} u_j$ has semisimple part $\tilde{s}^{x_i} \in \tilde{Z}$ and unipotent part u_j, and lies in the G-class containing $\tilde{s}u_j$; thus $(\tilde{s}u_j)^G \cap \tilde{Z}u_j = \{\tilde{s}^{x_i}u_j : i \in I_j\}$. Choose a subset $\tilde{Z}_j$ of $\tilde{Z}$ such that each element of $\tilde{Z}$ lies in the set $\{\tilde{s}^{x_i} : i \in I_j\}$ for precisely one element $\tilde{s}$ of $\tilde{Z}_j$; then for distinct elements $\tilde{s} \in \tilde{Z}_j$ the classes $(\tilde{s}u_j)^G$ are distinct, and by Lemma 2.20 we have

$$\sum_{\tilde{s}\in\tilde{Z}_j} R_{\boldsymbol{T},\theta}(\tilde{s}u_j) = \sum_{i'\in {I_j}'} Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) \sum_{\tilde{s}\in\tilde{Z}_j} \sum_{i\in I_j} \theta((\tilde{s}^{x_i})^{x_{i'}})$$

$$= \sum_{i'\in {I_j}'} Q^{C_{\boldsymbol{G}}(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j) \sum_{\tilde{s}\in\tilde{Z}} \theta(\tilde{s}^{x_{i'}}).$$

For each $i' \in {I_j}'$, by Lemma 2.18 the sum $\sum_{\tilde{s}\in\tilde{Z}} \theta(\tilde{s}^{x_{i'}})$ is a polynomial in q; since ${I_j}'$ is independent of q, and as stated in Section 2.3 we know that the Green functions here are also polynomials in q, we see that $\sum_{\tilde{s}\in\tilde{Z}_j} R_{\boldsymbol{T},\theta}(\tilde{s}u_j)$ is likewise a polynomial in q. Summing over j (and recalling that a is independent of q) we see that the sum of the values taken by $R_{\boldsymbol{T},\theta}$ over the H_m-classes of the type concerned is a polynomial in q as required.

Finally we consider degrees. Fix $j \le a$ and $i' \in {I_j}'$. The degree of the polynomial $\sum_{\tilde{s}\in\tilde{Z}} \theta(\tilde{s}^{x_{i'}})$ is at most $\dim \boldsymbol{Z} = \dim\big(Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle\big) = d_1$; moreover taking $\theta = 1$ and observing that $|\tilde{Z}| = |I_j| \cdot |\tilde{Z}_j|$ we see that d_1 equals the degree of the polynomial giving the number of H_m-classes of the given type. Since by Lemma 2.7 the degree of $Q^{C_G(s)}_{{}^{x_{i'}}\boldsymbol{T}}(u_j)$ is at most d_2, it follows that the degree of each polynomial $\sum_{\tilde{s}\in\tilde{Z}_j} R_{\boldsymbol{T},\theta}(\tilde{s}u_j)$ is at most $d_1 + d_2$, whence the same is true of the polynomial at the end of the previous paragraph. □

To conclude this section we observe that if $s \in H_m$ is a regular semisimple element of $\boldsymbol{G}$, so that $\Phi(\boldsymbol{T})_s = \varnothing$ and $\boldsymbol{Z} = C_{\boldsymbol{G}}(s) = \boldsymbol{T}$, and we take $j \le a$ and $i' \in {I_j}'$, then if $\theta \ne 1$ the degree of the polynomial $\sum_{\tilde{s}\in\tilde{Z}} \theta(\tilde{s}^{x_{i'}})$ is strictly less than $\dim \boldsymbol{T}$ (whereas if $\theta = 1$ it equals $\dim \boldsymbol{T}$): this follows from Lemmas 2.5

and 2.18, because if $\theta \neq 1$ then $\sum_{\check{s}\in Z} \theta(\check{s}^{x_{i'}}) = 0$, while for each nonempty root subsystem Ψ' the sum $\sum_{\check{s}\in Z_{\Psi'}} \theta(\check{s}^{x_{i'}})$ is a polynomial in q of degree less than $\dim \boldsymbol{T}$. It follows that if $s \in H_m$ is regular, the bound $d_1 + d_2$ (which then equals $\dim \boldsymbol{T}$) is not achieved if $\theta \neq 1$.

3. The permutation characters ${1_{H_m}}^G$

In this section we employ the method of Section 2 to decompose the permutation characters ${1_{H_m}}^G$. In Section 3.1 we identify the relevant types of H_m-class in H_m, and give information about them and the notation for them which we will use throughout. In Section 3.2 we treat the geometric conjugacy class of unipotent characters; in Section 3.3 we deal with the other geometric conjugacy classes; finally in Section 3.4 we combine the results to obtain the full decomposition of ${1_{H_m}}^G$.

3.1. ***Relevant types of H_m-class in H_m.*** As explained in Section 2.2, in calculating scalar products of ${1_{H_m}}^G$ with generalized Deligne–Lusztig characters it suffices to consider contributions from types of H_m-class in H_m which are relevant, meaning that the degree bound given in Claim 1 is greater than or equal to the degree in q of the order of the H_m-centralizer. We find that most types of H_m-class in H_m are irrelevant.

We take as an example the type of semisimple H_m-class $[s]$ for $m \in \{1, 2\}$ denoted by t_{15} in [15]; according as $p \neq 2$ or $p = 2$ such H_m-classes lie in ones called h_{37} in [25] or h_6 in [24], and they contain elements $\left(1, 1, \lambda\mu, \frac{\lambda}{\mu}; \lambda\right)$ for $\lambda, \mu \in k^*$ satisfying $\lambda^{q-1} = \mu^{q-1} = 1$ and $\lambda^2, \mu^2, \lambda\mu^{\pm 1} \neq 1$. The number of such H_m-classes is $\frac{1}{8}(q-3)(q-5)$ if $p \neq 2$ and $\frac{1}{8}(q-2)(q-4)$ if $p = 2$, so that $\dim Z(C_{\boldsymbol{G}}(s)) = 2$; the H_m-centralizer is $\langle T_0, U_{\pm(1-2)}, U_{\pm(1+2)}\rangle$, of type ${A_1}^2$, while the $\boldsymbol{G}$-centralizer is $\langle \boldsymbol{T}_0, \boldsymbol{U}_{\pm(1-2)}, \boldsymbol{U}_{\pm 2}\rangle$ of type C_2. Write $\epsilon = 1$ or -1 according as $m = 1$ or 2. There are four types of H_m-class $[su]$, depending on whether the projection of u on each A_1 factor of $C_{H_m}(s)$ is trivial or not. If both are trivial then $u = 1$, and so $\dim C_{\boldsymbol{G}}(su) = 12$ while $|C_{H_m}(su)| = q^2(q^2-1)^2(q-1)(q-\epsilon)$. If exactly one is trivial then it is clear that u lies in the C_2-class labeled $W(1) + V(2)$ in [18, Table 8.1a] (this is the class with Bala–Carter label A_1), and so $\dim C_{\boldsymbol{G}}(su) = 8$ while $|C_{H_m}(su)| = q^2(q^2-1)(q-1)(q-\epsilon)$. Finally if neither is trivial then consideration of Jordan blocks on the natural C_2-module shows that u lies in the C_2-class labeled $W(2)$ (if $p \neq 2$) or $V(2)^2$ (if $p = 2$) in [18, Table 8.1a], and so $\dim C_{\boldsymbol{G}}(su) = 6$ while $|C_{H_m}(su)| = q^2(q-1)(q-\epsilon)$. Thus the bound $d_1 + d_2 = \dim\big(Z(C_{\boldsymbol{G}}(s)) \cap \boldsymbol{H}_m\langle\tau_{\bar{r}}\rangle\big) + \frac{1}{2}(\dim C_{\boldsymbol{G}}(su) - \dim \boldsymbol{T})$ given in Claim 1 is 6, 4 or 3, respectively, while the degree of the order of the H_m-centralizer is 8, 6 or 4, respectively.

Treating all types of H_m-class in this manner, we find that for a type of H_m-class to be relevant the root system of the $\boldsymbol{G}$-centralizer of the semisimple part of an element in one of the classes concerned must be C_3, $A_1\tilde{A}_2$, $\tilde{A}_2$, $\tilde{A}_1$ or $\varnothing$; moreover, if it is $\tilde{A}_2$ the nonnegligible part of the contribution can be of degree 1 in q, but in the remaining cases it can only be a constant. We shall deal with each of these five possibilities in turn.

In order to do so we require some further notation. Firstly, there are 25 conjugacy classes in W; in [15, Section 4] the following representatives $w_{(1)}, \dots, w_{(25)}$ are listed.

$w_{(1)} = 1$	$w_{(10)} = w_2w_3w_4$	$w_{(18)} = w_1w_{3-4}w_{2-3}w_{++++}$
$w_{(2)} = w_{3-4}$	$w_{(11)} = w_{3-4}w_{2-3}w_{1-2}$	$w_{(19)} = w_1w_2w_4w_{3-4}$
$w_{(3)} = w_4$	$w_{(12)} = w_{1-2}w_4w_{3-4}$	$w_{(20)} = w_1w_2w_4w_{+-+-}$
$w_{(4)} = w_3w_4$	$w_{(13)} = w_4w_{3-4}w_{+--+}$	$w_{(21)} = w_3w_{2-3}w_{1-2}w_4$
$w_{(5)} = w_{1-2}w_4$	$w_{(14)} = w_4w_{3-4}w_{2-3}$	$w_{(22)} = w_2w_{1-2}w_4w_{3-4}$
$w_{(6)} = w_{3-4}w_{2-3}$	$w_{(15)} = w_{1-2}w_4w_{++--}$	$w_{(23)} = w_4w_{3-4}w_{2-3}w_{1-2}$
$w_{(7)} = w_4w_{+---}$	$w_{(16)} = w_1w_{3-4}w_{2-3}$	$w_{(24)} = w_2w_{2-3}w_{3-4}w_{+---}$
$w_{(8)} = w_4w_{3-4}$	$w_{(17)} = w_1w_2w_3w_4$	$w_{(25)} = w_3w_{2-3}w_{3+4}w_{+---}$
$w_{(9)} = w_1w_2w_{3-4}$		

Next, for $w \in W$ we write $T_w = \{s \in \boldsymbol{T}_0 : (s^F)^w = s\}$; moreover we write $T_{(n)}$ for $T_{w_{(n)}}$. The tori $T_{(1)}, \dots, T_{(25)}$ are also given in [15, Section 4]; for reasons of space we shall not reproduce them here. For each $n \in \{1, \dots, 25\}$ we may choose $g \in \boldsymbol{G}$ satisfying $(g^F)^{-1}g \in \boldsymbol{N}$ and $(g^F)^{-1}g\boldsymbol{T}_0 = w_{(n)}$, and define $\boldsymbol{T}_{(n)} = {}^g\boldsymbol{T}_0$; then $\boldsymbol{T}_{(n)}$ is an F-stable maximal torus of $\boldsymbol{G}$, and we have ${\boldsymbol{T}_{(n)}}^F = {}^gT_{(n)}$. The $\boldsymbol{T}_{(n)}$ are representatives of the G-classes of F-stable maximal tori of $\boldsymbol{G}$, with $\boldsymbol{T}_{(n)}$ being obtained from $\boldsymbol{T}_0$ by twisting with $w_{(n)}$. (In fact as things stand the twisting element is ${w_{(n)}}^{-1}$, but this is only defined up to F-conjugacy in W, which is simply conjugacy because the map F fixes each element of W, and all conjugacy classes in W are self-inverse, as is evident from the fact that the character table of W given in [10] has only real entries.)

Now for each of the five possibilities mentioned above, we shall concentrate on the semisimple classes. We shall give the notation used in [25] (for $p \neq 2$) and [24] (for $p = 2$) for the semisimple G-classes (and shall hereafter employ the former, which is of the form 'h_ℓ' for some ℓ); we shall give the form of elements within them (where instead of an actual G-class representative we will provide an element of $\boldsymbol{T}_0$ lying in the appropriate $\boldsymbol{G}$-class, as is customary); we shall state how such a G-class meets H_m, giving the notation used in [15] for the corresponding class in $B_4(q)$ in the cases of H_1 and H_2, and that used in [5] in the case of H_3; we shall specify the centralizers in G and H_m; we shall give as much information as we shall need on the number of H_m-classes (as noted in Section 2.4, this number is a polynomial in q, and we shall give the leading one or two terms as appropriate, with

[25]	[24]	[15]	[5]	ϵ	# H_m-classes	n									
h_7	h_2	t_{10}	s_3	1	$\frac{1}{2}q+\cdots$	1	2	3	4	5	7	8	9	12	13
h_8	h_{14}	t_{37}	s_7	-1	$\frac{1}{2}q+\cdots$	17	9	10	4	5	20	19	2	12	15

Table 5. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system C_3.

'$+\cdots$' denoting the presence of lower-degree terms which may be ignored); and we shall indicate the tori $T_{(n)}$ which meet the classes (again, up to $\boldsymbol{G}$-conjugacy). Finally, for each such type of semisimple H_m-class we shall specify (if this is not obvious) the corresponding types of H_m-class which are relevant.

In what follows we often write $\epsilon = \pm 1$; we set $A_2^1(q) = A_2(q)$ and $A_2^{-1}(q) = {}^2A_2(q)$, and we write T_m^ϵ for a maximal torus of $A_2^\epsilon(q)$ of order $(q-\epsilon)^2$, q^2-1 or $q^2+\epsilon q+1$ according as $m = 1$, 2 or 3.

There are two types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system C_3; they contain elements $\left(1, \lambda, \frac{1}{\lambda}, 1; 1\right)$ for appropriate λ. Each such class meets H_m in a single class. The centralizers in G and H_m are $C_3(q).T_1$ and $A_1(q^m).A_1(q)^{3-m}.T_1$, respectively, where T_1 is a torus of order $q-\epsilon$. The notation used for the type of class in [25], [24], [15] and [5], the value of ϵ, the leading term in the number of H_m-classes, and the ten values of n such that the class meets the torus $T_{(n)}$ are given in Table 5. For each such h_ℓ, there are two types of relevant H_m-class: the H_m-classes concerned are the semisimple classes themselves, and the classes of regular elements of H_m whose semisimple parts are of type h_ℓ, in which the unipotent part lies in the class in $C_3(q)$ labeled $W(2)+V(2)$ in [18, Table 8.2a].

There are two types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $A_1\tilde{A}_2$; they contain elements $\left(\lambda, \frac{1}{\lambda}, \lambda^2, 1; \lambda\right)$ for appropriate λ. Each such class meets H_m in a single class. The centralizers in G and H_m are $A_1(q).A_2^\epsilon(q).T_1$ and $A_1(q).T_m^\epsilon.T_1$, respectively, where T_1 is a torus of order $q-\epsilon$. The notation used for the type of class in [25], [24], [15] and [5], the value of ϵ, the leading term in the number of H_m-classes, and the six values of n such that the class meets the torus $T_{(n)}$ are given in Table 6. For each such h_ℓ, there is only one type of relevant H_m-class: the H_m-classes concerned contain regular elements of H_m with semisimple parts of type h_ℓ, in which the unipotent part projects nontrivially on the $A_1(q)$ factor and trivially on the $A_2^\epsilon(q)$ factor.

[25]	[24]	[15]	[5]	ϵ	# H_m-classes	n					
h_9	h_5	t_{23}	s_5	1	$\frac{1}{2}q+\cdots$	1	2	3	5	7	15
h_{10}	h_{17}	t_{50}	s_{10}	-1	$\frac{1}{2}q+\cdots$	17	9	10	5	20	13

Table 6. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $A_1\tilde{A}_2$.

[25]	[24]	[15]	[5]	ϵ	$\|T\|$	$\#\,H_m$-classes	n
h_{31}	h_9	t_{27}	s_6	1	$(q-1)^2$	$\frac{1}{12}q^2-\frac{2}{3}q+\cdots$	1 3 7
h_{32}	h_{21}	t_{54}	s_{15}	-1	$(q+1)^2$	$\frac{1}{12}q^2-\frac{1}{3}q+\cdots$	17 10 20
h_{33}	h_{28}	t_{86}	s_8	1	q^2-1	$\frac{1}{4}q^2-\frac{1}{2}q+\cdots$	2 5 15
h_{34}	h_{42}	t_{92}	s_{11}	-1	q^2-1	$\frac{1}{4}q^2-\frac{1}{2}q+\cdots$	9 5 13
h_{35}	h_{45}	t_{108}	s_{12}	1	q^2+q+1	$\frac{1}{6}q^2+\frac{1}{6}q+\cdots$	6 16 18
h_{36}	h_{48}	t_{112}	s_{13}	-1	q^2-q+1	$\frac{1}{6}q^2-\frac{1}{6}q+\cdots$	21 14 25

Table 7. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $\tilde{A}_2$.

For each of the remaining three possibilities, the semisimple elements concerned are regular in H_m; thus the centralizer in H_m is a maximal torus, and for each h_ℓ the only H_m-classes containing elements with semisimple parts of type h_ℓ are the semisimple H_m-classes themselves.

There are six types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $\tilde{A}_2$; they contain elements $\big(\lambda,\mu,\frac{\lambda}{\mu},1;\lambda\big)$ for appropriate λ and μ. Each such class meets H_m in a single class. The centralizers in G and H_m are $A_2^\epsilon(q).T$ and $T_m^\epsilon.T$, respectively, where T is a torus. The notation used for the type of class in [25], [24], [15] and [5], the values of ϵ and $|T|$, the leading terms in the number of H_m-classes, and the three values of n such that the class meets the torus $T_{(n)}$ are given in Table 7.

There are ten types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $\tilde{A}_1$; they contain elements $\big(\lambda,\mu,\frac{\nu^2}{\lambda\mu},1;\nu\big)$ for appropriate λ, μ and ν. Each such class fails to meet H_3, but meets $B_4(q)$ in two classes, one of which is regular in $B_4(q)$; this regular class in $B_4(q)$ fails to meet H_2, but meets H_1 in two classes, while the other class in $B_4(q)$ meets both H_1 and H_2 in a single class. The number of classes of a given type in H_m is thus f times the number of such classes in H_2. The centralizers in G and H_m are $A_1(q).T$ and $T_1.T$, respectively, where T is a torus and T_1 is a torus of order $q-\epsilon$. The notation used for the type of class in [25], [24] and [15], the values of ϵ and $|T|$, the leading term in the number of H_m-classes, and the two values of n such that the class meets the torus $T_{(n)}$ are given in Table 8; in each case the first t_j listed in the third column meets both H_1 and H_2, while the second meets just H_1.

Finally, there are 25 types of semisimple G-class whose $\boldsymbol{G}$-centralizer has root system $\varnothing$; they contain elements $\big(\lambda,\mu,\frac{\nu^2}{\lambda\mu\pi},\pi;\nu\big)$ for appropriate λ, μ, ν and π. Each such class meets H_m for exactly one value of m, and does so in six H_m-classes; this is because $W_{\boldsymbol{H}}$ is normal in W, with quotient $\langle W_{\boldsymbol{H}}w_4, W_{\boldsymbol{H}}w_{+---}\rangle \cong S_3$, so that each class in W meets exactly one of the cosets $W_{\boldsymbol{H}}$, $W_{\boldsymbol{H}}w_4$ and $W_{\boldsymbol{H}}w_4w_{+---}$. The centralizer in both G and H of such an element is a torus T. The notation used for the type of class in [25], [24] and either [15] or [5], the value of m such that the class

[25]	[24]	[15]	ϵ	$\lvert T\rvert$	# H_m-classes	n	
h_{66}	h_{10}	t_{28}, t_{29}	1	$(q-1)^3$	$\frac{1}{48} f q^3 + \cdots$	1	3
h_{67}	h_{22}	t_{55}, t_{56}	-1	$(q+1)^3$	$\frac{1}{48} f q^3 + \cdots$	17	10
h_{68}	h_{33}	t_{67}, t_{122}	1	$(q^2-1)(q-1)$	$\frac{1}{16} f q^3 + \cdots$	4	3
h_{69}	h_{37}	t_{74}, t_{123}	-1	$(q^2-1)(q+1)$	$\frac{1}{16} f q^3 + \cdots$	4	10
h_{70}	h_{29}	t_{87}, t_{88}	1	$(q^2-1)(q-1)$	$\frac{1}{8} f q^3 + \cdots$	2	5
h_{71}	h_{43}	t_{93}, t_{94}	-1	$(q^2-1)(q+1)$	$\frac{1}{8} f q^3 + \cdots$	9	5
h_{72}	h_{54}	t_{99}, t_{129}	1	$(q^2+1)(q-1)$	$\frac{1}{8} f q^3 + \cdots$	11	8
h_{73}	h_{59}	t_{102}, t_{130}	-1	$(q^2+1)(q+1)$	$\frac{1}{8} f q^3 + \cdots$	11	19
h_{74}	h_{46}	t_{109}, t_{110}	1	q^3-1	$\frac{1}{6} f q^3 + \cdots$	6	16
h_{75}	h_{49}	t_{113}, t_{114}	-1	q^3+1	$\frac{1}{6} f q^3 + \cdots$	21	14

Table 8. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $\tilde{A}_1$.

meets H_m, the value of $|T|$, the leading term in the number of H_m-classes, and the single value of n such that the class meets the torus $T_{(n)}$ are given in Table 9. (Note that the W-classes containing $w_{(4)}$ and $w_{(11)}$ each split into two $W(B_4)$-classes, so that there are two types of semisimple class given in [15]; in each case the second of the two $W(B_4)$-classes splits further into two $W_{\boldsymbol{H}}$-classes, so that there are in fact three types of semisimple class in H_1 in these cases.)

As stated above, from now on we shall refer to the types of semisimple class described above by the notation of [25]. Thus we have h_7 and h_8; h_9 and h_{10}; $h_{31}, \ldots, h_{36}$; $h_{66}, \ldots, h_{75}$; and $h_{76}, \ldots, h_{100}$. Of these five collections, for the first three each semisimple G-class meets each H_m in a single class; in the fourth, each G-class meets H_m in f classes; in the fifth, each G-class meets just one H_m and does so in 6 classes. In all cases, in the calculations to follow we shall need to consider the classes of regular elements of H_m whose semisimple parts are of the types concerned; for h_7 and h_8 we must also treat the semisimple classes themselves.

3.2. ***Unipotent characters.*** We first consider the geometric conjugacy class of unipotent characters of G; the $R_{\boldsymbol{T},\theta}$ lying in this class are those with $\theta = 1$. We write $R_{(n)}$ for $R_{\boldsymbol{T}_{(n)},1}$.

We begin with types h_7 and h_8; we must treat the semisimple classes and those containing regular elements of H_m. We shall take as an example the contribution from elements with semisimple part s of type h_ℓ to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ in the case $(\ell, n) = (7, 1)$. The number of classes is $\frac{1}{2}q + \cdots$; there are $\frac{1152}{48} = 24$ distinct conjugates of s lying in $T_{(1)}$. For the semisimple classes, the Green function value is

$$\frac{(q^2-1)(q^4-1)(q^6-1)}{(q-1)^3} = (q^4+q^2+1)(q^2+1)(q+1)^3,$$

[25]	[24]	[15] or [5]	m	$\|T\|$	# H_m-classes	n
h_{76}	h_{12}	t_{30}	1	$(q-1)^4$	$\frac{1}{192}q^4+\cdots$	1
h_{77}	h_{24}	t_{57}	1	$(q+1)^4$	$\frac{1}{192}q^4+\cdots$	17
h_{78}	h_{30}	t_{89}	1	$(q^2-1)(q-1)^2$	$\frac{1}{16}q^4+\cdots$	2
h_{79}	h_{36}	t_{68}	2	$(q^2-1)(q-1)^2$	$\frac{1}{48}q^4+\cdots$	3
h_{80}	h_{39}	t_{81}, t_{124}	1	$(q^2-1)^2$	$\frac{3}{32}q^4+\cdots$	4
h_{81}	h_{44}	t_{105}	2	$(q^2-1)^2$	$\frac{1}{8}q^4+\cdots$	5
h_{82}	h_{47}	t_{111}	1	$(q^3-1)(q-1)$	$\frac{1}{6}q^4+\cdots$	6
h_{83}	h_{68}	s_6	3	$(q^3-1)(q-1)$	$\frac{1}{12}q^4+\cdots$	7
h_{84}	h_{56}	t_{100}	2	$(q^2+1)(q-1)^2$	$\frac{1}{16}q^4+\cdots$	8
h_{85}	h_{57}	t_{95}	1	$(q^2-1)(q+1)^2$	$\frac{1}{16}q^4+\cdots$	9
h_{86}	h_{58}	t_{75}	2	$(q^2-1)(q+1)^2$	$\frac{1}{48}q^4+\cdots$	10
h_{87}	h_{60}	t_{107}, t_{131}	1	q^4-1	$\frac{3}{8}q^4+\cdots$	11
h_{88}	h_{62}	t_{125}	2	q^4-1	$\frac{1}{8}q^4+\cdots$	12
h_{89}	h_{72}	s_{11}	3	$(q^3+1)(q-1)$	$\frac{1}{4}q^4+\cdots$	13
h_{90}	h_{51}	t_{117}	2	$(q^3+1)(q-1)$	$\frac{1}{6}q^4+\cdots$	14
h_{91}	h_{73}	s_8	3	$(q^3-1)(q+1)$	$\frac{1}{4}q^4+\cdots$	15
h_{92}	h_{52}	t_{116}	2	$(q^3-1)(q+1)$	$\frac{1}{6}q^4+\cdots$	16
h_{93}	h_{74}	s_{12}	3	$(q^2+q+1)^2$	$\frac{1}{24}q^4+\cdots$	18
h_{94}	h_{64}	t_{103}	2	$(q^2+1)(q+1)^2$	$\frac{1}{16}q^4+\cdots$	19
h_{95}	h_{71}	s_{15}	3	$(q^3+1)(q+1)$	$\frac{1}{12}q^4+\cdots$	20
h_{96}	h_{50}	t_{115}	1	$(q^3+1)(q+1)$	$\frac{1}{6}q^4+\cdots$	21
h_{97}	h_{63}	t_{128}	1	$(q^2+1)^2$	$\frac{1}{16}q^4+\cdots$	22
h_{98}	h_{65}	t_{132}	2	q^4+1	$\frac{1}{4}q^4+\cdots$	23
h_{99}	h_{76}	s_{14}	3	q^4-q^2+1	$\frac{1}{4}q^4+\cdots$	24
h_{100}	h_{75}	s_{13}	3	$(q^2-q+1)^2$	$\frac{1}{24}q^4+\cdots$	25

Table 9. Semisimple G-classes with $\boldsymbol{G}$-centralizer having root system $\varnothing$.

while the order of the centralizer in H_m is a polynomial in q with leading term q^{10}; thus the contribution here is

$$\frac{24\cdot(q^4+q^2+1)(q^2+1)(q+1)^3\cdot\left(\frac{1}{2}q+\cdots\right)}{q^{10}+\cdots},$$

whose nonnegligible part is 12. For the classes of regular elements of H_m, the Green function value is a polynomial in q with leading term $3q^3$ (as may be seen from [11; 22]), while the order of the centralizer in H_m is $q^4+\cdots$; thus the contribution here is

$$\frac{24 \cdot (3q^3 + \cdots) \cdot \left(\frac{1}{2}q + \cdots\right)}{q^4 + \cdots},$$

with nonnegligible part 36. Treating the other cases (ℓ, n) similarly gives the following, in which the two values listed in each case correspond to the semisimple and regular classes of H_m, respectively.

h_7			h_8		
$R_{(1)}$	12	36	$R_{(17)}$	−12	−36
$R_{(2)}$	−3	3	$R_{(9)}$	3	−3
$R_{(3)}$	−6	−6	$R_{(10)}$	6	6
$R_{(4)}$	2	−2	$R_{(4)}$	−2	2
$R_{(5)}$	1	1	$R_{(5)}$	−1	−1
$R_{(7)}$	3	0	$R_{(20)}$	−3	0
$R_{(8)}$	2	−2	$R_{(19)}$	−2	2
$R_{(9)}$	−1	−3	$R_{(2)}$	1	3
$R_{(12)}$	−1	1	$R_{(12)}$	1	−1
$R_{(13)}$	−1	0	$R_{(15)}$	1	0

Next we consider types h_9 and h_{10}; here we treat only the classes of regular elements of H_m. As an example, take the contribution from such elements su with s of type h_ℓ to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ in the case $(\ell, n) = (9, 1)$. The number of classes is $\frac{1}{2}q + \cdots$; there are $\frac{1152}{12} = 96$ distinct conjugates of s lying in $T_{(1)}$. Since the projections of u in the $A_1(q)$ and $A_2(q)$ factors of $C_G(s)$ are regular and trivial, respectively, the Green function value at u is

$$\frac{1 \cdot (q^2 - 1)(q^3 - 1)}{(q-1)^2} = (q^2 + q + 1)(q + 1);$$

and $|C_{H_m}(su)|$ is a polynomial in q with leading term q^4. Thus the contribution is

$$\frac{96 \cdot (q^2 + q + 1)(q + 1) \cdot \left(\frac{1}{2}q + \cdots\right)}{q^4 + \cdots},$$

whose nonnegligible part is 48. Dealing similarly with the other pairs (ℓ, n) gives the following.

h_9 :

$R_{(1)}$	$R_{(2)}$	$R_{(3)}$	$R_{(5)}$	$R_{(7)}$	$R_{(15)}$
48	4	−12	−2	3	1

h_{10} :

$R_{(17)}$	$R_{(9)}$	$R_{(10)}$	$R_{(5)}$	$R_{(20)}$	$R_{(13)}$
−48	−4	12	2	−3	−1

For types $h_{31}, \dots, h_{36}$ the calculations are a little more delicate; consideration of leading terms alone is insufficient, since here nonnegligible parts of contributions are linear polynomials in q rather than merely constants. As an example, take the contribution from semisimple elements s of type h_ℓ to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ in the case $(\ell, n) = (31, 1)$. The number of classes is $\frac{1}{12}q^2 - \frac{2}{3}q + \cdots$; there are $\frac{1152}{6} = 192$ distinct conjugates of s lying in $T_{(1)}$. The Green function value is

$$\frac{(q^2-1)(q^3-1)}{(q-1)^2} = (q^2+q+1)(q+1);$$

and

$$|C_{H_m}(s)| = (q-1)^2 |T_m^1| = q^4 - (f+1)q^3 + \cdots .$$

Thus the contribution is

$$\frac{192 \cdot (q^3 + 2q^2 + \cdots) \cdot \frac{1}{12}(q^2 - 8q + \cdots)}{q^4 - (f+1)q^3 + \cdots} = 16(q + (f-5) + \cdots).$$

Dealing similarly with the other pairs (ℓ, n) gives the following.

h_{31}	$R_{(1)}$	$16q+16f-80$	$R_{(3)}$	$-4q-4f+28$	$R_{(7)}$	$q+f-8$
h_{32}	$R_{(17)}$	$-16q+16f+112$	$R_{(10)}$	$4q-4f-20$	$R_{(20)}$	$-q+f+4$
h_{33}	$R_{(2)}$	$4q+4f-4$	$R_{(5)}$	$-2q-2f+6$	$R_{(15)}$	$q+f-4$
h_{34}	$R_{(9)}$	$-4q+4f+12$	$R_{(5)}$	$2q-2f-2$	$R_{(13)}$	$-q+f$
h_{35}	$R_{(6)}$	$q+f+1$	$R_{(16)}$	$-q-f+1$	$R_{(18)}$	$4q+4f-8$
h_{36}	$R_{(21)}$	$-q+f+1$	$R_{(14)}$	$q-f+1$	$R_{(25)}$	$-4q+4f-8$

The remaining types h_ℓ are more easily handled. For types $h_{66}, \dots, h_{75}$ the Green function value concerned is always $\pm q + 1$. Proceeding as above we obtain the following.

h_{66}	$R_{(1)}$	$12f$	$R_{(3)}$	$-f$
h_{67}	$R_{(17)}$	$-12f$	$R_{(10)}$	f
h_{68}	$R_{(4)}$	$-2f$	$R_{(3)}$	$3f$
h_{69}	$R_{(4)}$	$2f$	$R_{(10)}$	$-3f$
h_{70}	$R_{(2)}$	$6f$	$R_{(5)}$	$-f$

h_{71}	$R_{(9)}$	$-6f$	$R_{(5)}$	f
h_{72}	$R_{(11)}$	$-f$	$R_{(8)}$	$2f$
h_{73}	$R_{(11)}$	f	$R_{(19)}$	$-2f$
h_{74}	$R_{(6)}$	$3f$	$R_{(16)}$	$-f$
h_{75}	$R_{(21)}$	$-3f$	$R_{(14)}$	f

Finally, the types $h_{76}, \dots, h_{100}$ contain regular semisimple elements of G, so that the Green function value is just 1. For each such type h_ℓ, the classes meet $T_{(n)}$ for just one value of n, and H_m for just one value of m. It follows that for this choice of n and m the nonnegligible part of the contribution from classes of type h_ℓ

	H_1	H_2	H_3
$R_{(1)}$	$16q+106$	$16q+44$	$16q+16$
$R_{(2)}$	$4q+40$	$4q+14$	$4q+4$
$R_{(3)}$	$-4q-2$	$-4q+4$	$-4q+4$
$R_{(4)}$	6	0	0
$R_{(5)}$	-8	2	4
$R_{(6)}$	$q+19$	$q+5$	$q+1$
$R_{(7)}$	$q+1$	$q-1$	$q+1$
$R_{(8)}$	6	4	0
$R_{(9)}$	$-4q+4$	$-4q+2$	$-4q+4$
$R_{(10)}$	$4q-14$	$4q$	$4q+4$
$R_{(11)}$	6	0	0
$R_{(12)}$	0	2	0
$R_{(13)}$	$-q+1$	$-q-1$	$-q+1$
$R_{(14)}$	$q+1$	$q+3$	$q+1$
$R_{(15)}$	$q+1$	$q-1$	$q+1$
$R_{(16)}$	$-q-5$	$-q+1$	$-q+1$
$R_{(17)}$	$-16q+34$	$-16q+20$	$-16q+16$
$R_{(18)}$	$4q+4$	$4q-4$	$4q-5$
$R_{(19)}$	-6	0	0
$R_{(20)}$	$-q+1$	$-q-1$	$-q+1$
$R_{(21)}$	$-q+1$	$-q-1$	$-q+1$
$R_{(22)}$	6	0	0
$R_{(23)}$	0	2	0
$R_{(24)}$	0	0	3
$R_{(25)}$	$-4q+4$	$-4q-4$	$-4q-5$

Table 10. Scalar products $({1_{H_m}}^G, R_{(n)})_G$.

to the scalar product of ${1_{H_m}}^G$ with $R_{(n)}$ is simply $|C_W(w_{(n)})|$ times the coefficient of q^4 in the number of such classes, as given above; this value is always 6, 2 or 3 according as $m = 1$, 2 or 3.

Summing all of the above nonnegligible parts gives the values in Table 10 for the scalar products $({1_{H_m}}^G, R_{(n)})_G$.

There are now two steps in forming the irreducible unipotent characters of G from the Deligne–Lusztig generalized characters $R_{(n)}$. Firstly, for each irreducible character ϕ of W we form

$$R_\phi = \sum_{n=1}^{25} \frac{\phi(w_{(n)})R_{(n)}}{|C_W(w_{(n)})|}.$$

The class functions obtained in this way are among the set of almost characters. The second step is to transform the almost characters by nonabelian Fourier transform matrices to obtain the irreducible characters.

For the first of these steps, the character table of W is given in [10], and we have

$$(1_{H_m}{}^G, R_\phi)_G = \sum_{n=1}^{25} \frac{\phi(w_{(n)})}{|C_W(w_{(n)})|}(1_{H_m}{}^G, R_{(n)})_G;$$

it is therefore straightforward to calculate the scalar products of $1_{H_m}{}^G$ with those almost characters of the form R_ϕ. We find that the only such scalar products which are nonzero for some m are as follows.

ϕ	$(1_{H_m}{}^G, R_\phi)_G$
$\phi_{1,0}, \phi_{9,2}, \phi_{12,4}$	1
$\phi''_{8,3}$	$q+f$
$\phi''_{8,9}$	q
$\phi_{4,1}, \phi''_{2,4}, \phi''_{9,6}$	f
$\phi''_{6,6}$	$f-1$
$\phi''_{1,12}$	$3, -1, 0$ according as $m = 1, 2, 3$

The second step is complicated by the fact that not all almost characters are of the form R_ϕ; however, it turns out that it is still possible to deduce the scalar products of $1_{H_m}{}^G$ with all irreducible unipotent characters. The almost characters R_ϕ for $\phi = \phi_{1,0}, \phi_{9,2}, \phi''_{8,3}$ and $\phi''_{8,9}$ all lie in families of size one, and are thus themselves irreducible characters χ_ϕ; they therefore appear in $1_{H_m}{}^G$ with multiplicities 1, 1, $q+f$ and q, respectively. The almost characters R_ϕ for $\phi = \phi_{4,1}$ and $\phi''_{2,4}$ lie in a family of size four, whose other members are $R_{\phi'_{2,4}}$ and a class function Y_0, say; the corresponding unipotent characters are $\chi_{\phi_{4,1}}, \chi_{\phi''_{2,4}}, \chi_{\phi'_{2,4}}$ and $\chi_{B_2,1}$ (in the notation of [4, 13.9]), and we have

$$\begin{aligned}
\chi_{\phi_{4,1}} &= \tfrac{1}{2}(R_{\phi_{4,1}} + R_{\phi''_{2,4}} + R_{\phi'_{2,4}} + Y_0),\\
\chi_{\phi''_{2,4}} &= \tfrac{1}{2}(R_{\phi_{4,1}} + R_{\phi''_{2,4}} - R_{\phi'_{2,4}} - Y_0),\\
\chi_{\phi'_{2,4}} &= \tfrac{1}{2}(R_{\phi_{4,1}} - R_{\phi''_{2,4}} + R_{\phi'_{2,4}} - Y_0),\\
\chi_{B_2,1} &= \tfrac{1}{2}(R_{\phi_{4,1}} - R_{\phi''_{2,4}} - R_{\phi'_{2,4}} + Y_0).
\end{aligned}$$

If we let $(1_{H_m}{}^G, Y_0)_G = y_0$, then the values above imply that the scalar products of $1_{H_m}{}^G$ with the four irreducible characters are

$$f + \tfrac{1}{2}y_0, \quad f - \tfrac{1}{2}y_0, \quad -\tfrac{1}{2}y_0, \quad \tfrac{1}{2}y_0;$$

as these must all be nonnegative, it follows that $y_0 = 0$ and the scalar products are $f, f, 0, 0$.

The remaining almost characters R_ϕ listed above, those for which $\phi = \phi_{12,4}$, $\phi''_{9,6}$, $\phi''_{6,6}$ and $\phi''_{1,12}$, all lie in a single family of size 21; of the other 17 almost characters in this family, only 7 are also of the form R_ϕ. The corresponding nonabelian Fourier transform matrix is given in [4, p. 456] and repeated in the Appendix of [12]. We shall use an analysis similar to that employed in [12] to determine the scalar products with irreducible characters. We number the rows and columns of the 21×21 matrix in the order in which they appear in [4, p. 456]; for $1 \le i, j \le 21$ let m_i and n_j be the scalar products of ${1_{H_m}}^G$ with the i-th almost character and j-th irreducible character in the family, respectively. Each n_j is therefore the linear combination of the m_i with coefficients given by the j-th column of the matrix. The values calculated above imply that $m_i = 0$ for $i = 2, 3, 6, 9, 13, 15, 18$. In addition, there are two pairs of irreducible characters, and correspondingly two pairs of almost characters, which are complex conjugates of each other; since the values taken by ${1_{H_m}}^G$ are all real, it follows that the scalar products concerned must be equal, so that $m_{16} = m_{17}$, $m_{20} = m_{21}$, $n_{16} = n_{17}$ and $n_{20} = n_{21}$. Note that the scalar product with $R_{\phi''_{1,12}}$ means that the cases with $m = 1$, 2 and 3 must be handled separately.

First assume that $m = 1$; we thus have $m_1 = 1$, $m_5 = 2$, and $m_{10} = m_{12} = 3$. By adding together the appropriate columns of the matrix we see that

$$\begin{aligned} n_{15} + n_{16} + n_{17} &= -1 + m_4, \\ n_{18} + n_{19} + n_{20} + n_{21} &= 1 - m_4; \end{aligned}$$

thus $m_4 = 1$, and $n_{15} = \cdots = n_{21} = 0$. Since we then have $n_{20} = -\frac{1}{4}m_{11}$ and $n_{16} = \frac{1}{3}m_{16}$, it follows that $m_{11} = m_{16} = 0$. Next,

$$n_6 = -\tfrac{1}{2}m_7 \quad \text{and} \quad n_7 = \tfrac{1}{2}m_7,$$

so that $m_7 = n_6 = n_7 = 0$. Similarly,

$$n_8 = \tfrac{1}{2}m_8 \quad \text{and} \quad n_9 = -\tfrac{1}{2}m_8,$$

so that $m_8 = n_8 = n_9 = 0$;

$$n_{18} = -\tfrac{1}{2}m_{19} \quad \text{and} \quad n_{19} = \tfrac{1}{2}m_{19},$$

so that $m_{19} = n_{18} = n_{19} = 0$;

$$n_{14} = \tfrac{1}{2}m_{14} \quad \text{and} \quad n_2 + n_3 = -\tfrac{1}{2}m_{14},$$

so that $m_{14} = n_{14} = n_2 = n_3 = 0$; and

$$n_{11} = -\tfrac{1}{2}m_{20} \quad \text{and} \quad n_{13} = \tfrac{1}{2}m_{20},$$

so that $m_{20} = n_{11} = n_{13} = 0$. All the m_i having now been determined, the remaining n_j may be found; we obtain $n_1 = n_4 = 1$, $n_5 = 2$, $n_{10} = n_{12} = 3$. It follows that the irreducible constituents of ${1_{H_1}}^G$ lying in this family are $\chi_{\phi_{12,4}}$, $\chi_{\phi''_{9,6}}$, $\chi_{\phi''_{1,12}}$, $\chi_{\phi''_{6,6}}$, $\chi_{F_4^{\mathrm{II}}[1]}$, with multiplicities 1, 3, 3, 2, 1, respectively.

The analyses for $m = 2$ and $m = 3$ are very similar. In both cases we find that the irreducible constituents of ${1_{H_m}}^G$ lying in the family all have multiplicity 1; for $m = 2$ they are $\chi_{\phi''_{4,7}}$, $\chi_{\phi_{16,5}}$, $\chi_{B_2,\epsilon'}$ and $\chi_{F_4[-1]}$, while for $m = 3$ they are $\chi_{\phi'_{6,6}}$, $\chi_{F_4[\theta]}$ and $\chi_{F_4[\theta^2]}$. This completes the treatment of the geometric conjugacy class of unipotent characters of G.

3.3. ***Other geometric conjugacy classes.*** We now turn to the other geometric conjugacy classes, which contain the $R_{\boldsymbol{T},\theta}$ with $\theta \neq 1$. In most such instances, the contribution to the scalar product $({1_{H_m}}^G, R_{\boldsymbol{T},\theta})_G$ from a relevant type of H_m-class will have zero nonnegligible part; this is because adding together the roots of unity $\theta(s)$, as s runs through the elements of $\boldsymbol{T}$ lying in H_m-classes of the type concerned, usually results in a cancellation of terms. In fact, this is exactly what happens for all $\theta \neq 1$ in the case of types of H_m-class containing regular semisimple elements of G, as noted at the end of Section 2.4; we shall therefore not need to consider types $h_{76}, \ldots, h_{100}$ any further here.

We shall use notation akin to that of [15] for the geometric conjugacy classes of characters of G. Recall that there is a bijective correspondence between geometric conjugacy classes of G and semisimple classes of the dual group, which in this case is isomorphic to G itself; in [15] we have provided for each n an explicit correspondence, involving certain roots of unity ξ_i in k^* and ζ_i in $\mathbb{C}^*$, between the linear characters of the torus $T_{(n)}$ and the elements of the dual torus. We shall say that a geometric conjugacy class is of type κ_c if the corresponding semisimple class is termed h_c in [25]. Writing $\mathbb{Z}_n$ for the integers modulo n for appropriate $n \in \mathbb{N}$, we shall define a set S_c, and an equivalence relation $\sim$ on it, such that the set $\bar{S}_c$ of equivalence classes $[i]$ for $i \in S_c$ parametrizes the semisimple classes of type h_c; accordingly an individual geometric conjugacy class of type κ_c will be denoted by $\kappa_{c,[i]}$. (This notation differs slightly from that employed in [15], where such geometric conjugacy classes were indexed by the element i of S_c rather than the equivalence class $[i]$. The only such equivalence relation required there was that defined by $i \sim -i$, so that the list of all such geometric conjugacy classes could be obtained as $\kappa_{c,i}$ as i ran through the first half of the set S_c; here by contrast more complicated equivalence relations will be needed, so that it will be clearer to index geometric conjugacy classes explicitly by equivalence classes.) The irreducible characters lying in a given geometric conjugacy class are parametrized by the unipotent characters of the centralizer of an element of the corresponding semisimple class; an irreducible character lying in the geometric conjugacy class $\kappa_{c,[i]}$ will be written in the form $\chi^*_{\kappa_{c,[i]}}$, where the superscript indicates the corresponding unipotent character.

We shall refrain from giving full details, because there are several types requiring consideration, and the calculation for each is fairly involved (as may be surmised from the treatment of the unipotent geometric conjugacy class above). Instead we shall deal at some length with two types of geometric conjugacy class, κ_{31} and κ_7, and indicate briefly how the behavior for other classes is related to one or other of these. Note that it was only in handling semisimple classes of types $h_{76}, \ldots, h_{100}$ that H_1, H_2 and H_3 had to be treated separately in the calculations above; since these classes play no further role, at all stages it will be possible to work with all three permutation characters ${1_{H_m}}^G$ simultaneously.

For convenience we repeat from [15] the notation used for roots of unity, as this will be needed in much of what follows. We let $\xi \in k^*$ be a primitive $(q^4+1)(q^{12}-1)$-st root of unity, and for $s = 1, \ldots, 15$ we set $\xi_s = \xi^{r_s}$, where r_s and $(q^4+1)(q^{12}-1)/r_s = o(\xi_s)$ (the order of ξ_s in the multiplicative group k^*) are

$$
\begin{aligned}
r_1 &= (q+1)(q^2+1)(q^4+1)(q^8+q^4+1), & o(\xi_1) &= q-1,\\
r_2 &= (q-1)(q^2+1)(q^4+1)(q^8+q^4+1), & o(\xi_2) &= q+1,\\
r_3 &= (q^2+1)(q^4+1)(q^8+q^4+1), & o(\xi_3) &= q^2-1,\\
r_4 &= (q^2-1)(q^4+1)(q^8+q^4+1), & o(\xi_4) &= q^2+1,\\
r_5 &= (q-1)(q^3+1)(q^4+1)(q^6+1), & o(\xi_5) &= q^2+q+1,\\
r_6 &= (q+1)(q^3-1)(q^4+1)(q^6+1), & o(\xi_6) &= q^2-q+1,\\
r_7 &= (q^3+1)(q^4+1)(q^6+1), & o(\xi_7) &= q^3-1,\\
r_8 &= (q^3-1)(q^4+1)(q^6+1), & o(\xi_8) &= q^3+1,\\
r_9 &= (q+1)(q^4+1)(q^8+q^4+1), & o(\xi_9) &= (q^2+1)(q-1),\\
r_{10} &= (q-1)(q^4+1)(q^8+q^4+1), & o(\xi_{10}) &= (q^2+1)(q+1),\\
r_{11} &= (q^4+1)(q^8+q^4+1), & o(\xi_{11}) &= q^4-1,\\
r_{12} &= q^{12}-1, & o(\xi_{12}) &= q^4+1,\\
r_{13} &= (q^2-q+1)(q^4+1)(q^6+1), & o(\xi_{13}) &= (q^3-1)(q+1),\\
r_{14} &= (q^2+q+1)(q^4+1)(q^6+1), & o(\xi_{14}) &= (q^3+1)(q-1),\\
r_{15} &= (q^4+q^2+1)(q^8-1), & o(\xi_{15}) &= q^4-q^2+1.
\end{aligned}
$$

Likewise we write

$$\zeta = e^{2\pi i/(q^4+1)(q^{12}-1)} \in \mathbb{C}^*,$$

and for $s = 1, \ldots, 15$ we set $\zeta_s = \zeta^{r_s}$.

3.3.1. *Geometric conjugacy classes of type κ_{31}.* The geometric conjugacy classes of type κ_{31} correspond to semisimple classes in G containing elements

$$({\xi_1}^{i+j}, {\xi_1}^{i}, {\xi_1}^{j}, 1; {\xi_1}^{i+j}) \quad \text{with } {\xi_1}^{i}, {\xi_1}^{j}, {\xi_1}^{i\pm j}, {\xi_1}^{2i+j}, {\xi_1}^{i+2j} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 7; the number of classes is $\frac{1}{12}(q^2 - 8q + 10 + 3d + 2y)$. To parametrize these geometric conjugacy classes, we set

$$S_{31} = \{(i, j) \in {\mathbb{Z}_{q-1}}^2 : i, j, i \pm j, 2i + j, i + 2j \neq 0\},$$

so that $(i, j) \in S_{31}$ corresponds to the element given; we define an equivalence relation on S_{31} by

$$(i, j) \sim (j, i) \sim (-i, -j) \sim (i + j, -j),$$

and let $\bar{S}_{31}$ be the set of equivalence classes $[(i, j)]$. The geometric conjugacy classes of type κ_{31} are in bijective correspondence with $\bar{S}_{31}$; we shall write $\kappa_{31,[(i,j)]}$ for the class corresponding to $[(i, j)] \in \bar{S}_{31}$.

There are three distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{31,[(i,j)]}$; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as

$$(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00j}), \quad (\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0j}), \quad (\boldsymbol{T}_{(6)}, \theta^{(6)}_{-(q^2+q+1)i,2i+j}).$$

For convenience of reference, we repeat the definition of the characters θ here:

$$\theta^{(1)}_{i00j}({\xi_1}^a, {\xi_1}^b, {\xi_1}^c, {\xi_1}^{2d-a-b-c}; {\xi_1}^d) = {\zeta_1}^{ia+jd},$$
$$\theta^{(2)}_{i0j}({\xi_1}^a, {\xi_1}^{2c-a-b}, {\xi_3}^b, {\xi_3}^{qb}; {\xi_1}^c) = {\zeta_1}^{ia+jc},$$
$$\theta^{(6)}_{-(q^2+q+1)i,2i+j}({\xi_1}^{2b-a}, {\xi_7}^a, {\xi_7}^{qa}, {\xi_7}^{q^2a}; {\xi_1}^b) = {\zeta_1}^{-ia+(2i+j)b}.$$

We shall take each character $R_{\boldsymbol{T},\theta}$ in turn, and find the scalar product with ${1_{H_m}}^G$; to do this we shall take each type of class handled above containing elements with semisimple parts lying in the torus concerned, and calculate the nonnegligible part of its contribution. We begin with the pair $(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00j})$.

The semisimple classes of type h_7 contain elements $(1, {\xi_1}^a, {\xi_1}^{-a}, 1; 1)$ with ${\xi_1}^{2a} \neq 1$; each such element has 24 distinct conjugates in $T_{(1)}$. Of these, 6 are of the form $(1, *, *, *; 1)$, and are thus sent to 1 by $\theta^{(1)}_{i00j}$; the remaining 18 are sent to various other powers of ζ_1. On summing over the $\frac{1}{2}(q - 2 - d)$ classes of such elements, the values obtained from the conjugates of the form $(1, *, *, *; 1)$ combine to give a sum of $3(q - 2 - d)$; on the other hand the values obtained from the conjugates not of the form $(1, *, *, *; 1)$ cancel each other out to give a sum of $9(d - 1)$, which is too small to affect the rest of the calculation. We may now proceed as in the unipotent case already treated, multiplying by the Green function value and dividing by the order of the centralizer in H_m; since the leading term is obtained from only 6 of the 24 values taken by $\theta^{(1)}_{i00j}$ on each class, each nonnegligible part is one quarter of the corresponding value in the unipotent case.

Thus the nonnegligible part of the contribution from the semisimple classes is 3, while that from the regular classes is 9.

The classes with semisimple part of type h_9 behave in a very similar fashion. The semisimple elements here are $({\xi_1}^a, {\xi_1}^{-a}, {\xi_1}^{2a}, 1; {\xi_1}^a)$ with ${\xi_1}^{2a}, {\xi_1}^{3a} \neq 1$; each such element has 96 distinct conjugates in $T_{(1)}$. Again, 6 of these are of the form $(1, *, *, *; 1)$, and are thus sent to 1 by $\theta^{(1)}_{i00j}$, while the remainder are sent to various other powers of ζ_1. Only the former need be considered, because summing over the $\frac{1}{2}(q-1-d-y)$ classes of such elements produces cancellation among the latter. Since the proportion of roots of unity which are 1 is $\frac{6}{96}$, multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a rational polynomial whose leading term is one sixteenth of that obtained in the unipotent case; thus the nonnegligible part of the contribution from the classes of regular elements of H_m with semisimple part of type h_9 is 3.

The semisimple classes of type h_{31} require somewhat more care. The elements are $({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$ with ${\xi_1}^a, {\xi_1}^b, {\xi_1}^{a\pm b}, {\xi_1}^{2a+b}, {\xi_1}^{a+2b} \neq 1$ (as above); each such element has 192 distinct conjugates in $T_{(1)}$. However, it is not sufficient here simply to count how many are sent to 1 by $\theta^{(1)}_{i00j}$; because the calculation will involve not just the leading term, but the next term as well, we must consider the sum of roots of unity more carefully than was necessary in the previous two paragraphs.

Let s be the element given. The 192 distinct conjugates of s in $T_{(1)}$ are obtained as $s^{w'w''w'''}$, where $w' \in \langle w_1, w_2, w_3\rangle$, $w'' \in \{1, w_{1-2}, w_{1-3}, w_{1-4}\}$ and $w''' \in \langle w_{2-3}, w_{3-4}\rangle$. Since the effect of w''' is simply to permute the second, third and fourth coefficients, we have

$$\theta^{(1)}_{i00j}(s^{w'w''w'''}) = \theta^{(1)}_{i00j}(s^{w'w''});$$

moreover, conjugation by $w_2w_3w_4$ inverts s. Thus the sum of the values taken by $\theta^{(1)}_{i00j}$ on the conjugates of s is

$$\sum_{w',w''} 6(\theta^{(1)}_{i00j}(s^{w'w''}) + \theta^{(1)}_{i00j}(s^{w'w''})^{-1}),$$

where w' is restricted to run over $\{1, w_1, w_2, w_3\}$. For each of the 16 possibilities for the pair (w', w''), we consider the sum of the values over the different classes of type h_{31}. There are $\frac{1}{12}(q^2-8q+10+3d+2y)$ such classes; the conditions which a and b must satisfy mean that we must sum over the square $0 \leq a, b \leq q-2$, subtract the sums over the six lines $a=0$, $b=0$, $a-b=0$, $a+b=0$, $2a+b=0$ and $a+2b=0$ (where all equalities are of course taken modulo $q-1$), and then divide by 12 to allow for the fact that the points (a, b), (b, a), $(-a, -b)$ and $(a, a-b)$ all give the same class (for points (a, b) lying on more than one such line a further compensation is really required, but this is too small to affect the nonnegligible part of the contribution).

If $(w', w'') = (w_1, w_{1-4})$ then $\theta^{(1)}_{i00j}(s^{w'w''}) = 1$; thus the sum over the different classes here is $q^2 - 8q + \cdots$ (note that we require only the terms of positive degree in q here, since any remaining terms are small enough to be ignored). If $(w', w'') = (w_1, w_{1-2})$ we have $\theta^{(1)}_{i00j}(s^{w'w''}) = \zeta_1{}^{ia}$; the sum of these values over the square and five of the lines is zero, but the sum over the line $a = 0$ is $q - 1$, so that the sum over the classes is $-q + \cdots$. If $(w', w'') = (w_3, w_{1-4})$ or (w_3, w_{1-2}) we have $\theta^{(1)}_{i00j}(s^{w'w''}) = \zeta_1{}^{ja}$ or $\zeta_1{}^{(i+j)a}$, respectively, which behave entirely similarly. Likewise if $(w', w'') = (w_1, w_{1-3})$, (w_2, w_{1-4}) or (w_2, w_{1-3}) we have $\theta^{(1)}_{i00j}(s^{w'w''}) = \zeta_1{}^{ib}$, $\zeta_1{}^{jb}$ or $\zeta_1{}^{(i+j)b}$, respectively, while if $(w', w'') = (w_1, 1)$, $(1, w_{1-4})$ or $(1, 1)$ we have $\theta^{(1)}_{i00j}(s^{w'w''}) = \zeta_1{}^{-i(a+b)}$, $\zeta_1{}^{j(a+b)}$ or $\zeta_1{}^{(i+j)(a+b)}$, respectively; thus there are nine pairs (w', w'') giving a sum $-q + \cdots$. If however $(w', w'') = (w_2, w_{1-2})$ then $\theta^{(1)}_{i00j}(s^{w'w''}) = \zeta_1{}^{-ia+jb}$; the sum of these values over the square and all six of the lines is zero. The remaining five pairs behave similarly. We therefore obtain a total sum of $q^2 - 17q + \cdots$. Multiplying by the Green function value and dividing by the order of the centralizer in H_m gives $q + f - 14$ as the nonnegligible part of the contribution from the semisimple classes of type h_{31}.

Finally, we treat the semisimple classes of type h_{66}, which contain elements $(\xi_1{}^{a+b}, \xi_1{}^{a+c}, \xi_1{}^{b+c}, 1; \xi_1{}^{a+b+c})$ with both $\xi_1{}^a, \xi_1{}^b, \xi_1{}^c, \xi_1{}^{a\pm b}, \xi_1{}^{a\pm c}, \xi_1{}^{b\pm c} \neq 1$ and $\xi_1{}^{a+b+c} \neq 1, \xi_1{}^{-a}, \xi_1{}^{-b}, \xi_1{}^{-c}$. Each such element has 576 distinct conjugates in $T_{(1)}$, but none of these is of the form $(1, *, *, *; 1)$. In this case, therefore, summing over all classes of such elements produces cancellation in all instances; accordingly, the nonnegligible part of the contribution from these classes is 0. This completes the consideration of the pair $(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00j})$.

We turn to the pair $(\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0j})$. The semisimple classes of type h_7 are as above; the elements lying in $T_{(2)}$ are $(1, 1, \xi_1{}^a, \xi_1{}^a; \xi_1{}^a)$, $(\xi_1{}^a, \xi_1{}^a, 1, 1; \xi_1{}^a)$ and $(\xi_1{}^a, \xi_1{}^{-a}, 1, 1; 1)$, together with their inverses. None is of the form $(1, *, *, *; 1)$, so that the values taken by $\theta^{(2)}_{i0j}$ are all roots of unity other than 1. Summing over the classes therefore produces cancellation as above, and so the nonnegligible part of the contribution from both the semisimple classes and the regular classes is 0.

The semisimple classes of type h_8 contain elements $(1, \xi_2{}^a, \xi_2{}^{-a}, 1; 1)$ with $\xi_2{}^{2a} \neq 1$; the elements lying in $T_{(2)}$ are $(1, 1, \xi_2{}^a, \xi_2{}^{-a}; 1)$ and its inverse. Since each of these is sent to 1 by $\theta^{(2)}_{i0j}$, the contributions here are the same as in the unipotent case: the nonnegligible parts are 1 from the semisimple classes and 3 from the regular classes.

The semisimple classes of type h_9 are again as above. The element given has 8 conjugates lying in $T_{(2)}$, of which only $(1, \xi_1{}^{-2a}, \xi_1{}^a, \xi_1{}^a; 1)$ and its inverse are of the form $(1, *, *, *; 1)$; thus the nonnegligible part of the contribution from the classes of regular elements of H_m with semisimple part of type h_9 is one quarter of that in the unipotent case, and therefore is 1.

The semisimple classes of type h_{33} contain elements $(\xi_1{}^a, \xi_3{}^a, \xi_3{}^{qa}, 1; \xi_1{}^a)$ with $\xi_1{}^a, \xi_2{}^a \neq 1$; there are $\frac{1}{4}(q^2 - 2q + d)$ such classes. As with type h_{31} above, simply counting the number of roots of unity equal to 1 is insufficient here. If we let s be the element given, then to cover all classes we must sum from $a=0$ to q^2-2, subtract the sums of the terms in which a is a multiple of $q+1$ or $q-1$, and divide by 4 (again, an adjustment should really be made for terms with $a=0$ or $a=\frac{1}{2}(q^2-1)$, but this is too small to affect the final nonnegligible parts). There are 16 conjugates of s in $T_{(2)}$. Of these, 4 are of the form $(1,*,*,*;1)$, so that the value taken by $\theta^{(2)}_{i0j}$ is 1; summing over the classes here gives q^2-2q+d. For the remaining 12 conjugates, the value taken by $\theta^{(2)}_{i0j}$ is $\zeta_1{}^{\pm ia}$, $\zeta_1{}^{\pm ja}$ or $\zeta_1{}^{\pm(i+j)a}$; summing from $a=0$ to q^2-2 gives zero, as does the sum of the terms with a a multiple of $q+1$, but the terms with a a multiple of $q-1$ sum to $q+1$. Thus the total sum is $q^2-5q+\cdots$; multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a contribution with nonnegligible part $q+f-4$.

Finally, we treat the semisimple classes of type h_{70}, which contain elements $(\xi_1{}^{2b-a},\xi_3{}^a,\xi_3{}^{qa},1;\xi_1{}^b)$ with $\xi_3{}^{(q\pm1)a},\xi_1{}^b,\xi_1{}^{a-b},\xi_1{}^{a-2b}\neq1$. No conjugate in $T_{(2)}$ is of the form $(1,*,*,*;1)$, so that summing over all classes of such elements produces cancellation in all instances; accordingly, the nonnegligible part of the contribution from these classes is 0. This completes consideration of the pair $(\boldsymbol{T}_{(2)},\theta^{(2)}_{i0j})$.

The third and final pair is $(\boldsymbol{T}_{(6)},\theta^{(6)}_{-(q^2+q+1)i,2i+j})$; there are only two types of class requiring consideration. For the semisimple classes of type h_{35}, the elements are $(\xi_5{}^a,\xi_5{}^{-qa},\xi_5{}^{(q+1)a},1;\xi_5{}^a)$ with $\xi_5{}^{3a}\neq1$; all 6 conjugates lying in $T_{(6)}$ are sent to 1 by $\theta^{(6)}_{-(q^2+q+1)i,2i+j}$, so the contribution is the same as in the unipotent case, with nonnegligible part $q+f+1$. For the semisimple classes of type h_{74}, the elements are $(\xi_7{}^{(q+1)a},\xi_7{}^{(q^2+q)a},\xi_7{}^{(q^2+1)a},1;\xi_7{}^{(q^2+q+1)a})$ with $\xi_5{}^a,\xi_1{}^a\neq1$; here none of the conjugates is sent to 1 by $\theta^{(6)}_{-(q^2+q+1)i,2i+j}$, so the contribution has nonnegligible part 0.

Table 11 lists these nonnegligible parts of contributions to scalar products with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{31,[(i,j)]}$, where for types h_7 and h_8 the first row relates to the semisimple classes and the second row to the classes of regular elements of H_m. It follows that the scalar product of $1_{H_m}{}^G$ with each of the three characters $R_{\boldsymbol{T},\theta}$ treated here is $q+f+1$. Taking linear combinations given by the character table of the Weyl group of type A_2, we see that the only constituent of $1_{H_m}{}^G$ lying in the geometric conjugacy class $\kappa_{31,[(i,j)]}$ is the semisimple character, whose multiplicity is $q+f+1$; we shall call this character $\chi^1_{\kappa_{31,[(i,j)]}}$.

3.3.2. *Geometric conjugacy classes of types κ_{32}, κ_{33}, κ_{34}, κ_{35} and κ_{36}.* The geometric conjugacy classes of types κ_{32}, κ_{33}, κ_{34}, κ_{35} and κ_{36} correspond to semisimple classes in G containing elements as follows:

$$\kappa_{32}:(\xi_2{}^{i+j},\xi_2{}^i,\xi_2{}^j,1;\xi_2{}^{i+j})\qquad \text{with } \xi_2{}^i,\xi_2{}^j,\xi_2{}^{i\pm j},\xi_2{}^{2i+j},\xi_2{}^{i+2j}\neq1;$$

$$\kappa_{33}:(\xi_1{}^i,\xi_3{}^i,\xi_3{}^{qi},1;\xi_1{}^i)\qquad \text{with } \xi_1{}^i,\xi_2{}^i\neq1;$$

	$\boldsymbol{T}_{(1)}$	$\boldsymbol{T}_{(2)}$	$\boldsymbol{T}_{(6)}$
h_7	3 9	0 0	– –
h_8	– –	1 3	– –
h_9	3	1	–
h_{31}	$q+f-14$	–	–
h_{33}	–	$q+f-4$	–
h_{35}	–	–	$q+f+1$
h_{66}	0	–	–
h_{70}	–	0	–
h_{74}	–	–	0
	$q+f+1$	$q+f+1$	$q+f+1$

Table 11. Contributions to scalar products $(1_{H_m}{}^G, R_{\boldsymbol{T},\theta})_G$ for $R_{\boldsymbol{T},\theta}$ in $\kappa_{31,[(i,j)]}$.

$$\kappa_{34} : ({\xi_2}^i, {\xi_3}^{-i}, {\xi_3}^{qi}, 1; {\xi_2}^i) \quad \text{with } {\xi_1}^i, {\xi_2}^i \neq 1;$$
$$\kappa_{35} : ({\xi_5}^i, {\xi_5}^{-qi}, {\xi_5}^{(q+1)i}, 1; {\xi_5}^i) \quad \text{with } {\xi_5}^{3i} \neq 1;$$
$$\kappa_{36} : ({\xi_6}^i, {\xi_6}^{qi}, {\xi_6}^{-(q-1)i}, 1; {\xi_6}^i) \quad \text{with } {\xi_6}^{3i} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 7; the number of classes is $\frac{1}{12}(q^2-4q-2+3d+2z)$, $\frac{1}{4}(q^2-2q+d)$, $\frac{1}{4}(q^2-2q+d)$, $\frac{1}{6}(q^2+q+1-y)$ or $\frac{1}{6}(q^2-q+1-z)$, respectively. To parametrize these geometric conjugacy classes, we set

$$S_{32} = \{(i, j) \in {\mathbb{Z}_{q+1}}^2 : i, j, i \pm j, 2i + j, i + 2j \neq 0\},$$
$$S_{33} = S_{34} = \{i \in \mathbb{Z}_{q^2-1} : (q \pm 1)i \neq 0\},$$
$$S_{35} = \{i \in \mathbb{Z}_{q^2+q+1} : 3i \neq 0\},$$
$$S_{36} = \{i \in \mathbb{Z}_{q^2-q+1} : 3i \neq 0\},$$

so that $(i, j) \in S_{32}$ or $i \in S_c$ for $c = 33, \ldots, 36$ corresponds to the element given; define an equivalence relation on the set S_{32} by

$$(i, j) \sim (j, i) \sim (-i, -j) \sim (i + j, -j),$$

and similarly define equivalence relations on S_{33}, S_{34}, S_{35} and S_{36} by

$$i \sim -i \sim qi.$$

Let $\bar{S}_{32}$ denote the set of equivalence classes $[(i, j)]$ of S_{32}, and $\bar{S}_c$ denote the set of equivalence classes $[i]$ of S_c for $c = 33, \dots, 36$. The geometric conjugacy classes of type κ_c are then in bijective correspondence with the set $\bar{S}_c$ for $c = 32, \dots, 36$; we shall write $\kappa_{32,[(i,j)]}$ for the class corresponding to $[(i, j)] \in \bar{S}_{32}$, and $\kappa_{c,[i]}$ for the class corresponding to $[i] \in \bar{S}_c$ for $c = 33, \dots, 36$.

We shall give less detail for these types, as the behavior in each case is very similar to κ_{31}. For each κ_c there are three characters $R_{\boldsymbol{T},\theta}$ in each geometric conjugacy class; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as follows:

$$\begin{aligned}
\kappa_{32,[(i,j)]} &: (\boldsymbol{T}_{(17)}, \theta^{(17)}_{i00j}),\ (\boldsymbol{T}_{(9)}, \theta^{(9)}_{-ij0}),\ (\boldsymbol{T}_{(21)}, \theta^{(21)}_{-(q^2-q+1)i,2i+j});\\
\kappa_{33,[i]} &: (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00i}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0i}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q^2+q+1)i});\\
\kappa_{34,[i]} &: (\boldsymbol{T}_{(10)}, \theta^{(10)}_{00i}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{-ii}),\ (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q^2-q+1)i});\\
\kappa_{35,[i]} &: (\boldsymbol{T}_{(7)}, \theta^{(7)}_{(q-1)i0}),\ (\boldsymbol{T}_{(15)}, \theta^{(15)}_{(q^2-1)i}),\ (\boldsymbol{T}_{(18)}, \theta^{(18)}_{i0});\\
\kappa_{36,[i]} &: (\boldsymbol{T}_{(20)}, \theta^{(20)}_{(q+1)i0}),\ (\boldsymbol{T}_{(13)}, \theta^{(13)}_{(q^2-1)i}),\ (\boldsymbol{T}_{(25)}, \theta^{(25)}_{i0}).
\end{aligned}$$

In each case, we find as with κ_{31} that the scalar product with ${1_{H_m}}^G$ is the same for all three; thus the only constituent of ${1_{H_m}}^G$ lying in each such geometric conjugacy class is the semisimple character, which we call $\chi^1_{\kappa_{32,[(i,j)]}}$ or $\chi^1_{\kappa_{c,[i]}}$ for $c = 33, \dots, 36$. The multiplicities of these constituents are $q - f - 1$ for $\kappa_{32,[(i,j)]}$; $q + f - 1$ for $\kappa_{33,[i]}$; $q - f + 1$ for $\kappa_{34,[i]}$; $q + f - 1$ for $\kappa_{35,[i]}$; and $q - f + 2$ for $\kappa_{36,[i]}$.

3.3.3. *Geometric conjugacy classes of types κ_9 and κ_{10}.* The geometric conjugacy classes of types κ_9 and κ_{10} correspond to semisimple classes in G containing elements as follows:

$$\begin{aligned}
\kappa_9 &: ({\xi_1}^i, {\xi_1}^{-i}, {\xi_1}^{2i}, 1; {\xi_1}^i) \quad \text{with } {\xi_1}^{2i}, {\xi_1}^{3i} \neq 1,\\
\kappa_{10} &: ({\xi_2}^i, {\xi_2}^{-i}, {\xi_2}^{2i}, 1; {\xi_2}^i) \quad \text{with } {\xi_2}^{2i}, {\xi_2}^{3i} \neq 1.
\end{aligned}$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 6; the number of classes is $\frac{1}{2}(q - 1 - d - y)$ or $\frac{1}{2}(q + 1 - d - z)$, respectively. To parametrize these geometric conjugacy classes, we write $\epsilon = 1$ for κ_9 and $\epsilon = -1$ for κ_{10}, and for $c = 9, 10$ we set

$$S_c = \{i \in \mathbb{Z}_{q-\epsilon} : 2i, 3i \neq 0\},$$

so that $i \in S_c$ corresponds to the element given; we define an equivalence relation on S_c by

$$i \sim -i,$$

and let $\bar{S}_c$ be the set of equivalence classes $[i]$. The geometric conjugacy classes of type κ_c are in bijective correspondence with $\bar{S}_c$; we shall write $\kappa_{c,[i]}$ for the class corresponding to $[i] \in \bar{S}_c$. There are six distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{c,[i]}$; by a temporary abuse of notation we may say that $\kappa_{9,[i]}$ is the union of $\kappa_{31,[(i,i)]}$ and $\kappa_{33,[(q+1)i]}$, while $\kappa_{10,[i]}$ is the union of $\kappa_{32,[(i,i)]}$ and $\kappa_{34,[(q-1)i]}$ (by which we mean that the characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{9,[i]}$ are obtained from those lying in $\kappa_{31,[(i,j)]}$ and $\kappa_{33,[i]}$ by setting $j=i$ in the former and replacing i by $(q+1)i$ in the latter, and similarly for $\kappa_{10,[i]}$).

We find that the calculations to find nonnegligible parts of contributions proceed almost exactly as for types κ_{31} and κ_{33}, or κ_{32} and κ_{34}; the only differences occur with semisimple classes of types h_9 and h_{31} for κ_9, or h_{10} and h_{32} for κ_{10}. For type h_9, it was found in the treatment of κ_{31} above that only 6 of the 96 conjugates of $({\xi_1}^a, {\xi_1}^{-a}, {\xi_1}^{2a}, 1; {\xi_1}^a)$ were sent to 1 by $\theta^{(1)}_{i00j}$; here we find that setting $j=i$ means that an extra 12 conjugates of the form $({\xi_1}^{-a}, *, *, *; {\xi_1}^a)^{\pm 1}$ are sent to 1, and thus the nonnegligible part of the contribution from these classes increases by 6 from 3 to 9. For type h_{31}, on the other hand, the treatment above divided into a consideration of 16 cases, in 6 of which the sum of the roots of unity concerned was too small to affect matters; here we find that these 6 behave in the same manner as the other 9 where the root of unity is not simply 1, and accordingly the nonnegligible part of the contribution from these classes decreases by 6 from $q+f-14$ to $q+f-20$. Since these two changes cancel each other out (and the same is true for κ_{10} with types h_{10} and h_{32}), the scalar products with ${1_{H_m}}^G$ of the $R_{\boldsymbol{T},\theta}$ are the same as above: for $\kappa_{9,[i]}$ we have

$$\begin{aligned} q+f+1 &\quad \text{for } (\boldsymbol{T}_{(1)}, \theta^{(1)}_{i00i}),\ (\boldsymbol{T}_{(2)}, \theta^{(2)}_{i0i}),\ (\boldsymbol{T}_{(6)}, \theta^{(6)}_{-(q^2+q+1)i,3i}), \\ -q-f+1 &\quad \text{for } (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00(q+1)i}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0(q+1)i}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q^2+q+1)(q+1)i}), \end{aligned}$$

while for $\kappa_{10,[i]}$ we have

$$\begin{aligned} -q+f+1 &\quad \text{for } (\boldsymbol{T}_{(17)}, \theta^{(17)}_{i00i}), (\boldsymbol{T}_{(9)}, \theta^{(9)}_{-ii0}), (\boldsymbol{T}_{(21)}, \theta^{(21)}_{-(q^2-q+1)i,3i}), \\ q-f+1 &\quad \text{for } (\boldsymbol{T}_{(10)}, \theta^{(10)}_{00(q-1)i}), (\boldsymbol{T}_{(5)}, \theta^{(5)}_{-(q-1)i,(q-1)i}), (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q^2-q+1)(q-1)i}). \end{aligned}$$

Taking linear combinations given by the character table of the Weyl group of type A_2A_1, we see that there are two constituents of ${1_{H_m}}^G$ lying in each such geometric conjugacy class. The first constituent, which has multiplicity 1, is the semisimple character. The second constituent, which has multiplicity $q+\epsilon f$, corresponds to the unipotent character of the centralizer $A_1(q).A_2^{\epsilon}(q).T_1$ whose restrictions to the $A_1(q)$ and $A_2^{\epsilon}(q)$ factors are the Steinberg and the trivial characters, respectively. We shall call these characters $\chi^{1,1}_{\kappa_{c,[i]}}$ and $\chi^{\mathrm{St},1}_{\kappa_{c,[i]}}$.

3.3.4. *Geometric conjugacy classes of types κ_3 and κ_4.* The geometric conjugacy classes of types κ_3 and κ_4 correspond to semisimple classes in G containing elements as follows:

$$\kappa_3 : ({\xi_1}^i, {\xi_1}^{-i}, {\xi_1}^{-i}, 1; {\xi_1}^i) \quad \text{with } {\xi_1}^i \neq 1 = {\xi_1}^{3i},$$
$$\kappa_4 : ({\xi_2}^i, {\xi_2}^{-i}, {\xi_2}^{-i}, 1; {\xi_2}^i) \quad \text{with } {\xi_2}^i \neq 1 = {\xi_2}^{3i}.$$

If we write $\epsilon = 1$ for κ_3 and $\epsilon = -1$ for κ_4, the centralizer in G is a product of two groups $A_2^\epsilon(q)$ (one factor involving long root groups and the other short), while the number of such classes is $\frac{1}{2}(y-1)$ for κ_3 and $\frac{1}{2}(z-1)$ for κ_4. Since there is at most one such class, we shall simply call it κ_c for $c = 3, 4$, with the understanding that the class does not exist unless q is congruent to ϵ modulo 3. There are nine distinct characters $R_{\boldsymbol{T},\theta}$ lying in κ_c; by a temporary abuse of notation similar to that above we may say that κ_3 is the union of $\kappa_{9,[(q-1)/3]}$ and $\kappa_{35,[(q^2+q+1)/3]}$, while κ_4 is the union of $\kappa_{10,[(q+1)/3]}$ and $\kappa_{36,[(q^2-q+1)/3]}$.

We find that the calculations to find nonnegligible parts of contributions proceed exactly as for types κ_9, κ_{10}, κ_{35} and κ_{36} above. Thus the scalar products with ${1_{H_m}}^G$ of the $R_{\boldsymbol{T},\theta}$ are the same as above: for κ_3, writing q_i^+ for $(q^i-1)/3$ we have

$$\begin{aligned}
q+f+1 &\quad \text{for } (\boldsymbol{T}_{(1)}, \theta^{(1)}_{q_1^+00q_1^+}),\ (\boldsymbol{T}_{(2)}, \theta^{(2)}_{q_1^+0q_1^+}),\ (\boldsymbol{T}_{(6)}, \theta^{(6)}_{q_3^+0}),\\
-q-f+1 &\quad \text{for } (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00q_2^+}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0q_2^+}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q+1)q_3^+}),\\
q+f-2 &\quad \text{for } (\boldsymbol{T}_{(7)}, \theta^{(7)}_{q_3^+0}),\ (\boldsymbol{T}_{(15)}, \theta^{(15)}_{(q+1)q_3^+}),\ (\boldsymbol{T}_{(18)}, \theta^{(18)}_{(q^2+q+1)/3,0}),
\end{aligned}$$

while for κ_4, writing q_i^- for $(q^i-(-1)^i)/3$ we have

$$\begin{aligned}
-q+f+1 &\quad \text{for } (\boldsymbol{T}_{(17)}, \theta^{(17)}_{q_1^-00q_1^-}),\ (\boldsymbol{T}_{(9)}, \theta^{(9)}_{-q_1^-q_1^-0}),\ (\boldsymbol{T}_{(21)}, \theta^{(21)}_{q_3^-0}),\\
q-f+1 &\quad \text{for } (\boldsymbol{T}_{(10)}, \theta^{(10)}_{00q_2^-}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{-q_2^-q_2^-}),\ (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q-1)q_3^-}),\\
-q+f-2 &\quad \text{for } (\boldsymbol{T}_{(20)}, \theta^{(20)}_{q_3^-0}),\ (\boldsymbol{T}_{(13)}, \theta^{(13)}_{(q-1)q_3^-}),\ (\boldsymbol{T}_{(25)}, \theta^{(25)}_{(q^2-q+1)/3,0}).
\end{aligned}$$

Taking linear combinations given by the character table of the Weyl group of type A_2A_2, we see that there are again two constituents of ${1_{H_m}}^G$ lying in each such geometric conjugacy class. To describe them, we shall say that the $A_2^\epsilon(q)$ factors of the centralizer involving long and short root groups are the long and short factors, respectively. The first constituent, which has multiplicity 1, corresponds to the unipotent character of the centralizer whose restrictions to the long and short factors are ρ and the trivial character, respectively (where ρ is the third unipotent character of the long factor after the trivial and Steinberg characters). The second constituent, which has multiplicity $q+\epsilon(f-1)$, corresponds to the unipotent character of the centralizer whose restrictions to the long and short factors are the Steinberg and the trivial characters, respectively. We shall call these characters $\chi_{\kappa_c}^{\rho,1}$ and $\chi_{\kappa_c}^{\mathrm{St},1}$.

3.3.5. *Geometric conjugacy classes of type κ_7.* The geometric conjugacy classes of type κ_7 correspond to semisimple classes in G containing elements

$$(1, {\xi_1}^i, {\xi_1}^{-i}, 1; 1) \quad \text{with } {\xi_1}^{2i} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 5; the number of classes is $\frac{1}{2}(q-2-d)$. To parametrize these geometric conjugacy classes, we set

$$S_7 = \{i \in \mathbb{Z}_{q-1} : 2i \neq 0\},$$

so that $i \in S_7$ corresponds to the element given; we define an equivalence relation on S_7 by

$$i \sim -i,$$

and let $\bar{S}_7$ be the set of equivalence classes $[i]$. The geometric conjugacy classes of type κ_7 are in bijective correspondence with $\bar{S}_7$; we shall write $\kappa_{7,[i]}$ for the class corresponding to $[i] \in \bar{S}_7$.

There are ten distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as

$$(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i000}), \quad (\boldsymbol{T}_{(2)}, \theta^{(2)}_{i00}), \quad (\boldsymbol{T}_{(3)}, \theta^{(3)}_{i00}), \quad (\boldsymbol{T}_{(4)}, \theta^{(4)}_{i00}),$$
$$(\boldsymbol{T}_{(5)}, \theta^{(5)}_{-(q+1)i,(q+1)i}), \quad (\boldsymbol{T}_{(6)}, \theta^{(6)}_{0i}), \quad (\boldsymbol{T}_{(8)}, \theta^{(8)}_{i0}),$$
$$(\boldsymbol{T}_{(10)}, \theta^{(10)}_{00(q+1)i}), \quad (\boldsymbol{T}_{(11)}, \theta^{(11)}_{q(q^2+1)(q+1)i/2,i}), \quad (\boldsymbol{T}_{(14)}, \theta^{(14)}_{(q^3+1)i}).$$

We shall give rather less detail here than for κ_{31}, both to save space and because the calculations are similar to those that have already been seen; we shall therefore not bother to repeat from [15] the definitions of the characters θ here.

To begin with, in all instances involving semisimple classes of types other than $h_{31}, \ldots, h_{36}$ the nonnegligible part of the contribution is a constant; as was seen in the treatment of κ_{31}, the value concerned is then determined by the proportion of conjugates sent to 1. As an example, take the pair $(\boldsymbol{T}_{(1)}, \theta^{(1)}_{i000})$ and semisimple classes of type h_7; as has been stated, the elements are $(1, {\xi_1}^a, {\xi_1}^{-a}, 1; 1)$ with ${\xi_1}^{2a} \neq 1$. Of the 24 conjugates of such an element in $T_{(1)}$, there are 12 of the form $(1, *, *, *; *)$, which are therefore sent to 1 by $\theta^{(1)}_{i000}$. Thus the nonnegligible part of the contribution from classes with semisimple part of type h_7 is one half of the value in the unipotent case, and is therefore 6 for the semisimple classes and 18 for the regular classes. All other such cases are similar; we thus need say no more about these contributions.

We must therefore consider the semisimple classes of types $h_{31}, \ldots, h_{36}$. For some of the instances where the classes of such a type h_ℓ meet one of the $T_{(n)}$ involved here, the character θ is such that all conjugates are taken to 1; as a result the contribution is the same as in the unipotent case. This occurs when the pair

(ℓ, n) is (32, 10), (34, 5), (35, 6) or (36, 14); the pairs left to be considered are (31, 1), (31, 3), (33, 2) and (33, 5).

We begin with $(\ell, n) = (31, 1)$. When dealing with this instance in the treatment of κ_{31}, we divided into 16 cases. Of these, we find that 4 are such that the conjugates are sent to 1 by $\theta^{(1)}_{i000}$, giving a sum over the different classes of $4q^2 - 32q + \cdots$; the remaining 12 each give a sum $-q + \cdots$. The total sum is therefore $4q^2 - 44q + \cdots$; multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a contribution with nonnegligible part $4q + 4f - 32$.

The instance where $(\ell, n) = (31, 3)$ is a little different. As before, we let $s = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$; there are then 48 conjugates of s in $T_{(3)}$. None of these is of the form $(1, *, *, *; *)$ to be sent to 1 by $\theta^{(3)}_{i00}$; instead, the sum over the conjugates of the values taken by $\theta^{(3)}_{i00}$ is

$$8({\zeta_1}^{ia} + {\zeta_1}^{-ia} + {\zeta_1}^{ib} + {\zeta_1}^{-ib} + {\zeta_1}^{i(a+b)} + {\zeta_1}^{-i(a+b)}).$$

As before, we sum over the square $0 \le a, b \le q - 2$, subtract the sums over the six lines $a = 0$, $b = 0$, $a - b = 0$, $a + b = 0$, $2a + b = 0$ and $a + 2b = 0$ (where all equalities are once more taken modulo $q - 1$), and then divide by 12. For each of the six powers of ζ_1, the sums over the square and five of the lines are zero, but the sum over the other line is $q - 1$. Thus the total sum is $-4q + \cdots$; proceeding as usual then gives a contribution with nonnegligible part 4.

Next we take $(\ell, n) = (33, 2)$. Of the 16 conjugates lying in $T_{(2)}$ mentioned in the treatment of κ_{31}, we find that 8 are sent to 1 by $\theta^{(2)}_{i00}$, giving a sum of $2q^2 - 4q + 2d$; the remaining 8 give a sum $-2q - 2$. The total sum is therefore $2q^2 - 6q + \cdots$; multiplying by the Green function value and dividing by the order of the centralizer in H_m gives a contribution with nonnegligible part $2q + 2f - 4$.

Lastly we turn to $(\ell, n) = (33, 5)$. As before, we let $s = ({\xi_1}^a, {\xi_3}^a, {\xi_3}^{qa}, 1; {\xi_1}^a)$; there are 8 conjugates of s in $T_{(5)}$. None of these is sent to 1 by $\theta^{(5)}_{-(q+1)i,(q+1)i}$; instead, the sum over the conjugates of the values taken by $\theta^{(5)}_{-(q+1)i,(q+1)i}$ is

$$4({\zeta_1}^{ia} + {\zeta_1}^{-ia}).$$

As before, we then sum from $a = 0$ to $q^2 - 2$, subtract the sums of the terms in which a is a multiple of $q + 1$ or $q - 1$, and then divide by 4; this gives a total sum of $-2q + \cdots$, and proceeding as usual then gives a contribution with nonnegligible part 2.

Table 12 lists all nonnegligible parts of contributions to scalar products with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$; again, for types h_7 and h_8 the first row relates to the semisimple classes and the second to classes of regular elements of H_m. We have therefore found the scalar products of ${1_{H_m}}^G$ with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$. On taking linear combinations given by the character table of the Weyl group C_3, we obtain nonzero scalar products with four of the resulting characters. Two of these are

	$T_{(1)}$	$T_{(2)}$	$T_{(3)}$	$T_{(4)}$	$T_{(5)}$	$T_{(6)}$	$T_{(8)}$	$T_{(10)}$	$T_{(11)}$	$T_{(14)}$
h_7	6 18	-1 1	-2 -2	0 0	1 1	– –	– –	– –	– –	– –
h_8	– –	1 3	– –	-2 2	-1 -1	– –	– –	6 6	– –	– –
h_9	12	2	0	–	0	–	–	–	–	–
h_{10}	–	–	–	–	2	–	–	12	–	–
h_{31}	$4q+4f-32$	–	4	–	–	–	–	–	–	–
h_{32}	–	–	–	–	–	–	–	$4q-4f-20$	–	–
h_{33}	–	$2q+2f-4$	–	–	2	–	–	–	–	–
h_{34}	–	–	–	–	$2q-2f-2$	–	–	–	–	–
h_{35}	–	–	–	–	–	$q+f+1$	–	–	–	–
h_{36}	–	–	–	–	–	–	–	–	–	$q-f+1$
h_{66}	f	–	0	–	–	–	–	–	–	–
h_{67}	–	–	–	–	–	–	–	f	–	–
h_{68}	–	–	f	0	–	–	–	–	–	–
h_{69}	–	–	–	f	–	–	–	0	–	–
h_{70}	–	f	–	–	0	–	–	–	–	–
h_{71}	–	–	–	–	f	–	–	–	–	–
h_{72}	–	–	–	–	–	–	f	–	0	–
h_{73}	–	–	–	–	–	–	–	–	f	–
h_{74}	–	–	–	–	–	f	–	–	–	–
h_{75}	–	–	–	–	–	–	–	–	–	f
	$4q+5f+4$	$2q+3f+2$	f	f	$2q-f+2$	$q+2f+1$	f	$4q-3f+4$	f	$q+1$

Table 12. Contributions to scalar products $({1_{H_m}}^G, R_{T,\theta})_G$ for $R_{T,\theta}$ in $\kappa_{7,[i]}$.

irreducible characters: they correspond to the trivial character and the unipotent character $\chi_{1,2}$, and the multiplicities are $q+f+1$ and $q+1$, respectively. (Here for a pair of partitions (α, β) with $|\alpha|+|\beta|=3$ we write $\chi_{\alpha,\beta}$ for the corresponding unipotent character of $C_3(q)$ lying in the principal series, as in [4, Section 13.8].) The remaining two lie in a family of size four, and the scalar products are both f; an analysis using the nonabelian Fourier transform matrix which is entirely similar to that employed with the unipotent characters $\chi_{\phi_{4,1}}$ and $\chi_{\phi''_{2,4}}$ above shows that there are two other constituents, each with multiplicity f, corresponding to the unipotent characters $\chi_{2,1}$ and $\chi_{-,3}$. We shall call these four characters $\chi_{\kappa_{7,[i]}}^{1}$, $\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}$, $\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}$ and $\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}$.

3.3.6. *Geometric conjugacy classes of type κ_8.* The geometric conjugacy classes of type κ_8 correspond to semisimple classes in G containing elements

$$(1, {\xi_2}^{i}, {\xi_2}^{-i}, 1; 1) \quad \text{with } {\xi_2}^{2i} \neq 1.$$

The centralizer in G of these elements is given in the paragraph of Section 3.1 relating to Table 5; the number of classes is $\frac{1}{2}(q-d)$. To parametrize these geometric conjugacy classes, we set

$$S_8 = \{i \in \mathbb{Z}_{q+1} : 2i \neq 0\},$$

so that $i \in S_8$ corresponds to the element given; we define an equivalence relation on S_8 by

$$i \sim -i,$$

and let $\bar{S}_8$ be the set of equivalence classes $[i]$. The geometric conjugacy classes of type κ_8 are in bijective correspondence with $\bar{S}_8$; we shall write $\kappa_{8,[i]}$ for the class corresponding to $[i] \in \bar{S}_8$.

There are ten distinct characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{8,i}$; in the notation of [15] we may take the pairs $(\boldsymbol{T}, \theta)$ as

$$(\boldsymbol{T}_{(17)}, \theta^{(17)}_{i000}),\ (\boldsymbol{T}_{(9)}, \theta^{(9)}_{i00}),\ (\boldsymbol{T}_{(10)}, \theta^{(10)}_{i00}),\ (\boldsymbol{T}_{(4)}, \theta^{(4)}_{0i0}),\ (\boldsymbol{T}_{(5)}, \theta^{(5)}_{0(q-1)i}),\ (\boldsymbol{T}_{(21)}, \theta^{(21)}_{0i}),$$
$$(\boldsymbol{T}_{(19)}, \theta^{(19)}_{i0}),\ (\boldsymbol{T}_{(3)}, \theta^{(3)}_{00(q-1)i}),\ (\boldsymbol{T}_{(11)}, \theta^{(11)}_{q(q^2+1)(q-1)i/2,i}),\ (\boldsymbol{T}_{(16)}, \theta^{(16)}_{(q^3-1)i}).$$

The working is very similar to that of κ_7; we again find that there are four constituents of ${1_{H_m}}^G$ in each such geometric conjugacy class. They correspond to the trivial character and the unipotent characters $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{-,3}$ of the centralizer, and the multiplicities are $q+f-1$, $q-1$, f and f, respectively. We shall call these four characters $\chi^{1}_{\kappa_{8,[i]}}$, $\chi^{\chi_{1,2}}_{\kappa_{8,[i]}}$, $\chi^{\chi_{2,1}}_{\kappa_{8,[i]}}$ and $\chi^{\chi_{-,3}}_{\kappa_{8,[i]}}$.

3.3.7. *The geometric conjugacy class of type κ_1.* The geometric conjugacy class of type κ_1 occurs only in odd characteristic, when it corresponds to the semisimple class in G containing the involution

$$(1, -1, -1, 1; 1).$$

The centralizer in G is the product of groups $C_3(q)$ and $A_1(q)$. Since there is at most one such class, we shall simply call it κ_1, with the understanding that the class does not exist unless q is odd. There are twenty distinct characters $R_{\boldsymbol{T},\theta}$ lying in κ_1; by a temporary abuse of notation similar to those employed previously we may say that κ_1 is the union of $\kappa_{7,[(q-1)/2]}$ and $\kappa_{8,[(q+1)/2]}$.

As before, the calculations to find nonnegligible parts of contributions proceed exactly as for types κ_7 and κ_8. We find that there are six constituents of ${1_{H_m}}^G$. In two cases, the restriction to the $A_1(q)$ factor of the corresponding unipotent character of the centralizer is the trivial character; the restrictions to the $C_3(q)$ factor are the trivial character and the unipotent character $\chi_{1,2}$, and both multiplicities are 1. In the other four cases, the restriction to the $A_1(q)$ factor is the Steinberg character; the restrictions to the $C_3(q)$ factor are the trivial character and the unipotent characters $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{-,3}$, and the multiplicities are $q+f$, q, f and f, respectively. We shall call these six characters $\chi_{\kappa_1}^{1,1}$, $\chi_{\kappa_1}^{\chi_{1,2},1}$, $\chi_{\kappa_1}^{1,\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}}$ and $\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}}$.

3.4. *The complete decomposition of* $\mathbf{1}_{H_m}{}^{G}$. If we now add together the degrees of the constituents of $1_{H_m}{}^G$ found so far, taken with multiplicity, we obtain

$$\begin{array}{ll} q^{12}(q^8+q^4+1)(q^4+1) & \text{if } m=1, \\ q^{12}(q^{12}-1) & \text{if } m=2, \\ q^{12}(q^8-1)(q^4-1) & \text{if } m=3, \end{array}$$

which in each case is equal to $|G:H_m|$. We have therefore proved the following.

Proposition 3.1. *If $G=F_4(q)$ and $H_m={}^mD_4(q)$, the decomposition of $1_{H_m}{}^G$ into irreducible characters is*

$$\begin{aligned}
&\chi_{\phi_{1,0}}+\chi_{\phi_{9,2}}+(q+f)\chi_{\phi''_{8,3}}+q\chi_{\phi''_{8,9}}+f\chi_{\phi_{4,1}}+f\chi_{\phi''_{2,4}} \\
&\quad+\left\{\begin{array}{ll} \chi_{\phi_{12,4}}+3\chi_{\phi''_{9,6}}+3\chi_{\phi''_{1,12}}+2\chi_{\phi''_{6,6}}+\chi_{F_4{}^{\mathrm{II}}[1]} & \textit{if } m=1 \\ \chi_{\phi''_{4,7}}+\chi_{\phi_{16,5}}+\chi_{B_2,\epsilon'}+\chi_{F_4[-1]} & \textit{if } m=2 \\ \chi_{\phi'_{6,6}}+\chi_{F_4[\theta]}+\chi_{F_4[\theta^2]} & \textit{if } m=3 \end{array}\right\} \\
&\quad+\chi_{\kappa_1}^{1,1}+\chi_{\kappa_1}^{\chi_{1,2},1}+(q+f)\chi_{\kappa_1}^{1,\mathrm{St}}+q\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}}+f\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}}+f\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}} \\
&\quad+(q+f-1)\chi_{\kappa_3}^{\mathrm{St},1}+\chi_{\kappa_3}^{\rho,1}+(q-f+1)\chi_{\kappa_4}^{\mathrm{St},1}+\chi_{\kappa_4}^{\rho,1} \\
&\quad\quad+\sum_{[i]\in\bar{S}_7}((q+f+1)\chi_{\kappa_{7,[i]}}^{1}+(q+1)\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}+f\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}+f\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}) \\
&\quad\quad+\sum_{[i]\in\bar{S}_8}((q+f-1)\chi_{\kappa_{8,[i]}}^{1}+(q-1)\chi_{\kappa_{8,[i]}}^{\chi_{1,2}}+f\chi_{\kappa_{8,[i]}}^{\chi_{2,1}}+f\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}) \\
&\quad\quad+\sum_{[i]\in\bar{S}_9}(\chi_{\kappa_{9,[i]}}^{1,1}+(q+f)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1})+\sum_{[i]\in\bar{S}_{10}}(\chi_{\kappa_{10,[i]}}^{1,1}+(q-f)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1}) \\
&\quad\quad+\sum_{[(i,j)]\in\bar{S}_{31}}(q+f+1)\chi_{\kappa_{31,[(i,j)]}}^{1}+\sum_{[(i,j)]\in\bar{S}_{32}}(q-f-1)\chi_{\kappa_{32,[(i,j)]}}^{1} \\
&\quad\quad+\sum_{[i]\in\bar{S}_{33}}(q+f-1)\chi_{\kappa_{33,[i]}}^{1}+\sum_{[i]\in\bar{S}_{34}}(q-f+1)\chi_{\kappa_{34,[i]}}^{1} \\
&\quad\quad+\sum_{[i]\in\bar{S}_{35}}(q+f-2)\chi_{\kappa_{35,[i]}}^{1}+\sum_{[i]\in\bar{S}_{36}}(q-f+2)\chi_{\kappa_{36,[i]}}^{1}.
\end{aligned}$$

It is now straightforward to calculate the ranks of the three actions as the sums of the squares of the multiplicities of the constituents; we obtain the following.

Corollary 3.2. *The rank of the action of $G = F_4(q)$ on cosets of $H_m = {}^mD_4(q)$ is*

$$\begin{array}{ll} q^4+q^3+9q^2+17q+24 & \text{if } m=1,\\ q^4+q^3+q^2+q+4 & \text{if } m=2,\\ q^4+q^3-q+3 & \text{if } m=3.\end{array}$$

Indeed, this is confirmed by a separate calculation of $({1_{H_m}}^G, {1_{H_m}}^G)_G$, which does not require Deligne–Lusztig theory. If H is any subgroup of G, then, for $g \in G$,

$$ {1_H}^G(g) = \sum_{[h]\subseteq[g]} \frac{|C_G(h)|}{|C_H(h)|}, $$

where the sum is over the H-classes $[h]$ lying in the G-class $[g]$; thus

$$ ({1_H}^G, {1_H}^G)_G = \sum_{[g]\subset G} \frac{{1_H}^G(g)^2}{|C_G(g)|} = \sum_{[g]\subset G} |C_G(g)| \biggl(\sum_{[h]\subseteq[g]} \frac{1}{|C_H(h)|} \biggr)^2. $$

Knowledge of the fusion of classes from H into G enables this to be calculated; applying this to each H_m gives the values above.

Before concluding this section we note that the three permutation characters ${1_{H_1}}^G$, ${1_{H_2}}^G$ and ${1_{H_3}}^G$ are given by a formula which, outside a single family of unipotent characters, is linear in the parameter f. As the values taken by f are 3, 1 and 0, respectively, another way of saying this is that if we define the generalized character

$$ \psi(G; H) = {1_{H_1}}^G - 3.{1_{H_2}}^G + 2.{1_{H_3}}^G, $$

then the coefficient in $\psi(G; H)$ of any irreducible character lying outside this family is zero.

The reason for this may be traced back to the contributions to scalar products $({1_{H_m}}^G, R_{\boldsymbol{T},\theta})$ from regular semisimple classes of types $h_{76}, \dots, h_{100}$ (these were the only contributions whose nonnegligible parts could not be expressed as linear polynomials in f): for each such type h_ℓ, the nonnegligible part was nonzero only when $\theta = 1$, so that $R_{\boldsymbol{T},\theta} = R_{(n)}$ for some n, and only for one value of m, when its value was 6, 2 or 3 according as $m = 1$, 2 or 3. From this we see that for all n we have $(\psi(G; H), R_{(n)})_G = 6(-1)^{m-1}$ where the regular semisimple elements in the torus $T_{(n)}$ lie in H_m; it follows that $\psi(G; H) = 6R_{\phi''_{1,12}}$, where $R_{\phi''_{1,12}}$ is an almost character of degree q^{12}, after which applying the appropriate nonabelian Fourier transform matrix produces the observed linear combination of irreducible unipotent characters.

If we successively remove long simple roots to reduce from $\boldsymbol{G}$ to $\boldsymbol{C}$ and then to $\boldsymbol{A}$, in each case replacing $\boldsymbol{H}$ by its intersection with the reduced group (first a group ${A_1}^3$, then a 2-dimensional torus), the behavior is very similar. The groups $(H \cap C)_m$ are $A_1(q)^3$, $A_1(q^2)A_1(q)$ and $A_1(q^3)$, respectively, while the groups $(H \cap A)_m$ are tori of order $(q-1)^2$, q^2-1 and q^2+q+1, respectively. Using the notation of [4, Section 13.8], we find that $\psi(C;\, H\cap C)=6R_{\phi_{111,-}}$, where $R_{\phi_{111,-}}$ is an almost character of degree q^6; the family here is of size 4, and applying the appropriate nonabelian Fourier transform matrix gives

$$6R_{\phi_{111,-}}=3\chi_{\phi_{111,-}}+3\chi_{\phi_{1,11}}-3\chi_{\phi_{-,21}}-3\chi_{B_2,\epsilon}.$$

Likewise $\psi(A;\, H\cap A)=6R_{\phi_{111}}$, where $R_{\phi_{111}}$ is an almost character of degree q^3; this time the family is of size 1, so $6R_{\phi_{111}}=6\chi_{\phi_{111}}$. (In fact in this last case the unipotent character $\chi_{\phi_{111}}$ is the Steinberg character St; indeed it follows from [4, Proposition 7.5.4, Corollary 7.6.5] that $\psi(A;\, H\cap A)=6\,\mathrm{St}$.)

4. Extensions of H_m by graph automorphisms

In this section we shall decompose the permutation characters ${1_{H_m.\Gamma}}^G$, where Γ is a nontrivial group of graph automorphisms of H_m. Recall that the cases to be considered are as follows: $H_m.2=H_m\langle\tau_2\rangle$ for $m=1,2$; $H_m.3=H_m\langle\tau_3\rangle$ for $m=1,3$; and $H_1.S_3=H_1\langle\tau_2,\tau_3\rangle$. Note that each constituent of ${1_{H_m.\Gamma}}^G$ is also one of ${1_{H_m}}^G$, so we need only consider the types of geometric conjugacy class treated in Section 3.

For convenience, writing $r=|\Gamma|$ we shall define an integer q_r which is close to $\frac{q}{r}$, and express multiplicities in ${1_{H_m.\Gamma}}^G$ in terms of q_r. Although the details of the calculations to follow will depend upon whether or not the characteristic p divides r, the use of the notation q_r means that in each case it will still be possible to give a single expression for the decomposition of ${1_{H_m.\Gamma}}^G$ which is valid for all characteristics.

In contrast to the results obtained in Section 3, we shall see that the multiplicity of a constituent may depend upon the particular geometric conjugacy class in which it lies, rather than simply being determined by the type of the geometric conjugacy class. We shall therefore require some further notation: we set

$$\epsilon^2_{i,j}=\begin{cases}1 & \text{if } 2\mid(q-1) \text{ and either } 2\nmid i \text{ or } 2\nmid j \text{ (or both)},\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^2_{i}=\epsilon^2_{i,i}=\begin{cases}1 & \text{if } 2\mid(q-1) \text{ and } 2\nmid i,\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^{3,+}_{i}=\begin{cases}1 & \text{if } 3\mid(q-1) \text{ and } 3\nmid i,\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^{3,-}_{i}=\begin{cases}1 & \text{if } 3\mid(q+1) \text{ and } 3\nmid i,\\ 0 & \text{otherwise;}\end{cases}$$

$$\epsilon^{4}=\begin{cases}1 & \text{if } 4\mid(q-1),\\ 0 & \text{otherwise.}\end{cases}$$

We shall again use the method described in Section 2 to determine scalar products $(1_{H_m\langle\tau_r\rangle}{}^G, R_{T,\theta})$ as sums of contributions from types of H_m-class. For those classes lying in H_m the value is that obtained in Section 3, multiplied by a factor of $|H_m|/|H_m\langle\tau_r\rangle| = 1/r$; we therefore consider types of H_m-class lying in $H_m\tau_r$. Much as in Section 3, we begin in Section 4.1 by determining the relevant types of H_m-class in $H_m\tau_r$; we then decompose the permutation characters $1_{H_m.\Gamma}{}^G$, firstly in Section 4.2 where $\Gamma = \langle\tau_2\rangle$ and $m \in \{1, 2\}$, and secondly in Section 4.3 where $\Gamma = \langle\tau_3\rangle$ and $m \in \{1, 3\}$; finally in Section 4.4 we combine these results to decompose the permutation character $1_{H_m.\Gamma}{}^G$ where $\Gamma = \langle\tau_2, \tau_3\rangle$ and $m = 1$.

4.1. *Relevant types of H_m-class in $H_m\tau_r$ for $r \in \{2, 3\}$.* Using the information on H_m-classes in $H_m\tau_r$ described in Section 2.1, as in Section 3.1 we treat all types of H_m-class in $H_m\tau_r$ to determine those which are relevant. We find that the number of such types is small; we describe them here.

In each case the elements concerned are simply of the form $s\tau_r$, and the nonnegligible parts of contributions are simply constants; as we shall explain, the H_m-classes are related to those of types $h_{31}, \dots, h_{36}$ or $h_{66}, \dots, h_{75}$, and we shall refer to the information given in Tables 7 and 8 of Section 3.1 concerning numbers of H_m-classes and the tori $T_{(n)}$ which meet them. In particular we note the following concerning what is stated there. For all these types we may take s of the form $\left(\lambda, \mu, \frac{\nu^2}{\lambda\mu}, 1; \nu\right)$. For types h_ℓ with $\ell \in \{31, \dots, 36\}$ (which here occur for both $r = 2$ and $r = 3$), we have $\nu = \lambda$ so that the $\boldsymbol{G}$-centralizer has root system $\tilde{A}_2$, with Weyl group $\langle w_4, w_{+---}\rangle$; there exists $w_{[\ell]} \in \langle w_{2-3}, w_{1-2}w_{3-4}w_{3+4}\rangle$ such that $s^{F_m} = {}^{w_{[\ell]}}s$, so that s lies in the three tori $T_{w_{[\ell]}}$, $T_{w_{[\ell]}w_4}$ and $T_{w_{[\ell]}w_4w_{+---}}$, and up to conjugacy they are listed in that order in Table 7. For types h_ℓ with $\ell \in \{66, \dots, 75\}$ (which here occur only for $r = 2$), we have $\nu \neq \lambda$ so that the $\boldsymbol{G}$-centralizer has root system $\tilde{A}_1$, with Weyl group $\langle w_4\rangle$; there exists $w_{[\ell]} \in \langle w_{1-2}, w_{2-3}, w_{3-4}w_{3+4}\rangle$ such that $s^{F_m} = {}^{w_{[\ell]}}s$, so that s lies in the two tori $T_{w_{[\ell]}}$ and $T_{w_{[\ell]}w_4}$, and up to conjugacy they are listed in that order in Table 8.

If $p = r$, the expression for ${\tau_r}^{y_r}$ given in Section 1 shows that τ_r lies in the unipotent class $\tilde{A}_1$ or $\tilde{A}_2$ according as $r = 2$ or 3. If the type of the H_m-class containing s is h_ℓ, we shall denote the type of the H_m-class containing $s\tau_r$ by $h_\ell\tau_r$; in each case the number of H_m-classes and the tori containing s are as given in Tables 7 and 8 (if we replace f by 1 in the latter table).

If $p \neq r$, the classes all consist of semisimple elements whose $\boldsymbol{H}$-centralizer is a torus, of dimension $5-r$. This time the expression for ${\tau_r}^{y_r}$ given in Section 1 shows that $s{\tau_r}^{y_r} = s(-1, -1, -1, -1; -1)$ or $s(1, 1, 1, \omega^2; \omega)$ according as $r = 2$ or 3. The situation here is however somewhat more complicated than in the above paragraph.

First suppose $r = 2$, and s lies in an H_m-class of type h_ℓ for some $\ell \in \{31, \dots, 36\}$. Here things are simple: we shall denote the type of the H_m-class containing $s\tau_2$

by $h_\ell \tau_2$; the corresponding element $s{\tau_2}^{y_2}$ is of the form $\bigl(-\lambda, -\mu, -\frac{\lambda}{\mu}, -1; -\lambda\bigr)$, its $\boldsymbol{G}$-centralizer has root system $\tilde{A}_1$, up to conjugacy it lies in the first and second tori given in Table 7, and the number of H_m-classes is as given there.

Next suppose $r = 2$, and s lies in an H_m-class of type h_ℓ for some $\ell \in \{66, \dots, 75\}$. We have $(({\tau_2}^{y_2})^{F_m})^{w_{[\ell]}} = {\tau_2}^{y_2}$; thus $((s{\tau_2}^{y_2})^{F_m})^{w_{[\ell]}} = s{\tau_2}^{y_2}$ so that $s{\tau_2}^{y_2} \in T_{w_{[\ell]}}$, and we obtain a type of H_m-class in $H_m\tau_2$ which we may call $h_\ell\tau_2$; however, the number of H_m-classes is one half of that given in Table 8 (replacing f by 1), because conjugation by w_4 multiplies $s{\tau_2}^{y_2}$ by $(1, 1, 1, 1; -1)$ which lies in the torus $T_{w_{[\ell]}}$. To compensate for this we may take $s^* \in \boldsymbol{T}_2$ satisfying

$$((s^*)^{F_m})^{w_{[\ell]}} = s^*(1, 1, 1, 1; -1)$$

(so that s^* depends only on ℓ and not on s), and then $((s^*s{\tau_2}^{y_2})^{F_m})^{w_{[\ell]}w_4} = s^*s{\tau_2}^{y_2}$, so that $s^*s{\tau_2}^{y_2} \in T_{w_{[\ell]}w_4}$, and we obtain another type of H_m-class in $H_m\tau_2$ which we may call $h_\ell{}'\tau_2$; again the number of H_m-classes is one half of that given in Table 8 (replacing f by 1). The elements $s{\tau_2}^{y_2}$ and $s^*s{\tau_2}^{y_2}$ are of the form $\bigl(-\lambda, -\mu, -\frac{\nu^2}{\lambda\mu}, -1; -\nu\bigr)$, their $\boldsymbol{G}$-centralizer has root system $\varnothing$, and up to conjugacy they lie in the first and second tori, respectively, given in Table 8.

Finally suppose $r = 3$, and s lies in an H_m-class of type h_ℓ for some $\ell \in \{31, \dots, 36\}$. Recall that we take $e \in \{0, \pm 1\}$ such that $q \equiv e \pmod 3$ (so here $e = \pm 1$ since $p \neq 3$); temporarily take $\ell' \in \{0, 1\}$ such that $\ell \equiv \ell' \pmod 2$. The details from now on depend on the pair (e, ℓ'). The element ${\tau_3}^{y_3}$ is fixed or inverted by F_m according as $e = 1$ or -1, and is fixed or inverted by $w_{[\ell]}$ according as $\ell' = 1$ or 0; and it is inverted by w_4. First assume $(e, \ell') = (1, 0)$ or $(-1, 1)$. Then $(({\tau_3}^{y_3})^{F_m})^{w_{[\ell]}} = ({\tau_3}^{y_3})^{-1}$, so $(({\tau_3}^{y_3})^{F_m})^{w_{[\ell]}w_4} = {\tau_3}^{y_3}$; thus $((s{\tau_3}^{y_3})^{F_m})^{w_{[\ell]}w_4} = s{\tau_3}^{y_3}$ so that $s{\tau_3}^{y_3} \in T_{w_{[\ell]}w_4}$, and we obtain a type of H_m-class in $H_m\tau_3$ which we may call $h_\ell\tau_3$, whose elements lie in the second torus given in Table 7, with the number of H_m-classes being as given there. Now assume $(e, \ell') = (1, 1)$ or $(-1, 0)$. Then $(({\tau_3}^{y_3})^{F_m})^{w_{[\ell]}} = {\tau_3}^{y_3}$; thus $((s{\tau_3}^{y_3})^{F_m})^{w_{[\ell]}} = s{\tau_3}^{y_3}$ so that $s{\tau_3}^{y_3} \in T_{w_{[\ell]}}$, and we obtain a type of H_m-class in $H_m\tau_3$ which we may call $h_\ell\tau_3$; however, the number of H_m-classes is one third of that given in Table 7, because conjugation by w_4w_{+---} multiplies $s{\tau_3}^{y_3}$ by $(\omega^2, \omega, \omega, 1; \omega^2)$ which lies in the torus $T_{w_{[\ell]}}$. To compensate for this we may take $s^* \in \boldsymbol{T}_3$ satisfying

$$((s^*)^{F_m})^{w_{[\ell]}} = s^*(\omega, \omega^2, \omega^2, 1; \omega)$$

(so that s^* depends only on ℓ and not on s), and then

$$((s^*s{\tau_3}^{y_3})^{F_m})^{w_{[\ell]}w_4w_{+---}} = s^*s{\tau_3}^{y_3},$$

so that $s^*s{\tau_3}^{y_3} \in T_{w_{[\ell]}w_4w_{+---}}$, and we obtain another type of H_m-class in $H_m\tau_3$ which we may call $h_\ell{}'\tau_3$; this time the number of H_m-classes is two thirds of that given in Table 7, because replacing w_4w_{+---} by its inverse $w_{+---}w_4$ gives another

type	# H_m-classes	n		
$h_{31}\tau_2$	$\frac{1}{12}q^2+\cdots$	1	3	7
$h_{32}\tau_2$	$\frac{1}{12}q^2+\cdots$	17	10	20
$h_{33}\tau_2$	$\frac{1}{4}q^2+\cdots$	2	5	15
$h_{34}\tau_2$	$\frac{1}{4}q^2+\cdots$	9	5	13
$h_{35}\tau_2$	$\frac{1}{6}q^2+\cdots$	6	16	18
$h_{36}\tau_2$	$\frac{1}{6}q^2+\cdots$	21	14	25

Table 13. H_m-classes in $H_m\tau_2$ related to those of types $h_{31}, \ldots, h_{36}$.

collection of H_m-classes and the maximal tori $T_{w_{[\ell]}w_4w_{+---}}$ and $T_{w_{[\ell]}w_{+---}w_4}$ are conjugate. The elements $s\tau_3{}^{y_3}$ and $s^*s\tau_3{}^{y_3}$ are of the form $\left(\lambda, \mu, \frac{\lambda}{\mu}, \omega^2; \omega\lambda\right)$, their $\boldsymbol{G}$-centralizer has root system $\varnothing$, and up to conjugacy they lie in the first and third tori, respectively, given in Table 7.

We conclude this section by summarizing the notation and the information about numbers of H_m-classes and the tori $T_{(n)}$ containing the semisimple parts of elements therein. Recall that we take $d \in \{0, 1\}$ with $q \equiv d \bmod 2$, as well as $e \in \{0, \pm 1\}$ with $q \equiv e \bmod 3$ as in the previous paragraph. We observe that, for fixed ℓ and r, the number of H_m-classes of type $h_\ell\tau_r$, combined with those of type $h_{\ell}{}'\tau_r$ if they exist, is given by the same polynomial in q for both values of d (if $r = 2$) or for all three values of e (if $r = 3$). If $r = 2$ we have H_m-classes as given in Table 13, where the last entries in the final column are to be ignored if $d = 1$, and Table 14; if $r = 3$ we have H_m-classes as given in Table 15.

4.2. ***The characters $\mathbf{1}_{H_m.2}{}^{G}$ for $m = 1, 2$.*** Recall that we take $d \in \{0, 1\}$ with $q \equiv d \bmod 2$. We define

$$q_2 = \tfrac{1}{2}(q+d), \qquad f_2 = \tfrac{1}{2}(f+1) = \begin{cases} 2 & \text{if } m = 1, \\ 1 & \text{if } m = 2. \end{cases}$$

We proceed as in Section 3, determining nonnegligible parts of contributions to the scalar product $(1_{H_m.2}{}^{G}, R_{\boldsymbol{T},\theta})$ from the various types of class. Contributions from classes lying in H_m are of course exactly as already calculated, except for a factor of $\frac{1}{2}$; it remains to consider the classes in $H_m\tau_2$.

4.2.1. *Unipotent characters.* We begin with the classes of type $h_\ell\tau_2$ with $\ell \in \{31, \ldots, 36\}$; for example, we take $\ell = 31$, in which case the elements concerned have semisimple parts lying in tori $T_{(n)}$ for $n = 1$, 3 and 7 (the last of which is absent if $d = 1$). If $d = 0$, we see from the Appendix of [8] that the Green function value is $2q+1$, 1 or $-q+1$ according as $n = 1$, 3 or 7; thus the contributions to

	$d=0$		$d=1$	
type	$\#\,H_m$-classes	n	$\#\,H_m$-classes	n
$h_{66}\tau_2$	$\frac{1}{48}q^3+\cdots$	1 3	$\frac{1}{96}q^3+\cdots$	1
$h_{66}{}'\tau_2$			$\frac{1}{96}q^3+\cdots$	3
$h_{67}\tau_2$	$\frac{1}{48}q^3+\cdots$	17 10	$\frac{1}{96}q^3+\cdots$	17
$h_{67}{}'\tau_2$			$\frac{1}{96}q^3+\cdots$	10
$h_{68}\tau_2$	$\frac{1}{16}q^3+\cdots$	4 3	$\frac{1}{32}q^3+\cdots$	4
$h_{68}{}'\tau_2$			$\frac{1}{32}q^3+\cdots$	3
$h_{69}\tau_2$	$\frac{1}{16}q^3+\cdots$	4 10	$\frac{1}{32}q^3+\cdots$	4
$h_{69}{}'\tau_2$			$\frac{1}{32}q^3+\cdots$	10
$h_{70}\tau_2$	$\frac{1}{8}q^3+\cdots$	2 5	$\frac{1}{16}q^3+\cdots$	2
$h_{70}{}'\tau_2$			$\frac{1}{16}q^3+\cdots$	5
$h_{71}\tau_2$	$\frac{1}{8}q^3+\cdots$	9 5	$\frac{1}{16}q^3+\cdots$	9
$h_{71}{}'\tau_2$			$\frac{1}{16}q^3+\cdots$	5
$h_{72}\tau_2$	$\frac{1}{8}q^3+\cdots$	11 8	$\frac{1}{16}q^3+\cdots$	11
$h_{72}{}'\tau_2$			$\frac{1}{16}q^3+\cdots$	8
$h_{73}\tau_2$	$\frac{1}{8}q^3+\cdots$	11 19	$\frac{1}{16}q^3+\cdots$	11
$h_{73}{}'\tau_2$			$\frac{1}{16}q^3+\cdots$	19
$h_{74}\tau_2$	$\frac{1}{6}q^3+\cdots$	6 16	$\frac{1}{12}q^3+\cdots$	6
$h_{74}{}'\tau_2$			$\frac{1}{12}q^3+\cdots$	16
$h_{75}\tau_2$	$\frac{1}{6}q^3+\cdots$	21 14	$\frac{1}{12}q^3+\cdots$	21
$h_{75}{}'\tau_2$			$\frac{1}{12}q^3+\cdots$	14

Table 14. H_m-classes in $H_m\tau_2$ related to those of types $h_{66},\dots,h_{75}$.

the scalar product with $R_{(n)}$ are

$$\frac{192\cdot(2q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},\quad \frac{48\cdot 1\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},\quad \frac{12\cdot(-q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},$$

having nonnegligible parts 16, 0 and $-\frac{1}{2}$, respectively. If however $d=1$, the Green function value is $q+1$ or $-q+1$ according as $n=1$ or 3; thus the contributions to the scalar product with $R_{(n)}$ are

$$\frac{576\cdot(q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},\quad \frac{48\cdot(-q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)},$$

having nonnegligible parts 24 and -2, respectively. The other instances may be dealt with similarly.

	$e=0$		$e=1$		$e=-1$	
type	# H_m-classes	n	# H_m-classes	n	# H_m-classes	n
$h_{31}\tau_3$	$\frac{1}{12}q^2+\cdots$	1 3 7	$\frac{1}{36}q^2+\cdots$	1	$\frac{1}{12}q^2+\cdots$	3
$h_{31}{}'\tau_3$			$\frac{1}{18}q^2+\cdots$	7		
$h_{32}\tau_3$	$\frac{1}{12}q^2+\cdots$	17 10 20	$\frac{1}{12}q^2+\cdots$	10	$\frac{1}{36}q^2+\cdots$	17
$h_{32}{}'\tau_3$					$\frac{1}{18}q^2+\cdots$	20
$h_{33}\tau_3$	$\frac{1}{4}q^2+\cdots$	2 5 15	$\frac{1}{12}q^2+\cdots$	2	$\frac{1}{4}q^2+\cdots$	5
$h_{33}{}'\tau_3$			$\frac{1}{6}q^2+\cdots$	15		
$h_{34}\tau_3$	$\frac{1}{4}q^2+\cdots$	9 5 13	$\frac{1}{4}q^2+\cdots$	5	$\frac{1}{12}q^2+\cdots$	9
$h_{34}{}'\tau_3$					$\frac{1}{6}q^2+\cdots$	13
$h_{35}\tau_3$	$\frac{1}{6}q^2+\cdots$	6 16 18	$\frac{1}{18}q^2+\cdots$	6	$\frac{1}{6}q^2+\cdots$	16
$h_{35}{}'\tau_3$			$\frac{1}{9}q^2+\cdots$	18		
$h_{36}\tau_3$	$\frac{1}{6}q^2+\cdots$	21 14 25	$\frac{1}{6}q^2+\cdots$	14	$\frac{1}{18}q^2+\cdots$	21
$h_{36}{}'\tau_3$					$\frac{1}{9}q^2+\cdots$	25

Table 15. H_m-classes in $H_m\tau_3$ related to those of types $h_{31},\dots,h_{36}$.

Combining the two cases, we obtain the following table of nonnegligible parts.

$h_{31}\tau_2$	$R_{(1)}$	$8(d+2)$	$R_{(3)}$	$-2d$	$R_{(7)}$	$\frac{1}{2}(d-1)$
$h_{32}\tau_2$	$R_{(17)}$	$-8(d+2)$	$R_{(10)}$	$2d$	$R_{(20)}$	$-\frac{1}{2}(d-1)$
$h_{33}\tau_2$	$R_{(2)}$	$2(d+2)$	$R_{(5)}$	$-d$	$R_{(15)}$	$\frac{1}{2}(d-1)$
$h_{34}\tau_2$	$R_{(9)}$	$-2(d+2)$	$R_{(5)}$	d	$R_{(13)}$	$-\frac{1}{2}(d-1)$
$h_{35}\tau_2$	$R_{(6)}$	$\frac{1}{2}(d+2)$	$R_{(16)}$	$-\frac{1}{2}d$	$R_{(18)}$	$2(d-1)$
$h_{36}\tau_2$	$R_{(21)}$	$-\frac{1}{2}(d+2)$	$R_{(14)}$	$\frac{1}{2}d$	$R_{(25)}$	$-2(d-1)$

Note that the coefficients of d above are precisely the same, apart from the factor of $\frac{1}{2}$ already mentioned, as those of q obtained from consideration of types $h_{31},\dots,h_{36}$ in Section 3.2. This means that when the sets of contributions are added and q is replaced by $2q_2-d$, the terms in d cancel to leave linear polynomials in q_2.

We now consider the classes of type $h_\ell\tau_2$ or $h_\ell{}'\tau_2$ with $\ell\in\{66,\dots,75\}$; for example, we take $\ell=66$, in which case the elements concerned have semisimple parts lying in tori $T_{(n)}$ for $n=1$ and 3. In all cases the element is regular in H_m, so the Green function value is 1. If $d=0$ the contributions to the scalar product with $R_{(n)}$ are

$$\frac{576 \cdot 1 \cdot \left(\frac{1}{48}q^3 + \cdots\right)}{2(q^3 + \cdots)}, \quad \frac{48 \cdot 1 \cdot \left(\frac{1}{48}q^3 + \cdots\right)}{2(q^3 + \cdots)},$$

having nonnegligible parts 6 and $\frac{1}{2}$, respectively. If however $d = 1$, the contributions to the scalar product with $R_{(n)}$ are

$$\frac{1152 \cdot 1 \cdot \left(\frac{1}{96}q^3 + \cdots\right)}{2(q^3 + \cdots)}, \quad \frac{96 \cdot 1 \cdot \left(\frac{1}{96}q^3 + \cdots\right)}{2(q^3 + \cdots)}$$

(the former from $h_{66}\tau_2$ and the latter from $h_{66}{}'\tau_2$), having nonnegligible parts 6 and $\frac{1}{2}$, respectively. Once more, the other instances may be dealt with similarly, to produce the following table of nonnegligible parts.

$h_{66}\tau_2$ or $h_{66}{}'\tau_2$	$R_{(1)}$	6	$R_{(3)}$	$\frac{1}{2}$	$h_{71}\tau_2$ or $h_{71}{}'\tau_2$	$R_{(9)}$	3	$R_{(5)}$	$\frac{1}{2}$
$h_{67}\tau_2$ or $h_{67}{}'\tau_2$	$R_{(17)}$	6	$R_{(10)}$	$\frac{1}{2}$	$h_{72}\tau_2$ or $h_{72}{}'\tau_2$	$R_{(11)}$	$\frac{1}{2}$	$R_{(8)}$	1
$h_{68}\tau_2$ or $h_{68}{}'\tau_2$	$R_{(4)}$	1	$R_{(3)}$	$\frac{3}{2}$	$h_{73}\tau_2$ or $h_{73}{}'\tau_2$	$R_{(11)}$	$\frac{1}{2}$	$R_{(19)}$	1
$h_{69}\tau_2$ or $h_{69}{}'\tau_2$	$R_{(4)}$	1	$R_{(10)}$	$\frac{3}{2}$	$h_{74}\tau_2$ or $h_{74}{}'\tau_2$	$R_{(6)}$	$\frac{3}{2}$	$R_{(16)}$	$\frac{1}{2}$
$h_{70}\tau_2$ or $h_{70}{}'\tau_2$	$R_{(2)}$	3	$R_{(5)}$	$\frac{1}{2}$	$h_{75}\tau_2$ or $h_{75}{}'\tau_2$	$R_{(21)}$	$\frac{3}{2}$	$R_{(14)}$	$\frac{1}{2}$

Summing the nonnegligible parts gives the values in Table 16 for the scalar products $(1_{H_m.2}{}^G, R_{(n)})_G$.

	H_1	H_2		H_1	H_2
$R_{(1)}$	$16q_2+75$	$16q_2+44$	$R_{(14)}$	q_2+1	q_2+2
$R_{(2)}$	$4q_2+27$	$4q_2+14$	$R_{(15)}$	q_2	q_2-1
$R_{(3)}$	$-4q_2+1$	$-4q_2+4$	$R_{(16)}$	$-q_2-2$	$-q_2+1$
$R_{(4)}$	5	2	$R_{(17)}$	$-16q_2+7$	$-16q_2$
$R_{(5)}$	-3	2	$R_{(18)}$	$4q_2$	$4q_2-4$
$R_{(6)}$	q_2+12	q_2+5	$R_{(19)}$	-2	1
$R_{(7)}$	q_2	q_2-1	$R_{(20)}$	$-q_2+1$	$-q_2$
$R_{(8)}$	4	3	$R_{(21)}$	$-q_2+1$	$-q_2$
$R_{(9)}$	$-4q_2+1$	$-4q_2$	$R_{(22)}$	3	0
$R_{(10)}$	$4q_2-5$	$4q_2+2$	$R_{(23)}$	0	1
$R_{(11)}$	4	1	$R_{(24)}$	0	0
$R_{(12)}$	0	1	$R_{(25)}$	$-4q_2+4$	$-4q_2$
$R_{(13)}$	$-q_2+1$	$-q_2$			

Table 16. Scalar products $(1_{H_m.2}{}^G, R_{(n)})_G$.

We may now proceed as before to find the scalar products of ${1_{H_m.2}}^G$ with irreducible unipotent characters. On taking linear combinations given by the character table of W, we find that the only scalar products of ${1_{H_m.2}}^G$ with almost characters R_ϕ which are nonzero for some m are as follows.

ϕ	$({1_{H_m.2}}^G, R_\phi)_G$
$\phi_{1,0}, \phi_{9,2}$	1
$\phi''_{8,3}$	$q_2 + f_2$
$\phi''_{8,9}$	q_2
$\phi_{4,1}, \phi''_{2,4}$	f_2
$\phi_{12,4}, \phi''_{4,7}, \phi_{16,5}$	$\frac{1}{2}$
$\phi''_{6,6}$	$f_2 - 1$
$\phi''_{9,6}$	$f_2 - \frac{1}{2}$
$\phi''_{1,12}$	$2f_2 - \frac{5}{2}$

For $\phi = \phi_{1,0}, \phi_{9,2}, \phi''_{8,3}$ and $\phi''_{8,9}$ we again have irreducible characters χ_ϕ, appearing in ${1_{H_m.2}}^G$ with multiplicities 1, 1, $q_2 + f_2$ and q_2, respectively. For $\phi = \phi_{4,1}$ and $\phi''_{2,4}$, in the family of size four, as in Section 3.2 we obtain two irreducible characters $\chi_{\phi_{4,1}}$ and $\chi_{\phi''_{2,4}}$, each appearing with multiplicity f_2. Separate analyses of the family of size 21 for the two values of m lead to the following: if $m = 1$ we have constituents $\chi_{\phi_{12,4}}, \chi_{\phi''_{9,6}}, \chi_{\phi''_{1,12}}$ and $\chi_{\phi''_{6,6}}$, with multiplicities 1, 2, 1 and 1, respectively; if $m = 2$ we have constituents $\chi_{\phi''_{4,7}}$ and $\chi_{\phi_{16,5}}$, each with multiplicity 1. This completes the treatment of unipotent characters.

4.2.2. *Other geometric conjugacy classes.* We begin with the geometric conjugacy classes of type κ_{31}; here the only types of H_m-class we need consider are $h_\ell\tau_2$ for $\ell \in \{31, 33, 35\}$. We briefly deal with type $h_{31}\tau_2$, containing elements $s\tau_2$ with $s = ({\xi_1}^{a+b}, {\xi_1}^a, {\xi_1}^b, 1; {\xi_1}^{a+b})$; the other types are very similar. The calculations are in some ways simpler than those in Section 3.3.1, because the nonnegligible parts are constants and it therefore suffices to consider leading terms of polynomials; on the other hand, some of the details depend on the precise geometric conjugacy class rather than just its type. Recall that there are three distinct characters $R_{\boldsymbol{T},\theta}$ lying in the geometric conjugacy class $\kappa_{31,[(i,j)]}$, and the one with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ (in which s lies) has $\theta = \theta^{(1)}_{i00j}$.

If $d = 0$, we saw in Section 3.3.1 that of the 192 conjugates of s lying in $T_{(1)}$, only 12 were of the form $(1, *, *, *; 1)$ and thus sent to 1 by $\theta^{(1)}_{i00j}$, with the values taken at other elements producing cancellation. Since the Green function value at τ_2 is $2q + 1$, the contribution to the scalar product from these elements is

$$\frac{12\cdot(2q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)}=1+\cdots.$$

If instead $d=1$, we have $(s\tau_2)^{y_2}=(-{\xi_1}^{a+b},-{\xi_1}^{a},-{\xi_1}^{b},-1;-{\xi_1}^{a+b})$; of the conjugates of this element in $T_{(1)}$, we need only consider those of the form $(\pm1,*,*,*;\pm1)$, since the values obtained from the others will again produce cancellation. There are 12 conjugates of the form $(-1,*,*,*;1)$, where the value taken by $\theta^{(1)}_{i00j}$ is $(-1)^i$; likewise there are 12 each of the forms $(1,*,*,*;-1)$ and $(-1,*,*,*;-1)$, where the value is $(-1)^j$ or $(-1)^{i+j}$, respectively. Since $(-1)^i+(-1)^j+(-1)^{i+j}=3-4\epsilon_{i,j}^2$, and the Green function value here is $q+1$, we obtain a contribution to the scalar product of

$$\frac{12(3-4\epsilon_{i,j}^2)\cdot(q+1)\cdot\left(\frac{1}{12}q^2+\cdots\right)}{2(q^3+\cdots)}=\tfrac{3}{2}-2\epsilon_{i,j}^2+\cdots.$$

Thus for both values of d the extra nonnegligible part is $1+\frac{1}{2}d-2\epsilon_{i,j}^2$. We saw in Section 3.3.1 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(1)}$ is $q+f+1$; halving and adding the nonnegligible part just found gives $q_2+f_2+1-2\epsilon_{i,j}^2$.

We find that the additional contributions from H_m-classes of type $h_{33}\tau_2$ and $h_{35}\tau_2$ to the scalar product with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(2)}$ and $\boldsymbol{T}_{(6)}$, respectively, are the same as those just calculated; thus the scalar product of ${1_{H_m.2}}^G$ with each of the three characters $R_{\boldsymbol{T},\theta}$ treated here is $q_2+f_2+1-2\epsilon_{i,j}^2$, which is therefore the multiplicity of the semisimple character $\chi^1_{\kappa_{31,[(i,j)]}}$ in ${1_{H_m.2}}^G$.

The geometric conjugacy classes of types $\kappa_{32},\dots,\kappa_{36}$ behave very similarly. For κ_{33} and κ_{35} the types of class requiring attention are as for κ_{31}; for κ_{32}, κ_{34} and κ_{36} they are $h_\ell\tau_2$ for $\ell\in\{32,34,36\}$. For κ_{35} and κ_{36}, the additional contribution is in fact 0 if $d=1$, because the elements $(s\tau_2)^{y_2}$ do not lie in the tori concerned; if however $d=0$, the extra nonnegligible part is $\pm\frac{1}{2}$. On the other hand, for κ_{33} and κ_{34} the nonnegligible part of the additional contribution is 0 if $d=0$, because the Green function in each case is merely 1 and is thus too small to affect matters; however if $d=1$ we have an extra term $\pm\frac{1}{2}(-1)^i$. Upon combining these terms with those already found, and taking linear combinations as before, we obtain the following multiplicities in ${1_{H_m.2}}^G$:

$$\begin{aligned}\chi^1_{\kappa_{32,[(i,j)]}}&:q_2-f_2+1-2\epsilon_{i,j}^2,\\ \chi^1_{\kappa_{33,[i]}}&:q_2+f_2-1-\epsilon_i^2,\\ \chi^1_{\kappa_{34,[i]}}&:q_2-f_2+1-\epsilon_i^2,\\ \chi^1_{\kappa_{35,[i]}}&:q_2+f_2-2,\\ \chi^1_{\kappa_{36,[i]}}&:q_2-f_2+1.\end{aligned}$$

Next we turn to the geometric conjugacy classes of types κ_9, κ_{10}, κ_3 and κ_4. As described in Sections 3.3.3 and 3.3.4, these may loosely be regarded as the unions of certain of those of types $\kappa_{31}, \ldots, \kappa_{36}$ just considered, for appropriate values of the parameters. The details of the calculations of additional nonnegligible parts are just as above; note that if $d = 1$ then $\epsilon^2_{(q\pm1)i}$ and $\epsilon^2_{(q\pm1)/3}$ are both zero because the subscripts are even. Upon taking linear combinations to obtain the irreducible characters, we obtain the following multiplicities in ${1_{H_m.2}}^G$:

$$\begin{aligned} \chi^{\mathrm{St},1}_{\kappa_{9,[i]}} &: q_2 + f_2 - \epsilon_i^2, & \chi^{1,1}_{\kappa_{9,[i]}} &: 1 - \epsilon_i^2, \\ \chi^{\mathrm{St},1}_{\kappa_{10,[i]}} &: q_2 - f_2 + 1 - \epsilon_i^2, & \chi^{1,1}_{\kappa_{10,[i]}} &: \epsilon_i^2, \\ \chi^{\mathrm{St},1}_{\kappa_3} &: q_2 + f_2 - 1, & \chi^{\rho,1}_{\kappa_3} &: 1, \\ \chi^{\mathrm{St},1}_{\kappa_4} &: q_2 - f_2 + 1, & \chi^{\rho,1}_{\kappa_4} &: 0. \end{aligned}$$

We now treat the geometric conjugacy classes of type κ_7, where there are ten distinct characters $R_{\boldsymbol{T},\theta}$ lying in the geometric conjugacy class $\kappa_{7,[i]}$; the one with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ has $\theta = \theta^{(1)}_{i000}$, and we must consider H_m-classes of types $h_{31}\tau_2$ and $h_{66}\tau_2$. For the former we may again take $s = ({\xi_1}^{a+b}, {\xi_1}^{a}, {\xi_1}^{b}, 1; {\xi_1}^{a+b})$ and argue much as above. If $d = 0$ there are 48 conjugates of s of the form $(1, *, *, *; *)$, so the contribution to the scalar product is

$$\frac{48 \cdot (2q+1) \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{2(q^3 + \cdots)} = 4 + \cdots .$$

If instead $d = 1$, there are 48 conjugates of $(s\tau_2)^{y_2}$ of the form $(1, *, *, *; *)$ and 96 of the form $(-1, *, *, *; *)$, so the contribution is

$$\frac{(48 + 96(-1)^i) \cdot (q+1) \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{2(q^3 + \cdots)} = 2 + 4(-1)^i + \cdots .$$

For H_m-classes of type $h_{66}\tau_2$ we may take $s = ({\xi_1}^{a+b}, {\xi_1}^{a+c}, {\xi_1}^{b+c}, 1; {\xi_1}^{a+b+c})$; as the element is regular the Green function value is 1. If $d = 0$, there are again 48 conjugates of s of the form $(1, *, *, *; *)$, so the contribution is

$$\frac{48 \cdot 1 \cdot \left(\frac{1}{48}q^3 + \cdots\right)}{2(q^3 + \cdots)} = \tfrac{1}{2} + \cdots .$$

If instead $d = 1$, there are 96 conjugates of $(s\tau_2)^{y_2}$ of the form $(-1, *, *, *; *)$ (and none of the form $(1, *, *, *; *)$), so the contribution is

$$\frac{96(-1)^i \cdot 1 \cdot \left(\frac{1}{96}q^3 + \cdots\right)}{2(q^3 + \cdots)} = \tfrac{1}{2}(-1)^i + \cdots .$$

Thus for both values of d the extra nonnegligible part is $2d + \frac{9}{2} - 9\epsilon_i^2$. We saw in Section 3.3.5 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ is $4q + 5f + 4$; halving and adding the nonnegligible part just found gives $4q_2 + 5f_2 + 4 - 9\epsilon_i^2$.

For each of the nine other characters $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$, there is precisely one type $h_\ell \tau_2$ or $h_\ell{}' \tau_2$ with $\ell \in \{66, \dots, 75\}$ giving a contribution with nonnegligible part $\frac{1}{2}$ or $\frac{1}{2}(-1)^i$ according as $d = 0$ or 1. In addition, for the character with $\boldsymbol{T} = \boldsymbol{T}_{(2)}$ we obtain a contribution from type $h_{33}\tau_2$ with nonnegligible part 2 or $1 + 2(-1)^i$ according as $d = 0$ or 1; likewise for the character with $\boldsymbol{T} = \boldsymbol{T}_{(6)}$ we obtain a contribution from type $h_{35}\tau_2$ with nonnegligible part 1 or $\frac{1}{2} + (-1)^i$ according as $d = 0$ or 1. (There are also various other types where the contribution has zero nonnegligible part.) Combining with the values found in Section 3.3.5 we obtain the following scalar products of ${1_{H_m.2}}^G$ with the $R_{\boldsymbol{T},\theta}$ lying in $\kappa_{7,[i]}$:

$$
\begin{aligned}
\boldsymbol{T} = \boldsymbol{T}_{(1)} : &\quad 4q_2 + 5f_2 + 4 - 9\epsilon_i^2,\\
\boldsymbol{T} = \boldsymbol{T}_{(10)} : &\quad 4q_2 - 3f_2 + 4 - \epsilon_i^2,\\
\boldsymbol{T} = \boldsymbol{T}_{(2)} : &\quad 2q_2 + 3f_2 + 2 - 5\epsilon_i^2,\\
\boldsymbol{T} = \boldsymbol{T}_{(5)} : &\quad 2q_2 - f_2 + 2 - \epsilon_i^2,\\
\boldsymbol{T} = \boldsymbol{T}_{(6)} : &\quad q_2 + 2f_2 + 2 - 3\epsilon_i^2,\\
\boldsymbol{T} = \boldsymbol{T}_{(14)} : &\quad q_2 + 1 - \epsilon_i^2,\\
\boldsymbol{T} = \boldsymbol{T}_{(3)}, \boldsymbol{T}_{(4)}, \boldsymbol{T}_{(8)}, \boldsymbol{T}_{(11)} : &\quad f_2 - \epsilon_i^2.
\end{aligned}
$$

Using the character table of the Weyl group C_3 and the appropriate nonabelian Fourier transform matrix as in Section 3.3.5 shows that the irreducible characters $\chi_{\kappa_{7,[i]}}^{1}$ and $\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}$ have multiplicities in ${1_{H_m.2}}^G$ equal to $q_2 + f_2 + 1 - 2\epsilon_i^2$ and $q_2 + 1 - \epsilon_i^2$, respectively, while both $\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}$ and $\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}$ have multiplicity $f_2 - \epsilon_i^2$.

The geometric conjugacy classes of type κ_8 behave entirely similarly; here we find that the irreducible characters $\chi_{\kappa_{8,[i]}}^{1}$ and $\chi_{\kappa_{8,[i]}}^{\chi_{1,2}}$ have multiplicities in ${1_{H_m.2}}^G$ equal to $q_2 + f_2 - 1$ and $q_2 - \epsilon_i^2$, respectively, while both $\chi_{\kappa_{8,[i]}}^{\chi_{2,1}}$ and $\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}$ have multiplicity $f_2 - 1 + \epsilon_i^2$.

The geometric conjugacy class κ_1 may again be treated in similar fashion; we find that the irreducible characters $\chi_{\kappa_1}^{1,1}$ and $\chi_{\kappa_1}^{\chi_{1,2},1}$ both have multiplicity ϵ^4 in ${1_{H_m.2}}^G$, while $\chi_{\kappa_1}^{1,\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}}$, $\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}}$ and $\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}}$ have multiplicities $q_2 + f_2 - 1 + \epsilon^4$, q_2, $f_2 - 1 + \epsilon^4$ and $f_2 - 1 + \epsilon^4$, respectively.

4.2.3. *The complete decomposition of* ${1_{H_m.2}}^G$ *for* $m = 1, 2$. Combining the multiplicities obtained above gives the complete decomposition of ${1_{H_m.2}}^G$ for $m = 1, 2$ as follows.

Proposition 4.1. *If $G = F_4(q)$ and $H_m = {}^mD_4(q)$ for $m = 1, 2$, the decomposition of ${1_{H_m.2}}^G$ into irreducible characters is*

$$\begin{aligned}
&\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + (q_2+f_2)\chi_{\phi''_{8,3}} + q_2\chi_{\phi''_{8,9}} + f_2\chi_{\phi_{4,1}} + f_2\chi_{\phi''_{2,4}} \\
&\quad + \left\{\begin{matrix} \chi_{\phi_{12,4}} + 2\chi_{\phi''_{9,6}} + \chi_{\phi''_{1,12}} + \chi_{\phi''_{6,6}} & \text{if } m=1 \\ \chi_{\phi''_{4,7}} + \chi_{\phi_{16,5}} & \text{if } m=2 \end{matrix}\right\} \\
&\quad + \epsilon^4\chi_{\kappa_1}^{1,1} + \epsilon^4\chi_{\kappa_1}^{\chi_{1,2},1} + (q_2+f_2-1+\epsilon^4)\chi_{\kappa_1}^{1,\mathrm{St}} + q_2\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}} \\
&\qquad + (f_2-1+\epsilon^4)\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}} + (f_2-1+\epsilon^4)\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}} \\
&\quad + (q_2+f_2-1)\chi_{\kappa_3}^{\mathrm{St},1} + \chi_{\kappa_3}^{\rho,1} + (q_2-f_2+1)\chi_{\kappa_4}^{\mathrm{St},1} \\
&\quad + \sum_{[i]\in\bar{S}_7}\Big((q_2+f_2+1-2\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{1} + (q_2+1-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{1,2}} \\
&\qquad\qquad + (f_2-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{2,1}} + (f_2-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}\Big) \\
&\quad + \sum_{[i]\in\bar{S}_8}\Big((q_2+f_2-1)\chi_{\kappa_{8,[i]}}^{1} + (q_2-\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{1,2}} \\
&\qquad\qquad + (f_2-1+\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{2,1}} + (f_2-1+\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}\Big) \\
&\quad + \sum_{[i]\in\bar{S}_9}((1-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{1,1} + (q_2+f_2-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1}) \\
&\quad + \sum_{[i]\in\bar{S}_{10}}(\epsilon_i^2\chi_{\kappa_{10,[i]}}^{1,1} + (q_2-f_2+1-\epsilon_i^2)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1}) \\
&\quad + \sum_{[(i,j)]\in\bar{S}_{31}}(q_2+f_2+1-2\epsilon_{i,j}^2)\chi_{\kappa_{31,[(i,j)]}}^{1} + \sum_{[(i,j)]\in\bar{S}_{32}}(q_2-f_2+1-2\epsilon_{i,j}^2)\chi_{\kappa_{32,[(i,j)]}}^{1} \\
&\quad + \sum_{[i]\in\bar{S}_{33}}(q_2+f_2-1-\epsilon_i^2)\chi_{\kappa_{33,[i]}}^{1} + \sum_{[i]\in\bar{S}_{34}}(q_2-f_2+1-\epsilon_i^2)\chi_{\kappa_{34,[i]}}^{1} \\
&\quad + \sum_{[i]\in\bar{S}_{35}}(q_2+f_2-2)\chi_{\kappa_{35,[i]}}^{1} + \sum_{[i]\in\bar{S}_{36}}(q_2-f_2+1)\chi_{\kappa_{36,[i]}}^{1}.
\end{aligned}$$

Again we may now calculate the ranks of the actions; we obtain the following.

Corollary 4.2. *The rank of the action of $G = F_4(q)$ on cosets of $H_m.2 = {}^mD_4(q).2$ for $m = 1, 2$ is*

$$\begin{aligned}
&\tfrac{1}{4}(q^4+q^3+12q^2+20q+28) && \text{if } m=1 \text{ and } d=0, \\
&\tfrac{1}{4}(q^4+q^3+12q^2+27q+39) && \text{if } m=1 \text{ and } d=1, \\
&\tfrac{1}{4}(q^4+q^3+4q^2+4q+8) && \text{if } m=2 \text{ and } d=0, \\
&\tfrac{1}{4}(q^4+q^3+4q^2+7q+11) && \text{if } m=2 \text{ and } d=1.
\end{aligned}$$

4.3. ***The characters $\mathbf{1}_{H_m.3}{}^G$ for $m = 1, 3$.*** Recall that we take $e \in \{0, \pm 1\}$ with $q \equiv e \bmod 3$. We define

$$q_3 = \tfrac{1}{3}(q + 2e), \qquad f_3 = \tfrac{1}{3} f = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 3. \end{cases}$$

Again we proceed as in Section 3. This time contributions from classes lying in H_m are as already calculated, except for a factor of $\frac{1}{3}$; it remains to consider the classes in $H_m \tau_3$ and $H_m {\tau_3}^2$ (clearly the contributions from the two outer cosets will be complex conjugates of each other).

4.3.1. *Unipotent characters.* We must consider the H_m-classes of type $h_\ell \tau_3$ or $h_\ell{}' \tau_3$ with $\ell \in \{31, \ldots, 36\}$; for example, we again take $\ell = 31$, so that the elements concerned have semisimple parts lying in tori $T_{(n)}$ for $n = 1$, 3 and 7. In all cases the elements are regular, so the Green function value is 1. Thus if $e = 0$ the contributions to the scalar product with $R_{(n)}$ are

$$\frac{192 \cdot 1 \cdot \left(\frac{1}{12} q^2 + \cdots\right)}{3(q^2 + \cdots)}, \quad \frac{48 \cdot 1 \cdot \left(\frac{1}{12} q^2 + \cdots\right)}{3(q^2 + \cdots)}, \quad \frac{12 \cdot 1 \cdot \left(\frac{1}{12} q^2 + \cdots\right)}{3(q^2 + \cdots)},$$

having nonnegligible parts $\frac{16}{3}$, $\frac{4}{3}$ and $\frac{1}{3}$, respectively; if $e = 1$ the contributions to the scalar product with $R_{(1)}$ and $R_{(7)}$ are

$$\frac{1152 \cdot 1 \cdot \left(\frac{1}{36} q^2 + \cdots\right)}{3(q^2 + \cdots)}, \quad \frac{36 \cdot 1 \cdot \left(\frac{1}{18} q^2 + \cdots\right)}{3(q^2 + \cdots)}$$

(the latter from $h_{31}{}' \tau_3$), having nonnegligible parts $\frac{32}{3}$ and $\frac{2}{3}$, respectively; if $e = -1$ the contribution to the scalar product with $R_{(3)}$ is

$$\frac{96 \cdot 1 \cdot \frac{1}{12}(q^2 - \cdots)}{3(q^2 + \cdots)},$$

having nonnegligible part $\frac{8}{3}$. The other instances are precisely similar.

Combining the three possibilities for e gives the following table of nonnegligible parts.

$h_{31}\tau_3$ or $h_{31}{}'\tau_3$	$R_{(1)}$	$\frac{16}{3}(1+e)$	$R_{(3)}$	$\frac{4}{3}(1-e)$	$R_{(7)}$	$\frac{1}{3}(1+e)$
$h_{32}\tau_3$ or $h_{32}{}'\tau_3$	$R_{(17)}$	$\frac{16}{3}(1-e)$	$R_{(10)}$	$\frac{4}{3}(1+e)$	$R_{(20)}$	$\frac{1}{3}(1-e)$
$h_{33}\tau_3$ or $h_{33}{}'\tau_3$	$R_{(2)}$	$\frac{4}{3}(1+e)$	$R_{(5)}$	$\frac{2}{3}(1-e)$	$R_{(15)}$	$\frac{1}{3}(1+e)$
$h_{34}\tau_3$ or $h_{34}{}'\tau_3$	$R_{(9)}$	$\frac{4}{3}(1-e)$	$R_{(5)}$	$\frac{2}{3}(1+e)$	$R_{(13)}$	$\frac{1}{3}(1-e)$
$h_{35}\tau_3$ or $h_{35}{}'\tau_3$	$R_{(6)}$	$\frac{1}{3}(1+e)$	$R_{(16)}$	$\frac{1}{3}(1-e)$	$R_{(18)}$	$\frac{4}{3}(1+e)$
$h_{36}\tau_3$ or $h_{36}{}'\tau_3$	$R_{(21)}$	$\frac{1}{3}(1-e)$	$R_{(14)}$	$\frac{1}{3}(1+e)$	$R_{(25)}$	$\frac{4}{3}(1-e)$

	H_1	H_3
$R_{(1)}$	$16q_3+46$	$16q_3+16$
$R_{(2)}$	$4q_3+16$	$4q_3+4$
$R_{(3)}$	$-4q_3+2$	$-4q_3+4$
$R_{(4)}$	2	0
$R_{(5)}$	0	4
$R_{(6)}$	q_3+7	q_3+1
$R_{(7)}$	q_3+1	q_3+1
$R_{(8)}$	2	0
$R_{(9)}$	$-4q_3+4$	$-4q_3+4$
$R_{(10)}$	$4q_3-2$	$4q_3+4$
$R_{(11)}$	2	0
$R_{(12)}$	0	0
$R_{(13)}$	$-q_3+1$	$-q_3+1$

	H_1	H_3
$R_{(14)}$	q_3+1	q_3+1
$R_{(15)}$	q_3+1	q_3+1
$R_{(16)}$	$-q_3-1$	$-q_3+1$
$R_{(17)}$	$-16q_3+22$	$-16q_3+16$
$R_{(18)}$	$4q_3+4$	$4q_3+1$
$R_{(19)}$	-2	0
$R_{(20)}$	$-q_3+1$	$-q_3+1$
$R_{(21)}$	$-q_3+1$	$-q_3+1$
$R_{(22)}$	2	0
$R_{(23)}$	0	0
$R_{(24)}$	0	1
$R_{(25)}$	$-4q_3+4$	$-4q_3+1$

Table 17. Scalar products $(1_{H_m.3}{}^G, R_{(n)})_G$.

Much as before, note that the coefficients of e above are precisely the same, apart from the factor of $\frac{1}{3}$ already mentioned, as those of q obtained from consideration of types $h_{31}, \ldots, h_{36}$ in Section 3.2. This means that when the sets of contributions are added and q is replaced by $3q_3 - 2e$, the terms in e cancel to leave linear polynomials in q_3.

Summing the nonnegligible parts gives the values in Table 17 for the scalar products $(1_{H_m.3}{}^G, R_{(n)})_G$.

We may now proceed as before to find the scalar products of $1_{H_m.3}{}^G$ with irreducible unipotent characters. On taking linear combinations given by the character table of W, we find that the only scalar products of $1_{H_m.3}{}^G$ with almost characters R_ϕ which are nonzero for some m are as follows.

ϕ	$(1_{H_m.3}{}^G, R_\phi)_G$
$\phi_{1,0}, \phi_{9,2}$	1
$\phi''_{8,3}$	q_3+f_3
$\phi''_{8,9}$	q_3
$\phi_{4,1}, \phi''_{2,4}, \phi''_{9,6}, \phi''_{1,12}$	f_3
$\phi_{12,4}$	$\frac{1}{3}$
$\phi''_{6,6}$	$f_3-\frac{1}{3}$
$\phi'_{6,6}$	$\frac{2}{3}$

For $\phi = \phi_{1,0}$, $\phi_{9,2}$, $\phi''_{8,3}$ and $\phi''_{8,9}$ we again have irreducible characters χ_ϕ, appearing in ${1_{H_m.3}}^G$ with multiplicities 1, 1, $q_3 + f_3$ and q_3, respectively. For $\phi = \phi_{4,1}$ and $\phi''_{2,4}$, in the family of size four, as in Section 3.2 we obtain two irreducible characters $\chi_{\phi_{4,1}}$ and $\chi_{\phi''_{2,4}}$, each appearing with multiplicity f_3. Separate analyses of the family of size 21 for the two values of m lead to the following: if $m = 1$ we have constituents $\chi_{\phi_{12,4}}$, $\chi_{\phi''_{9,6}}$, $\chi_{\phi''_{1,12}}$ and $\chi_{F_4{}^{\mathrm{II}}[1]}$, each with multiplicity 1; if $m = 3$ we have a constituent $\chi_{\phi'_{6,6}}$ with multiplicity 1. This completes the treatment of unipotent characters.

4.3.2. *Other geometric conjugacy classes.* We begin again with geometric conjugacy classes of type κ_{31}; we consider H_m-classes of type $h_{31}\tau_3$, containing elements $s\tau_3$ with $s = ({\xi_1}^{a+b}, {\xi_1}^{a}, {\xi_1}^{b}, 1; {\xi_1}^{a+b})$. These contribute if $e = 0$ or 1, but not if $e = -1$. As in Section 4.2.2 we recall that there are three distinct characters $R_{\boldsymbol{T},\theta}$ lying in the geometric conjugacy class $\kappa_{31,[(i,j)]}$, and the one with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ (in which s lies) has $\theta = \theta^{(1)}_{i00j}$.

If $e = 0$, we saw in Section 3.3.1 that 12 of the conjugates of s lying in $T_{(1)}$ were of the form $(1, *, *, *; 1)$ and thus sent to 1 by $\theta^{(1)}_{i00j}$, with the values taken at other elements producing cancellation. Thus the contribution to the scalar product from these elements is

$$\frac{12 \cdot 1 \cdot \left(\frac{1}{12}q^2 + \cdots\right)}{3(q^2 + \cdots)} = \tfrac{1}{3} + \cdots .$$

If $e = 1$, we have $(s\tau_2)^{y_2} = ({\xi_1}^{a+b}, {\xi_1}^{a}, {\xi_1}^{b}, \omega^2; \omega{\xi_1}^{a+b})$; of the conjugates of this element in $T_{(1)}$, we need only consider those of the form $(\omega^{\pm 1}, *, *, *; \omega^{\pm 1})$, since the values obtained from the others will again produce cancellation. If we write $\bar{\zeta}$ for the cube root of unity in $\mathbb{C}$ corresponding to ω in k, there are 36 conjugates of the form $(\omega^2, *, *, *; \omega)$, where the value taken by $\theta^{(1)}_{i00j}$ is $\bar{\zeta}^{-i+j}$, and 36 of the form $(\omega, *, *, *; \omega^2)$, where the value taken is $\bar{\zeta}^{i-j}$. Since

$$\bar{\zeta}^{-i+j} + \bar{\zeta}^{i-j} = 2 - 3\epsilon^{3,+}_{i-j},$$

we obtain a contribution to the scalar product of

$$\frac{36(2 - 3\epsilon^{3,+}_{i-j}) \cdot 1 \cdot \left(\frac{1}{36}q^2 + \cdots\right)}{3(q^2 + \cdots)} = \tfrac{2}{3} - \epsilon^{3,+}_{i-j} + \cdots .$$

Thus for all three values of e the extra nonnegligible part is $\frac{1}{3}(1 + e) - \epsilon^{3,+}_{i-j}$. We saw in Section 3.3.1 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T} = \boldsymbol{T}_{(1)}$ is $q + f + 1$; dividing by three and adding twice the nonnegligible part just found gives $q_3 + f_3 + 1 - 2\epsilon^{3,+}_{i-j}$.

We find that the additional contributions from H_m-classes of type $h_{33}\tau_3$ and $h_{35}\tau_3$ to the scalar product with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T} = \boldsymbol{T}_{(2)}$ and $\boldsymbol{T}_{(6)}$, respectively, are the same as those just calculated; thus the scalar product of ${1_{H_m.3}}^G$ with each of the three characters $R_{\boldsymbol{T},\theta}$ treated here is $q_3+f_3+1-2\epsilon_{i-j}^{3,+}$, which is therefore the multiplicity of the semisimple character $\chi^1_{\kappa_{31,[(i,j)]}}$ in $1_{H_m.3}{}^G$.

The geometric conjugacy classes of types $\kappa_{32},\ldots,\kappa_{36}$ behave very similarly; we obtain the following multiplicities in $1_{H_m.3}{}^G$:

$$\begin{aligned}
\chi^1_{\kappa_{32,[(i,j)]}} &: q_3-f_3-1+2\epsilon_{i-j}^{3,-},\\
\chi^1_{\kappa_{33,[i]}} &: q_3+f_3-1+2\epsilon_{i}^{3,-},\\
\chi^1_{\kappa_{34,[i]}} &: q_3-f_3+1-2\epsilon_{i}^{3,+},\\
\chi^1_{\kappa_{35,[i]}} &: q_3+f_3-2\epsilon_{i}^{3,+},\\
\chi^1_{\kappa_{36,[i]}} &: q_3-f_3+2\epsilon_{i}^{3,-}.
\end{aligned}$$

Next we turn to the geometric conjugacy classes of types κ_9, κ_{10}, κ_3 and κ_4. As before, these may loosely be regarded as the unions of certain of those of types $\kappa_{31},\ldots,\kappa_{36}$ just considered, for appropriate values of the parameters. The details of the calculations of additional nonnegligible parts are just as above; we obtain the following multiplicities in $1_{H_m.3}{}^G$:

$$\begin{aligned}
\chi^{\mathrm{St},1}_{\kappa_{9,[i]}} &: q_3+f_3, & \chi^{1,1}_{\kappa_{9,[i]}} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_{10,[i]}} &: q_3-f_3, & \chi^{1,1}_{\kappa_{10,[i]}} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_3} &: q_3+f_3-1, & \chi^{\rho,1}_{\kappa_3} &: 1,\\
\chi^{\mathrm{St},1}_{\kappa_4} &: q_3-f_3+1, & \chi^{\rho,1}_{\kappa_4} &: 1.
\end{aligned}$$

For the geometric conjugacy classes of type κ_7, we again treat H_m-classes of type $h_{31}\tau_3$, containing elements $s\tau_2$ with $s=({\xi_1}^{a+b},{\xi_1}^{a},{\xi_1}^{b},1;{\xi_1}^{a+b})$. These contribute if $e=0$ or 1, but not if $e=-1$; the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(1)}$ has $\theta=\theta^{(1)}_{i000}$.

If $e=0$, we saw in Section 3.3.5 that 48 of the conjugates of s lying in $T_{(1)}$ were of the form $(1,*,*,*;*)$ and thus sent to 1 by $\theta^{(1)}_{i000}$, with the values taken at other elements producing cancellation. Thus the contribution to the scalar product from these elements is

$$\frac{48\cdot 1\cdot\left(\frac{1}{12}q^2+\cdots\right)}{3(q^2+\cdots)}=\tfrac{4}{3}+\cdots.$$

If instead $e=1$, we have $(s\tau_2)^{y_2}=({\xi_1}^{a+b},{\xi_1}^{a},{\xi_1}^{b},\omega^2;\omega{\xi_1}^{a+b})$; there are 144 conjugates of the form $(\omega^2,*,*,*;*)$, where the value taken by $\theta^{(1)}_{i000}$ is $\bar{\zeta}^{-i}$, and 144 of the form $(\omega,*,*,*;*)$, where the value taken is $\bar{\zeta}^{i}$. Since

$$\bar{\zeta}^{-i}+\bar{\zeta}^{i}=2-3\epsilon_i^{3,+},$$

we obtain a contribution to the scalar product of

$$\frac{144(2-3\epsilon_i^{3,+})\cdot 1\cdot\left(\frac{1}{36}q^2+\cdots\right)}{3(q^2+\cdots)}=\tfrac{8}{3}-4\epsilon_i^{3,+}+\cdots.$$

Thus for all three values of e the extra nonnegligible part is $\frac{4}{3}(1+e)-4\epsilon_i^{3,+}$. We saw in Section 3.3.5 that the scalar product of ${1_{H_m}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(1)}$ is $4q+5f+4$; dividing by three and adding twice the nonnegligible part just found gives $4q_3+5f_3+4-8\epsilon_i^{3,+}$.

There are seven other pairs (ℓ,n) such that the geometric conjugacy class contains a character $R_{\boldsymbol{T},\theta}$ with $\boldsymbol{T}=\boldsymbol{T}_{(n)}$ and the classes of type $h_\ell\tau_3$ contain elements whose semisimple parts lie in the torus $T_{(n)}$. For $(\ell,n)=(31,3)$ or $(33,5)$ we find that all roots of unity concerned produce cancellation, so the nonnegligible part is 0 (these are the two pairs where the classes meet the torus if $e=-1$). For the other five pairs the calculations are very similar to the above: the extra nonnegligible part is $\frac{1}{3}(1+e)-\epsilon_i^{3,+}$ times the coefficient of $q+1$ in the value obtained in Section 3.3.5, and it follows that the scalar product of ${1_{H_m.3}}^G$ with the appropriate $R_{\boldsymbol{T},\theta}$ is obtained from that value by replacing $q+1$ by $q_3+1-2\epsilon_i^{3,+}$ and f by f_3. Accordingly the irreducible characters $\chi^1_{\kappa_{7,[i]}}$ and $\chi^{\chi_{1,2}}_{\kappa_{7,[i]}}$ have multiplicities in ${1_{H_m.3}}^G$ equal to $q_3+f_3+1-2\epsilon_i^{3,+}$ and $q_3+1-2\epsilon_i^{3,+}$, respectively, while both $\chi^{\chi_{2,1}}_{\kappa_{7,[i]}}$ and $\chi^{\chi_{-,3}}_{\kappa_{7,[i]}}$ have multiplicity f_3.

Again the geometric conjugacy classes of type κ_8 behave entirely similarly; here we find that the irreducible characters $\chi^1_{\kappa_{8,[i]}}$ and $\chi^{\chi_{1,2}}_{\kappa_{8,[i]}}$ have multiplicities in ${1_{H_m.3}}^G$ equal to $q_3+f_3-1+2\epsilon_i^{3,-}$ and $q_3-1+2\epsilon_i^{3,-}$, respectively, while both $\chi^{\chi_{2,1}}_{\kappa_{8,[i]}}$ and $\chi^{\chi_{-,3}}_{\kappa_{8,[i]}}$ have multiplicity f_3.

Finally the geometric conjugacy class κ_1 may again be treated in similar fashion; we find that the irreducible characters $\chi^{1,1}_{\kappa_1}$ and $\chi^{\chi_{1,2},1}_{\kappa_1}$ both have multiplicity 1 in ${1_{H_m.3}}^G$, while $\chi^{1,\mathrm{St}}_{\kappa_1}$, $\chi^{\chi_{1,2},\mathrm{St}}_{\kappa_1}$, $\chi^{\chi_{2,1},\mathrm{St}}_{\kappa_1}$ and $\chi^{\chi_{-,3},\mathrm{St}}_{\kappa_1}$ have multiplicities q_3+f_3, q_3, f_3 and f_3, respectively.

4.3.3. *The complete decomposition of* ${1_{H_m.3}}^G$ *for* $m=1,3$. Combining the multiplicities obtained above gives the complete decomposition of ${1_{H_m.3}}^G$ for $m=1,3$ as follows.

Proposition 4.3. *If* $G=F_4(q)$ *and* $H_m={}^mD_4(q)$ *for* $m=1,3$, *the decomposition of* ${1_{H_m.3}}^G$ *into irreducible characters is*

$$\chi_{\phi_{1,0}}+\chi_{\phi_{9,2}}+(q_3+f_3)\chi_{\phi''_{8,3}}+q_3\chi_{\phi''_{8,9}}+f_3\chi_{\phi_{4,1}}+f_3\chi_{\phi''_{2,4}}$$
$$+\begin{Bmatrix}\chi_{\phi_{12,4}}+\chi_{\phi''_{9,6}}+\chi_{\phi''_{1,12}}+\chi_{F_4{}^{\mathrm{II}}[1]} & \text{if } m=1\\ \chi_{\phi'_{6,6}} & \text{if } m=3\end{Bmatrix}$$
$$+\chi^{1,1}_{\kappa_1}+\chi^{\chi_{1,2},1}_{\kappa_1}+(q_3+f_3)\chi^{1,\mathrm{St}}_{\kappa_1}+q_3\chi^{\chi_{1,2},\mathrm{St}}_{\kappa_1}+f_3\chi^{\chi_{2,1},\mathrm{St}}_{\kappa_1}+f_3\chi^{\chi_{-,3},\mathrm{St}}_{\kappa_1}\quad\text{(continues)}$$

$$+\ (q_3+f_3-1)\chi_{\kappa_3}^{\mathrm{St},1}+\chi_{\kappa_3}^{\rho,1}+(q_3-f_3+1)\chi_{\kappa_4}^{\mathrm{St},1}+\chi_{\kappa_4}^{\rho,1}$$

$$+\sum_{[i]\in\bar{S}_7}\big((q_3+f_3+1-2\epsilon_i^{3,+})\chi_{\kappa_{7,[i]}}^{1}+(q_3+1-2\epsilon_i^{3,+})\chi_{\kappa_{7,[i]}}^{\chi_{1,2}}+f_3\chi_{\kappa_{7,[i]}}^{\chi_{2,1}}+f_3\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}\big)$$

$$+\sum_{[i]\in\bar{S}_8}\big((q_3+f_3-1+2\epsilon_i^{3,-})\chi_{\kappa_{8,[i]}}^{1}+(q_3-1+2\epsilon_i^{3,-})\chi_{\kappa_{8,[i]}}^{\chi_{1,2}}+f_3\chi_{\kappa_{8,[i]}}^{\chi_{2,1}}+f_3\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}\big)$$

$$+\sum_{[i]\in\bar{S}_9}(\chi_{\kappa_{9,[i]}}^{1,1}+(q_3+f_3)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1})+\sum_{[i]\in\bar{S}_{10}}(\chi_{\kappa_{10,[i]}}^{1,1}+(q_3-f_3)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1})$$

$$+\sum_{[(i,j)]\in\bar{S}_{31}}(q_3+f_3+1-2\epsilon_{i-j}^{3,+})\chi_{\kappa_{31,[(i,j)]}}^{1}+\sum_{[(i,j)]\in\bar{S}_{32}}(q_3-f_3-1+2\epsilon_{i-j}^{3,-})\chi_{\kappa_{32,[(i,j)]}}^{1}$$

$$+\sum_{[i]\in\bar{S}_{33}}(q_3+f_3-1+2\epsilon_i^{3,-})\chi_{\kappa_{33,[i]}}^{1}+\sum_{[i]\in\bar{S}_{34}}(q_3-f_3+1-2\epsilon_i^{3,+})\chi_{\kappa_{34,[i]}}^{1}$$

$$+\sum_{[i]\in\bar{S}_{35}}(q_3+f_3-2\epsilon_i^{3,+})\chi_{\kappa_{35,[i]}}^{1}+\sum_{[i]\in\bar{S}_{36}}(q_3-f_3+2\epsilon_i^{3,-})\chi_{\kappa_{36,[i]}}^{1}.$$

Again we may now calculate the ranks of the actions; we obtain the following.

Corollary 4.4. *The rank of the action of $G=F_4(q)$ on cosets of $H_m.3={}^mD_4(q).3$ for $m=1,3$ is*

$$\begin{array}{ll}\frac{1}{9}(q^4+q^3+13q^2+21q+36) & \textit{if } m=1 \textit{ and } e=0 \textit{ or } 1,\\ \frac{1}{9}(q^4+q^3+13q^2+21q+44) & \textit{if } m=1 \textit{ and } e=-1,\\ \frac{1}{9}(q^4+q^3+4q^2+3q+9) & \textit{if } m=3 \textit{ and } e=0 \textit{ or } 1,\\ \frac{1}{9}(q^4+q^3+4q^2+3q+17) & \textit{if } m=3 \textit{ and } e=-1.\end{array}$$

4.4. ***The character $1_{H_1.S_3}{}^G$.*** We define

$$q_6=\tfrac{1}{6}(q+3d+2e)=\tfrac{1}{3}(q_2+d+e)=\tfrac{1}{2}(q_3+d).$$

If we write $1_{H_1}{}^{H_1.S_3}=1+\epsilon+2\rho$ where ϵ is linear and ρ has dimension 2, then $1_{H_1.2}{}^{H_1.S_3}=1+\rho$ and $1_{H_1.3}{}^{H_1.S_3}=1+\epsilon$, whence

$$1_{H_1.S_3}{}^{H_1.S_3}=1_{H_1.2}{}^{H_1.S_3}-\tfrac{1}{2}(1_{H_1}{}^{H_1.S_3}-1_{H_1.3}{}^{H_1.S_3});$$

inducing up from $H_1.S_3$ to G gives

$$1_{H_1.S_3}{}^G=1_{H_1.2}{}^G-\tfrac{1}{2}(1_{H_1}{}^G-1_{H_1.3}{}^G).$$

It is therefore now easy to calculate the decomposition of $1_{H_1.S_3}{}^G$ to be as follows.

Proposition 4.5. *If $G = F_4(q)$ and $H_1 = D_4(q)$, the decomposition of ${1_{H_1.S_3}}^G$ into irreducible characters is*

$$\begin{aligned}
&\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + (q_6+1)\chi_{\phi''_{8,3}} + q_6\chi_{\phi''_{8,9}} + \chi_{\phi_{4,1}} + \chi_{\phi''_{2,4}} + \chi_{\phi_{12,4}} + \chi_{\phi''_{9,6}} \\
&+ \epsilon^4\chi_{\kappa_1}^{1,1} + \epsilon^4\chi_{\kappa_1}^{\chi_{1,2},1} + (q_6+\epsilon^4)\chi_{\kappa_1}^{1,\mathrm{St}} + q_6\chi_{\kappa_1}^{\chi_{1,2},\mathrm{St}} + \epsilon^4\chi_{\kappa_1}^{\chi_{2,1},\mathrm{St}} + \epsilon^4\chi_{\kappa_1}^{\chi_{-,3},\mathrm{St}} \\
&\qquad + q_6\chi_{\kappa_3}^{\mathrm{St},1} + \chi_{\kappa_3}^{\rho,1} + q_6\chi_{\kappa_4}^{\mathrm{St},1} \\
&+ \sum_{[i]\in\bar{S}_7} \big((q_6+2-\epsilon_i^{3,+}-2\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{1} + (q_6+1-\epsilon_i^{3,+}-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{1,2}} \\
&\qquad + (1-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{2,1}} + (1-\epsilon_i^2)\chi_{\kappa_{7,[i]}}^{\chi_{-,3}}\big) \\
&+ \sum_{[i]\in\bar{S}_8} ((q_6+\epsilon_i^{3,-})\chi_{\kappa_{8,[i]}}^{1} + (q_6+\epsilon_i^{3,-}-\epsilon_i^2)\chi_{\kappa_{8,[i]}}^{\chi_{1,2}} + \epsilon_i^2\chi_{\kappa_{8,[i]}}^{\chi_{2,1}} + \epsilon_i^2\chi_{\kappa_{8,[i]}}^{\chi_{-,3}}) \\
&+ \sum_{[i]\in\bar{S}_9} ((1-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{1,1} + (q_6+1-\epsilon_i^2)\chi_{\kappa_{9,[i]}}^{\mathrm{St},1}) + \sum_{[i]\in\bar{S}_{10}} (\epsilon_i^2\chi_{\kappa_{10,[i]}}^{1,1} + (q_6-\epsilon_i^2)\chi_{\kappa_{10,[i]}}^{\mathrm{St},1}) \\
&+ \sum_{[(i,j)]\in\bar{S}_{31}} (q_6+2-\epsilon_{i-j}^{3,+}-2\epsilon_{i,j}^2)\chi_{\kappa_{31,[(i,j)]}}^{1} + \sum_{[(i,j)]\in\bar{S}_{32}} (q_6+\epsilon_{i-j}^{3,-}-2\epsilon_{i,j}^2)\chi_{\kappa_{32,[(i,j)]}}^{1} \\
&+ \sum_{[i]\in\bar{S}_{33}} (q_6+\epsilon_i^{3,-}-\epsilon_i^2)\chi_{\kappa_{33,[i]}}^{1} + \sum_{[i]\in\bar{S}_{34}} (q_6-\epsilon_i^{3,+}-\epsilon_i^2)\chi_{\kappa_{34,[i]}}^{1} \\
&+ \sum_{[i]\in\bar{S}_{35}} (q_6-\epsilon_i^{3,+})\chi_{\kappa_{35,[i]}}^{1} + \sum_{[i]\in\bar{S}_{36}} (q_6-1+\epsilon_i^{3,-})\chi_{\kappa_{36,[i]}}^{1}.
\end{aligned}$$

Yet again we may now calculate the rank of the action; we obtain the following.

Corollary 4.6. *The rank of the action of $G = F_4(q)$ on cosets of $H_1.S_3 = D_4(q).S_3$ is*

$$\begin{aligned}
&\tfrac{1}{36}(q^4+q^3+28q^2+48q+84) && \text{if } d=0 \text{ and } e=1,\\
&\tfrac{1}{36}(q^4+q^3+28q^2+48q+92) && \text{if } d=0 \text{ and } e=-1,\\
&\tfrac{1}{36}(q^4+q^3+28q^2+75q+99) && \text{if } d=1 \text{ and } e=0,\\
&\tfrac{1}{36}(q^4+q^3+28q^2+75q+111) && \text{if } d=1 \text{ and } e=1,\\
&\tfrac{1}{36}(q^4+q^3+28q^2+75q+119) && \text{if } d=1 \text{ and } e=-1.
\end{aligned}$$

5. Contribution to classification programs

In this final section we consider the part played by D_4-type subgroups of $F_4(q)$ in the classification programs mentioned in Section 1, namely those of primitive actions which are multiplicity-free, or have all suborbits self-paired, or arise from a distance-transitive graph. Recall that the primitive actions (in which the action is on the cosets of a subgroup which is maximal) are those where the subgroup is either $D_4(q).S_3$ or ${}^3D_4(q).3$.

5.1. *Primitive multiplicity-free actions.* From the decompositions of Section 4 we can see that the permutation character ${1_H}^{F_4(q)}$, where H is either $D_4(q).S_3$ or ${}^3D_4(q).3$, is multiplicity-free if and only if $q = 2$. Indeed, the multiplicity of the constituent $\chi_{\phi''_{8,3}}$ in the former case is $q_6 + 1$, which is greater than 1 for all q apart from 2, and in the latter case is q_3, which is greater than 1 for all q apart from 2, 3 and 5; if q is 3 or 5 the multiplicity of the constituent $\chi^1_{\kappa_{34,[1]}}$ in the latter case is $q_3 + 1$, which is greater than 1.

However, if $q = p^a$ for some $a > 1$, the possibility arises of extending both $F_4(q)$ and H by field automorphisms. Write ϕ for the field automorphism which for each $\alpha \in \Phi$ and $\lambda \in k$ sends $x_\alpha(\lambda)$ to $x_\alpha(\lambda^p)$; then $F_4(q).\langle\phi\rangle$ acts on cosets of $H.\langle\phi\rangle$, and we have the corresponding permutation character ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$. Given an irreducible character χ of $F_4(q)$, there exist $a', a'' \geq 1$ with $a = a'a''$ such that applying ϕ fuses together a' irreducible characters of $F_4(q)$ including χ and the resulting character extends to a'' distinct characters of the group $F_4(q).\langle\phi\rangle$. If the multiplicity of χ as a constituent of the permutation character ${1_H}^{F_4(q)}$ is greater than a'' (which in particular is true if it is greater than a), then at least one of the extensions to $F_4(q).\langle\phi\rangle$ must have multiplicity greater than 1 in ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$.

We claim that, if $q = p^a$ with $a > 1$, then ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$ is not multiplicity-free. For most such values of q we may see this by again considering the multiplicity of the constituent $\chi_{\phi''_{8,3}}$. Indeed, if $H = D_4(q).S_3$ we have $q_6 + 1 > a$ for all q apart from 4, 8 and 16, while if $H = {}^3D_4(q).3$ we have $q_3 > a$ for all q apart from 4 and 8. We are therefore left with five pairs (H, q) to treat.

Two of these five pairs may be settled by considering a single constituent. If $H = D_4(q).S_3$ and $q = 16$, the multiplicity in ${1_H}^{F_4(q)}$ of the constituent $\chi^1_{\kappa_{7,[3]}}$ is equal to $q_6 + 2 = 5 > 4 = a$. Likewise if $H = {}^3D_4(q).3$ and $q = 8$, the multiplicity in ${1_H}^{F_4(q)}$ of the constituent $\chi^1_{\kappa_{36,[1]}}$ is equal to $q_3 + 2 = 4 > 3 = a$. Thus in neither case is ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$ multiplicity-free.

Another two may be settled by considering how ϕ fuses geometric conjugacy classes. Suppose $H = D_4(q).S_3$ and $q = 8$; here the constituents $\chi^1_{\kappa_{8,[i]}}$ for $i = 1, 2, 4$ all have multiplicity $q_6 + 1 = 2$. However, these three characters are fused by ϕ, because the semisimple classes corresponding to the geometric conjugacy classes $\kappa_{8,[i]}$ for $i = 1, 2, 4$ contain elements $(1, {\xi_2}^i, {\xi_2}^{-i}, 1; 1)$, and as ϕ squares entries in root elements $x_\alpha(\lambda)$, and therefore in torus elements $(\mu_1, \mu_2, \mu_3, \mu_4; \nu)$, it fuses these semisimple classes; so $a' = 3$ and hence $a'' = 1$. Likewise suppose $H = {}^3D_4(q).3$ and $q = 4$; here the constituents $\chi^1_{\kappa_{36,[i]}}$ for $i = 1, 2$ both have multiplicity $q_3 = 2$. However, these two characters are fused by ϕ, because the semisimple classes corresponding to the geometric conjugacy classes $\kappa_{36,[i]}$ for $i = 1, 2$ contain elements $({\xi_6}^i, {\xi_6}^{qi}, {\xi_6}^{-(q-1)i}, 1; {\xi_6}^i)$, and ϕ similarly fuses these semisimple classes; so $a' = 2$ and hence $a'' = 1$. Thus in each case the single extension of the fused character has multiplicity 2 in the permutation character ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$.

This leaves the pair where $H = D_4(q).S_3$ and $q = 4$. Here the rank is 29; there are two constituents of multiplicity 2, namely $\chi_{\phi''_{8,3}}$ and $\chi^{1}_{\kappa_{7,[1]}}$, and all other constituents have multiplicity 1. There is therefore no fusion among the constituents of multiplicity greater than 1; consequently the methods used until now are insufficient to determine whether or not either multiplicity persists in ${1_{H.\langle\phi\rangle}}^{F_4(q).\langle\phi\rangle}$. In this case we refer to [17], which calculates $(P, D_4(q))$-double cosets in $F_4(q)$ (where P is the maximal parabolic subgroup whose Levi subgroup has derived group $B_3(q)$), and in fact concludes that if $q > 2$ then the action of $F_4(q)$ on the cosets of $D_4(q).S_3$ is never multiplicity-free, even if field automorphisms are applied.

(A word is in order regarding the relationship of the current article to [17]. As stated in the very first paragraph here, the project on D_4-type subgroups of $F_4(q)$ is essentially a continuation of [15], which was published at the end of the previous millennium. Almost all of the work on D_4-type subgroups was completed more than twenty years ago; in particular it became clear that something beyond the character decompositions presented here was required to settle the case of the preceding paragraph, and the double coset calculations of [17] were performed to do this. However, some issues remained unresolved and the material ended up being set aside. A few years ago the opportunity arose of publishing the double coset material as a paper in its own right; but there seemed no prospect of applying similar methods to treat the action on cosets of ${}^3D_4(q).3$. More recently the issues which had prevented publication of the work as a whole were finally resolved, and the present article is the result; but [17] should really be regarded as an addendum to it. The author apologizes for the inordinate delay in completing the project, especially to those who have waited patiently for decades to see the results appear.)

Thus the only two primitive multiplicity-free actions here are those of $F_4(2)$ on cosets of $D_4(2).S_3$ and ${}^3D_4(2).3$. The first permutation character has rank 9 and decomposition

$$\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + \chi_{\phi''_{8,3}} + \chi_{\phi_{4,1}} + \chi_{\phi''_{2,4}} + \chi_{\phi_{12,4}} + \chi_{\phi''_{9,6}} + \chi^{1}_{\kappa_{8,[1]}} + \chi^{\chi_{1,2}}_{\kappa_{8,[1]}};$$

the constituents have degrees 1, 22932, 44200, 1377, 1105, 584766, 541450, 23205 and 1949220, respectively. The second has rank 7 and decomposition

$$\chi_{\phi_{1,0}} + \chi_{\phi_{9,2}} + \chi_{\phi'_{6,6}} + \chi^{\mathrm{St},1}_{\kappa_4} + \chi^{\rho,1}_{\kappa_4} + +\chi^{1}_{\kappa_{8,[1]}} + \chi^{\chi_{1,2}}_{\kappa_{8,[1]}};$$

the constituents have degrees 1, 22932, 519792, 2165800, 541450, 23205 and 1949220, respectively.

For the remainder of the paper we take $q = 2$; we have $G = F_4(2)$, and we write $H = D_4(2).S_3$ or ${}^3D_4(2).3$. As $T_0 = \{1\}$ we may identify W with N.

5.2. ***Subdegrees and pairing of suborbits.*** We recall that in the action of G on the left cosets of H, the suborbit containing the left coset gH is its orbit under the stabilizer H, and thus is the set of left cosets whose union is the (H, H)-double coset HgH. The size of the suborbit is called the subdegree, and is equal to $|HgH|/|H| = |H|/|H \cap {}^gH|$; the subgroup $H \cap {}^gH$ consists of the elements of G fixing both H and gH, and is known as the 2-point stabilizer. The sum of the subdegrees is the index $|G : H|$. For any double coset HgH, the set of the inverses of its elements is the double coset $Hg^{-1}H$; the corresponding suborbits are said to be paired, and if they are equal the suborbit is called self-paired.

Our goal here is to compute the subdegrees and show that all suborbits are self-paired (for which the multiplicity-freeness of the permutation character is a necessary condition).

As we shall be performing explicit calculations here, we need to know the exact location of the subgroup H of G. In the case where $H = D_4(2).S_3$ this is immediate, but in the other case where $H = {}^3D_4(2).3$ it depends on the choice of the element g_3, which we recall was chosen to lie in $\boldsymbol{A}$, commute with $\tau_3 = n_4 n_{+---}$ and satisfy ${g_3}^F.{g_3}^{-1} = \tau_3$. We begin with some comments which apply for any such choice of g_3, and in fact for all values of q.

Label the simple roots of G as

$$\alpha_1 = \epsilon_2 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_4, \quad \alpha_3 = \epsilon_4, \quad \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4);$$

then any root in Φ has the form $\sum_{i=1}^4 c_i \alpha_i$ with all $c_i \in \mathbb{Z}$. We may partition Φ according to the pair of values (c_1, c_2), and write the corresponding equivalence class as $[c_1 c_2]$. Roots in the equivalence class [00] lie in the $\tilde{A}_2$ subsystem $\Phi(\boldsymbol{A})$. There are two other types of equivalence class containing positive roots: each of [10], [13] and [23] is a singleton class containing a long root; by contrast each of [01], [11] and [12] is of size six, containing three short roots and three long roots, with the union of the class with $\Phi(\boldsymbol{A}) \cap \Phi^+$ being the set of positive roots of a C_3 subsystem (and the corresponding six root subgroups of G all commute with each other) — indeed in the case of [01] the C_3 subsystem is $\Phi(\boldsymbol{C})$. Taking similarly negative roots (and writing $-[c_1 c_2]$ for $[(-c_1)(-c_2)]$), we see that the ‘nonzero’ equivalence classes effectively form a root system $\overline{\Phi}$ of type G_2, with positive roots $\mathsf{a} = [01]$, $\mathsf{b} = [10]$, $\mathsf{a} + \mathsf{b} = [11]$, $2\mathsf{a} + \mathsf{b} = [12]$, $3\mathsf{a} + \mathsf{b} = [13]$ and $3\mathsf{a} + 2\mathsf{b} = [23]$. For each such $\mathsf{r} = [c_1 c_2] \in \overline{\Phi}$ there is a root subgroup U_{r} in ${}^3D_4(q)$, of order q^3 or q according as r is short or long.

Now return to the case where $q = 2$. Take $\lambda \in \mathbb{F}_8$ satisfying $\lambda^3 = \lambda + 1$. We choose

$$g_3 = x_4(\lambda) x_{+---}(\lambda^6) x_{+--+}(\lambda^5) h_4(\lambda^2) h_{+---}(\lambda) n_4 n_{+---} n_4 x_4(\lambda^6) x_{+---}(\lambda) x_{+--+}(\lambda^5);$$

a straightforward calculation in $\boldsymbol{A}$ shows that

$$ {g_3}^F = \tau_3 g_3 = g_3 \tau_3, $$

so that g_3 commutes with τ_3 and satisfies ${g_3}^F.{g_3}^{-1} = \tau_3$. We then have

$$^3D_4(2) = (\boldsymbol{H}^{g_3})^F = (\boldsymbol{H}^{F\tau_3})^{g_3},$$

and the group H is obtained by adjoining $\langle\tau_3\rangle$ to $^3D_4(2)$.

For each short root $\mathsf{r} \in \bar{\Phi}$, we see that τ_3 cycles both the three short roots and the three long roots in r; if $\alpha \in \Phi$ is one of the long roots, the corresponding elements of $\boldsymbol{H}^{F\tau_3}$ are $x_\alpha(\mu)x_{\tau_3{}^2(\alpha)}(\mu^2)x_{\tau_3(\alpha)}(\mu^4)$ for $\mu \in \mathbb{F}_8$. Conjugation by g_3 gives the root subgroup $U_\mathsf{r} = \{x_\mathsf{r}(t_1, t_2, t_3) : t_1, t_2, t_3 \in \mathbb{F}_2\}$ in $^3D_4(q)$, where

$$x_\mathsf{a}(t_1, t_2, t_3) = x_{3-4}(t_1)x_{3+4}(t_2)x_{1-2}(t_3)x_{+-+-}(t_1 + t_2)x_{+-++}(t_1 + t_3)x_3(t_2 + t_3),$$
$$x_{\mathsf{a+b}}(t_1, t_2, t_3) = x_{2-4}(t_1)x_{2+4}(t_2)x_{1-3}(t_3)x_{++--}(t_1 + t_2)x_{++-+}(t_1 + t_3)x_2(t_2 + t_3),$$
$$x_{\mathsf{2a+b}}(t_1, t_2, t_3) = x_{1+4}(t_1)x_{1-4}(t_2)x_{2+3}(t_3)x_{++++}(t_1 + t_2)x_{+++-}(t_1 + t_3)x_1(t_2 + t_3),$$

and $x_{-\mathsf{r}}(t_1, t_2, t_3)$ is obtained from $x_\mathsf{r}(t_1, t_2, t_3)$ by negating all roots. For each long root $\mathsf{r} \in \overline{\Phi}$, the corresponding root $\alpha \in \Phi$ is orthogonal to $\Phi(A)$, whence the root subgroup $\boldsymbol{U}_\alpha < \boldsymbol{H}$ commutes with both g_3 and τ_3; we therefore have the root subgroup $U_\mathsf{r} = \{x_\mathsf{r}(t) : t \in \mathbb{F}_2\}$ in $^3D_4(q)$, where

$$x_\mathsf{b}(t) = x_{2-3}(t), \quad x_{\mathsf{3a+b}}(t) = x_{1+3}(t), \quad x_{\mathsf{3a+2b}}(t) = x_{1+2}(t),$$

and $x_{-\mathsf{r}}(t)$ is obtained from $x_\mathsf{r}(t)$ by negating the root.

We may proceed as usual to obtain elements of the maximal torus of $^3D_4(2)$. This is a cyclic subgroup $\langle s\rangle$ of A of order 7, where

$$\begin{aligned}
s &= x_4(1)x_{+--+}(1)n_{+---}n_4x_4(1),\\
s^2 &= x_{+---}(1)x_4(1)n_4n_{+---}n_4,\\
s^3 &= x_{+---}(1)x_{+--+}(1)n_{+---}n_4x_{+--+}(1),\\
s^4 &= x_{+--+}(1)n_4n_{+---}x_{+---}(1)x_{+--+}(1),\\
s^5 &= n_4n_{+---}n_4x_4(1)x_{+---}(1),\\
s^6 &= x_4(1)n_4n_{+---}x_4(1)x_{+--+}(1);
\end{aligned}$$

according as $\mathsf{r} \in \overline{\Phi}$ is short or long, the A_1 subgroup $\langle U_\mathsf{r}, U_{-\mathsf{r}}\rangle$ either contains $\langle s\rangle$ or intersects it trivially, and either

$$^s x_\mathsf{r}(t_1, t_2, t_3) = x_\mathsf{r}(t_1 + t_2, t_1 + t_2 + t_3, t_2) \quad \text{or} \quad ^s x_\mathsf{r}(t) = x_\mathsf{r}(t).$$

We also obtain elements $n_\mathsf{r} \in {}^3D_4(q)$ for $\mathsf{r} \in \overline{\Phi}$, where

$$\begin{aligned}
n_\mathsf{a} &= n_{3-4}n_{3+4}n_{1-2}, & n_\mathsf{b} &= n_{2-3},\\
n_{\mathsf{a+b}} &= n_{2-4}n_{2+4}n_{1-3}, & n_{\mathsf{3a+b}} &= n_{1+3},\\
n_{\mathsf{2a+b}} &= n_{1+4}n_{1-4}n_{2+3}, & n_{\mathsf{3a+2b}} &= n_{1+2};
\end{aligned}$$

the subgroup $N^\dagger = \langle n_{\mathsf{r}} : \mathsf{r} \in \overline{\Phi} \rangle = \langle n_{\mathsf{a}}, n_{\mathsf{b}} \rangle$ of N is dihedral of order 12 and is the Weyl group of ${}^3D_4(q)$, and according as $\mathsf{r} \in \overline{\Phi}$ is short or long we have

$$^{n_{\mathsf{r}}}s = s^{-1} \quad \text{or} \quad ^{n_{\mathsf{r}}}s = s.$$

Finally the effect of τ_3 on all of these elements of ${}^3D_4(q)$ is as follows: it commutes with each n_{r}, and with $x_{\mathsf{r}}(t)$ if $\mathsf{r} \in \overline{\Phi}$ is long; if $\mathsf{r} \in \overline{\Phi}$ is short then

$$^{\tau_3}x_{\mathsf{r}}(t_1, t_2, t_3) = x_{\mathsf{r}}(t_3, t_1, t_2);$$

and

$$^{\tau_3}s = s^2.$$

5.2.1. *The action of $F_4(2)$ on cosets of $D_4(2).S_3$.* Here we take $H = D_4(2).S_3$. There are 9 subdegrees $|HgH|/|H|$, and they sum to $|G : H| = 3168256$; we shall prove the following.

Proposition 5.1. *In the action of $G = F_4(2)$ on cosets of $H = D_4(2).S_3$, the subdegrees are as given in Table 18, and all suborbits are self-paired.*

In the calculations in this section we shall frequently use Bruhat decomposition (see [3, Corollary 8.4.4]). Given $n \in N$, write $U_n = \prod \{U_\alpha : \alpha \in \Phi^+,\ n(\alpha) \notin \Phi^+\}$ (recall that we identify W with N). Then each element of G has a unique expression in the form unv, where $u \in U$, $n \in N$ and $v \in U_n$; and $unv \in H \iff u, v \in H$.

We shall work through the rows of Table 18 in turn. Taking $g = 1$ clearly gives the suborbit $HgH = H$, and the subdegree is 1.

Take $g = x_1(1)$, $x_{++++}(1)x_1(1)$ or $x_{+++-}(1)x_{++++}(1)x_1(1)$; then $g^2 = 1$, so the suborbit is self-paired. As g centralizes $U \cap H$, given $h = unv \in H$ we have $h \in {}^gH \iff (unv)^g \in H \iff n^g \in H \iff g^{-1}.{}^ng \in H$. We have ${}^ng = x_{n(1)}(1)$, $x_{n(++++)}(1)x_{n(1)}(1)$

g	$\lvert H \cap {}^gH \rvert$	$\frac{\lvert HgH \rvert}{\lvert H \rvert}$
1	1045094400	1
$x_1(1)$	2580480	405
$x_{++++}(1)x_1(1)$	172032	6075
$x_{+++-}(1)x_{++++}(1)x_1(1)$	73728	14175
$x_{++--}(1)x_{+-++}(1)$	10752	97200
$x_{++--}(1)x_{+-++}(1)x_2(1)$	1536	680400
$x_3(1)x_{++--}(1)x_{+-++}(1)x_2(1)$	1536	680400
$x_4(1)x_{+--+}(1)n_{+--+}x_4(1)x_{+---}(1)$	672	1555200
$x_{3+4}(1)x_{+---}(1)x_4(1)x_3(1)x_{+-+-}(1)$ $\times\, n_{3+4}n_{+--+}x_4(1)x_{+--+}(1)x_{3+4}(1)$	7776	134400
		3168256

Table 18. Suborbits and subdegrees for the action of $F_4(2)$ on cosets of $D_4(2).S_3$.

or $x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1)$, respectively; thus $g^{-1}.{}^n g \in H \iff {}^n g = g \iff n$ preserves the set $\{1\}$, $\{++++, 1\}$ or $\{+++-, ++++, 1\}$, respectively. In the first case this gives $n \in \langle n_{2-3}, n_{3-4}, n_4\rangle$; so

$$H \cap {}^g H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4}\rangle\langle U_{\pm(2-3)}, U_{\pm(3-4)}, U_{\pm(3+4)}\rangle\langle n_4\rangle,$$

of order $2q^{12}(q^2-1)(q^3-1)(q^4-1) = 2580480$, and the subdegree is 405. In the second case it gives $n \in \langle n_{2-3}, n_{3-4}, n_{+---}\rangle$; so

$$H \cap {}^g H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4}, U_{2+3}, U_{2+4}, U_{3+4}\rangle\langle U_{\pm(2-3)}, U_{\pm(3-4)}\rangle\langle n_{+---}\rangle,$$

of order $2q^{12}(q^2-1)(q^3-1) = 172032$, and the subdegree is 6075. In the third case it gives $n \in \langle n_{2-3}, n_4, n_{+---}\rangle$; so

$$H \cap {}^g H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4}, U_{2\pm 4}, U_{3\pm 4}, U_{2+3}\rangle\langle U_{\pm(2-3)}\rangle\langle n_4, n_{+---}\rangle,$$

of order $6q^{12}(q^2-1) = 73728$, and the subdegree is 14175.

In the next few cases we shall find it helpful to define a symmetric relation on the set of short roots in Φ^+: given such roots α and β, we say that α is related to β, and write $\alpha \sim \beta$, if $\alpha + \beta$ is another short root in Φ^+.

Now take $g = x_{++--}(1)x_{+-++}(1)$; then $g^{-1} = g^{n_{2-3}n_4}$, so the suborbit is self-paired. Given $h = unv \in H$ we have $h \in {}^g H \iff (unv)^g \in H \iff (g^u)^{-1}.{}^n({}^v g) \in H$. Here g centralizes the root groups $U_{1\pm 2}$, $U_{1\pm 3}$, $U_{1\pm 4}$, U_{2+3}, U_{2+4}, U_{3-4}; so conjugating g by an element of $U \cap H$ gives an element of

$$x_{++--}(1)x_{+-++}(1)U_{++-+}U_{+++-}U_{++++}U_{1+4}U_{1+3}U_{1+2}.$$

Thus if we write

$$S = \{++--, +-++, ++-+, +++-, ++++\},$$

the condition $(g^u)^{-1}.{}^n({}^v g) \in H$ forces $n(++--), n(+-++) \in S$. As the only instance of roots in S being related is

$$++-- \sim +-++,$$

we see that n must either fix or interchange $++--$ and $+-++$; but if it interchanged them then $(g^u)^{-1}.{}^n({}^v g)$ would involve $x_1(1)$ which is not in H, so n must fix them, whence $n \in \langle n_{3-4}, n_{2+4}\rangle$. As $v \in U_n$, any short root elements appearing in ${}^n({}^v g)$ apart from those in g must lie in root subgroups U_α for α of height less than that of $++--$ or $+-++$, whereas any short root elements appearing in $(g^u)^{-1}$ apart from those in g must lie in root subgroups U_α for α of height greater than that of $++--$ or $+-++$; so both u and v must centralize g. Therefore we have

$$H \cap {}^g H = \langle U_{1\pm 2}, U_{1\pm 3}, U_{1\pm 4}\rangle\langle U_{\pm(3-4)}, U_{\pm(2+4)}\rangle,$$

of order $q^9(q^2-1)(q^3-1) = 10752$, and the subdegree is 97200.

Now take $g = x_{++--}(1)x_{+-++}(1)x_2(1)$; then $g^{-1} = g^{n_{+---}}$, so the suborbit is self-paired. Given $h = unv \in H$ we again have $h \in {}^gH \iff (unv)^g \in H \iff (g^u)^{-1}.{}^n({}^vg) \in H$. Here g centralizes the root groups U_{1+2}, $U_{1\pm3}$, $U_{1\pm4}$, U_{2+3}, U_{2+4}, U_{3-4}; so conjugating g by an element of $U \cap H$ gives an element of

$$x_{++--}(1)x_{+-++}(1)x_2(1)U_{++-+}U_{+++-}U_{++++}U_1U_{1+4}U_{1+3}U_{1+2}.$$

Thus if we write

$$S = \{++--, +-++, 2, ++-+, +++-, ++++, 1\},$$

the condition $(g^u)^{-1}.{}^n({}^vg) \in H$ forces $n(++--), n(+-++), n(2) \in S$. As the only instances of roots in S being related are

$$++-- \sim +-++ \sim 2,$$

we see that n must fix $+-++$, and either fix or interchange $++--$ and 2; thus $n \in \langle n_{3-4}, n_{+---}\rangle$, and as $v \in U_n \cap H$ we must have $v \in U_{3-4}$ so that v centralizes g. If $n \in \langle n_{3-4}\rangle$ then n also centralizes g, as then must u; if instead $n \in n_{+---}\langle n_{3-4}\rangle$ then as $x_{1-2}(1)x_{3+4}(1)n_{+---}$ centralizes g we see that $x_{1-2}(1)x_{3+4}(1)u$ must. Therefore we have

$$H \cap {}^gH = \langle U_{1+2}, U_{1\pm3}, U_{1\pm4}, U_{2+3}, U_{2+4}, x_{1-2}(1)x_{3+4}(1)n_{+---}\rangle\langle U_{\pm(3-4)}\rangle,$$

of order $q^9(q^2-1) = 1536$, and the subdegree is 680400.

Now take $g = x_3(1)x_{++--}(1)x_{+-++}(1)x_2(1)$; then $g^{-1} = g^{n_{2-3}n_4}$, so the suborbit is self-paired. Given $h = unv \in H$ we again have $h \in {}^gH \iff (unv)^g \in H \iff (g^u)^{-1}.{}^n({}^vg) \in H$. Here g centralizes the root groups U_{1+2}, U_{1+3}, $U_{1\pm4}$, U_{2+3}, U_{2+4}, U_{3-4}, while its commutator with $x_{1-2}(1)x_{1-3}(1)$ is $x_{1+3}(1)x_{1+2}(1)$; so conjugating g by an element of $U \cap H$ gives an element of

$$x_3(1)x_{++--}(1)x_{+-++}(1)x_2(1+t)x_{++-+}(t)U_{+++-}U_{++++}U_1U_{2+3}U_{1+4}U_{1+3}U_{1+2},$$

where the projection of the element of $U \cap H$ on the root group U_{2-3} is $x_{2-3}(t)$. Thus if we write

$$S = \{3, ++--, +-++, 2, ++-+, +++-, ++++, 1\},$$

the condition $(g^u)^{-1}.{}^n({}^vg) \in H$ forces $n(3), n(++--), n(+-++), n(\alpha) \in S$, where α is 2 or $++-+$ according as the projection of v on the root group U_{2-3} is trivial or not. As the only instances of roots in S being related are

$$++-+ \sim 3 \sim ++-- \sim +-++ \sim 2,$$

if α is 2 the chain $n(3), n(++--), n(+-++), n(2)$ must be $3, ++--, +-++, 2$ or its reverse, or $++-+, 3, ++--, +-++$ or its reverse, while if α is $++-+$ the same must be true of the chain $n(++-+), n(3), n(++--), n(+-++)$. In each case this uniquely determines n, which we find lies in $\langle n_{2-3}, n_4, n_{+---}\rangle$; as $v \in U_n$ this forces $v \in U_{2-3}$. If α is 2 we must then have $v = 1$; if however α is $++-+$ we must then have $v = x_{2-3}(1)$, which eliminates two of the four possibilities since they have $n \in \langle n_4, n_{+---}\rangle$. There are therefore six possibilities for n, and for each we may find an element u giving $(g^u)^{-1}.{}^n({}^v g) \in H$: for example, if $n = n_{2-3}n_4$ then $v = 1$ and we may take $u = x_{1-3}(1)x_{2-4}(1)x_{3+4}(1)$, while if $n = n_{2-3}n_{+---}n_4$ then $v = x_{2-3}(1)$ and we may take $u = 1$. Therefore we have

$$H \cap {}^gH = \langle U_{1+2}, U_{1+3}, U_{1\pm 4}, U_{2+3}, U_{2+4}, U_{3-4}, x_{1-2}(1)x_{1-3}(1)\rangle \\ .\langle x_{1-3}(1)x_{2-4}(1)x_{3+4}(1)n_{2-3}n_4, n_{2-3}n_{+---}n_4x_{2-3}(1)\rangle,$$

of order $6q^8 = 1536$, and the subdegree is 680400. Note that the center of the 2-point stabilizer here is trivial, while that of the 2-point stabilizer in the previous case is U_{1+2}; so the two suborbits must be distinct.

At this point we note that the remaining two subdegrees sum to 1689600, and of course each divides $|H| = 2^{13}3^6 5^2 7$. It is now a simple matter to determine the pairs of factors of $|H|$ with the correct sum (the larger must lie in the range [844800, 1689600), so given each of the 42 possibilities for the powers of 3, 5 and 7 there is at most one for the power of 2): we find they are (1658880, 30720), (1382400, 307200), (1555200, 134400), (860160, 829440), (1075200, 614400) and (1612800, 76800). (In fact by using [29, Theorem 30.1(C)] we could reduce this list of six pairs to the third and fifth, but we shall see that this is unnecessary.)

Now take $g = x_4(1)x_{+--+}(1)n_{+--+}x_4(1)x_{+---}(1)$; then $g^{-1} = gn_4$, so the suborbit is self-paired. Since $g \in A$ it is immediate that g commutes with $\langle U_{\pm(2-3)}, U_{\pm(1+3)}\rangle$, and it certainly commutes with its square n_4; moreover calculation in C shows that $(n_{1-2}n_{3-4}n_{3+4})^g = n_{1-2}n_3$. Thus we have

$$H \cap {}^gH \geq \langle U_{\pm(2-3)}, U_{\pm(1+3)}\rangle\langle n_{1-2}n_{3-4}n_{3+4}, n_4\rangle,$$

of order $4q^3(q^2-1)(q^3-1) = 672$, and the subdegree divides 1555200. Since none of the (nontrivial) 2-point stabilizers found to date contains a group $A_2(q)$ with a graph automorphism, the suborbit is one of the remaining two; as the only one of the twelve possible subdegrees given in the previous paragraph which divides 1555200 is 1555200 itself, we have equality in the previous sentence in both the 2-point stabilizer and the subdegree; moreover the remaining subdegree must be 134400, so the 2-point stabilizer must be of order 7776.

For the final suborbit take

$$g = x_{3+4}(1)x_{+---}(1)x_4(1)x_3(1)x_{+-+-}(1)n_{3+4}n_{+--+}x_4(1)x_{+--+}(1)x_{3+4}(1);$$

then $g^2 = 1$, so the suborbit is self-paired. Clearly g commutes with $\langle U_{\pm(1+2)}\rangle$; calculation shows that g also commutes with n_4, and that conjugation by g multiplies $x_{1-2}(1)x_{3-4}(1)x_{3+4}(1)$ by n_4, and it sends n_{+---} to $x_{3-4}(1)n_{3-4}n_{3+4}x_{3+4}(1)$, and

$x_{2-3}(1)x_{2-4}(1)x_{2+4}(1)x_{1-3}(1)x_{1+3}(1)$ to $x_{2-3}(1)x_{2+3}(1)x_{1-4}(1)x_{1+4}(1)x_{1+3}(1)$. As g has order 2, it follows that each of these elements lies in $H \cap {}^gH$. The last two of them generate a quaternion group of order 8 with center U_{1+2}; adding n_{1+2} extends it to a group $\mathrm{PSU}_3(2)$ of order 72. The group generated by $x_{3-4}(1)n_{3-4}n_{3+4}x_{3+4}(1)$ and its images under $\langle n_4, n_{+---}\rangle$ is $\mathbb{Z}_3 \times \mathbb{Z}_3$ of order 9; adding $x_{1-2}(1)x_{3-4}(1)x_{3+4}(1)$ extends it to $(\mathbb{Z}_3 \times \mathbb{Z}_3).2$ of order 18, and this group normalizes $\mathrm{PSU}_3(2)$. The group $\langle n_4, n_{+---}\rangle$ commutes with $\mathrm{PSU}_3(2)$ and normalizes $(\mathbb{Z}_3 \times \mathbb{Z}_3).2$, so we have a group of order $72 \cdot 18 \cdot 6 = 7776$. Since none of the (nontrivial) 2-point stabilizers found to date has order divisible by 7776, the suborbit is indeed the remaining one. As conjugation by $x_{1-2}(1)x_{3-4}(1)x_{3+4}(1)$ takes the first generator of the quaternion group to the inverse of the second, we have

$$H \cap {}^gH = \langle x_{2-3}(1)x_{2-4}(1)x_{2+4}(1)x_{1-3}(1)x_{1+3}(1), n_{1+2}, \\ x_{3-4}(1)n_{3-4}n_{3+4}x_{3+4}(1), x_{1-2}(1)x_{3-4}(1)x_{3+4}(1), n_4, n_{+---}\rangle,$$

of order 7776, and the subdegree is 134400.

This concludes the proof of Proposition 5.1.

5.2.2. *The action of $F_4(2)$ on cosets of ${}^3D_4(2).3$.* Here we take $H = {}^3D_4(2).3$. There are 7 subdegrees $|HgH|/|H|$, and they sum to $|G : H| = 5222400$; we shall prove the following.

Proposition 5.2. *In the action of $G = F_4(2)$ on cosets of $H = {}^3D_4(2).3$, the subdegrees are as given in Table 19, and all suborbits are self-paired.*

In the calculations in this section we shall again use Bruhat decomposition, but here we shall require a slightly different form. First set

$$U_A = U \cap A = \prod\{U_\alpha : \alpha \in \Phi^+ \cap \Phi(\boldsymbol{A})\}, \quad U' = \prod\{U_\alpha : \alpha \in \Phi^+ \setminus \Phi(\boldsymbol{A})\};$$

then $U = U_A U' = U' U_A$ and $U_A \cap U' = \{1\}$. Next recall that we have the Weyl groups $W_{\boldsymbol{A}}$ and $W_{\boldsymbol{H}}$, with $W = W_{\boldsymbol{A}} W_{\boldsymbol{H}} = W_{\boldsymbol{H}} W_{\boldsymbol{A}}$ and $W_{\boldsymbol{A}} \cap W_{\boldsymbol{H}} = \{1\}$. Now given

g	$\lvert H \cap {}^gH\rvert$	$\frac{\lvert HgH\rvert}{\lvert H\rvert}$
1	634023936	1
$x_{+++-}(1)x_{++++}(1)x_1(1)$	36864	17199
n_{+--+}	36288	17472
$x_{+++-}(1)x_{++++}(1)x_1(1)n_{+--+}$	576	1100736
$x_{++--}(1)x_{+-++}(1)$	768	825552
$x_4(1)n_{3-4}$	216	2935296
$x_{1-2}(1)x_4(1)n_{3-4}$	1944	326144
		5222400

Table 19. Suborbits and subdegrees for the action of $F_4(2)$ on cosets of ${}^3D_4(2).S_3$.

an element of $N = W$ we may write it uniquely as $n_A n$ with $n_A \in W_{\boldsymbol{A}}$, $n \in W_{\boldsymbol{H}}$; as before we have $U_{n_A n} = \prod \{U_\alpha : \alpha \in \Phi^+,\ n_A n(\alpha) \notin \Phi^+\}$. Set

$$\begin{aligned}(U_{n_A n})_A &= \prod \{U_\alpha : \alpha \in \Phi^+,\ n_A n(\alpha) \notin \Phi^+,\ n(\alpha) \in \Phi(\boldsymbol{A})\},\\ (U_n)' &= \prod \{U_\alpha : \alpha \in \Phi^+,\ n_A n(\alpha) \notin \Phi^+,\ n(\alpha) \notin \Phi(\boldsymbol{A})\}\\ &= \prod \{U_\alpha : \alpha \in \Phi^+,\ n(\alpha) \notin \Phi^+ \cup \Phi(\boldsymbol{A})\}\end{aligned}$$

(where the final line justifies the notation $(U_n)'$); then $U_{n_A n} = (U_{n_A n})_A (U_n)' = (U_n)'(U_{n_A n})_A$ and $(U_{n_A n})_A \cap (U_n)' = \{1\}$. Standard Bruhat decomposition now shows that any element of G may be uniquely written as $(uu_A)(n_A n)(v_A v)$ with $u \in U'$, $u_A \in U_A$, $n_A \in W_{\boldsymbol{A}}$, $n \in W_{\boldsymbol{H}}$, $v_A \in (U_{n_A n})_A$ and $v \in (U_n)'$; setting $a = u_A n_A{}^n v_A \in A$ we see that the element may be written as $uanv$ with $u \in U'$, $a \in A$, $n \in W_{\boldsymbol{H}}$ and $v \in (U_n)'$.

To see that this last expression is unique, suppose we have $uanv = \hat{u}\hat{a}\hat{n}\hat{v}$ with $u, \hat{u} \in U'$, $a, \hat{a} \in A$, $n, \hat{n} \in W_{\boldsymbol{H}}$ and $v, \hat{v} \in (U_n)'$. Writing the left side in standard Bruhat decomposition gives an expression $(uu_A)(n_A n)(v_A v)$, where $u_A \in U \cap A$, $n_A \in W_{\boldsymbol{A}}$ and $v_A \in U_{n_A n}$; doing the same with the right side and using uniqueness of standard Bruhat decomposition we see that $uu_A = \hat{u}\hat{u}_A \in U$ and $n_A n = \hat{n}_A \hat{n} \in N$, and then the factorizations of the previous paragraph give $u = \hat{u}$ and $n = \hat{n}$. Therefore $anv = \hat{a}n\hat{v}$, so $\hat{a}^{-1}a = n\hat{v}v^{-1}n^{-1}$; the left side here is in A and the right side is in ${}^n(U_n)'$, which is a product of negative root subgroups each lying outside A, so both sides must be the identity, whence $a = \hat{a}$ and $v = \hat{v}$. We therefore have uniqueness in the expression $uanv$. Since by [3, Proposition 13.5.3] each element of H may be written in the form $u_1 s^i n_1 v_1 \tau_3{}^j = u_1 s^i \tau_3{}^j n_1 v_1{}^{\tau_3{}^j}$, where $u_1 \in U \cap H < U'$, $s^i \tau_3{}^j \in A \cap H$, $n_1 \in N^\dagger < W_{\boldsymbol{H}}$ and $v_1{}^{\tau_3{}^j} \in U_{n_1} \cap H < (U_{n_1})'$, we see that $uanv \in H \iff u, a, n, v \in H$.

In some cases our approach will require the following. We have the Weyl group $W_{\boldsymbol{C}}$ of $\boldsymbol{C}$, and $|W : W_{\boldsymbol{C}}| = 24$. Recall from [3, Theorem 2.5.8, Corollary 2.5.9] that there is a set of right coset representatives of $W_{\boldsymbol{C}}$ in W, each of which is of minimal length in its coset; we shall call these $n_{(1)}, \ldots, n_{(24)}$. (Note that [3] uses left cosets instead of right cosets, so our elements are the inverses of those there.) Likewise we have $|W_{\boldsymbol{H}} : W_{\boldsymbol{H}} \cap W_{\boldsymbol{C}}| = 24$, where $W_{\boldsymbol{H}} \cap W_{\boldsymbol{C}}$ is the Weyl group of $\boldsymbol{H} \cap \boldsymbol{C}$, which is a Levi subgroup of $\boldsymbol{H}$ of type $A_1{}^3$; if we regard $\epsilon_1 - \epsilon_2$, $\epsilon_2 - \epsilon_3$, $\epsilon_3 - \epsilon_4$, $\epsilon_3 + \epsilon_4$ as simple roots of $\boldsymbol{H}$ then again we have coset representatives of $W_{\boldsymbol{H}} \cap W_{\boldsymbol{C}}$ in $W_{\boldsymbol{H}}$ of minimal length, which we shall call $n_{(1)}', \ldots, n_{(24)}'$. We choose notation such that for all j we have $n_{(j)}' \in W_{\boldsymbol{C}} n_{(j)}$.

Now as $N^\dagger \cap W_{\boldsymbol{C}} = \langle n_{\mathsf{a}} \rangle$ we have $|N^\dagger : N^\dagger \cap W_{\boldsymbol{C}}| = 6$, so that just six of the cosets $W_{\boldsymbol{C}} n_{(j)}$ contain elements of $N^\dagger$; write $J = \{j \leq 24 : N^\dagger \cap W_{\boldsymbol{C}} n_{(j)} \neq \varnothing\}$. The elements $n_{(j)}'$ for $j \in J$ are $1, n_{\mathsf{b}}, n_{\mathsf{b}} n_{\mathsf{a}}, n_{\mathsf{a+b}}, n_{\mathsf{a+b}} n_{\mathsf{a}}, n_{\mathsf{3a+2b}}$, each of which in fact lies in $N^\dagger$; we have $n_{(j)}' = n_{(j)}$ in the first, second, fifth and sixth of these cases and ${n_{(j)}}' = n_{+--+}n_{(j)}$ in the third and fourth. As roots in $\Phi(\boldsymbol{A})$ are either fixed or negated by elements of $N^\dagger$, we see that for $j \in J$ we have either $U_{{n_{(j)}}'} = U_{n_{(j)}}$ or $U_{{n_{(j)}}'} = U_A U_{n_{(j)}}$, whence in either case $(U_{{n_{(j)}}'})' = U_{n_{(j)}}$.

Write $P = UC$, so that P is a maximal parabolic subgroup of G. For each $j \leq 24$ we have $Pn_{(j)} = P{n_{(j)}}'$; moreover G is the disjoint union of the double cosets $Pn_{(j)}U$ for $j \leq 24$, and for fixed j the double coset $Pn_{(j)}U$ is the disjoint union of the cosets $Pn_{(j)}v$ as v runs through $U_{n_{(j)}}$. Given such a coset $Pn_{(j)}v$, if $j \in J$ and $v \in H$ then evidently the coset meets H; we claim that the converse is also true.

Thus suppose we have $h \in Pn_{(j)}v \cap H$, and as above write $h = uanv_0$ with $u \in U' \cap H$, $a \in A \cap H$, $n \in N^\dagger$ and $v_0 \in (U_n)' \cap H$; then $Pn_{(j)}v = Ph = Pnv_0$, so $n \in W_{\boldsymbol{C}}n_{(j)} = W_{\boldsymbol{C}}{n_{(j)}}'$, whence $j \in J$ and n is either ${n_{(j)}}'$ or $n_{\mathsf{a}}{n_{(j)}}'$. In the former case we have $v_0 \in (U_{{n_{(j)}}'})' = U_{n_{(j)}}$, so as $Pn_{(j)}v = P{n_{(j)}}'v_0 = Pn_{(j)}v_0$ we must have $v = v_0 \in H$ as required. In the latter case we have $v_0 \in (U_{n_{\mathsf{a}}{n_{(j)}}'})'$, and we may write $nv_0 = n_{\mathsf{a}}{n_{(j)}}'v_0 = n_{\mathsf{a}}v_1.{n_{(j)}}'v_2$, where $v_1 \in \prod_{\alpha \in [01]} U_\alpha < P$ and $v_2 \in (U_{{n_{(j)}}'})' = U_{n_{(j)}}$; as ${v_1}^{{n_{(j)}}'}.v_2 = v_0 \in H$ and the sets of roots involved in ${v_1}^{{n_{(j)}}'}$ and v_2 are disjoint, both v_1 and v_2 must lie in H. Thus

$$Pn_{(j)}v = Pnv_0 = P{n_{(j)}}'v_2 = Pn_{(j)}v_2$$

with $v_2 \in U_{n_{(j)}}$, so we must have $v = v_2 \in H$ as required. We have thus shown that the converse is indeed true; we shall use this in some of the arguments in this section.

We shall work through the rows of Table 19 in turn. Taking $g = 1$ clearly gives the suborbit $HgH = H$, and the subdegree is 1.

Take $g = x_{+++-}(1)x_{++++}(1)x_1(1)$; then $g^2 = 1$, so the suborbit is self-paired. As g centralizes $U \cap H$, given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g \in H \iff (an)^g \in H \iff (g^a)^{-1}.{}^ng \in H$. We have ${}^ng = x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1)$, while τ_3 and g commute and $\langle s\rangle$ acts simply transitively on $U_{+++-}U_{++++}U_1 \setminus \{1\}$; thus $(g^a)^{-1}.{}^ng \in H \iff g^a = {}^ng = g \iff a \in \langle \tau_3 \rangle$ and n preserves the set $\{+++-, ++++, 1\}$, which forces $n \in \langle n_{\mathsf{b}} \rangle$. So

$$H \cap {}^gH = \langle U \cap H, n_{\mathsf{b}} \rangle \langle \tau_3 \rangle,$$

of order $3q^{12}(q^2 - 1) = 36864$, and the subdegree is 17199.

Take $g = n_{+--+}$; then $g^2 = 1$, so the suborbit is self-paired. As g preserves U', A and $N^\dagger$, given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g = u^g a^g n^g v^g \in H \iff u^g, a^g, n^g, v^g \in H$. For all $n \in N^\dagger$ we have $n^g = n$; calculation shows that $s^g \notin H$, while $(\tau_3)^g = {\tau_3}^{-1}$; and given $\mathsf{r} \in \overline{\Phi}$, if r is long then g commutes with U_{r}, while if r is short then g preserves the set of roots α lying in r, and ${U_{\mathsf{r}}}^g \cap H = \langle x_{\mathsf{r}}(1, 1, 1)\rangle$. So

$$H \cap {}^gH = \langle U_{\mathsf{b}}, U_{-\mathsf{b}}, x_{\mathsf{a}}(1, 1, 1), x_{-\mathsf{a}}(1, 1, 1)\rangle \langle \tau_3 \rangle,$$

of order $3q^6(q^2 - 1)(q^6 - 1) = 36288$, and the subdegree is 17472.

Now take $g = x_{+++-}(1)x_{++++}(1)x_1(1)n_{+--+}$; then $g^2 = 1$, so the suborbit is self-paired. Write $g' = x_{+++-}(1)x_{++++}(1)x_1(1)$; using the comments of the previous two paragraphs about g' and n_{+--+} centralizing and inverting certain elements, we see that given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g \in H \iff (u^{n_{+--+}})g'(a^{n_{+--+}})ng'v^{n_{+--+}} \in H$, with $(u^{n_{+--+}})g' \in U'$ and $a^{n_{+--+}} \in A$. If $n(1) \notin \Phi^+$ then as $n \in N^\dagger$ and $g'v^{n_{+--+}} \in (U_n)'$ the expression is in the desired form; if instead $n(1) \in \Phi^+$ we rewrite it as

$$(u^{n_{+--+}})g'(x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1))^{a^{n_{+--+}}}.a^{n_{+--+}}.n.v^{n_{+--+}}.$$

In either case the element of A is $a^{n_{+--+}}$; for this to be in H we require $a \in \langle\tau_3\rangle$, so $a^{n_{+--+}} \in \langle\tau_3\rangle$. Now observe that if $\mathsf{r} \in \overline{\Phi}$ is a short root, then in each element of U_{r} the sum of the coefficients in the three short root subgroups is 0, and n_{+--+} permutes these coefficients. Thus $g'v^{n_{+--+}}$ cannot lie in H, so if $n(1) \notin \Phi^+$ we cannot have $(uanv)^g \in H$; assuming $n(1) \in \Phi^+$ we have $(uanv)^g \in H \iff u^{n_{+--+}}g'(x_{n(+++-)}(1)x_{n(++++)}(1)x_{n(1)}(1))^{a^{n_{+--+}}}, v^{n_{+--+}} \in H$, and the first of these implies that n preserves the set $\{+++-, ++++, 1\}$, which forces $n \in \langle n_{\mathsf{b}}\rangle$. It now follows that

$$H \cap {}^gH = \langle x_{\mathsf{a}}(1,1,1), x_{\mathsf{a+b}}(1,1,1), x_{\mathsf{2a+b}}(1,1,1)\rangle U_{\mathsf{3a+b}}U_{\mathsf{3a+2b}}\langle U_{\pm\mathsf{a}}\rangle\langle\tau_3\rangle,$$

of order $3q^6(q^2-1) = 576$, and the subdegree is 1100736.

Now take $g = x_{++--}(1)x_{+-++}(1)$; then $g^{-1} = g^{x_{\mathsf{a+b}}(0,1,1)}$, so the suborbit is self-paired. Given $h = uanv \in H$ we have $h \in {}^gH \iff (uanv)^g \in H \iff (g^{ua})^{-1}({}^n({}^vg)) \in H$. Here g centralizes the root groups $U_{\mathsf{2a+b}}, U_{\mathsf{3a+b}}, U_{\mathsf{3a+2b}}$; so conjugating g by an element of $U \cap H$ gives an element of

$$x_{++--}(1)x_{+-++}(1)U_{++-+}U_{+++-}U_{++++}U_1U_{1+4}U_{1+3}U_{1+2}.$$

Thus the projection of ${}^vg$ on the product of the root groups U_α for $\alpha \in \mathsf{a}$ or for $\alpha \in \mathsf{a+b}$ involves short root elements but no long root elements; so the condition $(g^{ua})^{-1}({}^n({}^vg)) \in H$ forces $n \in N^\dagger$ to send both a and $\mathsf{a+b}$ to positive roots in $\overline{\Phi}$, whence $n \in \langle n_{\mathsf{b}}\rangle$ and so $v \in U_{\mathsf{b}}$. A straightforward calculation shows that $h' = x_{\mathsf{a+b}}(1,0,1)x_{\mathsf{b}}(1)n_{\mathsf{b}}s^2\tau_3 \in H \cap {}^gH$. Thus if $n = n_{\mathsf{b}}$, according as $v = 1$ or $v = x_{\mathsf{b}}(1)$ we may multiply h on the right by h'^{-1} or h' to reduce to the case where $n = 1$; so we may assume $(g^{ua})^{-1}g \in H$. The projection of g^u on the product of the root groups whose roots lie in a is $x_{+-++}(1)$, which thus must be centralized by a; this forces $a \in \{1, s^3\tau_3, s^2{\tau_3}^2\}$. Likewise the projection of g^u on the product of the root groups whose roots lie in $\mathsf{a+b}$ is either $x_{++--}(1)$ or $x_{++--}(1)x_{++-+}(1)$; since conjugating each of these by either $s^3\tau_3$ or $s^2{\tau_3}^2$ gives a term $x_2(1)$, we must have $a = 1$, so $(g^u)^{-1}g \in H$. After calculation in $U \cap H$ it now follows that

$$H \cap {}^gH = \{x_{\mathsf{a}}(t_1,t_2,t_3)x_{\mathsf{a+b}}(t_2+t_3, t_1+t_2, t_1) : t_i \in \mathbb{F}_2\}U_{\mathsf{2a+b}}U_{\mathsf{3a+b}}U_{\mathsf{3a+2b}}$$
$$.\langle x_{\mathsf{a+b}}(1,0,1)x_{\mathsf{b}}(1)n_{\mathsf{b}}s^2\tau_3\rangle,$$

of order $3q^8 = 768$, and the subdegree is 825552.

Finally take $g = x_4(1)n_{3-4}$ or $x_{1-2}(1)x_4(1)n_{3-4}$; then $g^{-1} = g^{x_{\mathsf{a}}(0,0,1)}$, so the suborbit is self-paired. Here we shall use the approach involving cosets of P. Take $h = uanv \in H$ as usual and consider $(uanv)^g \in (Pnv)^g = Pnvn_{3-4}x_3(1)x_{1-2}(\delta)$ where $\delta = 0$ or 1. If v involves the term $x_{3-4}(1)$ we must have either $n \in \langle n_{\mathsf{a}}\rangle n_{\mathsf{b}}n_{\mathsf{a}}$ or $n \in \langle n_{\mathsf{a}}\rangle n_{\mathsf{a+b}}n_{\mathsf{a}}$. In the former possibility we have $v = x_{\mathsf{a}}(1, t_2, t_3)x_{\mathsf{3a+b}}(t_4)$ for some $t_i \in \mathbb{F}_2$: conjugating each term in v other than $x_{3-4}(1)$ by n_{3-4} and using the relation $n_{3-4}x_{3-4}(1)n_{3-4} = x_{3-4}(1)n_{3-4}x_{3-4}(1)$ gives the coset

$$\begin{aligned}&Pn_{2-3}n_{1-2}n_{3+4}x_{3-4}(1)n_{3-4}x_{3-4}(1)x_{3+4}(t_2)x_{1-2}(t_3)\\&\times x_{+--+}(1+t_2)x_{+-++}(1+t_3)x_4(t_2+t_3)x_{1+4}(t_4)x_3(1)x_{1-2}(\delta);\end{aligned}$$

now moving all possible terms to the left gives

$$\begin{aligned}&Pn_{2-3}n_{1-2}n_{3+4}n_{3-4}x_{3-4}(1)x_{3+4}(t_3)x_{1-2}(1+t_2+t_3+\delta)\\&\times x_{+-+-}(1+t_2)x_{+-++}(1+t_2t_3)x_3(1+t_2+t_3)x_{1+3}(t_4).\end{aligned}$$

The sum of the coefficients in the three short root subgroups is $1 + t_2t_3 + t_3$, which is 0 only when $t_3 = 1$ and $t_2 = 0$; but then the sum of the coefficients in the three root subgroups U_{3-4}, U_{3+4} and U_{+-+-} is not 0. Thus the product of the root elements with roots in a does not lie in H, so the coset $Pnvn_{3-4}x_3(1)$ contains no elements of H. In the latter possibility we have $v = x_{\mathsf{a}}(1, t_2, t_3)x_{\mathsf{2a+b}}(t_4, t_5, t_6)x_{\mathsf{3a+b}}(t_7)x_{\mathsf{3a+2b}}(t_8)$ for some $t_i \in \mathbb{F}_2$; although the expression is more complicated, the above approach gives on the right the same root elements with roots in a, and thus yields the same conclusion. Thus we may assume that v does not involve the term $x_{3-4}(1)$, so the coset is $Pnn_{3-4}(v^{n_{3-4}})x_3(1)x_{1-2}(\delta)$. Unless $n \in \langle n_{\mathsf{a}}\rangle$ or $\langle n_{\mathsf{a}}\rangle n_{\mathsf{3a+2b}}$ we see that nn_{3-4} is an element $n_{(j)}{}'$ for some $j \notin J$, so again the coset contains no elements of H; we have therefore reduced to the possibilities where $n \in \{1, n_{\mathsf{a}}, n_{\mathsf{3a+2b}}, n_{\mathsf{a}}n_{\mathsf{3a+2b}}\}$. From now on we treat the two cases $\delta = 0$ and $\delta = 1$ separately, although we shall see that there are considerable similarities between them.

First assume $\delta = 0$, so that $g = x_4(1)n_{3-4}$. If $n = 1$ then $v = 1$; a straightforward calculation shows that $\{ua : (ua)^g \in H\}$ is

$$Q = \langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(1, 1, 1)x_{\mathsf{2a+b}}(0, 0, 1), x_{\mathsf{a+b}}(1, 0, 0)x_{\mathsf{2a+b}}(1, 0, 1)x_{\mathsf{3a+b}}(1)\rangle,$$

a quaternion group with center $U_{\mathsf{3a+2b}}$. If $n = n_{\mathsf{a}}$ we obtain two further cosets of Q, containing the element $h_0 = x_{\mathsf{a}}(1, 1, 1)sn_{\mathsf{a}}x_{\mathsf{a}}(0, 1, 1)$ and its inverse (note that h_0 has order 3, and centralizes Q). If $n = n_{\mathsf{3a+2b}}$ then as $g \in C$ it commutes with n, so we obtain all elements q_1nq_2 with $q_1, q_2 \in Q$. Finally if $n = n_{\mathsf{a}}n_{\mathsf{3a+2b}}$ we obtain all elements $h_0{}^{\pm 1}q_1nq_2$. Thus

$$\begin{aligned}H \cap {}^gH = \big\langle &x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(1, 1, 1)x_{\mathsf{2a+b}}(0, 0, 1), x_{\mathsf{a+b}}(1, 0, 0)x_{\mathsf{2a+b}}(1, 0, 1)x_{\mathsf{3a+b}}(1),\\&x_{\mathsf{a}}(1, 1, 1)sn_{\mathsf{a}}x_{\mathsf{a}}(0, 1, 1), n_{\mathsf{3a+2b}}\big\rangle,\end{aligned}$$

of order $q^3(q^2-1)(q^3+1)=216$, and the subdegree is 2935296.

Now assume $\delta=1$, so that $g=x_{1-2}(1)x_4(1)n_{3-4}$. Here we first note that $(x_{\mathsf{a}}(0,1,1)s^3\tau_3)^g=s^3\tau_3$, so it suffices to work in ${}^3D_4(2)$. If $n=1$ then $v=1$; a straightforward calculation shows that $\{us^i : (us^i)^g \in H\}$ is

$$Q=\langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(0,0,1)x_{\mathsf{2a+b}}(0,0,1),\, x_{\mathsf{a+b}}(1,1,0)x_{\mathsf{2a+b}}(0,1,0)x_{\mathsf{3a+b}}(1)\rangle,$$

a quaternion group with center $U_{\mathsf{3a+2b}}$. If $n=n_{\mathsf{a}}$ we obtain eight further cosets of Q, containing the element $h_0=x_{\mathsf{a}}(1,0,0)n_{\mathsf{a}}x_{\mathsf{a}}(0,1,0)$ and its powers (note that h_0 has order 9, and normalizes Q). If $n=n_{\mathsf{3a+2b}}$ then as $g\in C$ it commutes with n, so we obtain all elements q_1nq_2 with $q_1,q_2\in Q$. Finally if $n=n_{\mathsf{a}}n_{\mathsf{3a+2b}}$ we obtain all elements ${h_0}^iq_1nq_2$ for $1\le i\le 8$. Thus

$$H\cap {}^gH=\big\langle x_{\mathsf{b}}(1)x_{\mathsf{a+b}}(0,0,1)x_{\mathsf{2a+b}}(0,0,1),\, x_{\mathsf{a+b}}(1,1,0)x_{\mathsf{2a+b}}(0,1,0)x_{\mathsf{3a+b}}(1),\\ x_{\mathsf{a}}(1,0,0)n_{\mathsf{a}}x_{\mathsf{a}}(0,1,0),\, n_{\mathsf{3a+2b}}\big\rangle\langle x_{\mathsf{a}}(0,1,1)s^3\tau_3\rangle,$$

of order $9q^3(q^2-1)(q^3+1)=1944$, and the subdegree is 326144.

This concludes the proof of Proposition 5.2.

5.3. *Distance-transitive graphs.* We recall further that in the action of G on the left cosets of a maximal subgroup H, given a self-paired suborbit corresponding to a double coset HgH which is not simply H itself, we may obtain a graph as follows: the vertices are the left cosets $g'H$ for $g'\in G$, and there is an edge between the vertices $g'H$ and $g''H$ if and only if $g'^{-1}g''\in HgH$ (note that this makes sense because the suborbit is self-paired). The graph is regular, of valency $|HgH|/|H|$; it is connected as H is a maximal subgroup of G; and G acts transitively on it.

If we consider the vertex H itself, the vertices at distance 1 from H are the cosets lying in HgH, those at distance 2 from H are those lying in $HgHgH$ which are not at distance 0 or 1, and so on. Writing r for the rank of the action, the graph is distance-transitive if, for each $i<r$, the left cosets at distance i from H form a single suborbit; in this case we may order the subdegrees $k_0,k_1,\dots,k_{r-1}$ so that the number of left cosets at distance i from H is k_i (so that $k_0=1$, and $k_1=|HgH|/|H|$).

Our goal here is to show that for no choice of suborbit HgH the graph is distance-transitive. To do this we shall make use of [2, Proposition 5.1.1], which among other things implies the following: if the graph as above is distance-transitive with $r\ge 4$,

(i) there exist h,l with $1\le h\le l\le r-1$ such that $1<k_1<\dots<k_h=\dots=k_l>\dots>k_{r-1}$, and

(ii) if $i<j$ and $i+j\le r-1$ then $k_i\le k_j$.

5.3.1. *The action of $F_4(2)$ on cosets of $D_4(2).S_3$.* Here we take $H=D_4(2).S_3$. We have $r=9$; our result is the following.

Proposition 5.3. *The action of $G = F_4(2)$ on cosets of $H = D_4(2).S_3$ gives rise to no distance-transitive graph.*

Proof. Suppose the statement is false. The smallest nontrivial subdegree is 405, corresponding to the suborbit HgH where $g = x_1(1)$. If we had $k_1 = 405$ then $HgHgH$ would contain only one double coset other than H or HgH; but we have

$$x_1(1)x_{++++}(1) = g.n_{+---}.g.n_{+---} \in HgHgH$$

and

$$x_{++--}(1)x_{+-++}(1) = n_{+-++}.g.n_{+-++}n_{++--}.g.n_{++--} \in HgHgH,$$

and the two left-hand sides lie in the double cosets corresponding to subdegrees 6075 and 97200. Thus by (i) above we must have $k_8 = 405$. The next smallest subdegree is 6075, corresponding to the suborbit HgH where $g = x_1(1)x_{++++}(1)$. By (ii) above we cannot have $k_7 = 6075$ as this would force $k_1 > k_7$, so by (i) above we must have $k_1 = 6075$; but

$$x_1(1) = n_{+-++}x_{-3-4}(1).g.x_{-3-4}(1).g.x_{1+2}(1)n_{+-++} \in HgHgH,$$

so that $x_1(1)H$ would be at distance 2 from H instead of 8. This contradiction proves the result. □

5.3.2. *The action of $F_4(2)$ on cosets of ${}^3D_4(2).3$.* Here we take ${}^3D_4(2).3$. We have $r = 7$; our result is the following.

Proposition 5.4. *The action of $G = F_4(2)$ on cosets of $H = {}^3D_4(2).3$ gives rise to no distance-transitive graph.*

Proof. Suppose the statement is false. The smallest nontrivial subdegree is 17199, corresponding to the suborbit HgH where $g = x_{+++-}(1)x_{++++}(1)x_1(1)$; as already mentioned, $\langle s\rangle$ acts simply transitively on $U_{+++-}U_{++++}U_1 \setminus \{1\}$, and indeed $g^{s^2} = x_1(1)$. We have $x_3(1) = x_1(1)^{n_\mathsf{a} n_\mathsf{b}} \in HgH$, and then

$$n_{3-4} = x_{-\mathsf{a}}(1, 1, 1)s.x_3(1).x_\mathsf{a}(1, 1, 1)n_\mathsf{a} s x_\mathsf{a}(1, 1, 1) \in HgH;$$

thus $n_4(1)n_{3-4} = n_{3-4}n_3(1) \in HgHgH$. It follows that if we had $k_1 = 17199$ then we would have $k_2 = 2935296$; but this is the largest subdegree, so we would have $k_2 > k_3$, contrary to (ii) above. Thus by (i) above we must have $k_6 = 17199$. The next smallest subdegree is 17472, corresponding to the suborbit HgH where $g = n_{+--+}$. By (ii) above we cannot have $k_5 = 17472$ as this would force $k_1 > k_5$, so by (i) above we must have $k_1 = 17472$; but

$$x_{+++-}(1)x_{++++}(1)x_1(1) = s^5\tau_3.g.x_{2\mathsf{a}+\mathsf{b}}(0, 0, 1).g.x_{2\mathsf{a}+\mathsf{b}}(1, 0, 0)\tau_3{}^2 s^2 \in HgHgH,$$

so that $x_{+++-}(1)x_{++++}(1)x_1(1)H$ would be at distance 2 from H instead of 6. This contradiction proves the result. □

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CONSTRUCTIBLE REPRESENTATIONS AND CATALAN NUMBERS

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Dedicated to the memory of Gary Seitz

We establish a connection between constructible representations (arising in the study of left cells in Weyl groups) and Catalan numbers.

0. Introduction

0.1. The sequence of Catalan numbers is the sequence Cat_n, $n = 1, 2, 3, \ldots$, where $\mathrm{Cat}_n = (2n)!/(n!(n+1)!)$. According to [3], Catalan numbers first appeared in the work of Ming Antu (1692–1763). They were rediscovered by Euler (1707–1783). See also [13].

In this paper we give a new way in which Catalan numbers appear in connection with Lie theory.

0.2. Let G be a connected reductive algebraic group of adjoint type over $\mathbb{C}$ whose Weyl group W is assumed to be irreducible. Let $\widehat{W}$ be the set of (isomorphism classes of) irreducible representations (over $\mathbb{Q}$) of W.

In [4], a partition of $\widehat{W}$ into subsets called *families* was defined and in [6] a class of not necessarily irreducible representations (later called *constructible representations*, see [9]) of W with all components in a family c (which we now fix) was defined by an inductive procedure. Let $\mathrm{Con}(c)$ be the set of constructible representations (up to isomorphism) attached to c. In [6] it was conjectured that the representations in $\mathrm{Con}(c)$ are precisely the representations associated in [2] to the various left cells of W contained in the two-sided cell of W defined by c; this conjecture was proved in [7]. It is known that $|c| = 1$ if W is of type A, $|c| = \binom{D+1}{D/2}$ (with $D \in 2\mathbb{N}$) if W is of type B, C or D, and $|c|$ is one of 1, 2, 3, 4, 5, 11, 17 if W is of exceptional type.

0.3. We would like to find an explicit formula for $|\mathrm{Con}(c)|$.

If $|c|$ is one of 1, 2, 3, 4, 5, 11, 17 then $|\mathrm{Con}(c)|$ is 1, 1, 2, 2, 3, 5, 7 respectively.

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In the remainder of this paper we assume that:

(a) $|c| = \binom{D+1}{D/2}$ with $D = 2d \in 2\mathbb{N}$.

In Section 1 we prove the following result.

Theorem 0.4. *We have* $|\mathrm{Con}(c)| = \mathrm{Cat}_{d+1}$.

It is known (see [12, §2.13]) that if W is of type D then $|\mathrm{Con}(c)| = |\mathrm{Con}(c')|$ for some family c' in a Weyl group of type B or C. We will therefore assume in the rest of the paper that W is of type B or C.

0.5. According to [1, Corollary 4], we have

(a) $$\mathrm{Cat}_n = \sum_{p=1}^{n} N(n, p)$$

where
$$N(n, p) = \frac{1}{n}\binom{n}{p}\binom{n}{p-1}$$
are the Narayana numbers.

We denote by F the field with two elements.

In [8] a bijection between $\mathrm{Con}(c)$ and a certain collection X_c of subgroups of F^d is described. For each p, $1 \le p \le d+1$, let $X_{c,p}$ be the set of subgroups of cardinal 2^{p-1} in X_c. The following refinement of Theorem 0.4 is proved in Section 2.

Theorem 0.6. *We have* $|X_{c,p}| = N_{d+1,p}$.

0.7. In Section 3 we state a conjecture according to which Catalan numbers appear in connection with the study of Springer fibers for G.

0.8. For any $i \le j$ in $\mathbb{Z}$ we set $[i, j] = \{h \in \mathbb{Z}; i \le h \le j\}$.

1. Proof of Theorem 0.4

1.1. Let $D \in 2\mathbb{N}$. Let V_D be an F-vector space with a nondegenerate symplectic form $\langle \cdot, \cdot \rangle : V_D \times V_D \to F$ and with a given subset $\{e_1, e_2, e_3, \dots, e_D\}$ such that $\langle e_i, e_j \rangle = 1$ if $i - j = \pm 1$ and $\langle e_i, e_j \rangle = 0$ otherwise.

Assuming that $D \ge 2$ and $i \in [1, D]$, we define a linear (injective) map $T_i : V_{D-2} \to V_D$ by:

- $e_a \mapsto e_a$ if $a < i - 1$.
- $e_{i-1} \mapsto e_{i-1} + e_i + e_{i+1}$.
- $e_a \mapsto e_{a+2}$ if $a \ge i$.

(We regard V_{D-2} as a subspace of V_D in an obvious way.)

Let $\mathcal{F}(V_D)$ be the family of isotropic subspaces associated in [11, §1.17] to V_D and its basis $\{e_1, e_2, \dots, e_D\}$. (The characteristic functions of these subspaces form a basis of the $\mathbb{C}$-vector space of functions $V_D \to \mathbb{C}$.) We have a partition $\mathcal{F}(V_D) = \bigsqcup_{k\geq 0} \mathcal{F}^k(V_D)$. We will only give here the definition of $\mathcal{F}^0(V_D)$ and $\mathcal{F}^1(V_D)$. The definition is by induction on D. When $D = 0$, $\mathcal{F}^0(V_D)$ consists of 0 and $\mathcal{F}^1(V_D)$ is empty. Assume now that $D \geq 2$. A subspace E of V_D is said to be in $\mathcal{F}^0(V_D)$ if either $E = 0$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}^0(V_{D-2})$ such that $E = T_i(E') + Fe_i$ (this is a direct sum). A subspace E of V_D is said to be in $\mathcal{F}^1(V_D)$ if either $E = F(e_1 + e_2 + \cdots + e_D)$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}^1(V_{D-2})$ such that $E = T_i(E') + Fe_i$.

For example, if $D = 2$, $\mathcal{F}^0(V_D)$ consists of 0, Fe_1, Fe_2 and $\mathcal{F}^1(V_D)$ consists of $F(e_1+e_2)$. If $D = 4$, $\mathcal{F}^0(V_D)$ consists of

$$0,\ Fe_1,\ Fe_2,\ Fe_3,\ Fe_4,\ Fe_1+Fe_3,\ Fe_1+Fe_4,\ Fe_2+Fe_4,\ F(e_1+e_2+e_3)+Fe_2,\ F(e_2+e_3+e_4)+Fe_3$$

and $\mathcal{F}^1(V_D)$ consists of

$$F(e_1+e_2+e_3+e_4),\ F(e_1+e_2+e_3+e_4)+Fe_2,\ F(e_1+e_2+e_3+e_4)+Fe_3,\ F(e_1+e_2)+Fe_4,\ Fe_1+F(e_3+e_4).$$

We have

$$\mathcal{F}^0(V_D) = \mathcal{F}^0_{D/2}(V_D) \sqcup \mathcal{F}^0_{<D/2}(V_D)$$

where

$$\mathcal{F}^0_{D/2}(V_D) = \{E \in \mathcal{F}^0(V_D); \dim(E) = \tfrac{1}{2}D\},$$
$$\mathcal{F}^0_{<D/2}(V_D) = \{E \in \mathcal{F}^0(V_D); \dim(E) < \tfrac{1}{2}D\}.$$

1.2. Let $\mathcal{G}^0_D$ (resp. $\mathcal{G}^1_D$) be the set of lines in V_D of the form $F(e_a + e_{a+1} + \cdots + e_b)$ where $a \leq b$ in $[1, D]$ satisfy $b - a = 1 \pmod 2$ (resp. $b - a = 0 \pmod 2$). Let $\mathcal{G}_D = \mathcal{G}^0_D \sqcup \mathcal{G}^1_D$. For $E \in \mathcal{F}(V_D)$ let $B_E = \{L \in \mathcal{G}_D; L \subset E\}$. According to [12, §1.2(e), (f), (g)], if $E \in \mathcal{F}(V_D)$ then

(a) $$E = \bigoplus_{L \in B_E} L;$$

moreover we have $E \in \mathcal{F}^0(V_D)$ if and only if $B_E \subset \mathcal{G}^1_D$; we have $E \in \mathcal{F}^1(V_D)$ if and only if B_E contains a unique line L_E in $\mathcal{G}^0_D$.

It follows that if $E \in \mathcal{F}^1(V_D)$ we can write

(b) $E = E_0 \oplus L_E$ where $E_0 = \bigoplus_{L \in B_E; L \neq L_E} L$.

We show:

(c) $E_0 \in \mathcal{F}^0(V_D)$.

We argue by induction on D. If $D=0$ then $\mathcal{F}^1_D=\varnothing$ and there is nothing to prove. Assume now that $D\geq 2$. If $E=F(e_1+e_2+\cdots+e_D)$, then $E_0=0$ and (c) is obvious. If E is not of this form then there exists $i\in[1,D]$ and $E'\in\mathcal{F}^1_{D-2}$ such that $E=T_i(E')+Fe_i$. By the induction hypothesis we have $E'=E'_0\oplus L_{E'}$ where $E'_0\in\mathcal{F}^0_{D-2}$. We have $E=T_i(E'_0)+Fe_i+T_i(L_{E'})=\tilde{E}_0+\tilde{L}$ where $\tilde{E}_0=T_i(E'_0)+Fe_i\in\mathcal{F}^0(V_D)$ and $\tilde{L}=T_i(L_{E'})\in\mathcal{G}^0_D$ (from the definition of T_i). Since $\tilde{L}\subset E$ we must have $\tilde{L}=L_E$. We have $B_E=B_{\tilde{E}_0}\cup\{L_E\}$ (the union is disjoint since $B_{\tilde{E}_0}\subset\mathcal{G}^1_D$, $L_E\in\mathcal{G}^0_D$). Thus $B_{\tilde{E}_0}=B_E-\{L_E\}$. Since $\tilde{E}_0=\sum_{L\in B_{\tilde{E}_0}}=\sum_{L\in B_E-\{L_E\}}L=E_0$ we see that $E_0=\tilde{E}_0\in\mathcal{F}^0(V_D)$. This proves (c).

Note that in (c) we have $\dim(E)\leq\frac{1}{2}D$, $\dim(L_E)=1$, hence $\dim(E_0)<\frac{1}{2}D$. Thus we can define a map $\Xi_D:\mathcal{F}^1(V_D)\to\mathcal{F}^0_{<D/2}(V_D)$ by $E\mapsto E_0$ (notation of (c)).

We show:

(d) The map Ξ_D is surjective.

We argue by induction on D. If $D=0$ then $\mathcal{F}^0_{<D/2}(V_D)$ is empty and there is nothing to prove. Assume now that $D\geq 2$. Let $E_0\in\mathcal{F}^0_{<D/2}(V_D)$. If $E_0=0$ then $E=F(e_1+e_2+\cdots+e_D)$ is as required. Now assume that $E_0\neq 0$. Then there exists $i\in[1,D]$ and $E'_0\in\mathcal{F}^0(V_{D-2})$ such that $E_0=T_i(E'_0)\oplus Fe_i$. We see that $\dim(E'_0)=\dim(T_i(E'_0))=\dim(E_0)-1<\frac{1}{2}D-1=\frac{1}{2}(D-2)$ so that $E'_0\in\mathcal{F}^0_{<(D-2)/2}(V_{D-2})$. By the induction hypothesis there exists $L\in\mathcal{G}^0_{D-2}$ such that $E'_0+L\in\mathcal{F}^1(V_{D-2})$. Let $E=T_i(E'_0+L)+Fe_i$. We have $E\in\mathcal{F}^1(V_D)$ and $E=E_0+T_i(L)$. Note that $T_i(L)\in\mathcal{G}^0_D$ and is contained in E, hence it is equal to L_E (see (b)). It follows that $E_0=\Xi_D(E)$. This proves (d).

We show:

(e) Ξ_D is injective.

Assume that E,E' in $\mathcal{F}^1(V_D)$ satisfy $\Xi_D(E)=\Xi_D(E')$. We must show that $E=E'$.

We have $E=E_0\oplus L$, $E'=E_0\oplus L'$ where $E_0\in\mathcal{F}^0(V_D)$ and $L=F(e_a+e_{a+1}+\cdots+e_b)$, $L'=F(e_{a'}+e_{a'+1}+\cdots+e_{b'})$, where $a<b$ and $a'<b'$ in $[1,D]$ satisfy $b-a=1\pmod 2$, $b'-a'=1\pmod 2$. (In fact, from [11, §1.3(e), see (P_2)] we have that $a=1\pmod 2$, $b=0\pmod 2$, $a'=1\pmod 2$, $b'=0\pmod 2$.) Assume first that $a<a'$ so that $a\leq a'-2$. From [11, §1.3(e), see (P_2)] we see that there exist $1\leq c\leq c'\leq D$ such that $c\leq a\leq c'$ and such that the line $\mathcal{L}=F(e_c+e_{c+1}+\cdots+e_{c'})$ is contained in E_0, hence also in $\mathcal{G}^1_D$. But then the pair of distinct lines $\mathcal{L},L$ would violate [11, §1.3(e), see (P_0)]. We see that we must have $a\geq a'$. Similarly we have $a'\geq a$, hence $a'=a$.

Assume next that $b<b'$ so that $b+2\leq b'$. From [11, §1.3(e), see (P_2)] we see that there exist $1\leq c\leq c'\leq D$ such that $c\leq b'\leq c'$ and such that the line

$\mathcal{L} = F(e_c + e_{c+1} + \cdots + e_{c'})$ is contained in E_0 hence also in $\mathcal{G}^1_D$. But then the pair of distinct lines $\mathcal{L}$, L' would violate [11, §1.3(e), see (P_0)]. We see that we must have $b \geq b'$. Similarly we have $b' \geq b$, hence $b' = b$.

We see that $L = L'$, hence $E = E'$. This proves (e).

1.3. From §1.2(c), (d), (e) we see that $|\mathcal{F}^0_{<D/2}(V_D)| = |\mathcal{F}^1(V_D)|$; hence

$$|\mathcal{F}^0(V_D)| - |\mathcal{F}^0_{D/2}(V_D)| = |\mathcal{F}^1(V_D)|,$$

that is, $|\mathcal{F}^0_{D/2}(V_D)| = |\mathcal{F}^0(V_D)| - |\mathcal{F}^1(V_D)|$. According to [11, §1.27] we have

$$|\mathcal{F}^0(V_D)| = \binom{D+1}{D/2}, \quad |\mathcal{F}^1(V_D)| = \binom{D+1}{(D-2)/2}.$$

It follows that

$$|\mathcal{F}^0_{D/2}(V_D)| = \binom{D+1}{D/2} - \binom{D+1}{(D-2)/2} = \frac{(2d+2)!}{(d+1)!(d+2)!} = C_{d+1},$$

where $D = 2d$.

1.4. In [5] the set c is identified with a subset of V_D. Now any object in $\mathrm{Con}(c)$ is multiplicity-free, hence may be identified with a subset of c, hence with a subset of V_D. This subset is a Lagrangian subspace of V_D. Thus $\mathrm{Con}(c)$ is identified with a subset of the set of Lagrangian subspaces of V_D. This subset is the same as $\mathcal{F}^0_{D/2}(V_D)$ (see [10, §2.8(iii)]). We see that $|\mathrm{Con}(c)| = C_{d+1}$ and Theorem 0.4 is proved.

1.5. An alternative proof of Theorem 0.4 can be given using the parametrization of $\mathrm{Con}(c)$ in terms of "admissible arrangements" in [6, p. 220].

2. Proof of Theorem 0.6

2.1. We preserve the notation of V_D. We have $V_D = V^0_D \oplus V^1_D$ where V^0_D has basis $\{e_2, e_4, \ldots, e_D\}$ and V^1_D has basis $\{e_1, e_3, \ldots, e_{D-1}\}$. Assuming that $D \geq 2$ we define for any $i \in [1, D]$ a linear map $\mathcal{T}_i : V^1_{D-2} \to V^1_D$ by:

- $e_k \mapsto e_k$ if $k \leq i-2$.
- $e_k \mapsto e_{k+2}$ if $k \geq i$.
- $e_{i-1} \mapsto e_{i-1} + e_{i+1}$ if i is even.

Following [10, §2.3] we define a collection $\mathcal{C}(V^1_D)$ of subspaces of V^1_D by induction on D. If $D = 0$, $\mathcal{C}(V^1_D)$ consists of $\{0\}$. Assume now that $D \geq 2$. A subspace $\mathcal{E}$ of V^1_D is said to be in $\mathcal{C}(V^1_D)$ if either $\mathcal{E} = \{0\}$ or there exists $i \in [1, D]$ and $\mathcal{E}' \in \mathcal{C}(V^1_{D-2})$ such that:

- $\mathcal{E} = \mathcal{T}_i(\mathcal{E}') + Fe_i$ if i is odd.
- $\mathcal{E} = \mathcal{T}_i(\mathcal{E}')$ if i is even.

For example, $\mathcal{C}(V_2^1)$ consists of two subspaces: $0, Fe_1$; $\mathcal{C}(V_4^1)$ consists of five subspaces:

$$0,\ Fe_1,\ Fe_3,\ F(e_1+e_3),\ Fe_1+Fe_3;$$

$\mathcal{C}(V_6^1)$ consists of 14 subspaces:

$$0,\ Fe_1,\ Fe_3,\ Fe_5,\ F(e_1+e_3),\ F(e_3+e_5),\ F(e_1+e_3+e_5),\ Fe_1+Fe_3,$$
$$Fe_1+Fe_5,\ Fe_3+Fe_5,\ F(e_1+e_3)+Fe_5,\ Fe_1+F(e_3+e_5),\ F(e_1+e_3+e_5)+Fe_3,$$
$$Fe_1+Fe_3+Fe_5.$$

2.2. If $\mathcal{E} \in \mathcal{C}(V_D^1)$ we set $\mathcal{E}^! = \{x \in V_D^0; \langle x, \mathcal{E}\rangle = 0\}$. The following result appears in [10, §2.4].

(a) $\mathcal{E} \mapsto \mathcal{E} \oplus \mathcal{E}^!$ defines a bijection $\mathcal{C}(V_D^1) \xrightarrow{\sim} \mathcal{F}_{D/2}^0(V_D)$. The inverse bijection is given by $E \mapsto E \cap V_D^1$.

2.3. Let $\mathcal{Z}_D^*$ be the set of all elements of V_D^1 of the form

$$e_{a,b} = e_a + e_{a+2} + e_{a+4} + \cdots + e_b$$

for various numbers $a \le b$ in $\{1, 3, \dots, D-1\}$.

For any $s \ge 0$, let $\mathcal{Z}_D^s$ be the set of all finite unordered sequences

$$e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s}$$

in $\mathcal{Z}_D^*$ such that for any $n \ne m$ in $\{1, 2, \dots, s\}$ we have either

$$a_n \le b_n < a_m \le b_m \quad \text{or} \quad a_m \le b_m < a_n \le b_n,$$
$$\text{or} \quad a_n < a_m \le b_m < b_n \quad \text{or} \quad a_m < a_n \le b_n < b_m.$$

Let $\mathcal{Z}_D = \bigcup_{s\ge 0} \mathcal{Z}_D^s$ (a disjoint union).

For example, $\mathcal{Z}_2$ consists of the two sequences $\varnothing, \{e_1\}$; $\mathcal{Z}_4$ consists of the five sequences $\varnothing,\ \{e_1\},\ \{e_3\},\ \{e_1+e_3\},\ \{e_1, e_3\}$; and $\mathcal{Z}_6$ consists of 14 sequences:

$$\varnothing,\ \{e_1\},\ \{e_3\},\ \{e_5\},\ \{e_1+e_3\},\ \{e_3+e_5\},\ \{e_1+e_3+e_5\},$$
$$\{e_1, e_3\},\ \{e_1, e_5\},\ \{e_3, e_5\},\ \{e_1+e_3, e_5\},\ \{e_1, e_3+e_5\},\ \{e_1+e_3+e_5, e_3\},\ \{e_1, e_3, e_5\}.$$

Theorem 2.4. *The assignment*

$$\Theta_D : (e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s}) \mapsto Fe_{a_1,b_1} + Fe_{a_2,b_2} + \cdots + Fe_{a_s,b_s}$$

defines a bijection $\mathcal{Z}_D \xrightarrow{\sim} \mathcal{C}(V_D^1)$.

When $D \le 6$ this follows from §2.1, §2.3. Note that Theorem 2.4 gives an order-preserving bijection between the set of noncrossing partitions (see [13]) and $\mathcal{C}(V_D^1)$ (with the order given by inclusion).

2.5. Assuming that $D \ge 2$ we define for any $i \in [1, D]$ a map $\sigma_i : \mathcal{Z}_{D-2}^* \to \mathcal{Z}_D^*$ by:

- $e_{a,b} \mapsto e_{a+2,b+2}$ if $i \le a$.
- $e_{a,b} \mapsto e_{a,b+2}$ if $a < i \le b+1$.
- $e_{a,b} \mapsto e_{a,b}$ if $i > b+1$.

Note that

- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b})$ if i is even.
- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b})$ if i is even and $i \le a$ or $i > b$.
- $\sigma_i(e_{a,b}) = \mathcal{T}_i(e_{a,b}) + e_i$ if i is odd and $a < i \le b$.

2.6. Assume that $D \ge 2$ and $i \in [1, D]$. Let $e_{a,b}, e_{a',b'}$ be in $\mathcal{Z}_{D-2}^*$ and let $e_{\tilde{a},\tilde{b}} = \sigma_i(e_{a,b})$, $e_{\tilde{a}',\tilde{b}'} = \sigma_i(e_{a',b'})$. We show:

(i) If $b < a'$ then $\tilde{b} < \tilde{a}'$.

(ii) If $a < a'$ and $b' < b$ then $\tilde{a} < \tilde{a}'$ and $\tilde{b}' < \tilde{b}$.

(iii) If i is odd and $\tilde{a} \le i \le \tilde{b}$ then $\tilde{a} < i < \tilde{b}$.

In the setup of (i) assume that $\tilde{a}' \le \tilde{b}$. Then we have $a' \le b$ or $a'+2 \le b$ or $a'+2 \le b+2$ or $a' \le b+2$. The first three cases are clearly impossible; in the 4th case we have $b+2 = a'$ (since $b+2 \le a' \le b+2$), $b'+1 < i$ and $b+1 \ge i$, so that $b > b' \ge a'$, a contradiction.

In the setup of (ii) assume that $\tilde{a} \ge \tilde{a}'$. Then we have $a \ge a'$ or $a+2 \ge a'+2$ or $a \ge a'+2$ or $a+2 \ge a'$. The first three cases are clearly impossible; in the 4th case we have $a+2 = a'$ (since $a+2 \le a' \le a+2$), $a' < i$ and $a \ge i$, so that $a > a'$, a contradiction. Thus, $\tilde{a} < \tilde{a}'$.

Again, in the setup of (ii) assume that $\tilde{b}' \ge \tilde{b}$. Then we have $b' \ge b$ or $b'+2 \ge b+2$ or $b' \ge b+2$ or $b'+2 \ge b$. The first three cases are clearly impossible. In the 4th case we have $b'+2 = b$ (since $b \ge b'+2 \ge b$), $b+1 < i$ and $b'+1 \ge i$ so that $b' > b$, a contradiction. Thus, $\tilde{b}' < \tilde{b}'$.

In the setup of (iii) assume that $\tilde{a} = i$. We have $\tilde{a} = a$ or $\tilde{a} = a+2$. If $\tilde{a} = a$ we have $a = i$ and $b < i$, hence $b < \tilde{b}$ so that $\tilde{b} = b+2$; this implies $i \le b$, a contradiction. If $\tilde{a} = a+2$ we have $a+2 = i$, $i \le a$, a contradiction. Thus $\tilde{a} < i$.

In the setup of (iii) assume that $\tilde{b} = i$. We have $\tilde{a} = b$ or $\tilde{b} = b+2$. If $\tilde{b} = b$ we have $b = i$ and $b < i$, a contradiction. If $\tilde{b} = b+2$ we have $b+2 = i$ and either $a \ge i$ or $a < i \le b$. In the first case we have $a \ge b+2 > b$, a contradiction; in the second case we have $b+2 \le b$, a contradiction. Thus, $i < \tilde{b}$.

2.7. From §2.6(i)–(iii) we see that when $D \ge 2$ and $i \in [1, D]$, there is a well-defined map $\Sigma_i : \mathcal{Z}_{D-2} \to \mathcal{Z}_D$ given by

$$(e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s}) \mapsto \begin{cases} (\sigma_i(e_{a_1,b_1}), \sigma_i(e_{a_2,b_2}), \dots, \sigma_i(e_{a_s,b_s}), e_i) & \text{if } i \text{ is odd,} \\ (\sigma_i(e_{a_1,b_1}), \sigma_i(e_{a_2,b_2}), \dots, \sigma_i(e_{a_s,b_s})) & \text{if } i \text{ is even.} \end{cases}$$

2.8. Let $\epsilon \in \mathcal{Z}_D$, $\epsilon \neq \varnothing$. Let $e_{a,b} \in \epsilon$ be such that $b-a$ is minimum. If $b-a=0$ we set $i=a=b$; we have $i \in [1, D]$ and i is odd. If $b-a>0$ we define $i \in [1, D]$ by $a=i-1<i+1\le b$; then i is even. We will show that

(a) ϵ is in the image of $\Sigma_i : \mathcal{Z}_{D-2} \to \mathcal{Z}_D$.

If i is odd we can write $\epsilon = (e_{\tilde{a}_1,\tilde{b}_1}, e_{\tilde{a}_2,\tilde{b}_2}, \dots, e_{\tilde{a}_s,\tilde{b}_s}, e_i)$.

If i is even we can write $\epsilon = (e_{\tilde{a}_1,\tilde{b}_1}, e_{\tilde{a}_2,\tilde{b}_2}, \dots, e_{\tilde{a}_s,\tilde{b}_s})$, where $a_t = a,\ b_t = b$ for some t.

To $e_{\tilde{a}_t,\tilde{b}_t}$, $t = 1, 2, \dots, s$, we associate the element

$$\begin{aligned} e_{a_t,b_t} &= e_{\tilde{a}_t-2,\tilde{b}_t-2} && \text{if } i \le \tilde{a}_t - 2,\\ e_{a_t,b_t} &= e_{\tilde{a}_t,\tilde{b}_t-2} && \text{if } \tilde{a}_t < i \le \tilde{b}_t - 1,\\ e_{a_t,b_t} &= e_{\tilde{a}_t,\tilde{b}_t} && \text{if } \tilde{b}_t < i. \end{aligned}$$

(Note that we cannot have $i = \tilde{a}_t$ or $i = \tilde{b}_t$. Moreover when i is even we see from the definitions that we cannot have $i = \tilde{a}_t - 1$.) This element is in $\mathcal{Z}^*_{D-2}$.

Consider $n \neq m$ in $\{1, 2, \dots, s\}$. We set

$$(\tilde{a}_n, \tilde{b}_n, \tilde{a}_m, \tilde{b}_m) = (\tilde{a}, \tilde{b}, \tilde{a}', \tilde{b}'), \quad (a_n, b_n, a_m, b_m) = (a, b, a', b').$$

We show:

(i) If $\tilde{b} < \tilde{a}'$, then $b < a'$.

(ii) If $\tilde{a}' < \tilde{a} \le \tilde{b} < \tilde{b}'$, then $a' < a \le b < b'$.

In the setup of (i) assume that $a' \le b$. Then we have $\tilde{a}' \le \tilde{b}$ or $\tilde{a}' - 2 \le \tilde{b}$ or $\tilde{a}' - 2 \le \tilde{b} - 2$ or $\tilde{a}' \le \tilde{b} - 2$. The first three cases are clearly impossible. In the 4th case we have $\tilde{b} < \tilde{a}' \le \tilde{b} - 2$, hence $\tilde{b} < \tilde{b} - 2$, a contradiction. Thus $b < a'$.

In the setup of (ii), a', a, b, b' are as follows:

- $\tilde{a}' - 2,\ \tilde{a} - 2,\ \tilde{b} - 2,\ \tilde{b}' - 2$ if $i \le \tilde{a}' - 2$.
- $\tilde{a}',\ \tilde{a} - 2,\ \tilde{b} - 2,\ \tilde{b}' - 2$ if $\tilde{a}' < i \le \tilde{a} - 2$ (so that $\tilde{a}' < \tilde{a} - 2$).
- $\tilde{a}',\ \tilde{a},\ \tilde{b} - 2,\ \tilde{b}' - 2$ if $\tilde{a} < i \le \tilde{b} - 1$ (so that $\tilde{a} \le \tilde{b} - 2$).
- $\tilde{a}',\ \tilde{a},\ \tilde{b},\ \tilde{b}' - 2$ if $\tilde{b} < i \le \tilde{b}' - 2$ (so that $\tilde{b} < \tilde{b}' - 2$).
- $\tilde{a}',\ \tilde{a},\ \tilde{b},\ \tilde{b}'$ if $\tilde{b}' < i$.

Since i is distinct from each of $\tilde{a}',\ \tilde{a}' - 1,\ \tilde{a},\ \tilde{a} - 1,\ \tilde{b},\ \tilde{b}',\ \tilde{b}' - 1$ we see that we must be in one of the five cases above. Note that $a' < a \le b < b'$ in each case.

From (i), (ii) we see that $\epsilon' := (e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s})$ belongs to $\mathcal{Z}_{D-2}$. From the definitions we see that $\epsilon = \Sigma_i(\epsilon')$. Hence (a) holds.

2.9. We define a subset $\mathcal{Z}'_D$ of $\mathcal{Z}_D$ by induction on D. If $D = 0$, $\mathcal{Z}'_D$ consists of the empty sequence. Assume now that $D \ge 2$. A sequence $\epsilon \in \mathcal{Z}_D$ is said to be in $\mathcal{Z}'_D$ if either ϵ is the empty sequence or if there exists $i \in [1, D]$ and $\epsilon' \in \mathcal{Z}'_{D-2}$ such that $\epsilon = \Sigma_i(\epsilon')$. (Note that $\Sigma_i(\epsilon')$ is well defined.) Using §2.8(a) we see by induction on D that

(a) $\mathcal{Z}_D = \mathcal{Z}'_D$.

2.10. Assume that $D \geq 2$ and $i \in [1, D]$. For $\epsilon' \in \mathcal{Z}_{D-2}$ we show:

(a) $\Theta_D(\Sigma_i(\epsilon')) = \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i$ if i is odd.

(b) $\Theta_D(\Sigma_i(\epsilon')) = \mathcal{T}_i(\Theta_{D-2}\epsilon')$ if i is even.

We can write $\epsilon' = (e_{a_1,b_1}, e_{a_2,b_2}, \dots, e_{a_s,b_s})$. Then

$$\Theta_D(\Sigma_i(\epsilon')) = \begin{cases} F\sigma_i(e_{a_1,b_1}) + F\sigma_i(e_{a_2,b_2}) + \cdots + F\sigma_i(e_{a_s,b_s}) + Fe_i & \text{if } i \text{ is odd,} \\ F\sigma_i(e_{a_1,b_1}) + F\sigma_i(e_{a_2,b_2}) + \cdots + F\sigma_i(e_{a_s,b_s}) & \text{if } i \text{ is even.} \end{cases}$$

Using the definitions we see that

$$\begin{aligned} \Theta_D(\Sigma_i(\epsilon')) &= F\mathcal{T}_i(e_{a_1,b_1}) + F\mathcal{T}_i(e_{a_2,b_2}) + \cdots + F\mathcal{T}_i(e_{a_s,b_s}) + Fe_i \\ &= \mathcal{T}_i(Fe_{a_1,b_1} + Fe_{a_2,b_2} + \cdots + Fe_{a_s,b_s}) + Fe_i = \mathcal{T}_i(X_{D-1}(\epsilon')) + Fe_i \end{aligned}$$

if i is odd,

$$\begin{aligned} \Theta_D(\Sigma_i(\epsilon')) &= F\mathcal{T}_i(e_{a_1,b_1}) + F\mathcal{T}_i(e_{a_2,b_2}) + \cdots + F\mathcal{T}_i(e_{a_s,b_s}) \\ &= \mathcal{T}_i(Fe_{a_1,b_1} + Fe_{a_2,b_2} + \cdots + Fe_{a_s,b_s}) = \mathcal{T}_i(X_{D-1}(\epsilon')) \end{aligned}$$

if i is even. This proves (a), (b).

2.11. We prove the following part of Theorem 2.4.

(a) The map Θ_D in Theorem 2.4 is well defined (it has image contained in $\mathcal{C}(V_D^1)$).

We argue by induction on D. When $D = 0$, (a) is obvious. Assume now that $D \geq 2$. Let $\epsilon \in \mathcal{Z}_D$. If $\epsilon = \varnothing$ then $\Theta_D(\epsilon) = 0 \in \mathcal{F}_D$. Assume now that $\epsilon \neq \varnothing$. Using §2.8, we can find $i \in [1, D]$ and $\epsilon' \in \mathcal{Z}_{D-2}$ such that $\epsilon = \Sigma_i(\epsilon')$ so that $\Theta_D(\epsilon) = \Theta_D(\Sigma_i(\epsilon'))$. By the induction hypothesis we have $\Theta_{D-2}\epsilon' \in \mathcal{C}(V_{D-2}^1)$. By the definition of $\mathcal{C}(V_D^1)$ we then have

$$\begin{aligned} &\mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i \in \mathcal{C}(V_D^1) && \text{if } i \text{ is odd;} \\ &\mathcal{T}_i(\Theta_{D-2}\epsilon') \in \mathcal{C}(V_D^1) && \text{if } i \text{ is even.} \end{aligned}$$

Using §2.10, we can rewrite this as $\Theta_D(\epsilon) \in \mathcal{C}(V_D^1)$. This proves (a).

2.12. We prove the following part of Theorem 2.4.

(a) The map Θ_D in Theorem 2.4 (see §2.11(a)) is surjective.

We argue by induction on D. When $D = 0$, (a) is obvious. Assume now that $D \geq 2$. Let $\mathcal{E} \in \mathcal{C}(V_D^1)$. If $\mathcal{E} = 0$ then $\mathcal{E} = \Theta_D(\varnothing)$. Assume now that $\mathcal{E} \neq 0$. We can find $i \in [1, D]$ and $\mathcal{E}' \in \mathcal{C}(V_{D-2}^1)$ such that $\mathcal{E} = \mathcal{T}_i(\mathcal{E}') + Fe_i$ if i is odd and $\mathcal{E} = \mathcal{T}_i(\mathcal{E}')$ if

i is even. By the induction hypothesis we have $\mathcal{E}' = \Theta_{D-2}(\epsilon')$ for some $\epsilon' \in \mathcal{Z}_{D-2}$. Thus we have $\mathcal{E} = \mathcal{T}_i(\Theta_{D-2}\epsilon') + Fe_i$ if i is odd, $\mathcal{E} = \mathcal{T}_i(\Theta_{D-2}\epsilon')$ if i is even. Using §2.10 we can rewrite this as $\mathcal{E} = \Theta_D(\epsilon)$ where $\epsilon = \Sigma_i(\epsilon') \in \mathcal{Z}_D$. This proves (a).

2.13. We have $\mathcal{C}(V_D^1) = \bigsqcup_{s \in [0,d]} \mathcal{C}^s(V_D^1)$ where $\mathcal{C}^s(V_D^1) = \{\mathcal{E} \in \mathcal{C}(V_D^1); \dim \mathcal{E} = s\}$. Clearly, the map Θ in Theorem 2.4 restricts to a map $\Theta^s : \mathcal{Z}_D^s \to \mathcal{C}^s(V_D^1)$ for any $s \in [0, d]$. From §2.12(a) it follows that Θ^s is surjective for any $s \in [0, d]$. In [1] it is shown that $|\mathcal{Z}_D^s| = N_{d+1,s+1}$ (see §0.5) for any $s \in [0, d]$. Using this and §0.5(a) we see that

$$\mathrm{Cat}_{d+1} = \sum_{s \in [0,d]} N(d+1, s+1) = \sum_{s \in [0,d]} |\mathcal{Z}_D^s| = |\mathcal{Z}_D|.$$

We see that Θ_D is a surjective map from a set with cardinal $|\mathcal{Z}_D| = \mathrm{Cat}_{d+1}$ to a set with the same cardinal $|\mathcal{C}(V_D^1)| = |\mathcal{F}_{D/2}^0(V_D)| = \mathrm{Cat}_{d+1}$ (the first equality holds by §2.2(a); the second equality follows from Theorem 0.4). It follows that Θ is a bijection and Theorem 2.4 is proved.

This implies that $\Theta^s : \mathcal{Z}_D^s \to \mathcal{C}^s(V_D^1)$ is a bijection for any $s \in [0, d]$. We see that Theorem 0.6 holds. (We use that X_c in §0.5 is the same as $\mathcal{C}^s(V_D^1)$ if we identify $V_D^1 = F^d$.)

3. A conjecture

3.1. In this section we fix a unipotent element $u \in G$. We assume that either

- G is of type $C_{d(d+1)}$, $d \geq 1$ and u has Jordan blocks of sizes $2d$, $2d$, $2d-2$, $2d-2, \ldots, 2, 2$ or that
- G is of type $B_{d(d+1)}$, $d \geq 1$ and u has Jordan blocks of sizes $2d+1$, $2d-1$, $2d-1, \ldots, 1, 1$.

Let $\mathcal{B}_u$ be the variety of Borel subgroups of G that contain u and let $[\mathcal{B}_u]$ be the set of irreducible components of $\mathcal{B}_u$. Let $A(u)$ be the group of components of the centralizer of u in G. Note that $A(u)$ acts naturally by permutations on $[\mathcal{B}_u]$. For each $\xi \in [\mathcal{B}_u]$ we denote by $A(u)_\xi$ the stabilizer of ξ in $A(u)$. Let Δ_u be the set of subgroups of $A(u)$ of the form $A(u)_\xi$ for some $\xi \in [\mathcal{B}_u]$.

We assume that c is the family containing the Springer representation of W associated to u and to the unit representation of $A(u)$. We conjecture that

(a) *there exists an isomorphism* $A(u) \xrightarrow{\sim} V_D^1$, $D = 2d$ *which carries* Δ_u *to the collection* $\mathcal{C}(V_D^1)$ (*see* §2.1) *of subspaces of* V_D^1.

(This would imply that $|\Delta_u|$ is a Catalan number.)

We have verified that (a) is true when $d = 1, 2, 3$.

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A REDUCTION THEOREM FOR SIMPLE GROUPS WITH $e(G) = 3$

RICHARD LYONS AND RONALD SOLOMON

In memory of our dear friend, Gary Seitz

This paper highlights the use of transfer to analyze an unbalancing configuration in the GLS classification project for finite simple groups. The configuration occurs when $e(G) = 3$, $m_{2,p}(G) = m_p(G) = 3$ for some odd prime p, and a p-component of p-rank 1 is responsible for the unbalancing.

1. Introduction

An important step in the GLS approach to the classification of the $\mathcal{K}$-proper finite simple groups G of restricted even type with $e(G) \geq 3$, as it was for Michael Aschbacher [1981; 1983] in his classification of such groups of characteristic-2 type with $e(G) = 3$, is to establish a dichotomy quite analogous to the Gorenstein–Walter alternative for generic groups [Gorenstein et al. 2002, Theorem $\mathcal{C}_7^*$: Stage 1, Chapter 3, p. 49].

Throughout this paper, G is a $\mathcal{K}$-proper finite simple group of restricted even type.

For a suitable odd prime divisor p of G such that $m_{2,p}(G) = e(G)$, and a suitable subgroup $A \cong E_{p^n}$ of G, $n \geq 3$, the dichotomy is, roughly speaking:

(1A1) the p-layer $L_{p'}(C_G(a))$ of $C_G(a)$ is semisimple for all $a \in A^\#$; or

(1A2) G has a strong p-uniqueness subgroup M.

Without defining the technical term "strong p-uniqueness subgroup", we remark that the archetype of a strong p-uniqueness subgroup is a strongly p-embedded subgroup such that $O_{p'}(M)$ has even order. However, various weaker conditions also suffice to qualify a subgroup M as a strong p-uniqueness subgroup. See, for example, [Gorenstein et al. 1994, Chapter 2, Section 8] for the case $m_{2,p}(G) \geq 4$. And for the case $m_{2,p}(G) = 3 = e(G)$, GLS is using the condition $\Gamma^o_{P,2}(G) \leq M$ for some $P \in \mathrm{Syl}_p(G)$ instead of requiring strong p-embedding.

We shall call this the fundamental dichotomy, and its verification when $e(G) \geq 4$, along with the identification of G in (1A1), forms a large part of [Gorenstein et al.

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2002; 2018a; 2018b]. When $e(G) = 3 = m_{2,p}(G) < m_p(G)$ for some odd prime p, G has been classified in [Capdeboscq et al. 2023] (and without any reference to strong p-uniqueness subgroups).

In this paper, we shall consider the remaining case in which $e(G) = 3$, so that $m_p(G) = 3$ for any odd prime p such that $m_{2,p}(G) = 3$. To avoid complications that are of little interest, we shall assume for simplicity that there exists such a prime p for which $p > 3$. In Section 3, we will reduce the proof of the fundamental dichotomy to one very specific subcase, and in Section 4, we will outline our proof that the fundamental dichotomy holds in this subcase as well. As noted above, when G is of characteristic-2 type (a somewhat more restrictive condition than restricted even type) and $e(G) = 3$, this result was proved by Aschbacher. Our approach to the necessary signalizer analysis, however, is different from his, using elementary p-transfer arguments to reduce to the single subcase of Section 4, where signalizer functors seem to be unavoidable. We thank Michael for his assistance in the treatment of this subcase.

Throughout we shall focus on an odd prime divisor p of $|G|$. For any subgroup H of G such that $H = L_{p'}(H)$, we shall let

$$H^\dagger = H/O_{p'}(H).$$

In the Gorenstein–Walter alternative, the second option is that G is $\frac{3}{2}$-balanced with respect to A, and $\Theta_{3/2}(G; A)$ is a nontrivial p'-group. This option is then to be pursued further, to the existence of a strong p-uniqueness subgroup, the first approximation being $N_G(\Theta_{3/2}(G; A))$. A similar connection between $\frac{3}{2}$-balance and strong p-uniqueness subgroups can be seen in the context of this paper. Hence, given our purpose here, we shall assume the following proposition.

Proposition 1.1. *Let G be a $\mathcal{K}$-proper finite group of even type with $e(G) = 3$. Let $p > 3$ be a prime such that $m_{2,p}(G) = m_p(G) = 3$. Let $A \le G$ with $A \cong E_{p^3}$. Then either the fundamental dichotomy holds for G, p, and A, or A can be chosen such that for some $x, a, b \in A^\#$ and $L \le C_G(x)$, $A = \langle x, a, b\rangle$, L is an A-invariant p-component of $C_G(x)$, and*

$$[O_{p'}(C_{\mathrm{Aut}_{C_G(x)} L^\dagger}(b)), a] \ne 1.$$

The conclusion of this proposition, in the language of [Gorenstein et al. 2002, Chapter 2, 3.2–3.4], is that (x, L) *is a* $\frac{3}{2}$, *A-obstruction*, i.e., an obstruction to A's being $\frac{3}{2}$-balanced in G. In the absence of such obstructions, the fundamental dichotomy follows by applications of [Gorenstein et al. 1996, 20.7, 5.19(i)], and the standard construction (due to Aschbacher) of a $\frac{3}{2}$-balanced functor on A [Gorenstein et al. 1996, 21.9, 21.10]. Accordingly, as this functor is trivial or nontrivial, (1A1) or (1A2) can be proved. (The reader may have noticed that there is a second type of possible obstruction, due to failure of local 2-balance. But such obstructions cannot occur, it turns out, under our hypotheses that $p > 3$, $e(G) = 3$, and G is $\mathcal{K}$-proper.)

2. Statement of results

In view of the foregoing, we shall proceed under the following hypothesis:

(2A1) G is a $\mathcal{K}$-proper finite simple group of restricted even type with $e(G) = 3$.

(2A2) For all odd primes q such that $m_{2,q}(G) = 3$, we have $m_q(G) = 3$.

(2A3) There is a prime $p > 3$ such that $m_p(G) = m_{2,p}(G) = 3$.

(2A4) There is an elementary abelian p-subgroup A of G with $m_p(A) = 3$ and a $\frac{3}{2}$, A-obstruction (x, L) in $C_G(x)$ for some $x \in A^\#$.

As G is a $\mathcal{K}$-proper finite simple group, every proper simple section of G is a known simple group. The principal application of the $\mathcal{K}$-proper assumption (in conjunction with the assumption that $p > 3$ and $m_p(G) = 3$) will be via the following $\mathcal{K}$-group fact, which we accept here without proof. The assumption $p > 3$ is critical for this proposition.

Proposition 2.1. *Let G, p and A be as above with (x, L) a $\frac{3}{2}$, A-obstruction. Then $L^\dagger$ is a simple group of Lie type in characteristic $s \neq p$ with $m_p(L) = 1$ and $L \lhd N_G(\langle x \rangle)$. Also, $A = \langle x, a, b \rangle$, where $b \in L$ and a induces a field automorphism of order p on $L^\dagger$. Moreover, $[O_{p'}(C_{L^\dagger}(b)), a] \neq 1$ and $L_{p'}(C_{L^\dagger}(a)) \neq 1$.*

We will therefore assume that the following configuration holds (here we let $C_y := C_G(y)$ for all $y \in A^\#$):

(2B1) $x \in \mathcal{I}_p(G)$ ($\mathcal{I}_p(X)$ is the set of elements of X of order p).

(2B2) $P \in \mathrm{Syl}_p(G)$ and $T = C_P(x) \in \mathrm{Syl}_p(C_x)$.

(2B3) L is a p-component of C_x and $R = \langle r \rangle = T \cap L$ is cyclic with $\Omega_1(R) = \langle b \rangle$.

(2B4) $Q = C_T(L^\dagger)$.

(2B5) $\mathcal{I}_p(T - QR) \neq \varnothing$.

(2B6) For any $a \in \mathcal{I}_p(T - QR)$, $Q_a = C_Q(a)$, and $P_a \in \mathrm{Syl}_p(C_a)$ with $C_P(a) \leq P_a$.

(2B7) For any $a \in \mathcal{I}_p(T - QR)$, a induces a nontrivial field automorphism on $L^\dagger$, $L_a := L_{p'}(C_L(a)) \neq 1$, and K_a is the subnormal closure of L_a in C_a.

Our main aim in this paper is to prove:

Theorem 2.2. *Assume* (2A1)–(2A4) *and* (2B1)–(2B7). *Then*:

(a) $T = P$.

(b) $Q \cong R$.

(c) *P/QR is cyclic and embeds in* $\mathrm{Out}(L^\dagger)$; *the image of P/QR consists of images of field automorphisms.*

(d) *There exists $a \in \mathcal{I}_p(P - QR)$ such that $a^P = a\langle x, b\rangle = a\Omega_1(QR)$.*

(e) $L_{p'}(C_a) = L_1L_2$ *with* $L_1^\dagger \cong L_2^\dagger \cong L_a^\dagger$, $Q_a \in \mathrm{Syl}_p(L_1)$, *and* $R_a := C_R(a) \in \mathrm{Syl}_p(L_2)$.

(f) *There exists $t \in N_G(P)$ inverting a and interchanging $\langle x\rangle$ with $\langle b\rangle$, Q with R, and L_1 with L_2.*

As a consequence, GLS will obtain the following theorem in a forthcoming volume. We shall make a few remarks about the proof of this result in Section 4.

Theorem 2.3. *Suppose that (2A1)–(2A4) and (2B1)–(2B7) hold. Then G has a strongly p-embedded subgroup M such that $O_{p'}(M)$ has even order.*

3. Proof of Theorem 2.2

In this section we prove Theorem 2.2. We let

(3A)
$$\mathfrak{A} = \{a_0 \in \mathcal{I}_p(T - QR) \mid K_{a_0} \text{ is a nontrivial pumpup of } L_{a_0}\};$$
$$E = \langle x, b\rangle = A \cap QR.$$

Lemma 3.1. *The following conditions hold*:

(a) $L \lhd C_x$.

(b) *For every $a \in \mathfrak{A}$, K_a is a vertical pumpup of L_a.*

(c) $\Omega_1(T) \le Q\langle a\rangle \times \langle b\rangle$ *for any $a \in \mathcal{I}_p(T - PQ)$.*

(d) *Q_a is cyclic for any $a \in \mathcal{I}_p(T - PQ)$.*

(e) $E = \Omega_1(Z(T))$.

Proof. Proposition 2.1 contains (a). As $m_p(G) = 3$, (d) holds and K_a cannot be a diagonal pumpup of L_a. Then (b) holds by $L_{p'}$-balance and the definition of $\mathfrak{A}$. Since $\Omega_1(T/QR) = \langle QRa\rangle$, we have $\Omega_1(T/Q) \le QR\langle a\rangle/Q$. As R is cyclic, $\Omega_1(R\langle a\rangle) = \langle b, a\rangle$, and (c) follows. As $m_p(G) = 3$ and $L \lhd C_x$, we have $E \le \Omega_1(Z(T)) \le E\langle a\rangle$ for any $a \in \mathcal{I}_p(T - PQ)$. But as a induces a field automorphism of order p on $L^\dagger$, $[R, a] \ne 1$. This implies (e). □

Lemma 3.2. *We have that $m_p(Q) = 1$.*

Proof. Suppose not and choose $B \cong E_{p^2}$ with $B \lhd T$ and $B \le Q$.

Suppose first that $x^g \in E - \langle x\rangle$ for some $g \in G$. Then by Lemma 3.1(e) and Burnside's lemma, we may assume that $g \in N_G(T)$. Now, $E = E^g$ and so $E \cap L^g = (E \cap L)^g = \langle b^g\rangle$. Let $Q_0 = C_{Q\langle a\rangle}((L^g)^\dagger)$. As $Q\langle a\rangle$ normalizes L^g, we have $Q_0 \lhd Q\langle a\rangle$. Hence, if $Q_0 \ne 1$, then $Q_0 \cap \Omega_1(Z(Q\langle a\rangle)) \ne 1$, and so in view of Lemma 3.1(d), $x \in Q_0$. But then $E = \langle x^g, x\rangle \le C_T((L^g)^\dagger)$, contrary to

$E \cap L^g \neq 1$. Hence, $Q\langle a\rangle$ acts faithfully on $(L^g)^\dagger$, so $Q\langle a\rangle$ embeds in T/Q. But $\Omega_1(T/Q) \cong \Omega_1(R\langle a\rangle) = \langle a, b\rangle$, so $\Omega_1(Q\langle a\rangle) = \langle a, x\rangle$. But $B\langle a\rangle \leq \Omega_1(Q\langle a\rangle)$, a contradiction. Hence $x^G \cap E \subseteq \langle x\rangle$. Thus, $\langle x\rangle \lhd N_G(P)$, so $P = T$.

Since Q_a is cyclic, $[B, a] \neq 1$, so B shears a to $\langle x\rangle$, centralizing $\langle b\rangle$. Likewise R shears a to $\langle b\rangle$, centralizing $\langle x\rangle$.

Suppose that $a^g \in E$ for some $g \in G$. Then $P_a \in \mathrm{Syl}_p(G)$. Then E and $\Omega_1(Z(P_a))$ are commuting E_{p^2}-subgroups of P_a, so $E \cap \Omega_1(Z(P_a))$ contains some $d \neq 1$. As $a \notin E$, $\Omega_1(Z(P_a)) = \langle a, d\rangle$. But then by the previous paragraph, some p-element in $\mathrm{Aut}_G(\langle a, d\rangle)$ shears a to $\langle d\rangle$. As $\langle a, d\rangle$ is contained in a Sylow p-center, however, $\mathrm{Aut}_G(\langle a, d\rangle)$ is a p'-group, a contradiction. Hence $a^G \cap E = \varnothing$. Consequently, $\langle x\rangle$ is weakly closed in $\langle a, b, x\rangle$.

Now let d be an extremal conjugate of a in P. Then we may choose $g \in G$ such that $a^g = d$ and $C_P(a)^g \leq C_P(d)$. As $d \notin E$, we have $\langle b, x, d\rangle = \Omega_1(C_P(d))$, so $\langle a, b, x\rangle^g = \langle b, x, d\rangle$. By the weak closedness of $\langle x\rangle$, $g \in N := N_G(\langle x\rangle)$. As $QR \in \mathrm{Syl}_p(LC_N(L^\dagger))$, $LC_N(L^\dagger) \lhd N$, and the image of P/QR in $\mathrm{Out}(L^\dagger)$ is disjoint from $[\mathrm{Out}(L^\dagger), \mathrm{Out}(L^\dagger)]$, we must have $a^g \in QRa$. But P/QR is cyclic, so by the Thompson transfer lemma [Gorenstein et al. 1996, 15.15], $a \notin O^p(G)$, contradicting the simplicity of G. The proof is complete. □

We immediately deduce:

Lemma 3.3. *We have* $A = \Omega_1(T)$.

Lemma 3.4. *Suppose that* $a \in \mathfrak{A}$, *with* x *inducing a nontrivial field automorphism on* $K_a^\dagger$. *Then*:

(a) $C_Q(a) = \langle x\rangle$ *and* $|Q| \leq p^2$.

(b) *If* $|Q| = p^2$, *then* $T = QR\langle a\rangle$ *and* $\mathcal{E}_1(T) - \mathcal{E}_1(QR) \subseteq \langle a\rangle^{C_x}$.

(c) *There is a complement* F *to* QR *in* T *such that* F *faithfully induces field automorphisms on* $L^\dagger$.

Proof. As $C_Q(a)$ acts faithfully on $K_a^\dagger$ centralizing the image of $L_a = L_{p'}(C_{K_a}(x))$, we have $C_Q(a) = \langle x\rangle$. Therefore $|Q| \leq p^2$, so (a) holds. By the structure of $\mathrm{Aut}(L^\dagger)$, there is a complement F_1 to R in T with $a \in F_1$.

Suppose first that $Q = \langle x\rangle$. Then $F_1/\langle x\rangle$ is cyclic. If F_1 is cyclic, then $\Omega_1(T) = \langle x\rangle \times \Omega_1(R)$, contrary to $a \in \Omega_1(T)$. Hence, F_1 is noncyclic abelian with $F_1/\langle x\rangle$ cyclic, whence $F_1 = F \times \langle x\rangle$ for some F which induces field automorphisms on $L^\dagger$, so (c) holds in this case.

If Q has order p^2, then as $C_Q(a) = \langle x\rangle$, $[Q, a] \neq 1$. But T/QR is cyclic and $[QR, Q] = 1$, so T/QR embeds in $\mathrm{Aut}(Q)$ and hence has order p. Thus $T = QR\langle a\rangle$ and Q shears a to $\langle x\rangle$. As R shears a to $\langle b\rangle$, (b) and (c) hold and the proof is complete. □

Lemma 3.5. *If* $\mathfrak{A} = \varnothing$, *then the conclusions of Theorem 2.2 hold.*

Proof. If $\mathfrak{A} = \varnothing$, then L_a pumps up trivially in C_a for every $a \in A - E$. As $L = L_{p'}(C_x) \ncong L_a$ and $A = \Omega_1(T)$, $x^G \cap T \subseteq E$. If $\langle x \rangle$ is weakly closed in E, then $T = P$ and $\langle x \rangle$ is weakly closed in P; so by [Gorenstein et al. 1996, 16.20], $N_G(\langle x \rangle)$ controls G-fusion, hence G-transfer, in P. As $a \notin [N_G(\langle x \rangle), N_G(\langle x \rangle)]$, we conclude that $a \notin [G, G]$, contradicting the simplicity of G. So there exists $t \in N_G(T)$ such that $x_1 := x^t \in x^G \cap E - \langle x \rangle$. Let $L_0^t = L_{p'}(C_{L^t}(a))$ and let M_a be the (trivial) pumpup of L_0^t in C_a. As $C_E(K_a^\dagger) = \langle x \rangle \neq \langle x_1 \rangle = C_E(M_a^\dagger)$, while K_a and M_a are normal in $L_{p'}(C_a)$, we have $[K_a, M_a] \leq O_{p'}(C_a)$ with $\mho^1(R) \in \mathrm{Syl}_p(K_a)$ and $b^t \in M_a$. Now $\langle b^t \rangle \in C_E(K_a^\dagger) = \langle x \rangle$. But t was arbitrary in $N_G(T) - N_G(\langle x \rangle)$. Hence we must have $\langle x \rangle^{N_G(T)} = \{\langle x \rangle, \langle b \rangle\}$. In particular, $P = T$, as p is odd.

If $Q = Q_a$, then $\langle b \rangle = [T, \Omega_1(T)] \lhd N_G(T)$, contradicting $b^t = x$. Hence, $|Q : Q_a| = p$ and $a^T = Ea$. Also $K_a M_a = L_{p'}(C_G(a))$ as $m_p(G) = 3$. Also $Q \times R = J_a(T)$. In particular, $\langle a \rangle^{N_G(T)} = \langle a \rangle^T$, whence we may modify t by an element of T and assume that $t \in N_G(\langle a \rangle)$.

Indeed, there exists a p'-element $h \in N_L(QR)$ such that $[QR, h] = R$ and $C_{QR}(h) = Q$. As $T^t = T$, $QR = (QR)^t$ induces inner automorphisms on $(L^t)^\dagger = L_{p'}(C_G(b))^\dagger$, and QR is invariant under $h \in N_G(\langle b \rangle)$. Therefore the only two largest h-invariant cyclic subgroups of QR are $\{Q, R\} = \{C_{QR}(h), [QR, h]\} = \{C_{QR}((L^t)^\dagger), QR \cap L^t\}$, so $R = C_{QR}((L^t)^\dagger)$ and $Q = QR \cap L^t$. So t interchanges Q and R.

Now $E = Z(P)$ is the weak closure of $\langle x \rangle$ in P, so by the Hall–Wielandt theorem (see [Gorenstein et al. 1996, 15.27]) and the simplicity of G, we have $N_G(E) = O^p(N_G(E))$. However, $N_G(E) = \big(N_G(E) \cap N_G(\langle x \rangle)\big)\langle t \rangle$ with

$$t^2 \in N_G(\langle x \rangle) \cap N_G(\langle a \rangle) \leq C_G(a).$$

If $[t, a] = 1$, then $a \in N_G(E) - O^p(N_G(E))$, an impossibility. Hence as $t \in N_G(\langle a \rangle)$, t inverts a. All the other conclusions of Theorem 2.2 then follow easily, completing the proof of the lemma. □

Our remaining strategy for Theorem 2.2 is to consider various cases for the possible isomorphism types of $K_a^\dagger$, as a varies over $\mathfrak{A}$. By inspection of the possibilities, we reduce to the following cases.

Lemma 3.6. *Let $a \in \mathfrak{A}$. Then one of the following holds*:

(a) $m_p(K_a) = 1$ *and x induces a nontrivial field automorphism on* $K_a^\dagger$;

(b) $P_a \cap K_a$ *is abelian of rank* 2; *or*

(c) $P_a \cap K_a \cong p^{1+2}$, *and* $(p, K_a^\dagger) = (5, \mathrm{HS})$, $(5, \mathrm{Ru})$, *or* $(7, \mathrm{He})$.

In particular, $O_p(K_a^\dagger) = 1$.

Proof. If $m_p(K_a) = 1$, then as $L_a = L_{p'}(C_{K_a}(x))$, x induces an outer automorphism on $K_a^\dagger$. Then as $p > 3$, certainly $K_a^\dagger \in \mathcal{C}\mathrm{hev}(s)$, $s \neq p$, and (a) follows as $m_p(K_a) = 1$.

Suppose that $m_p(K_a) > 1$. Since $m_p(G) = 3$, $O_p(K_a^\dagger) = 1$ by inspection, and then $m_p(K_a) = 2$ and again $K_a^\dagger \in \mathcal{C}\mathrm{hev}(s) \cup \mathcal{A}\mathrm{lt} \cup \mathcal{S}\mathrm{por}$, $s \neq p$. Again by inspection since $p > 3$, p does not divide the order of the Weyl group of $K_a^\dagger$ if $K_a \in \mathcal{C}\mathrm{hev}(s)$, so $P_a \cap K_a$ is abelian and (b) holds in that case. Finally we may assume that $K_a \in \mathcal{A}\mathrm{lt} \cup \mathcal{S}\mathrm{por}$ and $P_a \cap K_a$ is not abelian, and (c) follows easily by inspection. □

We next eliminate case (b).

Lemma 3.7. *Let $a \in \mathfrak{A}$. Then $P_a \cap K_a$ is not abelian of rank* 2.

Proof. In the case that $K_a^\dagger \in \mathcal{C}\mathrm{hev}(s)$, as $p > 3$, we have that $s \neq p$, p does not divide $|\mathrm{Outdiag}(K_a^\dagger)|$, and $K_a^\dagger$ is simple. If x induces a field automorphism on $K_a^\dagger$, then $m_p(K_a) = m_p(L_a) = 1$, contradiction. Hence x induces an inner automorphism on $K_a^\dagger$, corresponding to an element $x_0 \in P_a \cap K_a$. This conclusion is also obvious if $K_a^\dagger \in \mathcal{A}\mathrm{lt} \cup \mathcal{S}\mathrm{por}$. In any case, since $A = \Omega_1(P) \leq P_a$, $A = A_0 \times \langle a \rangle$ where $A_0 = A \cap K_a$.

In all cases, $N_{K_a}(A)$ is irreducible on A_0 [Gorenstein and Lyons 1983, I-(11.1)]. But also $\mathrm{Aut}_R(A)$ contains a transvection shearing a onto $\langle b \rangle \leq A \cap L_a \leq A_0$ and centralizing x. If $\mathrm{Aut}_G(A)$ is irreducible on A, then by McLaughlin's theorem [1967], it contains $\mathrm{SL}(A)$, so it is transitive on $A^\#$. But $a \notin x^G$. Thus $\mathrm{Aut}_G(A)$ is reducible on A. As the irreducible constituents of $N_{K_a}(A)$ on A are A_0 and $\langle a \rangle$, $\mathrm{Aut}_G(A)$ embed in the maximal parabolic subgroup M of $\mathrm{GL}(A)$ stabilizing A_0, with $O_p(\mathrm{Aut}_G(A)) \cong E_{p^2}$ centralizing A_0 and $\mathrm{Aut}_G(A)$ irreducible on A_0. As the p^2 members of $\mathcal{E}_1(A) - \mathcal{E}_1(A_0)$ are permuted transitively by $\mathrm{Aut}_G(A)$ and $\langle a \rangle \not\leq A_0$, we must have $\langle x \rangle \leq A_0$. Hence, $A_0 = \langle x, b \rangle = E$.

Suppose that $|\mathrm{Aut}_G(A)|_p = p^3$. As $\mathrm{Aut}_G(A)$ is irreducible on E, it follows that $\mathrm{Aut}_G(E)$ covers $O^{p'}(M/O_p(M)) \cong \mathrm{SL}(E)$. In particular as

$$\Omega_1(Z(P)) \leq \Omega_1(Z(T)) = E,$$

every element of $E^\#$ is p-central in G and $T \in \mathrm{Syl}_p(G)$. But then p does not divide $|\mathrm{Aut}_G(E)|$, a contradiction.

Hence, $|\mathrm{Aut}_G(A)|_p = p^2$, so if $T \leq T^* \in \mathrm{Syl}_p(N_G(A))$, then

$$T^* = C_{T^*}(E) = T.$$

As $A = \Omega_1(T)$ char T, $T \in \mathrm{Syl}_p(G)$; and then $N_G(A)$ controls G-fusion in T by [Gorenstein et al. 1996, 16.20].

We have $E = \Omega_1(P_a \cap K_a)$. Suppose that $a^{N_G(A)} \cap \langle a \rangle = \{a\}$. As a is T-conjugate to every element of order p in Ea, i.e., in QRa, it follows that if $g \in G$ with $a^g \in T$, then $a^g \in QRa$. Then $V_{G \to T/QR}(a) \neq 1$, so $a \notin [G, G]$, a contradiction.

We conclude that $a^g \in \langle a \rangle - \{a\}$ for some $g \in N_G(A)$. Since a does not belong to $[N_G(\langle x \rangle), N_G(\langle x \rangle)]$, $N_G(\langle a \rangle) \cap N_G(\langle x \rangle) \leq C_a$, so

$$K_a\big(N_G(\langle a \rangle) \cap N_G(\langle x \rangle)\big) \leq C_a < N_G(\langle a \rangle). \tag{3B}$$

In particular, $N_{C_a}(P_a \cap K_a)$ does not control $N_{N_G(\langle a\rangle)}(P_a \cap K_a)$-fusion in $\mathcal{E}_1(E)$ (otherwise equality would hold in (3B), by a Frattini-type argument). It follows[1] that $K_a^\dagger \cong \mathrm{Sp}_4(2^n)$ for some $n > 1$, and p divides $2^{2n}-1$. Moreover, $N_{C_a}(P_a \cap K_a)$ maps into the subgroup X of $\mathrm{Out}(K_a^\dagger)$ of index 2 consisting of images of field automorphisms, whereas the image of $N_{N_G(\langle a\rangle)}(P_a \cap K_a)$ in $\mathrm{Out}(K_a^\dagger)$ does not lie in X. Therefore $N_G(\langle a\rangle)/C_a$ has even order, so there exists $g \in N_G(\langle a\rangle)$ inverting a. As $E = Z(T)$ is weakly closed and p-central we may take $g \in N_E := N_G(E)$. Also $N_0 := N_{K_a}(E) \le N_E$. Set $\bar{N}_E = N_E/O_{p'}(N_E)$. By the structure of C_x, $\bar{T} \lhd \bar{N}_E$. Since $[Q,a] \ne 1 \ne [R,a]$, we have $QR = J_a(T)$, so $\overline{QR} \lhd \bar{N}_E$. Now $\bar{N}_0 \cong D_8$ has equivalent absolutely irreducible representations on $\bar{E} = \Omega_1(\overline{QR})$ and $\overline{QR}/\Phi(\overline{QR})$, and one equivalence is the p^m-power mapping, where $Q \cong R \cong Z_{p^{m+1}}$. By absolute irreducibility, the mapping $\overline{QR}/\Phi(\overline{QR}) \to \bar{E}$ induced by commutation with a must also be a power mapping, and so it commutes with the conjugation action of g. That is,

$$[y^g, a] = [y,a]^g \quad \text{for all } y \in QR.$$

Hence $[y^g, a] = [y^g, a^g] = [y^g, a^{-1}]$ for all $y \in QR$, so $a^2 \in C_G(QR)$. As a has order p, $[QR, a] = 1$, a final contradiction. □

Lemma 3.8. *Let $a \in \mathfrak{A}$. Then $\langle b\rangle$ is weakly closed in A.*

Proof. Let $P_a \le S \in \mathrm{Syl}_p(G)$. Since R shears $\langle a\rangle$ to $\langle b\rangle$, $\langle b\rangle = Z(S)$ is weakly closed in E. Suppose that $d \in A - E$ and $\langle d\rangle \in \langle b\rangle^G$. Since d is p-central in G and $K_d \ne 1$, $O_p(K_d^\dagger) \ne 1$. As $O_p(L_d^\dagger) = 1$, we must have $d \in \mathfrak{A}$. But then by Lemma 3.6, $O_p(K_d^\dagger) = 1$, completing the proof. □

Finally, we eliminate Lemma 3.6(c).

Lemma 3.9. *Let $a \in \mathfrak{A}$. Then $m_p(K_a) = 1$, and x induces a nontrivial field automorphism on $K_a^\dagger$.*

Proof. Suppose not. Then $K_a^\dagger \cong \mathrm{Ru}$ or HS (with $p = 5$) or He (with $p = 7$). Let $S_0 = P_a \cap K_a \in \mathrm{Syl}_p(K_a)$, so that $S_0 \cong p^{1+2}$. Let $S_1 = \langle a\rangle \times S_0 \le S \in \mathrm{Syl}_p(G)$, so that $Z(S_1) = \langle a, b\rangle$ and $S_1 = \Omega_1(C_S(a))$, and set $S_2 = N_S(S_1)$. Now a is sheared to $\langle b\rangle$ by R, and that fusion must occur in S_2 as well. So $\langle a\rangle \in \mathrm{Syl}_3(C(a, K_a))$ and $S_1 \in \mathrm{Syl}_p(C_a)$, and we may assume that $S_2 = S_1 R$.

Let $N_0 = N_G(S_1)$ and $N_a = N_{N_0}(\langle a\rangle)$. Let $\bar{S}_1 = S_1/\langle b\rangle \cong E_{p^3}$ and $\widetilde{S}_1 = \bar{S}_1/\langle \bar{a}\rangle \cong S_1/Z(S_1)$. Then as $\langle b\rangle$ is not conjugate to $\langle a\rangle$, but $a^R = a\langle b\rangle$, N_0 acts on $\bar{S}_1$ and $\widetilde{S}_1$. Moreover, $a^{N_0} \supseteq a\langle b\rangle$, so $|N_0 : N_a| = p$ and $N_0 = N_a R_0$ for some $R_0 \in \mathrm{Syl}_p(N_0)$.

[1] By the following $\mathcal{K}$-group lemma, whose proof we omit. Suppose K is a known finite simple group with $e(K) \le 3$, and $R \in \mathrm{Syl}_p(K)$ is abelian of rank 2 for some prime $p > 3$. Suppose also that $L_{p'}(C_K(x)) \ne 1$ for some $x \in P^\#$. Let $g \in N_{\mathrm{Aut}(K)}(R)$ and suppose that for some $X \in \mathcal{E}_1(R)$, $X^g \notin X^{N_K(R)}$. Then for some $q = 2^n > 2$, $K \cong \mathrm{Sp}_4(q)$, p divides $q^2 - 1$, and the image of g in $\mathrm{Out}(K)$ is not the image of a field automorphism.

Let $X_0 = \mathrm{Aut}_{N_0}(\bar{S}_1)$ and $X_a = \mathrm{Aut}_{N_a}(\bar{S}_1)$, and $Y_0 = \mathrm{Aut}_{N_0}(\widetilde{S}_1)$ and $Y_a = \mathrm{Aut}_{N_a}(\widetilde{S}_1)$. Thus $|X_0 : X_a|$ divides p, as does $|Y_0 : Y_a|$. From [Gorenstein et al. 1998, 5.3], $|Y_a| = 16m$, $18m$, or 32 according as $K_a \cong \mathrm{HS}$, He, or Ru. Here $m = 1$ or 2. In any case, $|\mathrm{SL}_2(p)|$ does not divide $p\,|Y_a|$, so Y_0 does not contain $\mathrm{SL}_2(p)$. But Y_a is irreducible on $\widetilde{S}_1$, so p does not divide $|Y_0|$. Therefore $Y_0 = Y_a$.

The image of R_0 in X_0 therefore stabilizes the chain

$$\bar{S}_1 > \langle \bar{a} \rangle > 1.$$

But the stabilizer of this chain in $\mathrm{Aut}(\bar{S}_1) \cong \mathrm{GL}_3(p)$ is of order p^2 and isomorphic to $\widetilde{S}_1$ as $\mathrm{GL}(\widetilde{S}_1) \times \mathrm{GL}(\langle \bar{a} \rangle)$-module. In particular, X_a is irreducible on $O_p(X_0)$. As $|O_p(X_0)| \le p$, $O_p(X_0) = 1$. Hence $X_0 = X_a$ stabilizes $S_0/\langle b \rangle$ and S_0, and it follows that $S_1 R_0$ is extraspecial of order p^{1+4} and exponent p^2, with $S_1 = \Omega_1(S_1 R_0)$. Thus, $N_S(S_1 R_0) \le N_S(S_1)$, forcing $S = S_1 R_0$. As $R_0 \cong Z_{p^2}$, $N_G(S)$ has equivalent representations on S/S_1 and $Z(S)$, so $[S/C_S(R_0), N_G(S)] = 1$. By theorems of Yoshida or Wielandt [Gorenstein et al. 1996, 15.19, 15.20], G is not simple, a final contradiction. □

For the rest of this section, we assume for a contradiction that

(3C1) $\mathfrak{A} \neq \varnothing$;

(3C2) for all $a \in \mathfrak{A}$, $m_p(K_a) = 1$; and

(3C3) x induces a nontrivial field automorphism on $K_a^\dagger$.

We first prove:

Lemma 3.10. *Assume (3C1)–(3C3). Then*:

(a) $\langle x \rangle^{N_G(T)} = \mathcal{E}_1(E) - \{\langle b \rangle\}$.

(b) $T < P$ *and* $\Omega_1(Z(P)) = \langle b \rangle$.

Proof. As $b \in L_a \le K_a$, and $m_p(K_a) = 1$ with x inducing a nontrivial field automorphism on $K_a^\dagger$, we have $\mathcal{E}_1(E) - \langle b \rangle = \langle x \rangle^S$ for some $S \in \mathrm{Syl}_p(C_{K_a}(b))$. In particular, p divides $|\mathrm{Aut}_G(E)|$. As $E = \Omega_1(Z(T))$, we have $T < P$ and $\Omega_1(Z(P)) \le C_{\Omega_1(Z(T))}(S) = \langle b \rangle R$, so (b) holds. But then x is not p-central in G so (a) follows as well. □

Lemma 3.11. *Assume (3C1)–(3C3). Then $\mathfrak{A} = A - E$, and (2B1)–(2B7) are satisfied with any element of $A - \langle b \rangle$ in place of a.*

Let $a_1 \in \mathcal{I}_p(T - \langle b \rangle)$ and set $K_1 = L_{p'}(C_G(a_1))$. Then $K_1^\dagger \cong L^\dagger$. Let $T_1 \in \mathrm{Syl}_p(C_G(a_1))$ and $Q_1 = C_{T_1}(L_1^\dagger)$. Then $|Q_1| \le p^2$. Finally, if $a_1 \notin E$, then $C_{Q_1}(x) = \langle a_1 \rangle$ and x acts as a nontrivial field automorphism on $K_1^\dagger$.

Proof. If $a_1 \in E - \langle b \rangle$, then $\langle a_1 \rangle \in \langle x \rangle^G$ by Lemma 3.10 and the lemma holds by Lemma 3.4(a). Suppose then that the lemma fails for some $a_1 \in \langle a, x \rangle - \langle x \rangle$ with $L_{a_1} := L_{p'}(C_L(a_1))$ and with K_{a_1} the trivial pumpup of L_{a_1} in $C_G(a_1)$.

Let $x_1 \in x\langle b\rangle - \{x\}$. By Lemma 3.10, $x_1 \in x^{C_L(b)}$, and we take $g \in C_L(b)$ with $x_1 = x^g$. Let $L_2 := L_{p'}(C_{L^g}(a_1))$ and let K_2 be the pumpup of L_2 in $C_G(a_1)$. Then $L_2 \le L_{p'}(C_G(a_1))$ with $b \in K_{a_1} \cap K_2$, so $K_2 = [\langle b\rangle, K_{a_1}] \le K_{a_1}$. Thus K_2 is a trivial pumpup of L_2, so $K_2^\dagger$ is centralized by both x_1 and (like $K_{a_1}^\dagger$) x. Hence, $K_2^\dagger$ is centralized by b, a contradiction.

Therefore, K_{a_1} is a vertical pumpup of L_{a_1}, and x acts as a nontrivial field automorphism on $L_{a_1}^\dagger$ by (3C1)–(3C3). We then apply Lemma 3.4 to a_1, L_{a_1}, x, T_1, and Q_1 in place of x, L, a, T, and Q to obtain the conclusions of the lemma. □

We now fix $U \lhd P$ with $U \cong E_{p^2}$, and set $P_0 = C_P(U)$.

Lemma 3.12. *Assume* (3C1)–(3C3). *Then the following conditions hold for some* $x_0 \in A - \langle b\rangle$ *satisfying* (2B1)–(2B7) (*in the role of* x).

(a) *One of the following holds*:

 (1) $\langle a\rangle \in \mathrm{Syl}_p(C(a, K_a))$ *for all* $a \in A - E$; *or*

 (2) $\mathcal{E}_1(A) - \{\langle b\rangle\}$ *is completely fused in* G.

(b) $x_0 \in U \le A$.

(c) $|P : T_{x_0}| = p$ *for some* $T_{x_0} \in \mathrm{Syl}_p(C_G(x_0))$.

Proof. Suppose that (a1) fails, so that for every $x_0 \in A - \langle b\rangle$, there is $y \in A - \langle x_0, b\rangle$ such that $|C(y, K_y)|_p = p^2$. Then we may assume we started with $x_0 \in A - \langle b\rangle$ such that $|C(x_0, K_{x_0})| = p^2$. By Lemma 3.4, y is sheared to $\langle x_0\rangle$. Applying Lemma 3.4 again in $C_G(y)$, we see that x_0 is sheared to $\langle y\rangle$. Hence $\mathcal{E}_1(\langle x_0, y\rangle)$ is completely fused in G. As x_0 and y are each fused to $\langle x_0 y\rangle$ in the centralizer of the other, (a2) holds, and (a) follows.

Next, assume that x_0 has been chosen so that (a) holds. Let $a_0 \in A - \langle x_0, b\rangle$. Since $E_{p^2} \cong U \lhd P$, there exists $1 \ne y \in C_{\langle x_0, a_0\rangle}(U)$. Then $U \le C_G(y)$. If (a2) holds, then we may vary x_0 to satisfy (b). We have $y \in U$, and taking a Sylow p-subgroup T_y of $C_G(y)$ containing $C_P(y) \ge \langle U, A\rangle$, we have $U \le \Omega_1(T_y) = A$, so (b) holds for y. So assume that (a1) holds (for x_0). Let $Q_{x_0} \in \mathrm{Syl}_p(C(x_0, K_{x_0}))$. If $|Q_{x_0}| = p$, then $\langle a\rangle \in \mathrm{Syl}_p(C(a, K_a))$ for all $a \in A - \langle b\rangle$. (We set $K_x = L$ and $K_{x'} = L^g$ for any $x' \in \langle x, b\rangle - \langle x\rangle - \langle b\rangle$, where $g \in N_P(\langle x, b\rangle)$ is such that $x^g = x'$.) Hence we can again vary x_0 to satisfy (b). We again use $y \in C_{\langle x_0, a_0\rangle}(U)$ in place of x_0 to get (b). If, on the other hand, $|Q_{x_0}|_p = p^2$, then $\langle x_0\rangle = \Phi(Q_{x_0}) \le C_P(U)$ so $U \le C_P(x_0) =: T_{x_0}$. If $U \ne \langle x_0, b\rangle$ then by Lemma 3.4 again, $\langle x_0\rangle = [Q_{x_0}, U] \le U$ so $U = \langle x_0, b\rangle$, contradiction. Thus (b) holds in all cases. Finally $\langle b\rangle$ is 3-central and weakly closed in A, so x_0 must be half p-central, proving (c) and the lemma. □

We replace our original x by x_0 as in Lemma 3.12.

Lemma 3.13. *Let* $a \in \mathfrak{A}$. *Then* $Q = \langle x\rangle$ *and* $T = \langle x\rangle \times R\langle a\rangle$.

Proof. Suppose first that $|Q| = p^2$, and let $a \in \mathfrak{A}$. Note that if $y \in P - T$, then $C_Q(y) = 1$. Also $C_{Q\times R}(a) = \langle x\rangle \times \Phi(R)$. So $Q \times R$ is the unique largest abelian subgroup of P. By Lemma 3.4(b), $T = QR\langle a\rangle$. Consider $N := N_G(QR)$, $C = C_G(QR)$, and $\bar{N} = N/C$. Then $C_{\bar{N}}(E) \le O_p(\bar{N})$ and so $C_{\bar{N}}(E) = \langle\bar{a}\rangle$. Let $\widetilde{N} = N/C_N(E)$. Then $\widetilde{N}$ embeds in $\mathrm{GL}(E)$ and $|\widetilde{N} : N_{\widetilde{N}}(\langle x\rangle)| = p$. It follows that $\widetilde{N} = \widetilde{P} N_{\widetilde{N}}(\langle x\rangle)$, and so $\bar{N} = \bar{P} N_{\bar{N}}(\langle x\rangle)$. As $|\bar{P}| = p^2$ and $[\bar{a}, N_{\bar{N}}(\langle x\rangle)] = 1$, we have $\bar{a} \notin [\bar{N}, \bar{N}]$. By the generalized Hall–Wielandt theorem of Yoshida [Gorenstein et al. 1996, 15.27] applied to the weakly closed subgroup QR of P, $a \notin [G, G]$, contrary to the simplicity of G. Therefore, $Q = \langle x\rangle$.

Now, there is a complement F to R in T containing $\langle x\rangle$. Then $F/\langle x\rangle$ is cyclic, so $F = \langle x\rangle \times F_1$ for some F_1 as $m_p(C_x) = 3$. Write $\Omega_1(F_1) = \langle a\rangle$, so that $a \in \mathfrak{A}$. Then $F_1/\langle a\rangle$ acts on K_a and faithfully induces field automorphisms on $L_a^\dagger \cong L_{p'}(C_{K_a^\dagger}(x))$. As x induces a field automorphism of order p on $K_a^\dagger$, $F_1/\langle a\rangle = 1$. Hence $T = \langle x\rangle \times R\langle a\rangle$, as claimed. □

Lemma 3.14. $\mathfrak{A} = \varnothing$.

Proof. Suppose this is false and continue the above argument. Again let $a \in \mathfrak{A}$ and set $N = N_G(A)$, $C = O_{p'}(C_G(A))$, and $\bar{N} = N/C$. Let $Z = C_R(a) = \Phi(R)$. Thus, $C_G(A) = (\langle a, x\rangle \times Z)C$ and $|\mathrm{Aut}_G(A)|_p = p^2$, by Lemma 3.13. Moreover a generator r of R acts as a transvection on A.

Now, $\mathrm{Aut}_G(A)$ is p-closed.[2] As $C_G(A)$ is p-nilpotent, $\bar{P} \lhd \bar{N} = \bar{N}_0$ where $N_0 = N_{N_G(A)}(P)$. As $A \lhd T_1$ for some $T_1 \in \mathrm{Syl}_p(C_a)$, $C_P(a)$ contains some r_1 centralizing $\langle a, b\rangle$ and shearing x to b. Modifying r_1 by an element of $\langle a, x\rangle$, we may choose $r_1 \in K_a$. Then $P \cap K_a \ge \langle r_1, Z\rangle =: R_1$, so $R_1 = P \cap K_a = \langle r_1\rangle$ and $P = \langle Z, a, x, r, r_1\rangle$. Hence, $Z \le Z(P)$.

We consider the structure of

$$P/Z = \langle aZ, xZ, rZ, r_1Z\rangle = \Omega_1(P/Z).$$

Now $|P : C_P(a)| = |P : C_P(x)| = p$, so $[r, r_1] \in C_P(A) = ZA$ and $[P/Z, P/Z] = \langle [r, r_1]Z\rangle \le Z(P/Z)$. Thus, either $[P, P] = \langle b\rangle$ and $P/[P, P] \cong E_{p^4}$, or, by Lemma 3.12(a), we may choose notation so that $[r, r_1] = x$ and $[P, P] = E$.

Suppose first that $[P, P] = E$. Then E char P and so $\bar{E} \lhd \bar{N}_0$. Hence $\langle x\rangle^{N_0} = \langle x\rangle^P$ and so $N_0 = P N_{N_0}(\langle x\rangle)$. Also $P/Z \cong Z_p \times p^{1+2}$ with $[P/Z, P/Z] = \langle xZ\rangle$. Now, $\langle b\rangle \lhd N_0$ and so $\widetilde{N}_0 := N_0/C_{N_0}(E)$ embeds in a Borel subgroup of $\mathrm{GL}(E)$. Also, there is a p'-element $t \in N_{L_a}(\langle b\rangle)$ such that for some integer $\lambda \not\equiv 1 \pmod{p}$, $b^t = b^\lambda$. Then if $\widetilde{H}$ is a complement to $\widetilde{R}_1$ in $\widetilde{N}_0$ containing $\tilde{t}$, then $\widetilde{H}$ normalizes $C_E(t) = \langle x\rangle$. Let H be a Hall p'-subgroup of N_0 mapping onto $\widetilde{H}$. Then H normalizes $T = C_P(E)$ and as $TH \le N_G(\langle x\rangle)$, we have $[T, H] \le T \cap [N_G(\langle x\rangle), N_G(\langle x\rangle)] \le R\langle x\rangle$. Then $RE \lhd N_0$ and $[H, a] \le RE$. Set $\widehat{N}_0 = N_0/RE$. Then $\widehat{P} = \langle\hat{a}, \hat{r}_1\rangle \lhd \widehat{N}_0$

[2] As is any $H \le \mathrm{GL}_3(p)$ with $|H|_p = p^2$ [Gorenstein et al. 1998, 6.5.3]

with $\widehat{P} \cong E_{p^2}$ and $\hat{a} \notin [\widehat{P}, \widehat{H}]$. Hence $O^p(N_0) \cap P < P$. Now, $Z = Z(P)$ and $Z(P/Z) = \langle aZ, xZ \rangle$. Thus $Z_2(P) = \langle a, x, Z \rangle$ and $A = \Omega_1(Z_2(P))$ char P. So $N_0 = N_G(P)$. By Yoshida's transfer theorem [Gorenstein et al. 1996, 15.19], P has a quotient $P/Y \cong Z_p \wr Z_p$. As $|[P, P]| = p^2$ and $p > 3$, this is impossible. This proves that $[P, P] = \langle b \rangle$.

It follows that $RR_1 \leq P$ with RR_1 having distinct cyclic maximal subgroups R and R_1. Thus $RR_1 = R\Omega_1(RR_1) = R\langle e \rangle$ where e has order p. Now the eigenvalues of t on $R/\Phi(R)$, $\langle b \rangle = \Omega_1(R) = \Omega_1(R_1)$, and $R_1/\Phi(R_1)$ are all equal, so $[RR_1, t] = RR_1$. But if RR_1 is not abelian, then $[R, e] = \Omega_1(R)$, and conjugating by t we see that $[e, t] = 1$, whence $[RR_1, t] = R$, a contradiction.

It follows that $RR_1 = R \times \langle e \rangle$ for some $e \in \mathcal{I}_p(P)$. Then $(\langle x \rangle \times R)\langle e \rangle = C_P(R) \in \mathrm{Syl}_p(C_G(R))$. So, $C_P(R) = \langle x, e \rangle * R$ with $\langle x, e \rangle \cong p^{1+2}$. Similarly $C_P(R_1) = \langle a, e \rangle * R_1$ for some $a \in \mathfrak{A}$. Hence e normalizes A and $S := A\langle e \rangle \cong Z_p \times p^{1+2}$. Let $e_1 \in Z(S) - \langle b \rangle$. As $e_1 \in C_P(A)$, $e_1 \in A$ and $[e_1, e] = 1$. Let $S \leq P_1 \in \mathrm{Syl}_p(C_G(e_1))$. As $e_1 \in A - \langle b \rangle$, $A = \Omega_1(P_1)$, so $\Omega_1(S) \leq A$, contrary to $S = \Omega_1(S)$, a final contradiction. □

Now Lemmas 3.5 and 3.14 prove Theorem 2.2.

4. Remarks on Theorem 2.3

As the hypotheses of Theorem 2.3 yield that $\Omega_1(P) = A \cong E_{p^3}$, the theorem will be proved once we prove that $\Gamma_{A,1}(G) \leq M$ for some $M \leq G$ such that $O_{p'}(M)$ has even order. Using Theorem 2.2 and the hypothesis that G has even type, we prove that $L^\dagger \in \mathcal{C}\mathrm{hev}(2)$. (Recall that $L := L_{p'}(C_G(x))$.) We are then able to prove that for all $e \in E^\#$, $C_G(e)$ is p-solvable unless $e \in \langle x \rangle \cup \langle b \rangle$, in which case $L_{p'}(C_G(e)) \cong L$. Then, as $e(G) = 3$, we can show that $L/O_2(L)$ is quasisimple, and for every $a \in A - E$, each 2-component of $L_{p'}(C_G(a))/O_2(L_{p'}(C_G(a))$ is quasisimple as well. Next we follow Aschbacher [1981] and use a functor $\Theta^{(2)}_{3/2}$ which is an analogue of the standard $\frac{3}{2}$-balanced functor, but with O_2 replacing $O_{p'}$. The standard $\frac{3}{2}$-balanced functor does not exist in this case because L is not weakly locally balanced with respect to A; but the saving grace is that the obstructing cores are of odd order. We then set $\Sigma = \Theta^{(2)}_{3/2}(G; A)$. If $\Sigma \neq 1$, it is immediate that $\Gamma_{A,2}(G) \leq N_G(\Sigma) =: M < G$. On the other hand, if $\Sigma = 1$, then $L_{p'}(C_G(a))$ is semisimple for all $a \in A^\#$, and setting $\Gamma(B) = \langle L_{p'}(C_G(a)) \mid a \in B^\# \rangle$ for any hyperplane B of A, we deduce that $\Gamma(B) = LL^t$ (for t as in Theorem 2.2(f)) and $\Gamma_{A,2}(G) \leq N_G(LL^t) =: M < G$. This step uses Gary Seitz's fundamental generation theorem for finite groups of Lie type [Seitz 1982]. Further applications of that result yield in both cases that $\Gamma_{A,1}(G) \leq M$. Finally, since $e(G) = 3$, A normalizes a nontrivial 2-subgroup of G, which fairly quickly leads to a contradiction if $O_{p'}(M)$ has odd order, completing the proof of Theorem 2.3.

5. Concluding remarks

We take this opportunity to recognize Gary Seitz's fundamental contributions to the classification of finite simple groups in the 1970s and early 1980s, when he was also a player in the concurrent effort to classify the finite doubly transitive groups. This of course represents only a small fraction of his complete mathematical work. Alone or with collaborators, he proved many landmark results that each served to propel these classification efforts. We give several examples.

On doubly transitive groups, he worked with Christoph Hering, Bill Kantor, and Mike O'Nan [Kantor and Seitz 1971; 1972; Hering et al. 1972; Kantor et al. 1972], in particular completing the classification of finite split BN-pairs of rank 1. Seitz and Paul Fong then classified finite split BN-pairs of rank 2 [1973; 1974].

He made major contributions to the classification of finite simple groups of component type. With Michael Aschbacher he classified simple groups with a known quasisimple standard subgroup centralized by a four-group [1976b; 1981]. Partly with Bob Griess and David Mason he classified most simple groups with a standard subgroup in $\mathcal{C}$hev(2) whose centralizer has a cyclic Sylow 2-subgroup [Seitz 1979a; 1979b; 1979c; Griess et al. 1978].

On top of all these, Gary proved many useful results illuminating the structure of finite groups of Lie type. Outstanding examples are the classification of involutions in groups in $\mathcal{C}$hev(2) [Aschbacher and Seitz 1976a], and the generation properties of r' elementary abelian subgroups of odd order acting on groups in $\mathcal{C}$hev(r) [Seitz 1982]. The latter has been mentioned in the above discussion of Theorem 2.3, and in general is vital for the ultimate success of the signalizer functor method.

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DECOMPOSITION NUMBERS IN THE PRINCIPAL BLOCK AND SYLOW NORMALISERS

GUNTER MALLE AND NOELIA RIZO

To the memory of Gary Seitz

If G is a finite group and p is a prime number, we investigate the relationship between the p-modular decomposition numbers of characters of height zero in the principal p-block of G and the p-local structure of G. In particular we prove that, under certain conditions on the nonabelian composition factors of G, $d_{\chi 1_G} \neq 0$ for all irreducible characters χ of degree prime to p in the principal p-block of G if, and only if, the normaliser of a Sylow p-subgroup of G has a normal p-complement.

1. Introduction

Let G be a finite group, p a prime number, $\mathrm{Irr}(G)$ the set of irreducible ordinary characters of G and $\mathrm{IBr}(G)$ the set of irreducible p-Brauer characters of G. Then the restriction χ° of any $\chi \in \mathrm{Irr}(G)$ to the set of p-regular elements of G can be written as

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

where the $d_{\chi\varphi}$ are uniquely determined nonnegative integers. These are called the p-modular *decomposition numbers of* G, and a great deal of literature is devoted to understanding them.

Navarro and Tiep [2020; 2022] initiated the investigation on relations between p-decomposition numbers and properties of Sylow p-normalisers, considering two different settings. In [Navarro and Tiep 2020] they conjectured that if $p > 3$ then $d_{\chi 1_G} \neq 0$ for all $\chi \in \mathrm{Irr}_{p'}(G)$, that is, for all irreducible characters of G of degree prime to p, if and only if G has self-normalising Sylow p-subgroups, and that this happens if and only if $d_{\chi 1_G} = 1$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. Note that irreducible characters

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of degree prime to p lie in p-blocks of maximal defect, so this situation can only happen when the principal block is the only Brauer p-block of maximal defect of G. It is then natural to wonder what can be said when this is not the case and one restricts attention to the principal p-block $B_0(G)$. In this sense, in [Navarro and Tiep 2022] they conjectured that $d_{\chi 1_G} \neq 0$ for all $\chi \in \mathrm{Irr}(B_0(G))$ if and only if G has a normal p-complement. In this paper we consider the intersection of the two conditions in [Navarro and Tiep 2020] and [2022], namely what happens if $d_{\chi 1_G} = 1$ just for all $\chi \in \mathrm{Irr}_{p'}(B_0(G))$. Our main result is:

Theorem A. *Let G be a finite group and $p > 3$ be a prime. Assume that all nonabelian simple composition factors of G of order divisible by p satisfy Property $(*)$ below. The following are equivalent*:

(i) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} \neq 0$.*

(ii) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} = 1$.*

(iii) *For $P \in \mathrm{Syl}_p(G)$ we have $\boldsymbol{N}_G(P) = P \times K$ for some $K \leq G$.*

Moreover, *if G is p-solvable*, *this equivalence holds for every prime p.*

Notice that, as pointed out in [Navarro and Tiep 2020], all irreducible characters of odd degree of the alternating group $\mathfrak{A}_5$ contain the trivial character in their 2-modular reduction, and similarly, all irreducible characters of the Ree group ${}^2G_2(27)$ of nonzero 3-defect contain the trivial character in their 3-modular reduction, while the respective Sylow p-normalisers have no normal p-complement, so the equivalence in Theorem A fails for nonsolvable groups with $p \leq 3$.

Theorem A involves the following property that a finite nonabelian simple group might, or might not, satisfy:

Property $(*)$**.** Let S be nonabelian simple and $p > 3$ a prime dividing $|S|$. Then for all almost simple groups H with socle S and $|H : S|$ a p-power, there exists $\chi \in \mathrm{Irr}_{p'}(B_0(H))$ such that $d_{\chi 1_H} = 0$.

We prove Property $(*)$ does hold for many simple groups: for sporadic groups, alternating groups, and simple groups of Lie type in characteristic different from p. We also show it for some groups of Lie type in characteristic p. (The general case of groups of Lie type in their defining characteristic was also left open in [Navarro and Tiep 2020; 2022].) As the knowledge on p-decomposition numbers for groups of Lie type in their own characteristic is too weak at present we refrain from making a general conjecture and just leave it as a question as to whether Property $(*)$ holds for all nonabelian finite simple groups.

Structure of the paper. In Section 2 we prove Property $(*)$ for S a sporadic simple group, an alternating group, a simple group of Lie type in characteristic different from p as well as for some groups of Lie type in characteristic p. In Section 3 we show that Theorem A holds for p-solvable groups and in Theorem 3.4 we reduce the general case to the validity of Property $(*)$ on composition factors, thus completing the proof of Theorem A.

2. Almost simple groups

In this section, we discuss instances of Property $(*)$ from the introduction.

Alternating and sporadic groups. We start out with simple groups not of Lie type.

Proposition 2.1. *Property $(*)$ holds for S a sporadic simple group or the Tits group.*

Proof. Let S be as in the assumption and $H \geq S$ as in Property $(*)$. Since $p > 2$ this means $H = S$. By [Navarro and Tiep 2022, Proposition 3.2] there exists $\chi \in \mathrm{Irr}(B_0(H))$ such that $d_{\chi 1_H} = 0$. If S has abelian Sylow p-subgroups, by one direction of Brauer's height zero conjecture [Kessar and Malle 2013], we then have $\chi \in \mathrm{Irr}_{p'}(B_0(H))$ as required. Now assume Sylow p-subgroups are nonabelian. By [Navarro and Tiep 2020, Lemma 3.2], there is $\chi \in \mathrm{Irr}_{p'}(H) = \mathrm{Irr}_{p'}(B_0(H))$ with $d_{\chi 1_H} = 0$, and again we are done since by inspection in [GAP 2020], S has just one p-block of maximal defect. □

Proposition 2.2. *Property $(*)$ holds for S an alternating group.*

Proof. Let $S = \mathfrak{A}_n$ with $n \geq 5$. As $p > 2$ again we have $H = S$. Set $G := \mathfrak{S}_n$. Recall that the irreducible characters of $\mathfrak{S}_n$ are naturally labelled by partitions of n. If p divides n, then let $\chi = \chi^{(n-2,1^2)} \in \mathrm{Irr}(B_0(G))$, as in [Navarro and Tiep 2022, Proposition 3.1]. Then χ° is the sum of two irreducible Brauer characters of degrees $n-2$ and $\frac{1}{2}(n-2)(n-3)$ and hence it is of degree prime to p and does not contain any irreducible Brauer character of degree 1. Now, any $\theta \in \mathrm{Irr}(S)$ under χ lies in $\mathrm{Irr}_{p'}(B_0(S))$ and $d_{\theta 1_S} = 0$, as wanted.

So we may assume that p does not divide n. Let $n = a_k p^k + \cdots + a_1 p + a_0$ be the p-adic expansion of n. Since p does not divide n, we have $a_0 > 0$. Suppose first that $a_0 > 1$. Consider $\chi = \chi^{(a_0, 1^{n-a_0})}$. By Peel's theorem (see [James 1978, Theorem 24.1]), $\chi^\circ \in \mathrm{IBr}(G)$. Hence, it is enough to show that $\chi(1) \neq 1$, $\chi(1)$ is p', and $\chi \in \mathrm{Irr}(B_0(G))$. By the hook length formula, χ has degree

$$\chi(1) = \frac{n!}{n \cdot (a_0 - 1)!(n - a_0)!} = \binom{n-1}{n-a_0}.$$

Then $\chi(1) \neq 1$ since $1 < a_0 < n$. Moreover

$$n - 1 = a_k p^k + \cdots + a_1 p + (a_0 - 1)$$

and

$$n - a_0 = a_k p^k + \cdots + a_1 p,$$

so by Lucas' theorem we have

$$\chi(1) = \binom{n-1}{n-a_0} \equiv \prod_{i=1}^{k} \binom{a_i}{a_i}\binom{a_0-1}{0} \equiv 1 \pmod{p}.$$

Thus, $\chi(1)$ is not divisible by p. Finally since the p-core of $\lambda = (a_0, 1^{n-a_0})$ is (a_0), we have that χ lies in the principal p-block by the Nakayama conjecture, as desired. Now take $\theta \in \mathrm{Irr}(S)$ under χ, so $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ and $d_{\theta 1_S} = 0$.

Finally, suppose that $a_0 = 1$, so p divides $n-1$. In this case consider $\chi = \chi^{(n-3,2,1)}$, so χ lies in $B_0(G)$. This is the character in the proof of [Navarro and Tiep 2020, Lemma 3.1(ii)], so we are done in this case as well. □

Groups of Lie type in nondefining characteristic. For groups of Lie type in cross characteristic we consider the following setup. Let $\boldsymbol{G}$ be a simple linear algebraic group of adjoint type over an algebraically closed field of characteristic r and $F : \boldsymbol{G} \to \boldsymbol{G}$ a Steinberg map, with group of fixed points $G := \boldsymbol{G}^F$. It is well known that any simple group of Lie type can be obtained as $S = [G, G]$ for $\boldsymbol{G}$, F chosen suitably. Moreover, if $\boldsymbol{G}_{\mathrm{sc}}$ denotes a simply connected covering of $\boldsymbol{G}$, with corresponding Steinberg map also denoted F, then $S \cong \boldsymbol{G}_{\mathrm{sc}}^F/\boldsymbol{Z}(\boldsymbol{G}_{\mathrm{sc}}^F)$, if S is not the Tits simple group, which was already discussed in Proposition 2.1. Let $(\boldsymbol{G}^*, F)$ be dual to $(\boldsymbol{G}, F)$ and $G^* := \boldsymbol{G}^{*F}$.

We let $\boldsymbol{B} \le \boldsymbol{G}$ denote an F-stable Borel subgroup of $\boldsymbol{G}$, and set $B := \boldsymbol{B}^F$.

We recall that outer automorphisms of prime order $p \ge 5$ of simple groups of Lie type are either field automorphisms, or diagonal automorphisms for groups of types $\mathrm{PSL}_n(\epsilon q)$, with $\epsilon \in \{\pm 1\}$.

Proposition 2.3. *Property (∗) holds for S as above if $|B|$ is prime to p.*

Proof. By assumption p divides $|S|$, hence also $|G| = |G^*|$. Let $1 \ne s \in G^*$ be a (semisimple) p-element in the centre of a Sylow p-subgroup of G^*, and let $\chi \in \mathrm{Irr}(G)$ be the semisimple character in the Lusztig series $\mathcal{E}(G, s)$, unique since $\boldsymbol{G}$ has connected centre, see [Geck and Malle 2020, Definition 2.6.9]. By the degree formula for Jordan decomposition [Geck and Malle 2020, Corollary 2.6.6], $\chi(1)$ is then prime to p, and also $\chi(1) > 1$ as p does not divide $|\boldsymbol{Z}(G^*)|$ and so $C_{G^*}(s) < G^*$. Furthermore, by [Hiss 1990b, Corollary 3.4], the semisimple character χ lies in the same p-block of G as the semisimple character in $\mathcal{E}(G, 1)$, i.e., the trivial character, so in the principal p-block. Since p does not divide $|B|$, the permutation module 1_B^G is projective, and thus contains the projective cover of the trivial module. But all constituents of 1_B^G are unipotent, so lie in $\mathcal{E}(G, 1)$; see [Geck and Malle 2020, Example 3.2.6]. Hence, by Brauer reciprocity, 1_G° does not occur as a constituent of χ°. Taking for θ any character of S below χ we see that $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ and $d_{\theta 1_S} = 0$.

Next observe that the order of any outer diagonal automorphism of S divides the order of B. Thus, H is an extension of S by a p-power order field automorphism γ. Note that any such automorphism of S extends to G and then also induces a dual automorphism on G^*, which we denote γ^*; see, e.g., [Taylor 2018, §5.6]. Now choose s more precisely to be also γ^*-invariant (which is possible as γ^* is a p-element, so necessarily has nontrivial fixed points on the centre of a γ^*-stable Sylow p-subgroup of G^*). Then χ is also γ-invariant by [Taylor 2018, Proposition 7.2]. Since the index $|G:S|$ is a divisor of $|B|$, hence prime to p, there exists a γ-invariant $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ below χ, with $d_{\theta 1_S} = 0$. Let $\tilde{\theta}$ be an extension of θ to $H = S\langle\gamma\rangle$ in $B_0(H)$. Then $\tilde{\theta}$ is as desired. □

Proposition 2.4. *Property (∗) holds for S as above if F is a Frobenius map with respect to an $\mathbb{F}_q$-structure and p divides $q-1$.*

Proof. Let W be the Weyl group of G, that is, the F-fixed points of the Weyl group of $\boldsymbol{G}$. As p divides $q-1$, the p-decomposition matrix of the group algebra of W embeds into the p-modular decomposition matrix of G; see [Dipper 1990, Corollary 4.10]. Let $\epsilon \in \mathrm{Irr}_{p'}(W)$ be the (linear) sign character. Then $\epsilon^\circ \neq 1_W^\circ$ since $p > 2$, whence $d_{\epsilon 1_W} = 0$. Now ϵ corresponds to the Steinberg character St of G. Then $d_{\mathrm{St} 1_G} = d_{\epsilon 1_W} = 0$, St lies in the principal p-block of G (e.g., by [Enguehard 2000, Theorem A]) and its degree is a power of the defining characteristic, so prime to p. Now note that any p-automorphism of S is realised inside the extension of G (which induces all diagonal automorphisms) by a generator γ of the cyclic group of p-power order field automorphisms, so we may assume $H \le \tilde{G} := G\langle\gamma\rangle$. By [Geck and Malle 2020, Theorem 4.5.11], St is invariant under γ. Let $\widetilde{\mathrm{St}}$ be an extension of St to $\tilde{G}$ in $B_0(\tilde{G})$, so $d_{\widetilde{\mathrm{St}} 1_{\tilde{G}}} = 0$. Then $\widetilde{\mathrm{St}}|_H$ is irreducible, since St restricts irreducibly to S, hence lies in $B_0(H)$ and so is as required. □

Theorem 2.5. *Property (∗) holds for S of Lie type when p is not the defining characteristic.*

Proof. By Proposition 2.3 we may assume that p divides the order of a Borel subgroup B of G. If F is a Frobenius map with respect to an $\mathbb{F}_q$-structure and p divides $q-1$, we are done by Proposition 2.4. If G is a Suzuki or Ree group and p divides $|B|$, then $p \mid (q^2-1)$ where F^2 defines an $\mathbb{F}_{q^2}$-structure, and the exact same arguments as in the proof of Proposition 2.4 apply.

We are reduced to the case that F is a Frobenius map with respect to an $\mathbb{F}_q$-structure and p divides $|B|$ but not $q-1$. Since p is not the defining prime, this implies that G is a twisted group of Lie type ${}^2A_{n-1}$, ${}^2D_n$, or ${}^2E_6$ and p divides $q+1$, respectively of type ${}^3D_4$ and p divides q^2+q+1. Let $d = 2, 3$ in the respective cases. Then the centraliser of a Sylow d-torus of G is a maximal torus, so has a unique d-cuspidal unipotent character. Thus, by [Enguehard 2000, Theorem A] there is a unique unipotent block of G of maximal defect, the principal block, which hence contains the Steinberg character St of G. By [Hiss 1990a, Theorem B] we have $d_{\mathrm{St}\,1_G} = 0$ in our case unless $G = \mathrm{PGU}_3(q)$. Except for that latter case, we can now argue as in the proof of Proposition 2.4 to conclude. For $G = \mathrm{PGU}_3(q)$ let χ be the cuspidal unipotent character of degree $q(q-1)$, prime to p. By [Geck 1990, Theorem 4.3(a)] it lies in $B_0(G)$ and satisfies $d_{\chi 1_G} = 0$. Again, χ restricts irreducibly to S and is invariant under all automorphisms, so we can argue as before. □

Groups of Lie type in defining characteristic. We do not see how to approach Property (∗) for groups of Lie type in their defining characteristic in general. All characters of positive defect lie in the principal block and decomposition numbers tend to be large and little is known.

Proposition 2.6 (Navarro and Tiep). *Property (∗) holds if $S = \mathrm{PSp}_{2n}(p^f)$, $n \geq 1$, with either $p > 3$ or $p = 3$ and n even.*

Proof. In this case the principal block of S is the only p-block of positive defect. Let H be almost simple with H/S a p-group. By [Navarro 1998, Corollary 9.6] there is just one p-block of H covering $B_0(S)$, necessarily the principal block $B_0(H)$. Now the irreducible character $\chi \in \mathrm{Irr}_{p'}(H)$ with $d_{\chi 1_H} = 0$ constructed in [Navarro and Tiep 2020, Proposition 3.11] lies above a character of S of positive defect, hence in the principal p-block of H and we are done. □

Observe that this does not extend to $p = 3$ and n odd: the group $H = \mathrm{PSp}_2(3^3).3$ has no irreducible character $\chi \in \mathrm{Irr}_{3'}(H)$ with $d_{\chi 1_H} = 0$.

[Navarro and Tiep 2022] also contains results for special linear and unitary groups but these are not applicable here as the considered characters are not of p'-degree. Nevertheless, we can follow their general approach.

For $G = \mathrm{SL}_n(q)$ we let τ_j, $j = 1, \ldots, q-2$ denote the nonunipotent *Weil characters* of degree $(q^n - 1)/(q-1)$, ordered so that τ_j is trivial on the centre $Z(\mathrm{SL}_n(q))$ of order $z := \gcd(n, q-1)$ if and only if $z \mid j$.

Proposition 2.7. *Let $S = \mathrm{PSL}_n(q)$ with $q = p^f$, $p \neq 2$ and $n \geq 3$.*

(a) *If either* $\gcd(p-1, (q-1)/\gcd(n, q-1)) > 1$ *or* $2^f < (q-1)/z - 1$ *then there is* $\chi \in \mathrm{Irr}_{p'}(B_0(S))$ *such that* $d_{\chi 1_S} = 0$.

(b) *Write* $f = p^a f'$ *with* $\gcd(p, f') = 1$ *and set* $q' := p^{f'}$. *If*

$$a = 0 \quad \text{or} \quad 2^f < (q'-1)/\gcd(q'-1, n) - 1$$

then Property (∗) holds for $S = \mathrm{PSL}_n(q)$.

Proof. Let $G := \mathrm{SL}_n(q)$ and set $z = \gcd(n, q-1)$. We are interested in the characters τ_j of G that are trivial on $Z(G)$, that is, for which $j = zj'$ for some integer $1 \leq j' \leq (q-1-z)/z$. By [Zalesski and Suprunenko 1990, Theorem 1.11] the decomposition number $d_{\tau_j 1_G}$ equals the number of solutions $x_s \in \{0, 1\}$ of the congruence

$$n(p-1)\sum_{s=0}^{f-1} x_s p^s \equiv j \pmod{(q-1)}.$$

Dividing by z, we need to count solutions to

$$\frac{n}{z(p-1)}\sum_{s=0}^{f-1} x_s p^s \equiv j' \pmod{(q-1)/z}.$$

If there is a prime ℓ dividing $p-1$ and $(q-1)/z$, then reducing modulo ℓ we see there is no solution for $j' = 1$. Also, the left hand side can take at most 2^f distinct values. Since there are $(q-1)/z - 1$ admissible values for j', there is j' with no solutions whenever $2^f < (q-1)/z - 1$. Thus, under either of our assumptions we find j' with $\tau_j = \tau_{zj'} \in \mathrm{Irr}_{p'}(G)$ with $d_{\tau_j 1_G} = 0$. Since G has a single p-block of positive defect, τ_j lies in the principal block. Furthermore, by construction $Z(\mathrm{SL}_n(q))$ lies in the kernel of τ_j and hence τ_j deflates to a character of $S = \mathrm{PSL}_n(q)$. Thus we get (a).

(b) Since $p > 2$ does not divide $q-1$, the p-power order automorphisms of S are field automorphisms, of order dividing p^a where $f = p^a f'$ is as in the statement. If $a = 0$, Property (∗) follows from (a) since necessarily $H = S$. For $a > 0$ let γ be a field automorphism of G (and hence of S) of order p^a. There are exactly $q' - 2$ Weil characters of G invariant under γ, which hence extend to $G\langle\gamma\rangle$. Of these, $(q'-1)/\gcd(q'-1, n) - 1$ are trivial on $Z(G)$, so define characters in $\mathrm{Irr}_{p'}(S)$ invariant under γ. By the argument above, if this number is bigger than 2^f then there exists such a character χ with $d_{\chi 1_S} = 0$. Hence any character of H in $B_0(H)$ lying above it verifies Property (∗). □

Note that the case $n = 2$ is addressed in Proposition 2.6. Observe that the condition in Proposition 2.7(a) is satisfied if $\gcd(n, q-1) = 1$, for example. It also holds when $p > n+1$ (since $p-1$ always divides $q-1$), or if $q > n(2^f+1)$. Thus, Proposition 2.7 extends and complements [Navarro and Tiep 2022, Proposition 3.3(ii)].

Corollary 2.8. *Let $S = \mathrm{PSL}_n(q)$ with $q = p^f$, $p \neq 2$ and $3 \le n \le 9$. Then there is $\chi \in \mathrm{Irr}_{p'}(B_0(S))$ with $d_{\chi 1_S} = 0$ unless possibly S is one of*

$$\mathrm{PSL}_4(5),\ \mathrm{PSL}_6(3),\ \mathrm{PSL}_6(7),\ \mathrm{PSL}_8(3),\ \mathrm{PSL}_8(9),\ \mathrm{PSL}_8(5),\ \mathrm{PSL}_8(25).$$

Proof. For the groups $\mathrm{PSL}_n(q)$, $n \le 9$, considered here, either the conditions in Proposition 2.7(a) are satisfied, or if not, a direct checking with the Zalesski–Suprunenko formula shows the claim, except for the groups listed in the conclusion and for $\mathrm{PSL}_4(3)$. The decomposition matrix of $\mathrm{PSL}_4(3)$ is available in [GAP 2020] from which the claim can be verified for that group. □

For $G = \mathrm{SU}_n(q)$ we let τ_j, $j = 1, \dots, q$, denote the nonunipotent Weil characters, constructed by Seitz [1975], of degree $(q^n - (-1)^n)/(q+1)$, again ordered so that τ_j is trivial on the centre $Z(\mathrm{SU}_n(q))$ of order $z := \gcd(n, q+1)$ if and only if $z \mid j$.

Proposition 2.9. *Let $S = \mathrm{PSU}_n(q)$ with $q = p^f$, $p \neq 2$ and $n \geq 3$.*

(a) *If $2^f < (q+1)/z - 1$ then there is $\chi \in \mathrm{Irr}_{p'}(B_0(S))$ such that $d_{\chi 1_S} = 0$.*

(b) *Write $f = p^a f'$ with $\gcd(p, f') = 1$ and set $q' := p^{f'}$. If*

$$a = 0 \quad or \quad 2^f < (q'+1)/\gcd(q'+1, n) - 1$$

then Property (∗) holds for $S = \mathrm{PSU}_n(q)$.

Proof. The argument is very similar to the one for the special linear groups. Let $G = \mathrm{SU}_n(q)$ and set $z = \gcd(n, q+1) = |Z(G)|$. Again, we consider characters τ_j trivial on $Z(G)$, that is, for which $j = zj'$ for some integer $1 \leq j' \leq (q+1-z)/z$. By [Zalesski 1990, Main Theorem] the decomposition number $d_{\tau_j 1_G}$ equals the number of solutions $x_s \in \{0, 1\}$ of the congruence

$$n\left((p-1)\sum_{s=0}^{f-1} x_s p^s - 1\right) \equiv j \pmod{(q+1)}.$$

(In fact, [Zalesski 1990] has an additional summand of $\frac{1}{2}(q+1)$ on the right-hand side, but this disappears here due to a different numbering of the τ_j, see [Navarro and Tiep 2022, p. 612]; in any case, this difference will not matter for our argument here.) Since the left-hand side can take at most 2^f distinct values, while there are $(q+1)/z - 1$ admissible values for j' the assertion in (a) follows. For (b) we can argue exactly as in the proof of Proposition 2.7. □

As in [Navarro and Tiep 2020; 2022] we have no general results for orthogonal or exceptional type groups in their defining characteristic.

3. The reduction

In this section we prove Theorem A. We will need the following results, which we collect here for the reader's convenience.

Lemma 3.1 [Murai 1994, Lemma 4.3]. *Let $N \triangleleft G$ and let $\theta \in \mathrm{Irr}_{p'}(B_0(N))$. Suppose that θ extends to PN, where $P \in \mathrm{Syl}_p(G)$. Then there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ satisfying $[\theta^G, \chi] \neq 0$.*

The following argument is inside the proof of [Navarro and Tiep 2020, Theorem 2.6].

Lemma 3.2. *Let G be a finite group and suppose that $d_{\chi 1_G} \neq 0$ for every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$. Let $M \triangleleft G$ and let $P \in \mathrm{Syl}_p(G)$. Then $d_{\psi 1_{MP}} \neq 0$ for every $\psi \in \mathrm{Irr}_{p'}(B_0(MP))$.*

Proof. Since MP/M is a p-group, we have that $\psi_M = \tau \in \mathrm{Irr}_{p'}(B_0(M))$. By Lemma 3.1 there exists $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ lying over τ. By hypothesis, $d_{\chi 1_G} \neq 0$, and then χ_M° contains 1_M, so $d_{\tau 1_M} \neq 0$. Since MP/M is a p-group, we have that $(MP)^\circ = M^\circ$ and then $d_{\psi 1_{MP}} \neq 0$, as wanted. □

We next prove Theorem A for p-solvable groups.

Theorem 3.3. *Let G be a p-solvable group. Then the following are equivalent*:

(i) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} \neq 0$.*

(ii) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} = 1$.*

(iii) *For $P \in \mathrm{Syl}_p(G)$ we have $\boldsymbol{N}_G(P) = P \times K$ for some $K \leq G$.*

Proof. We first prove (iii) implies (ii). By [Navarro et al. 2014, Theorem 3.2] we have $K \subseteq \boldsymbol{O}_{p'}(G) =: X$. Now, let $\overline{G} = G/X$ and $\overline{P} = PX/X$, then (iii) implies that $\boldsymbol{N}_{\overline{G}}(\overline{P}) \cong \overline{P}$. Since $\mathrm{Irr}_{p'}(B_0(G)) = \mathrm{Irr}_{p'}(B_0(G/X))$ we know by [Navarro and Tiep 2020, Theorem B] that (i) and (ii) hold.

Since (ii) implies (i) trivially, we just need to show that (i) implies (iii). We proceed by induction on $|G|$. Let $N = \boldsymbol{O}_{p'}(G)$ and use the bar notation. Since $\mathrm{Irr}_{p'}(B_0(G)) = \mathrm{Irr}_{p'}(B_0(\overline{G}))$, if $N > 1$, we have by induction that $\boldsymbol{N}_{\overline{G}}(\overline{P}) = \overline{P} \times \overline{K}$. By [Navarro et al. 2014, Theorem 3.2] we have that $\overline{K} \subseteq \boldsymbol{O}_{p'}(\overline{G}) = 1$ so $\overline{K} = 1$ and hence $\boldsymbol{N}_{\overline{G}}(\overline{P}) = \overline{P}$. This implies that $\boldsymbol{N}_G(P) = P \times \boldsymbol{C}_N(P)$ and we are done. So we may assume that $N = 1$. But in this case the principal p-block is the only p-block of G, and we are done by [Navarro and Tiep 2020, Theorem B]. □

We finally prove Theorem A.

Theorem 3.4. *Let G be a finite group. Assume that $p > 3$ and all nonabelian composition factors of G of order divisible by p satisfy Property $(*)$. Then the following are equivalent*:

(i) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} \neq 0$.*

(ii) *For every $\chi \in \mathrm{Irr}_{p'}(B_0(G))$ we have $d_{\chi 1_G} = 1$.*

(iii) *For $P \in \mathrm{Syl}_p(G)$ we have $\boldsymbol{N}_G(P) = P \times K$ for some $K \leq G$.*

Proof. Since [Navarro et al. 2014, Theorem 3.2] holds for odd primes, we can argue as in the first part of the proof of Theorem 3.3 to show that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) (notice that, as happens in [Navarro and Tiep 2020], (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is always true if $p > 3$, with no extra conditions on the composition factors of G). So we just need to prove that (i) implies (iii). We work by induction on $|G|$. Arguing as in the second paragraph of the proof of Theorem 3.3 we may then assume that $\boldsymbol{O}_{p'}(G) = 1$.

Let $M \lhd G$ be the largest p-solvable normal subgroup of G. We claim that $M = 1$. Let $\overline{G} = G/M$ and use the bar notation. Suppose $M > 1$. Since $\mathrm{Irr}_{p'}(B_0(\overline{G}))$ is contained in $\mathrm{Irr}_{p'}(B_0(G))$ we have by induction that $\boldsymbol{N}_{\overline{G}}(\overline{P}) = \overline{P} \times \overline{K}$. By [Navarro and Tiep 2020, Theorem 3.2] this implies that $\overline{K} \subseteq \boldsymbol{O}_{p'}(\overline{G})$. Since M is the largest normal p-solvable subgroup of G, $\boldsymbol{O}_{p'}(\overline{G})$ is trivial and hence $\boldsymbol{N}_{\overline{G}}(\overline{P}) = \overline{P}$. By [Guralnick et al. 2004, Theorem 1.1] this forces $\overline{G}$ to be solvable. Hence G is p-solvable and we are done by Theorem 3.3.

Let N be a minimal normal subgroup of G; thus $N = S_1 \times \cdots \times S_t$ where $S_i \cong S$ is a nonabelian simple group of order divisible by p and the S_i are transitively permuted by G. Let $X/N = \boldsymbol{O}_{p'}(G/N)$ and let $H = \bigcap \boldsymbol{N}_G(S_i)$. We claim that $G = (H \cap X)P$. Write $Y = (H \cap X)P$. We first show that $\boldsymbol{O}_{p'}(Y) = 1$. Indeed, since $\boldsymbol{O}_{p'}(G)$, we have that $\boldsymbol{O}_{p'}(Y) \cap (H \cap X) \subseteq \boldsymbol{O}_{p'}(H \cap X) = 1$. Now, $\boldsymbol{O}_{p'}(Y) \cong (H \cap X)\boldsymbol{O}_{p'}(Y)/(H \cap X)$ is a p-group, so $\boldsymbol{O}_{p'}(Y) = 1$ as wanted. By Lemma 3.2 applied to $H \cap X$ in place of M, we have that Y satisfies (i). Since the nonabelian composition factors of Y are composition factors of G, if $Y < G$, by induction this gives $\boldsymbol{N}_Y(P) = P \times K$ for some K. But then $K \subseteq \boldsymbol{O}_{p'}(Y)$ by [Navarro et al. 2014, Theorem 3.2], so $K = 1$. This means that $\boldsymbol{N}_Y(P) = P$ and then by [Guralnick et al. 2004, Theorem 1.1] the group Y is solvable. But then N is solvable, a contradiction. Hence $Y = G$, as wanted.

Since $G = (H \cap X)P$ and H acts trivially on $\{S_1, \ldots, S_t\}$, P must act transitively on the set $\{S_1, \ldots, S_t\}$. Write $S = S_1$ and, for $i = 2, \ldots, t$, write $S_i = S^{x_i}$ with $x_i \in P$. We proceed now as in the proof of [Navarro and Tiep 2020, Theorem 2.6]. Let $R = \boldsymbol{N}_P(S)$. If $SR = G$, then $S = N$ and $\boldsymbol{C}_G(S)$ is a normal p-subgroup of G. Since there are no nontrivial p-solvable normal subgroups of G, we conclude that $\boldsymbol{C}_G(S) = 1$ and hence G is almost simple with socle S and $|G : S|$ is a power of p. Since S satisfies Property $(*)$ by assumption, we have a contradiction. Hence we may assume that $SR < G$.

Let $Q = P \cap N$ and let $R_1 = R \cap S = Q \cap S = P \cap S \in \mathrm{Syl}_p(S)$. Let $\gamma \in \mathrm{Irr}_{p'}(B_0(SR))$ and notice that $\gamma_S = \psi \in \mathrm{Irr}_{p'}(B_0(S))$ since SR/S is a p-group. For $i = 2, \ldots, t$, let $\psi_i = \psi^{x_i} \in \mathrm{Irr}_{p'}(B_0(S_i))$ and let $\eta = \psi \times \psi_2 \times \cdots \times \psi_t \in \mathrm{Irr}_{p'}(B_0(N))$, which is P-invariant by [Navarro et al. 2007, Lemma 4.1(ii)]. Then η extends to PN. By Lemma 3.1 and hypothesis we have $d_{\eta 1_N} \neq 0$. By [Navarro and Tiep 2020, Lemma 2.3] we have that $d_{\psi 1_S} \neq 0$ and then, since SR/S is a p-group, we conclude that $d_{\gamma 1_{SR}} \neq 0$. Since $SR < G$ and S is a composition factor of G, this implies $\boldsymbol{N}_{SR}(R) = R \times K$ for some K. Then $K \subseteq \boldsymbol{O}_{p'}(SR)$. Arguing as before, we have that $\boldsymbol{O}_{p'}(SR) = 1$, so $K = 1$ and then $\boldsymbol{N}_{SR}(R) = R$. Now by [Guralnick et al. 2004, Theorem 1.1], SR is solvable, and hence S is solvable, which is our final contradiction. □

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LEVI DECOMPOSITIONS OF LINEAR ALGEBRAIC GROUPS AND NONABELIAN COHOMOLOGY

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To the memory of Gary Seitz (1943–2023)

Let k be a field, and let G be a linear algebraic group over k for which the unipotent radical U of G is defined and split over k. Consider a finite, separable field extension ℓ of k and suppose that the group G_ℓ obtained by base-change has a *Levi decomposition* (over ℓ). We continue here our study of the question previously investigated (*Arch. Math.* 100:1 (2013), 7–24): does G have a *Levi decomposition* (over k)?

Using nonabelian cohomology we give some condition under which this question has an affirmative answer. On the other hand, we provide an(other) example of a group G as above which has no Levi decomposition over k.

1. Introduction

Let k be a field, and let G be a linear algebraic group over k. Thus G is a group scheme which is smooth and affine over k.

If k_{alg} denotes an algebraic closure of k, the *unipotent radical* of $G_{k_{\text{alg}}}$ is the maximal connected, unipotent, normal subgroup. The unipotent radical of G is defined over k if G has a k-subgroup U such that $U_{k_{\text{alg}}}$ is the unipotent radical of $G_{k_{\text{alg}}}$.

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Definition 1.1. We say that G *satisfies condition **(R)*** if the unipotent radical U of G is *defined and split* over k. (See Definition 2.1 for the notion of split unipotent group). Write $\pi : G \to G/U$ for the quotient morphism; we say that G/U is the *reductive quotient* of G. Thus G/U is a (not necessarily connected) reductive algebraic group.

Remark 1.2. If k is perfect then **(R)** holds for any linear algebraic group G over k. Indeed, the unipotent radical U is defined over k by Galois descent. Every (smooth) connected unipotent group over a perfect field k is k-split; see Remark 2.2.

Definition 1.3. Suppose that G satisfies condition **(R)**. The group G has a *Levi decomposition* (over k) if there is a closed k-subgroup scheme M of G such that the restriction of the quotient mapping determines an isomorphism

$$\pi_{|M} : M \xrightarrow{\sim} G/R.$$

The subgroup M is then a *Levi factor* of G.

If G satisfies **(R)** and if M is a Levi factor of G then Proposition 2.7 below shows that G may be identified with the semidirect product $U \rtimes M$ as algebraic groups.

Remark 1.4. When k has characteristic 0, Mostow showed that G always has a Levi decomposition; see, e.g., [McNinch 2010, §3.1]. For any field k of characteristic $p > 0$, there are linear algebraic groups G over k with no Levi factor; see, e.g., Conrad, Gabber, and Prasad's work [Conrad et al. 2015, A.6] for a construction.

We now fix a linear algebraic k-group G satisfying **(R)**. Suppose that ℓ is a finite, separable field extension of k, and suppose that G_ℓ has a Levi decomposition. We pose this question:

($\diamond$) If G_ℓ has a Levi decomposition (over ℓ), does G have a Levi decomposition (over k)?

This question about descent of Levi factors was already considered in [McNinch 2013] whose main result gave the following partial answer:

Theorem 1.5. *Assume that ℓ is a finite, Galois field extension of k with Galois group $\Gamma = \mathrm{Gal}(\ell/k)$, and assume that G_ℓ has a Levi decomposition. If $|\Gamma|$ is invertible in k then G has a Levi decomposition.*

We introduce the nonabelian cohomology set $H^1_{\mathrm{coc}}(M, U)$ in Section 3, and in Section 4 we prove the following result providing a different partial answer to ($\diamond$):

Theorem 1.6. *If ℓ is a finite separable extension of k, suppose*

(*a*) *G_ℓ has a Levi decomposition,*

(*b*) *the group scheme ${U_\ell}^{M_\ell}$ is trivial, and*

(*c*) $H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$.

Then G has a Levi decomposition.

We also prove Corollary 4.5 which gives a reformulation of Theorem 1.6 using a filtration of U. After some preliminaries in Sections 5 and 6, we prove the following related result in Section 7:

Theorem 1.7. *Suppose*

(*a*) G_ℓ *has a Levi decomposition*,

(*b*) $\mathrm{Inn}(U_\ell)^{M_\ell}$ *is trivial*,

(*c*) *the center Z of U is a vector group on which G acts linearly*, *and*

(*d*) $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$.

Then G has a Levi decomposition.

The reader should compare these results with [McNinch 2010, Theorem 5.2]. This older result shows that a certain condition involving the vanishing of second cohomology H^2 unconditionally guarantees the existence of a Levi factor. These newer results — Theorems 1.6, 1.7 and Corollary 4.5 — instead give conditions using vanishing of (some form of) first cohomology to descend Levi factors over finite separable field extensions.

We note that *some* additional hypotheses are required to answer the question (◇). Indeed, Section 8 provides an example of an algebraic group G satisfying condition **(R)** for which G_ℓ has a Levi factor for some cyclic Galois extension ℓ of degree p over k, but G has no Levi factor over k.

Every example currently known to the author of a group G satisfying **(R)** for which (◇) has a negative answer is *not connected*. This suggests the following natural problem for which a solution would be desirable:

Problem 1.8. Let ℓ be a finite, separable field extension of k and G a connected linear algebraic group over k satisfying **(R)**. Either find a proof of the assertion "G_ℓ has a Levi factor implies that G has a Levi factor" or find an example of a group for which this condition fails.

2. Preliminaries

We fix an arbitrary field k. Throughout the paper, G will denote a linear algebraic group over k. Thus G is a group scheme which is smooth, affine, and of finite type over k.

If V is a linear representation of G, then for $i \geq 0$, $H^i(G, V)$ denotes the i-th (*Hochschild*) *cohomology* group of V; see, for instance, [Jantzen 2003, I.4].

Automorphism group functors. By a k-group functor, we mean a functor from the category of commutative k-algebras to the category of groups. Of course, any group scheme — and in particular, any linear algebraic group — over k is a fortiori a k-group functor, but we will consider a few group functors which are in general not representable (i.e., which fail to be group schemes).

For a linear algebraic group G over k, we write $\mathrm{Aut}(G)$ for the k-group functor which assigns to a commutative k-algebra Λ the group $\mathrm{Aut}(G)(\Lambda) = \mathrm{Aut}(G(\Lambda))$.

If Z denotes the (scheme-theoretic) center of G, there is a natural homomorphism of k-group functors $\mathrm{Inn} : G/Z \to \mathrm{Aut}(G)$ whose image determines a normal k-subgroup functor $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$; see [SGA 3_{III} 2011, XXIV §1.1].

Now, the k-group functor $\mathrm{Out}(G)$ is defined for each Λ by the rule

$$\mathrm{Out}(G)(\Lambda) = \mathrm{Aut}(G)(\Lambda)/\,\mathrm{Inn}(G)(\Lambda)).$$

The quotient mappings $\mathrm{Aut}(G(\Lambda)) \to \mathrm{Aut}(G(\Lambda))/\,\mathrm{Inn}(G(\Lambda))$ determine a homomorphism of k-group functors

$$\Psi : \mathrm{Aut}(G) \to \mathrm{Out}(G). \tag{2-1}$$

Unipotent groups. Recall from [Borel 1991, §15.1] the following:

Definition 2.1. A connected, unipotent linear algebraic group U over k is said to be *k-split* provided that there is a sequence

$$1 = U_0 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

of closed, connected, normal k-subgroups of U such that $U_{i+1}/U_i \simeq \boldsymbol{G}_{a/k}$ for $i = 0, \ldots, m-1$, where $\boldsymbol{G}_a = \boldsymbol{G}_{a/k}$ is the additive group.

Remark 2.2. When k is not a perfect field, there are connected unipotent k-groups which are not k-split; see, for example, [Serre 2002, III.§2.1, Exercise 3]. On the other hand, if k is perfect, every connected unipotent k-group is k-split; see [Borel 1991, Corollary 15.5(ii)].

Proposition 2.3. *Let U be a k-split unipotent group. If V is a normal k-subgroup of U, then U/V is again a k-split unipotent group.*

Proof. The assertion follows from [Borel 1991, Theorem 15.4(i)]. □

A substantial reason for our focus on split unipotent groups is the following result of Rosenlicht:

Proposition 2.4. *Suppose that U is a connected, k-split unipotent subgroup of G and write $\pi : G \to G/U$ for the quotient morphism. Then there is a morphism of k-varieties*

$$\sigma : G/U \to G,$$

which is a section to π — that is, $\pi \circ \sigma$ is the identity. In particular, the mapping $\pi : G(k) \to (G/U)(k)$ on k-points is surjective.

Proof. See [Springer 2009, Theorem 14.2.6]. □

Extensions, group actions and semidirect products. Let A and M be linear algebraic k-groups.

Definition 2.5. An *extension* of M by A is a linear algebraic k-group E together with a sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} M \to 1, \tag{2-2}$$

where i and π are morphisms of algebraic groups over k, i determines an isomorphism of A onto $\ker \pi$, and the homomorphism π is faithfully flat.

Definition 2.6. If A and M are linear algebraic groups, we say that A is *an M-group* provided that there is a morphism of k-group functors $M \to \mathrm{Aut}(A)$.

If A is an M-group via the homomorphism of k-group functors

$$\alpha : M \to \mathrm{Aut}(A)$$

then we can form the semidirect product $A \rtimes_\alpha M$; it is an extension of M by A. (We omit the subscript α from $\rtimes_\alpha$ when it is clear from context).

If E is an extension (2-2), observe that the conjugation action of E determines a morphism of group functors $\mathrm{Inn} : E \to \mathrm{Aut}(A)$.

We record the following two results; their proofs are straightforward and left to the reader:

Proposition 2.7. *Let A and M be linear algebraic k-groups and consider an extension* (2-2)

$$1 \to A \to G \xrightarrow{\pi} M \to 1.$$

If $s : M \to G$ is a group homomorphism that is a section to π then the multiplication mapping $(x, m) \mapsto xm$ induces an isomorphism

$$A \rtimes_\phi M \xrightarrow{\sim} G$$

of algebraic k-groups, where $\phi : M \to \mathrm{Aut}(A)$ is the composite $\mathrm{Inn} \circ s$.

Proposition 2.8. *Let A and M be linear algebraic k-groups and consider an extension* (2-2)

$$1 \to A \to G \xrightarrow{\pi} M \to 1.$$

There is a unique homomorphism of k-group functors $\phi : M \to \mathrm{Out}(A)$ such that for any section $s_0 : M \to G$ to π as in Proposition 2.4, for any commutative k-algebra Λ, and for any $m \in M(\Lambda)$, $\phi(m)$ is the class of the inner automorphism $\mathrm{Inn}(s_0(m))$ *in* $\mathrm{Out}(A)$.

Remark 2.9. A unipotent k-group U is *wound* if every mapping $\mathbf{A}^1 \to U$ of k-schemes is constant. A connected, wound unipotent group of positive dimension is not k-split. If M is a connected and reductive k-group and if U is a wound unipotent k-group, then

(∗) any homomorphism of k-group functors $M \to \operatorname{Aut}(U)$ is trivial.

If M is a torus then (∗) follows from [Conrad et al. 2015, Corollary B.44]. Now (∗) follows in general since the connected reductive group M is generated by its maximal k-tori — see [Springer 2009, Theorem 13.3.6].

Observation (∗) provides some partial justification for our focus on groups satisfying **(R)**.

Linear actions. Let G and U be linear algebraic groups, suppose that U is connected and unipotent, and suppose that U is a G-group.

Definition 2.10. If U is a vector group, the action of G on U is said to be *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \operatorname{Lie}(U)$.

Definition 2.11. A filtration

$$1 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

by G-invariant closed k-subgroups U_i with U_i normal in U_{i+1} for each i is a *linear filtration* for the action of G if U_{i+1}/U_i is a vector group on which G acts linearly for each $i = 0, \dots, m-1$.

A linear filtration is a *central linear filtration* if U_{i+1}/U_i is central in U/U_i for each $i \ge 0$.

The following result was proved already in [Stewart 2013] under the assumption that k is algebraically closed.

Theorem 2.12. *Assume that the unipotent radical U of G is defined and split over k.*

(a) *If G is connected, there is a linear filtration of U for the action of G.*

(b) *If U has a linear filtration for the action of $U \rtimes G$ then it has a central linear filtration.*

Proof. (a) is the main result of [McNinch 2014].

To see (b), suppose that the subgroups U_i form a linear filtration of U for the action of $U \rtimes G$. We may clearly refine this filtration to arrange that $\operatorname{Lie}(U_i)/\operatorname{Lie}(U_{i+1})$ is an irreducible representation of $U \rtimes G$ for each i.[1] We claim that this refined filtration is central. We proceed by induction on the length m of the linear filtration. If $m = 1$ then U is abelian and the result is immediate.

Suppose now that $m > 1$ and that one knows that any linear filtration of U for the action of $U \rtimes G$ of length $< m$ for which the factors of consecutive terms form irreducible $U \rtimes G$-representations is central.

[1] Since U is unipotent, an irreducible representation of $U \rtimes G$ amounts to an irreducible representation of G.

Now, the conjugation action of U on U_1 is a *linear* action; thus, the fixed points for the conjugation action of U on U_1 form a G-invariant subgroup scheme which is smooth over k. Since $U_1 \simeq \mathrm{Lie}(U_1)$ is an irreducible G-representation, it follows that U acts trivially on U_1; thus U_1 is central in U. Now, it is clear that

$$1 \subset U_2/U_1 \subset \cdots \subset U_m/U_1 = U/U_1 \tag{2-3}$$

forms a linear filtration of U/U_1 for the action of G for which the factors of consecutive terms form irreducible $U \rtimes G$-representations. Thus by induction (2-3) is a central linear filtration; this completes the proof. □

Remark 2.13. In the proof of Theorem 2.12, we constructed a central linear filtration by arranging that the action of $U \rtimes G$ on each quotient U_{i+1}/U_i is irreducible. This condition is sufficient, but not necessary — in general, there are central linear filtrations for which $\mathrm{Lie}(U_{i+1})/\mathrm{Lie}(U_i)$ is a reducible G-representation for some i.

Galois cohomology. Write $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ for the absolute Galois group of k where k_{sep} is a separable closure of k.

Let G be a k-group functor satisfying the conditions spelled out in [Serre 2002, II.§1.1]. Then Γ acts continuously on the group $G(k_{\mathrm{sep}})$ and we may consider the Galois cohomology set $H^1(k, G) := H^1(\Gamma, G(k_{\mathrm{sep}}))$ [Serre 2002, II.§5.1].

Proposition 2.14. *Let U be a connected, split unipotent algebraic group over k. Then the Galois cohomology set satisfies $H^1(k, U) = 1$.*

Proof. The necessary tools are recalled in [McNinch 2004, Proposition 30]. □

3. Nonabelian cohomology

Let A and M be linear algebraic k-groups and suppose that A is an M-group. Following [Demarche 2015, §2.1], we introduce the cohomology set $H^1_{\mathrm{coc}}(M, A)$ as follows. Let $Z^1_{\mathrm{coc}}(M, A)$ denote the set of regular maps $f : M \to A$ such that for each commutative k-algebra Λ and each $x, y \in M(\Lambda)$, the 1-cocycle condition

$$f(xy) = f(x) \cdot {}^x f(y) \tag{3-1}$$

holds. Two cocycles $f, f' \in Z^1_{\mathrm{coc}}(M, A)$ are *cohomologous* provided there is $u \in U(k)$ such that for each Λ and each $x \in M(\Lambda)$ we have

$$f(x) = u^{-1} \cdot f'(x) \cdot {}^x u.$$

This defines an equivalence relation on $Z^1_{\mathrm{coc}}(M, A)$ and we write $H^1_{\mathrm{coc}}(M, A)$ for the quotient set.

We view $H^1_{\mathrm{coc}}(M, A)$ as a *pointed set*; the marked point $1 \in H^1_{\mathrm{coc}}(M, A)$ is the class of the cocycle in $Z^1_{\mathrm{coc}}(M, A)$ which takes the constant value 1. The pointed set $H^1_{\mathrm{coc}}(M, A)$ is trivial if $H^1_{\mathrm{coc}}(M, A) = \{1\}$; we often indicate this condition by the shorthand $H^1_{\mathrm{coc}}(M, A) = 1$.

One interpretation or application of this cohomology set arises from examination of a semidirect product $G = A \rtimes M$. Consider a linear algebraic group G with normal subgroup A and a quotient mapping $\pi : G \to M = G/A$. We suppose that there is a group homomorphism $s_0 : M \to G$ which is a section to π. According to Proposition 2.7, s_0 determines an isomorphism $G \simeq A \rtimes M$.

Definition 3.1. Consider the set of all homomorphisms of k-groups $M \to G$ which are sections to π; two such homomorphisms s, s' will be considered *equivalent* if there is $a \in A(k)$ such that $s = as'a^{-1}$. Then $\mathrm{Sect}(G \xrightarrow{\pi} M)$ denotes the quotient of the set of all such homomorphisms by this equivalence relation.

Proposition 3.2. *Write $\mu : G \times G \to G$ for the multiplication mapping. For a given homomorphism $s_0 : M \to G$ which is a section to π, the assignment*

$$f \mapsto \mu \circ (f, s_0),$$

— where $(f, s_0) : M \to M \times G$ is the mapping $m \mapsto (f(m), s_0(m))$ — determines a bijection

$$A_{s_0} : H^1_{\mathrm{coc}}(M, A) \to \mathrm{Sect}(G \xrightarrow{\pi} M).$$

Proof. As already observed above, the choice of s_0 determines an isomorphism of linear algebraic groups $G \simeq A \rtimes M$; see Proposition 2.7. Now the result follows from [Demarche 2015, Proposition 2.2.2]. □

Remark 3.3. $H^1_{\mathrm{coc}}(M, A)$ is a pointed set — i.e., a set with a distinguished element. That distinguished element is the class of the trivial mapping $(x \mapsto 1) : G \to A$. In the bijection of Proposition 3.2 the section corresponding to the trivial class is s_0.

Remark 3.4. When Z is a vector group with a linear action of M, $H^1_{\mathrm{coc}}(M, Z)$ coincides with the usual Hochschild cohomology group $H^1(M, Z) \simeq H^1(M, \mathrm{Lie}(Z))$. In that case $H^1_{\mathrm{coc}}(M, Z)$ is a k-vector space.

Suppose now that $A = U$ is a split unipotent M-group and that $Z \subset U$ is a central k-subgroup that is M-invariant. Then U/Z is a split unipotent M-group, and there is a mapping

$$\Delta : H^1_{\mathrm{coc}}(M, U/Z) \to H^2(M, Z), \tag{3-2}$$

where $H^2(M, Z)$ denotes the second Hochschild cohomology; it is defined as follows. First, use Rosenlicht's result Proposition 2.4 to choose a regular mapping $s : U/Z \to U$ which is a section to the quotient homomorphism $U \to U/Z$. Let $\alpha = [f] \in H^1_{\mathrm{coc}}(M, A/Z$ with $f \in Z^1_{\mathrm{coc}}(M, A/Z)$.

As in [Demazure and Gabriel 1970, II, Subsection 3.2.3] — see also [McNinch 2010, §4.4] — the rule $(g, h) \mapsto s(f(g))s(f(h))s(f(gh))^{-1}$ determines a Hochschild 2-cocycle whose class in $H^2(G, Z)$ we denote by $\Delta(\alpha)$.

Proposition 3.5. *Let U be a split unipotent M-group, and let Z be a central, closed and smooth k-subgroup of U that is M-invariant. Write $i : Z \to U$ and $\pi : U \to U/Z$ for the inclusion and quotient mappings, respectively.*

(*a*) *The sequence of pointed sets*

$$H^1(M, Z) \xrightarrow{i_*} H^1_{\mathrm{coc}}(M, U) \xrightarrow{\pi_*} H^1_{\mathrm{coc}}(M, U/Z) \xrightarrow{\Delta} H^2(M, Z)$$

is exact.

(*b*) *If $(U/Z)^M = 1$ then i_* is injective.*

Sketch. (a) The proof of the corresponding statement for cohomology of pro-finite groups given in [Serre 2002, I.§5.7] may be applied here mutatis mutandum. The main required adaptation is the definition (given above) of the mapping Δ (which required the existence of a regular section $U/Z \to U$).

(b) Suppose that $f_1, f_2 : M \to Z$ are 1-cocycles and that $i_*([f_1]) = i_*([f_2])$. Thus f_1, f_2 are cohomologous in $Z^1_{\mathrm{coc}}(M, U)$, so there is $u \in U(k)$ such that

$$f_1(x) = u^{-1} \cdot f_2(x) \cdot xu$$

for every commutative k-algebra Λ and every $x \in M(\Lambda)$. Passing to the quotient U/Z we see that $1 = u^{-1}xu$, so that the class of u lies in $(U/Z)^M(\Lambda)$. □

Remark 3.6. Assume that ℓ is a finite, Galois extension of k with Galois group $\Gamma = \mathrm{Gal}(\ell/k)$. Then Γ acts on the Galois cohomology $H^1(M_\ell, A_\ell)$ through its action on regular mappings $M_\ell \to A_\ell$.

If A is a vector group on which M acts linearly, then $H^1(M_\ell, A_\ell)$ may be identified with $H^1(M, A) \otimes_k \ell$. In that case $H^1(M, A)$ may be identified with $H^1(M_\ell, A_\ell)^\Gamma$.

This observation prompts several questions. Suppose U is a split unipotent M-group and that U has a central linear filtration for the action of M.

(a) Under what conditions is it true that $H^1_{\mathrm{coc}}(M, U) = H^1_{\mathrm{coc}}(M_\ell, U_\ell)^\Gamma$?

(b) Under what conditions is it true that the condition $H^1_{\mathrm{coc}}(M, U) = 1$ is equivalent to the condition $H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$?

4. Descent of Levi factors, I

Proof of Theorem 1.6. Recall that G is a linear algebraic group satisfying condition **(R)**, U is the unipotent radical and $M = G/U$ is the reductive quotient. Moreover,

ℓ is a finite, separable field extension of k. We must show that under assumptions (a), (b), and (c), the group G has a Levi decomposition.

First, note that the assumptions are unaffected if we pass to a finite separable extension of ℓ. Thus, we may and will suppose that ℓ is Galois over k; write $\Gamma = \mathrm{Gal}(\ell/k)$ for the Galois group.

According to (a), G_ℓ has a Levi decomposition. Thus we may choose a homomorphism $s : M_\ell \to G_\ell$ which is a section to π. According to (c), we have $H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$. Together with Proposition 3.2, this shows that $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$ contains a single element. In particular, every homomorphism $u : M_\ell \to G_\ell$ which is a section to π differs from s by conjugation with an element of $U(\ell)$.

There is a natural action of Γ on homomorphisms $M_\ell \to G_\ell$ which determines in turn an action of Γ on $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$. For each $\gamma \in \Gamma$, we thus find an element $u_\gamma \in U(\ell)$ such that ${}^\gamma s = u_\gamma^{-1} \cdot s \cdot u_\gamma$.

We now contend that (♣): u_γ is a 1-cocycle on Γ with values in $U(\ell)$. Well, for $\gamma, \tau \in \Gamma$ we see that

$$ {}^{\gamma\tau}s = u_{\gamma\tau}^{-1} \cdot s \cdot u_{\gamma\tau}, \tag{4-1}$$

while on the other hand

$$ {}^{\gamma\tau}s = {}^{\gamma}(u_\tau^{-1} \cdot s \cdot u_\tau) = {}^{\gamma}u_\tau^{-1} \cdot {}^{\gamma}s \cdot {}^{\gamma}u_\tau = {}^{\gamma}u_\tau^{-1} \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot {}^{\gamma}u_\tau. \tag{4-2}$$

Now, assumption (b) guarantees that $U_\ell^{M_\ell}$ is trivial, and it follows that the stabilizer in U_ℓ of the section s is trivial. Thus together (4-1) and (4-2) imply that

$$u_{\gamma\tau} = u_\gamma \cdot {}^{\gamma}u_\tau.$$

This confirms (♣). Since U is a split unipotent k-group, $H^1(k, U) = 1$; see Proposition 2.14. Thus there is $u \in U(\ell)$ such that

$$u_\gamma = u^{-1} \cdot {}^{\gamma}u \tag{4-3}$$

for each $\gamma \in \Gamma$; that is, ${}^{\gamma}u = uu_\gamma$.

Now set $s_0 = u \cdot s \cdot u^{-1} \in \mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$. We claim that s_0 is a k-homomorphism. It is enough to argue that s is fixed by the Galois group Γ. For $\gamma \in \Gamma$ we note that

$${}^{\gamma}s_0 = {}^{\gamma}u \cdot s \cdot u^{-1} = {}^{\gamma}u \cdot {}^{\gamma}s \cdot {}^{\gamma}u^{-1} = u \cdot u_\gamma \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot u_\gamma^{-1} \cdot u = usu^{-1} = s_0.$$

Thus $s_0 : M \to G$ is a k-morphism which is a section to π; this shows that G has a Levi factor as required. □

In the remainder of this section, we are going to formulate a variant of Theorem 1.6 using a filtration of U. We are going to *assume that U has a central linear filtration*

$$1 = Z_0 \subset Z_1 \subset \cdots Z_m = U$$

for the action of G; see Definition 2.11. Note that such a filtration always exists in the case G is connected; see Theorem 2.12.

Proposition 4.1. *For each $n \geq 0$ the homomorphism of k-group functors*

$$\phi_0 : M \to \mathrm{Out}(U)$$

of Proposition 2.8 determines an action of M on the quotient Z_{n+1}/Z_n.

Proof. Since Z_{n+1}/Z_n is abelian, $\mathrm{Out}(Z_{n+1}/Z_n) = \mathrm{Aut}(Z_{n+1}/Z_n)$. For each natural number n, ϕ_0 determines by restriction and passage to the quotient a homomorphism of k-group functors

$$\phi_{0|Z_{n+1}} : M \to \mathrm{Out}(Z_{n+1}/Z_n) = \mathrm{Aut}(Z_{n+1}/Z_n),$$

i.e., an action of M on Z_{n+1}/Z_n. □

Lemma 4.2. *Suppose that $H^1(M, Z_{i+1}/Z_i) = 0$ for each $i = 0, \ldots, m-1$. Then*

$$H^1_{\mathrm{coc}}(M, U) = 1 \quad \textit{and} \quad H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1.$$

Proof. First observe that for a linear representation V of G, $H^1(G, V) = 0$ if and only if $H^1(G_\ell, V_\ell) = 0$. Now the result follows from Proposition 3.5. □

Remark 4.3. Viewing a finite-dimensional linear representation V of M as an algebraic group, the scheme-theoretic fixed-point subgroup V^M coincides with the vector group given by the M-fixed points on the linear representation V. In particular, if V is an irreducible representation of M, the group scheme V^M is equal to $\{0\}$.

Lemma 4.4. *Suppose that $(Z_{i+1}/Z_i)^M = \{1\}$ for each $i = 0, \ldots, m-1$. Then $U^M = \{1\}$ is the trivial group scheme.*

Proof. We proceed by induction on m, the length of the central linear filtration of U. If $m = 0$, $U = 1$ and the result is immediate.

Now suppose that $m > 0$ and that the result is known for connected and split unipotent M-groups having a central linear filtration of length $< m$. Thus by induction we know $(U/Z_1)^M = \{1\}$. Thus U^M is contained in the kernel of the quotient mapping $U \to U/Z_1$, that is, U^M is contained in Z_1. Since $(Z_1)^M$ is the trivial group scheme, the proof is complete. □

We now obtain a corollary to Theorem 1.6:

Corollary 4.5. *Assume that U has a central linear filtration for the action of G and suppose*

(a) *G_ℓ has a Levi decomposition (over ℓ),*

(bb) *the group scheme $(Z_{i+1}/Z_i)^M$ is trivial for $i = 0, \ldots, m-1$, and*

(cc) *$H^1(M, Z_{i+1}/Z_i) = 0$ for $i = 0, \ldots, m-1$.*

Then G has a Levi decomposition.

Proof. Note that according to Lemma 4.4, condition (bb) implies hypothesis (b) of Theorem 1.6. Similarly, according to Lemma 4.2 (cc) implies hypothesis (c) of Theorem 1.6. Thus the result follows from Theorem 1.6. □

5. Automorphisms of extensions

Let A and M be linear algebraic groups over k, and let E and E' be extensions of M by A as in Definition 2.5.

Definition 5.1. A *morphism of extensions* $\phi : E \to E'$ is a morphism of algebraic groups for which the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{\pi} & M & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & A & \xrightarrow{i'} & E' & \xrightarrow{\pi} & M & \longrightarrow & 1 \end{array}$$

Remark 5.2. If $\phi : E \to E'$ is a morphism of extensions, then ϕ is necessarily an isomorphism of algebraic groups $E \xrightarrow{\sim} E'$.

Write $\mathrm{Autext}(E)$ for the group of automorphisms of E. Let Z be the (schematic) center of A. Since Z is characteristic in A, E acts on Z by conjugation. Since A acts trivially on Z, the action of E on Z factors through $M \simeq E/A$.

Write $Z^1_{\mathrm{coc}}(M, Z)$ for the Hochschild 1-cocycles as in Section 3. Since Z is commutative, $Z^1_{\mathrm{coc}}(M, Z)$ is a group. The following result is a consequence of [Florence and Lucchini Arteche 2019, Proposition 2.3].

Proposition 5.3. *There is a canonical isomorphism of groups* $Z^1_{\mathrm{coc}}(M, Z) \xrightarrow{\sim} \mathrm{Autext}(E)$.

Now suppose that ℓ is a finite, separable field extension of k.

Theorem 5.4. *Assume that the center Z of A is a vector group and that the action of M on Z is linear. If the extensions E_ℓ and E'_ℓ of M_ℓ by A_ℓ are isomorphic, then E and E' are isomorphic extensions of M by A.*

Proof. Write k_{sep} for a separable closure of k containing ℓ and write $\mathscr{E}$ for the set of isomorphism classes of extensions of M by A over k which after scalar extension to k_{sep} become isomorphic to the extension $E_{k_{\mathrm{sep}}}$ of $M_{k_{\mathrm{sep}}}$ by $A_{k_{\mathrm{sep}}}$.

As in [Serre 2002, III.§1], one knows that there is a bijection

$$\mathscr{E} \xrightarrow{\sim} H^1(k, \mathrm{Autext}(E)) := H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), \mathrm{Autext}(E_{k_{\mathrm{sep}}})). \tag{5-1}$$

Thus, the theorem will follow if we argue that the Galois cohomology set $H^1(k, \mathrm{Autext}(E))$ is trivial — i.e., contains a unique element.

By assumption, Z is a vector group with linear action of M, so that $Z^1(M, Z)$ is a k-vector space (possibly of infinite dimension). Now Proposition 5.3 shows that $\mathrm{Autext}(E) = Z^1(M, Z)$ is a k-vector space; it follows from "additive Hilbert 90" that

$$H^1(k, \mathrm{Aut}(E)) \simeq H^1(k, Z^1(A, Z))$$

is trivial; see, for example, [McNinch 2013, (4.1.2)]. $\square$

6. Automorphisms and cohomology

Let A and M be a linear algebraic k-groups, and suppose that A is a M-group via the mapping

$$\phi : M \to \mathrm{Aut}(A).$$

Let Z denote the center of A as a group scheme. Then $\mathrm{Inn}(A) \simeq A/Z$ is also an M-group via ϕ; for $h \in \mathrm{Inn}(A)(\Lambda)$ and $g \in M(\Lambda)$, we have ${}^g h = \phi(g)h\phi(g)^{-1}$.

Denote by $\phi_0 = \Psi \circ \phi$ the homomorphism of group functors

$$M \xrightarrow{\phi} \mathrm{Aut}(A) \xrightarrow{\Psi} \mathrm{Out}(A),$$

where $\Psi : \mathrm{Aut}(A) \to \mathrm{Out}(A)$ is the natural map of (2-1).

Consider those homomorphisms of k-group functors $\theta : M \to \mathrm{Aut}(A)$ satisfying

$$(*) \qquad \Psi \circ \theta_1 = \phi_0.$$

We say that two such homomorphisms θ_1 and θ_2 are equivalent if they are conjugate by $\mathrm{Inn}(A)(k)$; that is, if there is $h \in \mathrm{Inn}(A)(k)$ for which

$$\theta_1(g) = h^{-1}\theta_2(g)h$$

for each commutative k-algebra Λ and each $g \in M(\Lambda)$. We write $\mathrm{Lift}(\phi_0)$ for the quotient of the set of all homomorphisms $M \to \mathrm{Aut}(A)$ satisfying $(*)$ by the equivalence relation just described.

Proposition 6.1. *Write* $\mu : \mathrm{Aut}(A) \times \mathrm{Aut}(A) \to \mathrm{Aut}(A)$ *for the group operation. For* $f \in Z^1_{\mathrm{coc}}(M, A)$, *define* $\Phi_f : M \to \mathrm{Aut}(A)$ *by the rule*

$$\Phi_f = \mu \circ (f, \phi) : M \to \mathrm{Aut}(A) \times \mathrm{Aut}(A) \to \mathrm{Aut}(A).$$

Then the assignment $f \mapsto \Phi_f$ *determines a bijection*

$$\Phi : H^1_{\mathrm{coc}}(G, \mathrm{Inn}(A)) \to \mathrm{Lift}(\phi_0).$$

Proof. For any 1-cocycle $f \in Z^1_{\mathrm{coc}}(G, M)$, one checks that the mapping $\Phi_f : G \to \mathrm{Aut}(A)$ is homomorphism of k-group functors contained in $\mathrm{Lift}(f\phi)$.

We now claim for $f_1, f_2 \in Z^1_{\mathrm{coc}}(M, A)$ that f_1 and f_2 are cohomologous if and only if Φ_{f_1} and Φ_{f_2} are equivalent.

($\Rightarrow$): By assumption there is $h \in \mathrm{Inn}(U)(k)$ such that for each commutative k-algebra Λ and each $g \in M(\Lambda)$, the following identity holds:

$$f_1(g) = h^{-1} f_2(g)^g h.$$

Now observe that

$$\begin{aligned}\Phi_{f_1}(g) &= f_1(g)\phi(g) = h^{-1} f_2(g)^g h \cdot \phi(g) \\ &= h^{-1} f_2(g)\phi(g)h\phi(g)^{-1}\phi(g) = h^{-1} f_2(g)\phi(g)h = h^{-1}\Phi_{f_2}(g)h,\end{aligned}$$

so that Φ_{f_1} and Φ_{f_2} are equivalent.

($\Leftarrow$): By assumption there is $h \in \mathrm{Inn}(A)(k)$ for which

$$\Phi_{f_1} = h^{-1}\Phi_{f_2}h.$$

Then for each commutative k-algebra Λ and each $g \in M(\Lambda)$ we have

$$\begin{aligned}f_1(g) &= \Phi_{f_1}(g) \cdot \phi(g)^{-1} = h^{-1}\Phi_{f_2}(g)h \cdot \phi(g)^{-1} \\ &= h^{-1}\Phi_{f_2}(g)\phi(g)^{-1\,g}h = h^{-1} f_2(g)^g h,\end{aligned}$$

so that f_1 and f_2 are cohomologous.

It now follows that $f \mapsto \Phi_f$ determines a well-defined injective mapping

$$\Phi : H^1_{\mathrm{coc}}(M, \mathrm{Inn}(A)) \to \mathrm{Lift}(\phi_0).$$

To see that Φ is surjective, suppose $\theta : M \to \mathrm{Aut}(A)$ represents a class in $\mathrm{Lift}(\phi_0)$. For each commutative k-algebra Λ and each $g \in M(\Lambda)$, we have $\theta(g)\phi(g)^{-1} \in \mathrm{Inn}(A)(\Lambda)$. Thus we have a morphism of k-functors $f : M \to \mathrm{Inn}(A)$ given by

$$f(g) = \theta(g)\phi(g)^{-1}.$$

By the Yoneda lemma, the assignment f is a morphism of varieties, and a calculation confirms that f is a 1-cocycle for the action of M on $\mathrm{Inn}(A)$ determined by ϕ. Then $[\theta] = [\Phi_f] = \Phi([f])$ which proves that Φ is surjective. □

7. Descent of Levi factors, II

We are going to prove Theorem 1.7. We first prove the following:

Lemma 7.1. *Let M, A be linear algebraic groups, and suppose that A is an M-group via the homomorphism $\phi : M \to \mathrm{Aut}(A)$ of k-group functors. Let $x \in A(k)$ and consider the mapping $\phi_1 : M \to \mathrm{Aut}(A)$ given for each commutative k-algebra Λ and each $g \in M(\Lambda)$ by the rule $\phi_1(g) = \mathrm{Inn}(x)\phi(g)\,\mathrm{Inn}(x)^{-1}$. Then there is a k-isomorphism of extensions of M by A:*

$$A \rtimes_\phi M \simeq A \rtimes_{\phi_1} M.$$

Proof. Write $G = A \rtimes_\phi M$ for the semidirect product constructed using the action defined by ϕ. Now, the mapping $\phi : M \to \mathrm{Aut}(A)$ may be identified with the composite

$$M \xrightarrow{m \mapsto (1,m)} G = A \rtimes_\phi M \xrightarrow{\mathrm{Inn}} \mathrm{Aut}(A)$$

and $\phi_1 : M \to \mathrm{Aut}(A)$ identifies with the composite

$$M \xrightarrow{m \mapsto (x,1)(1,m)(x,1)^{-1}} A \rtimes_\phi M \xrightarrow{\mathrm{Inn}} \mathrm{Aut}(A).$$

Write $s_1 : M \to G = A \rtimes_\phi M$ for the section given by the rule

$$s_1(m) = (x, 1)(1, m)(x, 1)^{-1}.$$

It now follows from Proposition 2.7 that the product mapping

$$((a, m) \mapsto a \cdot s_1(m)) : A \times M \to G$$

determines an isomorphism $A \rtimes_{\phi_1} M \xrightarrow{\sim} G = A \rtimes_\phi M$ of extensions, as required. □

We now prove Theorem 1.7 from Section 1:

Proof. By assumption (a), G_ℓ has a Levi factor M_ℓ; this choice determines a homomorphism

$$\phi : M_\ell \to \mathrm{Aut}(U_\ell)$$

such that $\phi_{0,\ell} = \Psi \circ \phi$ where $\phi_0 : M \to \mathrm{Out}(U)$ is the mapping determined by Proposition 2.8 and $\Psi : \mathrm{Aut}(U) \to \mathrm{Out}(U)$ is the natural mapping of (2-1).

There is a natural action of the Galois group Γ on $\mathrm{Aut}(U_\ell)$ and on $\mathrm{Out}(U_\ell)$ for which Ψ is equivariant. For any $\gamma \in \Gamma$ it follows that

$$\Psi \circ {}^\gamma\phi = \phi_0,$$

i.e., in the notation of Proposition 6.1, ${}^\gamma\phi$ determines a class in $\mathrm{Lift}(\phi_{0,\ell})$.

According to Proposition 6.1 there is a bijection $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) \xrightarrow{\sim} \mathrm{Lift}(\phi_0)$. Since $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$ it follows that classes of the automorphisms ${}^\gamma\phi$ in $\mathrm{Lift}(\phi_0)$ all coincide; that is, all ${}^\gamma\phi$ are equivalent.

By the definition of the equivalence relation defining $\mathrm{Lift}(\phi_0)$, we find for each $\gamma \in \Gamma$ an element $h_\gamma \in \mathrm{Inn}(U)(\ell)$ such that

$${}^\gamma\phi = h_\gamma^{-1} \cdot \phi \cdot h_\gamma.$$

If $\gamma, \tau \in \Gamma$ we see that

$$(7\text{-}1) \qquad {}^{\gamma\tau}\phi = h_{\gamma\tau}^{-1} \cdot \phi \cdot h_{\gamma\tau},$$

while on the other hand

$$(7\text{-}2) \qquad {}^\gamma({}^\tau\phi) = {}^\gamma(h_\tau^{-1} \cdot \phi \cdot h_\tau) = {}^\gamma h_\tau^{-1} \cdot {}^\gamma\phi \cdot {}^\gamma h_\tau = {}^\gamma h_\tau^{-1} \cdot h_\gamma^{-1}\phi \cdot h_\gamma \cdot {}^\gamma h_\tau.$$

By assumption (b) we know that the stabilizer in $\mathrm{Inn}(U)$ of the automorphism ϕ is trivial. Thus taken together (7-1) and (7-2) imply that

$$h_{\gamma\tau} = h_\gamma {}^\gamma h_\tau;$$

i.e., h_γ is a 1-cocycle on Γ with values in $\mathrm{Inn}(U)(\ell)$. Since U is connected and split unipotent, so is $\mathrm{Inn}(U)$; see Proposition 2.3. Thus $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$ by Proposition 2.14.

It follows that the cocycle h_γ is trivial. Thus there is $h \in \mathrm{Inn}(U)(\ell)$ such that for each $\gamma \in \Gamma$ we have

$$h_\gamma = h^{-1} \cdot {}^\gamma h.$$

We now claim that the mapping $\phi_1 : M_\ell \to \mathrm{Aut}(U_\ell)$ defined by

$$\phi_1 = h \cdot \phi \cdot h^{-1}$$

is Γ-stable. For $\gamma \in \Gamma$ we have

$${}^\gamma\phi_1 = {}^\gamma(h \cdot \phi \cdot h^{-1}) = {}^\gamma h \cdot {}^\gamma\phi \cdot {}^\gamma h^{-1} = hh_\gamma \cdot h_\gamma^{-1}\phi h_\gamma \cdot h_\gamma^{-1}h^{-1} = \phi_1.$$

Thus ϕ_1 is Γ-stable and hence defines a morphism $\phi_1 : M \to \mathrm{Aut}(U)$ of k-group functors which we may use to define a semidirect product $G_1 = U \rtimes_{\phi_1} M$ over k.

The center Z of U is a connected and split unipotent group; thus $H^1(\ell, Z) = 1$. It follows that the mapping $U(\ell) \to \mathrm{Inn}(U)(\ell)$ is surjective, so we may choose an element $u \in U(\ell)$ for which $\mathrm{Inn}(u) = h \in \mathrm{Inn}(U)(\ell)$.

Thus we have

$$\phi_1 = \mathrm{Inn}(u) \cdot \phi \cdot \mathrm{Inn}(u)^{-1}.$$

It now follows from Lemma 7.1 that there is an isomorphism of extensions

$$G_\ell = U_\ell \rtimes_\phi M_\ell \simeq G_{1,\ell} = U_\ell \rtimes_{\phi_1} M_\ell$$

of M_ℓ by U_ℓ.

According to Theorem 5.4, assumption (c) implies that the extension G_ℓ has a unique k-form. Since G and G_1 are both k-forms of this extension, it follows that $G \simeq G_1$ are k-isomorphic extensions and in particular are k-isomorphic algebraic groups; since G_1 has a Levi factor over k, we conclude that G has a Levi factor over k as well. □

8. An example

In [McNinch 2013, §5] we gave an example of an extension

$$1 \to W \to E \to \mathbb{Z}/p\mathbb{Z} \to 1$$

with E commutative and W a connected, commutative unipotent group of exponent p^2. The group E was constructed by *twisting*, and it provided a negative answer to the question (◇) from Section 1. Namely, for a suitable finite Galois extension ℓ of k the group E_ℓ has a Levi factor, but E had no Levi factor.

We conclude with another example of a linear algebraic group over k which provides a negative answer to the question ($\diamondsuit$).

The example below gives a noncommutative extension of a finite abelian p-group by a connected, noncommutative unipotent group; in this case, the construction of the extension is perhaps slightly more straightforward.

Suppose that the characteristic of k is $p > 2$. Consider the additive polynomial $X^p - X \in k[X]$ defining the *Artin–Schreier* mapping $\mathscr{P}$: for any commutative k-algebra Λ, this mapping $\mathscr{P} : \Lambda \to \Lambda$ is given by the rule $x \mapsto x^p - x$.

Recall that if $s \in k$ is not in the image of $\mathscr{P} : k \to k$ then the polynomial $F(X) = X^p - X - s \in k[X]$ is irreducible. If α is a root of $F(X)$ in an extension field of k then $\ell = k(\alpha)$ is a Galois extension of k with $\mathrm{Gal}(\ell/k) \simeq \mathbb{Z}/p\mathbb{Z}$.

Let V be a vector space of dimension 2 over k with a basis e, f, and write $\beta : V \times V \to k$ for the unique nondegenerate symplectic form satisfying $\beta(e, f) = 1 = -\beta(f, e)$. Viewing $\mathscr{P} \circ \beta$ as a *factor system*, we define a unipotent group H as an extension of V by $\boldsymbol{G}_a$; see [Serre 1988, VII.§1.4]. Explicitly, for a commutative k-algebra Λ we have

$$H(\Lambda) = \Lambda \times V \otimes_k \Lambda$$

with operation

$$\begin{aligned}(t, v) \cdot (s, w) &= (t + s + \mathscr{P}(\beta(v, w)), v + w)\\ &= (t + s + \beta(v, w)^p - \beta(v, w), v + w)\end{aligned}$$

for $v, w \in V \otimes \Lambda$ and $s, t \in \Lambda$.

Thus H is the nonabelian central extension

$$0 \to \boldsymbol{G}_a \xrightarrow[t \mapsto (0,t)]{i} H \xrightarrow[(v,t) \mapsto v]{} V \to 0. \tag{8-1}$$

Write Z for the center of H; then $Z \simeq \boldsymbol{G}_a$ is the image of the mapping i of (8-1).

Fix $t \in k$ and let $V_{0,t} = \langle te, f \rangle \subset V$, so that $V_{0,t} \simeq (\mathbb{Z}/pZ)^2$. Let μ_t be the central extension of $V_{0,t}$ by $Z \simeq \boldsymbol{G}_a$ defined by β (*not* by $\mathscr{P} \circ \beta$)). Thus there is an exact sequence

$$0 \to \boldsymbol{G}_a \to \mu_t \to V_{0,t} = (\mathbb{Z}/p\mathbb{Z})^2 \to 0$$

and the group operation is given by $(a, v) \cdot (b, w) = (a + b + \beta(v, w), v + w)$, for $v, w \in V_{0,t} \otimes \Lambda = V_{0,t}$ and $a, b \in \Lambda$.

Write E for the fiber product $E = H \times_{\boldsymbol{G}_a} \mu_t$; thus E is an extension of $V_{0,t} \simeq (\mathbb{Z}/p\mathbb{Z})^2$ by H. By the definition of the fiber product, there is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{G}_a & \longrightarrow & \mu_t & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H & \longrightarrow & H \times_{\boldsymbol{G}_a} \mu_t & \xrightarrow{\pi} & (\mathbb{Z}/p\mathbb{Z})^2 & \longrightarrow & 0 \end{array}$$

Proposition 8.1. *If $X^p - X + t$ has no root in k, then the group $E = H \times_{\mathbf{G}_a} \mu_t$ has no Levi factor over k. If α is a root $X^p - X + t$ and $\ell = k(\alpha)$ then E_ℓ has a Levi factor.*

Sketch. We may represent elements of $E(k)$ as tuples (a, v, w) where $v \in V_{0,t}$, $w \in V$ and $a \in k$. We have

$$(a, v, w) \cdot (a', v', w') = (a + a' + \beta(v, v') + \mathscr{P}\beta(w, w)', v + v', w + w').$$

Now, any elements $\tilde{e}$, $\tilde{f}$ of $E(k)$ mapping to te, $f \in V_{0,t}$ via π must have the form $\tilde{e} = (a, te, v)$ for some $v \in V$ and $a \in k$ and $\tilde{f} = (b, f, w)$ for some $w \in V$ and $b \in k$.

We see that

$$\tilde{e} \cdot \tilde{f} = (a, te, v) \cdot (b, f, w) = (a + b + t + \mathscr{P}\beta(v, w), te + f, v + w)$$

while

$$\tilde{f} \cdot \tilde{e} = (b, f, w) \cdot (a, te, v) = (a + b + -t - \mathscr{P}\beta(v, w), te + f, v + w)$$

Since the characteristic of k is not 2, $\tilde{e} \cdot \tilde{f} = \tilde{f} \cdot \tilde{e}$ if and only if

$$0 = \mathscr{P}\beta(v, w) + t = \beta(v, w)^p - \beta(v, w) + t.$$

If $X^p - X + t$ has no root in k, it follows that the group $\langle \tilde{e}, \tilde{f} \rangle$ is nonabelian for any choice of $\tilde{e}$, $\tilde{f}$. This shows that E has no Levi factor.

On the other hand, E_ℓ always has a Levi factor since we may take $\tilde{e} = (0, te, \alpha e)$ and $\tilde{f} = (0, f, f)$; then $\langle \tilde{e}, \tilde{f} \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^2$, so that $\langle \tilde{e}, \tilde{f} \rangle$ provides a Levi factor. □

Remark 8.2. The group E of Proposition 8.1 fails to satisfy hypotheses (b) and (c) of Theorem 1.6: Let $M = E/H \simeq (\mathbb{Z}/p\mathbb{Z})^2$ be the reductive quotient of E. Then:

- M_ℓ acts trivially on H_ℓ. Thus, $H_\ell^{M_\ell} = H_\ell \neq \{1\}$, so that condition (b) fails.
- The cohomology group $H^1_{\mathrm{coc}}(\mathbb{Z}/p\mathbb{Z}, \mathbf{G}_a)$ is nontrivial. Using a Künneth formula, we see that $H^1_{\mathrm{coc}}(M, \mathbf{G}_a) \neq 1$. Now use Proposition 3.5 to conclude that $H^1_{\mathrm{coc}}(M, H) \neq 1$ and $H^1_{\mathrm{coc}}(M_\ell, H_\ell) \neq 1$. Thus condition (c) fails.

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ON THE INTERSECTION OF PRINCIPAL BLOCKS

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To the memory of Gary Seitz

If p and q are two primes, G is a finite group, and $B_p(G)$ is the set of complex irreducible characters in the principal p-block of G, we study $|B_p(G) \cap B_q(G)|$ and its possible relation with some local subgroup.

1. Introduction

It is not often the case that one finds interactions between the representation theory of finite groups with respect to different primes p and q. If G is a finite group and $B_p(G)$ is the set of the complex irreducible characters of G in the principal p-block of G, the p-block containing the principal character 1_G, relations between the sets $B_p(G)$ and $B_q(G)$ are an exception. For instance, it was proved in [Bessenrodt et al. 2007] that $B_p(G) = B_q(G)$ implies that $p = q$ (following pioneering work in [Navarro and Willems 1997]). The study of the other end case, when $B_p(G) \cap B_q(G) = \{1_G\}$, has led to the main conjecture in [Liu et al. 2020] (and its strengthening in [Navarro et al. 2022]): G should have a Sylow p-subgroup P and a Sylow q-subgroup Q such that $xy = yx$ for all $x \in P$ and $y \in Q$ (in other words, $[P, Q] = 1$). This conjecture, which in turn would generalize the main theorem in [Bessenrodt and Zhang 2008], has been reduced to almost simple groups in [Liu et al. 2020] but, as surprising as it may seem, the values of the characters of the almost simple groups are not yet understood well enough in order to solve this problem. The local condition $[P, Q] = 1$ is not isolated in character theory and has already appeared in the so-called Brauer's height zero conjecture for two primes,

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formulated in [Malle and Navarro 2020], and solved recently in [Liu et al. 2024]. (See also [Beltrán et al. 2016].) The global condition $B_p(G) \cap B_q(G) = \{1_G\}$ is definitely less transparent. (For a recent group-theoretical characterization of this condition, see [Robinson 2024].)

Our aim is to propose a new counting global/local conjecture which at the same time implies a characterization of the trivial intersection block property.

Conjecture A. *Let p and q be primes, and let G be a finite group.*

(a) *Assume that there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$, and let $N = \boldsymbol{N}_G(PQ)$. Then*

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|.$$

(b) *We have that $B_p(G) \cap B_q(G) = \{1_G\}$ if and only if there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ and $B_p(N) \cap B_q(N) = \{1_N\}$, where $N = \boldsymbol{N}_G(PQ)$.*

As we will prove below (see Lemma 2.2(ii)), we have that $B_p(N) \cap B_q(N) = \{1_N\}$ (for $N = \boldsymbol{N}_G(PQ)$) if and only if $N = KL$, where K and L are normal subgroups of N of order not divisible by p and q, respectively. This would give, together with Conjecture A(b), another purely group-theoretical characterization of the trivial intersection block property.

We will give solid evidence that Conjecture A is true, but we cannot be optimistic about a general proof of it at the present time. As we have already mentioned, the "only if" implication of Conjecture A(b) seems out of reach. On the other hand, notice that Conjecture A(a) for $p = q$ implies the Alperin–McKay conjecture for principal blocks and abelian Sylow p-subgroups, a conjecture which is still open (for odd primes) in its full generality. Interestingly enough, Conjecture A(a) does not seem implied by it or by the inductive Alperin–McKay condition [Späth 2013]. A reduction theorem of Conjecture A along the lines of the global/local inductive conditions will appear elsewhere.

The main result is the following.

Theorem B. *Conjecture A is true for quasisimple groups, symmetric groups and p-solvable groups.*

We also prove Conjecture A for several almost (quasi)simple groups, including $\mathrm{GL}_n(q_0)$ and $\mathrm{GU}_n(q_0)$ for any prime power q_0. (See Theorem 4.2 and Corollary 4.3.)

2. Preliminaries

Our notation for blocks follows [Navarro 1998], and for characters [Navarro 2018]. To simplify notation, we have chosen to write $B_p(G)$ for what we usually write $\mathrm{Irr}(B_p(G))$. Also, when there is no possible confusion, we write $1 = 1_X$ to be the principal character of any finite group X, and we write $B_p(X) \cap B_q(X) = 1_X$ when $B_p(X) \cap B_q(X) = \{1_X\}$.

As usual, we identify the irreducible characters of G/N with the irreducible characters of G having N in their kernel. Although the following is well-known, it seems convenient to write it down.

Lemma 2.1. *Let G be a finite group, $N \trianglelefteq G$, and suppose that $\chi \in \mathrm{Irr}(G)$ has N in its kernel. Let $\hat{\chi} \in \mathrm{Irr}(G/N)$ be the character of G/N naturally associated to χ.*

(i) *If $\hat{\chi} \in B_p(G/N)$, then $\chi \in B_p(G)$. Therefore $B_p(G/N) \subseteq B_p(G)$.*

(ii) *If N is a central p-subgroup of G, then $\chi \in B_p(G)$ if and only if $\hat{\chi} \in B_p(G/N)$.*

(iii) *If N is a p'-subgroup of G, then $\chi \in B_p(G)$ if and only if $\hat{\chi} \in B_p(G/N)$. Therefore $B_p(G/N) = B_p(G)$.*

(iv) *G is a p'-group if and only if $B_p(G) = 1$.*

Proof. By the comments in pages 198 and 199 of [Navarro 1998], we have that $\bar{B} = B_p(G/N)$ is contained in a unique block B of G. Also, $\bar{B}$ is contained in B if and only if $\mathrm{Irr}(\bar{B}) \cap \mathrm{Irr}(B) \neq \varnothing$. Since the principal character 1_G belongs to $\mathrm{Irr}(\bar{B}) \cap \mathrm{Irr}(B_p(G))$, then (i) is done. For the second part, recall that the map $G^0 \to (G/N)^0$ given by $x \mapsto xN$ is a bijection. Then $\sum_{x \in G^0} \hat{\chi}(xN) = \sum_{x \in G^0} \chi(x)$. The proof is complete using Corollary 3.25 of [Navarro 1998]. The third part follows from Theorem 9.9(c) of [Navarro 1998].

If G is a p'-group, then $B_p(G) = 1$ by part (iii). For the converse, apply, for instance, weak block orthogonality Corollary 3.7 of [Navarro 1998]. □

Of course, the converse of Lemma 2.1(i) does not hold: if $\chi \in \mathrm{Irr}(B_p(G))$ and $N \subseteq \ker(\chi)$, it is not true that $\hat{\chi} \in B_p(G/N)$. (For instance, the sign character in S_3 for $p = 3$.) In what follows, $\boldsymbol{O}_{p'}(G)$ denotes the largest normal subgroup of G of order not divisible by p.

Lemma 2.2. *Let G be a finite group, and let p and q be primes.*

(i) *If $B_p(G) \cap B_q(G) = 1$ and $N \trianglelefteq G$, then $B_p(G/N) \cap B_q(G/N) = 1$.*

(ii) *If $\boldsymbol{O}_{p'}(G)\boldsymbol{O}_{q'}(G) = G$, then $B_p(G) \cap B_q(G) = 1$. If G is p-solvable and q-solvable, then the converse holds.*

(iii) *Let $Q \in \mathrm{Syl}_q(G)$, $N \trianglelefteq G$, and let $M = N\boldsymbol{C}_G(Q \cap N)$. Then $B_q(G)$ is the only block covering $B_q(M)$, and $\mathrm{Irr}(G/M) \subseteq B_q(G)$. Therefore, if $B_p(G) \cap B_q(G) = 1$, then $B_p(M) \cap B_q(M) = 1$ and G/M is a p'-group.*

(iv) *Suppose that $N \trianglelefteq G$, and that $|N|$ is not divisible by p nor q. Then*

$$B_p(G) \cap B_q(G) = B_p(G/N) \cap B_q(G/N).$$

(v) *Suppose that Z is a central subgroup of G and $p \neq q$. Then*

$$B_p(G) \cap B_q(G) = B_p(G/Z) \cap B_q(G/Z).$$

If $p \neq q$ and Conjecture A(a) holds for G/Z, then it also holds for G, and similarly for Conjecture A(b).

Proof. Part (i) follows from Lemma 2.1(i).

We prove (ii). Recall that if $N, M \trianglelefteq G$ and $G = NM$, then the only irreducible character of G lying over 1_N and 1_M is 1_G. If $\chi \in B_p(G)$, then $\boldsymbol{O}_{p'}(G) \subseteq \ker(\chi)$ by Lemma 2.1(iii), and the first part easily follows. For the second, recall that $B_p(G) = \mathrm{Irr}(G/\boldsymbol{O}_{p'}(G))$ if G is p-solvable, by Theorem 10.20 of [Navarro 1998]. Let $L = \boldsymbol{O}_{p'}(G)\boldsymbol{O}_{q'}(G)$. Hence, $\mathrm{Irr}(G/L) \subseteq B_p(G) \cap B_q(G)$, and (ii) is easily completed.

For (iii), notice that $M \trianglelefteq G$, using the Frattini argument. We have that $Q \cap N \subseteq Q \cap M$, and therefore $\boldsymbol{C}_G(Q \cap M) \subseteq M$. Then the first part follows from Lemma 9.20 and Theorem 9.19 of [Navarro 1998]. Now assume that $B_p(G) \cap B_q(G) = 1$. Let $\tau \in B_p(M) \cap B_q(M)$. Since $B_p(G)$ covers $B_p(M)$, there is some $\chi \in B_p(G)$ over τ (by Theorem 9.4 of [Navarro 1998]). Now χ lies in some q-block that covers $B_q(M)$, by Theorem 9.2 of [Navarro 1998]. By the previous part, $\chi \in B_q(G)$, and then $\chi = 1$. Thus $\tau = 1$. Finally, let $\gamma \in B_p(G/M) \subseteq B_p(G)$. Then γ lies over 1_M, and therefore the q-block of γ covers the q-block of M. It follows that γ lies in the principal q-block of G, and therefore $\gamma = 1$, by hypothesis. Thus G/M is a group with $B_p(G/M) = 1$, and this implies that G/M is a p'-group by Lemma 2.1(iv). This finishes (iii).

Part (iv) is obvious since $B_p(G) = B_p(G/N)$ if $|N|$ is not divisible by p.

Now, we prove part (v). Arguing by induction on $|G : Z|$, we may assume that $|Z|$ is a prime. Using part (iv), we may assume that $|Z| = p$. Let $\chi \in B_p(G) \cap B_q(G)$. Since $q \neq p$ and $\chi \in B_q(G)$, we have that $Z \subseteq \ker(\chi)$. Also, $\chi \in B_q(G/Z)$, by Lemma 2.1(iii). By Lemma 2.1(ii), we have that $\chi \in B_p(G/Z)$. This finishes the proof of the first statement.

For the second statement, let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, so that $PZ/Z \in \mathrm{Syl}_p(G/Z)$ and $QZ/Z \in \mathrm{Syl}_q(G/Z)$. Observe that $[P, Q] = 1$ if and only if $[PZ/Z, QZ/Z] = 1$. (Indeed, if $x \in P$ and $y \in Q$ commute modulo Z, then $xyx^{-1}y^{-1} = z \in Z$. Now $xyx^{-1} = yz$ is a q-element, and so z is a q-element. Similarly, z is a p-element, and thus $[x, y] = 1$.) If $[P, Q] = 1$ then $\boldsymbol{N}_{G/Z}(PQZ/Z) = N/Z$ for $N = \boldsymbol{N}_G(PQ)$. (Indeed, writing $Z = \boldsymbol{O}_p(Z) \times \boldsymbol{O}_q(Z) \times Z_1$ and $Z_1 = \boldsymbol{O}_{\{p,q\}'}(Z)$, we have $PQZ = PQ \times Z_1$ and $PQ = \boldsymbol{O}_{\{p,q\}}(PQZ)$. Hence if $h \in G$ normalizes PQZ, then it normalizes PQ.) Now apply the first statement. □

Remark 2.3. Suppose that G is a finite group and $N \trianglelefteq G$. If $B_p(N) \cap B_q(N) = 1$ and $B_p(G/N) \cap B_q(G/N) = 1$, then it is false in general that $B_p(G) \cap B_q(G) = 1$, as shown by A_4, with $p = 2$ and $q = 3$. Also, if $B_p(G) \cap B_q(G) = 1$, then it is false in general that $B_p(N) \cap B_q(N) = 1$, as shown by $G = \mathsf{D}_{10} \times \mathsf{S}_3$ for $p = 3$ and $q = 5$.

We will use the Alperin–Dade theorem on isomorphic blocks several times.

Theorem 2.4. (Alperin–Dade) *Let p be a prime. Suppose that S is a normal subgroup of a finite group G such that p does not divide $|G/S|$, and let $P \in \mathrm{Syl}_p(S)$. If $G = S\boldsymbol{C}_G(P)$, then restriction from G to S defines a bijection $B_p(G) \to B_p(S)$.*

Proof. See Theorem 2.4 of [Navarro et al. 2022], for instance. □

Now we work towards proving that Conjecture A holds if G is p-solvable.

Lemma 2.5. *Suppose that G has a normal p'-subgroup K containing $Q \in \mathrm{Syl}_q(G)$. Then restriction of characters defines a natural bijection*

$$B_p(G) \cap B_q(G) \to B_p(\boldsymbol{N}_G(Q)) \cap B_q(\boldsymbol{N}_G(Q)).$$

Also,

$$B_p(G) \cap B_q(G) = \mathrm{Irr}(G/K\boldsymbol{C}_G(Q)),$$

and therefore $B_p(G) \cap B_q(G) = 1$ if and only if $G = K\boldsymbol{C}_G(Q)$.

Proof. Let $H = \boldsymbol{N}_G(Q)$. By the Frattini argument, we have that $G = KH$. Thus restriction defines a bijection $\mathrm{Irr}(G/K) \to \mathrm{Irr}(H/\boldsymbol{N}_K(Q))$.

Let $M = K\boldsymbol{C}_G(Q) \trianglelefteq G$. Let $V = \boldsymbol{O}_{q'}(H) \subseteq \boldsymbol{C}_G(Q)$. Since $Q \in \mathrm{Syl}_q(G)$, we have that $\boldsymbol{C}_G(Q) = \boldsymbol{Z}(Q) \times V$, using the Schur–Zassenhaus theorem. Since H is q-solvable, we have that $B_q(H) = \mathrm{Irr}(H/V)$, by Theorem 10.20 of [Navarro 1998]. Also, $M = KV$. By Alperin–Dade Theorem 2.4, we have that restriction defines a bijection $B_q(M) \to B_q(K)$.

We claim that $B_p(G) \cap B_q(G) = \mathrm{Irr}(G/M)$. Let $\chi \in B_p(G) \cap B_q(G)$. Then $\chi \in \mathrm{Irr}(G/K)$ by Lemma 2.1(iii). Let $\eta \in \mathrm{Irr}(M)$ be under χ. Then $\eta \in B_q(M)$ and since η lies over 1_K, we have that $\eta = 1_M$, by Alperin–Dade. Hence $\chi \in \mathrm{Irr}(G/M)$. Conversely, if $\chi \in \mathrm{Irr}(G/M)$, then $\chi \in B_p(G) \cap B_q(G)$, using Lemma 2.2(iii) and Lemma 2.1(iii). This proves the claim. The claim, applied now in H (with respect to its normal subgroup $\boldsymbol{N}_K(Q)$), proves that

$$B_p(H) \cap B_q(H) = \mathrm{Irr}(H/\boldsymbol{N}_K(Q)V).$$

The proof of the lemma now easily follows. □

Lemma 2.6. *Suppose that $P \in \mathrm{Syl}_p(G)$ is normal in G. Let q be any prime.*

(i) *Suppose that $Q \in \mathrm{Syl}_q(G)$ centralizes P. Then*

$$|B_p(G) \cap B_q(G)| = \big|B_p(\boldsymbol{N}_G(Q)) \cap B_q(\boldsymbol{N}_G(Q))\big|.$$

(ii) *We have that $B_p(G) \cap B_q(G) = 1$ if and only if there is $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ and $B_p(\boldsymbol{N}_G(Q)) \cap B_q(\boldsymbol{N}_G(Q)) = 1$.*

Proof. Both parts of the lemma are trivial if $p = q$. So we may assume that $p \neq q$. Let $C = \boldsymbol{C}_G(P)$, $Z = \boldsymbol{Z}(P)$ and $K = \boldsymbol{O}_{p'}(G)$. Then $C = Z \times K$.

If we assume that $[Q, P] = 1$, then $Q \subseteq K$ and (i) follows from Lemma 2.5. So we prove (ii). Now, we assume that $B_p(G) \cap B_q(G) = 1$ and we prove that there exists $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ by induction on $|G|$. Since G is p-solvable, we know that $B_p(G) = \mathrm{Irr}(G/K)$. Assume that $K > 1$. Then $B_p(G/K) \cap B_q(G/K) = 1$ and we have that there is $Q \in \mathrm{Syl}_p(G)$ such that $[P, Q] \subseteq K$. Since $P \trianglelefteq G$, $[P, Q] \subseteq P$, and thus $[P, Q] = 1$. Hence, we may assume that $K = 1$. Therefore $B_p(G) = \mathrm{Irr}(G)$ and therefore $B_q(G) = 1$. Thus q does not divide $|G|$ by Lemma 2.1(iv). Hence, in (ii), we may assume that $[P, Q] = 1$, where $Q \in \mathrm{Syl}_q(G)$. Then $Q \subseteq K$, and we apply Lemma 2.5. □

Theorem 2.7. *Suppose that G is p-solvable.*

(i) *Suppose that there are $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then $|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|$, where $N = \boldsymbol{N}_G(PQ)$.*

(ii) *We have that $B_p(G) \cap B_q(G) = 1$ if and only if there are $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$ and $B_p(N) \cap B_q(N) = 1$, where $N = \boldsymbol{N}_G(PQ)$.*

Proof. Suppose first that $p = q$. In this case, the second part is trivial, using Lemma 2.1(iv). For the first part, this is the Alperin–McKay conjecture for p-solvable groups with abelian Sylow p-subgroups, which is nearly trivial. (By the Hall–Higman 1.2.3 lemma, we have that $KP \trianglelefteq G$, where $K = \boldsymbol{O}_{p'}(G)$. Since we may also assume that $K = 1$ by Lemma 2.1(iii), the result easily follows.)

We may assume that $p \neq q$. We start with the first part, which we argue by induction on $|G|$. Let $K = \boldsymbol{O}_{p'}(G)$. Since $[P, Q] = 1$, then we have that $Q \subseteq K$ by the Hall–Higman 1.2.3 lemma. Let $H = \boldsymbol{N}_G(Q)$. By Lemma 2.5, we have that $|B_p(G) \cap B_q(G)| = |B_p(H) \cap B_q(H)|$. Since $\boldsymbol{N}_G(PQ) = \boldsymbol{N}_G(P) \cap H$, by induction, we may assume that $Q \trianglelefteq G$. By Lemma 2.6, we conclude that

$$|B_p(G) \cap B_q(G)| = |B_p(\boldsymbol{N}_G(P)) \cap B_q(\boldsymbol{N}_G(P))|$$

and this proves (ii).

Finally we prove (ii). The "if" part is immediate from (i). Assume now that G is p-solvable and that $B_p(G) \cap B_q(G) = 1$. By Theorem 1.4 of [Liu et al. 2020] we know that there exists $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then we apply part (i). □

3. (Quasi)simple groups

For a finite group G and a prime p, continue to let $B_p(G)$ denote the irreducible complex characters in the principal p-block of G. The goal of this section is to show that for quasisimple groups, the following holds:

Theorem 3.1. *Let p and q be two primes (not necessarily distinct) and let S be a finite quasisimple group. Let $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_q(S)$. Assume that $[P, Q] = 1$. Then*

$$|B_p(S) \cap B_q(S)| = |B_p(N) \cap B_q(N)|,$$

where $N := \boldsymbol{N}_S(PQ)$.

We remark that when $p = q$, the statement is the Alperin–McKay conjecture together with Brauer's height zero conjecture for principal blocks of quasisimple groups with abelian Sylow p-subgroups.

As a corollary, we obtain Theorem B for quasisimple groups:

Corollary 3.2. *Conjecture A holds for finite quasisimple groups.*

Proof. Thanks to Theorem 3.1, it remains to show that if S is a quasisimple group such that $B_p(S) \cap B_q(S) = 1$, then $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_q(S)$. We assume $p \neq q$, as otherwise this is trivially true. Let $\bar{S} = S/\boldsymbol{Z}(S)$ be the corresponding simple group, and note that $B_p(\bar{S}) \subseteq B_p(S)$ and $B_q(\bar{S}) \subseteq B_q(S)$. Then $|B_p(\bar{S}) \cap B_q(\bar{S})| = 1$, which implies $(\bar{S}, p, q)$ is one of very few possibilities thanks to [Brough et al. 2021], and these satisfy $[\bar{P}, \bar{Q}] = 1$ for some $\bar{P} \in \mathrm{Syl}_p(\bar{S})$ and $\bar{Q} \in \mathrm{Syl}_q(\bar{S})$ (see the discussion after Conjecture 1.3 in [Liu et al. 2020]). Hence by [Malle and Navarro 2020, Lemma 3.1] (or Lemma 2.2(v)), we also have $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_q(S)$, completing the proof. □

Given a connected reductive group $\boldsymbol{G}$ and Steinberg endomorphism $F: \boldsymbol{G} \to \boldsymbol{G}$, we write $\boldsymbol{G}^F$ for the corresponding finite group of Lie type obtained as the fixed points under F. If p is a prime and n an integer prime to p, we write $d_p(n)$ for the order of n modulo p if p is odd and modulo 4 if $p = 2$.

The main results of [Malle and Navarro 2020, Section 3] yield:

Proposition 3.3 (Malle–Navarro). *Let G be a finite quasisimple group and $p \neq q$ two primes dividing $|G|$. Assume that there are $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] = 1$. Then one of the following holds:*

(i) *$G = J_1$ and $\{p, q\} = \{3, 5\}$.*

(ii) *$G = J_4$ and $\{p, q\} = \{5, 7\}$.*

(iii) *$S = G/\boldsymbol{Z}(G)$ is a simple group of Lie type defined in characteristic r distinct from p and q. In this case, if $G = \boldsymbol{G}^F$ is a quasisimple group of Lie type, then:*

- *$d_p(r^f) = d_q(r^f) =: d$, where G is defined over $\mathbb{F}_{r^f}$.*
- *p and q are odd.*
- *P and Q are abelian, and $PQ \leq \boldsymbol{S}$ for an (F-stable) Sylow d-torus $\boldsymbol{S}$ of $\boldsymbol{G}$.*
- *p and q are good for $\boldsymbol{G}$, larger than 3 if G is of type ${}^3\mathrm{D}_4$, and do not divide $|\boldsymbol{Z}(\boldsymbol{G})^F : (\boldsymbol{Z}(\boldsymbol{G})^\circ)^F| \cdot |\boldsymbol{Z}(\boldsymbol{G}^*)^F : (\boldsymbol{Z}(\boldsymbol{G}^*)^\circ)^F|$, where $(\boldsymbol{G}^*, F)$ is dual to $(\boldsymbol{G}, F)$.*

Proof. This is [Malle and Navarro 2020, Propositions 3.2–3.5]. The last item of (iii) follows from [Malle 2014, Lemma 2.1 and Proposition 2.2]. □

Before working on simple groups of Lie type, we settle Conjecture A (when $p \neq q$) for sporadic groups:

Lemma 3.4. *If $p \neq q$, then Conjecture A holds for sporadic quasisimple groups.*

Proof. By Lemma 2.2(v), we need only consider sporadic simple groups G.

On the one hand, by [Brough et al. 2021, Theorem 1.2], $B_p(G) \cap B_q(G) = 1$ precisely when (G, p, q) occurs in Proposition 3.3(i) and (ii). On the other hand, assume G admits commuting Sylow p- and q-subgroups P and Q. Then we are in (i) or (ii) of Proposition 3.3. Using the information in [GAP 2018] on centralizer size and power fusion of elements of order p, q, and pq in G, one can show that $N = \boldsymbol{N}_G(PQ) = PQ \rtimes \boldsymbol{O}_{\{p,q\}'}(N)$ is $\mathsf{S}_3 \times \mathsf{D}_{10}$ in the case of Proposition 3.3(i), and $(\mathsf{C}_5 \rtimes \mathsf{C}_4) \times (\mathsf{C}_7 \rtimes \mathsf{C}_3)$ in the case of Proposition 3.3(ii). It follows from Lemma 2.2(ii) that $B_p(N) \cap B_p(N) = 1$. □

Proposition 3.5. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group such that $G = \boldsymbol{G}^F$ is a quasisimple group of Lie type. Assume $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ for primes p and q (not necessarily distinct) satisfying assertion* (iii) *of Proposition 3.3 and let $\boldsymbol{L} := \boldsymbol{C}_{\boldsymbol{G}}(\boldsymbol{S})$. Then $\boldsymbol{L} = \boldsymbol{C}_{\boldsymbol{G}}(P) = \boldsymbol{C}_{\boldsymbol{G}}(Q)$ and $\boldsymbol{N}_G(PQ) = \boldsymbol{N}_G(Q) = \boldsymbol{N}_G(P) = \boldsymbol{N}_G(\boldsymbol{L})$. Further, $\boldsymbol{L}^F = P \times O^p(\boldsymbol{L}^F) = Q \times O^q(\boldsymbol{L}^F)$.*

Proof. Note that $\boldsymbol{N}_G(PQ) \leq \boldsymbol{N}_G(P) \cap \boldsymbol{N}_G(Q)$. Then the statements follow from [Malle 2014, Propositions 2.3 and 2.4]. □

Lemma 3.6. *Keep the situation of Proposition 3.5. Write $N := \boldsymbol{N}_G(PQ)$ and $L := \boldsymbol{L}^F \lhd N$. Then $B_q(N)$ is the unique q-block of N covering $B_q(L)$ and $B_p(N)$ is the unique p-block of N covering $B_p(L)$.*

Proof. This follows from Proposition 3.5 and [Navarro 1998, Corollary (9.21)]. □

For $G = \boldsymbol{G}^F$ a finite reductive group, let $(\boldsymbol{G}^*, F^*)$ be dual to $(\boldsymbol{G}, F)$ and write $G^* := (\boldsymbol{G}^*)^{F^*}$. Then $\mathrm{Irr}(G)$ is partitioned into rational Lusztig series $\mathcal{E}(G, s)$, where s ranges over semisimple elements of G^*, up to G^*-conjugacy. (See, for example, [Cabanes and Enguehard 2004, Theorem 8.24].) We will write $\mathrm{UCh}(G) := \mathcal{E}(G, 1)$ for the set of unipotent characters, and will similarly denote by $\mathrm{UCh}(B_p(G))$ the set $\mathcal{E}(G, 1) \cap B_p(G)$ of unipotent characters in the principal p-block for a given prime p.

Proposition 3.7. *Keep the situation of Proposition 3.5, but now assume $p \neq q$ are distinct primes. Then we have a bijection between the sets $B_p(G) \cap B_q(G)$ and* $\mathrm{Irr}(N/L)$ *and between the sets $B_p(N) \cap B_q(N)$ and* $\mathrm{Irr}(N/L)$.

In particular, we have

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|.$$

Proof. First, by [Cabanes and Enguehard 2004, Theorem 9.12], and using that neither p nor q is the defining prime for G, we have $B_p(G)$ contains only characters in rational series $\mathcal{E}(G, s)$ with $|s|$ a power of p. Similarly, $B_q(G)$ is comprised only of characters in series with s a power of q. This means that $B_p(G) \cap B_q(G) = \mathrm{UCh}(B_p(G)) \cap \mathrm{UCh}(B_q(G))$ contains only unipotent characters.

Recall that P is abelian, so every character in $B_p(G)$ has degree prime to p by the "if" direction of Brauer's height zero conjecture [Kessar and Malle 2013]. Further, note that the last item of Proposition 3.3 implies that both p and q satisfy the necessary hypotheses for [Cabanes and Enguehard 1994, Theorem]. Now, by [Cabanes and Enguehard 1994, Theorem], $\mathrm{UCh}(B_p(G)) = \mathrm{UCh}(B_q(G))$ is comprised of those characters $\chi \in \mathrm{Irr}(G)$ in the d-Harish–Chandra series of $(\boldsymbol{L}, 1_L)$, with $\boldsymbol{L}$ as in Proposition 3.5 and $L := \boldsymbol{L}^F$. Note that $\boldsymbol{L}$ is as stated due to [Malle 2007, Corollary 6.6].

Then we have $B_p(G) \cap B_q(G) = \mathrm{UCh}(B_p(G)) = \mathrm{UCh}(B_q(G))$. Further, this set is in bijection with the irreducible characters of the so-called relative Weyl group $W_G(\boldsymbol{L}) = N/L$ by [Broué et al. 1993, Theorem 3.2], completing the claim for G.

Now, note that every character in $B_p(N) \cap B_q(N)$ lies above some character in $B_p(L) \cap B_q(L)$. Recall from Proposition 3.5 that $L = P \times X = Q \times Y$ for some p', respectively q', subgroups $X, Y \lhd L$, so $\mathrm{Irr}(B_p(L)) = \mathrm{Irr}(P) \otimes \{1_X\}$ and similar for q. Then this forces $B_p(L) \cap B_q(L) = \{1_L\}$, and hence every character of $B_p(N) \cap B_q(N)$ lies above the trivial character of L.

Conversely, by Lemma 3.6, $B_p(N)$, respectively $B_q(N)$, is the unique block above $B_p(L)$, respectively $B_q(L)$. Hence, the characters of N above 1_L are exactly the members of $B_p(N) \cap B_q(N)$. By Gallagher's theorem, this set is in bijection with $\mathrm{Irr}(N/L)$, completing the proof. □

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $S = G/Z$ with $Z \leq \boldsymbol{Z}(G)$, where G is the Schur cover of the simple group $\bar{S} := S/\boldsymbol{Z}(S)$, and write P, Q as $P = \widehat{P}Z/Z$ and $Q = \widehat{Q}Z/Z$ for $\widehat{P} \in \mathrm{Syl}_p(G)$ and $\widehat{Q} \in \mathrm{Syl}_q(G)$.

First, assume that $p \neq q$. Then [Malle and Navarro 2020, Lemma 3.1] yields that $[P, Q] = 1$ implies $[\widehat{P}, \widehat{Q}] = 1$. It follows that G is of one of the forms in Proposition 3.3. In cases (i) and (ii), the result is Lemma 3.4. In case (iii), we may apply Lemma 2.2(v) to replace the Schur cover with a group of Lie type G of the form in Proposition 3.5. Then every character in $B_p(\boldsymbol{N}_G(\widehat{P}\widehat{Q})) \cap B_q(\boldsymbol{N}_G(\widehat{P}\widehat{Q}))$ is trivial on $\boldsymbol{Z}(G) \leq L$ by the second-to-last paragraph of the proof of Proposition 3.7. Further, we have $N = \boldsymbol{N}_S(PQ) = \boldsymbol{N}_G(\widehat{P}\widehat{Q})/Z$. Using, e.g., [Cabanes and Enguehard 2004, Lemma 17.2], we have $B_p(N)$ is the set of characters in $B_p(\boldsymbol{N}_G(\widehat{P}\widehat{Q}))$ lying above 1_Z and similar for q. Then we have, by Proposition 3.7,

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|.$$

For the same reason, we have $B_p(S)$ is the set of characters in $B_p(G)$ lying above 1_Z and similar for $B_q(S)$. But recall from the proof of Proposition 3.7 that $B_p(G) \cap B_q(G) = \mathrm{UCh}(B_p(G))$. Since unipotent characters are trivial on $\boldsymbol{Z}(G)$,

$$|B_p(S) \cap B_q(S)| = |B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|,$$

which completes the proof when $p \neq q$.

Now let $p = q$ and assume that $P \in \mathrm{Syl}_p(S)$ is abelian. Since the Alperin–McKay conjecture is known when $p = 2$ by [Ruhstorfer $\geq$ 2025] (alternatively, we can use Walter's classification of simple groups with abelian Sylow 2-subgroups [1969]), we assume p is odd. By [Koshitani and Späth 2016], we may further assume that P is not cyclic.

If S is a covering group of an alternating group A_n, then the Alperin–McKay conjecture holds for S by the main result of [Michler and Olsson 1990]. If $\bar{S}$ is a sporadic simple group or the Tits group, the Alperin–McKay conjecture has been checked by T. Breuer [$\geq$ 2025]. In the cases where a Sylow p-subgroup of $\bar{S}$ is abelian, we note that the Schur multiplier is a p'-group, so we are done by Lemma 2.1(iii). Similarly, when $\bar{S}$ is a group of Lie type with an exceptional Schur multiplier and an abelian noncyclic Sylow p-subgroup, we have that the Schur multiplier has size not divisible by p, except for the case $\mathrm{PSL}_4(3)$, which can be checked in [GAP 2018]. Hence in these cases, we may work with a quasisimple group of Lie type G such that $G/\boldsymbol{Z}(G) = \bar{S}$, rather than the full Schur multiplier.

So we now assume $G = \boldsymbol{G}^F$ is a quasisimple group of Lie type and $\bar{S} = S/\boldsymbol{Z}(S) = G/\boldsymbol{Z}(G)$ is not isomorphic to any A_n. If p is the defining prime for G, then the assumption that P is abelian yields that $\bar{S} = \mathrm{PSL}_2(p^a)$ for some positive integer a. In this case, $|\boldsymbol{Z}(G)|$ is prime to p, and $B_p(S) = B_p(\bar{S}) = \mathrm{Irr}_{p'}(\bar{S}) = \mathrm{Irr}(\bar{S}) \setminus \{\mathrm{St}_S\}$ and the result follows from the considerations in [Isaacs et al. 2007, Section (15F)] (see also the proof of [Späth 2013, Theorem 8.4]). Hence, we may assume that p is not the defining characteristic for G.

If $p \geq 5$, we have that $\widehat{P}$ is also abelian (see the discussion after [Malle 2014, Proposition 2.2]), and we are again in the situation of Proposition 3.3(iii). As $p \nmid |\boldsymbol{Z}(G)|$, we then have $B_p(G) = B_p(S)$ and $B_p(\boldsymbol{N}_G(\widehat{P})) = B_p(N)$, by [Navarro 1998, Theorem 9.9]. Recall that $B_p(L) = \mathrm{Irr}(\widehat{P}) \otimes \{1_X\}$ where X is the p'-group such that $L = \boldsymbol{C}_G(\widehat{P}) = \widehat{P} \times X$ guaranteed by Proposition 3.5. Note that every $\psi \in \mathrm{Irr}(\widehat{P})$ extends to its inertia group in $\boldsymbol{N}_G(\widehat{P}) = \boldsymbol{N}_G(\boldsymbol{L})$ by [Isaacs 2006, Theorem (6.26)]. Then [Malle 2014, Theorem 2.9] yields a bijection between $B_p(G)$ and $B_p(\boldsymbol{N}_G(\widehat{P}))$, and hence between $B_p(S)$ and $B_p(N)$, as long as $p \geq 5$.

Suppose $p = 3$. Then the assumption that P is abelian implies that $\bar{S} = \mathrm{PSL}_2(q_0)$, $\bar{S} = \mathrm{PSL}_n^{\epsilon}(q_0)$ with $3 \leq n \leq 5$ and $\epsilon \in \{\pm 1\}$, or $\bar{S} = \mathrm{PSp}_4(q_0)$, considering the order polynomials and using [Malle 2014, Proposition 2.2] and the discussion after. If $\widehat{P}$ is abelian, then the same considerations from before hold, noting that in this situation for the principal blocks, the conclusions of [Malle 2014, Proposition 2.7 and Theorem 2.8] continue to hold. That is, $B_p(G)$ and $B_p(N_G(\widehat{P}))$ are both in bijection with pairs (ψ, ϕ) with $\psi \in \mathrm{Irr}(P)$ and $\phi \in \mathrm{Irr}(N_G(L)_\psi/L)$.

Then we assume $\widehat{P}$ is nonabelian, and hence $S = \mathrm{PSL}_3^\epsilon(q_0)$ with $(q_0-\epsilon)_3 = 3$. Recall that by the "if" direction of BHZ [Kessar and Malle 2013], we have $B_3(S)$ and $B_3(N)$ consist only of $3'$-characters. From here, the proof of [Malle 2008, Corollary 3.9 and Theorem 3.12] yields the result. Indeed, note that the group N is described in loc. cit., and its construction in GAP shows that it contains a unique block of maximal defect. On the other hand, the six members of $\mathrm{Irr}_{p'}(S)$ are the deflations of three unipotent characters of G and the three irreducible constituents of the restriction to G of a semisimple character of $\mathrm{GL}_3^\epsilon(q_0)$ corresponding to a semisimple element with eigenvalues $\{\omega, \omega^{-1}, 1\}$, where $|\omega| = 3$. The three unipotent characters of $\mathrm{GL}_3^\epsilon(q_0)$ and the stated semisimple character all lie in $B_3(\mathrm{GL}_3^\epsilon(q_0))$, using [Fong and Srinivasan 1982, Theorem 7A]. Hence these six characters all lie in $B_3(S)$. □

4. Almost simple groups

Proposition 4.1. (a) *Let $p \neq q$ be primes. Then Conjecture A holds for any finite group G with a central subgroup Z such that $G/Z \cong \mathsf{A}_n$ or S_n with $n \geq 5$.*

(b) *Let $p = q$ be any prime. Then Conjecture A holds for covering groups G of A_n and S_n with $n \geq 5$.*

Proof. (a) By Lemma 2.2(v), we may assume that $G \cong \mathsf{A}_n$ or S_n, and that $n \geq p > q$. By [Beltrán et al. 2016, Lemma 2.4], see also Proposition 3.3, G has no Hall $\{p, q\}$-subgroup if $q > 2$. Furthermore, by [Beltrán et al. 2016, Theorem 2.2], A_n contains a p-element x such that $C_{\mathsf{A}_n}(x)$ contains no Sylow 2-subgroup of A_n. It follows that G has no Hall $\{p, 2\}$-subgroup. Thus G does not admit any pair (P, Q) of commuting Sylow p-subgroup P and Sylow q-subgroup Q. Finally, [Bessenrodt and Zhang 2008, Proposition 3.2] and its proof show that $B_p(G) \cap B_q(G) \neq 1_G$.

(b) If $p \neq 2$ then the statement follows from [Michler and Olsson 1990]. If $p = 2$, then $B_p(G) \neq 1_G$, and G can have abelian Sylow 2-subgroups P only when $G = \mathsf{A}_5$, in which case $|B_p(G)| = 4 = |B_p(N_G(P))|$. □

Our next result proves Conjecture A for almost simple Lie-type groups of adjoint type.

Theorem 4.2. *Let p and q be (not necessarily distinct) primes, and let $\boldsymbol{G}_{\mathrm{ad}}$ be a simple algebraic group of adjoint type over a field of positive characteristic, with a Steinberg endomorphism F such that $\boldsymbol{G}_{\mathrm{ad}}^F$ is almost simple. Then Conjecture A holds for any G with $[\boldsymbol{G}_{\mathrm{ad}}^F, \boldsymbol{G}_{\mathrm{ad}}^F] \lhd G \leq \boldsymbol{G}_{\mathrm{ad}}^F$.*

Proof. Since Conjecture A holds trivially if p or q does not divide $|G|$, we will assume p and q both divide $|G|$. Now, as shown in the proof of [Brough et al. 2021, Lemma 5.4], $B_p(G) \cap B_q(G) \neq 1_G$ if $p \neq q$. If $p = q$, then $B_p(G) \neq 1_G$ because $p \mid |G|$.

It remains to show that if G admits $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with $[P, Q] = 1$, then

$$|B_p(G) \cap B_q(G)| = |B_p(N) \cap B_q(N)|. \tag{4-1}$$

Inspecting the orders of G and the simple subgroup $S = [G, G]$, we see that p and q both divide $|S|$.

(a) Here we consider the case $p \neq q$. As S admits the Hall $\{p, q\}$-subgroup $(P \cap S)(Q \cap S)$, we can apply Proposition 3.3 and Theorem 3.1 to S. In particular, $\boldsymbol{G}_{\mathrm{ad}}$ is defined over a field of characteristic $r \neq p, q$.

Assume first that

$$G = \boldsymbol{G}_{\mathrm{ad}}^F.$$

To work on the local side, we identify $\boldsymbol{G}_{\mathrm{ad}} = \boldsymbol{G}/Z$ where $\boldsymbol{G}$ is a simple, simply connected algebraic group and Z a finite central subgroup, and $S = \boldsymbol{G}^F Z/Z \cong \boldsymbol{G}^F/Z^F$ for a suitable Steinberg endomorphism $F : \boldsymbol{G} \to \boldsymbol{G}$. As in the proof of Lemma 2.2(v), $P \cap S = P_1 Z/Z$ and $Q \cap S = Q_1 Z/Z$ for some $P_1 \in \mathrm{Syl}_p(\boldsymbol{G}^F)$ and $Q_1 \in \mathrm{Syl}_q(\boldsymbol{G}^F)$ with $[P_1, Q_1] = 1$. By Proposition 3.3, we have $p, q > 2$ and $P_1 Q_1 \leq \boldsymbol{T}^F$ for an F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$. We also observe that

$$\gcd(pq, |G/S|) = 1. \tag{4-2}$$

Indeed, since $p, q > 2$, the claim follows unless $G = \mathrm{PGL}_n^\epsilon(q_0)$ or $G = E_6^\epsilon(q_0)_{\mathrm{ad}}$ with $\epsilon = \pm$. In the former case, it was shown in the proof of [Beltrán et al. 2016, Proposition 2.7] that the existence of the commuting pair (P_1, Q_1) in $\boldsymbol{G}^F = \mathrm{SL}_n^\epsilon(q_0)$ in the case $\gcd(pq, q_0 - \epsilon) > 1$ implies that $n < \min(p, q)$, and so (4-2) follows as $|G/S| = \gcd(n, q_0 - \epsilon)$. In the latter case, [Beltrán et al. 2016, Proposition 2.12(a)] shows that $p, q > 3$, and so (4-2) follows as $|G/S| = \gcd(3, q_0 - \epsilon)$.

Thus $P, Q \leq S$, and so $P, Q \leq T := (\boldsymbol{T}/Z)^F$. Next we note that

$$G = ST. \tag{4-3}$$

(Indeed, suppose $gZ \in G = (\boldsymbol{G}/Z)^F$ for $g \in \boldsymbol{G}$. Then $g^{-1}F(g) \in Z$. Since $Z \leq \boldsymbol{T}$, by the Lang–Steinberg theorem, there is $t \in \boldsymbol{T}$ such that $t^{-1}F(t) = g^{-1}F(g)$. It follows that $tZ \in (\boldsymbol{T}/Z)^F$, $gt^{-1} \in \boldsymbol{G}^F$, and $gZ = (gt^{-1}Z)(tZ) \in (\boldsymbol{G}^F Z/Z)(\boldsymbol{T}/Z)^F$.) Let

$$N := \boldsymbol{N}_G(PQ), \quad C := \boldsymbol{C}_G(PQ) = PQ \times \boldsymbol{O}_{\{p,q\}'}(C), \quad N_1 := \boldsymbol{N}_S(PQ).$$

As T centralizes PQ, (4-3) shows that $G = SC$ and $N = N_1 C$; in particular,

$$G = S\boldsymbol{C}_G(P) = S\boldsymbol{C}_G(Q), \quad N = N_1 \boldsymbol{C}_N(P) = N_1 \boldsymbol{C}_N(Q). \tag{4-4}$$

It therefore follows from Theorem 2.4 that

$$|B_p(G)\cap B_q(G)| = |B_p(S)\cap B_q(S)|, \quad |B_p(N)\cap B_q(N)| = |B_p(N_1)\cap B_q(N_1)|.$$

Together with Theorem 3.1 applied to S, these equalities yield (4-1).

In the general case $S \lhd G \le \boldsymbol{G}_{\mathrm{ad}}^F$, note that (4-2) and (4-4) still hold for any such G, and hence we are again done by using Theorems 2.4 and 3.1.

(b) Now we consider the case $p = q$. Arguing as in the proof of Theorem 3.1, we may assume that $p > 2$ and that $\boldsymbol{G}_{\mathrm{ad}}$ is defined over a field of characteristic $r \ne p, q$. Suppose in addition that both assertion (iii) of Proposition 3.3 (with $p = q$) and (4-2) hold. (Note that $|G_1| = |G|$ for $G_1 = \boldsymbol{G}^F$, so $p \nmid |G_1|/|S|$ in this case as well.) Then the arguments in (a) apply to show that (4-4) holds, and so we are done again by using Theorems 2.4 and 3.1.

The same considerations as in the proof of Theorem 3.1 show that the only remaining case is when $p=3$, $S=\mathrm{PSL}_3^\epsilon(q_0)$, $G=\mathrm{PGL}_3^\epsilon(q_0)$, $\epsilon=\pm$, and $(q_0-\epsilon)_3=3$. In this case, $H \cong \boldsymbol{O}_{3'}(\boldsymbol{Z}(H)) \times G$ for $H := \mathrm{GL}_3^\epsilon(q_0)$. Hence, by Lemma 2.1(iii), (4-1) follows from the main result of [Michler and Olsson 1983] applied to H. □

Corollary 4.3. *Conjecture A holds for* $\mathrm{GL}_n(q_0)$ *and* $\mathrm{GU}_n(q_0)$, *for any* $n \ge 2$ *and any prime power* q_0.

Proof. It suffices to consider the case $G = \mathrm{GL}_n^\epsilon(q_0)$ is nonsolvable. In this case, $G/\boldsymbol{Z}(G) = \mathrm{PGL}_n^\epsilon(q_0)$ satisfies Conjecture A by Theorem 4.2 when $p \ne q$. Hence we are done using Lemma 2.2(v) if $p \ne q$, and by the main results of [Michler and Olsson 1983], which prove the Alperin–McKay conjecture for these groups, when $p = q$. □

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HESSELINK STRATA IN SMALL CHARACTERISTIC AND LUSZTIG–XUE PIECES

ALEXANDER PREMET

In memory of Gary Seitz

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and $\mathfrak{g} = \mathrm{Lie}(G)$. We show that the nilpotent pieces $\mathrm{LX}(\Delta)$ introduced by Lusztig form a partition of the nilpotent cone of $\mathfrak{g}$ and hence coincide with the Hesselink strata $\mathcal{H}(\Delta)$ where Δ runs through the set of all weighted Dynkin diagrams of G. Thanks to earlier results obtained by Lusztig, Xue and Voggesberger this boils down to describing the pieces $\mathrm{LX}(\Delta)$ for groups of type E_7 in characteristic 2 and for groups of type E_8 in characteristic 2 and 3. Our arguments are computer-free, but rely very heavily on the results of Liebeck and Seitz (2012).

1. Introduction

Let G be a connected reductive algebraic group of rank ℓ over an algebraically closed field $\Bbbk$ and T a maximal torus of G. Let Σ be the root system of G with respect to T and Π a basis of simple roots of Σ. Write $X(T)$ (resp. $X_*(T)$) for the lattice of rational characters (resp. cocharacters) of T and $X_*^+(T)$ for the intersection of $X_*(T)$ with the dual Weyl chamber of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ associated with Π. Each rational cocharacter $\lambda \in X_*(G)$ gives rise to a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\lambda, i), \qquad \mathfrak{g}(\lambda, i) = \{x \in \mathfrak{g} \mid (\mathrm{Ad}\,\lambda(t))x = t^i x \text{ for all } t \in \Bbbk^\times\},$$

of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. For $d \in \mathbb{Z}$, we put $\mathfrak{g}(\lambda, \geq d) := \bigoplus_{i \geq d} \mathfrak{g}(\lambda, i)$ and $\mathfrak{g}(\lambda, < d) := \bigoplus_{i < d} \mathfrak{g}(\lambda, i)$, and denote by $P(\lambda) = L(\lambda) R_u(\lambda)$ the parabolic subgroup of G associated with λ. Here $L(\lambda) = Z_G(\lambda)$ is a Levi subgroup of G. Recall that $\mathrm{Lie}(P(\lambda)) = \mathfrak{g}(\lambda, \geq 0)$ and $\mathrm{Lie}(L(\lambda)) = \mathfrak{g}(\lambda, 0)$.

We let $\mathcal{N}(\mathfrak{g})$ denote the nilpotent cone of $\mathfrak{g}$, the variety of all $(\mathrm{Ad}\, G)$-unstable vectors of $\mathfrak{g}$, and write $\mathfrak{D}_G$ for the set of all *Dynkin labels* attached to the nilpotent orbits of a complex Lie algebra with root system Σ. As explained in [4], the

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Hesselink strata of $\mathcal{N}(\mathfrak{g})$ are parametrised by the set of cocharacters $\tau_\Delta \in X_*^+(T)$ with $\Delta \in \mathfrak{D}_G$ and form a partition of $\mathcal{N}(\mathfrak{g})$, so that

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta\in\mathfrak{D}_G} \mathcal{H}(\Delta).$$

The cocharacter τ_Δ can be read off the weighted Dynkin diagram $(a_1, \dots, a_\ell)$ associated with the complex nilpotent orbit with label Δ as follows: if x is a root vector of $\mathfrak{g}$ corresponding to a simple root $\alpha_i \in \Pi$ then $(\operatorname{Ad}\tau_\Delta(t))(x) = t^{a_i}x$ for all $t \in \Bbbk^\times$. The Hesselink stratum attached to τ_Δ has the form

$$\mathcal{H}(\Delta) = (\operatorname{Ad} G)(\mathcal{V}(\tau_\Delta, 2)_{ss} + \mathfrak{g}(\tau_\Delta, \geq 3)),$$

where $\mathcal{V}(\tau_\Delta, 2)_{ss} \neq \varnothing$ is the set of all $(\operatorname{Ad} L(\tau_\Delta))$-semistable vectors of $\mathfrak{g}(\tau_\Delta, 2)$ (see [4] for more detail).

Given $\Delta \in \mathfrak{D}_G$ we write $\mathfrak{g}_2^{\Delta,!}$ for the set of all $x \in \mathfrak{g}(\tau_\Delta, 2)$ such that $G_x \subset P(\tau_\Delta)$ where $G_x = Z_G(x)$ is the stabiliser of x in G. As explained in [4, Remark 7.3] each set $\mathfrak{g}_2^{\Delta,!}$ contains $\mathcal{V}(\tau_\Delta, 2)_{ss}$, a nonempty Zariski open subset of $\mathfrak{g}(\tau_\Delta, 2)$. The set

$$\mathrm{LX}(\Delta) := (\operatorname{Ad} G)(\mathfrak{g}_2^{\Delta,!} + \mathfrak{g}(\tau_\Delta, \geq 3))$$

containing $\mathcal{H}(\Delta)$ will be referred to as the *Lusztig–Xue piece* of $\mathcal{N}(\mathfrak{g})$ associated with Δ. The pieces $\mathrm{LX}(\Delta)$ and their analogues for $\mathcal{N}(\mathfrak{g}^*)$ and for the unipotent variety of G were introduced by Lusztig. Viability of these pieces has to do with the fact that $\mathfrak{g}_2^{\Delta,!}$ is defined in a more transparent fashion than its elusive subset $\mathcal{V}(\tau_\Delta, 2)_{ss}$.

In [11, Appendix A], Lusztig and Xue proved that the pieces $\mathrm{LX}(\Delta)$ form a partition of $\mathcal{N}(\mathfrak{g})$ in the case where G is a classical group. Very recently, the same property was established by Voggesberger for groups of type G_2, F_4 and E_6 (see our discussion in Section 2.1 for more detail). These results imply that $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ provided that G is not of type E_7 or E_8.

The partition property of the coadjoint analogues of $\mathrm{LX}(\Delta)$ was established by Lusztig [10] and Xue [18] in all cases where G is a simple algebraic group and $p = \operatorname{char}(\Bbbk)$ equals the ratio of the squared lengths of long and short roots in Σ. In all other cases there is a G-equivariant bijection between $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{g}^*)$ which enables one to identify the nilpotent coadjoint orbits and pieces of $\mathfrak{g}^*$ with those of $\mathfrak{g}$; see [13, Section 5.6].

It was conjectured in [17] that $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ should also hold for all $\Delta \in \mathfrak{D}_G$ in the case where G is a group of type E_7 or E_8. Our goal is to prove this conjecture.

The orbits $\mathcal{O}(e) = (\operatorname{Ad} G)e$ with $e \in \mathcal{N}(\mathfrak{g})$ will be denoted by their Dynkin labels Δ or their variants $(\Delta)_p$. The latter are attached to a small number of new nilpotent orbits which appear when $(\Sigma, p) \in \{(\mathrm{G}_2, 3), (\mathrm{F}_4, 2), (\mathrm{E}_7, 2), (\mathrm{E}_8, 2), (\mathrm{E}_8, 3)\}$; see [9] for detail. Combining our results obtained in Section 2 with the results of Lusztig, Xue and Voggesberger mentioned above we obtain the following:

Theorem 1.1. *Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Then $\mathcal{H}(\Delta) = \mathrm{LX}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ and hence*

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathrm{LX}(\Delta).$$

Since proving Theorem 1.1 reduces quickly to the case where Σ is an irreducible root system, we may assume without loss of generality that our algebraic group G is simple and simply connected.

We use Steinberg's notation $x_\alpha(t)$ for elements of the unipotent root subgroups U_α of G; see [15, §3]. Simple root vectors e_{α_i} with $\alpha_i \in \Pi$ are denoted by e_i, and we always use Bourbaki's numbering [2] of simple roots. We assume that root vectors $e_\gamma \in \mathfrak{g}_\gamma$ come from a Chevalley basis of an admissible lattice $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}_\mathbb{C}$, where $\mathfrak{g}_\mathbb{C}$ is a complex Lie algebra with root system Σ. Since we mostly work over fields of characteristic 2, the signs of structure constants do not really affect our computations.

The Weyl group of Σ is denoted by W and we write $\langle \cdot\,, \cdot \rangle$ for the canonical pairing between $X(T)$ and $X_*(T)$ with values in $\mathbb{Z}$. We fix a W-invariant $\mathbb{Q}$-valued inner product $(\cdot \mid \cdot)$ on $X_*(T)_\mathbb{Q} = X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$ which enables one to identify $X_*(T)_\mathbb{Q}$ with the dual vector space $X(T)_\mathbb{Q} = X(T) \otimes_\mathbb{Z} \mathbb{Q}$.

When we need to specify a particular root vector, we sometimes follow the conventions of [9]. For example, a root vector e_γ with $\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ is denoted by $e_{234^25^26}$. When confusion is unlikely, we prefer standard conventions for specifying root vectors.

We occasionally use the $(\mathrm{Ad}\, G)$-invariant $\mathbb{Z}$-valued bilinear form κ on a minimal admissible lattice $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}_\mathbb{C}$ introduced in [4, 7.2]. Its reduction modulo p will be denoted by the same symbol. When describing certain cocharacters $\tau \in X_*(T)$ we often specify their effect on the root vectors e_i where $1 \leq i \leq \ell$. More precisely, if $(\mathrm{Ad}\, \tau(t))e_i = t^{r_i} e_i$ for all $t \in \Bbbk^\times$, then we write $\tau = (r_1, \ldots, r_\ell)$. This will cause no confusion since $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is a $\mathbb{Q}$-basis of $X(T)_\mathbb{Q}$.

If $\mathrm{char}(\Bbbk) = p > 0$ then $\mathfrak{g} = \mathrm{Lie}(G)$ carries a canonical restricted Lie algebra structure $\mathfrak{g} \ni x \mapsto x^{[p]} \in \mathfrak{g}$ equivariant under the adjoint action of G. It is well known that the nilpotent cone $\mathcal{N}(\mathfrak{g})$ coincides with the set of all $x \in \mathfrak{g}$ such that $x^{[p]^N} = 0$ for $N \gg 0$. A restricted Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is called $[p]$-*nilpotent* (resp. *toral*) if $\mathfrak{a} \subseteq \mathcal{N}(\mathfrak{g})$ (resp. if the $[p]$-mapping $x \mapsto x^{[p]}$ is one-to-one on $\mathfrak{a}$). Given $x \in \mathfrak{g}$ we write $\mathfrak{g}_x$ for the centraliser of x in the Lie algebra $\mathfrak{g}$, and we often use the fact that $\mathrm{Lie}(G_x) \subseteq \mathfrak{g}_x$. If $x \in \mathfrak{g}(\lambda, r)$ for some $\lambda \in X_*(G)$ and $r \in \mathbb{Z}$ then $\mathfrak{g}_x = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_e(\lambda, i)$ where $\mathfrak{g}_e(\lambda, i) = \mathfrak{g}_e \cap \mathfrak{g}(\lambda, i)$. We say that $x \in \mathfrak{g}$ is *toral* if $x^{[p]} = x$.

2. Hesselink strata and Lusztig–Xue pieces

2.1. Let $\tau = \tau_\Delta \in X_*(T)$ be the cocharacter associated with $\Delta \in \mathfrak{D}_G$ and write $\mathfrak{g}_2^{\Delta,!}$ for the set of all $x \in \mathfrak{g}(\tau, 2)$ such that $G_x \subset P(\tau)$. Since the parabolic subgroup $P(\tau)$

is optimal for all $x \in \mathcal{V}(\tau, 2)_{ss}$ in the sense of Kempf–Rousseau theory, we have the inclusion $\mathcal{V}(\tau, 2)_{ss} \subseteq \mathfrak{g}_2^{\Delta,!}$. From [4, Theorem 6.1(iii)] it follows that $\mathcal{V}(\tau, 2)_{ss}$ is a nonempty Zariski open subset of $\mathfrak{g}(\tau, 2)$. The set

$$\mathrm{LX}(\Delta) := (\mathrm{Ad}\, G)(\mathfrak{g}_2^{\Delta,!} + \mathfrak{g}(\tau, \geq 3))$$

is called the *Lusztig–Xue piece* of $\mathcal{N}(\mathfrak{g})$ associated with Δ. By the above, $\mathrm{LX}(\Delta)$ contains $\mathcal{H}(\Delta)$ for every $\Delta \in \mathfrak{D}_G$.

The pieces $\mathrm{LX}(\Delta)$ were first introduced by Lusztig [11] in the Lie algebra case. In [10], the definition was extended to cover the coadjoint nullcone $\mathcal{N}(\mathfrak{g}^*)$ and the unipotent variety $\mathfrak{U}(G)$ of G. In [11, Appendix A], Lusztig and Xue proved that the decomposition

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathrm{LX}(\Delta) \tag{1}$$

holds for all groups of type A, B, C, D, the key point being that the union is disjoint. Very recently, the same property was established by L. Voggesberger for groups of type G_2, F_4 and E_6 with the help of Magma; see [17, Theorem 1.1].

Note that if (1) holds for G then $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$; see [4, Remark 7.3.1] for detail. It was conjectured in [17, Conjecture 1.2] that (1) should also hold for the $\Bbbk$-groups of type E_7 and E_8. A similar expectation (covering $\mathcal{N}(\mathfrak{g})$, $\mathcal{N}(\mathfrak{g}^*)$ and $\mathfrak{U}(G)$) was expressed by Lusztig in [10, 2.3].

We aim to confirm Voggesberger's conjecture. Lusztig's expectation related to partitioning the unipotent variety of G will be discussed in Section 2.12. For completeness, we also provide a new proof for groups of type G_2, F_4 and E_6.

Our task will become much simpler if we establish the following:

$$\text{If } e \in \mathfrak{g}(\tau_\Delta, 2) \text{ and } G_e \subset P(\tau_\Delta) \text{ then } e \in \mathcal{H}(\Delta). \tag{2}$$

Proving statement (2) will occupy the main body of the paper. From now on we assume that the group G is simple, simply connected, and has type E_6, E_7, E_8, F_4 or G_2. Let κ denote the normalised Killing form introduced in [4, 7.3]. If all roots of Σ have the same length then the radical of κ coincides with the central toral subalgebra of $\mathfrak{g}$; see [4, Lemma 7.3]. We start with a reduction lemma which will also explain why (1) always holds in good characteristic.

Lemma 2.1. *Suppose G is a group of type* E *and let $\tau = \tau_\Delta$, where $\Delta \in \mathfrak{D}_G$. If $e \in \mathfrak{g}(\tau, 2)$ is such that $G_e \subset P(\tau)$ and $\mathfrak{g}_e = \mathrm{Lie}(G_e)$, then $e \in \mathcal{H}(\Delta)$.*

Proof. As G is simply connected, the centre of the Lie algebra $\mathfrak{g}$ is spanned by a toral element $z \in \mathfrak{g}(\tau, 0)$ which is nonzero if and only if (Σ, p) is one of $(\mathrm{E}_6, 3)$ or $(\mathrm{E}_7, 2)$. For $i \geq 0$, we let $[e, \mathfrak{g}(\tau, i)]^\perp$ denote the set of all $x \in \mathfrak{g}(\tau, -i-2)$ such that $\kappa\big(x, [e, \mathfrak{g}(\tau, i)]\big) = 0$. Since κ is $(\operatorname{Ad} G)$-invariant we have that

$$[e, \mathfrak{g}(\tau, i)]^{\perp} = \{x \in \mathfrak{g}(-i-2) \mid [e, x] \in \operatorname{Rad}\kappa\}.$$

As $\operatorname{Rad}\kappa = \Bbbk z \subset \mathfrak{g}(\tau, 0)$, we get $[e, \mathfrak{g}(\tau, i)]^{\perp} = \mathfrak{g}_e(\tau, -i-2)$ for $i \geq 1$ and $[e, \mathfrak{g}(\tau, 0)]^{\perp} = \{x \in \mathfrak{g}(\tau, -2) \mid [e, x] \in \Bbbk z\}$.

By our assumption, $\mathfrak{g}_e = \operatorname{Lie}(G_e) \subset \operatorname{Lie}(P(\tau)) = \bigoplus_{i \geq 0} \mathfrak{g}(\tau, i)$. Since $\mathfrak{g}_e = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_e(\tau, i)$ it follows that $\mathfrak{g}_e(\tau, i) = 0$ for $i \leq -1$, forcing $\mathfrak{g}(\tau, i+2) = [e, \mathfrak{g}(\tau, i)]$ for all $i \geq 1$. Also, $\mathfrak{g}(\tau, 2) = [e, \mathfrak{g}(\tau, 0)]$ when $z = 0$. If $z \neq 0$ we can only say at this point that the subspace $[e, \mathfrak{g}(\tau, 0)]$ has codimension ≤ 1 in $\mathfrak{g}(\tau, 2)$.

Since $\mathfrak{g}_e = \operatorname{Lie}(G_e)$ we have that

$$\begin{aligned} \dim(\operatorname{Ad} G)e = \dim \mathfrak{g} - \dim \mathfrak{g}_e &= \sum_{i \in \mathbb{Z}} \dim \mathfrak{g}(\tau, i) - \sum_{i \geq 0} \dim \mathfrak{g}_e(\tau, i) \\ &= \sum_{i < 0} \dim \mathfrak{g}(\tau, i) + \sum_{i \geq 0} \dim\, [e, \mathfrak{g}(\tau, i)] \\ &= \dim\, [e, \mathfrak{g}(\tau, 0)] + \sum_{i \notin \{0,1,2\}} \dim \mathfrak{g}(\tau, i). \end{aligned}$$

Since it follows from [7, Theorem 2] that $\dim\, (\operatorname{Ad} G)e$ and $\sum_{i \notin \{0,1\}} \dim \mathfrak{g}(\tau, i)$ are even numbers, it must be that $\dim\, [e, \mathfrak{g}(\tau, 0)] \equiv \dim \mathfrak{g}(\tau, 2) \bmod 2$. In view of our earlier remarks this yields $\mathfrak{g}(\tau, 2) = [e, \mathfrak{g}(\tau, 0)]$.

Let $L(\tau) = Z_G(\tau)$, a Levi subgroup of $P(\tau)$ with Lie algebra $\mathfrak{g}(\tau, 0)$. Since $[\mathfrak{g}(\tau, 0), e]$ is contained in the tangent space $T_e\big((\operatorname{Ad} L(\tau))e\big)$, the $(\operatorname{Ad} L(\tau))$-orbit of e is Zariski open in $\mathfrak{g}(\tau, 2)$ and hence intersects with $\mathcal{V}(\tau, 2)_{ss} \neq \varnothing$. This yields $e \in \mathcal{H}(\Delta)$. □

2.2. Given a connected reductive $\Bbbk$-group H and a nilpotent element $e \in \operatorname{Lie}(H)$ (i.e., an unstable vector of the $(\operatorname{Ad} H)$-module $\operatorname{Lie}(H)$), we write $\hat{\Lambda}_H(e)$ for the set of all cocharacters in $X_*(H)$ optimal for e in the sense of Kempf–Rousseau theory. By [5, Theorem 7.2], this set of cocharacters does not depend on the choice of an H-invariant $\mathbb{R}_{\geq 0}$-valued norm mapping on $X_*(H)$.

Let $\tau = \tau_\Delta$, where $\Delta \in \mathfrak{D}_G$, and let $e \in \mathfrak{g}(\tau, 2)$ be such that $G_e \subset P(\tau)$. As e is a G-unstable vector of $\mathfrak{g}$ it affords an optimal cocharacter $\tau' \in X_*(G)$ with the property that $e \in \mathfrak{g}(\tau', \geq 2)$; see [4] for detail. Since $\operatorname{Ad} \tau(\Bbbk^\times)$ preserves the line $\Bbbk e$ and the cocharacter τ' is optimal for any nonzero scalar multiple of e, it follows from the main results of Kempf–Rousseau theory that $\operatorname{Ad} \tau(\Bbbk)$ normalises the optimal parabolic subgroup $P(\tau')$ of e; see [12, Theorem 2.1(iv)], for example. Since the latter is self-normalising and contains G_e we have $\tau(\Bbbk^\times)G_e \subset P(\tau')$. Hence $N_e := N_G(\Bbbk e) = \tau(\Bbbk^\times)G_e$ is a subgroup of $P(\tau')$. It follows from [1, 11.14(2)] (applied to tori) that for any maximal torus D of G_e there is a maximal torus $\tilde{D}$ of N_e such that $D \subseteq \tilde{D} \cap G_e$. Since $\tau(\Bbbk^\times)$ is contained in a maximal torus of N_e and all maximal tori of N_e are conjugate, this shows that G_e contains a maximal torus which commutes with $\tau(\Bbbk^\times)$; we call it T_0.

As $N_e \subset P(\tau')$, there is a maximal torus T' of $P(\tau')$ which contains the maximal torus $\tau(\Bbbk^\times)T_0$ of N_e. Since $L := Z_G(T_0)$ is a Levi subgroup of G, there exists $g \in G$ such that gLg^{-1} is a standard Levi subgroup L of G. Replacing e and τ by $(\mathrm{Ad}\, g)e$ and $g\tau g^{-1}$ we may assume without loss of generality that L is a standard Levi subgroup of G. By the main results of Kempf–Rousseau theory (and the description of Hesselink strata in [4]), there exists a unique cocharacter $\tau'' \in \hat{\Lambda}_G(e) \cap X_*(T')$ conjugate to τ' under $P(\tau')$ and such that $e = \sum_{i \geq 2} e_i$ where $e_i \in \mathfrak{g}(\tau'', i)$ and $e_2 \in \mathcal{V}(\tau, 2)_{ss}$; see [12, Theorem 2.1(iii)].

By construction, T' is a maximal torus of L containing $\tau(\Bbbk^\times)$ and $\tau''(\Bbbk^\times)$. Furthermore, $e \in \mathfrak{g}^{\mathrm{Ad}\, T_0} = \mathrm{Lie}(L)$. Let $L' = \mathcal{D}L$, the derived subgroup of L, and $\mathfrak{l}' = \mathrm{Lie}(L')$. It is immediate from Jacobson's formula for $[p]$-th powers in the restricted Lie algebra $(\mathfrak{l}, [p])$ that $e \in \mathfrak{l}'$.

2.3. Let $d = \dim T_0$ and $r = \ell - d$ where $\ell = \mathrm{rk}\, G$. We begin to investigate the case where $d > 0$, that is, the case where e is not distinguished in $\mathfrak{g}$. The group $L' = \mathcal{D}L$ is semisimple and $T' \cap L'$ contains a maximal torus of L'; we call it T_1. The subtorus $T_0 \cdot T_1$ of T' being self-centralising, it must be that $T' = T_0 \cdot T_1$. As the central subgroup $T_0 \cap L'$ of L' is finite, we have a direct sum decomposition

$$X(T')_{\mathbb{Q}} = X(T_0)_{\mathbb{Q}} \oplus X(T_1)_{\mathbb{Q}} \tag{3}$$

of the $\mathbb{Q}$-spans of $X(T_0)$ and $X(T_1)$ in $X(T')_{\mathbb{Q}} = X(T') \otimes_{\mathbb{Z}} \mathbb{Q}$. This shows that

$$\mathrm{rk}\, L' = \dim T_1 = \ell - \dim_{\mathbb{Q}} X(T_0)_{\mathbb{Q}} = \ell - d = r.$$

Since we identify the dual spaces $X(T')_{\mathbb{Q}}$ and $X_*(T')_{\mathbb{Q}} = X_*(T') \otimes_{\mathbb{Z}} \mathbb{Q}$ by means of a W-invariant inner product $(\,\cdot \mid \cdot\,)$ we have that $\eta(\mu(t)) = t^{(\mu|\eta)}$ for all $\eta \in X(T)$, $\mu \in X_*(T')$, and $t \in \Bbbk^\times$. As the subgroup $N_L(T')/T'$ of W acts trivially on $X(T_0)$ and has no nonzero fixed points on $X(T_1)$, the $\mathbb{Q}$-spans $X_*(T_0)_{\mathbb{Q}}$ and $X_*(T_1)_{\mathbb{Q}}$ are orthogonal to each other with respect to $(\,\cdot \mid \cdot\,)$.

Lemma 2.2. *Suppose $\nu \in X_*(T')$ is such that $e \in \mathfrak{g}(\nu, 2)$ and $(G_e)^\circ \subset P(\nu)$. If the group $(G_e)^\circ/R_u(G_e)$ is semisimple then $\nu = \tau$.*

Proof. As $T_0 \subseteq T' \cap G_e$ is a maximal torus of G_e it must be that $T_0 = (T' \cap G_e)^\circ$. As $\nu^{-1}(t) \cdot \tau(t) \in T' \cap G_e$ for all $t \in \Bbbk^\times$ we have that $\nu - \tau \in X_*(T_0)$. As $R_u(G_e)$ is a unipotent group, the torus $T_0 \subset G_e$ maps isomorphically onto a maximal torus of the semisimple group $S_e := (G_e)^\circ/R_u(G_e)$. It follows that $\mathrm{rk}\, S_0 = \dim T_0 = d$. We identify T_0 with its image in S_e. As $\nu(\Bbbk^\times) \subset N_e$ normalises $R_u(G_e)$, it acts on S_e by rational automorphisms.

It is straightforward to see that the connected algebraic group $\tilde{S}_e := \nu(\Bbbk^\times)S_e$ is reductive and $\nu(\Bbbk^\times)T_0$ is a maximal torus of $\tilde{S}_e$. So, if $\bar{\gamma} \in X(\nu(\Bbbk^\times)T_0)$ is a root of $\tilde{S}_e$ then so is $-\bar{\gamma}$. On the other hand, our assumption on ν implies that $\operatorname{Ad}\nu$ has nonnegative weights on $\operatorname{Lie}(S_e)$. This entails that $\nu(\Bbbk^\times)$ is a central torus of $\tilde{S}_e$. Repeating this argument with τ in place of ν we deduce that the torus $\nu(\Bbbk^\times)$ is central in $\tilde{S}_e$ as well.

Since $\operatorname{rk} S_e = d$ there are $\mathbb{Q}$-independent weights $\gamma_1, \dots, \gamma_d \in X(T_0)$ which serve as a basis of simple roots for the root system of $\tilde{S}_e$ with respect to T_0. As $\dim X(T_0)_{\mathbb{Q}} = d$, they must form a basis of the vector space $X(T_0)_{\mathbb{Q}}$. On the other hand, the preceding discussion shows that $\gamma_i(\nu(t)) = \gamma_i(\tau(t)) = 1$ for all $t \in \Bbbk^\times$ and $i \le d$. Our identification of $X(T')_{\mathbb{Q}}$ and $X_*(T')_{\mathbb{Q}}$ now yields that $\nu - \tau$ is orthogonal to $X(T_0)_{\mathbb{Q}}$ with respect to $(\,\cdot\mid\cdot\,)$. Since $\nu - \tau \in X_*(T_0) \subseteq X(T_0)_{\mathbb{Q}}$ this forces $\nu - \tau = 0$. □

2.4. Recall from Section 2.2 that our nilpotent element $e \in \mathfrak{l}'$ affords an optimal cocharacter $\tau'' \in \hat{\Lambda}_G(e) \cap X_*(T')$ such that $e = \sum_{i\ge 2} e_i$ with $e_i \in \mathfrak{l}'(\tau'', i)$ and $e_2 \in \mathcal{V}(\tau'', 2)_{ss}$. Identifying $X_*(T')_{\mathbb{Q}}$ and $X(T')_{\mathbb{Q}}$ as in Section 2.3 and using (3) we get $\tau'' = \tau''_0 + \tau''_1$ where $\tau''_0 \in X(T_0)_{\mathbb{Q}}$ and $\tau''_1 \in X(T_1)_{\mathbb{Q}}$. Since

$$(\tau'' \mid \tau'') = (\tau''_0 \mid \tau''_0) + (\tau''_1, \tau''_1) \ge (\tau''_0 \mid \tau''_0)$$

and $(\tau''_0 \mid \gamma) = 0$ for all $\gamma \in X(T_1)$, it follows from the optimality of τ'' that $\tau''_0 = 0$; see [12, p. 348] for a similar (characteristic-free) argument.

As $\tau'' \in X_*(T')$ we thus obtain that τ'' is an optimal cocharacter for e contained in $X_*(T_1)$. Therefore, the orbit $\mathcal{O}_L(e) := (\operatorname{Ad} L)e$ is contained in the Hesselink stratum $\mathcal{H}_L(\tau'')$ of the nilpotent cone $\mathcal{N}(\mathfrak{l}')$, the variety of all $(\operatorname{Ad} L)$-unstable vectors of $\mathfrak{l}'$. Since T_0 is a maximal torus of G_e the group $(L_e)^\circ$ is unipotent. In other words, e is a distinguished nilpotent element of $\mathfrak{l}'$.

Lemma 2.3. *If the* $(\operatorname{Ad} L)$*-orbit* $\mathcal{O}_L(e)$ *coincides with its stratum* $\mathcal{H}_L(\tau'')$ *and the group* $(G_e)^\circ/R_u(G_e)$ *is semisimple, then* τ *is* L*-conjugate to* τ'' *and* $e \in \mathcal{H}(\Delta)$.

Proof. Recall that $e = \sum_{i\ge 2} e_i$ where $e_i \in \mathfrak{l}'(\tau'', i)$ and $e_2 \in \mathcal{V}(\tau'', 2)_{ss}$. As one of the $(\operatorname{Ad} L)$-orbits of $\mathcal{H}_L(\tau'')$ intersects with $\mathfrak{l}'(\tau'', 2)$, our assumption on $\mathcal{H}_L(\tau'')$ implies that e is $(\operatorname{Ad} L)$-conjugate to e_2. Since $\tau'' \in \hat{\Lambda}_G(e_2)$ we have $G_{e_2} \subset P(\tau'')$. As the group $(G_{e_2})^\circ/R_u(G_{e_2}) \cong (G_e)^\circ/R_u(G_e)$ is semisimple, it follows from Lemma 2.2 that τ is L-conjugate to τ''. As τ'' is G-conjugate to τ_Δ by our discussion in Section 2.2, we deduce that $e_2 \in \mathcal{H}(\Delta)$. But then $e \in \mathcal{H}(\Delta)$ as wanted. □

Let $e \in \mathcal{N}(\mathfrak{g})$ and write S_e for the factor group $(G_e)^\circ/R_u(G_e)$. When $\operatorname{char}(\Bbbk) = 0$, one can use the tables in [3, pp. 401–407] to quickly compile a full list of the nilpotent orbits $\mathcal{O}(e) = (\operatorname{Ad} G)e$ for which S_e is a semisimple group. Then one can use [8] to find out that the same list is still valid in good characteristic. Since we are mainly concerned with the case where $p = \operatorname{char}(\Bbbk)$ is very bad for G, we must rely instead on the following important classification result obtained in [9].

Proposition 2.4 [9]. *Suppose G is exceptional and e is not distinguished in $\mathfrak{g}$. Then either S_e is a semisimple group or $\mathcal{O}(e)$ has one of the following labels*:

Type E_6: A_1^2, $\mathrm{A}_2\mathrm{A}_1$, $\mathrm{A}_2\mathrm{A}_1^2$, A_3, $\mathrm{A}_3\mathrm{A}_1$, $\mathrm{D}_4(\mathrm{a}_1)$, A_4, $\mathrm{A}_4\mathrm{A}_1$, $\mathrm{D}_5(\mathrm{a}_1)$, D_5.

Type E_7: $\mathrm{A}_2\mathrm{A}_1$, $\mathrm{A}_3\mathrm{A}_2$ $(p \neq 2)$, A_4, $\mathrm{A}_4\mathrm{A}_1$, $\mathrm{D}_5(\mathrm{a}_1)$, $\mathrm{E}_6(\mathrm{a}_1)$.

Type E_8: $\mathrm{A}_3\mathrm{A}_2$ $(p \neq 2)$, $\mathrm{A}_4\mathrm{A}_1$, $\mathrm{A}_4\mathrm{A}_1^2$, $\mathrm{D}_5\mathrm{A}_2$ $(p \neq 2)$, $\mathrm{D}_7(\mathrm{a}_2)$, $\mathrm{E}_6(\mathrm{a}_1)\mathrm{A}_1$, $\mathrm{D}_7(\mathrm{a}_1)$ $(p \neq 2)$.

If G is of type G_2 *or* F_4 *then S_e is a semisimple group for any $e \in \mathcal{N}(\mathfrak{g})$.*

Proof. The statement is obtained by examining Tables 22.1.1–22.1.5 in [9]. One also observes in the process that if a nilpotent orbit $\mathcal{O}(e)$ coincides with its Hesselink stratum then the type of S_e is independent of the characteristic of $\Bbbk$. This is curious, but will not be required in what follows. □

2.5. It is well known that $\dim \mathfrak{g}_e \geq \dim G_e$ for any $e \in \mathfrak{g}$, and if $\dim \mathfrak{g}_e = \dim G_e$, the orbit $\mathcal{O}(e)$ is called *smooth*. For all exceptional types, the smooth nilpotent orbits of $\mathfrak{g}$ can be determined from Stewart's tables [16] which record the Jordan blocks of all $\operatorname{ad} e$ with $e \in \mathcal{N}(\mathfrak{g})$. We note that the representatives of nilpotent orbits used in Stewart's tables are compatible with those in [9, Tables 12.1, 13.3 and 14.1].

Remark 2.5. Suppose G is exceptional. Although Liebeck and Seitz do not discuss Hesselink strata in [9], they can be spotted as follows. For each representative $e \in \mathcal{N}(\mathfrak{g})$ listed in the tables of [9] there exists a cocharacter $\mu \in X_*(G)$ conjugate to τ_Δ with $\Delta \in \mathfrak{D}_G$ and such that $e \in \mathfrak{g}(\mu, \geq 2)$ and $G_e \subset P(\mu)$. Of course, in characteristic 2 and 3 there are a few special cases where two representatives of different orbits, say e and $\tilde{e}$, are attached to the same μ (and the same Δ). If there is no $\tilde{e}$, then e is homogeneous (that is, lies in $\mathfrak{g}(\mu, 2)$) and $\overline{(\operatorname{Ad} P(\mu))e} = \mathfrak{g}(\mu, \geq 2)$. Then the orbit $(\operatorname{Ad} Z_G(\mu))e$ is dense in $\mathfrak{g}(\mu, 2)$ and the description of Hesselink strata in [4] implies that e is $Z_G(\mu)$-semistable. This is an excellent case since $\mathcal{H}(\Delta) = \mathcal{O}(e)$ is a single orbit and $\dim G_e = \dim \mathfrak{g}(\mu, 0) + \dim \mathfrak{g}(\mu, 1)$ is independent of p.

If $\tilde{e}$ does exist then one may assume without loss of generality that $e \in \mathfrak{g}(\mu, 2)$ and $\tilde{e} = e + e_\beta$ for some root vector $e_\beta \in \mathfrak{g}(\mu, d)$ with $d \geq 2$. Moreover, $\tilde{e}$ always lies in the new orbit with label $(\Delta)_p$ and $(\operatorname{Ad} P(\mu))\tilde{e}$ is Zariski dense in $\mathfrak{g}(\mu, \geq 2)$. Then $\dim G_{\tilde{e}} = \dim \mathfrak{g}(\mu, 0) + \dim \mathfrak{g}(\mu, 1)$, but $\dim G_e > \dim G_{\tilde{e}}$.

If $d > 2$ then it is still true that the orbit $(\operatorname{Ad} Z_G(\mu))e$ is dense in $\mathfrak{g}(\mu, 2)$. Therefore, both e and $\tilde{e}$ lie in the stratum $\mathcal{H}(\Delta)$. In fact, it follows from the main results of [9] that $\mathcal{H}(\Delta) = \mathcal{O}(e) \cup \mathcal{O}(\tilde{e})$. So this case is not too bad either.

In order to describe the strata of $\mathcal{H}(\Delta)$ explicitly, one has to clarify the remaining (problematic) case where $\tilde{e} = e + e_\beta$ and $d = 2$. Here $\mathfrak{g}(\mu, 2)$ contains both e and $\tilde{e}$, but the orbit $(\operatorname{Ad} Z_G(\mu))e$ is no longer dense in $\mathfrak{g}(\mu, 2)$. So we cannot conclude at this point that e is $Z_G(\mu)$-semistable. However, since μ is G-conjugate to τ_Δ

and it is shown in [9] that $G_e \subset P(\mu)$, we see that $e \in \mathrm{LX}(\Delta)$. Therefore, the main results of [9] together with Theorem 1.1 provide a very satisfactory description of the Hesselink strata of $\mathcal{N}(\mathfrak{g})$ for G exceptional. Namely, if there is no $\tilde{e}$ then $\mathcal{H}(\Delta) = \mathcal{O}(e)$ and if $\tilde{e}$ does exist then $\mathcal{H}(\Delta) = \mathcal{O}(e) \cup \mathcal{O}(\tilde{e})$.

Remark 2.6. Suppose e is as in Remark 2.5 and $\mathcal{O}(e) = \mathcal{H}(\Delta)$. Then $\dim G_e = \dim \mathfrak{g}(\mu, 0) + \dim \mathfrak{g}(\mu, 1)$ is independent of the characteristic of $\Bbbk$. Since the representatives e used in [16] agree with those of [9] one can compute $\dim G_e$ by counting the number of Jordan blocks of $\operatorname{ad} e$ under the assumption that $p \gg 0$. If the number obtained coincides with the actual number of Jordan blocks of $\operatorname{ad} e \in \operatorname{End} \mathfrak{g}$ then the orbit $(\operatorname{Ad} G)e$ is smooth.

Thanks to Proposition 2.4 and Lemma 2.1 we can reduce proving statement (2) to a much smaller number cases.

Lemma 2.7. *Suppose $\tau \in X_*(G)$ is conjugate to τ_Δ with $\Delta \in \mathfrak{D}_G$ and $e \in \mathfrak{g}(\tau, 2)$ is such that $G_e \subset P(\tau)$. If the group S_e is not semisimple, then $e \in \mathcal{H}(\Delta)$.*

Proof. Since S_e is not semisimple, e lies in one of the orbits listed in Proposition 2.4. Our discussion in Remark 2.5 (based on results of [9]) shows that any such orbit $\mathcal{O}(\Delta)$ coincides with the corresponding Hesselink stratum $\mathcal{H}(\Delta)$. Using [16, Tables 10, 11, 12] and the method described in Remark 2.6 one now checks directly that $\dim \mathfrak{g}_e = \dim G_e$ unless G is of type E_6, $p = 2$, and $\mathcal{O}(e)$ is one of $\mathcal{O}(\mathrm{A}_3\mathrm{A}_1)$ or $\mathcal{O}(\mathrm{D}_5)$. If $\dim \mathfrak{g}_e = \dim G_e$ then applying Lemma 2.1 gives $e \in \mathcal{H}(\Delta)$.

Now suppose G is of type E_6 and $p = 2$. In the remaining two cases e is a regular nilpotent element of a Levi subalgebra of type $\mathrm{A}_3\mathrm{A}_1$ or D_5, hence no generality will be lost by assuming that e is as in [8, pp. 85, 91].

We first suppose that $e \in \mathcal{O}(\mathrm{A}_3\mathrm{A}_1)$. Then $e = e_1 + e_3 + e_4 + e_6$. Since $e \in \mathfrak{g}(\tau, 2)$ and all maximal tori of N_e are conjugate we may also assume that $\tau = (2, r, 2, 2, s, 2)$ for some $r, s \in \mathbb{Z}$. If $\tilde{\alpha} = 122321$, the highest root of Σ with respect to Π, then $\{x_{\pm\tilde{\alpha}}(t) \mid t \in \Bbbk\}$ generate a subgroup of type A_1 in G_e. Since $G_e \subset P(\tau)$ it must be that $(\tau \mid \tilde{\alpha}) = 0$ forcing $14 + 2(r + s) = 0$. Therefore, $\tau = (2, r, 2, 2, -7 - r, 2)$. Since $x_{-\alpha_5}(t) \in G_e$ for all $t \in \Bbbk$ we also have $7 + r \geq 0$. Let $\beta = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $\beta' = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$. Then $\beta \pm \beta' \notin \Sigma$ and $x_\beta(t)x_{\beta'}(t) \in G_e$ for all $t \in \Bbbk$ yielding $e_\beta + e_{\beta'} \in \operatorname{Lie}(G_e)$. Hence $6 - (7 + r) \geq 0$, i.e., $r \leq -1$. As a result, $r \in \{-7, -6, -5, -4, -3, -2, -1\}$. For r in this set we denote by τ_r the W-conjugate of $\tau = (2, r, 2, 2, -7 - r, 2)$ contained in $X_*^+(T)$.

It is straightforward to check that $\tau_{-7} = (0, 1, 1, 0, 1, 2)$, $\tau_{-6} = (1, 1, 0, 0, 1, 2)$, $\tau_{-5} = (1, 0, 0, 1, 0, 2)$, $\tau_{-4} = (1, 1, 0, 0, 1, 1)$, $\tau_{-3} = (0, 1, 1, 0, 1, 0)$, $\tau_{-2} = (1, 1, 1, 0, 0, 1)$, and $\tau_{-1} = (2, 0, 0, 1, 0, 1)$. Using [3, p. 402] we now observe that only τ_{-3} has form τ_Δ with $\Delta \in \mathfrak{D}_G$. Furthermore, $\Delta = \mathrm{A}_3\mathrm{A}_1$. Since $\mathcal{O}(\mathrm{A}_3\mathrm{A}_1) = \mathcal{H}(\mathrm{A}_3\mathrm{A}_1)$ by Remark 2.5, we get $e \in \mathcal{H}(\Delta)$.

Finally, suppose $e \in \mathcal{O}(\mathrm{D}_5)$. Then $e = e_1 + e_2 + e_3 + e_4 + e_5$. As $e \in \mathfrak{g}(\tau, 2)$ and all maximal tori of N_e are conjugate we may assume that $\tau = (2, 2, 2, 2, 2, r)$ for some $r \in \mathbb{Z}$. Let $\gamma = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $\gamma' = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$. Then $\gamma \pm \gamma' \notin \Sigma$ and $x_{-\gamma}(t)x_{-\gamma'}(t) \in G_e$ for all $t \in \Bbbk$. It follows that $e_{-\gamma} + e_{-\gamma'} \in \mathrm{Lie}(G_e)$. Hence $r \le -6$. Now let $\delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$ and $\delta' = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. Then again $\delta \pm \delta' \notin \Sigma$ and $x_\delta(t)x_{\delta'}(t) \in G_e$ for all $t \in \Bbbk$, so that $e_\delta + e_{\delta'} \in \mathrm{Lie}(G_e)$. It follows that $r \ge -14$.

As a result, $r \in \{-14, -13, -12, -11, -10, -9, -8, -7, -6\}$. Let τ_r be the W-conjugate of $\tau = (2, 2, 2, 2, 2, r)$ contained in $X_*^+(T)$. Direct computations show that $\tau_{-14} = (2, 0, 2, 2, 2, 2)$, $\tau_{-13} = (2, 1, 2, 1, 1, 1)$, $\tau_{-12} = (2, 2, 2, 0, 2, 0)$, $\tau_{-11} = (2, 2, 1, 1, 1, 1)$, $\tau_{-10} = (2, 2, 0, 2, 0, 2)$, $\tau_{-9} = (1, 2, 1, 1, 1, 2)$, $\tau_{-8} = (0, 2, 2, 0, 2, 2)$, $\tau_{-7} = (1, 1, 1, 1, 2, 2)$, $\tau_{-6} = (2, 0, 0, 2, 2, 2)$. Using [3, p. 402] we observe that only τ_{-10} has the form τ_Δ with $\Delta \in \mathfrak{D}_G$. Moreover, $\Delta = \mathrm{A}_5$. Since $\mathcal{O}(\mathrm{D}_5) = \mathcal{H}(\mathrm{D}_5)$ by Remark 2.5, we get $e \in \mathcal{H}(\Delta)$, completing the proof. □

2.6. We are now in a position to reduce proving statement (2) to the case where $e \in \mathfrak{g}(\tau, 2)$ is distinguished in $\mathfrak{g}$ and $\mathcal{O}(e)$ is a proper subset of its Hesselink stratum.

Suppose $e \in \mathfrak{g}(\tau, 2)$ is not distinguished in $\mathfrak{g}$. By Section 2.2, the group G_e contains a maximal torus T_0 commuting with $\tau(\Bbbk^\times)$. We also know that the centraliser $L = Z_G(T_0)$ is a standard Levi subgroup of G containing a maximal torus T' such that $\tau(\Bbbk^\times)T_0 \subseteq T'$. Let $L' = \mathcal{D}L$ and $\mathfrak{l}' = \mathrm{Lie}(L')$. By our discussion in Section 2.2, there exists $\tau'' \in \hat{\Lambda}_G(e) \cap X_*(T')$ such that $e = \sum_{i \ge 2} e_i$ where $e_i \in \mathfrak{l}'(\tau'', i)$ and $e_2 \ne 0$. If e is $(\mathrm{Ad}\, L)$-conjugate to e_2 then $e \in \mathcal{H}(\Delta)$ by Lemma 2.3. Therefore, we may assume that e and e_2 lie in different orbits of $\mathcal{H}_L(\tau'')$. Thanks to Section 2.4 we may also assume that the group S_e is semisimple.

If follows from [6, Table 4] that if H is a group of type D_r with $r \in \{4, 5\}$ then every Hesselink stratum of $\mathrm{Lie}(H)$ is a single $(\mathrm{Ad}\, H)$-orbit. Since the same holds for H of type E_6 and A_r with $r \ge 1$, the proper Levi subgroup L of G must have a component of type D_6, D_7, E_7, B_2, B_3 or C_3, where in the last three cases G is a group of type F_4. Thanks to [9, Tables 22.1.1–22.1.5] we now see that the case we consider can occur only when $p = 2$ and G is not of type E_6 or G_2.

Suppose $p = 2$ and G is of type E_7. As $\mathcal{H}_L(\tau'')$ contains more than one orbit, the group L' must have type D_6. By [6, Table 4], the Lie algebra $\mathfrak{l}'$ has a unique new distinguished nilpotent orbit, and the above discussion shows that this new orbit, labelled $(\mathrm{A}_3\mathrm{A}_2)_2$ in [9], must coincide with $\mathcal{O}_L(e)$. By [9, Lemma 12.6], there exists $\mu \in X_*(L')$ such that $e \in \mathfrak{l}'(\mu, 2)$ and the orbit $((\mathrm{Ad}\, P_L(\mu))e$ is dense in $\mathfrak{l}'(\mu, \ge 2)$. From this it is immediate that e is a $Z_L(\mu)$-semistable vector of $\mathfrak{l}'(\mu, 2)$, so that $\mu \in \hat{\Lambda}_L(e)$. Since $\hat{\Lambda}_L(e)$ contains $\hat{\Lambda}_G(e) \cap X^*(T')$ and $e_2 \ne 0$, the cocharacters μ and τ'' are L-conjugate. But then $G_e \subset P(\nu)$ and applying Lemma 2.2 with $\nu = \mu$ we get $\mu = \tau$. As a result, $e \in \mathcal{H}(\Delta)$.

Suppose $p = 2$ and G is of type E_8. Then L' has type D_6, D_7 of E_7. If L' has type D_6 we can repeat verbatim the argument from the previous paragraph to conclude that $e \in \mathcal{H}(\Delta)$. Suppose L' is of type D_7. By [6, Table 4], the Lie algebra $\mathfrak{l}'$ has two new orbits, denoted by $(\mathrm{A}_3\mathrm{A}_2)_2$ and $(\mathrm{D}_4\mathrm{A}_2)_2$ in [9, Table 15.3]. Since the orbit $\mathcal{O}_{L'}((\mathrm{A}_3\mathrm{A}_2)_2)$ is not distinguished in $\mathfrak{l}'$ it must be that $e \in \mathcal{O}_{L'}((\mathrm{D}_4\mathrm{A}_2)_2)$. As [9, Lemma 12.6] is also applicable for $e \in \mathcal{O}_{L'}((\mathrm{D}_4\mathrm{A}_2)_2)$, we can argue as in the second part of the previous paragraph to conclude that $e \in \mathcal{H}(\Delta)$.

2.7. Retain the assumptions of Section 2.6 and suppose that $p = 2$ and L' is of type E_7. Since e is distinguished in $\mathfrak{l}'$ and $\mathcal{O}_{L'}(e) \neq \mathcal{O}_{L'}(e_2)$, it follows from [9, Tables 22.1.2] that $e \in \mathcal{O}_{L'}((\mathrm{A}_6)_2)$. Conjugating e by a suitable element of L' we may assume that

$$e = e_{56} + e_{67} + e_{134} + e_{234} + e_{345} + e_{245} + e_{123^24^25}, \tag{4}$$

where all summands e_γ involved in (4) are root vectors of L' with respect to T'; see [9, Table 14.1]. Let $W' = N_{L'}(T')/T'$, the Weyl group of L'. It is easy to check that the cocharacter $\mu = (-2, -2, -2, 6, -2, 4, -2)$ has the property that $e \in \mathfrak{l}'(\mu, 2)$. Since e is distinguished in $\mathfrak{l}'$, the group L'_e is unipotent and $N_{L'}(\Bbbk e) = \mu(\Bbbk^\times) Z_{L'}(e)$, so that $\mu(\Bbbk^\times)$ is a maximal torus of $N_{L'}(\Bbbk e)$. Since $e \in \mathfrak{g}(\tau, 2)$ and $\tau \in X_*(T')$ this entails that

$$(\mathrm{Ad}\,\mu(t))x = (\mathrm{Ad}\,\tau(t))x \quad \text{for all } x \in \mathfrak{l}' \text{ and } t \in \Bbbk^\times.$$

A direct computation shows that μ is W'-conjugate to $\tau_{\Delta'} = (2, 0, 0, 2, 0, 0, 2)$ which corresponds to the distinguished L'-orbit with label $\mathrm{E}_7(\mathrm{a}_4)$. The latter coincides with its Hesselink stratum $\mathcal{H}_{L'}(\Delta')$. From this it is immediate that $\mathfrak{l}'(\mu, 2)$ contains a Zariski open $Z_{L'}(\mu)$-orbit consisting of $Z_{L'}(\mu)$-semistable vectors. Since $e \notin \mathcal{H}_{L'}(\Delta')$ the orbit $(\mathrm{Ad}\, Z_{L'}(\mu)e$ is not dense in $\mathfrak{l}'(\mu, 2)$. Let

$$U_{L'}(\mu) := R_u(P_{L'}(\mu)).$$

Then $P_{L'}(\mu) = Z_{L'}(\mu)U_{L'}(\mu)$. Furthermore, we have $\mathrm{Lie}(U_{L'}(\mu)) = \mathfrak{l}'(\mu, \geq 2)$ and $\mathrm{Lie}(Z_{L'}(\mu)) = \mathfrak{l}'(\mu, 0)$. In type E_7, the present case was investigated in [9, pp. 209]. It was shown there that L'_e is a connected unipotent group of dimension 19 and $A := L'_e \cap Z_{L'}(\mu)$ is a 1-dimensional connected subgroup of L'_e with the property that $\mathrm{Lie}(A) = \mathfrak{l}'_e(\mu, 0)$.

Straightforward computations show that $[e, \mathfrak{l}'(\mu, 4)]$ has codimension 1 in $\mathfrak{l}'(\mu, 6)$ and $[e, \mathfrak{l}'(\mu, 2i)] = \mathfrak{l}'(\mu, 2i+2)$ for $i = 1$ and all $i \geq 3$. Since the group $U_{L'}(\mu)$ is generated by the root elements $x_\alpha(t) \in L'$ with $(\mu \mid \alpha) \geq 2$ we have that $(\mathrm{Ad}\, U_{L'}(\mu))e \subseteq e + \mathfrak{l}'(\mu, \geq 4)$. Since $[e, \mathfrak{l}'(\mu, 2i)] \subset T_e(\mathrm{Ad}\, U_{L'}(\mu)e)$ for all $i \geq 1$, the preceding remark yields that $T_e(\mathrm{Ad}\, U_{L'}(\mu)e)$ has codimension ≤ 1 in $\mathfrak{l}'(\mu, \geq 4)$. Therefore, $\dim U_{L'}(\mu)_e = \dim \mathrm{Lie}(U_{L'}(\mu)) - \dim T_e(\mathrm{Ad}\, U_{L'}(\mu)e) \leq \dim \mathfrak{l}'(\mu, 2) + 1 = 18$ (one should keep in mind that $P_{L'}(\mu)$ is a distinguished parabolic subgroup of L' and $\mathfrak{l}'(\mu,0)$ has dimension 17). Hence $A\cdot U_{L'}(\mu)_e\subseteq P_{L'}(\mu)_e$ has dimension 18 or 19.

Since L'_e is a connected group of dimension 19 there are two possibilities one of which would be very bad for us: either $L'_e\subset P_{L'}(\mu)$ or $L'_e\not\subset P_{L'}(\mu)$ and $T_e(\operatorname{Ad}U_{L'}(\mu)e)=\mathfrak{l}'(\mu,\geq 4)$. In the second case we would have $(\operatorname{Ad}U_{L'}(\mu))e=e+\mathfrak{l}'(\mu,\geq 4)$ by Rosenlicht's theorem [14, Theorem 2].

Once again our main source of reference comes to the rescue: it is proved in [9, p. 208] that L'_e contains the 1-parameter unipotent subgroup

$$U=\{x_{\alpha_1}(c)x_{\alpha_1+\alpha_3}(c^2)x_{\alpha_2}(c)x_{\alpha_5}(c)x_{\alpha_7}(c)\mid c\in\Bbbk\}.$$

The Lie algebra of U is spanned by $v=e_1+e_2+e_5+e_7\in\mathfrak{l}'(\mu,-2)$ (and one can check directly that $[e,v]=0$). Therefore, $L'_e\not\subset P_{L'}(\mu)$. As a byproduct we obtain that the orbit $(\operatorname{Ad}P_{L'}(\mu))e$ has codimension 1 on $\mathfrak{l}'(\mu,\geq 2)$.

In type E_8 the present case was investigated in [9, pp. 247, 248] where the element in (4) was replaced by its $(\operatorname{Ad}G)$-conjugate

$$e'=e_1+e_3+e_4+e_5+e_6+e_7+e_{1234^25^267}.$$

The cocharacter μ used above was replaced by $\mu'=(2,-14,2,2,2,2,2,-3)$. (In [9, p. 248], the torus $\mu'(\Bbbk^\times)$ is denoted by $\widetilde{T}$.) Direct computations show that there is $w\in W(\mathrm{E}_8)$ such that $w(\mu')=(0,0,0,1,0,1,0,2)=\tau_{\Delta'}$ where $\Delta'=\mathrm{E}_7(\mathrm{a}_4)$. Let $\tilde\alpha=23465432$, the highest root of Σ with respect to Π, and denote by $\tilde\alpha^\vee$ the corresponding coroot in $X_*(T)$, so that $(\operatorname{Ad}\tilde\alpha^\vee(t))(e_\gamma)=t^{\langle\gamma,\tilde\alpha^\vee\rangle}e_\gamma$ for all $t\in\Bbbk^\times$ and $\gamma\in\Sigma$. The adjoint action of $\tilde\alpha^\vee(\Bbbk^\times)$ endows $\mathfrak{g}$ with a short $\mathbb{Z}$-grading $\mathfrak{g}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2$ such that $\dim\mathfrak{g}_{\pm2}=\Bbbk e_{\pm\tilde\alpha}$ and $\mathfrak{g}_0=\operatorname{Lie}(\tilde L)$ where $\tilde L=Z_G(\tilde\alpha^\vee)$.

The derived subgroup $\tilde L'$ of $\tilde L$ has type E_7 and $\tilde L=T_0\tilde L'$ where $T_0=\tilde\alpha^\vee(\Bbbk^\times)$ is a 1-dimensional central torus of $\tilde L$. The type-A_1 subgroup S generated by $x_{\pm\tilde\alpha}(\Bbbk)$ and T_0 commutes with $\tilde L'$. We pick $\sigma\in N_S(\tilde\alpha^\vee(\Bbbk^\times))$ such that $\sigma(\tilde\alpha^\vee)=-\tilde\alpha^\vee$. Then $(\operatorname{Ad}\sigma)(\mathfrak{g}_1)=\mathfrak{g}_{-1}$ implying that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic as $(\operatorname{Ad}\tilde L')$-modules (both modules have dimension 56 and are irreducible over $\tilde L'$). The parabolic subgroup $P(\tilde\alpha^\vee)=\tilde LQ$, where $Q=R_u(P(\tilde\alpha^\vee))$, has the property that $\operatorname{Lie}(Q)=\mathfrak{g}_1\oplus\mathfrak{g}_2$ and $\mathcal{D}Q=x_{\tilde\alpha}(\Bbbk)$.

Computations in [9, p. 248] show that $\widetilde{T}=\mu'(\Bbbk^\times)\subset T\cap\tilde L'$, and $Q\cap G_{e'}$ contains an 8-dimensional abelian connected unipotent subgroup $V=\prod_{i=1}^8 V_i$ such that each V_i is 1-dimensional subgroup of V normalised by the torus $\mu'(\Bbbk^\times)$. The V_i's are described explicitly in [loc. cit.] and it is straightforward to check that $\mu'(\Bbbk^\times)$ acts on V_1, V_2, V_3, V_4, V_5, V_6, V_7 and V_8 with weights 11, 5, 3, 9, 9, 1, 3 and 7, respectively. As $\mu'(\Bbbk^\times)\subset\tilde L'$ commutes with S and $e'\in\operatorname{Lie}(\tilde L')$, the group $(\operatorname{Ad}\sigma)(V)=\prod_{i=1}^8(\operatorname{Ad}\sigma)(V_i)\subset G_{e'}$ has the same properties. As a consequence, both V and $(\operatorname{Ad}\sigma)(V)$ are contained in $P(\mu')_{e'}$.

As $e' \in \mathrm{Lie}(\tilde{L}')$ we also have that $S \subset P(\mu')_{e'}$. Since e and e' are $(\mathrm{Ad}\, G)$-conjugate, our discussion at the beginning of this subsection shows that $P_{\tilde{L}'}(\mu')_{e'}$ has codimension 1 in $\tilde{L}'_{e'}$. In particular, $G_{e'} \not\subset P(\mu')$.

On the other hand, we have that

$$\mathrm{Lie}(V) \oplus (\mathrm{Ad}\,\sigma)(\mathrm{Lie}(V)) \oplus \mathrm{Lie}(S) \oplus \mathrm{Lie}(P_{\tilde{L}'}(\mu')_{e'}) \subseteq \mathrm{Lie}(P(\mu')_{e'}),$$

and the subspace on the left has dimension $8+8+3+18=37$. As $G_{e'}$ is a group of dimension 38 by [9, Table 22.1.1], we have that $P(\mu')_e$ has codimension 1 in $G_{e'}$ This, in turn, implies that the orbit $(\mathrm{Ad}\, P(\mu')e'$ has codimension 1 in $\mathfrak{g}(\mu', \geq 2)$ and $(\mathrm{Ad}\, R_u(P(\mu'))e' = e' + \mathfrak{g}(\mu', \geq 4)$ (since this will not be required in what follows we omit the details).

We mention for completeness that if $\nu \in X_*(T)$ is such that $e' \in \mathfrak{g}(\nu, 2)$ and $G_{e'} \subset P(\nu)$ then the root vectors $e_{\pm\tilde{\alpha}} \in \mathrm{Lie}(G_{e'})$ must have weight 0 with respect to ν. Since $G_{e'} \not\subset P(\mu')$, we have thus excluded the only available option for ν, namely, $\nu = \mu'$. Therefore, $\mathcal{O}((\mathrm{A}_6)_2) \cap \left(\bigcup_{\Delta \in \mathfrak{D}_G} \mathfrak{g}_2^{\Delta,!}\right) = \varnothing$, that is, the present case cannot occur in types E_7 and E_8.

2.8. Retain the assumptions of Section 2.6 and suppose that $p = 2$ and L' is of type F_4. Since this case has already been treated in [17], our goal here is to offer a different proof. In view of our remarks in Section 2.6 we may assume that the Hesselink stratum of $\mathcal{N}(\mathfrak{l}')$ has more than one orbit and L' is of type B_2, B_3 or C_3. The tables in [6] show that in these cases $\mathfrak{l}'$ has a unique new distinguished nilpotent orbit.

The corresponding nilpotent orbits of $\mathfrak{g}$ are denoted in [9, Table 22.1.4] by $(\tilde{\mathrm{A}}_1)_2$, $(\mathrm{B}_2)_2$ and $(\tilde{\mathrm{A}}_2)_2$, respectively. Parts (i) and (iii) of the proof Lemma 16.9 in [9] show that in the first two cases there exists $\mu \in X_*(L')$ such that $e \in \mathfrak{l}'(\mu, 2)$ and the orbit $((\mathrm{Ad}\, P_L(\mu))e$ is dense in $\mathfrak{l}'(\mu, \geq 2)$. As before, this enables us to deduce that e is $Z_L(\mu)$-semistable in $\mathfrak{l}'(\mu, 2)$, so that $\mu \in \hat{\Lambda}_L(e)$. As $\hat{\Lambda}_L(e)$ contains $\hat{\Lambda}_G(e) \cap X^*(T')$ and $e \in \mathfrak{g}(\tau, 2)$, applying Lemma 2.2 gives $\mu = \tau$. Hence $e \in \mathcal{H}(\Delta)$.

Suppose $e \in \mathcal{O}((\tilde{\mathrm{A}}_2)_2)$. Then we may assume that $e = e_{0121} + e_{1111} + e_{2342}$ and T' is the torus used in the proof of part (ii) of [9, Lemma 16.9]. Let $\tau = (a_1, a_2, a_3, a_4)$ where $a_i \in \mathbb{Z}$. As $e \in \mathfrak{g}(\mu, 2)$ we have $a_2 + 2a_3 + a_4 = 2$, $a_1 + a_2 + a_3 + a_4 = 2$ and $2a_1 + 3a_2 + 4a_3 + 2a_4 = 2$. Solving this system of linear equations gives $\tau = (r, -2-r, r, 4)$ where $r \in \mathbb{Z}$. By [9, p. 274], the group G_e contains $x_{\pm\alpha_2}(t)$ and $x_{\alpha_1}(t)x_{\alpha_3}(t)$ for all $t \in \Bbbk$. Therefore, $\mathrm{Lie}(G_e)$ contains $e_{\pm\alpha_2}$ and $e_{\alpha_1} + e_{\alpha_3}$. Since $G_e \subset P(\tau)$, we must have $-2-2r = 0$ and $r \geq 0$. This shows that τ does not exist, that is, the present case cannot occur.

2.9. From now on we may assume that $e \in \mathfrak{g}(\tau_\Delta, 2)$ is a distinguished nilpotent element of $\mathfrak{g}$. If $\mathcal{O}(e) = \mathcal{H}(\Delta')$ then there is $\mu \in \hat{\Lambda}_G(e)$ such that $e \in \mathfrak{g}(\mu, 2)$. As Lemma 2.2 is still applicable in the present case we get $\mu = \tau_\Delta$ forcing $e \in \mathcal{H}(\Delta)$. In other words, statement (2) holds for e.

This shows that we may assume further that $\mathcal{O}(e) \subsetneq \mathcal{H}(\Delta)$. Thanks to the classification results of [9] it remains to consider the case where $e \in \mathcal{N}(\mathfrak{g})$ is distinguished and nonstandard (see Remark 2.5 for more detail). No such orbits exist when G has type E_6 and when G is of type G_2 and $p = 2$.

If G is of type E_7 then $p = 2$ and $\mathcal{O}(e) = \mathcal{O}((\mathrm{A}_6)_2)$. In Section 2.7, we have shown that this case cannot occur in our situation.

Lemma 2.8. *Suppose $e \in \mathfrak{g}(\tau_\Delta, 2)$ is distinguished in $\mathcal{N}(\mathfrak{g})$ and $G_e \subset P(\tau_\Delta)$. If $e' \in \mathcal{O}(e) \cap \mathfrak{g}(\mu, 2)$, where $\mu \in X_*(G)$, then there is $g \in G$ such that $e' = (\mathrm{Ad}\, g)(e)$ and $\mu = g\tau_\Delta g^{-1}$. Consequently, $G_{e'} \subset P(\mu)$.*

Proof. Let $g \in G$ be such that $(\mathrm{Ad}\, g)e = e'$. Then $e' \in \mathfrak{g}(g\tau_\Delta g^{-1}, 2)$ and $G_{e'} \subset P(g\tau_\Delta g^{-1})$. Since the group $(G_{e'})^\circ$ is unipotent and all maximal tori of $N_G(\Bbbk e')^\circ = \tau_{\Delta'}(\Bbbk^\times)(G_{e'})^\circ$ are conjugate, there is $u \in (G_{e'})^\circ$ such that $ug\tau_\Delta g^{-1}u^{-1} = \mu$. Hence $(\mathrm{Ad}\, ug)(e) = (\mathrm{Ad}\, u)(e') = e'$ and $G_{e'} = uG_{e'}u^{-1} \subset P(ug\tau_\Delta g^{-1}u^{-1}) = P(\mu)$. □

It remains to investigate the case where (Σ, p) is one of $(\mathrm{E}_8, 3)$, $(\mathrm{E}_8, 2)$, $(\mathrm{F}_4, 2)$ or $(\mathrm{G}_2, 3)$. If $(\Sigma, p) = (\mathrm{E}_8, 3)$ then $\mathcal{O}(e) = \mathcal{O}((\mathrm{A}_7)_3)$. Recall our standing assumption that $e \in \mathfrak{g}(\tau, 2)$, where $\tau = \tau_\Delta$, and $G_e \subset P(\tau)$. By [9, Table 4.1], we may assume that e is $(\mathrm{Ad}\, G)$-conjugate to

$$e'' = e_{567} + e_{1234} + e_{1345} + e_{3456} + e_{2456} + e_{234^25} + e_{678} + e_{45678}.$$

The cocharacter $\mu = (0, 2, 2, -2, 2, 0, 0, 2) \in X_*(T)$ has the property that $e \in \mathfrak{g}(\mu, 2)$ and is W-conjugate to $\tau_{\Delta'} = (0, 0, 0, 2, 0, 0, 0, 2)$ with $\Delta' = \mathrm{E}_8(\mathrm{b}_6)$; see [9, p. 210]. Replacing e'' by its $N_G(T)$-conjugate, e' say, we may assume that $e' \in \mathfrak{g}(\tau_{\Delta'}, 2)$. By Lemma 2.8, there is $g \in G$ such that $e' = (\mathrm{Ad}\, g)(e)$ and $\tau_{\Delta'} = g\tau_\Delta g^{-1}$. Since both $\tau = \tau_\Delta$ and $\tau_{\Delta'}$ lie in $X_*^+(T)$ it must be that $\Delta = \Delta'$.

We claim that contrary to our standing assumption on τ the group G_e is not contained in $P(\tau) = P(\tau_{\Delta'})$. Indeed, it follows from [9, Theorem 3.2] that the nonempty open subset $\mathcal{V}(\tau, 2)_{ss}$ of $\mathfrak{g}(\tau, 2)$ contains the open $Z_G(\tau)$-orbit of $\mathfrak{g}(\tau, 2)$, we call it V. It has the property that $(\mathrm{Ad}\, P(\tau))v$ is dense in $\mathfrak{g}(\tau, \geq 2)$ for every $v \in V$. If $x \in V$ then $\dim G_x = 28$, whilst $\dim G_{e'} = 30$ by [9, Table 22.1.1]. Since $e' \in \mathcal{O}((\mathrm{A}_7)_3)$ it must be that $e \notin \mathcal{V}(\tau, 2)_{ss}$. But then the orbit $(\mathrm{Ad}\, Z_G(\tau))e$ is not dense in $\mathfrak{g}(\tau, 2)$ implying that $[e, \mathfrak{g}(\tau, 0)]$ is a proper subspace of $\mathfrak{g}(\tau, 2)$.

In the present case, the normalised Killing form κ is nondegenerate and induces a perfect pairing between $\mathfrak{g}(\tau, 2)$ and $\mathfrak{g}(\tau, -2)$. This yields that $\mathfrak{g}_e(\tau, -2) = [e, \mathfrak{g}(\tau, 0)]^\perp$ is nonzero (a different proof can be found in [9, p. 211]). On the other hand, $\dim \mathfrak{g}_e = \dim G_e = 30$ by [16, Table 10]. If $G_e \subset P(\tau)$ then $\mathfrak{g}_e = \mathrm{Lie}(G_e)$ is contained in $\mathfrak{g}(\tau, \geq 0) = \mathrm{Lie}(P(\tau))$. As $\mathfrak{g}_e(-2) \neq 0$, we reach a contradiction. This shows that the present case cannot occur. In other words, $\mathcal{O}((\mathrm{A}_7)_3)$ has no elements contained in the union of $\mathfrak{g}_2^{\Delta,!}$ with $\Delta \in \mathfrak{D}_G$.

2.10. Suppose $(\Sigma, p) = (\mathrm{E}_8, 2)$. In this case we have to consider the new distinguished nilpotent orbits, namely, $\mathcal{O}((\mathrm{D}_5\mathrm{A}_2)_2)$, $\mathcal{O}((\mathrm{D}_7(\mathrm{a}_1)_2)$ and $\mathcal{O}((\mathrm{D}_7)_2)$. Thanks to [9, Table 14.1] we may choose, in the respective cases, the representatives

$$e' = e_{12345} + e_{2342^25} + e_{13456} + e_{23456} + e_{34567} + e_{24567} + e_{78} + e_{678},$$
$$e' = e_5 + e_{45} + e_{2342^2567} + e_{13} + e_{2456} + e_{3456} + e_{78} + e_8,$$
$$e' = e_1 + e_{234} + e_{345} + e_{245} + e_{456} + e_{567} + e_{678} + e_{12345678}.$$

It is easy to check that in the first case $e' \in \mathfrak{g}(\tau_{\Delta'}, 2)$ where $\tau_{\Delta'} = (0, 0, 0, 0, 2, 0, 0, 2)$ is attached to the orbit $\mathcal{O}(\mathrm{D}_5\mathrm{A}_2)$. By Lemma 2.8, there is $g \in G$ be such that $e' = (\operatorname{Ad} g)e$ and $\tau_{\Delta'} = g\tau g^{-1}$. Since $\dim G_{e'} = 34 = \dim \mathfrak{g}(\tau_{\Delta'}, 0)$ by [9, Table 22.1.1] and $G_{e'} \subset P(\tau_{\Delta'})$ by Lemma 2.8, the orbit $(\operatorname{Ad} P(\tau_{\Delta'}))e$ must be open in $\mathfrak{g}(\tau_{\Delta'}, \geq 2)$. But then e must lie in the open $(\operatorname{Ad} Z_G(\tau_{\Delta'}))$-orbit of $\mathfrak{g}(\tau_{\Delta'}, 2)$. As a consequence, e belongs to the nonempty open subset $\mathcal{V}(\tau_{\Delta'}, 2)_{ss}$ of $\mathfrak{g}(\tau_{\Delta'}, 2)$. Since $\tau = \tau_\Delta$ and $\tau_{\Delta'}$ are G-conjugate and lie in $X_*^+(T)$ we conclude that $\Delta' = \Delta$. Hence $e \in \mathcal{H}(\Delta)$.

In the second case, one checks that $e' \in \mathfrak{g}(\tau_{\Delta'}, 2)$ where $\tau_{\Delta'} = (2, 0, 0, 0, 2, 0, 0, 2)$ is attached to the orbit $\mathcal{O}(\mathrm{D}_7(\mathrm{a}_1))$. By [9, Table 22.1.1], we have $\dim G_e = 26 = \dim \mathfrak{g}(\tau_{\Delta'}, 0)$. Since $(G_e)^\circ$ is unipotent, applying Lemma 2.8 and arguing as in the previous case we deduce that $\tau = \tau_\Delta$ and $\tau_{\Delta'}$ are G-conjugate and $e \in \mathcal{V}(\tau_{\Delta'}, 2)_{ss}$. Therefore, $e \in \mathcal{H}(\Delta)$ as wanted.

The case where $e \in \mathcal{O}((\mathrm{D}_7)_2)$ is more complicated. First we note that $e' \in \mathfrak{g}(\mu, 2)$ where $\mu = (2, -4, -4, 10, -4, -4, 10, -4) \in X_*(T)$. One checks directly that μ is W-conjugate to $(2, 0, 0, 2, 0, 0, 2, 2) = \tau_{\Delta''}$, where $\Delta'' = \mathrm{E}_8(\mathrm{b}_4)$. Since both $\tau = \tau_\Delta$ and $\tau_{\Delta''}$ lie in $X_*^+(T)$, it follows from Lemma 2.8 that $\tau = \tau_{\Delta''}$ and there is $v \in \mathcal{O}((\mathrm{D}_7)_2) \cap \mathfrak{g}(\tau, 2)$ such that $G_v \subset P(\tau)$. Let

$$v' = e_{13} + e_{234} + e_{345} + e_{245} + e_{567} + e_{456} + e_7 + e_8,$$

an element of $\mathfrak{g}(\tau, 2)$. By [9, Tables 13.3 and 22.1.1], we have that $v' \in \mathcal{O}(\mathrm{E}_8(\mathrm{b}_4))$ and $\dim G_{v'} = 18 = \mathfrak{g}(\tau, 0)$. Since $G_{v'} \subset P(\tau)$ by [9, Theorem 15.1(ii)] the orbit $(\operatorname{Ad} P(\tau)v'$ is open in $\mathfrak{g}(\tau, \geq 2)$. It follows that the orbit $V' := (\operatorname{Ad} Z_G(\tau))v'$ is open in $\mathfrak{g}(\tau, 2)$. As $v \notin \mathcal{O}(v')$ we have that $v \notin V'$. Hence $\dim (\operatorname{Ad} Z_G(\tau))v < \dim \mathfrak{g}(\tau, 2)$ and, as a consequence, $[v, \mathfrak{g}(\tau, 0)]$ is a proper subspace of $\mathfrak{g}(\tau, 2)$.

By [9, Table 13.4], the maps $\operatorname{ad} v' : \mathfrak{g}(\tau, 4) \to \mathfrak{g}(\tau, 6)$ and $\operatorname{ad} v' : \mathfrak{g}(\tau, 8) \to \mathfrak{g}(\tau, 10)$ are not surjective (this also follows from the fact that $0 \neq (v')^{[2]} \in \mathfrak{g}_{v'}(\tau, 4)$ and $0 \neq (v')^{[4]} \in \mathfrak{g}_{v'}(\tau, 8)$ which is easy to see directly by applying Jacobson's formula for $[p]$-th powers with $p = 2$). Using the perfect pairings between $\mathfrak{g}(\tau, i)$ and $\mathfrak{g}(\tau, -i)$ induced by the normalised Killing form κ (which is nondegenerate in the present case) one observes that $\mathfrak{g}_{v'}(\tau, r) \neq 0$ for $r \in \{-6, -10\}$. As $v \in \mathfrak{g}(\tau, 2)$ lies in the Zariski closure of $(\operatorname{Ad} Z_G(\tau))v'$, the semicontinuity of the nullity of a rectangular matrix yields that $\mathfrak{g}_v(\tau, -6) \neq 0$ and $\mathfrak{g}_v(\tau, -10) \neq 0$, whilst our earlier remarks entail that $\mathfrak{g}_v(\tau, -2) = [v, \mathfrak{g}(\tau, 0)]^\perp \neq 0$. Therefore, $\dim \mathfrak{g}_v(\tau, < 0) \geq 3$.

As $v \in \mathcal{O}(e) = \mathcal{O}((\mathrm{D}_7)_2)$ it follows from [16, Table 10] that $\dim \mathfrak{g}_v = 24$. If $G_v \subset P(\tau)$ then [9, Table 22.1.1] shows that $\mathrm{Lie}(G_v)$ is a Lie subalgebra of dimension 22 in $\mathfrak{g}(\tau, \geq 0)$. But then

$$\dim \mathfrak{g}_v = \dim \mathfrak{g}_v(\tau, <0) + \dim \mathfrak{g}_v(\tau, \geq 0) \geq 3 + 22 = 25.$$

This contradiction shows that this case does not occur, that is, $e \in \mathcal{O}((\mathrm{D}_7)_2)$ cannot appear as an element of $\mathfrak{g}_2^{\Delta,!}$ with $\Delta \in \mathfrak{D}$. We thus conclude that (2) holds in type E_8.

2.11. Suppose (Σ, p) is one of $(\mathrm{F}_4, 2)$ or $(\mathrm{G}_2, 3)$. These cases have been treated in [17] by computational methods. The argument below will provide an alternative proof. In type F_4, we only need to consider the nonstandard distinguished orbits with labels $(\tilde{\mathrm{A}}_2\mathrm{A}_1)_2$, $(\mathrm{C}_3(\mathrm{a}_1))_2$ and $(\mathrm{C}_3)_2$. Indeed, [9, Theorem 16.1(ii)] implies that every standard distinguished orbit in $\mathcal{N}(\mathfrak{g})$ has a representative $e \in \mathfrak{g}(\mu, 2)$ such that the orbit $(\mathrm{Ad}\, P(\mu))e$ is open in $\mathfrak{g}(\mu, \geq 2)$, thereby forcing $e \in \mathcal{V}(\mu, 2)_{ss}$. Thanks to Lemma 2.8 we also know that μ is G-conjugate to $\tau = \tau_\Delta$. This yields $e \in \mathcal{H}(\Delta)$.

In view of [9, Table 14.1] we may assume that e is $(\mathrm{Ad}\, G)$-conjugate to one of the elements $e(i)$ with $i \in \{1, 2, 3\}$, where

$$\begin{aligned} e(1) &= e_{234} + e_{1121} + e_{1220} + e_{0122} && \text{in type } (\tilde{\mathrm{A}}_2\mathrm{A}_1)_2, \\ e(2) &= e_{123} + e_{0122} + e_{0120} + e_{1222} && \text{in type } (\mathrm{C}_3(\mathrm{a}_1))_2, \\ e(3) &= e_{123} + e_{0120} + e_4 + e_{1222} && \text{in type } (\mathrm{C}_3)_2. \end{aligned}$$

By Lemma 2.8, there is a unique $\tau_i \in X_*(T)$ conjugate to τ and such that $e(i) \in \mathfrak{g}(\tau_i, 2)$ and $G_{e(i)} \subset P(\tau_i)$. Direct computations show that $\tau_1 = (2, 2, -2, 2)$, $\tau_2 = (2, -2, 2, 0)$ and $\tau_3 = (6, -10, 6, 2)$. Using [2, Planche VIII] one checks directly that in the last two cases $G_{e(i)}$ contains $x_{\alpha_2}(t)$ for every $t \in \Bbbk$. Then the simple root vector e_2 lies in $\mathrm{Lie}(G_e) \cap \mathfrak{g}(\tau, <0)$. As noted in [9, p. 215] we have $x_{-\alpha_1}(t)x_{\alpha_3}(t) \in G_{e(1)}$ for all $t \in \Bbbk$ which yields $\mathrm{Lie}(G_{e(1)}) \cap \mathfrak{g}(\tau, -2) \neq 0$. So none of the three cases can occur in our situation.

Finally, suppose $(\Sigma, p) = (\mathrm{G}_2, 3)$. Thanks to [9, Proposition 13.5] we only need to consider the orbit $\mathcal{O}((\tilde{\mathrm{A}}_1)_3)$ which has a nice representative $e' = e_{21} + e_{32}$; see [9, Table 14.1]. As e is distinguished and $e' \in \mathfrak{g}(\mu, 2)$, where $\mu = (2 - 2) \in X_*(T)$, it follows from Lemma 2.8 that μ is G-conjugate to τ and $G_{e'} \subset P(\mu)$. But $x_{\alpha_2}(t) \in G_{e'}$ for all $t \in \Bbbk$, forcing $\mathrm{Lie}(G_{e'}) \cap \mathfrak{g}(\mu, -2) \neq 0$. This contradiction shows that this case cannot occur either.

Summarising, we have proved that statement (2) holds for all simple algebraic groups of exceptional types over algebraically closed fields of characteristic $p \geq 0$. This means that $\mathrm{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$. Since $\mathcal{N}(\mathfrak{g}) \subset \mathrm{Lie}(\mathcal{D}G)$ and $Z(G)$ acts trivially on $\mathfrak{g}$, proving Theorem 1.1 reduces quickly to the case where G is a simple algebraic group. For G exceptional, the theorem is a direct consequence of statement (2). For G classical, the result is known from [11, Theorem A.2]. The groups of type G_2, F_4 and E_6 were treated earlier in [17].

2.12. We would like to finish this paper by a brief discussion of the unipotent analogues of the Lusztig–Xue pieces, $\mathrm{LX}_u(\Delta)$, introduced by Lusztig in [11, 2.3].

Let $\mathfrak{U}(G)$ denote the unipotent variety of G, the set of all $(\mathrm{Ad}\, G)$-unstable elements of G. The Hesselink stratification

$$\mathfrak{U}(G) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathcal{H}_u(\Delta)$$

is described in [4] as follows. Let $\tau = \tau_\Delta$ and write $P(\tau) = Z_G(\tau)U(\tau)$ where $U(\tau) = R_u(P(\tau))$. Given $k \in \mathbb{Z}_{>0}$ we denote by $U_{\geq k}(\tau)$ the connected normal subgroup of $U(\tau)$ generated by all $x_\gamma(t)$ with $t \in \Bbbk$ and all $\gamma \in \Sigma$ such that $\langle \gamma, \tau \rangle \geq k$. It is well known that the factor group $U_{\geq 2}(\tau)/U_{\geq 3}(\tau)$ is endowed with a natural vector space structure over $\Bbbk$ and $\mathrm{Ad}\, Z_G(\tau)$ acts $\Bbbk$-linearly on $U_{\geq 2}(\tau)/U_{\geq 3}(\tau)$. Furthermore, $U_{\geq 2}(\tau)/U_{\geq 3}(\tau) \cong \mathfrak{g}(\tau, 2)$ as $(\mathrm{Ad}\, Z_G(\tau))$-modules, and there is a module isomorphism $\bar{\pi}_\Delta : U_{\geq 2}(\tau)/U_{\geq 3}(\tau) \xrightarrow{\sim} \mathfrak{g}(\tau, 2)$ sending a coset $\prod_{i=1}^r x_{\beta_i}(t_i)U_{\geq 3}(\tau)$ with $\langle \beta_i, \tau \rangle = 2$ to $\sum_{i=1}^r t_i e_{\beta_i}$, where e_{β_i} are root vectors independent of the choice of $(t_1, \ldots, t_r) \in \Bbbk^r$; see [4, 3.6], [9, Lemma 18.1] or [11, 2.2]. Composing $\bar{\pi}_\Delta$ with the canonical homomorphism $U_{\geq 2}(\tau) \to U_{\geq 2}(\tau)/U_{\geq 3}(\tau)$ we obtain a natural surjection $\pi_\Delta : U_{\geq 2}(\tau) \twoheadrightarrow \mathfrak{g}(\tau, 2)$.

It follows from [4, Theorems 3.6 and 5.2] that

$$\mathcal{H}_u(\Delta) = (\mathrm{Ad}\, G)\big(\pi_\Delta^{-1}(\mathcal{V}(\tau_\Delta, 2)_{ss})\big)$$

for every $\Delta \in \mathfrak{D}_G$. In [11, 2.3], the unipotent pieces $\mathrm{LX}_u(\Delta)$ are defined in a similar fashion except that $\mathcal{V}(\tau_\Delta, 2)_{ss}$ is replaced by an a priori larger set $\mathfrak{g}_2^{\Delta,!}$. More precisely,

$$\mathrm{LX}_u(\Delta) = (\mathrm{Ad}\, G)(\pi_\Delta^{-1}(\mathfrak{g}_2^{\Delta,!})).$$

In [11, Theorem 2.4], Lusztig proved that

$$\mathfrak{U}(G) = \bigsqcup_{\Delta \in \mathfrak{D}_G} \mathrm{LX}_u(\Delta) \tag{5}$$

when G is a simple algebraic group of type A, B, C or D, and he expected that (5) would continue to hold for all connected reductive groups. Our next result shows that this expectation was correct.

Corollary 2.9. *Let G be a connected reductive group over an algebraically closed field. Then $\mathrm{LX}_u(\Delta) = \mathcal{H}_u(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ and (5) holds for $\mathfrak{U}(G)$.*

Proof. Theorem 1.1 in conjunction with the preceding discussion shows that $\mathrm{LX}_u(\Delta) = \mathcal{H}_u(\Delta)$ for all $\Delta \in \mathfrak{D}_G$. Since the Hesselink strata $\mathcal{H}_u(\Delta)$ with $\Delta \in \mathfrak{D}_G$ form a partition of $\mathfrak{U}(G)$ by [4, Theorem 5.2], we deduce that (5) holds for $\mathfrak{U}(G)$. □

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MULTIPLICITY-FREE REPRESENTATIONS OF THE PRINCIPAL A_1-SUBGROUP IN A SIMPLE ALGEBRAIC GROUP

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We dedicate this paper to the memory of the esteemed mathematician, Gary Seitz, whose work and mentorship have a continuing impact on the field and on our lives

Let G be a simple algebraic group defined over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. For $p \geq h$, the Coxeter number of G, any regular unipotent element of G lies in an A_1-subgroup of G; there is a unique G-conjugacy class of such subgroups and any member of this class is a so-called "principal A_1-subgroup of G". Here we classify all irreducible $\Bbbk G$-modules whose restriction to a principal A_1-subgroup of G has no repeated composition factors, extending the work of Liebeck, Seitz and Testerman which treated the same question when $\Bbbk$ is replaced by an algebraically closed field of characteristic zero.

1. Introduction

We consider a question in the representation theory and subgroup structure of simple algebraic groups defined over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The main aim of our work is to generalise the results of [Liebeck et al. 2015; 2022; 2024], where the authors consider so-called "multiplicity-free subgroups" of simple algebraic groups defined over an algebraically closed field K of characteristic zero. More precisely, the authors consider triples (X, Y, V) where X and Y are simple algebraic groups defined over K with X a closed subgroup of Y, and V is an irreducible KY-module such that the KX-module V, obtained by restricting the action of Y to the subgroup X, is a sum of nonisomorphic irreducible KX-modules (a so-called "multiplicity-free" KX-module). The above cited articles provide a complete classification of such triples when either X has rank 1 and does not lie in a

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proper parabolic subgroup of Y, or Y is a classical group with natural module W and X is of type A_ℓ acting irreducibly on W. Note that the case where X acts irreducibly (and hence multiplicity freely) on V was settled by Dynkin [1952] in characteristic zero, and by Seitz [1987] and Testerman [1988] in positive characteristic.

The ultimate far-reaching aim of what we undertake in this paper would be to investigate the "multiplicity-free" triples (X, Y, V) as in [Liebeck et al. 2015; 2022; 2024], described above, replacing the field K by the field $\Bbbk$ of positive characteristic p, and considering composition factors rather than summands. The proofs in [Liebeck et al. 2022; 2024] use induction on the rank of the group X; the case where X is simple of rank 1 is considered in [Liebeck et al. 2015]. Here we treat the rank-1 case for the groups defined over $\Bbbk$, but consider a slightly more general setting than would strictly speaking be required for use in an inductive set-up. Namely, we consider all simple algebraic groups G (classical and exceptional), defined over $\Bbbk$, and A a closed A_1-subgroup of G containing a regular unipotent element of G, which we will call a "principal A_1-subgroup of G". (Such subgroups exist precisely when $p \geq h$, the Coxeter number of G; see [Testerman 1995, Corollary 0.5 and Theorem 0.1]. In addition, there is at most one conjugacy class of principal A_1-subgroups in G; see [Seitz 2000, Theorem 1.1].) We then determine all irreducible $\Bbbk G$-modules V such that the set of composition factors of the $\Bbbk A$-module V consists of nonisomorphic $\Bbbk A$-modules, and obtain a classification analogous to [Liebeck et al. 2015, Theorem 1]. Much of the analysis follows the same line of reasoning as that used in [Liebeck et al. 2015]; the main differences and difficulties arise from the lack of precise knowledge about the dimensions of irreducible $\Bbbk G$-modules and the multiplicities of their weights. In addition, while irreducible $\Bbbk A_1$-modules are completely understood, the description of the set of weights is not as simple as in characteristic zero. In [Liebeck et al. 2022; 2024], another essential ingredient of the proof is the work of Stembridge [2003], where he determines when the tensor product of two irreducible modules for a simple algebraic group defined over the field K is a direct sum of nonisomorphic irreducible modules. There has been recent progress on the analogous question for the simple groups defined over fields of positive characteristic in [Gruber 2021] and [Gruber and Mancini 2024]. The combination of the rank-1 theorem proven here and the work of Gruber and Mancini lays the foundation for the study of multiplicity-free subgroups of higher rank for groups defined over fields of positive characteristic.

In order to state our main result, we introduce some notation; further notation will be set up in Section 2. Fix G a simply connected simple algebraic group of rank $\ell \geq 2$ defined over the algebraically closed field $\Bbbk$. We fix a maximal torus T of G, a

Borel subgroup B of G with $T \subset B$, the root system Φ of G with respect to T, and a base $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of Φ, associated with the choice of Borel subgroup B. Let Φ^+ be the associated set of positive roots. Let $X(T)$ denote the associated weight lattice, with fundamental dominant weights $\{\omega_1, \dots, \omega_\ell\}$ defined by the choice of Π. (We label Dynkin diagrams as in [Bourbaki 2002].) Throughout, we fix $\lambda \in X(T)$ a dominant weight and set $V = L(\lambda)$, the irreducible $\Bbbk G$-module with highest weight λ. Assume that $p \geq h$, so that each regular unipotent element of G lies in an A_1-subgroup of G. Let $A \subset G$ be a principal A_1-subgroup of G. Fix a maximal torus T_A of A with $T_A \subset T$ and $T_A U_\alpha$, a Borel subgroup of A, with root group U_α, lying in B. For α the unique positive root of A (with respect to the given choices), we have $T_A = \alpha^\vee(\Bbbk^*)$, the image of the coroot $\alpha^\vee$. Henceforth, we will write $V{\downarrow}H$ for the kH-module obtained by restricting the action of G to a subgroup H. We say that *$V{\downarrow}A$ is MF* if all composition factors in the restriction are nonisomorphic.

We will also require a notation for the corresponding modules and subgroups for the groups defined over the algebraically closed field K of characteristic zero. We write G_K for a simply connected simple algebraic group defined over the field K, with root system of type Φ, and A_K for a principal A_1-subgroup of G_K (see [Jacobson 1951; Morozov 1942] for the proofs of existence and conjugacy of A_1-subgroups of G_K intersecting the class of regular unipotent elements). For the weight λ as above, we write $\Delta_K(\lambda)$ for the corresponding irreducible G_K-module. We will use the same terminology of "MF" for the action of A_K on $\Delta_K(\lambda)$. Our main result is:

Theorem 1. *Suppose that λ is p-restricted. Then $L(\lambda){\downarrow}A$ is MF if and only if one of the following holds*:

(i) *We have that $p > (\lambda{\downarrow}T_A)$ and $\Delta_K(\lambda){\downarrow}A_K$ is MF.*

(ii) *The group G is of type A_2, $\lambda = \omega_1 + \omega_2$ and $p = 3$.*

(iii) *The group G is of type B_2, $\lambda = 2\omega_1$ and $p = 5$.*

Corollary 2. *Let $\lambda = \sum_{i=0}^{t} p^i \lambda_i$ where each λ_i is a p-restricted dominant weight. Then $L(\lambda){\downarrow}A$ is MF if and only if one of the following holds*:

(i) *The module $\Delta_K(\lambda_i){\downarrow}A_K$ is MF and $p > (\lambda_i{\downarrow}A)$, for all $0 \leq i \leq t$.*

(ii) *The group G is of type A_2, $p=3$ and there exists $0 \leq i \leq t$ such that $\lambda_i = \omega_1+\omega_2$. For all $0 \leq j \leq t$ we have $\lambda_j \in \{0, \omega_1 + \omega_2, \omega_1, \omega_2\}$ and if $\lambda_j = \omega_1 + \omega_2$ for some $0 \leq j \leq t-1$, then $\lambda_{j+1} = 0$.*

(iii) *The group G is of type B_2, $p = 5$ and there exists $0 \leq i \leq t$ such that $\lambda_i = 2\omega_1$. For all $0 \leq j \leq t$ we have $\lambda_j \in \{0, 2\omega_1, \omega_1, \omega_2\}$ and if $\lambda_j = 2\omega_1$ for some $0 \leq j \leq t-1$, then $\lambda_{j+1} \in \{0, \omega_2\}$.*

G_K	weight λ
A_ℓ	$\omega_1,\ \omega_2,\ 2\omega_1,\ \omega_1+\omega_\ell$
	$\omega_3\ (5 \leq \ell \leq 7)$
	$3\omega_1\ (\ell \leq 5),\ 4\omega_1\ (\ell \leq 3),\ 5\omega_1\ (\ell \leq 3)$
A_3	110
A_2	$c1,\ c0$
B_ℓ	$\omega_1,\ \omega_2,\ 2\omega_1$
	$\omega_\ell\ (\ell \leq 8)$
B_3	$101,\ 002,\ 300$
B_2	$b0,\ 0b\ (1 \leq b \leq 5),\ 11,\ 12,\ 21$
C_ℓ	$\omega_1,\ \omega_2,\ 2\omega_1$
	$\omega_3\ (3 \leq \ell \leq 5)$
	$\omega_\ell\ (\ell = 4,\ 5)$
C_3	300
$D_\ell\ (\ell \geq 4)$	$\omega_1,\ \omega_2\ (\ell = 2m+1),\ 2\omega_1\ (\ell = 2m)$
	$\omega_\ell\ (\ell \leq 9)$
E_6	ω_1, ω_2
E_7	ω_1, ω_7
E_8	ω_8
F_4	ω_1, ω_4
G_2	$10,\ 01,\ 11,\ 20,\ 02,\ 30$

Table 1. Multiplicity-free restrictions in characteristic zero.

For the reader's convenience and for completeness, we list in Table 1 the nonzero weights λ for which $\Delta_K(\lambda)\downarrow A_K$ is MF, as obtained in [Liebeck et al. 2015].

We conclude the introduction with a few remarks about the proof. We first note that if $p > (\lambda\downarrow T_A)$, then one can show that the Weyl module with highest weight λ is an irreducible $\Bbbk G$-module (see [Korhonen 2018, Corollary 2.7.6]), and then the considerations of [Liebeck et al. 2015] for the groups defined over K yield the result (see Proposition 2.3). The arguments therefore focus on the cases where $p \leq (\lambda\downarrow T_A)$. Many aspects of the proof follow closely the arguments used in [Liebeck et al. 2015]. In particular, we use the fact that all irreducible $\Bbbk A_1$-modules have multiplicity-one weight spaces and therefore considering the set of T_A-weights and their multiplicities in V can directly be used to detect multiplicities of composition factors of $V\downarrow A$. Moreover, there are certain dimension bounds which must be respected by an MF-module. Thus, many of our preliminary lemmas are inspired by the results in [Liebeck et al. 2015, Section 2]. In addition, we rely on a result from [Hague and McNinch 2013] where the authors prove that certain tilting modules for G have a filtration by tilting modules for a principal A_1-subgroup A of G. Since reducible indecomposable tilting modules for groups of type A_1 necessarily have repeated composition factors, this result is quite useful for showing that many $\Bbbk G$-modules are not MF as $\Bbbk A$-modules (see Lemma 2.4).

2. Preliminary lemmas

Let us fix additional notation to be used throughout the paper.

Recall that G is a simply connected simple algebraic group with principal A_1-subgroup A. We assume throughout that $\ell \geq 2$, respectively 2, 3, 4, for G of type A_ℓ, respectively B_ℓ, C_ℓ, D_ℓ. For $1 \leq i \leq \ell$, let s_i denote the simple reflection associated to the root α_i. For $\lambda \in X(T)$, a dominant weight, we write $\Delta_G(\lambda)$ for the Weyl module for G of highest weight λ, and $L_G(\lambda)$ for the irreducible module for G of highest weight λ. We will suppress the G in this notation if there is no ambiguity. For a $\Bbbk G$-module V and $\mu \in X(T)$, we write V_μ for the μ-weight space with respect to T of the module V. When we say that roots are adjacent, or end-nodes, we mean with respect to the Dynkin diagram associated to the root system Φ.

For a group of type A_1, we identify the weight lattice of a fixed maximal torus with the ring $\mathbb{Z}$ and for a nonnegative integer s we write (s) for the irreducible $\Bbbk A_1$-module of highest weight s. If we want to underline that we are talking about the fixed principal A_1-subgroup A, we may write as well $L_A(s)$. Similarly, we write $\Delta(s)$ for the Weyl module of highest weight s and $T(s)$ for the indecomposable tilting module of highest weight s. For a $\Bbbk A$-module (s), we write $(s)^{(p^i)}$ for the module whose structure is induced by the composition of the p^i-Frobenius map on A and the morphism defining the module structure on (s). For A_K, the principal A_1-subgroup of G_K, and s a nonnegative integer, we will write $\Delta_{A_K}(s)$ for the irreducible KA_K-module of highest weight s.

Here and in Sections 3 and 4, we fix a p-restricted dominant weight $\lambda \in X(T)$ and set $V = L_G(\lambda)$. Throughout the paper, set $r = \lambda{\downarrow}T_A$, that is, $\lambda(\alpha^\vee(c)) = c^r$, for all $c \in \Bbbk^*$. The cocharacter $\alpha^\vee : \mathbb{G}_m \to T$, which defines the maximal torus of A, satisfies $\alpha_i(\alpha^\vee(c)) = c^2$ for all $c \in \Bbbk^*$; that is, $\alpha_i{\downarrow}T_A = 2$ for all $1 \leq i \leq \ell$. The value for r can then be determined by writing λ as a linear combination of simple roots and then using that each simple root takes value 2 on T_A. We list the values of r in Table 2.

Recall that the existence of a principal A_1-subgroup in G implies that $p \geq h$, the Coxeter number of G, a hypothesis which allows us to apply the following proposition, a consequence of [Premet 1987, Theorem 1].

G	r
A_ℓ	$\sum_1^\ell i(\ell+1-i)c_i$
B_ℓ	$\sum_1^{\ell-1} i(2\ell+1-i)c_i + \frac{\ell(\ell+1)}{2}c_\ell$
C_ℓ	$\sum_1^\ell i(2\ell-i)c_i$
D_ℓ	$\sum_1^{\ell-2} i(2\ell-1-i)c_i + \frac{\ell(\ell-1)}{2}c_{\ell-1} + \frac{\ell(\ell-1)}{2}c_\ell$
G_2	$6c_1+10c_2$
F_4	$22c_1+42c_2+30c_3+16c_4$
E_6	$16c_1+22c_2+30c_3+42c_4+30c_5+16c_6$
E_7	$34c_1+49c_2+66c_3+96c_4+75c_5+52c_6+27c_7$
E_8	$92c_1+136c_2+182c_3+270c_4+220c_5+168c_6+114c_7+58c_8$

Table 2. Values of $r = \lambda{\downarrow}T_A$ for $\lambda = \sum_1^\ell c_i\omega_i$.

Proposition 2.1. *Let $p \geq h$ and let μ be a p-restricted weight for G. Then the irreducible $\Bbbk G$-module $L(\mu)$ has precisely the same set of weights as the $\Bbbk G$-module $\Delta(\mu)$.*

Proof. This follows from [Premet 1987, Theorem 1] since the parameter $e(\Phi)$ appearing in the statement of [loc. cit.] is the maximum of the squares of the ratios of the lengths of the roots in Φ. □

We now introduce a shorthand notation for weights of V. For $\lambda - \sum_{i=1}^\ell a_i\alpha_i$, we write $\lambda - i_1^{a_{i_1}} \cdots i_m^{a_{i_m}}$, where $a_j = 0$ for $j \notin \{i_1, \dots, i_m\}$, and suppress those a_j with $a_j = 1$; for example, the weight $\lambda - \alpha_2 - 2\alpha_3 - \alpha_5$ will be written as $\lambda - 23^25$. For G of rank 2, we write $\lambda - ab$ for the weight $\lambda - a\alpha_1 - b\alpha_2$.

The following result is Corollary 2.7.6 from [Korhonen 2018]; we include a sketch of the proof for completeness.

Lemma 2.2. *If $p > r$, then $\Delta(\lambda)$ is irreducible.*

Proof. By the Jantzen sum formula [2003, Part II, 8.19], it suffices to prove that for all $\alpha \in \Phi^+$, we have $r \geq \langle \lambda + \delta, \alpha \rangle - 1$, where $\delta = \sum_{i=1}^\ell \omega_i$. It is easy to see that $\langle \lambda, \alpha \rangle$ is maximal when α is the highest root of the dual root system $\Phi^\vee$, i.e., when α is the highest short root β of Φ. By [Serre 1994, Proposition 5], we have $\langle \lambda + \delta, \beta \rangle \leq 1 + \sum_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle$. It is therefore sufficient to show that $r = \sum_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle$, which is a simple calculation using the fact that for a simple root α_i we have $\sum_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \langle \alpha_i, \alpha \rangle = \sum_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \langle s_i(\alpha_i), s_i(\alpha) \rangle = \sum_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \langle -\alpha_i, \alpha \rangle$ and so $\sum_{\alpha \in \Phi^+ \setminus \{\alpha_i\}} \langle \alpha_i, \alpha \rangle = 0$. □

The next proposition establishes Theorem 1 when $p > r$.

Proposition 2.3. *Assume $p > r$. Then $V{\downarrow}A$ is MF if and only if $\Delta_K(\lambda){\downarrow}A_K$ is MF.*

Proof. By Lemma 2.2, the Weyl module is irreducible and therefore $V = \Delta(\lambda)$. We have $\Delta_K(\lambda){\downarrow}A_K = \sum_0^k \Delta_{A_K}(r_i)$ for some integers $r_0 \geq r_1 \geq \cdots \geq r_k \geq 0$ with $r_0 = r$. Since $p > r$, a comparison of characters gives $\Delta(\lambda){\downarrow}A = \sum_0^k \Delta_A(r_i) = \sum_0^k (r_i)$, which then implies the result. □

As the next result shows, in many cases when $\Delta(\lambda)$ is irreducible and $r \geq p$, we can still directly conclude that $V{\downarrow}A$ is not MF.

Lemma 2.4. *Assume that $\Delta(\lambda)$ is irreducible, $r \geq p$, and $r \not\equiv -1 \bmod p$. If G is of type B_ℓ, respectively D_ℓ, and λ does not lie in the root lattice of G, assume that $p > \binom{\ell+1}{2}$, respectively $p > \binom{\ell}{2}$. Then $V{\downarrow}A$ is not MF.*

Proof. Here we use [Hague and McNinch 2013, Theorems 4.1.2, 4.1.4 and 4.2.1] to see that A is a so-called "good filtration" subgroup, which then implies that the irreducible Weyl module $\Delta(\lambda) = V$ has an A-filtration by both Weyl modules and by induced modules. So in particular, $V{\downarrow}A$ is a tilting module. Furthermore, since r is the highest T_A-weight in V, the module $T(r)$ is a summand of $V{\downarrow}A$. The hypotheses on r imply that the indecomposable tilting module $T(r)$ is reducible (see [Carter and Cline 1976, Theorem 1.2]). Since tilting modules for A are self-dual, no reducible indecomposable tilting module is MF, which then concludes the proof. □

We now turn to a sequence of definitions and lemmas which provide tools for studying the set of composition factors of $V{\downarrow}A$ based upon knowing the set of weights of V.

Definition 2.5. For $n \in \mathbb{Z}$, let n_d be the multiplicity of the T_A-weight $r-2d$ in $V{\downarrow}A$ and let m_d be the multiplicity of the composition factor $(r-2d)$ in $V{\downarrow}A$. Also, let $\boldsymbol{S}_d$ denote the multiset of composition factors whose highest weight is greater than $r-2d$ and in which $r-2d$ does not occur as a weight, and let s_d denote the cardinality of $\boldsymbol{S}_d$.

Lemma 2.6. *Assume that $V{\downarrow}A$ is MF. Then $n_d \leq d+1$.*

Proof. Let $\boldsymbol{B}$ be the multiset of composition factors of $V{\downarrow}A$ where $r-2d$ occurs as a weight. Since $V{\downarrow}A$ is MF, we have $\boldsymbol{B} \subseteq \{(r), (r-2), \ldots, (r-2d)\}$. Therefore $|\boldsymbol{B}| \leq d+1$ and we can conclude that $n_d \leq d+1$. □

Lemma 2.7. *For all $0 \leq d \leq r$ we have*

$$m_d = n_d - n_{d-1} + s_d - s_{d-1}. \quad (1)$$

Proof. We prove this by induction on d. If $d = 0$ the statement holds. Indeed, $m_0 = n_0 = 1$ since r is the highest weight and is afforded only by λ, and $n_{-1} = s_{-1} =$

$s_0 = 0$. Assume that (1) holds up to an arbitrary d. In general, the multiplicity m_{d+1} can be determined by taking the difference between n_{d+1} and the number of times the T_A-weight $r-2(d+1)$ appears in composition factors with greater highest weight. Thus,

$$m_{d+1} = n_{d+1} - \Big(\sum_{0 \leq k \leq d} m_k - s_{d+1} \Big).$$

By the inductive hypothesis $m_k = n_k - n_{k-1} + s_k - s_{k-1}$ for all $k \leq d$. Substituting we get

$$m_{d+1} = n_{d+1} + s_{d+1} - \sum_{0 \leq k \leq d} (n_k - n_{k-1} + s_k - s_{k-1}) = n_{d+1} - n_d + s_{d+1} - s_d,$$

concluding the proof. □

Lemma 2.8. *For all $1 \leq d < p$ we have $\boldsymbol{S}_{d-1} \subseteq \boldsymbol{S}_d$. In particular $s_d \geq s_{d-1}$.*

Proof. We begin by analysing what weights occur in an arbitrary irreducible module (t). We will write $[a_0, a_1, \dots, a_m]$ to denote the integer $\sum_{i=0}^{m} a_i p^i$. Then $t = [a_0, a_1, \dots, a_m]$ where the a_i's are the coefficients in the p-adic expansion of t. By Steinberg's tensor product theorem, we have

$$(t) \cong (a_0) \otimes (a_1)^{(p)} \otimes \cdots \otimes (a_m)^{(p^m)}.$$

The weights occurring in (t) are therefore of the form $[a_0 - 2i_0, a_1 - 2i_1, \dots, a_m - 2i_m]$ where $0 \leq i_j \leq a_j$. Let $t - 2q \geq 0$, with $q \in \mathbb{N}$, be an integer denoting a weight not occurring in (t). Then $t - 2q$ lies in an open interval (δ, γ) with

$$\begin{aligned} \delta &= [a_0, \dots, a_j, a_{j+1} - 2i_{j+1} - 2, \dots, a_m - 2i_m], \\ \gamma &= [-a_0, \dots, -a_j, a_{j+1} - 2i_{j+1}, \dots, a_m - 2i_m], \end{aligned}$$

where $0 \leq i_{j+1} < a_{j+1}$ and $0 \leq i_k \leq a_k$ for $k > j+1$. Conversely, any integer $t - 2q$ lying in such an interval corresponds to a weight not occurring in (t). We call these intervals the *gaps* of (t), so that a composition factor (t) is in $\boldsymbol{S}_d$ if and only if $r - 2d$ is in a gap of (t).

Assume for a contradiction that $(t) \in \boldsymbol{S}_{d-1} \setminus \boldsymbol{S}_d$ for some $t \leq r$. Then $t > r - 2d + 2$ and the composition factor (t) has a gap (δ, γ) as above containing $r - 2d + 2$, but not containing $r - 2d$. This means that

$$r - 2d = [a_0, \dots, a_j, a_{j+1} - 2i_{j+1} - 2, \dots, a_m - 2i_m],$$

implying that

$$\begin{aligned} 2d - (r - t) = t - (r - 2d) &= t - [a_0, \dots, a_j, a_{j+1} - 2i_{j+1} - 2, \dots, a_m - 2i_m] \\ &= 2p^{j+1}[i_{j+1} + 1, i_{j+2}, \dots, i_m] \geq 2p. \end{aligned}$$

This contradicts the assumption that $d < p$. □

Lemma 2.9. *Let* $1 \leq d < \min\{\lfloor \frac{r+2}{2} \rfloor, p\}$.

(i) *If* $n_d - n_{d-1} = 1$ *then* $r - 2d$ *is a composition factor of* $V{\downarrow}A$.

(ii) *If* $n_d - n_{d-1} \geq 2$ *then* $m_d \geq 2$ *and* $V{\downarrow}A$ *is not MF.*

(iii) *If* $\lambda = c\omega_i$ *and* $n_d \geq d+1$ *then* $V{\downarrow}A$ *is not MF.*

(iv) *If* $n_d - n_{d-1} = 1$ *and* $\mathbf{S}_{d-1} \neq \mathbf{S}_d$, *then* $m_d \geq 2$ *and* $V{\downarrow}A$ *is not MF.*

Proof. Parts (i), (ii) and (iv) follow directly from combining Lemmas 2.7 and 2.8. If $\lambda = c\omega_i$ then $n_1 = 1$, and since $n_d \geq d+1$, there exists $2 \leq d' \leq d$ such that $n_{d'} - n_{d'-1} \geq 2$, concluding by part (ii). □

We can often deduce the value n_d from the characteristic-zero case.

Lemma 2.10. *Assume that* $V \cong \Delta(\lambda)$. *Then* $n_d = \dim(\Delta_K(\lambda){\downarrow}A_K)_{r-2d}$.

Proof. This follows from [Jantzen 2003, Part II, 5.8], since T_A is uniquely determined by the property $\alpha_i{\downarrow}T_A = 2$ for all $1 \leq i \leq \ell$. □

We now establish two dimension bounds for multiplicity-free $\Bbbk A_1$-modules.

Definition 2.11. Given $r \in \mathbb{N}$, define $B(r)$ and $B_K(r)$ as

$$B(r) = \sum_{r-2k \geq 0} \dim L_A(r-2k) \quad \text{and} \quad B_K(r) = \sum_{r-2k \geq 0} \dim \Delta_{A_K}(r-2k).$$

In particular $B_K(r)$ is either $\left(\frac{r}{2}+1\right)^2$ or $\frac{r+1}{2}\frac{r+3}{2}$ according to whether r is even or odd, respectively.

Lemma 2.12. *We have* $B(r) \leq B_K(r)$ *and if* $V{\downarrow}A$ *is MF, then* $\dim V \leq B(r)$.

Proof. We have $B(r) \leq B_K(r)$ immediately since $\dim L_A(r-2k) \leq \dim \Delta_{A_K}(r-2k)$ for all k such that $r-2k \geq 0$. Now if $V{\downarrow}A$ is MF, it can have at most one composition factor $(r-2d)$, i.e., $m_d = 1$, for every $0 \leq d \leq \lfloor \frac{r}{2} \rfloor$. Therefore $\dim V \leq B(r)$. □

Lemma 2.13. *Suppose that* $\lambda = a\omega_i$ *and* $r \not\equiv 0 \bmod p$. *If* $V{\downarrow}A$ *is MF, then* $\dim V \leq B(r) - \dim(r-2)$.

Proof. The T_A-weight $r-2$ occurs with multiplicity 1 in V, and since $r \not\equiv 0 \bmod p$, it occurs as a weight in the composition factor (r). Therefore $r-2$ does not afford a composition factor of $V{\downarrow}A$, i.e., $m_1 = 0$. Since $V{\downarrow}A$ is MF, we have $m_d \leq 1$ for all $d \geq 0$ such that $r-2d \geq 0$. This proves that $\dim V \leq B(r) - \dim(r-2)$. □

The following result is our main reduction tool, showing that if $V{\downarrow}A$ is MF, then λ satisfies some highly restrictive conditions. The proof follows closely that of [Liebeck et al. 2024, Lemma 2.6].

Proposition 2.14. *Let* $\lambda = \sum_{i=1}^{\ell} c_i \omega_i$. *Assume that there exist* $i < j$ *with* $c_i \neq 0 \neq c_j$ *and that* $V \downarrow A$ *is MF. Then*:

(i) $c_k = 0$ *for* $k \neq i, j$.

(ii) *If* α_i *and* α_j *are nonadjacent*, *then* $c_i = c_j = 1$.

(iii) *If* α_i *and* α_j *are nonadjacent then they are both end-nodes.*

(iv) *Either* α_i *or* α_j *is an end-node.*

(v) *If both* $c_i > 1$ *and* $c_j > 1$, *then* G *has rank* 2 *and* $\lambda - ij$ *has multiplicity* 1.

(vi) *If either* $c_i > 1$ *or* $c_j > 1$, *then either* G *has rank* 2, *or* α_i *is adjacent to* α_j *and* $\lambda - ij$ *has multiplicity* 1.

Proof. We will use Proposition 2.1 throughout the proof, without direct reference.

(i) If $c_k \geq 1$ for $k \neq i, j$, we have $n_1 \geq 3$ as the T_A-weight $r-2$ is afforded by $\lambda - i$, $\lambda - j$ and $\lambda - k$. This contradicts Lemma 2.6.

(ii) Suppose α_i and α_j are not adjacent and that $c_i \geq 2$. Let $k \neq i, j$ such that α_k is adjacent to α_i and let $k' \neq i, j$ such that $\alpha_{k'}$ is adjacent to α_j. Then $n_2 \geq 4$, as the T_A-weight $r-4$ is afforded by $\lambda - i^2$, $\lambda - ik$, $\lambda - jk'$, $\lambda - ij$. This contradicts Lemma 2.6.

(iii) Assume that α_i and α_j are nonadjacent and that α_i is not an end-node. Then there exist distinct simple roots α_k, α_l, both adjacent to α_i, and a simple root $\alpha_m \neq \alpha_i$ adjacent to α_j. Then $n_2 \geq 4$, as the T_A-weight $r-4$ is afforded by $\lambda - ik$, $\lambda - il$, $\lambda - jm$ and $\lambda - ij$. This contradicts Lemma 2.6.

(iv) Assume that neither α_i nor α_j is an end-node. Then by (iii), the roots α_i and α_j are adjacent. Let $1 \leq k, l \leq \ell$ be distinct indices such that $\{i, j\} \cap \{k, l\} = \varnothing$ and such that α_i is adjacent to α_k and α_j is adjacent to α_l. Then the T_A-weight $r-8$ is afforded by $\lambda - kijl$, $\lambda - kij^2$, $\lambda - i^2j^2$, $\lambda - i^2jl$, $\lambda - ki^2j$ and $\lambda - ij^2l$. Therefore $n_4 \geq 6$, contradicting Lemma 2.6.

(v) If both $c_i > 1$ and $c_j > 1$, then by (ii), the roots α_i and α_j are adjacent. If the rank of G is not 2 we can find $k \neq i, j$, such that α_k is adjacent to either α_i or α_j. But then the T_A-weight $r-4$ is afforded by $\lambda - ij$, $\lambda - i^2$, $\lambda - j^2$ and either $\lambda - ik$ or $\lambda - jk$. Therefore $n_2 \geq 4$, contradicting Lemma 2.6. In addition, $\lambda - ij$ has multiplicity 1, else $n_2 \geq 4$, again contradicting Lemma 2.6.

(vi) Assume $c_i \geq 2$ and that G has rank at least 3. Then α_i and α_j are adjacent by (ii), and $r-4$ is afforded by $\lambda - i^2$, $\lambda - ij$ and either $\lambda - ik$ or $\lambda - jk$ for some $k \neq i, j$. Therefore, Lemma 2.6 implies that $\lambda - ij$ has multiplicity 1, as claimed. $\square$

Lemma 2.15. *Assume that $\lambda = \omega_i$ and that there exist $\{\beta_{i-3}, \dots, \beta_{i+3}\} \subseteq \Pi$ such that for $i-3 \le s < t \le i+3$, $(\beta_s, \beta_t) \ne 0$ if and only if $t = s+1$. Then $V{\downarrow}A$ is not MF.*

Proof. Here $p > 7$, as $\mathrm{rank}(G) \ge 7$, and Table 2 shows that $r > 15$. It is now a simple check to see that $n_4 \ge 5$, concluding by Lemma 2.9(iii). □

Lemma 2.16. *Assume that $\lambda = b\omega_i$ with $b \ge 2$. If $V{\downarrow}A$ is MF, then α_i is an end-node.*

Proof. If α_i is not an end-node, it is easy to see that $n_2 \ge 3$. As $\mathrm{rank}(G) \ge 3$ we have $p > 3$, and Table 2 shows that $r > 7$, so Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF. □

Remark 2.17. In the previous two proofs, we have applied Lemma 2.9, and in each case it was straightforward to see that the condition $d < \min\{\lfloor \frac{r+2}{2} \rfloor, p\}$ is satisfied. In what follows, we will apply the lemma without systematically pointing out how we conclude that this hypothesis holds.

The following lemma provides a classification for the second possibility of Proposition 2.14(vi).

Lemma 2.18 [Testerman 1988, 1.35]. *Assume that $\lambda = c_i\omega_i + c_j\omega_j$ with α_i and α_j adjacent and $c_ic_j \ne 0$. Let $d = \dim V_{\lambda-ij}$. Then $1 \le d \le 2$ and the following hold:*

(i) *If $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$, then $d = 1$ if and only if $c_i + c_j = p-1$.*

(ii) *If $(\alpha_i, \alpha_i) = 2(\alpha_j, \alpha_j)$, then $d = 1$ if and only if $2c_i + c_j + 2 \equiv 0 \bmod p$.*

(iii) *If $(\alpha_i, \alpha_i) = 3(\alpha_j, \alpha_j)$, then $d = 1$ if and only if $3c_i + c_j + 3 \equiv 0 \bmod p$.*

Finally, we conclude this section with two further results on the dimensions of certain weight spaces in V.

Lemma 2.19 [Seitz 1987, 8.6]. *Let $G = A_\ell$. Suppose that $\lambda = c_i\omega_i + c_j\omega_j$ and $1 \le s \le i < j \le t \le k$, with $c_ic_j \ne 0$. Let $d = \dim V_{\lambda - s(s+1)\cdots(t-1)t}$. Then:*

(i) *If $a + b + j - i \not\equiv 0 \bmod p$, then $d = j - i + 1$.*

(ii) *If $a + b + j - i \equiv 0 \bmod p$, then $d = j - i$.*

Lemma 2.20 [Burness et al. 2016, Lemma 2.2.8]. *Let $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$ and let $\mu = \lambda - \sum_{\beta \in S} c_\beta \beta \in X(T)$ for some subset $S \subseteq \Pi$. Set $X = \langle U_{\pm\beta} \mid \beta \in S\rangle$, where for $\gamma \in \Phi$, U_γ is the T-root subgroup associated to γ, $\lambda' = \lambda{\downarrow}(T \cap X)$ and $\mu' = \mu{\downarrow}(T \cap X)$. Then $\dim V_\mu = V'_{\mu'}$, where $V' = L_X(\lambda')$.*

3. The case where G has rank 2

Here we establish Theorem 1 in the case where G has rank 2. Let us recall our setup. Throughout the section we assume that λ is a p-restricted dominant weight for G, and we let $r = \lambda\downarrow T_A$ and $V = L(\lambda)$. We write $\lambda = ab$ as shorthand notation for $\lambda = a\omega_1 + b\omega_2$, and $\lambda - ab$ for $\lambda - a\alpha_1 - b\alpha_2$. We assume that $p \leq r$, as Proposition 2.3 settles the case $r < p$, and we have that $p \geq h$ since we are assuming the existence of a principal A_1-subgroup in G.

3.1. ***The case where G is A_2.*** We begin with the case $G = A_2$, where $p \geq h = 3$. The following is the main result, which we will prove after a sequence of lemmas.

Proposition 3.1. *Let $G = A_2$ and assume $p \leq r$. Then $V\downarrow A$ is MF if and only if $\lambda = \omega_1 + \omega_2$ and $p = 3$.*

Lemma 3.2. *Let $\lambda = ab$. Then $\lambda - ij$, with $i + j \leq a + b$, is a weight of $\Delta(\lambda)$ if and only if one of the following holds*:

(i) $i \leq j$ *and* $j - i \leq b$.

(ii) $i \geq j$ *and* $i - j \leq a$.

Proof. By [Bourbaki 1975, VIII, §7, Proposition 10], the weights in $\Delta(\lambda)$ are precisely the same as those occurring in $\Delta(a0) \otimes \Delta(0b)$. Let $\lambda_1 = a0$ and $\lambda_2 = 0b$ and recall that $\Delta(c_i\omega_i)$, for $i = 1, 2$, is the c_i-th symmetric power of the natural, respectively, dual module for G. Hence, $\lambda_1 - i_1 j_1$ is a weight of $\Delta(\lambda_1)$ if and only if

$$i_1 + j_1 \leq 2a \quad \text{and} \quad 0 \leq i_1 - j_1 \leq a.$$

Similarly, $\lambda_2 - i_2 j_2$ is a weight of $\Delta(\lambda_2)$ if and only if

$$i_2 + j_2 \leq 2b \quad \text{and} \quad 0 \leq j_2 - i_2 \leq b.$$

By symmetry it suffices to show that the statement of the lemma is valid when $i \geq j$. First of all, it is clear that all weights $\lambda - ij$ of $\Delta(\lambda)$ satisfy $i - j \leq a$, since a weight $\lambda_2 - i_2 j_2$ of $\Delta(\lambda_2)$ satisfies $i_2 - j_2 \leq 0$, and a weight $\lambda_1 - i_1 j_1$ of $\Delta(\lambda_1)$ satisfies $i_1 - j_1 \leq a$.

For the converse, consider a pair (i, j), such that $i \geq j$, $i + j = d \leq a + b$ and $i - j \leq a$. If $d \leq a$, then $j \leq i \leq a$, so $\lambda_1 - ij$ is a weight of $\Delta(\lambda_1)$ and $\lambda_1 + \lambda_2 - ij$ is then a weight of $\Delta(\lambda)$. If $d > a$, write $d = a + k$, where $k \leq b$. To conclude we show that we can find (i_1, j_1) with $i_1 + j_1 = a$ and (i_2, j_2) with $i_2 + j_2 = k$, such that $\lambda_1 - i_1 j_1$ is a weight of $\Delta(\lambda_1)$, $\lambda_2 - i_2 j_2$ is a weight of $\Delta(\lambda_2)$ and $i_1 - j_1 + i_2 - j_2 = i - j$. Fix $i_1 + j_1 = a$ and $i_2 + j_2 = k$. Note that we are allowed to pick $i_1 - j_1$ to be any integer between a and 1 if a is odd, and between a

and 0 if a is even. Similarly, we are allowed to choose $i_2 - j_2$ between $-k$ and -1 if k is odd, and between $-k$ and 0 when k is even, concluding easily. □

Lemma 3.3. *Let $\lambda = ab$ with $a \geq b > 0$ and $a + b = p - 1$.*

(i) *For $0 \leq d \leq b$, we have $n_d = d + 1$.*

(ii) *For $b + 1 \leq d \leq a$, we have that n_d increases alternatingly by respectively 0 and 1 with respect to n_{d-1}.*

(iii) *For $a < d \leq a + b$, we have that n_d alternates between $\lceil \frac{a+b}{2} \rceil$ and $\lceil \frac{a+b+1}{2} \rceil$.*

Proof. Here we use the fact that all T-weights in V are of multiplicity 1. (See [Zalesskii and Suprunenko 1987, Proposition 2].) Hence, the proof consists of counting the pairs (i, j) with $i + j = d$ and satisfying the conditions of Lemma 3.2.

(i) Let $0 \leq d \leq b$. The statement then follows immediately from noting that $\lambda - i(d - i)$ is a weight for $0 \leq i \leq d$.

(ii) Let us start from $d = b + 1$, where the weights are given by $\lambda - (b - i + 1)i$ for $0 \leq i \leq b$. This means that $n_{b+1} = b + 1 = n_b$ by part (i). For $d = b + 2$, still assuming that $d \leq a$, we find weights of the form $\lambda - (b - i + 2)i$ for $0 \leq i \leq b + 1$. The same reasoning continues until $d = a$, proving the statement.

(iii) Let $a < d \leq a + b$. We must count the weights of the form $\lambda - i(d - i)$ where $a \geq 2i - d$ and $b \geq d - 2i$. The conditions on i are equivalent to the inequalities $\frac{d-b}{2} \leq i \leq \frac{a+d}{2}$. Considering the various possibilities for the evenness of the terms in the inequality gives the result. □

Lemma 3.4. *Let $\lambda = ab$ with $a \geq b > 0$ and $a + b = p - 1$. Then $V{\downarrow}A$ is MF if and only if $a = b = 1$.*

Proof. Note that $r = 2(a + b) < 2p$ and that $a - b$ is an even number. For clarity we split the proof into four cases, depending on whether $a - b \geq 6$, $a - b = 4$, $a - b = 2$ or $a = b$. Suppose first that $a - b \geq 6$. By Lemma 3.3, all weights of the form $r - 2d$ with $b + 1 \leq d \leq a$ follow the pattern in (ii) of the same lemma. Since $r - 2(b + 1) = 2a - 2 \geq a + b + 4 = p + 3$, and $r - 2a = 2b \leq b + a - 6 = p - 7$, this includes weights that restrict to $p + 3$, $p + 1$, $p - 1$, $p - 3$, $p - 5$. Therefore by Lemma 2.9(i) either $(p + 3)$ or $(p + 1)$ is a composition factor for $V{\downarrow}A$. In the first case $p - 5$ occurs with multiplicity 1 more than $p - 3$, and does not occur as a weight in the composition factor $(p + 3)$, while $p - 3$ does. Therefore by Lemma 2.9(iv) the module $V{\downarrow}A$ is not MF. In the second case $(p - 3)$ is similarly a repeated composition factor.

Now suppose that $a - b = 4$. We have $p + 3 = a + b + 4 = r - 2\left(\frac{a+b}{2} - 2\right) = r - 2b$. Therefore by Lemma 3.3(i), we have that $(p + 3)$ is a composition factor for $V{\downarrow}A$.

The weights $p+1$, $p-1$, $p-3$, $p-5$ follow the pattern described in Lemma 2.9(ii). Therefore we can conclude like in the previous case.

Now suppose that $a=b+2$. Then by Lemma 3.3 we know that $r-2k$ occurs with multiplicity $k+1$ for k ranging between 0 and b. In particular $(r-2b)=(p+1)$ is a composition factor by Lemma 2.9(i). Again by Lemma 3.3, the T_A-weight $r-2(b+1)=2b+2$ occurs with multiplicity $b+1$ and $r-2a=2b$ occurs with multiplicity $b+2$. Since $2b=p-5$ does not occur as a weight in the composition factor $(p+1)$, while $p-3=2b+2$ does, Lemma 2.9(iv) implies that $V\downarrow A$ is not MF.

Finally assume that $a=b$. Then by Lemmas 3.3 and 2.9(i), the weights $r=4a, 4a-2, \dots, 2a$ afford composition factors for $V\downarrow A$, with the last weight occurring with multiplicity $a+1$. If $a \geq 2$ we find that $2a-2$ occurs with multiplicity $\lceil \frac{a+b}{2} \rceil = a$ and $2a-4$ occurs with multiplicity $\lceil \frac{a+b+1}{2} \rceil = a+1$. Since $2a-4=p-5$ does not occur as a weight in the composition factor $(p+3)$, while $p-3$ does, Lemma 2.9(iv) implies that $V\downarrow A$ is not MF. On the other hand if $a=b=1$ we find that $V\downarrow A=(4)\oplus(2)$. □

Proof of Proposition 3.1. Suppose that $V\downarrow A$ is MF, with $\lambda=ab$ and $a \geq b$. Since the Weyl module $\Delta(c0)$ is irreducible, the assumption that $r=2a+2b \geq p > a$, together with Lemma 2.4, implies that $b \geq 1$. If $\dim V_{\lambda-11}=2$, then $a+b \neq p-1$ by Lemma 2.18, and $b=1$ by Proposition 2.14(v). In this case, using the Jantzen p-sum formula [2003, Part II, 8.19] (for example), one sees that $\Delta(\lambda)$ is irreducible, a contradiction by Lemma 2.4. If $\dim V_{\lambda-11}=1$, then by Lemma 2.18 we have $a+b=p-1$, and we conclude by Lemma 3.4. □

3.2. *The case where G is B_2.*

We proceed with the case $G=B_2$, where $p \geq h=4$. The main result is the following, which we shall prove after a series of lemmas.

Proposition 3.5. *Let $G=B_2$ and assume that $p \leq r$. Then $V\downarrow A$ is MF if and only if $\lambda=2\omega_1$ and $p=5$.*

We begin by recalling some information about the structure of B_2 Weyl modules (with p-restricted highest weights). Let $\lambda=ab$ be a p-restricted dominant weight; here α_1 is long.

We consider the following alcoves in which a p-restricted weight can lie:

- $C_0=\{\lambda \mid 2a+b+3<p\}$;
- $C_1=\{\lambda \mid a+b+2<p<2a+b+3\}$;
- $C_2=\{\lambda \mid b+1<p<a+b+2 \text{ and } 2a+b+3<2p\}$;
- $C_3=\{\lambda \mid 2a+b+3>2p \text{ and } \max\{b+1, a+1\}<p\}$.

Lemma 3.6. (i) *If $\lambda \in C_i$ for $i = 1, 2, 3$, then $\Delta(\lambda)$ has exactly two composition factors, namely V and $L(\mu)$, where $\mu = (p-a-b-3)\omega_1 + b\omega_2$, respectively $a\omega_1 + (2p-2a-b-4)\omega_2$, $(2p-a-b-3)\omega_1 + b\omega_2$, when $i = 1, 2, 3$.*

(ii) *For $\lambda = a\omega_1 + (p-1)\omega_2$ with $2a + (p-1) + 3 > 2p$ and $a < p-1$, we have that the module $\Delta(\lambda)$ has exactly two composition factors, V and $L(\mu)$ for $\mu = (p-a-2)\omega_1 + (p-1)\omega_2$.*

For λ a p-restricted dominant weight not lying in $\bigcup_{i=1}^{3} C_i$ and not of the form described in (ii) above, $\Delta(\lambda)$ is irreducible.

Proof. This follows from the Jantzen p-sum formula [2003, Part II, 8.19]. □

Remark 3.7. Recall that here $\omega_1 = \alpha_1 + \alpha_2$. It follows from Lemma 3.6 that for a p-restricted weight $\lambda = ab$, if $\Delta(\lambda)$ is reducible then the module $\Delta(\lambda)$ has exactly one composition factor in addition to the composition factor $L(\lambda)$. The highest weight of the second composition factor is of the form $(a-k)\omega_1 + b\omega_2$ or $a\omega_1 + (b-k)\omega_2$, for some $k \geq 1$. More precisely, for $\lambda \in C_i$, $i = 1, 2, 3$, and μ as in the statement of the lemma, we have $\mu = \lambda - (2a+b+3-p)(\alpha_1+\alpha_2)$, respectively $\lambda - (a+b+2-p)(\alpha_1 + 2\alpha_2)$, $\lambda - (2a+b+3-2p)(\alpha_1+\alpha_2)$. And in case (ii) of the lemma, $\mu = \lambda - (2a-p+2)(\alpha_1+\alpha_2)$.

We record for convenience the dimension of the Weyl module $\Delta(ab)$, namely

$$\dim \Delta(ab) = \tfrac{1}{6}(a+1)(b+1)(a+b+2)(2a+b+3).$$

Lemma 3.8. *Let $\lambda = c0$. Then $\lambda - ij$, with $i + j \leq 2c$, is a weight of V if and only if one of the following holds*:

(i) *$i \leq j$ and $j - i \leq i$.*

(ii) *$i \geq j$ and $i - j \leq c - \lfloor \frac{j+1}{2} \rfloor$.*

Proof. By Proposition 2.1, the set of weights of V is precisely the same as the set of weights of the corresponding KG_K-module $\Delta_K(\lambda)$. First we show that all weights satisfying either (i) or (ii) are weights of $\Delta_K(\lambda)$.

Suppose $i \leq j \leq 2i$. Since $i + j \leq 2c$, we have that $i \leq c$. In particular, $\lambda - i0$ is a weight of $\Delta_K(\lambda)$. Now the weight $s_{\alpha_2}(\lambda - i0) = \lambda - i(2i)$ is also a weight of $\Delta_K(\lambda)$ and by [Bourbaki 1975, VIII, §7, Proposition 3] for all $0 \leq m \leq 2i$ we have that $\lambda - im$ is a weight of $\Delta_K(\lambda)$. So in particular, $\lambda - ij$ is a weight of V.

Suppose now that $j \leq i \leq c + j - \lfloor \frac{1}{2}(j+1) \rfloor$. As $i + j \leq 2c$, we have that $j \leq c$ and so $\lfloor \frac{1}{2}(j+1) \rfloor \leq c$ and $\mu = \lambda - (\lfloor \frac{1}{2}(j+1) \rfloor)\alpha_1$ is a weight of V. Hence, $s_{\alpha_2}(\mu) = \mu - 2\lfloor \frac{1}{2}(j+1) \rfloor \alpha_2$ is a weight of $\Delta_K(\lambda)$, which again by [Bourbaki 1975, VIII, §7, Proposition 3] implies that $\nu = \lambda - (\lfloor \frac{1}{2}(j+1) \rfloor)j$ is a weight of $\Delta_K(\lambda)$. Further, we have that $\langle \nu, \alpha_1 \rangle = c + j - 2\lfloor \frac{1}{2}(j+1) \rfloor$ and using again [loc. cit.] for all $0 \le m \le c+j-2\lfloor \frac{1}{2}(j+1)\rfloor$, we have that $\nu - m\alpha_1 = \lambda - \big(m + \lfloor \frac{1}{2}(j+1)\rfloor\big) j$ is a weight of $\Delta_K(\lambda)$, giving that $\lambda - ij$ is a weight of V.

We now show that any weight $\lambda - ij$ with $i+j \le 2c$ satisfies either (i) or (ii). To this end, we use the fact that the set of weights of $\Delta_K(\lambda)$ is the same as the set of weights of the module $V_1^{\otimes c}$, the c-fold tensor product of the module V_1 with itself, where V_1 is the KG_K-module with highest weight ω_1. (See [Bourbaki 1975, VIII, §7, Proposition 10].) Let μ be a weight of $V_1^{\otimes c}$, so that

$$\begin{aligned}\mu &= c\omega_1 - a_1\alpha_1 - a_2(\alpha_1+\alpha_2) - a_3(\alpha_1+2\alpha_2) - a_4(2\alpha_1+2\alpha_2)\\ &= \lambda - (a_1+a_2+a_3+2a_4)\alpha_1 - (a_2+2a_3+2a_4)\alpha_2,\end{aligned}$$

with $a_i \in \mathbb{N}$ such that $a_1+a_2+a_3+a_4 \le c$.

There are two cases to consider; suppose first that $a_1 \le a_3$, so that

$$a_1+a_2+a_3+2a_4 \le a_2+2a_3+2a_4.$$

Then $a_2+2a_3+2a_4 \le 2a_1+2a_2+2a_3+4a_4 = 2(a_1+a_2+a_3+2a_4)$ and the weight μ satisfies the conditions of (i).

Now suppose $a_1 \ge a_3$, so that $a_1+a_2+a_3+2a_4 \ge a_2+2a_3+2a_4$ and as usual

$$a_1+a_2+a_3+2a_4+a_2+2a_3+2a_4 = a_1+2a_2+3a_3+4a_4 \le 2c, \tag{2}$$

and

$$a_1+a_2+a_3+a_4 \le c. \tag{3}$$

Note that $j - \lfloor \frac{j+1}{2} \rfloor = \lfloor \frac{j}{2} \rfloor$. If

$$a_1+a_2+a_3+2a_4 > c + \lfloor \tfrac{1}{2}(a_2+2a_3+2a_4)\rfloor = c + a_3 + a_4 + \lfloor \tfrac{1}{2}a_2 \rfloor,$$

then $a_1+a_2-\lfloor \frac{a_2}{2}\rfloor + a_4 > c$ and $2a_1+a_2+1+2a_4 > 2c$. If $2a_1+2a_2+2a_3+2a_4 \ge 2a_1+a_2+1+2a_4$ we obtain a contradiction to (3). Hence we may now assume $2a_2+2a_3 < a_2+1$, that is, $a_2 = 0 = a_3$. Now (3) becomes $a_1+a_4 \le c$ and so $a_1+2a_4 \le c + \lfloor \frac{2a_4}{2} \rfloor$ and the weight satisfies condition (ii). □

Lemma 3.9. *Let $\lambda = c1$, $c < p$. Then $\lambda - ij$, with $i+j \le 2c$, is a weight of V if and only if one of the following holds*:

(i) $i \le j$ *and* $j - i \le i+1$.

(ii) $i \ge j$ *and* $i - j \le c - \lfloor \frac{j}{2} \rfloor$.

Proof. By [Bourbaki 1975, VIII, §7, Proposition 10] and Proposition 2.1, the weights occurring in V are the same as the weights occurring in $c0 \otimes 01$. The statement then follows from Lemma 3.8. □

Lemma 3.10. *Let $\lambda = ab$ with $p > a \geq 1$, $p > b \geq 2$ and $2a+b+2 \equiv 0 \bmod p$. Then $V{\downarrow}A$ is not MF.*

Proof. Since $2a+b+2 \equiv 0 \bmod p$, by Lemma 3.6 we have

$$\dim V = \dim L(ab) = \dim \Delta(ab) - \dim L((a-1)b) \geq \dim \Delta(ab) - \dim \Delta((a-1)b).$$

Using the Weyl character formula, we have that

$$\dim V \geq \tfrac{1}{6}(1+b)(6+6a^2+5b+b^2+12a+6ab). \tag{4}$$

Since $p > a$ and $p > b$, there are exactly two possibilities for p, either $p = 2a+b+2$ and b is odd, or $p = a+1+\frac{b}{2}$ and b is even. Let us start with the first case, namely $p = 2a+b+2$. Assume that $b = 3$ and $a \geq 2$. Then $r = 4a+3b = 2p-1$ and

$$B(r) = 2\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + \sum_{k=1}^{\frac{p-1}{2}} 2k = \tfrac{1}{4}(3p^2+4p+1). \tag{5}$$

Plugging in $p = 2a+5$ and combining (5) with (4) gives $\dim V > B(r)$ and Lemma 2.12 implies that $V{\downarrow}A$ is not MF. The case $b = 3$, $a = 1$ and $p = 7$ can be handled directly; we observe that $n_1 = n_2 = 2$, while $n_3 = 4$, as the weight space $\lambda - 12$ is 2-dimensional (see [Lübeck 2018]). Then Lemma 2.9(ii) implies that $V{\downarrow}A$ is not MF.

Next assume that $b \geq 5$, in which case $r = 2p+(b-4) < 3p$. Then

$$B(r) = 3\sum_{k=1}^{\frac{b-3}{2}} 2k + 2\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + \sum_{k=1}^{\frac{p-1}{2}} 2k = \tfrac{1}{4}(3p^2+4p+10-12b+3b^2). \tag{6}$$

Plugging in $p = 2a+b+2$ and combining (6) with (4) gives

$$\dim V - B(r) \geq -39-36a-12a^2+5b+6a^2b-3b^2+6ab^2+b^3.$$

As $b \geq 5$ and $a \geq 1$, this means that $\dim V - B(r) > 0$, and Lemma 2.12 implies that $V{\downarrow}A$ is not MF.

We now consider the second case, where $p = a+1+\frac{b}{2}$. Here we have $r = 4p+b-4$. Suppose that $b = 2$, so that $a = p-2 \geq 3$ and $r = 3p+a < 4p$. If a is even,

$$\begin{aligned} B(r) &= 4\sum_{k=1}^{\frac{a+2}{2}}(2k-1) + 3\sum_{k=1}^{\frac{p-1}{2}} 2k + 2\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + \sum_{k=1}^{\frac{p-1}{2}} 2k \\ &= \tfrac{1}{2}(3p^2+2p+7+8a+2a^2). \end{aligned} \tag{7}$$

Plugging in $p = a+2$ and combining (7) with (4) gives

$$\dim V - B(r) \geq \tfrac{1}{2}(a^2+2a-3).$$

Therefore $\dim V - B(r) > 0$, and Lemma 2.12 implies that $V{\downarrow}A$ is not MF. If a is odd, we have

$$B(r) = 4\sum_{k=1}^{\frac{a+1}{2}} 2k + 3\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + 2\sum_{k=1}^{\frac{p-1}{2}} 2k + \sum_{k=1}^{\frac{p+1}{2}}(2k-1) = \tfrac{1}{2}(3p^2+4p+7+8a+2a^2). \tag{8}$$

Plugging in $p = a+2$ and combining (8) with (4) gives

$$\dim V - B(r) \geq \tfrac{1}{2}(a^2-7).$$

As $a \geq 3$, Lemma 2.12 implies that $V{\downarrow}A$ is not MF. Now suppose that $b \geq 4$, in which case $r = 4p+b-4 < 5p$. Then

$$B(r) = 5\sum_{k=1}^{\frac{b-2}{2}}(2k-1) + 4\sum_{k=1}^{\frac{p-1}{2}} 2k + 3\sum_{k=1}^{\frac{p+1}{2}}(2k-1) + 2\sum_{k=1}^{\frac{p-1}{2}} 2k + \sum_{k=1}^{\frac{p+1}{2}}(2k-1) = \tfrac{1}{4}(10p^2+8p+5b^2-20b+18). \tag{9}$$

Plugging in $p = a+1+\frac{b}{2}$ and combining (9) with (4) gives

$$\dim V - B(r) \geq \tfrac{1}{24}(-192-120a-36a^2+80b+12ab+24a^2b-21b^2+24ab^2+4b^3).$$

We can write this as

$$\dim V - B(r) \geq \tfrac{1}{24}(-192+80b-21b^2+4b^3+12a^2(-3+2b)+12a(-10+b+2b^2)).$$

Treating the right-hand side as a quadratic polynomial in a, it is easy to see that since $b \geq 4$, we must again have $\dim V - B(r) > 0$, concluding by Lemma 2.12. □

Lemma 3.11. *Let $\lambda = 1b$ with $b \geq 2$. Then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 3.6, we have that one of the following holds:

(i) $p > b+5$ and $V = \Delta(1b)$.

(ii) $b = p-4$.

(iii) $b = p-2$ and $\dim V \geq \dim \Delta(1b) - \dim \Delta(1(b-2))$.

In the first case, for $b \geq 3$, $\dim V$ exceeds $B_K(r)$ and Lemma 2.12 then implies that $V{\downarrow}A$ is not MF. For $b = 2$, we have $p \geq 11 > r$, contradicting our assumption that $p \leq r$.

The second case is covered by Lemma 3.10.

Finally, we consider the third case. Here $\dim V \geq 2(3+4b+b^2) = 2(p-1)(p+1)$, $r = 3p-2 > p$ and $b \geq 3$. In addition, we have $B(r) = \frac{1}{2}(p+1)(3p-1)$ so that $\dim V > B(r)$ and $V{\downarrow}A$ is not MF by Lemma 2.12. □

Lemma 3.12. *Let $\lambda = c1$ with $c \geq 1$ and $p = 2c+3$. Then $V{\downarrow}A$ is not MF.*

Proof. Note that all T-weight spaces of V are 1-dimensional, see [Zalesskii and Suprunenko 1987, Proposition 2]. By Lemma 3.9, for $1 \leq d \leq c$ and $0 \leq k \leq d$, we have that $\lambda-(d-k)k$ is a weight if and only if $0 \leq k \leq 2d-2k+1$. We then find that $n_d = \lfloor \frac{2d+1}{3} \rfloor + 1$, for $1 \leq d \leq c$. Since $\lambda-(c+1-k)k$ is a weight if and only if $1 \leq k \leq 2c+2-2k+1$, we find that $n_{c+1} = \lfloor \frac{2c}{3} \rfloor + 1$. Similarly $n_{c+2} = \lfloor \frac{2c+2}{3} \rfloor + 1$. There are now two cases to consider: either $c \equiv 1 \bmod 3$ or $c \equiv -1 \bmod 3$. In the first case $n_{c-1} > n_{c-2}$ (note that $n_{c-2} = 0$ if $c = 1$), and therefore by Lemma 2.9(i) we know that $(p+2)$ is a composition factor of $V{\downarrow}A$. We have $n_{c+2} = n_{c+1}+1$, and the T_A-weight $r-2(c+2) = p-4$ does not occur in the composition factor $(p+2)$, but the T_A-weight $p-2$ does. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF. The second case, when $c \equiv -1 \bmod 3$, follows similarly. □

Lemma 3.13. *Let $\lambda = c1$ with $2c+3 \neq p$ and $c \geq 1$. Then $V{\downarrow}A$ is not MF.*

Proof. Note that $r = 4c+3$. First assume $c = 1$, so that $r = 7$; hence $p = 7$ and the result follows from Lemma 2.4. When $c = 2$ we have $p \neq 7$ and by Lemma 3.6 $\Delta(\lambda)$ is irreducible. Since $p = 5$ or $p = 11$, the hypotheses of Lemma 2.4 are satisfied, and $V{\downarrow}A$ is not MF.

We henceforth assume that $c \geq 3$ and will show that $V{\downarrow}A$ is not MF. Suppose first that $p > 2c+4$, so that by Lemma 3.6 $\Delta(\lambda)$ is irreducible. Since $p \geq 5$ and $r = 4c+3$, the hypotheses of Lemma 2.4 are satisfied and $V{\downarrow}A$ is not MF.

Assume now that $p \leq 2c+4$, so that in fact $p \leq 2c+1$. Then by Lemma 3.6 and Remark 3.7, either $p-3 \leq c \leq p-1$ and $\Delta(\lambda)$ is irreducible which implies $\dim V = \dim \Delta(\lambda)$; or $c \leq p-4$ and $\dim V \geq \dim \Delta(c1) - \dim \Delta((c-k)1)$, for $k = 2c+4-p \geq 3$. In particular, $\dim V \geq \dim \Delta(c1) - \dim \Delta((c-3)1) = 4+6c+6c^2$. In both cases, one checks that $\dim V > B_K(r)$ so that $V{\downarrow}A$ is not MF by Lemma 2.12. □

Lemma 3.14. *Let $\lambda = c0$ with $c > 1$ and $p = 2c+1$. Then $V{\downarrow}A$ is MF if and only if $c = 2$.*

Proof. First note that $L(20){\downarrow}A = (8)+(4)$, since here $p = 5$. Now assume that $c \geq 3$, so that $p \geq 7$. By [Zalesskii and Suprunenko 1987, Proposition 2], all T-weight spaces of V are 1-dimensional. Let $1 \leq l \leq c$. Then $n_l = \lfloor \frac{2l}{3} \rfloor + 1$, since by Lemma 3.9 we have that $\lambda-(l-k)k$ is a weight if and only if $0 \leq k \leq 2l-2k$. Similarly $n_{c+1} = \lfloor \frac{2c+2}{3} \rfloor$, $n_{c+2} = \lfloor \frac{2c+1}{3} \rfloor$, $n_{c+3} = n_c$. Therefore by Lemma 2.9(i) either both $(p+1)$ and $(p+5)$ or both $(p+3)$ and $(p+5)$ are composition factors of $V{\downarrow}A$, and by Lemma 2.9(iii), the composition factor $(p-7)$ is repeated. □

Lemma 3.15. *Let $\lambda = c0$ with $c > 1$ and $p \neq 2c+1$. Then $V{\downarrow}A$ is not MF.*

Proof. First assume that $p > 2c+3$, so that Lemma 3.6 implies that $\Delta(c0)$ is irreducible. Since $r = 4c$ and by hypothesis $r \geq p$ and $p \geq 2c+4$, we have that p does not divide $r+1$ and Lemma 2.4 gives the result.

So we now assume that $p \leq 2c+3$. If $\Delta(\lambda)$ is irreducible, then one checks that $\dim V = \dim \Delta(\lambda)$ exceeds $B_K(r)$ for all $c \geq 9$. For $c \leq 8$ we combine the information from the tables in [Lübeck 2001] with the criteria of Lemma 2.4 to reduce to the case $c = 5$ and $p = 7$. But then we have $\dim V = 91$ and $B(r) = 88$, so we conclude by applying Lemma 2.12.

Now assume that $p \leq 2c+3$ and $\Delta(\lambda)$ is reducible. Then by Lemma 3.6, $c+2 < p$ and $2c+3 > p$, and so in particular, $4c = r > p$. By Remark 3.7, $\dim V = \dim \Delta(c0) - \dim \Delta((c-k)0)$ for $k = 2c+3-p$, so k is even. Assume that $c \geq k \geq 6$. We find that

$$\dim V - B_K(r) = c^2(-4+k) - c(4-3k+k^2) + \tfrac{1}{6}(-6+13k-9k^2+2k^3).$$

Treating this as a quadratic polynomial in c, we find that $\dim V - B_K(r)$ is certainly strictly positive if $-44+47k-16k^2+k^3 > 0$. Therefore if $k > 12$, by Lemma 2.12 we have that $V{\downarrow}A$ is not MF. If $k = 6$, 8, 10 or 12, we have $\dim V - B_K(r) > 0$ when $c \geq 10$. Therefore the only possibilities for (c, k), with $k \in \{6, 8, 10, 12\}$, are $(8, 6)$ with $p = 13$, $(7, 6)$ with $p = 11$, $(8, 8)$ with $p = 11$ or $(9, 6)$ with $p = 13$. In each of these cases, we find that $\dim V - B(r) > 0$, concluding by Lemma 2.12.

Note that $k \neq 2$ as $p \neq 2c+1$. So finally we consider the case $k = 4$ and $p = 2c-1$. Now $r = 4c = 2p+2$ and a direct computation shows that $\dim V = 4c^2 - 4c + 6$ while $B(r) = 3c^2 - 2c + 12$. Now we have $c \geq 4$ (since $k = 4$) and hence $\dim V$ exceeds $B(r)$, showing as before that $V{\downarrow}A$ is not MF. □

Lemma 3.16. *Let $\lambda = 0c$ with $c > 1$ and $p \leq r$. Then $V{\downarrow}A$ is not MF.*

Proof. We have $V = \Delta(0c)$ (see [Seitz 1987, Table 1]). One checks that $\dim V = \frac{1}{6}(1+c)(2+c)(3+c) > B_K(r)$ for $c \geq 9$; by Lemma 2.12, the module $V{\downarrow}A$ is not MF in these cases. For $2 \leq c \leq 8$, we may apply Lemma 2.4 to conclude that $V{\downarrow}A$ is not MF except for the pairs $(c, p) = (3, 5)$ and $(c, p) = (7, 11)$. Here we apply Lemma 2.13 to again conclude that $V{\downarrow}A$ is not MF. □

Proof of Proposition 3.5. Suppose that $V{\downarrow}A$ is MF, with $\lambda = ab$ and $r \geq p$. By Proposition 2.14 and Lemma 2.18, if $a, b \geq 2$ we must have $2a+b+2 \equiv 0 \bmod p$. Therefore Lemma 3.10 implies that either $a \leq 1$ or $b \leq 1$ and Lemmas 3.12 and 3.13 show that $\lambda \neq 11$. By Lemma 3.11 we conclude that if $a = 1$, then $b = 0$ contrary to our assumption that $r \geq p$, and another application of Lemmas 3.12 and 3.13 shows that if $b = 1$ then $a = 0$, again contrary to our assumption on p and r. We therefore reduce to the case $a=0$ or $b=0$, the first being ruled out by Lemma 3.16. If $b=0$, by Lemmas 3.14 and 3.15, and the above remarks, we conclude that $a=2$ with $p=5$, in which case $V\downarrow A$ is MF by Lemma 3.14. □

3.3. *The case where G is G_2.* We now move on to the final case where G has rank 2, i.e., $G=G_2$. Our main result, to be proven in a sequence of lemmas, is the following proposition.

Proposition 3.17. *Let $G=G_2$ and $\lambda=ab$ with $p\le r$. Then $V\downarrow A$ is not MF.*

Set $\lambda=ab$, with $0\le a,b<p$, where we take α_1 to be short, $(\alpha_2,\alpha_2)=1$. (This choice of root lengths is required for using the result [Seitz 1987, (6.2)] stated below in Lemma 3.18.) Here we have $r=6a+10b$, and $p\ge 7$ since $p\ge h$. We set $\mu=\lambda-11$ throughout the entire section and note that $\mu=(a+1)\omega_1+(b-1)\omega_2$. For $\alpha\in\Phi$, we let e_α, f_α denote the T-weight vectors in the Lie algebra of G associated with the root α, respectively $-\alpha$.

We will use a result from [Seitz 1987], which we state here only for the group G_2:

Lemma 3.18 [Seitz 1987, (6.2)]. *Assume $p>3$. Let ν be a dominant weight such that $L(\nu)$ affords a composition factor of $\Delta(\lambda)$. Then*

$$2(\lambda+\rho,\lambda-\nu)-(\lambda-\nu,\lambda-\nu)\in\tfrac{p}{6}\mathbb{Z}.$$

In view of applying Lemma 3.18, we record the results of some computations for particular subdominant weights in $\Delta(\lambda)$ in Table 3.

We note that since λ is p-restricted, V is irreducible as a module for the Lie algebra of G (see [Curtis 1960, Chapter II]). For the following lemmas, we let $v^+\in V_\lambda$, that is, v^+ is a highest weight vector in V. Then by [Testerman 1988, 1.29] we have that, for $\nu\le\lambda$, the weight space V_ν is spanned by vectors of the form $f_{\gamma_1}^{m_1}\cdots f_{\gamma_r}^{m_r}v^+$, where $\gamma_j\in\Phi^+$ and $m_j\in\mathbb{N}$ with $\lambda-\nu=\sum m_j\gamma_j$.

(a,b)	ν	$\dim\Delta(\lambda)_\nu$	$2(\lambda+\rho,\lambda-\nu)-(\lambda-\nu,\lambda-\nu)$
$a\ge 1, b\ge 1$	$\lambda-21$	$3-\delta_{a,1}$	$\frac{2a+3b+4}{3}$
$a\ge 1, b\ge 2$	$\lambda-12$	2	$\frac{a+6b}{3}$
$a\ge 1, b\ge 1$	$\lambda-22$	$4-\delta_{a,1}-\delta_{b,1}$	$\frac{2a+6b+4}{3}$
$a=0, b\ge 2$	$\lambda-22$	2	$\frac{6b+4}{3}$
$a\ge 1, b\ge 2$	$\lambda-13$	$2-\delta_{b,2}$	$\frac{a+9b-9}{3}$
$a\ge 1, b\ge 2$	$\lambda-32$	$7-2\delta_{a,1}-\delta_{a,2}$	$a+2b+2$
$a\ge 1, b\ge 3$	$\lambda-23$	$4-\delta_{a,1}$	$\frac{2a+9b-2}{3}$
$a=1, b\ge 3$	$\lambda-14$	$2-\delta_{b,3}$	$\frac{a+12b-24}{3}$

Table 3. Weight multiplicities for G_2-modules.

Lemma 3.19. *Assume $a \geq 2$, $b \geq 1$ and set $\nu = \lambda - 21$.*

(i) *If $a+3b+3 \not\equiv 0 \bmod p$ and $2a+3b+4 \not\equiv 0 \bmod p$, then $\dim V_\nu = 3$.*

(ii) *If $a+3b+3 \equiv 0 \bmod p$, then $\dim V_\nu = 2$.*

(iii) *If $2a+3b+4 \equiv 0 \bmod p$, then $\dim V_\nu = 2$.*

Proof. Using [Lübeck 2018] and [Cavallin 2017, Proposition A], one checks that $\dim \Delta(\lambda)_\nu = 3$. Note as well that if η is a dominant weight satisfying $\nu \prec \eta \prec \lambda$, then $\eta \in \{\lambda - 10, \lambda - 01 \text{ (if } b \geq 2), \lambda - 20 \text{ (if } a \geq 4), \mu\}$. The weight η does not afford a composition factor of $\Delta(\lambda)$, for $\eta \in X(T)$, $\eta \neq \mu$.

First consider the case where $a+3b+3 \not\equiv 0 \bmod p$. In this case, μ does not afford a composition factor of $\Delta(\lambda)$ (see Lemma 2.18) and the vectors $f_{\alpha_1+\alpha_2}v^+$ and $f_{\alpha_2}f_{\alpha_1}v^+$ are linearly independent. The weight space V_ν is spanned by the vectors $v_1 = f_{2\alpha_1+\alpha_2}v^+$, $v_2 = f_{\alpha_1+\alpha_2}f_{\alpha_1}v^+$ and $v_3 = f_{\alpha_2}f_{\alpha_1}^2 v^+$. Suppose $\sum_{i=1}^3 a_i v_i = 0$, for $a_i \in k$. Then applying e_{α_1} and e_{α_2} respectively, and using the fact that $f_{\alpha_1}^2 v^+ \neq 0$, we obtain the following system of equations:

$$2a_1 + aa_2 = 0, \quad 3a_2 + a_3(2a-2) = 0, \quad a_3(b+2) - a_2 = 0.$$

(These computations depend on a choice of structure constants; we have used those given in [Carter 1989, §12.5].) We then have that v_1, v_2, v_3 are linearly dependent if and only if $a_3 \neq 0$. If $a_3 \neq 0$, then we deduce that $2a+3b+4 \equiv 0 \bmod p$. Moreover, if $2a+3b+4 \equiv 0 \bmod p$ the three vectors are linearly dependent and it is easy to check that v_1 and v_2 are linearly independent. This gives (i).

Now consider the case where $a+3b+3 \equiv 0 \bmod p$, so that μ affords a composition factor of $\Delta(\lambda)$ and one checks that $bf_{\alpha_1+\alpha_2}v^+ + f_{\alpha_1}f_{\alpha_2}v^+ = 0$. Now if $a = p-1$ (so that $2a+3b+4 \equiv 0 \bmod p$), then ν does not occur in the composition factor afforded by μ. In addition, arguing as above, we see that $v_1 \in \langle v_2, v_3 \rangle$ and v_2 and v_3 are linearly independent, so that $\dim V_\nu = 2$. While if $a \neq p-1$, then ν occurs in the composition factor afforded by μ, with multiplicity 1. Moreover, $2a+3b+4 \not\equiv 0 \bmod p$, and Lemma 3.18 implies that the weight ν does not afford a composition factor of $\Delta(\lambda)$ and so $\dim V_\nu = 2$. These arguments give the conclusions of (ii) and (iii). □

Lemma 3.20. *Let $a = 1$, $b \geq 1$, and set $\nu = \lambda - 21$. Then $\dim V_\nu = 1$ if $3b+4 \equiv 0 \bmod p$ and $\dim V_\nu = 2$ otherwise.*

Proof. By [Lübeck 2018] and [Cavallin 2017, Proposition A], we have $\dim \Delta(\lambda)_\nu = 2$ and V_ν is spanned by $v_1 = f_{2\alpha_1+\alpha_2}v^+$ and $v_2 = f_{\alpha_1+\alpha_2}f_{\alpha_1}v^+$. If $3b+4 \equiv 0 \bmod p$, then by Lemma 2.18, μ affords a composition factor of $\Delta(\lambda)$ and ν occurs with multiplicity 1 there. So by Proposition 2.1, we have $\dim V_\nu = 1$.

If $3b+4 \not\equiv 0 \bmod p$, then μ does not afford a composition factor and $f_{\alpha_1+\alpha_2}v^+$ and $f_{\alpha_2}f_{\alpha_1}v^+$ are linearly independent. If $a_1v_1+a_2v_2=0$ for $a_i \in k$, then applying e_{α_1} and e_{α_2}, we deduce that $2a_1+a_2=0=3a_2$. Hence the two vectors are linearly independent and $\dim V_\nu = 2$. □

Lemma 3.21. *Assume* $b \geq 2$, $a \geq 1$, *and set* $\nu = \lambda - 12$. *Then* $\dim V_\nu = 1$ *if* $a+3b+3 \equiv 0 \bmod p$ *and* $\dim V_\nu = 2$ *otherwise.*

Proof. As in the preceding lemmas, we find that $\dim \Delta(\lambda)_\nu = 2$. If $a+3b+3 \equiv 0 \bmod p$, then μ affords a composition factor of $\Delta(\lambda)$ and using Proposition 2.1 we deduce that $\dim V_\nu = 1$.

So assume that $a+3b+3 \not\equiv 0 \bmod p$ and then $f_{\alpha_1+\alpha_2}v^+$ and $f_{\alpha_2}f_{\alpha_1}v^+$ are linearly independent. The ν weight space is spanned by $v_1 = f_{\alpha_1+\alpha_2}f_{\alpha_2}v^+$ and $v_2 = f_{\alpha_2}^2 f_{\alpha_1}v^+$. Suppose $a_1v_1+a_2v_2=0$ for $a_i \in k$. Applying e_{α_1} and e_{α_2} and using that $f_{\alpha_2}^2 v^+ \neq 0$, we deduce that $3a_1+aa_2=0$ and $a_1b=0$. Hence the two vectors are linearly independent, giving the result. □

We are now ready to prove the main proposition.

Proof of Proposition 3.17. We treat various cases separately below. In Cases 1 to 4, we use Proposition 2.14(v) and Lemma 2.18 to reduce to the case where $a+3b+3 \equiv 0 \bmod p$ (as otherwise $V{\downarrow}A$ is not MF and neither is $\Delta_K(\lambda){\downarrow}A_K$). Throughout we rely on the tables in [Lübeck 2018].

Case 1: $a \geq 3$ and $b \geq 3$. The T_A-weight $r-6$ is afforded by $\lambda-30$, $\lambda-03$, $\lambda-21$ and $\lambda-12$. An application of Lemma 3.19 then shows that $n_3 \geq 5$ and Lemma 2.6 then shows that $V{\downarrow}A$ is not MF.

Case 2: $a=2$ and $b \geq 4$. Here $3b+5 \equiv 0 \bmod p$, so Lemma 3.18 implies that neither of the weights $\lambda-21$ and $\lambda-12$ affords a composition factor of $\Delta(\lambda)$. Now, the T_A-weight $r-8$ is afforded by $\lambda-31$, $\lambda-22$, $\lambda-13$ and $\lambda-04$. By Lemma 3.18, none of these weights affords a composition factor of $\Delta(\lambda)$. Therefore $\dim V_{\lambda-31} = \dim V_{\lambda-22} = 2$, Indeed, for the weight $\lambda-31$ we note that the only dominant weights μ' with $\lambda-31 \prec \mu'$ are μ, $\lambda-01$, $\lambda-10$, $\lambda-21$. We have assumed that μ affords a composition factor of $\Delta(\lambda)$, but the second and third weights occur with multiplicity 1 in $\Delta(\lambda)$ and so do not afford a composition factor of $\Delta(\lambda)$ (and as mentioned above, neither does the fourth weight). This then allows us to determine $\dim V_{\lambda-31}$ and similarly for $V_{\lambda-22}$. We conclude that $n_4 \geq 6$ and apply Lemma 2.6 to see that $V{\downarrow}A$ is not MF.

Case 3: $a \geq 3$ and $b=2$. Here we have $a+9 \equiv 0 \bmod p$. Lemma 3.18 implies that neither of the weights $\lambda-21$, $\lambda-12$ affords a composition factor of $\Delta(\lambda)$. Now consider the T_A-weight $r-8$, afforded by $\lambda-31$, $\lambda-22$ and $\lambda-13$, none of which affords a composition factor of $\Delta(\lambda)$. Counting the occurrences of these weights in the irreducible $L(\mu)$, we see that $n_4 \geq 6$ and then use Lemma 2.6 to see that $V\downarrow A$ is not MF.

Case 4: $a = 2$ and $b \in \{2, 3\}$. Here, we have $a + 3b + 3 \equiv 0 \bmod p$. Consider first the weight $\lambda = 2\omega_1 + 2\omega_2$ with $p = 11$; here $\dim V = 295$ and $r = 32$. One then checks that $B(r) = 204$. For the weight $\lambda = 2\omega_1 + 3\omega_2$, with $p = 7$, we have $r = 42$, $\dim V = 532$ and $B(r) = 295$. In both cases, Lemma 2.12 then implies that $V\downarrow A$ is not MF.

We now turn to the cases where one or both of a and b is less than 2, in which case we no longer deduce that $\dim V_\mu = 1$.

Case 5: $a \geq 3$ and $b = 1$. If $a = p - 6$, then $n_2 = 2$, while the T_A-weight $r - 6$ is afforded by $\lambda - 21$, $\lambda - 12$ and $\lambda - 30$; using Lemma 3.18 we have that $n_3 = 4$ and so $V\downarrow A$ is not MF by Lemma 2.9.

Now suppose $a \neq p - 6$ so that $n_2 = 3$ and μ does not afford a composition factor of $\Delta(\lambda)$. Let $\nu = \lambda - 21$. Suppose first that $2a + 7 \equiv 0 \bmod p$; then by Lemma 3.19 we have $\dim V_\nu = 2$, which implies that the composition multiplicity $[\Delta(\lambda) : L(\nu)]$ is equal to 1 and $n_3 = 4$. Now count the occurrences of the T_A-weight $r - 8$ which is afforded by $\lambda - 31$, $\lambda - 22$ and $\lambda - 40$, the latter only if $a \geq 4$. If $a \geq 4$, Lemma 3.18 implies that $n_4 \geq 6$, giving the usual contradiction. The case where $2a + 7 \not\equiv 0 \bmod p$ is easier; here ν does not afford a composition factor of $\Delta(\lambda)$ and $n_3 = 5 = n_2 + 3$ (even if $a = 3$).

So we are left with the case $a = 3$, $b = 1$ and $p = 13$, where $\dim V = 259$ and $r = 28$. But as above, one checks that $\dim V > B(r)$, and Lemma 2.12 implies that $V\downarrow A$ is not MF.

Case 6: $a = 1$ and $b \geq 3$. Consider first the case where $3b + 4 \equiv 0 \bmod p$, when μ affords a composition factor of $\Delta(\lambda)$. Moreover, we note that $b \neq 4$. We claim that $n_4 = 4 - \delta_{b,3}$ and $n_5 \geq 6 - \delta_{b,3}$, which then shows that $V\downarrow A$ is not MF.

The T_A-weight $r - 8$ is afforded by $\lambda - 31$, $\lambda - 22$, $\lambda - 13$ and $\lambda - 04$ (the latter only if $b \geq 4$). The first of these is conjugate to μ and so has multiplicity 1 in V and the last of these has multiplicity $1 - \delta_{b,3}$. For the remaining two weights, we use repeatedly Lemma 3.18 and note that

(i) $\lambda - 21$ and $\lambda - 12$ do not afford composition factors of $\Delta(\lambda)$;

(ii) $\lambda - 22$ does not afford a composition factor of $\Delta(\mu)$ and so occurs with multiplicity 2 in $L(\mu)$; and

(iii) $\lambda - 13$ and $\lambda - 22$ do not afford composition factors of $\Delta(\lambda)$.

We then deduce that the weights $\lambda - 22$ and $\lambda - 13$ each occur with multiplicity 1 in V. Hence $n_4 = 4 - \delta_{b,3}$ as claimed.

Now we turn to n_5; the T_A-weight $r-10$ is afforded by $\lambda-41$, $\lambda-32$, $\lambda-23$, $\lambda-14$ and $\lambda-05$ (the latter only if $b \geq 5$). The first of these is conjugate to λ. We now argue that $\nu=\lambda-32$ has multiplicity 2 in V, which establishes the claim on n_5. Note that ν does not afford a composition factor of $\Delta(\lambda)$ nor of $\Delta(\mu)$. Applying Lemma 3.19, we deduce that $\nu=\mu-21$ has multiplicity 3 in $L(\mu)$ and so has multiplicity 2 in V, as claimed.

Now consider the case where $3b+4 \not\equiv 0 \bmod p$ and so $n_2=3$. By Lemmas 3.20 and 3.21 we have

$$\dim V_{\lambda-21} = \dim V_{\lambda-12} = 2,$$

which means that $n_3=5$, so that $V{\downarrow}A$ is not MF.

Case 7: $(a, b) \in \{(1, 1), (1, 2), (2, 1)\}$. Here we have $r=16$, respectively 26, 22. If $p \neq 7$, respectively $p \neq 7$, $p \neq 11$, the Weyl modules are irreducible and we may apply Lemma 2.4. For the primes $p=7, 7, 11$, respectively, an application of Lemma 2.13 shows that $V{\downarrow}A$ is not MF in the second and third cases. Now for the case $\lambda=\omega_1+\omega_2$ and $p=7$, we must argue more carefully. Here, one checks that the weights r, $r-2$, $r-4$, $r-6$ occur with multiplicities 1, 2, 1, 2 respectively. Since $r-6$ does not occur as a weight in (r), while $r-4$ does, by Lemma 2.9 we conclude that $V{\downarrow}A$ is not MF.

Case 8: $b=0$. Here we view G as a subgroup of B_3 via the 7-dimensional irreducible representation afforded by $L(\omega_1)$. Then we have that $A \subset G$ is the principal A_1-subgroup of B_3 and moreover the B_3-module $L_{B_3}(a\omega_1)$ remains irreducible upon restriction to G, and affords the module V. (See [Seitz 1987, Table 1].) Hence, we can use the B_3 analysis, which is given in Proposition 4.16, to conclude.

Case 9: $a=0$, $b \geq 4$. Here the T_A-weight $r-6$ is afforded by $\lambda-21$, $\lambda-12$ and $\lambda-03$, each of which has multiplicity 1 in $\Delta(\lambda)$ and so $n_3=3$. In particular, none of the listed weights affords a composition factor of $\Delta(\lambda)$, nor does $\lambda-11$. Now we separate into two cases. First suppose that $\lambda-22$ does not afford a composition factor of $\Delta(\lambda)$; then $n_4 \geq 5$ and $V{\downarrow}A$ is not MF.

Now suppose that $\nu=\lambda-22$ affords a composition factor of $\Delta(\lambda)$ and so by Lemma 3.18 we have $3b+2 \equiv 0 \bmod p$. We first treat the case where $b \geq 6$. We claim that $n_5=4$. The T_A-weight $r-10$ is afforded by $\lambda-23$, $\lambda-32$, $\lambda-14$, and $\lambda-05$. The first two occur in the composition factor afforded by ν, each with multiplicity 1, and using the multiplicities in the Weyl module and Proposition 2.1, we see that each of the four listed weights occurs with multiplicity 1 in V, establishing the claim. The T_A-weight $r-12$ is afforded by $\lambda-42$, $\lambda-33$, $\lambda-24$, $\lambda-15$ and $\lambda-06$. The second of these weights has multiplicity 4 in $\Delta(\lambda)$ and occurs with multiplicity 2 in $L(\nu)$. Moreover, this weight does not afford a composition factor of $\Delta(\lambda)$ (nor does any dominant weight $\eta \neq \nu$ with $\lambda - 33 \prec \eta \prec \lambda$) and so occurs with multiplicity 2 in V. Hence, $n_6 \geq 6$ and $V{\downarrow}A$ is not MF.

It remains to consider the cases $b = 4$ and $b = 5$ with $p = 7$, respectively $p = 17$ and $\dim V = 267$, respectively 546. In both cases, an application of Lemma 2.13 shows that $V{\downarrow}A$ is not MF.

Case 10: $a = 0$ and $1 \leq b \leq 3$. When $b = 1$, the Weyl module is irreducible and the result follows from Lemma 2.4. If $b = 2$ and $p \neq 7$, we may apply Lemma 2.4 to conclude. When $(b, p) = (2, 7)$, we use Lemma 2.10 and the proof of [Liebeck et al. 2015, Lemma 4.5] to deduce that $n_0 = 1, n_1 = 1, n_2 = 2, n_3 = 2, n_4 = 3, n_5 = 4$, so that $V{\downarrow}A$ has composition factors (20), (16) and (12). Since the T_A-weight 12 lies in the composition factor (16) but the T_A-weight 10 does not, Lemma 2.9 implies that $V{\downarrow}A$ is not MF.

Finally, we consider the case $b = 3$, where $r = 30$. Here the Weyl module is irreducible unless $p = 11$. If $p \neq 11$, the result follows from Lemma 2.4. If $p = 11$, we use [Lübeck 2018] to see that $n_0 = 1, n_2 = 1, n_2 = 2, n_3 = 3, n_4 = 3, n_5 = 3$. We then deduce that $V{\downarrow}A$ has no composition factor (22), nor (20). But then $\dim V = 148 > B(30) - \dim(20) - \dim(22)$, so that $V{\downarrow}A$ is not MF. □

4. The case where G has rank at least 3

We handle the case where G has rank at least 3, establishing the next proposition.

Proposition 4.1. *Suppose that G has rank at least 3 and $p \leq r$. Then $V{\downarrow}A$ is not MF.*

We assume throughout Section 4 that $p \leq r$. By Proposition 2.14(i) we only need to consider the case $\lambda = c_i\omega_i + c_j\omega_j$ (with c_i or c_j possibly 0), i.e., the weight λ has support on at most two nodes.

4.1. *The case c_ic_j not 0.*

We treat the case where $\lambda = c_i\omega_i + c_j\omega_j$ with $c_ic_j \neq 0$ in a sequence of lemmas.

Lemma 4.2. *Suppose that G has rank at least 4 and $\lambda = c_i\omega_i + c_j\omega_j$ with α_i and α_j adjacent and $c_i, c_j \geq 1$. Then $V{\downarrow}A$ is not MF.*

Proof. Since $p \geq h$ we have $p \geq 5, 11, 11, 7$ respectively for $G = A_\ell, B_\ell, C_\ell, D_\ell$ and $p \geq 13, 13, 19, 31$ respectively for $G = F_4, E_6, E_7, E_8$. By Proposition 2.14(iv) and (v), we can assume that $c_i = 1$ or $c_j = 1$, and α_i or α_j is an end-node. Recall that by Proposition 2.1, the set of weights of V is the same as the set of weights of $\Delta(\lambda)$. Using this, it is straightforward to see that if $c_i \geq 3$ or $c_j \geq 3$, then $V{\downarrow}A$ is not MF.

For example if $\lambda = c_1\omega_1 + \omega_2$ (so G is not of type E_ℓ) and $c_1 \geq 3$, the T_A-weight $r-6$ is afforded by $\lambda - 123$, $\lambda - 234$, $\lambda - 12^2$, $\lambda - 1^2 2$ and $\lambda - 1^3$. Therefore $n_3 \geq 5$, and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. Similarly, if $\lambda = \omega_1 + c_2\omega_2$, with $c_2 \geq 3$ (so again G is not of type E_ℓ), then the T_A-weight $r-6$ is afforded by $\lambda - 123$, $\lambda - 234$, $\lambda - 12^2$, $\lambda - 2^2 3$ and $\lambda - 2^3$. As before, $n_3 \geq 5$, and $V{\downarrow}A$ is not MF. If $c_i = 2$ or $c_j = 2$, by Lemma 2.18 we have $\dim V_{\lambda-\alpha_i-\alpha_j} > 1$, and by Proposition 2.14(vi) the module $V{\downarrow}A$ is not MF. Thus, we reduce to $c_i = c_j = 1$.

Consider the weight $\lambda = \omega_1 + \omega_2$. For G classical, the weights $\lambda - 123 = (\lambda-12)^{s_3}$, $\lambda-234$, $\lambda-1^2 2 = (\lambda-2)^{s_1}$, $\lambda-12^2 = (\lambda-1)^{s_2}$ occur with multiplicities 2, 1, 1, 1 respectively by Lemma 2.18. Therefore $n_3 \geq 5$ and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. The same argument, with the appropriate relabelling of indices, handles all remaining cases where $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$, including the cases $G = E_\ell$ and $\lambda \in \{\omega_1 + \omega_3, \omega_2 + \omega_4, \omega_{\ell-1} + \omega_\ell\}$. For the group of type F_4 and the weights $\omega_1 + \omega_2$ and $\omega_3 + \omega_4$, we use the weight space dimensions provided in [Lübeck 2018] to conclude again that $n_3 \geq 5$.

Therefore we reduce to $G = B_\ell$ or $G = C_\ell$ with $\lambda = \omega_{\ell-1} + \omega_\ell$. Suppose $G = B_\ell$. Since $p \geq h$, by Lemma 2.18 we have $\dim V_{\lambda-(\ell-1)\ell} = 2$. The T_A-weight $r-6$ is afforded by $\lambda - (\ell-1)\ell^2 = (\lambda - (\ell-1)\ell)^{s_\ell}$, $\lambda - (\ell-2)(\ell-1)\ell$ and $\lambda - (\ell-1)^2\ell$. If $\ell = 4$, the first two weight spaces have dimension 2 by [Lübeck 2018]. By Lemma 2.20, for any $\ell \geq 4$, we have $n_3 \geq 5$, and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. The C_ℓ case is handled similarly. □

Lemma 4.3. *Let $G = A_3$ and $\lambda = c\omega_1 + \omega_2$ or $\omega_1 + c\omega_2$, with $c > 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Proposition 2.14(vi) we can assume that the weight space $\lambda - 12$ is 1-dimensional. In particular we must have $c = p-2$ by Lemma 2.18. Let us start with $\lambda = \omega_1 + (p-2)\omega_2$. Since $p \geq 5$, the T_A-weight $r-6$ is afforded by $\lambda - 123$, $\lambda - 12^2$, $\lambda - 2^2 3$, $\lambda - 2^3$ and $\lambda - 1^2 2$. Therefore Lemma 2.6 implies that $V{\downarrow}A$ is not MF.

For the case $\lambda = (p-2)\omega_1 + \omega_2$, we will use a dimension argument. We refer to the discussion in [Jantzen 2003, Part II, 8.20], where the weight λ satisfies the conditions of the weight λ_2, with $s = 1 = t$ and $r = p-3$. Then one has

$$\dim V = \dim \Delta(\lambda) - \dim \Delta(\lambda - \alpha_1 - \alpha_2) + \dim \Delta(\lambda - 2\alpha_1 - 2\alpha_2 - \alpha_3).$$

Using the Weyl degree formula we find that $\dim V = \frac{(p-1)}{6}(p^2 + 7p + 18)$. We have $r = 3p-2$, and a simple calculation shows that $B(r) = \frac{(p+1)(3p-1)}{2}$. Since $B(r) < \dim V$ for all $p > 3$, by Lemma 2.12 we conclude that $V{\downarrow}A$ is not MF. □

Lemma 4.4. *Let $G = B_3$ or C_3 and let $\lambda \in \{c\omega_1 + \omega_2, \omega_1 + c\omega_2, c\omega_2 + \omega_3, \omega_2 + c\omega_3\}$, with $c > 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Proposition 2.14(vii) we can assume that the weight space $\lambda - ij$ is 1-dimensional, where $\lambda = c_i\omega_i + c_j\omega_j$. Note that $p \geq 7$ as $p \geq h$.

Case 1: $\lambda = c\omega_1 + \omega_2$. As the weight space $\lambda - ij$ is 1-dimensional, we have $c = p-2$ by Lemma 2.18. In particular $c \geq 5$. The T_A-weight $r-6$ is afforded by $\lambda - 1^3$, $\lambda - 1^2 2$, $\lambda - 12^2$, and $\lambda - 123$; in addition, for $G = B_3$, $r-6$ is afforded by $\lambda - 23^2$ and if $G = C_3$, by $\lambda - 2^2 3$. Hence $n_3 \geq 5$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 2: $\lambda = \omega_1 + c\omega_2$. As in the previous case, we reduce to $c = p-2$, so $c \geq 5$. The T_A-weight $r-6$ is afforded by $\lambda - 12^2$, $\lambda - 1^2 2$, $\lambda - 2^2 3$, $\lambda - 123$, and $\lambda - 2^3$. Therefore $n_3 \geq 5$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 3: $G = B_3$ and $\lambda = c\omega_2 + \omega_3$. The T_A-weight $r-8$ is afforded by $\lambda - 12^2 3$, $\lambda - 1^2 2^2$, $\lambda - 2^3 3$, $\lambda - 23^3$, $\lambda - 123^2$, $\lambda - 2^2 3^2$ which implies $n_4 \geq 6$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 4: $G = C_3$ and $\lambda = c\omega_2 + \omega_3$. By Lemma 2.18 we may assume that $c + 4 \equiv 0 \bmod p$, implying $c \geq 3$. The T_A-weight $r-6$ is afforded by $\lambda - 2^3$, $\lambda - 12^2$, $\lambda - 2^2 3$, $\lambda - 123$, $\lambda - 23^2$, which implies $n_3 \geq 5$ and $V{\downarrow}A$ is not MF by Lemma 2.6.

Case 5: $G = B_3$ and $\lambda = \omega_2 + c\omega_3$. As above we reduce to $c + 4 \equiv 0 \bmod p$ and so $c \geq 3$. The T_A-weight $r-6$ is afforded by $\lambda - 123$, $\lambda - 2^2 3$, $\lambda - 3^3$ and $\lambda - 23^2$. By Lemma 3.6, the Weyl module $\Delta_{B_2}(1c)$ has exactly two composition factors $L_{B_2}(1c)$ and $L_{B_2}(0c)$, the latter afforded by $\lambda - 11$. Therefore the multiplicity of the weight $\lambda - 23^2$ in V is 2 and so $n_3 \geq 5$, which by Lemma 2.6 implies that $V{\downarrow}A$ is not MF.

Case 6: $G = C_3$ and $\lambda = \omega_2 + c\omega_3$. This is entirely similar. Here we may assume $2c + 3 \equiv 0 \bmod p$. If $c \geq 4$, the T_A-weight $r-8$ is afforded by $\lambda - 12^2 3$, $\lambda - 2^3 3$, $\lambda - 23^3$, $\lambda - 2^2 3^2$, $\lambda - 123^2$, and $\lambda - 3^4$. Therefore $n_4 \geq 6$, and $V{\downarrow}A$ is not MF by Lemma 2.6. If $c \leq 3$, we must have $c = 2$ and $p = 7$. By [Lübeck 2018], all weight spaces of V are 1-dimensional. The T_A-weight $r-8$ is again afforded by the first five weights listed above. On the other hand, the T_A-weight $r-6$ is afforded precisely by $\lambda - 123$, $\lambda - 23^2$ and $\lambda - 2^2 3$, implying that $n_3 = 3$. Therefore $n_4 - n_3 \geq 2$, and by Lemma 2.9 we conclude that $V{\downarrow}A$ is not MF. □

Lemma 4.5. *Let $G = A_3$, B_3 or C_3 and $\lambda = \omega_1 + \omega_2$ or $\omega_2 + \omega_3$. Then $V{\downarrow}A$ is not MF.*

Proof. Consider $G = A_3$. Then $r = 7$ and $p = 5$ or $p = 7$ as $r \geq p \geq h$. In both cases the Weyl module is irreducible and the conditions for Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF.

Now consider $G = B_3$. Since $p \geq 7$, the Weyl module is irreducible. The conditions for Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF.

Finally consider $G = C_3$. If $(\lambda, p) \neq (\omega_1 + \omega_2, 7)$ we can conclude as for B_3. Therefore assume that $\lambda = \omega_1 + \omega_2$ with $p = 7$. By Lemma 2.18 we have $\dim V_{\lambda-123} = 2$, and it is straightforward to see that $n_3 \geq 5$, which by Lemma 2.6 implies that $V{\downarrow}A$ is not MF. □

Lemma 4.6. *Assume that G has rank at least 3 and $\lambda = \omega_i + \omega_j$ where α_i and α_j are end-nodes. Then $V{\downarrow}A$ is not MF.*

Proof. Consider first the case where $G = A_\ell$, where $p \geq h = \ell + 1$. By [Lübeck 2001] the Weyl module $\Delta(\lambda)$ is irreducible if and only if $p \neq \ell+1$. If $p \neq \ell+1$, the conditions of Lemma 2.4 are satisfied and therefore $V{\downarrow}A$ is not MF. We therefore reduce to the case $p = \ell+1$, where V is isomorphic to the quotient of $\Delta(\lambda)$ by a 1-dimensional trivial submodule. For $d < \ell$, it is straightforward to see that $n_d = d+1$ (where we use that $r = 2\ell = 2(p-1)$). Therefore by Lemma 2.9(i) we know that $(p+1)$ is a composition factor of $V{\downarrow}A$. Now the T_A-weight $p-3$ occurs with multiplicity one more than the T_A-weight $p-1$, and it does not occur as a weight in $(p+1)$, while $p-1$ does. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF.

If $G = B_\ell$ or C_ℓ and $\ell \geq 4$, the first paragraph of the proof of [Liebeck et al. 2015, Lemma 3.5] shows that $n_3 \geq 5$, so $V{\downarrow}A$ is not MF by Lemma 2.6. If $G = C_3$, we can apply Lemma 2.4 to conclude that $V{\downarrow}A$ is not MF.

Now assume $G = B_3$. We have $p = 7$ or $p = 11$, as $r = 12$. If $p = 11$, the Weyl module is irreducible by [Lübeck 2001], and the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. When $p = 7$, using [Lübeck 2018], we find that $n_2 = 3$ and $(r-4)$ is therefore a composition factor by Lemma 2.9(i). Furthermore, we have $n_3 = 3$, $n_4 = 4$ and the T_A-weight $r-8$ does not occur as a weight in $(r-4)$, while the T_A-weight $r-6$ does. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF.

Now consider $G = D_\ell$, with $\ell \geq 4$. If $\lambda = \omega_1 + \omega_{\ell-1}$, the T_A-weight $r-2(\ell-1)$ is afforded by $\lambda - 1 \cdots (\ell-1)$, $\lambda - 2 \cdots \ell$, and $\lambda - 1 \cdots (\ell-2)\ell$. Since $p \geq h$ we have $p > \ell$, and therefore by Lemmas 2.19 and 2.20 we have $\dim V_{\lambda-1\cdots(\ell-1)} = \ell - 1$. Therefore $n_{\ell-1} \geq \ell + 1$ and Lemma 2.6 implies that $V{\downarrow}A$ is not MF. It is also easy to see that if $\lambda = \omega_{\ell-1} + \omega_\ell$, we have $n_3 \geq 5$. Again Lemma 2.6 implies that $V{\downarrow}A$ is not MF.

Finally, if G is exceptional, the arguments used in the proof of [Liebeck et al. 2015, Lemma 3.6] in characteristic zero allow us to conclude, as [Lübeck 2018] shows that the relevant weight spaces in V have the same dimension as the corresponding weight spaces in $\Delta_K(\lambda)$. □

Proposition 4.7. *Suppose that G has rank at least 3 and $\lambda = c_i\omega_i + c_j\omega_j$ with $c_i, c_j \geq 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 4.2, if α_i and α_j are adjacent and G has rank at least 4, the module $V{\downarrow}A$ is not MF. If α_i and α_j are adjacent and G has rank 3, Lemmas 4.3–4.5 combine to imply that $V{\downarrow}A$ is not MF.

Now assume that α_i and α_j are not adjacent, in which case by Proposition 2.14 we can assume that $c_i = c_j = 1$ and α_i and α_j are both end-nodes. In this case, by Lemma 4.6 we conclude that $V{\downarrow}A$ is not MF. □

4.2. ***The case where $\lambda = b\omega_i$.*** We now consider the case $\lambda = b\omega_i$. Note that if G is classical, then $\lambda \neq \omega_1$, as we are assuming that $p \leq r$, and necessarily $p \geq h$.

Lemma 4.8. *Assume that $G = A_\ell, B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$. Let $\lambda = b\omega_1$, with $b \geq 2$. Then $V{\downarrow}A$ is not MF.*

Proof. We first consider the case $b = 2$ and start by assuming that $(G, p) \neq (B_\ell, 2\ell+1)$. By [Lübeck 2001] and since $p \geq h$, the Weyl module is irreducible. A simple check shows that the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. Consider now the case $G = B_\ell$ and $p = 2\ell+1$, where V is isomorphic to the quotient of $\Delta(\lambda)$ by a 1-dimensional trivial submodule. For all strictly positive weights $r-2d$, we have $n_d = \dim(\Delta_K(\lambda){\downarrow}A_K)_{r-2d}$. By [Liebeck et al. 2015, Lemma 4.2], we have $\Delta_K(\lambda){\downarrow}A_K = (4\ell) + (4\ell-4) + \cdots$, which implies $n_d = d+1$ for d even with $d < 2\ell$, and $n_{d+1} = n_d$ for d odd with $d+1 < 2\ell$. By Lemma 2.9(i), for all $0 \leq d < 2\ell$ we have that $(r-2d)$ is a composition factor of $V{\downarrow}A$. In particular, either $(p+1)$ or $(p+3)$ is a composition factor of $V{\downarrow}A$. In the first case, the T_A-weight $p-3$ occurs with multiplicity one more than the T_A-weight $p-1$, but does not occur as a weight in $(p+1)$. Therefore Lemma 2.9(iv) implies that $V{\downarrow}A$ is not MF. Similarly, if $(p+3)$ is a composition factor of $V{\downarrow}A$, then the T_A-weight $p-5$ (note that $p > 5$ since $\ell \geq 3$) occurs with multiplicity one more than the T_A-weight $p-3$, but does not occur as a weight in $(p+3)$, concluding in the same way.

Now consider the case $b \geq 3$. Start with $G = A_\ell$. Here $V = \Delta(\lambda)$ by [Seitz 1987, 1.14]. If $\ell \geq 6$, the first paragraph of the proof of [Liebeck et al. 2015, Lemma 4.4] shows that $n_6 \geq 7$, which by Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF. If $b = 4$ and $\ell = 4$ or $\ell = 5$, we similarly have $n_4 \geq 5$. If $b \geq 5$ and $(b, \ell) \neq (5, 3)$, we have $n_6 - n_5 \geq 2$. Therefore by Lemma 2.9(iii) and (ii), we reduce down to the cases $(b, \ell) = (4, 3), (5, 3), (3, 3), (3, 4), (3, 5)$. For these cases we can conclude using Lemma 2.4, unless $\ell = b = 3$ and $p = 5$. In this case $r = 9$ and the weights 9, 7, 5, 3 occur respectively with multiplicities 1, 1, 2, 3. By Lemma 2.9(i), we have that

(5) is a composition factor, and in addition $r-6$ does not occur as weight in this composition factor. Therefore by Lemma 2.9(iv) the module $V{\downarrow}A$ is not MF.

The C_ℓ case follows from the $A_{2\ell-1}$ case since $A < C_\ell < A_{2\ell-1}$ is a principal A_1-subgroup of $A_{2\ell-1}$ and $V = S^b(L_{C_\ell}(\omega_1)) = L_{A_{2\ell-1}}(b\omega_1){\downarrow}C_\ell$. Now consider $G = B_\ell$. If $b \geq 4$, it is straightforward to check that $n_4 \geq 5$, which by Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF. If $b = 3$ and $\ell \geq 4$, by the proof of [Liebeck et al. 2015, Lemma 4.4], we have $n_6 \geq 7$, concluding in the same way. If $\ell = b = 3$ we have $p \leq r = 18$ and by Lemma 2.4 the module $V{\downarrow}A$ is not MF. Finally, we consider the case where $G = D_\ell$ where we have $A \leq B_{\ell-1} < G$. Since $\Delta_{B_{\ell-1}}(b\omega_1)$ is a composition factor of $\Delta_{D_\ell}(b\omega_1)$, if $\Delta_{D_\ell}(b\omega_1){\downarrow}A$ is MF, so is $\Delta_{B_{\ell-1}}(b\omega_1)$. Therefore by the B_ℓ result, we conclude that $V{\downarrow}A$ is not MF. □

Lemma 4.9. *Assume that $G = B_\ell$, C_ℓ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$. Let $\lambda = b\omega_i$, with $i > 1$ and $b > 1$. Then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 2.16 we can assume that $\lambda = b\omega_\ell$. We will treat the case $G = D_\ell$ at the end of the proof.

Assume for now that $b \geq 3$. If $G = C_\ell$, the T_A-weight $r-6$ is afforded by $\lambda - \ell^3$, $\lambda - (\ell-2)(\ell-1)\ell$, $\lambda - (\ell-1)^2\ell$, $\lambda - (\ell-1)\ell^2$. If $G = B_\ell$, the T_A-weight $r-6$ is afforded by $\lambda - (\ell-1)\ell^2$, $\lambda - (\ell-2)(\ell-1)\ell$ and $\lambda - \ell^3$, and using the fact that the B_2-module $\Delta(b\omega_2)$ is irreducible, by Lemma 2.20, we have that the first of these weights has multiplicity 2. Hence for both of the groups C_ℓ and B_ℓ, we have $n_3 \geq 4$. By Lemma 2.9(iii), the module $V{\downarrow}A$ is not MF.

We now consider the case $b = 2$ when $G = C_\ell$ and first assume that $\ell \geq 5$. The T_A-weight $r-10$ is afforded by $\lambda - (\ell-1)^3\ell^2$, $\lambda - (\ell-2)(\ell-1)^2\ell^2$, $\lambda - (\ell-1)^2\ell^3$, $\lambda - (\ell-2)^2(\ell-1)^2\ell$, $\lambda - (\ell-3)(\ell-2)(\ell-1)\ell^2$ and $\lambda - (\ell-4)(\ell-3)(\ell-2)(\ell-1)\ell$. Again, by Lemma 2.9(iii) the module $V{\downarrow}A$ is not MF.

Now consider the cases C_ℓ, for $\lambda = 2\omega_\ell$ and $\ell = 3, 4$ where $r = 18$, respectively 32, and $p \geq 7$, respectively 11. In both cases we have that $\Delta(2\omega_\ell)$ is irreducible by [Lübeck 2001]. For $\ell = 3$, the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. If $\ell = 4$, by the first paragraph of [Liebeck et al. 2015, Lemma 4.3] we have $\dim(\Delta_K(\lambda){\downarrow}A_K)_{r-12} \geq \dim(\Delta_K(\lambda){\downarrow}A_K)_{r-10}+2$. Therefore by Lemma 2.10 we find that $n_6 - n_5 \geq 2$, concluding by Lemma 2.9(ii).

Turn now to the case $G = B_\ell$ and $b = 2$. Here the T_A-weight $r-8$ is afforded by $\lambda - (\ell-2)(\ell-1)\ell^2$, $\lambda - (\ell-1)^2\ell^2$, $\lambda - (\ell-1)\ell^3$ and if $\ell \geq 4$, $\lambda - (\ell-3)(\ell-2)(\ell-1)\ell$. The first of these is conjugate to $\lambda - (\ell-1)\ell^2$ and so has multiplicity 2 by the first paragraph of this proof. Thus, if $\ell \geq 4$, $V{\downarrow}A$ is not MF by Lemma 2.9(iii). So finally, we reduce to $b = 2$ and $\ell = 3$, where $p \geq 7$ and $r = 12$. The Weyl module is irreducible by [Lübeck 2001]. The conditions of Lemma 2.4 are satisfied, so $V{\downarrow}A$ is not MF.

Finally suppose that $G = D_\ell$ and $\lambda = b\omega_\ell$. Since $A \leq B_{\ell-1} \leq D_\ell$ and $V{\downarrow}B_{\ell-1} \cong L_{B_{\ell-1}}(b\omega_{\ell-1})$, we may use the $B_{\ell-1}$ result to conclude. □

Lemma 4.10. *If $G = E_\ell$ and $\lambda = b\omega_i$ with $b > 1$, then $V{\downarrow}A$ is not MF.*

Proof. This follows verbatim from the proof of [Liebeck et al. 2015, Lemma 4.6], unless $i = \ell$ and $G = E_7$ or E_8 with $b = 2$ or $b = 3$. In these remaining cases, it is not difficult to check that we have $n_6 \geq n_5 + 2$ (as stated in the proof of [Liebeck et al. 2015, Lemma 4.6]), as this count relies on 1-dimensional weight spaces. By Proposition 2.14 the module $V{\downarrow}A$ is not MF. □

Lemma 4.11. *If $G = F_4$ and $\lambda = b\omega_i$ with $b > 1$, then $V{\downarrow}A$ is not MF.*

Proof. By Lemma 2.16 the simple root α_i corresponds to an end-node of the Dynkin diagram. If $i = 1$ we can conclude as in the first paragraph of the proof of [Liebeck et al. 2015, Lemma 4.7].

Assume $i = 4$. If $b \geq 3$, like in [Liebeck et al. 2015, Lemma 4.7] we have $n_4 \geq 5$, concluding by Lemma 2.9(iii). If $b = 2$ we have $V = \Delta(\lambda)$ by [Lübeck 2001], and since $r = 32$ and $r \geq p > 11$, the conditions of Lemma 2.4 are satisfied. Thus, $V{\downarrow}A$ is not MF. □

It remains to consider the case $\lambda = \omega_i$. Recall that for G classical, we have $\lambda \neq \omega_1$.

Lemma 4.12. *Assume that G has rank at least 3, $\lambda = \omega_i$ and that one of the following holds*:

(i) *$G = A_\ell, B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$, and $4 \leq i \leq \ell - 3$.*

(ii) *$G = A_\ell$, $i = 3$, and $\ell \geq 5$.*

(iii) *$G = A_\ell, B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$ and $i = 2$.*

Then $V{\downarrow}A$ is not MF.

Proof. (i) Lemma 2.15 applies, except when $G = D_7$, and implies that the module $V{\downarrow}A$ is not MF. For the case $G = D_7$, where $i = 4$, it is straightforward to see that $n_4 \geq 5$ and then Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF.

(ii) Here $V = \bigwedge^3(L(\omega_1))$. Assume for now that $l \geq 8$. The T_A-weight $r - 12$ is afforded by the wedge of weight vectors in $L(\omega_1)$ for each of the following triples of T_A-weights: $\ell(\ell-2)(\ell-16)$, $\ell(\ell-4)(\ell-14)$, $\ell(\ell-6)(\ell-12)$, $\ell(\ell-8)(\ell-10)$, $(\ell-2)(\ell-4)(\ell-12)$, $(\ell-2)(\ell-6)(\ell-10)$, $(\ell-4)(\ell-6)(\ell-8)$. Therefore $n_6 \geq 7$, and Lemma 2.9(iii) implies that $V{\downarrow}A$ is not MF.

For the remaining cases, when $5 \leq l \leq 7$, we have $\Delta(\lambda) = V$ and a quick check shows that the conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF.

(iii) Here $\lambda = \omega_2$, and as $p > \ell$ we have $V = \Delta(\lambda)$ [Lübeck 2001, Table 2]. We have $r = 2\ell-2, 4\ell-2, 4\ell-4$ or $r = 4\ell-6$ according to whether $G = A_\ell, B_\ell, C_\ell$ or $G = D_\ell$. Furthermore we have p greater than $\ell, 2\ell-1, 2\ell, 2\ell-2$ respectively. It is then an easy check to see that the conditions of Lemma 2.4 are satisfied, implying that $V\downarrow A$ is not MF. □

Lemma 4.13. *Assume that $G = B_\ell, C_\ell$ with $\ell \geq 3$ or $G = D_\ell$ with $\ell \geq 4$, and that $\lambda = \omega_i$ for $i \geq 3$ and V is not a spin module for B_ℓ or D_ℓ. Then $V\downarrow A$ is not MF.*

Proof. If $G = B_\ell$ or D_ℓ, then $V = \bigwedge^i(\omega_1)$ by [Seitz 1987] and the result follows from Lemma 4.12(i)(ii) for $G = A_{2\ell}$ or $A_{2\ell-1}$. Indeed, if $G = B_\ell$, then A is regular in $A_{2\ell}$ and $V = L_{A_{2\ell}}(\omega_i)\downarrow G$, while if $G = D_\ell$, then $A < B_{\ell-1} < D_\ell$ and by the B_ℓ case there is a $B_{\ell-1}$-composition factor of V (namely $L_{B_{\ell-1}}(\omega_i)$) that is not multiplicity-free in its restriction to A, implying that $V\downarrow A$ is not MF.

We now consider the case $G = C_\ell$. By part (i) of Lemma 4.12 we can furthermore assume that $i = 3$ or $i > \ell-3$. If $i = \ell-2 > 3$, the T_A-weight $r-8$ has multiplicity at least 5 as it is afforded by five different weights as in the proof of [Liebeck et al. 2015, Lemma 5.3]. Therefore Lemma 2.9(iii) implies that $V\downarrow A$ is not MF.

Assume $i = \ell-1 > 3$. Because $\ell \geq 5$, the T_A-weight $r-12$ is afforded by $\lambda-(\ell-3)(\ell-2)(\ell-1)^3\ell$, $\lambda-(\ell-2)(\ell-1)^3\ell^2$, $\lambda-(\ell-2)^2(\ell-1)^3\ell$, $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)^2(\ell-1)^2\ell$. When $\ell = 5$, the last two weights have multiplicity 2 in V by [Lübeck 2018], and therefore the same holds for $\ell \geq 5$ by Lemma 2.20. Thus, $n_6 \geq 7$, and by Lemma 2.9(iii) the module $V\downarrow A$ is not MF.

Now assume $i = \ell > 3$. Start with $\ell = 4$ or 5. In both cases the Weyl module is irreducible. We have $r = 16$ if $\ell = 4$, and $r = 25$ if $\ell = 5$. If $(\ell, p) \neq (5, 13)$, the conditions of Lemma 2.4 are satisfied, showing that $V\downarrow A$ is not MF. In the remaining case (when $(\ell, p) = (5, 13)$), we find that $B(r) - \dim(r-2) = 118 < \dim V = 132$. Therefore by Lemma 2.13, the module $V\downarrow A$ is not MF.

Now suppose $\ell \geq 6$ with $\lambda = \omega_\ell$. Here the T_A-weight $r-10$ has multiplicity 4 as it is afforded precisely by $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)\ell$, $\lambda-(\ell-2)^2(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)(\ell-1)^2\ell$, $\lambda-(\ell-2)(\ell-1)^2\ell^2$. The T_A-weight $r-12$ has multiplicity at least 6 as it is afforded by $\lambda-(\ell-5)(\ell-4)(\ell-3)(\ell-2)(\ell-1)\ell$, $\lambda-(\ell-2)^2(\ell-1)^2\ell^2$, $\lambda-(\ell-4)(\ell-3)(\ell-2)(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)^2(\ell-1)^2\ell$, $\lambda-(\ell-3)(\ell-2)(\ell-1)^2\ell^2$, $\lambda-(\ell-2)(\ell-1)^3\ell^2$. By Lemma 2.9(ii) $V\downarrow A$ is not MF.

Finally, assume $i = 3$. If $\ell \geq 6$, we have $n_6 \geq 7$, since by the last paragraph of the proof of [Liebeck et al. 2015, Lemma 5.3] there are seven distinct weights of V affording the T_A-weight $r-12$. Therefore $V\downarrow A$ is not MF by Lemma 2.9(iii).

In the remaining cases, when $\ell \in \{3, 4, 5\}$, the Weyl module is irreducible and we can apply Lemma 2.4, unless $\ell = 5$ and $p = 11$, in which case $r = 21 \equiv -1 \bmod p$. In this case, we find that $B(r) - \dim(r-2) = 84 < \dim V = 110$. Therefore by Lemma 2.13, the module $V{\downarrow}A$ is not MF. □

Lemma 4.14. *Assume that V is a spin module for B_ℓ with $\ell \geq 3$ or D_ℓ with $\ell \geq 4$. Then $V{\downarrow}A$ is not MF.*

Proof. We have $V = \Delta(\lambda)$. If $G = D_\ell$, then $A \leq B_{\ell-1} < G$ and $V{\downarrow}B_{\ell-1}$ is the spin module for $B_{\ell-1}$; therefore it suffices to prove the lemma for $G = B_\ell$, where $r = \ell(\ell+1)/2$ and $\dim V = 2^\ell$. If $V{\downarrow}A$ is MF the dimension of V is at most $B_K(r)$, by Lemma 2.12. This implies that if $\ell \geq 10$, the module $V{\downarrow}A$ is not MF.

Now assume $\ell \leq 9$. Since $p > h = 2\ell$ we know that $p \nmid r$. Therefore if $V{\downarrow}A$ is MF the dimension of V is at most $B(r) - \dim(r-2)$, by Lemma 2.13. This then reduces our considerations to the pairs (n, p) in the list (5, 11), (5, 13), (6, 13), (6, 17), (6, 19), (7, 17), (7, 19), (7, 23), (8, 31). For every $3 \leq \ell \leq 8$, by Lemma 2.10 we can read the dimension of the T_A-weight space $r - 2k$ off the table in the proof of [Liebeck et al. 2015, Lemma 5.4]. In each case we apply part (iii) of Lemma 2.9 to find that $V{\downarrow}A$ is not MF. The first repeated composition factors are of highest weight respectively 5, 9, 9, 11, 15, 14, 14, 16, 24. □

Lemma 4.15. *Assume $G = E_\ell$ or F_4 and $\lambda = \omega_i$. Then $V{\downarrow}A$ is not MF.*

Proof. If $G = E_\ell$ and $\lambda = \omega_4$, the T_A-weight $r - 4$ is afforded by $\lambda - 34$, $\lambda - 24$, $\lambda - 45$. Therefore $n_2 \geq 3$, and by Lemma 2.9(iii), the module $V{\downarrow}A$ is not MF.

If $G = E_8$ and $\lambda = \omega_3$ or ω_6, then $r = 182$ respectively $r = 168$, giving $B_K(r) =$ 8464 and 7225 respectively. By [Lübeck 2001], we have $\dim V > B_K(r)$ and therefore by Lemma 2.12 the module $V{\downarrow}A$ is not MF. If $G = E_8$ and $\lambda = \omega_5$, by Lemma 2.15 the module $V{\downarrow}A$ is not MF.

In all remaining cases, by [Lübeck 2001] we have that $V = \Delta(\lambda)$. Lemma 2.12 then allows us to reduce to the case where V is the minimal module for G, or the adjoint module for E_6, E_7 or F_4. The conditions of Lemma 2.4 are satisfied, implying that $V{\downarrow}A$ is not MF. □

Proposition 4.16. *Suppose that G has rank at least 3 and $\lambda = b\omega_i$, with $b \geq 2$ for G classical and $b \geq 1$ for G of exceptional type. Then $V{\downarrow}A$ is not MF.*

Proof. If G is classical, this is a direct consequence of Lemmas 4.8 and 4.9. If G is exceptional and $b \geq 2$, we similarly conclude by Lemmas 4.10 and 4.11.

If $b = 1$, where G is exceptional, we reach the same conclusion by Lemmas 4.12, 4.13, 4.14, 4.15. □

Proof of Proposition 4.1. If V is MF, then by Proposition 2.14, the weight λ is of the form $c_i\omega_i + c_j\omega_j$. If $c_ic_j \neq 0$, then the conclusion follows from Proposition 4.7. If $c_ic_j = 0$, then the conclusion follows from Proposition 4.16. □

5. Proof of Corollary 2

We prove Corollary 2, thereby extending Theorem 1 to the case where λ is not p-restricted. The following lemma serves as an inductive tool.

Lemma 5.1. *Let $\lambda = \sum_{i=0}^{t} p^i\lambda_i$ where λ_i is a p-restricted dominant weight for all $0 \le i \le t$. Assume that for some s with $0 \le s < t$, we have $\left(\sum_{i=0}^{s} p^i\lambda_i\right)\downarrow T_A < p^{s+1}$. Then $V\downarrow A$ is MF if and only if*

(i) *$L\left(\sum_{i=0}^{s} p^i\lambda_i\right)\downarrow A$ is MF, and*

(ii) *$L\left(\sum_{i=s+1}^{t} p^i\lambda_i\right)\downarrow A$ is MF.*

Proof. Let $V_1 = L\left(\sum_{i=0}^{s} p^i\lambda_i\right)$ and $V_2 = L\left(\sum_{i=s+1}^{t} p^i\lambda_i\right)$, so that $V = V_1 \otimes V_2$. If $V_2 = L(0)$, the statement is trivial. Thus, assume $V_2 \neq L(0)$.

One direction is clear. If either $V_1\downarrow A$ or $V_2\downarrow A$ is not MF, then $V\downarrow A$ is not MF. Assume now that both $V_1\downarrow A$ and $V_2\downarrow A$ are MF, and let $V_1\downarrow A$ have composition factors $(r_0), (r_1), \ldots, (r_m)$, so that by the assumption on s, we have $p^{s+1} > r_0 > r_1 > \cdots > r_m$. Similarly let $V_2\downarrow A$ have composition factors $(v_0), (v_1), \ldots, (v_n)$ where $v_0 > v_1 > \cdots > v_n \ge p^{s+1}$. Then for all $0 \le i \le m$ and $0 \le j \le n$ we have

$$(r_i) \otimes (v_j) \cong (r_i + v_j),$$

since $r_i < p^{s+1}$ and $v_j \ge p^{s+1}$. This implies that the composition factors of $V_1 \otimes V_2$ are precisely of the form $(r_i + v_j)$, which are all clearly distinct. Therefore $V\downarrow A$ is MF. □

Let us restate, and prove, Corollary 2.

Corollary 5.2. *Let $\lambda = \sum_{i=0}^{t} p^i\lambda_i$ where each λ_i is a p-restricted dominant weight and set $r_i = \lambda_i\downarrow T_A$, for $0 \le i \le t$. Then $V\downarrow A$ is MF if and only if one of the following holds*:

(i) *We have $p > r_i$ and $\Delta_K(\lambda_i)\downarrow A_K$ is MF for all $0 \le i \le t$.*

(ii) *The group G is of type A_2, $p = 3$ and there exists $0 \le i \le t$ such that $\lambda_i = \omega_1 + \omega_2$. For all $0 \le j \le t$ we have $\lambda_j \in \{0, \omega_1 + \omega_2, \omega_1, \omega_2\}$ and if $\lambda_j = \omega_1 + \omega_2$ for some $0 \le j \le t - 1$, then $\lambda_{j+1} = 0$.*

(iii) *The group G is of type B_2, $p = 5$ and there exists $0 \le i \le t$ such that $\lambda_i = 2\omega_1$. For all $0 \le j \le t$ we have $\lambda_j \in \{0, 2\omega_1, \omega_1, \omega_2\}$ and if $\lambda_j = 2\omega_1$ for some $0 \le j \le t - 1$, then $\lambda_{j+1} \in \{0, \omega_2\}$.*

Proof. We use induction on t. If $t = 0$, then λ is p-restricted and the statement follows from Theorem 1. Suppose now that $t > 0$ and that the statement is valid for all $0 \leq t \leq N$ for some $N \in \mathbb{N}$. Let $t = N+1$ and $V_1 = L(\lambda_0)$, $V_2 = L\big(\sum_{i=1}^{t} p^i \lambda_i\big)$. If V_1 or V_2, is the trivial $\Bbbk G$-module, then we can conclude by the inductive assumption (since the Frobenius twist of a module M is MF if and only if the module M is MF). Therefore we can assume that V_1 and V_2 are nontrivial.

Suppose first that $V{\downarrow}A$ is MF. Then certainly $V_1{\downarrow}A$ and $V_2{\downarrow}A$ are both MF. If $r_0 < p$, by Lemma 5.1 and the inductive assumption, we conclude that (G, λ, p) is as in one of the three conclusions of the statement. Therefore assume that $r_0 \geq p$. By Theorem 1 we have $G = A_2$, $p = 3$ and $\lambda_0 = 11$, or $G = B_2$, $p = 5$ and $\lambda_0 = 20$.

Consider first the case $G = A_2$. By Theorem 1 and Table 1, we must have $\lambda_i \in \{0, 11, 10, 01\}$ for all $0 \leq i \leq t$. If $\lambda_1 = 0$, we conclude by the inductive assumption combined with Lemma 5.1 for $s = 1$. If $\lambda_1 \in \{\omega_1 + \omega_2, \omega_1, \omega_2\}$ then $V_1{\downarrow}A \otimes L(p\lambda_1){\downarrow}A$ has (4) as a repeated composition factor and so $V{\downarrow}A$ is not MF. For $G = B_2$, by Theorem 1 and Table 1, we have $\lambda_1 \in \{0, \omega_1, 2\omega_1, \omega_2\}$ and a straightforward computation shows that $V_1{\downarrow}A \otimes L(p\lambda_1){\downarrow}A$ has a repeated composition factor for $\lambda_1 \in \{\omega_1, 2\omega_1\}$.

Suppose now that (i) holds. Then $V{\downarrow}A$ is MF by the inductive assumption combined with Lemma 5.1.

If (ii) or (iii) holds, it is easy to verify that the conditions of Lemma 5.1 with $s = 1$ are satisfied, concluding again by the inductive assumption. □

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Gary Seitz