Pacific Journal of Mathematics

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Volume 337 No. 1 July 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 337, No. 1, 2025
https://doi.org/10.2140/pjm.2025.337.25

ON GAMMA FACTORS OF GENERIC REPRESENTATIONS OF $U_{2n+1} \times \operatorname{Res}_{E/F} GL_r$

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We develop fundamental properties of the Rankin–Selberg gamma factors for generic representations of $U_{2n+1} \times \operatorname{Res}_{E/F} GL_r$ over local fields under a natural assumption on the Archimedean local integrals. In particular, the gamma factors defined by the Rankin–Selberg method are shown to coincide with those defined by the Langlands–Shahidi method.

1. Introduction

Let F be a local field of characteristic zero, E be an étale F-algebra of rank 2 and ψ be a nontrivial additive character of F. Let U_{2n+1} ($n \geq 0$) be the quasisplit unitary group of $2n+1$ variables over F and $G_r = \operatorname{Res}_{E/F} GL_r$ (with $r \geq 1$) be the Weil restriction of GL_r from E to F. Let π and τ be irreducible admissible generic (complex) representations of $U_{2n+1}(F)$ and $G_r(F)$, respectively. Then one can define the gamma factor

$$\gamma(s, \pi \times \tau, \psi)$$

through the local Rankin–Selberg integrals (see Section 3) studied in [4; 28]. The aim of this article is to develop fundamental properties of $\gamma(s, \pi \times \tau, \psi)$ under a natural assumption (see Assumption 3.2) on the Archimedean local integrals. The properties that we develop in this article are some of the "ten commandments" given in [25]. In particular, our main result can be stated as follows (see Theorem 4.1 for the precise formulation):

Theorem A. *The gamma factor $\gamma(s, \pi \times \tau, \psi)$ satisfies a collection of fundamental properties that uniquely characterize it. These properties include behavior in unramified cases, dependence on ψ, multiplicativity, behavior in minimal cases, the Archimedean property and global identity.*

To the best of our knowledge, the definition of gamma factors via local Rankin–Selberg integrals first appeared in Zhang's work (see [41, Proposition 1.1]). The definition presented in this article differs slightly from Zhang's. While it is standard

MSC2020: primary 11F70; secondary 22E50.
Keywords: Rankin–Selberg integrals, gamma factors, unitary groups.

that gamma factors emerge as proportionality factors in local integrals, identifying the "correct" gamma factors from these proportionality factors is nontrivial. A key subtlety lies in determining the appropriate normalization of certain intertwining operators between induced representations. In this article, we provide a suitable normalization to ensure that the gamma factor satisfies the expected properties.

Some properties of the gamma factor, such as its behavior in the unramified cases and the global identity, are available in the literature. However, a key property — the multiplicativity of the gamma factor — is notably absent. A significant portion of this article is devoted to establishing this property.

In addition to the Rankin–Selberg method, the gamma factor attached to $\pi \times \tau$ and ψ can also be defined using the Langlands–Shahidi method (see [29; 32]). For applications, it is important to know whether these two definitions coincide, or at least differ only by exponential factors. This alignment is useful, for instance, in computing the Rankin–Selberg integrals attached to newforms of generic representations of U_{2n+1} [11] and in proving the irreducibility of certain local descents [37]. As a consequence of Theorem A, we obtain the following corollary:

Corollary A. *The gamma factor $\gamma(s, \pi \times \tau, \psi)$ coincides with the one defined by Shahidi in* [32] *for $\pi \times \tau$ and ψ.*

Remark 1.1. In [2; 19], it is shown that, for split classical groups, the gamma factors defined by the Langlands–Shahidi method coincide with those defined by the local Langlands correspondence. Similar arguments can be adapted to the case of quasisplit unitary groups, and together with Corollary A, we see that the gamma factors associated to generic representations of $\mathrm{U}_{2n+1} \times \mathrm{Res}_{E/F}\,\mathrm{GL}_r$ defined through these three different approaches are in fact all identical: the method of Rankin–Selberg integrals, the method of Langlands–Shahidi, and the local Langlands correspondence.

The global Rankin–Selberg integrals for $\mathrm{U}_{2n+1} \times \mathrm{Res}_{E/F}\,\mathrm{GL}_r$ were first constructed by Gelbart and Piatetski-Shapiro in the case $n = r = 1$ in [15]. Their results for local integrals were later refined by Baruch in [3]. In [38], Tamir extended the global constructions and results to arbitrary $n = r \geq 1$, following the methods in [16, Part B]. For general $n \geq r \geq 1$, a basic theory of the global integrals was developed by Ben-Artzi and Soudry in [4]. On the other hand, the global constructions for $n < r$ first appeared in a work of Wambach [40] when $n = 1$ and $r = 2$. For any $n < r$, the global constructions were due to Ginzburg, Rallis, and Soudry [18]; while the basic properties of the local integrals were established by Morimoto and Soudry in [28].

Similar constructions of Rankin–Selberg integrals for other classical groups can be found in [16, Part B; 17; 18; 34]. Analogous results to Theorem A were obtained by Morimoto in [27] for even unitary groups, and by Soudry [36] and Kaplan [24] for odd orthogonal groups and other classical groups. Our proof of multiplicativity of gamma factors closely resembles the approach presented by Soudry in [34; 36].

This article is organized as follows. In Section 2, we introduce the notation used throughout the article. The gamma factors are defined in Section 3. Specifically, we first introduce the induced representations involved in the definition of the local Rankin–Selberg integrals in Section 3.2. Next, we define the normalized intertwining operators among these induced representations in Section 3.3. The local Rankin–Selberg integrals are presented in Section 3.5, followed by the definition of the gamma factors in Section 3.6.

In Section 4, we introduce the main result of this article, namely, Theorem 4.1. Some assertions of the main result, specifically the functional equation for the gamma factors and their dependence on ψ, are proved in this section. We prepare the groundwork for proving the multiplicativity of the gamma factors in Section 5. The multiplicativity of the gamma factors is addressed in Sections 6 and 7. Specifically, we prove multiplicativity with respect to the first variable (resp. second variable) in Section 6 (resp. Section 7), which comprise a large part of this article. Finally, we address the minimal case in Section 8.

2. General notation

We present a list of the most frequently used notation in this article.

2.1. *Fields.* Throughout, F will always be a local field of characteristic zero, and E an étale F-algebra of rank 2. Thus E is either a quadratic field extension of F or $E = F \times F$. In the latter case, we always identify F as a subfield of E via the diagonal embedding. We set $\mathscr{E} = E$ if E is a field, and $\mathscr{E} = F$ otherwise.

Denote by $z \mapsto \bar{z}$ the action of the generator of $\mathrm{Aut}_F(E) \cong \{\pm 1\}$. Let

$$E^0 = \{z \in E \mid z + \bar{z} = 0\} \quad \text{and} \quad E^1 = \{z \in E^\times \mid z\bar{z} = 1\}.$$

Fix an element $\delta \in E^\times$ with $\bar{\delta} = -\delta$, and put $\Delta = \delta^2 \in F^\times$. We have $E^0 = F \cdot \delta$ and $\delta = (\delta_0, -\delta_0)$ for some $\delta_0 \in F^\times$ when $E = F \times F$.

Let $|\cdot|_F$ be the normalized absolute value on F. Then $|z|_E := |z\bar{z}|_F$ gives rise to an absolute value on E. Fix a nontrivial additive character ψ of F and put $\psi_E(z) = \psi(z + \bar{z})$ for $z \in E$. If $\alpha \in F^\times$, then we define ψ_α to be the character of F given by $\psi_\alpha(x) = \psi(\alpha x)$.

2.2. *Matrices.* Let R be a ring with identity. The notation $\mathrm{Mat}_{m,k}(R)$ represents the ring of m by k matrices with entries in R. For $A \in \mathrm{Mat}_{m,k}(R)$, A_{ij} denotes the (i, j)-entry of A, and ${}^tA$ is the transpose of A. The identity element of $\mathrm{Mat}_{n,n}(R)$ is denoted as I_n.

When $R = E$, the automorphism $z \mapsto \bar{z}$ and its notation extend naturally to an involution on $\mathrm{Mat}_{m,k}(E)$, specifically, $(\bar{A})_{ij} = \bar{A}_{ij}$. If $E = F \times F$, we identify $\mathrm{Mat}_{m,k}(F)$ as a subring of $\mathrm{Mat}_{m,k}(E) = \mathrm{Mat}_{m,k}(F) \times \mathrm{Mat}_{m,k}(F)$ through the diagonal embedding.

2.3. ***Groups.*** Let N be a positive integer. Denote by $\mathrm{G}_N = \mathrm{Res}_{E/F}\,\mathrm{GL}_N$ the Weil restriction of GL_N from E to F. Then $\mathrm{G}_N(F) = \mathrm{GL}_N(E)$ when E is a field, and we have $\mathrm{G}_N(F) = \mathrm{GL}_N(F) \times \mathrm{GL}_N(F)$ when $E = F \times F$.

Let $\mathrm{U}_N \subset \mathrm{G}_N$ be a quasisplit unitary group over F of N variables. When $E = F \times F$, we identify $\mathrm{U}_N(F)$ with $\mathrm{GL}_N(F)$ via the projection onto the first component. On the other hand, if E is a field, we use the following realization of $\mathrm{U}_N(F)$ within $\mathrm{GL}_N(E)$. Define $S_N = J_N$ if N is even and

$$S_N = \begin{pmatrix} & & J_n \\ & -2 & \\ J_n & & \end{pmatrix}$$

if $N = 2n+1$. Then we set

$$\mathrm{U}_N(F) = \{g \in \mathrm{GL}_N(E) \mid {}^t\bar{g} S_N g = S_N\}.$$

Let $G = \mathrm{GL}_N, \mathrm{U}_N$ or G_N (over F). We denote by $B_G = T_G V_G \subset G$ the upper triangular Borel subgroup of G, where T_G is the diagonal torus, and V_G is the unipotent radical of B_G. If E is a field and $a \in \mathrm{GL}_N(E)$, we define $a^* = J_N\, {}^t\bar{a}^{-1} J_N$, where $J_N \in \mathrm{GL}_N(F)$ is the antidiagonal matrix whose entries on the antidiagonal are all equal to 1.

2.4. ***Generic characters.*** Let $G = \mathrm{GL}_N, \mathrm{U}_N$ or G_N, and $z \in V_G(F)$. Fix a unique (up to conjugate by elements in $T_G(F)$) generic character ψ_{G} of $V_G(F)$ as follows. First, suppose that $G = \mathrm{GL}_N$. Then we define

$$\psi_{\mathrm{GL}_N}(z) = \psi(z_{12} + z_{23} + \cdots + z_{N-1,N}).$$

Next, suppose that $G = \mathrm{U}_N$ with $N = 2n+1$. Then we define

$$\psi_{\mathrm{U}_{2n+1}}(z) = \psi_E(z_{12} + z_{23} + \cdots + z_{n-1,n} + 2^{-1} z_{n,n+1})$$

when E is a field, and

$$\psi_{\mathrm{U}_{2n+1}}(z) = \psi(z_{12} + z_{23} + \cdots + z_{n-1,n} + 2^{-1} z_{n,n+1} + z_{n+1,n+2} - z_{n+2,n+3} - \cdots - z_{2n,2n+1})$$

when $E = F \times F$. We emphasis that when $E = F \times F$, the characters $\psi_{\mathrm{U}_{2n+1}}$ and $\psi_{\mathrm{GL}_{2n+1}}$ are different, although the groups $\mathrm{U}_{2n+1}(F)$ and $\mathrm{GL}_{2n+1}(F)$ are isomorphic. We hope this will not cause a serious confusion.

Finally, suppose that $G = \mathrm{G}_N$. Then we define

$$\psi_{\mathrm{G}_N}(z) = \psi_E(z_{12} + z_{23} + \cdots + z_{N-1,N})$$

when E is a field, and $\psi_{\mathrm{G}_N}(z, z') = \psi_{\mathrm{GL}_N}(z)\psi^{-1}_{\mathrm{GL}_N}(z')$ when $E = F \times F$.

2.5. *Representations.* If G is a connected reductive algebraic group over F, a representation of $G(F)$ in this article refers to a complex representation of finite length that is smooth. The term smooth carries its usual meaning when F is non-Archimedean (see [5]). When F is Archimedean, it means a smooth admissible Fréchet representation of moderate growth in the sense of Casselman and Wallach (see [9]).

If π is a representation of $G(F)$, we usually write $\mathcal{V}_\pi$ for its representation space, ω_π for its central character (if it exists) and $\tilde{\pi}$ for the contragredient representation. In the Archimedean case, $\tilde{\pi}$ is defined as the Casselman–Wallach completion of the contragredient of the Harish-Chandra module underlying π.

If G' is another group over F and $(\pi', \mathcal{V}_{\pi'})$ is a representation of $G'(F)$, then $\pi \boxtimes \pi'$ represents the external tensor product representation of $G(F) \times G'(F)$ on $\mathcal{V}_\pi \otimes \mathcal{V}_{\pi'}$ when F is non-Archimedean. In the Archimedean case, it denotes the representation of $G(F)\times G'(F)$ on the completed projective tensor product $\mathcal{V}_\pi \widehat{\otimes} \mathcal{V}_{\pi'}$.

If G is quasisplit, ψ_G is a generic character of the unipotent radical of a fixed Borel subgroup (over F) of G, and π is an irreducible ψ_G-generic representation of G, we denote by $\mathcal{W}(\pi, \psi_G)$ the Whittaker model of π with respect to ψ_G. If G admits a unique[1] generic character, then we simply say that π is generic.

If $(\eta, \mathcal{V}_\eta)$ is a representation of $\mathrm{GL}_N(E)$, then η^* is the representation of $\mathrm{GL}_N(E)$ on $\mathcal{V}_\eta$ with the action $\eta^*(a) = \eta(a^*)$. On the other hand, if $s \in \mathbb{C}$, then η_s stands for $\eta \otimes |\det|_E^{s-1/2}$ and we understand that $\eta_s^* = \eta^* \otimes |\det|_E^{s-1/2}$. Similar definition and notation work for representations of $\mathrm{GL}_N(F)$.

2.6. $\mathrm{GL}_a \times \mathrm{GL}_b$ *gamma factors.* Let K be a local field of characteristic zero, and let ψ_K be a nontrivial additive character of K. If ϑ and ϱ are irreducible generic representations of $\mathrm{GL}_a(K)$ and $\mathrm{GL}_b(K)$, respectively, we denote by $\gamma(s, \vartheta \times \varrho, \psi_K)$ the $\mathrm{GL}_a \times \mathrm{GL}_b$ gamma factors as defined by Jacquet, Piatetski-Shapiro, and Shalika [22] and Jacquet [20].

3. Rankin–Selberg gamma factors

In this section, we define the Rankin–Selberg gamma factors, the main focus of this article. The primary references of this section are [4; 28].

[1] Up to conjugate by torus elements in the fixed Borel subgroup that normalize the unipotent radical.

3.1. ***Embeddings.*** We have two embeddings

$$\jmath_{n,r} : \mathrm{U}_{2r}(F) \hookrightarrow \mathrm{U}_{2n+1}(F) \quad \text{and} \quad \jmath^{n,r} : \mathrm{U}_{2n+1}(F) \hookrightarrow \mathrm{U}_{2r}(F)$$

depending on the relative sizes of n and r, defined as follows.

If $n \geq r$, then

$$\jmath_{n,r}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & & b \\ & I_{2(n-r)+1} & \\ c & & d \end{pmatrix},$$

where $a, b, c, d \in \mathrm{Mat}_{r,r}(\mathscr{E})$. If instead $n < r$, then

$$\jmath^{n,r}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = Q \begin{pmatrix} a & & & b \\ & 1 & & 0 \\ & & I_{2(r-n-1)} & \\ c & 0 & & d \end{pmatrix} Q^{-1}, \quad \text{where } Q = \begin{pmatrix} I_n & & & & \\ & 1 & & 1 & \\ & & I_{2(r-n-1)} & & \\ & 1 & & -1 & \\ & & & & I_n \end{pmatrix}$$

with $a \in \mathrm{Mat}_{n\times n}(\mathscr{E})$, $b \in \mathrm{Mat}_{n,n+1}(\mathscr{E})$, $c \in \mathrm{Mat}_{n+1,n}(\mathscr{E})$ and $d \in \mathrm{Mat}_{n+1,n+1}(\mathscr{E})$.

Note that the image of $\jmath^{n,r}$ is precisely the subgroup of $\mathrm{U}_{2r}(F)$ fixing the vector

$$e = {}^t(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0) \in F^{2r},$$

where 1 appears in the $(n+1)$-th and $(2r-n)$-th entries of e. This explains the occurrence of -2 in S_N when E is a field and N is odd.

3.2. ***Induced representations.*** Let $Q_{2r} = M_{Q_{2r}} N_{Q_{2r}}$ be the parabolic subgroup of U_{2r} whose Levi subgroup $M_{Q_{2r}}$ is isomorphic to G_r. Here $N_{Q_{2r}}$ is the unipotent radical of Q_{2r}. We have

$$M_{Q_{2r}}(F) = \left\{ m_r(a) = \begin{pmatrix} a & \\ & a^* \end{pmatrix} \;\middle|\; a \in \mathrm{G}_r(F) \right\}$$

when E is a field, and

$$M_{Q_{2r}}(F) = \left\{ m_r(a,b) = \begin{pmatrix} a & \\ & b \end{pmatrix} \;\middle|\; (a,b) \in \mathrm{G}_r(F) \right\}$$

when $E = F \times F$.

Let τ be an irreducible generic representation of $\mathrm{G}_r(F)$. When $E = F \times F$, τ is of the form $\tau_1 \boxtimes \tau_2$ for some irreducible generic representations τ_1, τ_2 of $\mathrm{GL}_r(F)$. In this case, we set $\tau_s = \tau_{1,s} \boxtimes \tau^*_{2,1-s}$. Now let

$$\rho_{\tau,s} = \mathrm{Ind}_{Q_{2r}(F)}^{\mathrm{U}_{2r}(F)}(\tau_s)$$

be a normalized induced representation of $\mathrm{U}_{2r}(F)$, whose underlying space $\mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s)$ consists of smooth functions $\xi_s : \mathrm{U}_{2r}(F) \to \mathcal{V}_\tau$ satisfying

$$\xi_s(muh) = \delta_{Q_{2r}}^{\frac{1}{2}}(m)\tau_s(m)\xi_s(h) \quad \text{for } (m,u,h) \in M_{Q_{2r}}(F) \times N_{Q_{2r}}(F) \times \mathrm{U}_{2r}(F).$$

Here $\delta_{Q_{2r}}$ stands for the modulus character of Q_{2r}.

Let $\lambda_{\tau,\psi^{-1}}$ be a nonzero Whittaker functional of τ with respect to the generic character $\psi_{\mathrm{G}_r}^{-1}$ (see Section 2.4). For a given $\xi_s \in \mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s)$, define the associated scalar-valued function $f_{\xi_s}^{\psi^{-1}}(h; a)$ by

$$f_{\xi_s}^{\psi^{-1}}(h; a) = \lambda_{\tau,\psi^{-1}}(\tau(a)\xi_s(h)) \quad \text{for } (h, a) \in \mathrm{U}_{2r}(F) \times \mathrm{G}_r(F). \tag{3-1}$$

Note that for each $h \in \mathrm{U}_{2r}(F)$, the function $a \mapsto f_{\xi_s}^{\psi^{-1}}(h; a)$ belongs to $\mathcal{W}(\tau, \psi_{\mathrm{G}_r}^{-1})$.

3.3. ***Intertwining operators.*** We introduce an intertwining operator on $\mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s)$, which is essential in defining the Rankin–Selberg gamma factors. As pointed out in the introduction, we have to normalize this operator carefully so that the gamma factors satisfy expected properties. The basic references of this subsection are [29; 32].

Let

$$w_{r,r} = \begin{pmatrix} & I_r \\ I_r & \end{pmatrix} \in \mathrm{U}_{2r}(F)$$

and set $\tau^* = \tau_2^* \boxtimes \tau_1^*$ when $E = F \times F$. Then we have the intertwining map

$$A_\psi(w_{r,r}, \tau, s) : \mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s) \to \mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_{1-s}^*)$$

defined by the integral

$$A_\psi(w_{r,r}, \tau, s)\xi_s(h) = \int_{N_{Q_{2r}}(F)} \xi_s(w_{r,r}^{-1} u h)\, du$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation in general.

The Haar measure du on $N_{Q_{2r}}(F)$ is defined as the product measure through its root groups. More precisely, when $E = F \times F$, each root group is isomorphic to F, and we take the measure on F that is self-dual with respect to the pairing $F \times F \to \mathbb{C}^\times$; $(x, y) \mapsto \psi(2xy)$. Next, assume that E is a field. Then the root groups of $N_{Q_{2r}}$ are isomorphic to either E or E^0. In the first case, we take the measure on E that is self-dual with respect to the pairing $E \times E \to \mathbb{C}^\times$; $(x, y) \mapsto \psi_E(xy)$; while in the second case, the measure is chosen to be self-dual with respect to the pairing $E^0 \times E^0 \to \mathbb{C}^\times$; $(x, y) \mapsto \psi(2xy)$. This explains the dependence of the intertwining map on ψ.

We need a normalized version $\mathcal{A}_\psi(w_{r,r}, \tau, s)$ of $A_\psi(w_{r,r}, \tau, s)$. By the works in [9; 10; 29; 31], there exists a nonzero meromorphic function (the so-called local coefficient) $C_{\psi,\delta}(w_{r,r}, \tau, s)$ such that

$$\int_{N_{Q_{2r}}(F)} f_{\xi_s}^{\psi^{-1}}(w_{r,r} u; I_r')\psi'(u_{r,r+1})\, du = C_{\psi,\delta}(w_{r,r}, \tau, s) \int_{N_{Q_{2r}}(F)} f_{A_\psi(w_{r,r},\tau,s)\xi_s}^{\psi^{-1}}(w_{r,r} u; I_r')\psi'(u_{r,r+1})\, du \tag{3-2}$$

for $\xi_s \in \mathcal{V}_{Q_{2r}}^{U_{2r}}(\tau_s)$. Here $I'_r \in G_r(F)$ is the matrix given inductively via

$$I'_1 = (1) \quad \text{and} \quad I'_r = \begin{pmatrix} I'_{r-1} & \\ & (-1)^{r-1} \end{pmatrix} \tag{3-3}$$

and $\psi'(u_{r,r+1}) = \psi(2\delta u_{r,r+1})$ if E is a field, whereas $\psi'(u_{r,r+1}) = \psi(2\delta_0 u_{r,r+1})$ if $E = F \times F$. The choice of the Haar measure du (on both sides) in (3-2) clearly does not affect $C_{\psi,\delta}(w_{r,r}, \tau, s)$.

Now we define

$$\mathcal{A}_\psi(w_{r,r}, \tau, s) = \omega_\tau^{-1}(\delta)|\delta|_E^{-r(s-\frac{1}{2})} C_{\psi,\delta}(w_{r,r}, \tau, s) A_\psi(w_{r,r}, \tau, s), \tag{3-4}$$

where $\omega_\tau(\delta) = \omega_{\tau_1}(\delta_0)\omega_{\tau_2}(-\delta_0)$ if $E = F \times F$. As the notation suggested, the normalized one does not depend on the choice of δ. This fact may be found in the literature, but it can also be verified by applying Corollary 8.2, the multiplicativity of local coefficients and intertwining operators, and the standard global argument.

3.4. ***Local Gelfand–Graev models.*** We introduce specific Hom-spaces, the dimensions of which are at most one.[2] This property is instrumental both in defining the gamma factors and in proving their multiplicativity.

Assume first that $n \geq r$. Let Z'_r and $Y'_{n,r}$ be subgroups of U_{2n+1} given by

$$Z'_{n,r}(F) = \left\{ z' = \begin{pmatrix} I_r & & & & \\ & z_1 & & & \\ & & 1 & & \\ & & & z_2 & \\ & & & & I_r \end{pmatrix} \in U_{2n+1} \,\middle|\, z_1, z_2 \in V_{GL_{n-r}} \right\}$$

and

$$Y'_{n,r}(F) = \left\{ y' = \begin{pmatrix} I_r & 0 & 0 & b & 0 \\ a & I_{n-r} & x & c & b' \\ & & 1 & x' & 0 \\ & & & I_{n-r} & 0 \\ & & & a' & I_r \end{pmatrix} \in U_{2n+1} \right\}.$$

Then $Y_{n,r} := Z'_{n,r} Y'_{n,r}$ is a subgroup of U_{2n+1}, and we define a character $\psi_{Y_{n,r}}$ of $Y_{n,r}(F)$ by

$$\psi_{Y_{n,r}}(z'y') = \begin{cases} \psi_{G_{n-r}}(z_1)\psi_E(2^{-1}x_{1,n-r}) & \text{if } E \text{ is a field,} \\ \psi_{G_{n-r}}((z_1, z_2))\psi(2^{-1}x_{1,n-r} + x'_{1,1}) & \text{if } E = F \times F. \end{cases}$$

Observe that $Y_{n,r}(F)$ is normalized by $\jmath_{n,r}(U_{2r}(F))$, and

$$\psi_{Y_{n,r}}(\jmath_{n,r}(h) y \jmath_{n,r}(h)^{-1}) = \psi_{Y_{n,r}}(y)$$

[2] These models are also known in the literature as Bessel models; here, we adopt the terminology of Ginzburg, Rallis, and Soudry as used in [18].

for every $h\in\mathrm{U}_{2r}(F)$ and $y\in Y_{n,r}(F)$. Thus $\psi_{Y_{n,r}}$ extends to a character of

$$\jmath_{n,r}(\mathrm{U}_{2r}(F))\ltimes Y_{n,r}(F)\subset\mathrm{U}_{2n+1}(F),$$

which we again denote by $\psi_{Y_{n,r}}$.

Assume next that $n<r$. Let $Z'^{n,r}$ and $Y'^{n,r}$ be subgroups of U_{2r} given by

$$Z'^{n,r}(F)=\left\{z'=\begin{pmatrix}I_{n+1}&&&\\&z_1&&\\&&z_2&\\&&&I_{n+1}\end{pmatrix}\in\mathrm{U}_{2r}\;\middle|\;z_1,z_2\in V_{\mathrm{GL}_{r-n-1}}\right\}$$

and

$$Y'^{n,r}(F)=\left\{y'=\begin{pmatrix}I_{n+1}&c&0&0\\0&I_{r-n-1}&0&0\\a&b&I_{r-n-1}&c'\\0&a'&0&I_{n+1}\end{pmatrix}\in\mathrm{U}_{2r}\right\}.$$

Then $Y^{n,r}:=Z'^{n,r}Y'^{n,r}$ is a subgroup of U_{2r}, and we define the character $\psi_{Y^{n,r}}$ of $Y^{n,r}(F)$ by

$$\psi_{Y_{n,r}}(z'y')=\begin{cases}\psi^{-1}_{\mathrm{G}_{r-n-1}}(z_1)\psi_E(a_{n+1,r-n-1}) & \text{if } E \text{ is a field,}\\ \psi^{-1}_{\mathrm{G}_{r-n-1}}((z_1,z_2))\psi(c_{n+1,1}+c'_{r-n-1,1}) & \text{if } E=F\times F.\end{cases}$$

Similarly, $Y^{n,r}(F)$ is normalized by $\jmath^{n,r}(\mathrm{U}_{2n+1}(F))$, and

$$\psi_{Y^{n,r}}(\jmath^{n,r}(g)y\jmath^{n,r}(g)^{-1})=\psi_{Y^{n,r}}(y)$$

for every $g\in\mathrm{U}_{2n+1}(F)$ and $y\in Y^{n,r}(F)$. Thus $\psi_{Y^{n,r}}$ extends to a character of

$$\jmath^{n,r}(\mathrm{U}_{2n+1}(F))\ltimes Y^{n,r}(F)\subset\mathrm{U}_{2r}(F),$$

which we again denote by $\psi_{Y^{n,r}}$.

Let π and σ be irreducible representations of $\mathrm{U}_{2n+1}(F)$ and $\mathrm{U}_{2r}(F)$, respectively. If $n\geq r$, we denote by $\pi|_{\mathrm{U}_{2r}\ltimes Y_{n,r}}$ the restriction of π to $\jmath_{n,r}(\mathrm{U}_{2r}(F))\ltimes Y_{n,r}(F)$. Also, we extend σ trivially to an irreducible representation of $\mathrm{U}_{2r}(F)\ltimes Y_{n,r}(F)$.

If $n<r$, the roles of π and σ are reversed. Specifically, we denote by $\sigma|_{\mathrm{U}_{2n+1}\ltimes Y^{n,r}}$ the restriction of σ to the subgroup $\jmath^{n,r}(\mathrm{U}_{2n+1}(F))\ltimes Y^{n,r}(F)$. Similarly, π is extended trivially to an irreducible representation of $\mathrm{U}_{2n+1}(F)\ltimes Y^{n,r}(F)$.

Now we can state the following important result:

Theorem 3.1 [1; 13; 23]. *Let π and σ be irreducible representations of* $\mathrm{U}_{2n+1}(F)$ *and* $\mathrm{U}_{2r}(F)$, *respectively. If $n\geq r$, then the dimension of the Hom-space*

$$\mathrm{Hom}_{\mathrm{U}_{2r}(F)\ltimes Y_{n,r}(F)}(\pi|_{\mathrm{U}_{2r}\ltimes Y_{n,r}},\sigma)\tag{3-5}$$

is at most one. If $n < r$, then the dimension of the Hom-space

$$\mathrm{Hom}_{\mathrm{U}_{2n+1}(F)\ltimes Y^{n,r}(F)}(\pi, \sigma|_{\mathrm{U}_{2n+1}\ltimes Y^{n,r}}) \tag{3-6}$$

is also at most one.

Remark. The embeddings $\jmath_{n,r}$, $\jmath^{n,r}$, and the subgroups $Y_{n,r}$, $Y^{n,r}$ differ from those commonly found in the literature. However, these differences are merely the result of conjugation by certain Weyl elements in U_{2n+1} or U_{2r}, depending on the relative sizes of n and r. A similar remark applies to the local Rankin–Selberg integrals defined in the next subsection.

3.5. ***Local Rankin–Selberg integrals.*** In this subsection, we introduce the local Rankin–Selberg integrals as presented in [4; 28].

For $n \geq r$, let $\bar{X}_{n,r} \subset \mathrm{U}_{2n+1}$ be the unipotent subgroup given by

$$\bar{X}_{n,r}(F) = \left\{ \bar{u} = \begin{pmatrix} I_r &&&& \\ A & I_{n-r} &&& \\ 0 & 0 & 1 && \\ 0 & 0 & 0 & I_{n-r} & \\ 0 & 0 & 0 & B & I_r \end{pmatrix} \in \mathrm{U}_{2n+1}(F) \right\}.$$

For $n < r$, define $\bar{X}^{n,r} \subset \mathrm{U}_{2r}$ as the unipotent subgroup

$$\bar{X}^{n,r}(F) = \left\{ \bar{u} = \begin{pmatrix} I_{n+1} &&& \\ 0 & I_{r-n-1} && \\ A & B & I_{r-n-1} & \\ 0 & C & 0 & I_{n+1} \end{pmatrix} \in \mathrm{U}_{2r}(F) \right\}.$$

The group $\bar{X}^{n,r}(F)$ is equipped with a character $\psi_{\bar{X}^{n,r}}$ defined by

$$\psi_{\bar{X}^{n,r}}(\bar{u}) = \begin{cases} \psi_E(A_{n+1,r-n-1}) & \text{if } E \text{ is a field,} \\ \psi(A_{n+1,r-n-1} - C_{11}) & \text{if } E = F \times F. \end{cases}$$

Now let π and τ be irreducible generic representations of $\mathrm{U}_{2n+1}(F)$ and $\mathrm{G}_r(F)$, respectively. If $v \in \mathcal{V}_\pi$, denote by W_v^ψ the associated Whittaker function (with respect to $\psi_{\mathrm{U}_{2n+1}}$). The local Rankin–Selberg integral attached to $v \in \mathcal{V}_\pi$ and $\xi_s \in \mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s)$ is given by

$$\Psi_{n,r}(v \otimes \xi_s) = \int_{V_{\mathrm{U}_{2r}}(F)\backslash \mathrm{U}_{2r}(F)} \int_{\bar{X}_{n,r}(F)} W_v^\psi(\bar{u}\jmath_{n,r}(h)) f_{\xi_s}^{\psi^{-1}}(h; I_r)\, d\bar{u}\, dh \tag{3-7}$$

if $n \geq r$, and

$$\begin{aligned}\Psi_{n,r}(v \otimes \xi_s) &\\ = \int_{V_{\mathrm{U}_{2n+1}}(F)\backslash \mathrm{U}_{2n+1}(F)} &\int_{\bar{X}^{n,r}(F)} W_v^\psi(g) f_{\xi_s}^{\psi^{-1}}(\bar{u}\jmath^{n,r}(g); I_r)\psi_{\bar{X}^{n,r}}^{-1}(\bar{u})\, d\bar{u}\, dg\end{aligned} \tag{3-8}$$

if $n < r$.

By the results of [4] (for $n \geq r$) and [28] (for $n < r$), these integrals converge absolutely for $\mathrm{Re}(s) \gg 0$, and there exist v and ξ_s such that $\Psi_{n,r}(v \otimes \xi_s) \equiv 1$ when F is non-Archimedean, and such that $\Psi_{n,r}(v \otimes \xi_s)$ is holomorphic and is nonzero at a given s_0 when F is Archimedean. Furthermore, for non-Archimedean F, $\Psi_{n,r}(v \otimes \xi_s)$ has a meromorphic continuation to an element in $\mathbb{C}(q^{-s})$, and defines an element in the Hom-spaces (3-5) or (3-6), according to the relative sizes of n and r. Here q is the cardinality of the residue field of F.

For Archimedean F, however, it is unclear whether the integrals admit meromorphic continuation to the entire complex plane, though this property is expected. Therefore, we make the following assumption:

Assumption 3.2. When F is Archimedean, every integral we consider admits a meromorphic continuation to the entire complex plane. Furthermore, each such integral defines an element in the Hom-spaces (3-5) or (3-6), depending on the relative sizes of n and r.

Note that we not only assume that the integral admits a meromorphic continuation, but also that it defines an element in the Hom-spaces (3-5) or (3-6). This requirement is significant: while it is straightforward to verify that the integral satisfies the equivariant property associated with that Hom-space, demonstrating the continuity of the integral on $\mathcal{V}_\pi \widehat{\otimes} \mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s)$, which is also a defining property of the Hom-space, is more challenging. Our interest in these Hom-spaces arises from the necessity of utilizing them to define and establish the multiplicativity of the gamma factors, as will be demonstrated in the following sections.

Soudry addressed the aforementioned issues regarding local Rankin–Selberg integrals of $\mathrm{SO}_{2n+1} \times \mathrm{GL}_r$ in [35]. We expect that his arguments can also be adapted to the case where E is a field and $n \geq r$. However, for other cases, it remains unclear how to verify the assumption; new ideas may be necessary.

3.6. *Gamma factors.* We are now ready to define the Rankin–Selberg gamma factors. By Theorem 3.1 and Assumption 3.2 for Archimedean F, we deduce that, for almost all s, the local Rankin–Selberg integral defines an element in the Hom-spaces (3-5) or (3-6), each of which has dimension at most one. This leads to the functional equation

$$\Psi_{n,r}(v \otimes \mathcal{A}_\psi(w_r, \tau, s)\xi_s) = \Gamma(s, \pi \times \tau, \psi)\Psi_{n,r}(v \otimes \xi_s), \tag{3-9}$$

where $\Gamma(s, \pi \times \tau, \psi)$ is a nonzero meromorphic function.

Now the Rankin–Selberg gamma factor attached to π, τ and ψ is defined to be

$$\gamma(s, \pi \times \tau, \psi) = \omega_\pi(-1)^r \omega_\tau(-1)^n \lambda_{E/F}(\psi)^{(2n+1)r} \Gamma(s, \pi \times \tau, \psi), \tag{3-10}$$

where $\lambda_{E/F}(\psi)$ is the Langlands' λ-constant (see [21, Lemma 1.2]) and $\omega_\tau = \omega_{\tau_1}\omega_{\tau_2}$.

Remark. Compared to the normalization of the gamma factors used by Kaplan [24] and Morimoto [27], our normalization includes an extra factor of $\lambda_{E/F}(\psi)^{(2n+1)r}$. This difference arises from our result in the minimal case, as will be seen in the next section. We also note that the gamma factors defined by Kaplan and Morimoto coincide with those defined by Shahidi.

4. Main result

In this section, we provide a precise formulation of the main result. To do so, we introduce the following notation.

4.1. First, let π be an irreducible representation of $\mathrm{U}_{2n+1}(F)$. If E is a field, denote by $\mathrm{BC}(\pi)$ the irreducible representation of $\mathrm{GL}_{2n+1}(E)$ with an L-parameter given by the standard base change (see [26, Section 2.2]) of the L-parameter of π.

Next, let τ be an irreducible generic representation of $\mathrm{G}_r(F)$, so that τ is of the form $\tau_1 \boxtimes \tau_2$ when $E = F \times F$. Let s_0 be a complex number, and let $\tau\,|\det|_E^{s_0}$ be the representation of $\mathrm{G}_r(F)$ on the same space, with the action

$$\tau\,|\det|_E^{s_0}(a) = \begin{cases} \tau(a)|\det(a)|_E^{s_0} & \text{if } E \text{ is a field,} \\ \tau_1(a_1)|\det(a_1)|_F^{s_0} \otimes \tau_2(a_2)|\det(a_2)|_F^{s_0} & \text{if } E = F \times F \text{ and } a = (a_1, a_2). \end{cases}$$

Finally, let σ be an irreducible generic representation of $\mathrm{GL}_m(\mathscr{E})$ (recall that $\mathscr{E} = E$ or F according to whether E is a field or not). We define

$$\gamma(s, \sigma \times \tau, \psi) = \begin{cases} \lambda_{E/F}(\psi)^{mr}\gamma(s, \sigma \times \tau, \psi_E) & \text{if } E \text{ is a field,} \\ \gamma(s, \sigma \times \tau_1, \psi)\gamma(s, \sigma \times \tau_2, \psi) & \text{if } E = F \times F, \end{cases}$$

and (when $E = F \times F$)

$$\gamma(s, \tau \times \sigma, \psi) = \gamma(s, \tau_1 \times \sigma, \psi)\gamma(s, \tau_2 \times \sigma, \psi).$$

Note that in $\gamma(s, \sigma \times \tau, \psi_E)$ (so that E is a field), τ is viewed as a representation of the E-group $\mathrm{GL}_r(E)$.

It is well known that the $\mathrm{GL}_a \times \mathrm{GL}_b$ gamma factors coincide with those defined from the associated Weil–Deligne representations under the local Langlands correspondence for general linear groups. In particular, if E is a field and we regard σ as a representation of the F-group $\mathrm{G}_m(F)$, then $\gamma(s, \sigma \times \tau, \psi)$ is precisely the gamma factor associated with the tensor product of the Weil–Deligne representations corresponding to σ and τ under the local Langlands correspondences for the F-groups $\mathrm{G}_m(F)$ and $\mathrm{G}_r(F)$, respectively.

Theorem 4.1. *Let π and τ be irreducible generic representations of $\mathrm{U}_{2n+1}(F)$ and $\mathrm{G}_r(F)$, respectively, with $\tau = \tau_1 \boxtimes \tau_2$ for some irreducible generic representations τ_1,*

τ_2 of $\mathrm{GL}_r(F)$ when $E=F\times F$. The gamma factor $\gamma(s,\pi\times\tau,\psi)$ satisfies the following properties:

(1) <u>*Unramified twisting*</u>:

$$\gamma(s,\pi\times\tau\,|\mathrm{det}|_E^{s_0},\psi)=\gamma(s+s_0,\pi\times\tau,\psi).$$

Here s_0 is any complex number.

(2) <u>*Multiplicativity*</u>: *Let $\sigma_1\boxtimes\cdots\boxtimes\sigma_k\boxtimes\pi_0$ and $\varrho_1\boxtimes\cdots\boxtimes\varrho_h$ be irreducible generic representations of $\mathrm{G}_{m_1}(F)\times\cdots\times\mathrm{G}_{m_k}(F)\times\mathrm{U}_{2n_0+1}(F)$ and $\mathrm{G}_{r_1}(F)\times\cdots\times\mathrm{G}_{r_h}(F)$, respectively, with $m_1+\cdots+m_k+n_0=n$ and $r_1+\cdots+r_h=r$. Assume that π (resp. τ) is the irreducible generic quotient of the representation of $\mathrm{U}_{2n+1}(F)$ (resp. $\mathrm{G}_r(F)$) that is parabolically induced from $\sigma_1\boxtimes\cdots\boxtimes\sigma_k\boxtimes\pi_0$ (resp. $\varrho_1\boxtimes\cdots\boxtimes\varrho_h$). Then*

$$\gamma(s,\pi\times\tau,\psi)=\gamma(s,\pi_0\times\tau,\psi)\prod_{i=1}^{k}\gamma(s,\sigma_i\times\tau_1,\psi)\gamma(s,\tilde{\sigma}_i\times\tau_2,\psi)$$

and

$$\gamma(s,\pi\times\tau,\psi)=\prod_{j=1}^{h}\gamma(s,\pi\times\varrho_j,\psi).$$

Here we understand that $\tau=\tau_1=\tau_2$ when E is a field.

(3) <u>*Unramified case*</u>: *If all data are unramified, then*

$$\gamma(s,\pi\times\tau,\psi)=\frac{L(1-s,\tilde{\pi}\times\tilde{\tau})}{L(s,\pi\times\tau)}$$

when E is a field, and

$$\gamma(s,\pi\times\tau,\psi)=\frac{L(1-s,\tilde{\pi}\times\tilde{\tau}')L(1-s,\tilde{\pi}\times\tilde{\tau}'')}{L(s,\pi\times\tau')L(s,\pi\times\tau'')}$$

when $E=F\times F$. Here, the L-factors are defined from the associated Weil–Deligne representations under the local Langlands correspondence for unramified representations.

(4) <u>*Functional equation*</u>:

$$\gamma(s,\pi\times\tau,\psi)\gamma(1-s,\tilde{\pi}\times\tilde{\tau},\psi^{-1})=\omega_{E/F}(-1)^r.$$

Here $\omega_{E/F}$ is the quadratic character (possibly trivial) associated to E/F by the local classical field theory.

(5) <u>*Dependence on ψ*</u>: *Let $\alpha\in F^\times$. Then*

$$\gamma(s,\pi\times\tau,\psi_\alpha)=\omega_\tau(\alpha)^{2n+1}|\alpha|_E^{(2n+1)r\left(s-\frac{1}{2}\right)}\gamma(s,\pi\times\tau,\psi),$$

where $\omega_\tau=\omega_{\tau_1}\omega_{\tau_2}$ when $E=F\times F$.

(6) Minimal case: *Assume that $n = 0$ and $r = 1$. Then*

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \mathrm{BC}(\pi) \times \tau, \psi) = \lambda_{E/F}(\psi)\gamma(s, \mathrm{BC}(\pi) \times \tau, \psi_E)$$

if E is a field, and

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi \times \tau_1, \psi)\gamma(s, \pi^{-1} \times \tau_2, \psi)$$

if $E = F \times F$.

(7) Archimedean property: *Assume that F is Archimedean. Then*

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \mathrm{BC}(\pi) \times \tau, \psi) = \lambda_{E/F}(\psi)^{(2n+1)r}\gamma(s, \mathrm{BC}(\pi) \times \tau, \psi_E)$$

if E is a field, and

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi \times \tau_1, \psi)\gamma(s, \tilde{\pi} \times \tau_2, \psi)$$

if $E = F \times F$.

(8) Global property: *Let $\dot{E}/\dot{F}$ be a quadratic extension of number fields with the adeles $\mathbb{A}_{\dot{E}}/\mathbb{A}_{\dot{F}}$. Let Π and Υ be irreducible cuspidal automorphic globally generic representations of $\mathrm{U}_{2n+1}(\mathbb{A}_{\dot{F}})$ and $\mathrm{G}_r(\mathbb{A}_{\dot{F}})$, respectively. Let Ψ be a nontrivial additive character of $\mathbb{A}_{\dot{F}}/\dot{F}$. Then*

$$L^T(s, \Pi \times \Upsilon) = \Big(\prod_{v \in T} \gamma(s, \Pi_v \times \Upsilon_v, \Psi_v)\Big) L^T(1-s, \tilde{\Pi} \times \tilde{\Upsilon}).$$

Here T is a finite set of places of $\dot{F}$ such that for $v \notin T$, all data are unramified, and $L^T(s, \Pi \times \Upsilon) = \prod_{v \notin S} L(s, \Pi_v \times \Upsilon_v)$ denotes the partial L-function with respect to T, and similarly for $L^T(s, \tilde{\Pi} \times \tilde{\Upsilon})$.

Remark. Clearly, properties (1), (2), (3) and (5) are independent of Assumption 3.2 when F is non-Archimedean. Property (6) also does not rely on the assumption, even if F is Archimedean, because in the minimal case, the local integrals essentially reduce to local Tate integrals (see [39]). In contrast, property (4) does depend on the assumption, even for non-Archimedean F, as its proof involves a global argument (see Section 4.2).

By the standard global argument (see [25, Section 9; 32, Sections 5, 6]), properties (2), (3), (6), (7) and (8) uniquely determine $\gamma(s, \pi \times \tau, \psi)$. Since the Langlands–Shahidi gamma factors also satisfy these properties, Corollary A follows directly.

We now discuss the proofs of these properties. Property (1) follows directly from the definition of the local Rankin–Selberg integrals. Property (3) is established through the unramified computations found in [4, Section 8] (for $n \geq r$) and [28, Section 7] (for $n < r$). Note that when all data are unramified and $|\delta|_E = 1$, we have

$$C_{\delta,\psi}(w_{r,r}, \tau, s) = \gamma(2s - 1, \mathrm{As}(\tau), \psi)$$

when E is a field (see [33, Proposition 2.3.1]), where $\mathrm{As}(\tau)$ denotes the Asai transfer of τ (see [4, Section 1]), and

$$C_{\delta,\psi}(w_{r,r}, \tau, s) = \gamma(2s-1, \tau_1 \times \tau_2, \psi)$$

when $E = F \times F$ (see [30]).

The Archimedean property (7) follows from properties (2), (6) and Casselman's subrepresentation theorem. The global property (8) is derived from the functional equation of the Eisenstein series (see, e.g., [17, (1.6)]), and the product formula

$$\prod_v \lambda_{\dot{E}_v/\dot{F}_v}(\psi_v) = 1, \tag{4-1}$$

where v runs over all places of $\dot{F}$.

In the remainder of this section, we will prove properties (4) and (5). The proofs of properties (2) and (6), however, will be addressed in the subsequent sections.

4.2. ***Proof of Theorem 4.1*(4).** We follow Kaplan's approach, as presented in [24, Section 5], which relies on a global argument.

Since

$$\lambda_{E/F}(\psi)^2 = \omega_{E/F}(-1),$$

we obtain

$$\gamma(s, \sigma \times \tau, \psi)\gamma(1-s, \tilde{\sigma} \times \tilde{\tau}, \psi^{-1}) = \lambda_{E/F}(\psi)^{2kr} = \omega_{E/F}(-1)^{kr},$$

where σ is any irreducible generic representation of $\mathrm{G}_k(F)$. This result, along with properties (2) and (6), implies that if both π and τ are generic quotients of some principal series representations, then property (4) holds. In particular, property (4) is thus valid in the Archimedean case. Given this and the multiplicativity of the gamma factors, we see that to establish property (4), it suffices to consider the case where both π and τ are supercuspidal.

Let $\dot{E}/\dot{F}$ be a quadratic extension of number fields such that $\dot{E}_{v_0}/\dot{F}_{v_0} \cong E/F$ for some place v_0 of $\dot{F}$. Let $\dot{\mathrm{U}}_{2n+1}$ be a quasisplit unitary group over $\dot{F}$ such that $\dot{\mathrm{U}}_{2n+1}(\dot{F}_{v_0}) \cong \mathrm{U}_{2n+1}(F)$. Then by [32, Proposition 5.1], there exist irreducible cuspidal automorphic globally generic representations Π and Υ of $\dot{\mathrm{U}}_{2n+1}(\mathbb{A}_{\dot{F}})$ and $\mathrm{G}_r(\mathbb{A}_{\dot{F}})$, respectively, such that

(i) $\Pi_{v_0} \cong \pi$ and $\Upsilon_{v_0} \cong \tau$;

(ii) for all finite places $v \neq v_0$, the representations Π_v and Υ_v have nonzero vectors fixed by the corresponding Iwahori subgroups.

By (ii) and a result of Borel (see [7]), the representations Π_v and Υ_v, for all finite $v \neq v_0$, are generic quotients of some unramified principal series representations. We thus conclude that property (4) holds for all places $v \neq v_0$.

Fix a nontrivial additive character Ψ of $\mathbb{A}_{\dot F}/\dot F$. By the global property (8),

$$\prod_{v\in T}\gamma(s,\Pi_v\times\Upsilon_v,\Psi_v)=\frac{L^T(s,\Pi\times\Upsilon)}{L^T(1-s,\tilde\Pi\times\tilde\Upsilon)},$$
$$\prod_{v\in T}\gamma(1-s,\tilde\Pi_v\times\tilde\Upsilon_v,\Psi_v^{-1})=\frac{L^T(1-s,\tilde\Pi\times\tilde\Upsilon)}{L^T(s,\Pi\times\Upsilon)},$$

where T is a finite set of places of $\dot F$ such that for $v\notin T$, all data are unramified. Particularly, $v_0\in T$, and we have

$$\prod_{v\in T}\gamma(s,\Pi_v\times\Upsilon_v,\Psi_v)\gamma(1-s,\tilde\Pi_v\times\tilde\Upsilon_v,\Psi_v^{-1})=1.$$

Since property (4) holds for all $v\neq v_0$, we deduce that

$$\begin{aligned}\gamma(s,\pi\times\tau,\Psi_{v_0})\gamma(1-s,\tilde\pi\times\tilde\tau,\Psi_{v_0}^{-1})&=\prod_{v_0\neq v\in T}\omega_{\dot E_v/\dot F_v}(-1)^r\\&=\prod_{v_0\neq v}\omega_{\dot E_v/\dot F_v}(-1)^r=\omega_{E/F}(-1)^r.\end{aligned}$$

Finally, since $\Psi_{v_0}=\psi_\alpha$ for some $\alpha\in F^\times$, we can apply property (5) to obtain the desired identity. □

4.3. ***Proof of Theorem 4.1*(5).** We divide the proof into two cases, depending on whether E is a field.

4.3.1. *E is a field.* Assume first that E is a field. Define

$$d_\alpha=\mathrm{diag}(1,\alpha^{-1},\dots,\alpha^{-r+1})\in \mathrm{G}_r(F).$$

We claim

$$C_{\psi_\alpha,\delta}(w_{r,r},\tau,s)=c_0^{-1}\,|\det(d_\alpha)|_E^{1-2s}\,C_{\psi,\alpha\delta}(w_{r,r},\tau,s),\tag{4-2}$$

where c_0 is the constant (arising from the choice of measures) such that

$$A_{\psi_\alpha}(w_{r,r},\tau,s)=c_0\,A_\psi(w_{r,r,},\tau,s).$$

To see this, observe that

$$f_{\xi_s}^{\psi_\alpha}(h;a)=f_{\xi_s}^{\psi}(h;d_\alpha a)\quad\text{for }(h,a)\in \mathrm{U}_{2r}(F)\times \mathrm{G}_r(F),$$

where ξ_s is an element in $\mathcal{V}_{Q_{2r}}^{\mathrm{U}_{2r}}(\tau_s)$. From the above observation, we can write (3-2) (with ψ replaced by ψ_α) as

$$\begin{aligned}&\int_{N_{Q_{2r}}(F)}f_{\xi_s}^{\psi^{-1}}(w_{r,r}u;d_\alpha I_r')\psi(2\alpha\delta u_{r,r+1})\,du\\&\qquad=c_0\,C_{\psi_\alpha,\delta}(w_{r,r},\tau,s)\int_{N_{Q_{2r}}(F)}f_{A_\psi(w_{r,r},\tau,s)\xi_s}^{\psi^{-1}}(w_{r,r}u;d_\alpha I_r')\psi(2\alpha\delta u_{r,r+1})\,du.\end{aligned}$$

Using the identities

$$f^{\psi^{-1}}_{\xi_s}(w_{r,r}u;\, d_\alpha I'_r) = \delta_{Q_{2r}}^{-\frac{1}{2}}(m(d_\alpha))|\det(d_\alpha)|_E^{\frac{1}{2}-s}\, f^{\psi^{-1}}_{\rho_{\tau,s}(m(d_\alpha^*))\xi_s}(w_{r,r}u';\, I'_r)$$

and

$$f^{\psi^{-1}}_{A_\psi(w_{r,r},\tau,s)\xi_s}(w_{r,r}u;\, d_\alpha I'_r) = \delta_{Q_{2r}}^{-\frac{1}{2}}(m(d_\alpha))|\det(d_\alpha)|_E^{s-\frac{1}{2}}\, f^{\psi^{-1}}_{A_\psi(w_{r,r},\tau,s)\rho_{\tau,s}(m(d_\alpha^*))\xi_s}(w_{r,r}u';\, I'_r),$$

where $u' := m(d_\alpha^*)\, u\, m(d_\alpha^*)^{-1}$, and noting that $u'_{r,r+1} = u_{r,r+1}$, we obtain

$$\begin{aligned}\int_{N_{Q_{2r}}(F)} & f^{\psi^{-1}}_{\rho_{\tau,s}(m(d_\alpha^*))\xi_s}(w_{r,r}u;\, I'_r)\psi^{-1}(2\alpha\delta u_{r,r+1})\,du \\ &= c_0\, C_{\psi_\alpha,\delta}(w_{r,r},\tau,s)|\det(d_\alpha)|_E^{2s-1} \\ &\qquad \cdot \int_{N_{Q_{2r}}(F)} f^{\psi^{-1}}_{A_\psi(w_{r,r},\tau,s)\rho_{\tau,s}(m(d_\alpha^*))\xi_s}(w_{r,r}u;\, I'_r)\psi^{-1}(2\alpha\delta u_{r,r+1})\,du\end{aligned}$$

after changing the variable $u' \mapsto u$. From this, (4-2) follows immediately.

Now, by applying (4-2) to (3-4), we get

$$\begin{aligned}\text{(4-3)}\quad & \mathcal{A}_{\psi_\alpha}(w_{r,r},\tau,s) \\ &= \omega_\tau^{-1}(\delta)|\delta|_E^{-r\left(s-\frac{1}{2}\right)} C_{\psi_\alpha,\delta}(w_{r,r},\tau,s) A_{\psi_\alpha}(w_{r,r},\tau,s) \\ &= \omega_\tau(\alpha)|\alpha|_E^{r\left(s-\frac{1}{2}\right)}|\det(d_\alpha)|_E^{1-2s}\cdot \omega_\tau^{-1}(\alpha\delta)|\alpha\delta|_E^{-r\left(s-\frac{1}{2}\right)} \\ &\qquad\qquad \cdot C_{\psi,\alpha\delta}(w_{r,r},\tau,s) A_\psi(w_{r,r},\tau,s) \\ &= \omega_\tau(\alpha)|\alpha|_E^{r\left(s-\frac{1}{2}\right)}|\det(d_\alpha)|_E^{1-2s}\mathcal{A}_\psi(w_{r,r},\tau,s),\end{aligned}$$

where we use the fact that the normalization given in (3-4) is independent of δ.

To proceed, we put

$$\mathsf{d}_\alpha = \mathrm{diag}(\alpha^n,\ldots,\alpha,1,\alpha^{-1},\ldots,\alpha^{-n}) \in \mathrm{U}_{2n+1}(F).$$

Then as above, we have for $v \in \mathcal{V}_\pi$,

$$W_v^{\psi_\alpha}(g) = W_v^\psi(\mathsf{d}_\alpha g) \quad \text{for } g \in \mathrm{U}_{2n+1}(F).$$

In the following, we denote the local Rankin–Selberg integral defined using ψ_α by $\Psi^\alpha_{n,r}(v\otimes\xi_s)$ for clarity. Suppose that $n \ge r$. Then since

$$\bar{u}' := \mathsf{d}_\alpha\, \bar{u}\, \mathsf{d}_\alpha^{-1} \in \overline{X}_{n,r}(F)$$

for $\bar{u} \in \overline{X}_{n,r}(F)$,

$$\mathsf{d}_\alpha\, \jmath_{n,r}(h)\, \mathsf{d}_\alpha^{-1} = \jmath_{n,r}(m_r(\alpha^n d_\alpha)\, h\, m_r(\alpha^n d_\alpha)^{-1})$$

for $h \in \mathrm{U}_{2r}(F)$, and

$$f_{\xi_s}^{\psi^{-1}}(h;\alpha^n d_\alpha) = \delta_{Q_{2r}}^{-\frac{1}{2}}(m(\alpha^n d_\alpha))|\det(\alpha^n d_\alpha)|_E^{\frac{1}{2}-s} f_{\xi_s}^{\psi^{-1}}(m(\alpha^n d_\alpha)h; I_r), \tag{4-4}$$

we find that

$$\begin{aligned}
&\Psi_{n,r}^{\alpha}(v\otimes\xi_s)\\
&\quad= \int_{V_{\mathrm{U}_{2r}}(F)\backslash \mathrm{U}_{2r}(F)}\int_{\bar{X}_{n,r}(F)} W_v^{\psi_\alpha}(\bar{u}\jmath_{n,r}(h)) f_{\xi_s}^{\psi_\alpha^{-1}}(h; I_r)\,d\bar{u}\,dh\\
&\quad= \omega_\tau(\alpha)^{-n}\int_{V_{\mathrm{U}_{2r}}(F)\backslash \mathrm{U}_{2r}(F)}\int_{\bar{X}_{n,r}(F)} W_v^{\psi}(\mathrm{d}_\alpha\bar{u}\jmath_{n,r}(h)) f_{\xi_s}^{\psi^{-1}}(h;\alpha^n d_\alpha)\,d\bar{u}\,dh\\
&\quad= c_1\,\omega_\tau(\alpha)^{-n}\delta_{Q_{2r}}^{-\frac{1}{2}}(m(\alpha^n d_\alpha))|\det(\alpha^n d_\alpha)|_E^{\frac{1}{2}-s}\Psi_{n,r}\big(\pi(\mathrm{d}_\alpha)v\otimes\rho_{\tau,s}(m(\alpha^n d_\alpha))\xi_s\big)
\end{aligned}$$

after changing the variables $\bar{u}'\mapsto\bar{u}$ and $m_r(\alpha^n d_\alpha)\,h\,m_r(\alpha^n d_\alpha)^{-1}\mapsto h$, where c_1 is the constant such that $d\bar{u}=c_1 d\bar{u}'$.

By (4-3) and a similar calculation, we also derive

$$\begin{aligned}
&\Psi_{n,r}^{\alpha}(v\otimes\mathcal{A}_\psi(w_{r,r},\tau,s)\xi_s)\\
&\qquad= c_1\,\omega_\tau(\alpha)^{n+1}|\alpha|_E^{r(s-\frac{1}{2})}\delta_{Q_{2r}}^{-\frac{1}{2}}(m(\alpha^n d_\alpha))|\det(\alpha^n d_\alpha)|_E^{s-\frac{1}{2}}|\det(d_\alpha)|_E^{1-2s}\\
&\qquad\qquad\times\Psi_{n,r}^{\alpha}(\pi(\mathrm{d}_\alpha)v\otimes\mathcal{A}_\psi(w_{r,r},\tau,s)\rho_{\tau,s}(m(\alpha^n d_\alpha))\xi_s).
\end{aligned}$$

Together, we obtain

$$\gamma(s,\pi\times\tau,\psi_\alpha)=\omega_\tau(\alpha)^{2n+1}|\alpha|_E^{(2n+1)r(s-\frac{1}{2})}\gamma(s,\pi\times\tau,\psi),$$

which proves property (5) when E is a field and $n\ge r$.

Suppose that $n<r$. In this case, we have

$$\jmath^{n,r}(\mathrm{d}_\alpha\,g\,\mathrm{d}_\alpha^{-1}) = m_r(\alpha^n d_\alpha)\,\jmath^{n,r}(g)\,m_r(\alpha^n d_\alpha)^{-1}$$

for $g\in \mathrm{U}_{2n+1}(F)$, and

$$\bar{u}' := m_r(\alpha^n d_\alpha)\,\bar{u}\,m_r(\alpha^n d_\alpha)^{-1}\in \overline{X}^{n,r}(F)$$

for $\bar{u}\in\overline{X}^{n,r}(F)$. Note that

$$(\psi_\alpha)_{\overline{X}^{n,r}}(\bar{u}) = \psi_{\overline{X}^{n,r}}(\bar{u}').$$

Using these, along with (4-4) and (4-3), we obtain, by a similar computation as in the case $n\ge r$, that

$$\gamma(s,\pi\times\tau,\psi_\alpha)=\omega_\tau(\alpha)^{2n+1}|\alpha|_E^{(2n+1)r(s-\frac{1}{2})}\gamma(s,\pi\times\tau,\psi)$$

when E is a field and $n<r$.

4.3.2. $E = F \times F$. Assume that $E = F \times F$. Define

$$d_\alpha = \mathrm{diag}(1, \alpha^{-1}, \dots, \alpha^{-r+1}) \in \mathrm{GL}_r(F)$$

as above. Note that in this case,

$$f_\xi^{\psi_\alpha^{-1}}(h; a_1, a_2) = f_\xi^{\psi^{-1}}(h; d_\alpha a_1, d_\alpha^* a_2) \quad \text{for } (h, a_1, a_2) \in \mathrm{GL}_{2r}(F) \times \mathrm{GL}_r(F) \times \mathrm{GL}_r(F).$$

Then a similar argument to that used in the case where E is a field shows that

$$C_{\psi_\alpha,\delta}(w_{r,r}, \tau, s) = c_0^{-1}\, |\det(d_\alpha)|_E^{1-2s}\, C_{\psi,\alpha\delta}(w_{r,r}, \tau, s),$$

where c_0 is the constant (arising from the choice of measures) such that

$$A_{\psi_\alpha}(w_{r,r}, \tau, s) = c_0\, A_\psi(w_{r,r,}, \tau, s).$$

We thus obtain

$$\mathcal{A}_{\psi_\alpha}(w_{r,r}, \tau, s) = \omega_\tau(\alpha)|\alpha|_E^{r(s-\frac{1}{2})}|\det(d_\alpha)|_E^{1-2s}\mathcal{A}_\psi(w_{r,r}, \tau, s)$$

as before. Here $\omega_\tau = \omega_{\tau_1}\omega_{\tau_2}$.

Put

$$\mathsf{d}_\alpha = \mathrm{diag}(\alpha^n, \dots, \alpha, 1, \alpha^{-1}, \dots, \alpha^{-n}) \in \mathrm{GL}_{2n+1}(F).$$

Then, for $v \in \mathcal{V}_\pi$,

$$W_v^{\psi_\alpha}(g) = W_v^\psi(\mathsf{d}_\alpha g),$$

where $g \in \mathrm{GL}_{2n+1}(F)$.

As in the case where E is a field, we have

$$\bar{u}' := \mathsf{d}_\alpha\, \bar{u}\, \mathsf{d}_\alpha^{-1} \in \overline{X}_{n,r}(F)$$

for $\bar{u} \in \overline{X}_{n,r}(F)$,

$$\mathsf{d}_\alpha\, \jmath_{n,r}(h)\, \mathsf{d}_\alpha^{-1} = \jmath_{n,r}(m_r(\alpha^n d_\alpha)\, h\, m_r(\alpha^n d_\alpha)^{-1})$$

for $h \in \mathrm{GL}_{2r}(F)$ if $n \geq r$, and

$$\jmath^{n,r}(\mathsf{d}_\alpha\, g\, \mathsf{d}_\alpha^{-1}) = m_r(\alpha^n d_\alpha)\, \jmath^{n,r}(g)\, m_r(\alpha^n d_\alpha)^{-1}$$

for $g \in \mathrm{GL}_{2n+1}(F)$, and

$$\bar{u}' := m_r(\alpha^n d_\alpha)\, \bar{u}\, m_r(\alpha^n d_\alpha)^{-1} \in \overline{X}^{n,r}(F)$$

for $\bar{u} \in \overline{X}^{n,r}(F)$ if $n < r$. Note that when $n < r$, we again have

$$(\psi_\alpha)_{\overline{X}^{n,r}}(\bar{u}) = \psi_{\overline{X}^{n,r}}(\bar{u}').$$

Using these, we can perform similar computations to obtain

$$\gamma(s, \pi \times \tau, \psi_\alpha) = \omega_\tau(\alpha)^{2n+1}|\alpha|_E^{(2n+1)r(s-\frac{1}{2})}\gamma(s, \pi \times \tau, \psi).$$

This completes the proof for the case $E = F \times F$, and thus the proof of (5). □

5. Multiplicativity: preliminaries

In this section, we prepare the groundwork for proving the multiplicativity of the gamma factors. Using induction in stages, we reduce the proof to the following propositions:

Proposition 5.1. *Let π and τ be irreducible generic representations of $\mathrm{U}_{2n+1}(F)$ and $\mathrm{G}_r(F)$, respectively. Suppose that π is the generic quotient of a representation of $\mathrm{U}_{2n+1}(F)$ parabolically induced from an irreducible generic representation $\sigma \boxtimes \pi_0$ of $\mathrm{G}_k(F) \times \mathrm{U}_{2n_0+1}(F)$ for some integers $k \geq 1$ and $n_0 \geq 0$ with $k + n_0 = n$. Then*

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi_0 \times \tau, \psi)\gamma(s, \sigma \times \tau_1, \psi)\gamma(s, \tilde{\sigma} \times \tau_2, \psi).$$

Here τ_1, τ_2 are irreducible generic representations of $\mathrm{GL}_r(F)$ such that $\tau = \tau_1 \boxtimes \tau_2$ when $E = F \times F$, and we understand that $\tau_1 = \tau_2 = \tau$ when E is a field.

Proposition 5.2. *Let π and τ be irreducible generic representations of $\mathrm{U}_{2n+1}(F)$ and $\mathrm{G}_r(F)$, respectively. Suppose that τ is the generic quotient of a induced representation of $\mathrm{G}_r(F)$ parabolically induced from an irreducible generic representation $\tau' \boxtimes \tau''$ of $\mathrm{G}_{r'}(F) \times \mathrm{G}_{r''}(F)$ for some integers $r' > 0$ and $r'' > 0$ with $r' + r'' = r$. Then*

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi \times \tau', \psi)\gamma(s, \pi \times \tau'', \psi).$$

Recall that the meaning of the gamma factors on the right-hand sides is provided in Section 4.1. The proof of Propositions 5.1 and 5.2 will be given in Sections 6 and 7, respectively. The following simple lemma allows us to reduce the proof of Proposition 5.1 to the case $n > r$.

Lemma 5.3. *Assume Proposition 5.2. If Proposition 5.1 holds for $n > r$, then it also holds for $n \leq r$.*

Proof. We apply a trick from Soudry (see [36, Section 0]). Assume that $n \geq r$. Let ϱ be any irreducible generic representation of $\mathrm{G}_m(F)$ with $m > n$ such that the normalized induced representation

$$\Upsilon = \mathrm{Ind}_{P_{r,m}(F)}^{\mathrm{G}_{r+m}(F)}(\tau \boxtimes \varrho)$$

of $\mathrm{G}_{r+m}(F)$ is irreducible. Here $P_{r,m}$ is the standard parabolic subgroup of G_{r+m} whose Levi subgroup is isomorphic to $\mathrm{G}_r \times \mathrm{G}_m$.

On one hand, we have

$$\begin{aligned}\gamma(s, \pi \times \Upsilon, \psi) &= \gamma(s, \pi_0 \times \Upsilon, \psi)\gamma(s, \sigma \times \Upsilon, \psi)\gamma(s, \tilde{\sigma} \times \Upsilon, \psi) \\ &= \gamma(s, \pi_0 \times \tau, \psi)\gamma(s, \pi_0 \times \varrho, \psi)\gamma(s, \sigma \times \tau, \psi) \\ &\qquad \cdot \gamma(s, \sigma \times \varrho, \psi)\gamma(s, \tilde{\sigma} \times \tau, \psi)\gamma(s, \tilde{\sigma} \times \varrho, \psi) \\ &= \gamma(s, \pi_0 \times \tau, \psi)\gamma(s, \sigma \times \tau, \psi)\gamma(s, \tilde{\sigma} \times \tau, \psi) \cdot \gamma(s, \pi \times \varrho, \psi)\end{aligned}$$

by the multiplicativity of the $\mathrm{GL}_a \times \mathrm{GL}_b$ gamma factors and the fact that $m > n$.

On the other hand, since

$$\gamma(s, \pi \times \Upsilon, \psi) = \gamma(s, \pi \times \tau, \psi)\gamma(s, \pi \times \varrho, \psi)$$

and gamma factors are never zero, we obtain the desired factorization. □

In the remainder, we prepare for the proofs of Propositions 5.1 and 5.2 in Sections 5.1 and 5.2, respectively. Note that Sections 5.1 and 5.2 are logically independent. Readers primarily interested in the proof of Proposition 5.1 (resp. Proposition 5.2) can therefore safely skip Section 5.2 (resp. Section 5.1).

As in [27], we choose to prove Propositions 5.1 and 5.2 only when $E = F \times F$. This is motivated by two main reasons: the calculations for these two cases are quite similar; and when E is a field, the computations closely follow, almost word for word, the calculations in [34, Section 11]. Actually, our guideline for the proof of Propositions 5.1 and 5.2 when E is a field is that of Soudry in [34, Section 11]. For the case $E = F \times F$, we expect that the computations are similar to the case E is a field. This is the idea behind our proofs.

To simplify the notation, references to the field F are omitted from the notation in the remainder and the next two sections; thus, for instance, GL_r means $\mathrm{GL}_r(F)$. We hope this will not cause a serious confusion.

5.1. ***Preliminaries for Proposition 5.1 when $E = F \times F$.*** We start with recalling the functional equations of the local Rankin–Selberg integrals developed in [20; 22].

5.1.1. Let τ and σ be irreducible generic representations of GL_r and GL_k, respectively, with $r > k$. Let n_0, ℓ be nonnegative integers satisfying $n_0 + \ell = r - k - 1$. Let $W \in \mathcal{W}(\tau, \psi_{\mathrm{GL}_r})$ and $W' \in \mathcal{W}(\sigma, \psi^{-1}_{\mathrm{GL}_k})$. Then the local Rankin–Selberg integral attached to W, W', n_0 and a complex number s is given by

$$\int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{\mathrm{Mat}_{n_0,k}} W\left(\begin{pmatrix} a & & \\ x & I_{n_0} & \\ & & I_{\ell+1}\end{pmatrix}\right) W'(a)|\det(a)|^{s-\frac{r-k}{2}}\, dx\, da.$$

This integral converges absolutely when $\mathrm{Re}(s) \gg 0$, admits the meromorphic continuation to whole complex plane and satisfies the functional equation

$$\text{(5-1)}\quad \omega_\sigma(-1)^{r-1}\int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k}\int_{\mathrm{Mat}_{k,\ell}} W\left(\begin{pmatrix} & I_{n_0+1} & \\ & & I_\ell \\ a & & x\end{pmatrix}\right) \cdot W'(a)|\det(a)|^{(s-1)+\frac{r-k}{2}-\ell}\,dx\,da$$
$$= \gamma(s,\tau\times\sigma,\psi)\int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k}\int_{\mathrm{Mat}_{n_0,k}} W\left(\begin{pmatrix} a & & \\ x & I_{n_0} & \\ & & I_{\ell+1}\end{pmatrix}\right) \cdot W'(a)|\det(a)|^{s-\frac{r-k}{2}}\,dx\,da.$$

We also record

$$\text{(5-2)}\qquad \gamma(s,\tau\times\sigma,\psi)\gamma(1-s,\tilde{\tau}\times\tilde{\sigma},\psi^{-1}) = 1$$

and

$$\text{(5-3)}\qquad \gamma(s,\tau\times\sigma,\psi^{-1}) = \omega_\tau(-1)^k\omega_\sigma(-1)^r\gamma(s,\tau\times\sigma,\psi),$$

which will be used in the proofs.

5.1.2. Let π be an irreducible generic representation of GL_{2n+1}. We assume that π is the generic quotient of the (normalized) induced representation

$$\mathrm{Ind}_{\overline{P}}^{\mathrm{GL}_{2n+1}}(\sigma\boxtimes\pi_0)$$

where

- π_0 (resp. $\sigma = \sigma_1\boxtimes\sigma_2$) is an irreducible generic representation of GL_{2n_0+1} (resp. G_k) with $k+n_0 = n$; and
- $\overline{P} = M_{\overline{P}}N_{\overline{P}}$ is the parabolic subgroup of GL_{2n+1} with the Levi subgroup
$$M_{\overline{P}} \cong \mathrm{G}_k\times\mathrm{GL}_{2n_0+1}$$
and the unipotent radical $N_{\overline{P}}$ consisting of *lower* triangular matrices.

Since the local Rankin–Selberg integrals are defined in terms of the Whittaker functions of π, we may actually assume that π is the full induced representation, instead of merely the generic quotient.

We realize π_0 (resp. σ) in its Whittaker model $\mathcal{W}(\pi_0,\psi_{\mathrm{U}_{2n_0+1}})$[3] (resp. $\mathcal{W}(\sigma,\psi_{\mathrm{G}_k})$). Then the underlying space $\mathcal{V}_{\overline{P}}^{\mathrm{GL}_{2n+1}}(\pi,\psi)$ of π consists of smooth functions

$$\varphi : \mathrm{GL}_{2n+1}\times\mathrm{G}_k\times\mathrm{GL}_{2n_0+1}\to\mathbb{C}$$

[3]Note that we use $\psi_{\mathrm{U}_{2n_0+1}}$ here instead of $\psi_{\mathrm{GL}_{2n_0+1}}$, as these two characters differ according to our convention (see Section 2.4), despite the groups being isomorphic. A similar remark applies to the character $\psi_{\mathrm{U}_{2n+1}}$ below.

satisfying:

- For a fixed $(g_0, a) \in \mathrm{GL}_{2n+1} \times \mathrm{G}_k$, the function
$$g_0 \mapsto \varphi(g;\ a;\ g_0)$$
belongs to $\mathcal{W}(\pi_0, \psi_{\mathrm{U}_{2n_0+1}})$.
- For a fixed $(g, g_0) \in \mathrm{GL}_{2n+1} \times \mathrm{GL}_{2n_0+1}$, the function
$$a \mapsto \varphi(g;\ a;\ g_0)$$
belongs to $\mathcal{W}(\sigma, \psi_{\mathrm{G}_k})$.
- For $(g, h) \in \mathrm{GL}_{2n+1}(F) \times \bar{P}$, $g_0, h_0 \in \mathrm{GL}_{2n_0+1}$ and $a, b \in \mathrm{G}_k$,
$$\varphi(hg;\ a;\ g_0) = \delta_{\bar{P}}^{\frac{1}{2}}(h)\varphi(g;\ ab;\ g_0h_0),$$
where (b, h_0) is the "Levi part" of h under the Levi decomposition of $\bar{P}$, and $\delta_{\bar{P}}$ is the modulus function of $\bar{P}$.

The Whittaker function W_φ attached to $\varphi \in \mathcal{V}_{\bar{P}}^{\mathrm{GL}_{2n+1}}(\pi, \psi)$ can be realized by the Jacquet integral

$$W_\varphi(g) = \int_{N_P} \varphi(yg;\ I_k, I_k;\ I_{2n_0+1})\psi_{N_P}^{-1}(y)\, dy, \tag{5-4}$$

where N_P is the unipotent radical of the parabolic subgroup P of GL_{2n+1} that is opposite to $\bar{P}$, and ψ_{N_P} is the restriction of $\psi_{\mathrm{U}_{2n+1}}$ to N_P. Formally, we have
$$W_\varphi(ug) = \psi_{\mathrm{U}_{2n+1}}(u)W_\varphi(g) \quad \text{for } (u, g) \in V_{\mathrm{GL}_{2n+1}} \times \mathrm{GL}_{2n+1}\,.$$

This integral may not converge absolutely; to rectify this, let ζ be a complex number and replace σ by $\sigma_\zeta = \sigma \otimes |\det|^{-\zeta}$. The resulting induced representation is π_ζ and we take a holomorphic section φ_ζ instead of φ in (5-4). Now the integral defining W_{φ_ζ}, which converges absolutely for $\mathrm{Re}(\zeta) \gg 0$, admits a continuation to the holomorphic function on the whole complex plane. Moreover, for a given $W \in \mathcal{W}(\pi, \psi_{\mathrm{U}_{2n+1}})$, there is a standard section φ_ζ of π_ζ such that $W_{\varphi_0} = W$ (see [6]).

5.2. ***Preliminaries for Proposition 5.2 when $E = F \times F$.*** We begin with some notation.

5.2.1. For positive integers i, j, we define
$$w_{i,j} = \begin{pmatrix} & I_j \\ I_i & \end{pmatrix} \in \mathrm{GL}_{i+j} \quad \text{and} \quad N_{i,j} = \left\{ \begin{pmatrix} I_i & x \\ & I_j \end{pmatrix} \,\middle|\, x \in \mathrm{Mat}_{i,j} \right\}.$$
Note that $w_{i,j}^{-1} = w_{j,i}$.

If H is a subgroup of GL_r, we define
$$H^\triangle = \left\{ \begin{pmatrix} h & \\ & I_r \end{pmatrix} \,\middle|\, h \in H \right\} \quad \text{and} \quad H^\triangledown = \left\{ \begin{pmatrix} I_r & \\ & h \end{pmatrix} \,\middle|\, h \in H \right\}.$$

On the other hand, if H is a subgroup of $\mathrm{GL}_{2r''}$, we define

$$H^{\diamond} = \left\{ \begin{pmatrix} I_{r'} & & \\ & h & \\ & & I_{r'} \end{pmatrix} \middle| \, h \in H \right\}.$$

These are subgroups of GL_{2r}.

In this subsection and Section 7, if ϱ_j is a representation of GL_{r_j} for $j = 1, \ldots, h$, we denote by $\varrho_1 \times \cdots \times \varrho_h$ the normalized induced representation of $\mathrm{GL}_{r_1+\cdots+r_h}$ inducing from the representation $\varrho_1 \boxtimes \cdots \boxtimes \varrho_h$ of the standard parabolic subgroup of $\mathrm{GL}_{r_1+\cdots+r_m}$ whose Levi subgroup is isomorphic to $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_h}$.

5.2.2. Let $\tau = \tau_1 \boxtimes \tau_2$ be an irreducible generic representation of $\mathrm{G}_r = \mathrm{GL}_r \times \mathrm{GL}_r$. We assume that τ_j is the irreducible generic quotient of $\tau'_j \times \tau''_j$ for $j = 1, 2$, where τ'_1 and τ'_2 (resp. τ''_1 and τ''_2) are irreducible generic representations of $\mathrm{GL}_{r'}$ (resp. $\mathrm{GL}_{r''}$) for some integers $r' > 0$ and $r'' > 0$ such that $r' + r'' = r$. Since τ_j and $\tau'_j \times \tau''_j$ have the same Whittaker model for $j = 1, 2$, we may actually assume that

$$\tau_1 \cong \tau'_1 \times \tau''_1 \quad \text{and} \quad \tau_2 \cong \tau'_2 \times \tau''_2.$$

Then by induction in stages,

$$\rho_{\tau,s} = \tau_{1,s} \times \tau^*_{2,1-s} \cong \tau'_{1,s} \times \rho_{\tau'',s} \times \tau'^*_{2,1-s} \cong (\tau'_{1,s} \times \tau''_{1,s}) \times (\tau''^*_{2,1-s} \times \tau'^*_{2,1-s}),$$

where $\tau'' := \tau''_1 \boxtimes \tau''_2$ is an irreducible generic representation of $\mathrm{G}_{r''}$.

We now describe elements in the underlying spaces of these induced representations. Let $P = M_P N_P$ be parabolic subgroup of GL_{2r} consisting of block upper triangular matrices with the Levi subgroup

$$M_P \cong \mathrm{GL}_{r'} \times \mathrm{GL}_{r''} \times \mathrm{GL}_{r''} \times \mathrm{GL}_{r'}$$

and the unipotent radical N_P. Let

$$\Upsilon = \tau'_1 \boxtimes \tau''_1 \boxtimes \tau''^*_2 \boxtimes \tau'^*_2$$

be an irreducible generic representation of M_P, and ν_s be a character of M_P given by

$$\nu_s(a, b, c, d) = |\det(a)|^{s-\frac{1}{2}} |\det(b)|^{s-\frac{1}{2}} |\det(c)|^{\frac{1}{2}-s} |\det(d)|^{\frac{1}{2}-s}.$$

Define a generic character $\psi'_{\mathrm{GL}_{2r}}$ of $V_{\mathrm{GL}_{2r}}$ by

$$\psi'_{\mathrm{GL}_{2r}}(z) = \psi(z_{12} + \cdots + z_{r-1,r} - 2z_{r,r+1} - z_{r+1,r+2} - \cdots - z_{2r-1,2r})$$

and let ψ'_{M_P} be its restriction to $M_P \cap V_{\mathrm{GL}_{2r}}$.

The underlying space of $\tau'_{1,s} \times \rho_{\tau'',s} \times \tau'^*_{2,1-s}$ consists of the smooth functions

$$\phi_s : \mathrm{GL}_{2r} \times \mathrm{GL}_{2r''} \times M_P \to \mathbb{C}$$

such that for $h \in \mathrm{GL}_{2r}$, $h', h'' \in \mathrm{GL}_{2r''}$, $a, b \in \mathrm{GL}_{r'}$, $c, d \in \mathrm{GL}_{r''}$ and $m \in M_P$,

- $\phi_s\left(\begin{pmatrix} a & * & * \\ & h' & * \\ & & b \end{pmatrix} h;\, h'';\, m\right) = |\det(ab^{-1})|^{\frac{r-r'}{2}} \phi_s\left(h;\ h'h'';\ m \begin{pmatrix} a & & \\ & I_{2r''} & \\ & & b \end{pmatrix}\right)$;
- $\phi_s\left(h;\ \begin{pmatrix} c & * \\ & d \end{pmatrix} h';\, m\right) = |\det(cd^{-1})|^{-\frac{r''}{2}} \phi_s\left(h;\ h';\ m \begin{pmatrix} I_{r'} & & & \\ & c & & \\ & & d & \\ & & & I_{r'} \end{pmatrix}\right)$; and
- for each fixed $(h, h'') \in \mathrm{GL}_{2r} \times \mathrm{GL}_{2r''}$, the function $m \mapsto \phi_s(h, h'', m)$ belongs to $\mathcal{W}(\Upsilon, \psi'^{-1}_{M_P})$, the Whittaker model of Υ with respect to $\psi'^{-1}_{M_P}$.

For each such ϕ_s, define the smooth function

$$\xi_s = \xi_{\phi_s} : \mathrm{GL}_{2r} \times \mathrm{G}_r \times M_P \to \mathbb{C}$$

by

$$\xi_s(h;\ a_1, a_2;\ m) = \delta_{Q_{2r}}^{-\frac{1}{2}}\left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}\right) \phi_s\left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} h;\ I_{2r''};\ m\right) \tag{5-5}$$

for $h \in \mathrm{GL}_{2r}$, $a_1, a_2 \in \mathrm{GL}_r$ and $m \in M_P$. Therefore the underlying space of $(\tau'_{1,s} \times \tau''_{1,s}) \times (\tau''^{*}_{2,1-s} \times \tau'^{*}_{2,1-s})$ consists of the functions ξ_s as ϕ_s varies, and (5-5) defines an intertwining isomorphism.

5.2.3. In the following, we choose Haar measures on various unipotent subgroups of GL_N by defining them as product measures, where each root group is naturally isomorphic to F. Then on F, we use the measure that is self-dual with respect to the character ψ_2.

Recall the intertwining operator $A_\psi(w_{r,r}, \tau_1 \boxtimes \tau_2, s)$ and its normalized version $\mathcal{A}_\psi(w_{r,r}, \tau_1 \boxtimes \tau_2, s)$ from Section 3.3. Since τ_1 and τ_2 are induced representations, the normalized intertwining operator can be expressed as a composition of certain (normalized) intertwining maps.

To describe this decomposition, let ϕ_s be an element in the underlying space of $\tau'_{1,s} \times \rho_{\tau''_s} \times \tau'^{*}_{2,1-s}$ as above. Define an intertwining operator

$$A_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s) \in \mathrm{Hom}_{\mathrm{GL}^{\diamond}_{2r''}}(\tau'_{1,s} \times \rho_{\tau'',s} \times \tau'^{*}_{2,1-s},\ \tau'_{1,s} \times \rho_{\tau''^{*},1-s} \times \tau'^{*}_{2,1-s})$$

by the integral

$$A_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s)\phi_s(h;\ h';\ m) = \int_{N_{r'',r''}} \phi_s(h;\ w^{-1}_{r'',r''} u h';\ \hat{I}'_{2r''} \hat{w}^{-1}_{r'',r''} m \hat{w}_{r'',r''})\, du$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation in general. Here $h \in \mathrm{GL}_{2r}$, $h' \in \mathrm{GL}_{2r''}$, $m \in M_P$,

$$\hat{w}_{r'',r''} = \begin{pmatrix} I_{r'} & & \\ & w_{r'',r''} & \\ & & I_{r'} \end{pmatrix} \quad \text{and} \quad \hat{I}'_{2r''} = \begin{pmatrix} I_{r'} & & \\ & I'_{2r''} & \\ & & I_{r'} \end{pmatrix}.$$

Recall that $I'_{2r''} \in \mathrm{GL}_{2r''}$ is given by (3-3). The normalization $\mathcal{A}_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s)$ of $A_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s)$ is defined to satisfy

$$\int_{N_{r'',r''}} \phi_s(h;\ uh';\ I_{M_P})\psi_2^{-1}(u_{r'',r''+1})\,du$$
$$= \omega_{\tau''}(\delta)|\delta|^{r''(s-\frac{1}{2})} \int_{N_{r'',r''}} \mathcal{A}_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s)\phi_s(h;\ uh';\ I_{M_P})\psi_2^{-1}(u_{r'',r''+1})\,du.$$

Next, let ξ'_s be an element in the underlying space of $(\tau'_{1,s} \times \tau''^{*}_{2,1-s}) \times (\tau''_{1,s} \times \tau'^{*}_{2,1-s})$. We define

$$A_\psi(w, (\tau'_1 \boxtimes \tau''_2, \tau''_1 \boxtimes \tau'_2), s)$$
$$\in \mathrm{Hom}_{M_{Q_{2r}}}\big((\tau'_{1,s} \times \tau''^{*}_{2,1-s}) \times (\tau''_{1,s} \times \tau'^{*}_{2,1-s}), (\tau''^{*}_{2,1-s} \times \tau'_{1,s}) \times (\tau'^{*}_{2,1-s} \times \tau''_{1,s})\big)$$

by the integral

$$A_\psi(w, (\tau'_1 \boxtimes \tau''_2, \tau''_1 \boxtimes \tau'_2), s)\xi'_s(h;\ a_1, a_2;\ m)$$
$$= \int_{N_{r',r''}} \int_{N_{r'',r'}} \xi'_s(h, w^{-1}_{r',r''}u_1a_1;\ w^{-1}_{r'',r'}u_2a_2;\ \omega^{-1}_{r',r''}m\omega_{r',r''})\,du_1\,du_2$$

for $\mathrm{Re}(s) \gg 0$ and by the meromorphic continuation in general. Here $h \in \mathrm{GL}_{2r}$, $a_1, a_2 \in \mathrm{GL}_r$, $m \in M_P$, and

$$w = \begin{pmatrix} w_{r',r''} & \\ & w_{r'',r'} \end{pmatrix}. \tag{5-6}$$

The normalized intertwining operator $\mathcal{A}_\psi(w, (\tau'_1 \boxtimes \tau''_2, \tau''_1 \boxtimes \tau'_2), s)$ satisfies

$$\int_{N_{r',r''}} \int_{N_{r'',r'}} \xi'_s(h;\ w^{-1}_{r',r''}u_1;\ w^{-1}_{r'',r'}u_2;\ I_{M_P})\psi((u_1)_{r',r'+1} - (u_2)_{r'',r''+1})\,du_1\,du_2$$
$$= \int_{N_{r',r''}} \int_{N_{r'',r'}} \mathcal{A}_\psi(w, (\tau'_1 \boxtimes \tau''_2, \tau''_1 \boxtimes \tau'_2), s)\xi'_s(h;\ w^{-1}_{r',r''}u_1;\ w^{-1}_{r'',r'}u_2;\ I_{M_P})$$
$$\times\ \psi((u_1)_{r',r'+1} - (u_2)_{r'',r''+1})\,du_1\,du_2.$$

Now we have (see [29])

$$\begin{aligned} &\mathcal{A}_\psi(w_{r,r}, \tau_1 \boxtimes \tau_2, s) \\ &\quad = \mathcal{A}_\psi(w_{r',r'}, \tau'_1 \boxtimes \tau'_2, s)\mathcal{A}_\psi(w, (\tau'_1 \boxtimes \tau''_2, \tau''_1 \boxtimes \tau'_2), s)\mathcal{A}_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s), \end{aligned} \tag{5-7}$$

where $\tau' := \tau'_1 \boxtimes \tau'_2$, and

$$\mathcal{A}_\psi(w_{r',r'}, \tau'_1 \boxtimes \tau'_2, s) \in \mathrm{Hom}_{\mathrm{GL}^\diamond_{2r'}}(\tau''^{*}_{2,1-s} \times \rho_{\tau',s} \times \tau''_{1,s}, \tau''^{*}_{2,1-s} \times \rho_{\tau'^{*},1-s} x \tau''_{1,s})$$

is defined analogously to $\mathcal{A}_\psi(w_{r'',r''}, \tau''_1 \boxtimes \tau''_2, s)$.

5.3. ***Conventions.*** We adopt the following conventions for the proofs of Propositions 5.1 and 5.2 in the following two sections.

First, similar to [24; 27], we prove them using formal arguments,[4] neglecting the convergence issues. The convergence can be confirmed through standard methods, as demonstrated in [4, Section 4; 28, Section 4]. Indeed, by applying the Iwasawa decompositions, the problem is typically reduced to estimating Whittaker functions evaluated on torus elements, possibly multiplied by certain unipotent elements. These estimates are provided in [4, Lemmas 4.1, 4.3, and 4.6]. Similarly, we also require estimates of sections evaluated on torus elements, possibly multiplied by certain unipotent elements. These estimates are given in [28, pages 92–94]. Consequently, in the proof of Proposition 5.1, when considering the Whittaker function for $\varphi \in \mathcal{V}(\pi, \psi)$, we omit ζ and pretend that (5-4) converges absolutely.

Next, in the proofs, we often need to show that one integral equals another, even when their domains of absolute convergence may be disjoint. To establish their equality, we proceed as follows: Since both integrals admit meromorphic continuation to the entire complex plane, their equality can be understood in the sense of their meromorphic continuations. Furthermore, as these integrals define an element in the Hom-spaces (3-5) or (3-6), each of which is one-dimensional, the equality follows from the uniqueness of such elements. Because this argument is central to the proof, we will provide at least a brief explanation of it during the proof of Proposition 5.1 (see Section 6.1.4).

6. Multiplicativity: the first variable

In this section, we focus on proving Proposition 5.1. The preliminaries relevant to the proof are provided in Section 5.1. In particular, it suffices to consider the case where $n > r$ by Lemma 5.3. For convenience, let us define

$$\ell = r - n - 1$$

and $\nu(a) = |\det(a)|$ for $a \in \mathrm{GL}_N$.

6.1. ***Proof of Proposition 5.1 when n > r.*** Let

$$\varphi \in \mathcal{V}_{\overline{P}}^{U_{2n+1}}(\pi, \psi) \quad \text{and} \quad \xi_s \in \mathcal{V}_{Q_{2r}}^{U_{2r}}(\tau_s).$$

Recall that

$$\Psi_{n,r}(\varphi \otimes \xi_s) = \int_{V_{\mathrm{GL}_{2n+1}} \backslash \mathrm{GL}_{2n+1}} \int_{\overline{X}^{n,r}} W_\varphi^\psi(g) f_{\xi_s}(\bar{u} \jmath^{n,r}(g);\ I_r, I_r) \psi_{\overline{X}^{n,r}}^{-1}(\bar{u})\, d\bar{u}\, dg.$$

[4]For the rigorous arguments regarding the establishment of the multiplicativity of the gamma factors involving ζ, we refer to [34, Section 11].

The goal is to establish

$$\begin{aligned}\text{(6-1)}\quad &\omega_\sigma(-1)^{r-1}\gamma(s,\sigma_1\times\tau_1,\psi^{-1})\gamma(s,\tilde{\sigma}_2\times\tau_2,\psi^{-1})\Psi_{n,r}(\varphi\otimes\xi_s)\\ &=\int_{N_{\bar{P}}\hat{V}_{\mathrm{G}_k}\,\mathrm{GL}^{\Diamond}_{2n_0+1}\backslash\mathrm{GL}_{2n+1}}\int_{\mathrm{Mat}_{\ell,k}}\int_{\mathrm{Mat}_{k,\ell}}\int_{V_{\mathrm{GL}_{2n_0+1}}\backslash\mathrm{GL}_{2n_0+1}}\varphi(g;\ I_k,I_k;\ g_0)\\ &\quad\times\int_{\overline{X}^{n_0,r}}f_{\xi_s}\left(\bar{u}\jmath^{n_0,r}(g_0)\begin{pmatrix}\dot{x}&\\&\ddot{y}\end{pmatrix}\jmath^{n,r}(g);\ I_r,I_r\right)\psi^{-1}_{\overline{X}^{n_0,r}}(\bar{u})\,d\bar{u}\,dg_0\,dx\,dy\,dg.\end{aligned}$$

Here $\hat{V}_{\mathrm{G}_k}$ and $\mathrm{GL}^{\Diamond}_{2n_0+1}$ are the subgroups of GL_{2n+1} given by

$$\hat{V}_{\mathrm{G}_k}=\left\{\begin{pmatrix}z_1&&\\&I_{2n_0+1}&\\&&z_2\end{pmatrix}\,\middle|\,z_1,z_2\in V_{\mathrm{GL}_k}\right\},\ \mathrm{GL}^{\Diamond}_{2n_0+1}=\left\{\begin{pmatrix}1_k&&\\&g_0&\\&&i_k\end{pmatrix}\,\middle|\,g_0\in\mathrm{GL}_{2n_0+1}\right\}$$

and for $x\in\mathrm{Mat}_{k\times\ell}$ and $y\in\mathrm{Mat}_{\ell\times k}$, we put

$$\text{(6-2)}\qquad \dot{x}=\begin{pmatrix}&I_{n_0+1}&\\&&I_\ell\\I_k&&x\end{pmatrix}\quad\text{and}\quad\ddot{y}=\begin{pmatrix}&&I_k\\I_\ell&&y\\&I_{n_0+1}&\end{pmatrix}.$$

Note that for fixed g and x, y, the $d\bar{u}dg_0$-integrations in (6-1) correspond precisely to the local Rankin–Selberg integral attached to π_0 and τ. Using (6-1), we can follow the arguments in [34, Sections 11.1–11.4] to establish Proposition 5.1 for the case $n<r$. Specifically, from (6-1), we derive

$$\Gamma(s,\pi\times\tau,\psi)=\omega_\sigma(-1)^r\omega_\tau(-1)^k\Gamma(s,\pi_0\times\tau,\psi)\gamma(s,\sigma\times\tau_1,\psi)\gamma(s,\tilde{\sigma}\times\tau_2,\psi),$$

from which Proposition 5.1 for $n<r$ follows. Here $\omega_\sigma:=\omega_{\sigma_1}\omega_{\sigma_2}$.

We remark that when π and τ are unramified, a similar identity was obtained by Morimoto and Soudry in [28, Theorem 7.5].

6.1.1. Substituting (5-4) in $\Psi_{n,r}(\varphi\otimes\xi_s)$ (see (3-8) with v instead of φ), we obtain

$$\begin{aligned}\text{(6-3)}\quad &\int_{V_{\mathrm{GL}_{2n+1}}\backslash\mathrm{GL}_{2n+1}}\int_{N_P}\varphi(yg;I_k,I_k;I_{2n_0+1})\psi^{-1}_{N_P}(y)\\ &\qquad\qquad\times\int_{\overline{X}^{n,r}}f_{\xi_s}(\bar{u}\jmath^{n,r}(g);I_r,I_r)\psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\,d\bar{u}\,dy\,dg\\ &=\int_{V_{\mathrm{GL}_{2n+1}}\backslash\mathrm{GL}_{2n+1}}\int_{N_P}\varphi(yg;I_k,I_k;I_{2n_0+1})\\ &\qquad\qquad\times\int_{\overline{X}^{n,r}}f_{\xi_s}(\bar{u}\jmath^{n,r}(yg);I_r,I_r)\psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\,d\bar{u}\,dy\,dg\\ &=\int_{\hat{V}_{\mathrm{G}_k}V^{\Diamond}_{\mathrm{GL}_{2n_0+1}}\backslash\mathrm{GL}_{2n+1}}\varphi(g;I_k,I_k;I_{2n_0+1})\int_{\overline{X}^{n,r}}f_{\xi_s}(\bar{u}\jmath^{n,r}(g);I_r,I_r)\psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\,d\bar{u}\,dg\end{aligned}$$

where

$$V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}} = \left\{ \begin{pmatrix} I_k & & \\ & z_0 & \\ & & I_k \end{pmatrix} \,\middle|\, z_0 \in V_{\mathrm{GL}_{2n_0+1}} \right\}.$$

Here we have used

$$\int_{\overline{X}^{n,r}} f_{\xi_s}(\bar{u} \jmath^{n,r}(yg); I_r, I_r) \psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\, d\bar{u} = \psi^{-1}_{N_P}(y) \int_{\overline{X}^{n,r}} f_{\xi_s}(\bar{u} \jmath^{n,r}(g); I_r, I_r) \psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\, d\bar{u}$$

for $y \in N_P$.

6.1.2. To proceed, we need the integration formula:

$$\begin{aligned} &\int_{V_{\mathrm{G}_k} V_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} F(g)\, dg \\ &\quad = \int_{\overline{P}' V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} \int_{V_{\mathrm{GL}_k} \backslash \mathrm{GL}_k} \int_{V_{\mathrm{GL}_k} \backslash \mathrm{GL}_k} \int_{\mathrm{Mat}_{k,n_0}} \int_{\mathrm{Mat}_{n_0,k}} \\ &\qquad\qquad F\left(\begin{pmatrix} a & & & & \\ & I_{n_0} & & & \\ & & 1 & & \\ & & & I_{n_0} & \\ & & & & b \end{pmatrix} \begin{pmatrix} I_k & & & & \\ x & I_{n_0} & & & \\ & & 1 & & \\ & & & I_{n_0} & \\ & & & y & I_k \end{pmatrix} g \right) dx\, dy\, da\, db\, dg, \end{aligned}$$

where $\overline{P}' \subset \overline{P}$ is the subgroup given by

$$\bar{P}' = \left\{ \begin{pmatrix} a & & & & \\ x & I_{n_0} & & & \\ & & 1 & & \\ & & & I_{n_0} & \\ & & & y & b \end{pmatrix} \,\middle|\, a, b \in \mathrm{GL}_k,\ x \in \mathrm{Mat}_{n_0,k},\ y \in \mathrm{Mat}_{k,n_0} \right\}.$$

The integral "$\int_{\overline{P}' V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}}$" involves a slight abuse of notation; it should be interpreted in terms of the Iwasawa decomposition of GL_{2n+1} (see also [34, page 69]):

$$\int_{\overline{P}' V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} F(g)\, dg = \int_{\mathrm{GL}_{2n+1}(\mathfrak{o})} \int_{B_{n,k}} \int_{\hat{V}_{\mathrm{G}_k} \backslash \overline{P}'} F(pbk)\, dp\, db\, dk,$$

where $B_{n,k} \subset \mathrm{GL}_{2n+1}$ is the subgroup

$$B_{n,k} = \left\{ \begin{pmatrix} I_k & & & & \\ 0 & I_{n_0} & & & \\ e & 0 & 1 & & \\ v & 0 & 0 & I_{n_0} & \\ \zeta & v' & e' & 0 & I_k \end{pmatrix} \begin{pmatrix} I_k & & & & \\ & t_1 & & & \\ & & 1 & & \\ & & & t_2 & \\ & & & & I_k \end{pmatrix} \mid t_1, t_2 \in T_{\mathrm{GL}_{n_0}} \right\}.$$

Next, observe that for

$$\bar{p} = \begin{pmatrix} a &&&& \\ x & I_{n_0} &&& \\ && 1 && \\ &&& I_{n_0} & \\ &&& by & b \end{pmatrix} \in \bar{P}'$$

we have

- $\bar{u}' := J^{n,r}(\bar{p})^{-1}\bar{u}J^{n,r}(\bar{p}) \in \overline{X}^{n,r}$, $\psi_{\overline{X}^{n,r}}(\bar{u}') = \psi_{\overline{X}^{n,r}}(\bar{u})$ and $d\bar{u} = \nu(ab^{-1})^{-\ell}d\bar{u}'$;
- $\varphi(\bar{p}g;\ I_k, I_k;\ I_{2n_0+1}) = \nu(ab^{-1})^{\frac{2n+1-k}{2}}\varphi(g;\ a, b;\ I_{2n_0+1})$; and
- $f_{\xi_s}(\bar{u}J^{n,r}(\bar{p}g);\ I_r, I_r) = \nu(ab^{-1})^{s-\frac{r-1}{2}} f_{\xi_s}\left(\bar{u}'J^{n,r}(g);\ \begin{pmatrix} a && \\ x & I_{n_0} & \\ && I_{\ell+1} \end{pmatrix};\ \begin{pmatrix} I_{\ell+1} && \\ & I_{n_0} & \\ & by & b \end{pmatrix}\right)$.

These, together with integration formula, (6-3) becomes

$$\text{(6-4)} \quad \int_{\bar{P}'V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}}\backslash \mathrm{GL}_{2n+1}} \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar{u})$$
$$\times \int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{\mathrm{Mat}_{k,n_0}} \int_{\mathrm{Mat}_{n_0,k}} \varphi(g;\ a, b;\ I_{2n_0+1})$$
$$\times f_{\xi_s}\left(\bar{u}J^{n,r}(g);\ \begin{pmatrix} a && \\ x & I_{n_0} & \\ && I_{\ell+1} \end{pmatrix};\ \begin{pmatrix} I_{\ell+1} && \\ & I_{n_0} & \\ & y & b \end{pmatrix}\right)$$
$$\times \nu(a)^{s-\frac{r-k}{2}}\nu(b)^{-s+\frac{r-k}{2}-n_0}\, dx\, dy\, da\, db\, d\bar{u}\, dg$$

after changing the variable $by \mapsto y$.

6.1.3. Since

$$\begin{pmatrix} I_{\ell+1} && \\ & I_{n_0} & \\ & y & b \end{pmatrix} = \begin{pmatrix} & I_{\ell+1} & \\ && I_{n_0} \\ b && y \end{pmatrix} \begin{pmatrix} && I_k \\ I_{\ell+1} && \\ & I_{n_0} & \end{pmatrix}$$

we can multiply (6-4) by

$$\omega_\sigma(-1)^{r-1}\gamma(s, \sigma_1 \times \tau_1, \psi^{-1})\gamma(1-s, \sigma_2 \times \tilde{\tau}_2, \psi)^{-1}$$

and apply the functional equation (5-1) for $\mathrm{GL}_r \times \mathrm{GL}_k$ to obtain

$$\text{(6-5)} \quad \int_{\bar{P}'V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}}\backslash \mathrm{GL}_{2n+1}} \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar{u})$$
$$\times \int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{\mathrm{Mat}_{\ell,k}} \int_{\mathrm{Mat}_{k,\ell}} \varphi(g;\ a, b;\ I_{2n_0+1})$$
$$\times f_{\xi_s}\left(\bar{u}J^{n,r}(g);\ \begin{pmatrix} & I_{n_0+1} & \\ && I_\ell \\ a && x \end{pmatrix};\ \begin{pmatrix} b && \\ y & I_\ell & \\ && I_{n_0+1} \end{pmatrix} \begin{pmatrix} && I_k \\ I_{r-n} && \\ & I_{n_0} & \end{pmatrix}\right)$$
$$\times \nu(a)^{s-\frac{r+k}{2}+n}\nu(b)^{1-s+\frac{r-k}{2}}\, dx\, dy\, da\, db\, d\bar{u}\, dg.$$

Now because

$$\begin{pmatrix} b & & \\ y & I_\ell & \\ & & I_{n_0+1} \end{pmatrix}\begin{pmatrix} & & I_k \\ I_{r-n} & & \\ & I_{n_0} & \end{pmatrix} = \begin{pmatrix} b & & \\ y & I_\ell & \\ & & I_{n_0+1} \end{pmatrix}\begin{pmatrix} & & I_k \\ I_\ell & & \\ & I_{n_0+1} & \end{pmatrix} = \begin{pmatrix} & & b \\ I_\ell & & y \\ & I_{n_0+1} & \end{pmatrix}$$

the above integral can be written as

$$\begin{aligned}(6\text{-}6)\quad & \int_{\bar{P}' V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} \int_{\bar{X}^{n,r}} \psi^{-1}_{\bar{X}^{n,r}}(\bar{u}) \\ & \times \int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{V_{\mathrm{GL}_k}\backslash \mathrm{GL}_k} \int_{\mathrm{Mat}_{\ell,k}} \int_{\mathrm{Mat}_{k,\ell}} \varphi(g;\ a, b;\ I_{2n_0+1}) \\ & \times f_{\xi_s}\left(\bar{u} J^{n,r}(g);\ \begin{pmatrix} & I_{n_0+1} & \\ & & I_\ell \\ a & & x \end{pmatrix};\ \begin{pmatrix} & & b \\ I_\ell & & by \\ & I_{n_0+1} & \end{pmatrix}\right) \\ & \times \nu(a)^{s-\frac{r+k}{2}+n} \nu(b)^{1-s+\frac{r-k}{2}+\ell}\, dx\, dy\, da\, db\, d\bar{u}\, dg\end{aligned}$$

after changing the variable $y \mapsto by$.

In summary, we have shown that

$$\omega_\sigma(-1)^{r-1}\gamma(s, \sigma_1 \times \tau_1, \psi^{-1})\gamma(s, \sigma_2 \times \tilde{\tau}_2, \psi^{-1})\Psi_{n,r}(\varphi \otimes \xi_s)$$

is equal to (6-6), by taking (5-2) into account.

6.1.4. We pause here to explain the derivation from (6-4) to (6-5). First, it can be formally verified that the integrals (6-4) and (6-5) satisfy the equivariant property required in the definition of the Hom-space (3-6).

Next, by analyzing the asymptotic behaviors of Whittaker functions, one can verify that the integrals (6-4) and (6-5) are absolutely convergent in certain regions of the complex plane. Furthermore, when F is non-Archimedean, these integrals admit meromorphic continuations to the entire complex plane. Together with Assumption 3.2, this ensures that the integrals (6-4) and (6-5) define elements in the Hom-space (3-6), for almost all s.

Note that the regions of convergence for the integrals (6-4) and (6-5) may not overlap. However, since both integrals are elements of the Hom-space (3-6), whose dimension is at most one, they must differ by a nonzero meromorphic function.

The final task is to identify this meromorphic function explicitly as

$$\omega_\sigma(-1)^{r-1}\gamma(s, \sigma_1 \times \tau_1, \psi^{-1})\gamma(1-s, \sigma_2 \times \tilde{\tau}_2, \psi)^{-1}.$$

To achieve this, one computes the integrals (6-4) and (6-5) using carefully chosen test functions.

For further discussions about this type of argument, we refer the readers to [34, Section 11].

6.1.5. To continue, we write

$$\hat{a} = \begin{pmatrix} a & \\ & I_{r-k} \end{pmatrix} \quad \text{and} \quad \check{b} = \begin{pmatrix} I_{r-k} & \\ & b \end{pmatrix}$$

for $a, b \in \mathrm{GL}_k$, so that

$$\begin{pmatrix} & I_{n_0+1} & \\ & & I_\ell \\ a & & x \end{pmatrix} = \dot{x}\hat{a} \quad \text{and} \quad \begin{pmatrix} & & b \\ I_\ell & & by \\ & I_{n_0+1} & \end{pmatrix} = \ddot{y}\check{b},$$

where $\dot{x}$ and $\ddot{y}$ are given by (6-2).

Note that for

$$c := \begin{pmatrix} \hat{a} & \\ & \check{b} \end{pmatrix} \in \mathrm{GL}_{2r}$$

we have

- if $\bar{u} \in \overline{X}^{n,r}$ and $\bar{u}' := c\bar{u}c^{-1}$, then $\bar{u}' \in \overline{X}^{n,r}$, $\psi_{\overline{X}^{n,r}}(\bar{u}') = \psi_{\overline{X}^{n,r}}(\bar{u})$ and $d\bar{u} = \nu(ab^{-1})^\ell d\bar{u}'$;
- $\varphi(g;\ a, b;\ I_{2n_0+1}) = \nu(ab^{-1})^{\frac{2n+1-k}{2}} \varphi(cg;\ I_k, I_k;\ I_{2n_0+1})$;
- $f_{\xi_s}(\bar{u}\jmath^{n,r}(g);\ \dot{x}\hat{a};\ \ddot{y}\check{b}) = \nu(ab^{-1})^{-s-\frac{r-1}{2}} f_{\xi_s}\big(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix}\bar{u}'\jmath(cg);\ I_r, I_r\big)$.

All together, the integral (6-6) becomes

$$\begin{aligned} \text{(6-7)} \quad & \int_{\overline{P}'' V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} \varphi(g;\ I_k, I_k;\ I_{2n_0+1}) \\ & \times \int_{\overline{X}^{n,r}} \int_{\mathrm{Mat}_{\ell,k}} \int_{\mathrm{Mat}_{k,\ell}} f_{\xi_s}\left(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \bar{u}\jmath^{n,r}(g);\ I_r, I_r\right) \psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\, dx\, dy\, d\bar{u}\, dg, \end{aligned}$$

where $\overline{P}'' \subset \overline{P}'$ is the subgroup given by

$$\overline{P}'' = \left\{ \begin{pmatrix} z_1 & & & & \\ x & I_{n_0} & & & \\ & & 1 & & \\ & & & I_{n_0} & \\ & & & y & z_2 \end{pmatrix} \middle|\ z_1, z_2 \in \mathrm{GL}_k,\ x \in \mathrm{Mat}_{n_0,k},\ y \in \mathrm{Mat}_{k,n_0} \right\},$$

and we have applied the integration formula

$$\int_{\overline{P}'' \backslash \overline{P}'} F(\bar{p})\, d\bar{p} = \int_{V_{\mathrm{GL}_k} \backslash \mathrm{GL}_k} \int_{V_{\mathrm{GL}_k} \backslash \mathrm{GL}_k} F\left(\begin{pmatrix} a & & & & \\ & I_{n_0} & & & \\ & & 1 & & \\ & & & I_{n_0} & \\ & & & & b \end{pmatrix}\right) \nu(ab^{-1})^{n_0}\, da\, db.$$

6.1.6. The next step is to factor the dg-integral in (6-7) through the rest of the unipotent subgroup $N_{\bar{P}}$, namely, through the subgroup $N'_{\bar{P}} \subset N_{\bar{P}}$ of elements

$$u' = \begin{pmatrix} I_k &&&& \\ 0 & I_{n_0} &&& \\ d_1 & 0 & 1 && \\ d_2 & 0 & 0 & I_{n_0} & \\ e & f_2 & f_1 & 0 & I_k \end{pmatrix}.$$

This gives

$$\int_{\bar{P}'' V^{\diamond}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} \varphi(g;\; I_k, I_k;\; I_{2n_0+1})$$
$$\times \int_{N'_{\bar{P}}} \int_{\bar{X}^{n,r}} \int_{\mathrm{Mat}_{\ell,k}} \int_{\mathrm{Mat}_{k,\ell}} f_{\xi_s}\left(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \bar{u}\, \jmath^{n,r}(u'g);\; I_r, I_r\right) \psi^{-1}_{\bar{X}^{n,r}}(\bar{u})\, dx\, dy\, d\bar{u}\, du'\, dg.$$

We proceed to compute $\bar{u}\,\jmath^{n,r}(u')$. To do so, write

$$\jmath^{n,r}(u') = \begin{pmatrix} L & \\ & W \end{pmatrix} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix}$$

with

$$L = \begin{pmatrix} I_k &&& \\ 0 & I_{n_0} && \\ d_1 & 0 & 1 & \\ 0 & 0 & 0 & I_\ell \end{pmatrix},\quad W = \begin{pmatrix} I_\ell &&& \\ 0 & 1 && \\ 0 & 0 & I_{n_0} & \\ 0 & -\frac{1}{2} f_1 & 0 & I_k \end{pmatrix},\quad Z = \begin{pmatrix} 0 &&& \\ -d_1 & 0 && \\ d_2 & 0 & 0 & \\ e' & f_2 & \frac{1}{2} f_1 & 0 \end{pmatrix}, \tag{6-8}$$

where $e' = e - \frac{1}{2} f_1 d_1$.

On the other hand, every $\bar{u} \in \bar{X}^{n,r}$ can be written as

$$\bar{u} = \begin{pmatrix} I_r & \\ D & I_r \end{pmatrix} \quad \text{with } D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{6-9}$$

where $A \in \mathrm{Mat}_{\ell,n+1}$, $B \in \mathrm{Mat}_{\ell,\ell}$ and $C \in \mathrm{Mat}_{n+1,\ell}$.

Now a simple calculation gives

$$\bar{u}\,\jmath^{n,r}(u') = \begin{pmatrix} L & \\ & W \end{pmatrix} \begin{pmatrix} I_r & \\ D' & I_r \end{pmatrix} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix}, \quad \text{where } D' = \begin{pmatrix} AL & B \\ 0 & W^{-1}C \end{pmatrix}.$$

Since $(AL)_{n+1,\ell} = A_{n+1,\ell}$, $(W^{-1}C)_{1,1} = C_{1,1}$ and $N'_{\bar{P}} \bar{P}'' = N_{\bar{P}} \hat{V}_{\mathrm{G}_k}$, we find that (6-7) is equal to

$$\int_{N_{\bar{P}} \hat{V}_{\mathrm{G}_k} V^{\diamond}_{\mathrm{GL}_{2n_0+1}} \backslash \mathrm{GL}_{2n+1}} \varphi(g;\; I_k, I_k;\; I_{2n_0+1}) \tag{6-10}$$
$$\times \int_{N'_{\bar{P}}} \int_{\bar{X}^{n,r}} \int_{\mathrm{Mat}_{\ell,k}} \int_{\mathrm{Mat}_{k,\ell}} f_{\xi_s}\left(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \begin{pmatrix} L & \\ & W \end{pmatrix} \bar{u} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix} \jmath^{n,r}(g);\; I_r, I_r\right)$$
$$\times\, \psi^{-1}_{\bar{X}^{n,r}}(\bar{u})\, dx\, dy\, d\bar{u}\, du'\, dg$$

after changing the variables

$$AL \mapsto A \quad \text{and} \quad W^{-1}C \mapsto C.$$

Here we understand that $du' = dL\,dW\,dZ$.

6.1.7. Next we compute

$$\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \begin{pmatrix} L & \\ & W \end{pmatrix} \begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix}^{-1} = \begin{pmatrix} \dot{x}L\dot{x}^{-1} & \\ & \ddot{y}W\ddot{y}^{-1} \end{pmatrix}$$

with

$$\dot{x}L\dot{x}^{-1} = \begin{pmatrix} I_{n_0+1} & -dx & d \\ & I_\ell & 0 \\ & & I_k \end{pmatrix} \quad \text{and} \quad \ddot{y}W\ddot{y}^{-1} = \begin{pmatrix} I_k & 0 & f \\ & I_\ell & yf \\ & & I_{n_0+1} \end{pmatrix},$$

where

$$d = \begin{pmatrix} 0 \\ d_1 \end{pmatrix} \in \mathrm{Mat}_{n_0+1,k} \quad \text{and} \quad f = \begin{pmatrix} -\frac{1}{2}f_1 & 0 \end{pmatrix} \in \mathrm{Mat}_{k,n_0+1}.$$

It follows that

$$f_{\xi_s}\left(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \begin{pmatrix} L & \\ & W \end{pmatrix} \bar{u} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix} J^{n,r}(g);\ I_r, I_r\right)$$
$$= \psi\left(d_1x_1 - \tfrac{1}{2}y_\ell f_1\right) f_{\xi_s}\left(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \bar{u} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix} J^{n,r}(g);\ I_r, I_r\right),$$

where x_1 is the first column of x and y_ℓ is the last row of y.

From this, we see that (6-10) can be written as

$$\int_{N_{\bar{P}}\hat{V}_{\mathrm{G}_k}V^{\diamondsuit}_{\mathrm{GL}_{2n_0+1}}\backslash \mathrm{GL}_{2n+1}} \varphi(g, I_k, I_{2n_0+1}) \int_{N'_{\bar{P}}} \int_{\overline{X}^{n,r}} \int_{\mathrm{Mat}_{\ell,k}} \int_{\mathrm{Mat}_{k,\ell}} \psi\left(d_1x_1 - \tfrac{1}{2}y_\ell f_1\right)$$
$$\times f_{\xi_s}\left(\begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \bar{u} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix} J^{n,r}(g), I_r\right) \psi^{-1}_{\overline{X}^{n,r}}(\bar{u})\, dx\, dy\, d\bar{u}\, du'\, dg. \tag{6-11}$$

6.1.8. We continue to deal with the product

$$\bar{u}' = \begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix} \bar{u} \begin{pmatrix} I_r & \\ Z & I_r \end{pmatrix} \begin{pmatrix} \dot{x} & \\ & \ddot{y} \end{pmatrix}^{-1}$$

with $\bar{u} \in \overline{X}^{n,r}$ given by (6-9).

A direct computation shows

$$\bar{u}' = \begin{pmatrix} I_r & \\ H & I_r \end{pmatrix}, \quad \text{where } H = \ddot{y}(D+Z)\dot{x}^{-1} = \ddot{y}D\dot{x}^{-1} + \ddot{y}Z\dot{x}^{-1}.$$

We now compute H term by term. For the first one, let

$$D = \begin{pmatrix} A' & A & B \\ 0 & 0 & C \\ 0 & 0 & C' \end{pmatrix},$$

where $A' \in \mathrm{Mat}_{\ell,k}$, $A \in \mathrm{Mat}_{\ell,n_0+1}$, $B \in \mathrm{Mat}_{\ell,\ell}$, $C \in \mathrm{Mat}_{n_0+1,\ell}$ and $C' \in \mathrm{Mat}_{k,\ell}$. Then we have

$$\ddot{y} D \dot{x}^{-1} = \begin{pmatrix} 0 & C' & 0 \\ A & B' & A' \\ 0 & C & 0 \end{pmatrix}, \quad \text{where } B' = B - A'x + yC'.$$

For the second, recall that Z is given by (6-8). We have

$$\ddot{y} Z \dot{x}^{-1} = \begin{pmatrix} f' & -e'x & e' \\ yf' & -ye'x & ye' \\ 0 & -d'x & d' \end{pmatrix},$$

where

$$d' = \begin{pmatrix} -d_1 \\ d_2 \end{pmatrix} \in \mathrm{Mat}_{n_0+1,k} \quad \text{and} \quad f' = \begin{pmatrix} f_2 & \frac{1}{2} f_1 \end{pmatrix} \in \mathrm{Mat}_{k,n_0+1}.$$

It follows that

$$\bar{u}' = \begin{pmatrix} I_r & \\ H & I_r \end{pmatrix} \quad \text{with } H = \begin{pmatrix} f' & C' - e'x & e' \\ A + yf' & B' - ye'x & A' + ye' \\ 0 & C - d'x & d' \end{pmatrix}.$$

Now the observation is that, as $\bar{u}$ and u' vary, the subgroup of GL_{2r} consisting of the elements $\bar{u}'$, with x and y fixed, is exactly $\overline{X}^{n_0,r}$.

6.1.9. By changing the variables $B' \mapsto B$, $e' \mapsto e$, and then

$$A + yf' \mapsto A, \quad A' + ye \mapsto A', \quad B - yex \mapsto B, \quad C - dx' \mapsto C, \quad C' - ex \mapsto C'$$

in H, and noting that

$$(A + yf')_{\ell,n_0+1} = A_{\ell,n_0+1} + \tfrac{1}{2} y_\ell f_1 \quad \text{and} \quad (C - d'x)_{1,1} = C_{1,1} - d_1 x_1$$

so that $\psi^{-1}_{\overline{X}^{n,r}}(\bar{u})$ changes to

$$\psi\big(-d_1 x_1 + \tfrac{1}{2} y_\ell f_1\big) \psi^{-1}_{\overline{X}^{n_0,r}}(\bar{u}^0),$$

the integral (6-11) becomes

$$\int_{N_{\bar{P}}\hat{V}_{\mathrm{G}_k}V^{\Diamond}_{\mathrm{GL}_{2n_0+1}}\backslash\mathrm{GL}_{2n+1}} \varphi(g;\ I_k, I_k;\ I_{2n_0+1}) \times \int_{\bar{X}^{n_0,r}}\int_{\mathrm{Mat}_{\ell,k}}\int_{\mathrm{Mat}_{k,\ell}} f_{\xi_s}\left(\bar{u}'\begin{pmatrix}\dot{x} & \\ & \ddot{y}\end{pmatrix} J^{n,r}(g);\ I_r, I_r\right) \times \psi^{-1}_{\bar{X}^{n_0,r}}(\bar{u}')\,dx\,dy\,d\bar{u}'\,dg. \tag{6-12}$$

To complete the proof, it remains to factor the dg-integration in (6-12) through the subgroup $\mathrm{GL}^{\Diamond}_{2n_0+1}$. Let $g_0 \in \mathrm{GL}_{2n_0+1}$. Then since

$$\begin{pmatrix}\dot{x} & \\ & \ddot{y}\end{pmatrix} J^{n,r}\left(\begin{pmatrix}I_k & & \\ & g_0 & \\ & & I_k\end{pmatrix}\right) = J^{n_0,r}(g_0)\begin{pmatrix}\dot{x} & \\ & \ddot{y}\end{pmatrix}$$

we get from (6-12) the right-hand side of (6-1). This finishes the proof. □

7. Multiplicativity: the second variable

In this section, we aim to prove Proposition 5.2. The preliminaries relevant to the proof are provided in Section 5.2.

7.1. ***Proof of Proposition 5.2.*** Let ϕ_s be an element in the underlying space of $\tau'_{1,s} \times \rho_{\tau'',s} \times \tau'^{*}_{2,1-s}$, and $\xi_s = \xi_{\phi_s}$ be given by (5-5). The goal is to establish

$$\Gamma(s, \pi \times \tau'', \psi)\Psi_{n,r}(v \otimes \xi_s) = \Psi_{n,r}(v \otimes \xi'_s), \tag{7-1}$$

where $\xi'_s = \xi_{\phi'_s}$ is the one associated to ϕ'_s by the formula similar to (5-5), where

$$\mathcal{A}_\psi(w_{r',r'}, \tau'_1 \boxtimes \tau'_2, s)\phi_s = \phi'_s.$$

Note that it suffices to prove Proposition 5.2. Indeed, if ϕ''_s (resp, ϕ'''_s) is the element in the underlying space of $\tau''^{*}_{2,1-s} \times \rho_{\tau',s} \times \tau''_{1,s}$ (resp. $\tau''^{*}_{2,1-s} \times \rho_{\tau'^{*},1-s} \times \tau''_{1,s}$) such that

$$\mathcal{A}_\psi(w, (\tau'_1 \boxtimes \tau''_2, \tau''_1 \boxtimes \tau'_2), s)\xi_{\phi'_s} = \xi_{\phi''_s} \quad (\text{resp. } \mathcal{A}_\psi(w_{r',r'}, \tau'_1 \boxtimes \tau'_2, s)\phi''_s = \phi'''_s),$$

then (7-1) (with r'' replaced by r') gives

$$\Gamma(s, \pi \times \tau', \psi)\Psi_{n,r}(v \otimes \xi_{\phi''_s}) = \Psi_{n,r}(v \otimes \xi_{\phi'''_s}).$$

Since $\mathcal{A}_\psi(w_{r,r}, \tau_1 \boxtimes \tau_2, s)\xi_{\phi_s} = \xi_{\phi'''_s}$ by (5-7), we find that

$$\begin{aligned}&\Gamma(s, \pi \times \tau', \psi)\Gamma(s, \pi \times \tau'', \psi)\Psi_{n,r}(v \otimes \xi_{\phi_s})\\ &= \Gamma(s, \pi \times \tau', \psi)\Psi_{n,r}(v \otimes \xi_{\phi'_s}) = \Psi_{n,r}(v \otimes \xi_{\phi'''_s}) = \Psi_{n,r}(v \otimes \mathcal{A}_\psi(w_{r,r}, \tau_1 \boxtimes \tau_2, s)\xi_{\phi_s})\end{aligned}$$

and Proposition 5.2 follows.

The above argument has already been employed in [24, Section 8] to establish the multiplicativity of the gamma factors for other classical groups. Additionally, note that we have implicitly used the fact that the Rankin–Selberg integrals and their associated analytic properties can be defined and hold for the induced representation $(\tau'_{1,s} \times \tau''_{1,s}) \times (\tau''^{*}_{2,1-s} \times \tau'^{*}_{2,1-s})$, despite the fact that the representation $\tau'_{1,s} \times \tau''_{1,s}$ (resp. $\tau''^{*}_{2,1-s} \times \tau'^{*}_{2,1-s}$) is not of the form η_s (resp. η_{1-s}), for some representation η of GL_r.

7.1.1. To prove (7-1), let

$$
\begin{aligned}
(7\text{-}2)\quad & f_{\xi_s}(h;\, a_1, a_2) \\
&= \int_{N_{r'',r'}} \int_{N_{r',r''}} \xi_s(h;\, w_{r',r''}^{-1} u_1 a_1;\, w_{r'',r'}^{-1} u_2 a_2;\, I_{M_P}) \\
&\qquad\qquad \times \psi((u_1)_{r',r'+1} - (u_2)_{r'',r''+1})\, du_1\, du_2 \\
&= \int_{N_{P'}} \delta_P^{-\frac{1}{2}}\left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}\right) \phi_s\left(w^{-1} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} uh;\, I_{2r''};\, I_{M_P}\right) \psi^{-1}_{\hat{N}_{r'',r'}}(u)\, du
\end{aligned}
$$

for $h \in \mathrm{GL}_{2r}$ and $a_1, a_2 \in \mathrm{GL}_r$, where w is given by (5-6),

$$
(7\text{-}3)\qquad \hat{N}_{r'',r'} := \left\{ \begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix} \,\middle|\, u_1 \in N_{r'',r'},\ u_2 \in N_{r',r''} \right\}
$$

and $\psi_{\hat{N}_{r'',r'}}$ is the character of $\hat{N}_{r'',r'}$ defined by

$$
\psi_{\hat{N}_{r'',r'}}(u) = \psi(-u_{r',r'+1} + u_{r+r'',r+r''+1}).
$$

We note that f_{ξ_s} here is the same as the one given by (3-1) when $E = F \times F$.

The proof of (7-1) is divided into three cases depending on the sizes of n, r and r''.

7.2. *The case: $n < r''$.* We put $\ell'' = r'' - n - 1 \geq 0$. Note that $n < r''$ implies

$$
\ell = r - n - 1 = r' + \ell'' \geq r'.
$$

7.2.1. Recall that the integral $\Psi_{n,r}(v \otimes \xi_s)$ in this case is given by

$$
\Psi_{n,r}(v \otimes \xi_s) = \int_{V_{\mathrm{GL}_{2n+1}} \backslash \mathrm{GL}_{2n+1}} \int_{\bar{X}^{n,r}} W_v^{\psi}(g) f_{\xi_s}(\bar{u} \jmath^{n,r}(g);\ I_r, I_r) \psi^{-1}_{\bar{X}^{n,r}}(\bar{u})\, d\bar{u}\, dg.
$$

By (3-8) and (7-2), it becomes

$$
\begin{aligned}
(7\text{-}4)\quad & \int_{V_{\mathrm{GL}_{2n+1}} \backslash \mathrm{GL}_{2n+1}} W_v(g) \int_{\bar{X}^{n,r}} \psi^{-1}_{\bar{X}^{n,r}}(\bar{u}) \\
&\qquad \times \int_{\hat{N}_{r'',r'}} \phi_s(w^{-1} u \bar{u} \jmath^{n,r}(g);\ I_{2r''};\ I_{M_P}) \psi^{-1}_{\hat{N}_{r'',r'}}(u)\, du\, d\bar{u}\, dg.
\end{aligned}
$$

One checks directly that

$$\bar{u}' := u\bar{u}u^{-1} \in \overline{X}^{n,r} \quad \text{and} \quad \psi_{\overline{X}^{n,r}}(\bar{u}') = \psi_{\overline{X}^{n,r}}(\bar{u})$$

for $u \in \hat{N}_{r'',r'}$ and $\bar{u} \in \overline{X}^{n,r}$. Since $d\bar{u}' = d\bar{u}$, we can switch the order of u and $\bar{u}$ to obtain

$$\text{(7-5)} \quad \int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} W_v(g) \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar{u}) \times \int_{\hat{N}_{r'',r'}} \phi_s(w^{-1}\bar{u}u\jmath^{n,r}(g);\ I_{2r''};\ I_{M_P})\psi^{-1}_{\hat{N}_{r'',r'}}(u)\, d\bar{u}\, du\, dg.$$

7.2.2. At this point, let $N_{\overline{P}}$ be the unipotent radical of the parabolic subgroup $\overline{P}$ of GL_{2r} that is opposite to P, and $\psi_{N_{\overline{P}}}$ be the character of $N_{\overline{P}}$ given by

$$\psi_{N_{\overline{P}}}(\bar{n}) = \begin{cases} \psi(-\bar{n}_{r,1} + \bar{n}_{2r,r+1}) & \text{if } \ell'' > 0, \\ \psi(-\bar{n}_{r,1} + \bar{n}_{2r,r+1} + \bar{n}_{2r,r} - \bar{n}_{r+1,1}) & \text{if } \ell'' = 0. \end{cases}$$

We now compute $w^{-1}\bar{u}uw$ for $\bar{u} \in \overline{X}^{n,r}$ and $u \in \hat{N}_{r'',r'}$. Write

$$\bar{u} = \begin{pmatrix} I_{r'} & & & \\ 0 & I_{r''} & & \\ A & B & I_{r''} & \\ D & C & 0 & I_{r'} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} I_{r''} & x & 0 & 0 \\ & I_{r'} & 0 & 0 \\ & & I_{r'} & y \\ & & & I_{r''} \end{pmatrix}$$

for some $D \in \mathrm{Mat}_{r'',r''}$, $A, y \in \mathrm{Mat}_{r',r''}$, $B \in \mathrm{Mat}_{r',r'}$ and $C, x \in \mathrm{Mat}_{r'',r'}$. We have

$$w^{-1}\bar{u}uw = \begin{pmatrix} I_{r'} & & \\ & \bar{u}' & \\ & & I_{r'} \end{pmatrix} \begin{pmatrix} I_{r'} & & & \\ x & I_{r''} & & \\ C & 0 & I_{r''} & \\ B' & A & y & I_{r'} \end{pmatrix} = \begin{pmatrix} I_{r'} & & \\ & \bar{u}' & \\ & & I_{r'} \end{pmatrix} \bar{n},$$

where $\bar{u}' = \left(\begin{smallmatrix} I_{r''} & \\ D & I_{r''} \end{smallmatrix}\right) \in \overline{X}^{n,r''}$ and $B' = B + Ax$. Note that $\bar{n} \in N_{\overline{P}}$. Then since

$$\psi_{\overline{X}^{n,r}}(\bar{u})\psi_{\hat{N}_{r'',r'}}(u) = \psi_{\overline{X}^{n,r''}}(\bar{u}')\psi_{N_{\overline{P}}}(\bar{n})$$

the integral (7-5) can be written as

$$\int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} W_v(g) \int_{\overline{X}^{n,r''}} \psi^{-1}_{\overline{X}^{n,r''}}(\bar{u}') \int_{N_{\overline{P}}} \phi_s(\bar{n}w^{-1}\jmath^{n,r}(g);\, \bar{u}';\, I_{M_P})\psi^{-1}_{N_{\overline{P}}}(\bar{n})\, d\bar{n}\, d\bar{u}'\, dg.$$

7.2.3. To proceed, observe that

- $w^{-1}\jmath^{n,r}(g)w = \jmath^{n,r''}(g)$ for $g \in \mathrm{GL}_{2n+1}$;
- $N_{\overline{P}}$ is normalized by $\jmath^{n,r''}(g)$;
- $d\bar{n}' = d\bar{n}$ where $\bar{n}' = \jmath^{n,r''}(g)^{-1}\bar{n}\jmath^{n,r''}(g)$; and
- $\psi_{N_{\overline{P}}}(\bar{n}') = \psi_{N_{\overline{P}}}(\bar{n})$.

From these, the above integral further becomes

$$\int_{N_{\bar P}} \psi_{N_{\bar P}}^{-1}(\bar n) \int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} W_v(g) \int_{\bar X^{n,r''}} \phi_s(\bar n w^{-1}, \bar u' \jmath^{n,r}(g), I_{M_P}) \psi_{\bar X^{n,r''}}^{-1}(\bar u')\, d\bar u'\, dg\, d\bar n$$

after changing the variable $\bar n' \mapsto \bar n$.

As the inner integral is the Rankin–Selberg integral attached to π and τ'' for each fixed $\bar n \in N_{\bar P}$, the functional equation implies that

$$\Gamma(s, \pi \times \tau'', \psi) \Psi_{n,r}(v \otimes \xi_s)$$

can be transformed into

$$\int_{N_{\bar P}} \psi_{N_{\bar P}}^{-1}(\bar n) \int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} W_v(g) \int_{\bar X^{n,r''}} \phi'_s(\bar n w^{-1}; \bar u' \jmath^{n,r}(g); I_{M_P}) \psi_{\bar X^{n,r''}}^{-1}(\bar u')\, d\bar u'\, dg\, d\bar n,$$

which is $\Psi_{n,r}(v \otimes \xi'_s)$ by the above derivations (with ϕ_s replaced by ϕ'_s). Note that here we also employ the argument in Section 6.1.4 to establish the equality. This verifies (7-1) when $n < r''$.

7.3. *The case: $r'' \le n < r$.* Note that in this case, we have

$$\ell = r - n - 1 = r' + (r'' - n - 1) < r'.$$

To simplify the notation, we write $n'' = n - r''$.

Let $V'_{\mathrm{GL}_{2n+1}}$ be the subgroup of $V_{\mathrm{GL}_{2n+1}}$ consisting of the matrices of the form

$$\begin{pmatrix} z_1 & 0 & 0 & * & * \\ & I_{n''} & 0 & 0 & * \\ & & 1 & 0 & 0 \\ & & & I_{n''} & 0 \\ & & & & z_2 \end{pmatrix} \quad \text{with } z_1, z_2 \in V_{\mathrm{GL}_{r''}}.$$

Also, let $\hat N'_{r'',r'}$ and $\hat N''_{r'',r'}$ be the subgroups of $\hat N_{r'',r'}$ consisting of matrices of the form

$$\begin{pmatrix} I_{r''} & 0 & * & 0 & 0 & 0 \\ & I_{n''+1} & 0 & 0 & 0 & 0 \\ & & I_\ell & 0 & 0 & 0 \\ & & & I_\ell & 0 & * \\ & & & & I_{n''+1} & 0 \\ & & & & & I_{r''} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_{r''} & * & 0 & 0 & 0 & 0 \\ & I_{n''+1} & 0 & 0 & 0 & 0 \\ & & I_\ell & 0 & 0 & 0 \\ & & & I_\ell & 0 & 0 \\ & & & & I_{n''+1} & * \\ & & & & & I_{r''} \end{pmatrix},$$

respectively. Recall that $\hat N_{r'',r'}$ is defined in (7-3).

The proof of (7-1) when $r'' \le n < r$ is actually quite lengthy, and we divide it into three steps.

Step 1. As in the previous case, the Rankin–Selberg integral $\Psi_{n,r}(v\otimes\xi_s)$ is given by (7-4). The first step is to show that $\Psi_{n,r}(v\otimes\xi_s)$ can be written as

$$\int_{V'_{\mathrm{GL}_{2n+1}}V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}}\backslash\mathrm{GL}_{2n+1}} W_v(g)\int_{\overline{X}^{n,r}}\psi^{-1}_{\overline{X}^{n,r}}(\bar u) \times\int_{\hat N'_{r'',r'}}\phi_s(w^{-1}u'\bar u\jmath^{n,r}(g);\ I_{2r''};\ I_{M_P})\,du'\,d\bar u\,dg. \tag{7-6}$$

The idea is to eliminate $\hat N''_{r'',r'}$ using a technique known as "root exchange". Roughly speaking, this involves replacing $\hat N''_{r'',r'}$ with a subgroup of $V_{\mathrm{GL}_{2n+1}}$ consisting of elements of the form

$$\begin{pmatrix} I_{r''} & * & 0 & 0 & 0\\ & I_{n''} & 0 & 0 & 0\\ & & 1 & 0 & 0\\ & & & I_{n''} & *\\ & & & & I_{r''}\end{pmatrix}.$$

To do so, let $u\in\hat N_{r'',r'}$ in (7-4) be given by

$$u=\begin{pmatrix} I_{r''} & 0 & x' & 0 & 0 & 0\\ & I_{n''+1} & 0 & 0 & 0 & 0\\ & & I_\ell & 0 & 0 & 0\\ & & & I_\ell & 0 & y'\\ & & & & I_{n''+1} & 0\\ & & & & & I_{r''}\end{pmatrix}\begin{pmatrix} I_{r''} & x'' & 0 & 0 & 0 & 0\\ & I_{n''+1} & 0 & 0 & 0 & 0\\ & & I_\ell & 0 & 0 & 0\\ & & & I_\ell & 0 & 0\\ & & & & I_{n''+1} & y''\\ & & & & & I_{r''}\end{pmatrix}=u'u'' \tag{7-7}$$

with $u'\in\hat N'_{r'',r'}$ and $u''\in\hat N''_{r'',r'}$.

We note that $\bar u':=u''\bar u u''^{-1}$ is contained in $\overline{X}^{n,r}$. Indeed, if

$$\bar u=\begin{pmatrix} I_{r''} & & & & & \\ 0 & I_{n''+1} & & & & \\ 0 & 0 & I_\ell & & & \\ A' & A'' & B & I_\ell & & \\ 0 & 0 & C'' & 0 & I_{n''+1} & \\ 0 & 0 & C' & 0 & 0 & I_{r''}\end{pmatrix} \tag{7-8}$$

then

$$\bar u'=\begin{pmatrix} I_{r''} & & & & & \\ 0 & I_{n''+1} & & & & \\ 0 & 0 & I_\ell & & & \\ A' & A''' & B & I_\ell & & \\ 0 & 0 & C''' & 0 & I_{n''+1} & \\ 0 & 0 & C' & 0 & 0 & I_{r''}\end{pmatrix},$$

where $A'''=A''-A'x''$ and $C'''=C''+y''C'$.

Now if we write

$$x'' = (x_1''\ x_2''), \quad y'' = \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix}, \quad A' = \begin{pmatrix} A_1' \\ A_2' \end{pmatrix}, \quad C' = (C_1'\ C_2'), \tag{7-9}$$

where A_2' (resp. x_2'') is the last row (resp. last column) of A' (resp. x''), and C_1' (resp. y_1'') is the first column (resp. first row) of C' (resp. y''), then (7-4) becomes

$$\begin{aligned}\int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} & W_v(g) \\ & \times \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar u) \int_{\hat N'_{r'',r'}} \int_{\hat N''_{r'',r'}} \phi_s(w^{-1}u'\bar u u''\jmath^{n,r}(g);\ I_{2r''};\ I_{M_P}) \\ & \qquad \times \psi^{-1}_{\hat N_{r'',r'}}(u'')\psi^{-1}(A_2'x_2''+y_1''C_1')\,du'\,du''\,d\bar u\,dg\end{aligned} \tag{7-10}$$

after changing the variable $\bar u' \mapsto \bar u$.

Here we use the facts that $d\bar u' = d\bar u$ and $\psi_{\overline{X}^{n,r}}(\bar u)$ changes to

$$\psi_{\overline{X}^{n,r}}(\bar u)\psi(A_2'x_2''+y_1''C_1')$$

after changing the variable, as well as that $\psi_{\hat N_{r'',r'}}(u) = \psi_{\hat N_{r'',r'}}(u'')$.

7.3.1. To proceed, let

$$z = z(x_1'', x_2'', y_1'', y_2'') = \begin{pmatrix} I_{r''} & x_1'' & 2x_2'' & 0 & 0 \\ & I_{n''} & 0 & 0 & 0 \\ & & 1 & 0 & -y_1'' \\ & & & I_{n''} & y_2'' \\ & & & & I_{r''} \end{pmatrix} \in V_{\mathrm{GL}_{2n+1}}$$

with x_i'', y_j'' $(j = 1, 2)$ as in (7-9).

We have

$$\jmath^{n,r}(z) = \begin{pmatrix} I_{r''} & x_1'' & x_2'' & 0 & 0 & -x_2'' & 0 & 0 \\ & I_{n''} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & -y_1'' \\ & & & I_\ell & 0 & 0 & 0 & 0 \\ & & & & I_\ell & 0 & 0 & 0 \\ & & & & & 1 & 0 & y_1'' \\ & & & & & & I_{n''} & y_2'' \\ & & & & & & & I_{r''} \end{pmatrix} = \begin{pmatrix} I_r & Z \\ & I_r \end{pmatrix} u'',$$

where $u'' \in \hat N''_{r'',r'}$ is as before (see (7-7)), and

$$Z = Z(x_2'', y_2'') = \begin{pmatrix} 0 & -x_2'' & 0 & x_2''y_1'' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y_1'' \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{r,r}$$

with lower left corner $0 \in \mathrm{Mat}_{\ell,\ell}$.

Since

$$\psi_{\mathrm{U}_{2n+1}}(z)=\psi^{-1}_{\hat{N}_{r'',r'}}(u'') \quad\text{and}\quad u''J^{n,r}(g)=\begin{pmatrix} I_r & -Z\\ & I_r\end{pmatrix}J^{n,r}(zg)$$

the integral (7-10) can be written as

$$\text{(7-11)}\quad \int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} W_v(zg)\int_{\overline{X}^{n,r}}\psi^{-1}_{\overline{X}^{n,r}}(\bar{u})$$
$$\times\int_{\hat{N}'_{r'',r'}}\int_{\mathrm{Mat}_{r'',n''}}\int_{\mathrm{Mat}_{n'',r''}}\int_{\mathrm{Mat}_{r'',1}}\int_{\mathrm{Mat}_{1,r''}}\phi_s\left(w^{-1}u'\bar{u}\begin{pmatrix} I_r & -Z\\ & I_r\end{pmatrix}J^{n,r}(zg);I_{2r''};I_{M_P}\right)$$
$$\times\psi^{-1}(A'_2x''_2+y''_1C'_1)\,dy''_1\,dx''_2\,dy''_2\,dx''_1\,du'\,d\bar{u}\,dg.$$

7.3.2. We continue to compute (with $\bar{u}\in\overline{X}^{n,r}$ given by (7-8))

$$\begin{pmatrix} I_r & Z\\ & I_r\end{pmatrix}\bar{u}\begin{pmatrix} I_r & -Z\\ & I_r\end{pmatrix}=\begin{pmatrix} I_{n+1} & c & & \\ & I_\ell & & \\ & & I_\ell & a\\ & & & I_{n+1}\end{pmatrix}\begin{pmatrix} I_{n+1} & & & \\ 0 & I_\ell & & \\ A & B' & I_\ell & \\ 0 & C & 0 & I_{n+1}\end{pmatrix}=b\bar{u}'',$$

where

$$A=(A'\ A''),\quad C=\begin{pmatrix} C''\\ C'\end{pmatrix}\quad\text{and}\quad B'=B+A\begin{pmatrix} -x''_2 & 0 & x''_2y''_1\\ 0 & 0 & 0\\ 0 & 0 & -y''_1\end{pmatrix}C$$

with center $0\in\mathrm{Mat}_{n'',n''}$. Note that $\bar{u}''\in\overline{X}^{n,r}$.

To explain c and a, we further denote

$$A''=(A''_1\ A''_2)\quad\text{and}\quad C''=\begin{pmatrix} C''_1\\ C''_2\end{pmatrix},$$

where A''_2 is the last column of A'', and C''_1 is the first row of C''. Then

$$c=\begin{pmatrix} -x''_2C'+x''_2y''_1C''_1\\ 0\\ -y''_1C'\end{pmatrix}\quad\text{and}\quad a=(A'x''_2\ \ 0\ \ A''_2x''_1-A'x''_2y''_1),$$

where the center zeros of c and a belong to $\mathrm{Mat}_{n'',\ell}$ and $\mathrm{Mat}_{\ell,n''}$, respectively.

Now we write

$$b=\begin{pmatrix} I_{r''} & 0 & 0 & c' & 0 & 0 & 0 & 0\\ & I_{n''} & 0 & 0 & 0 & 0 & 0 & 0\\ & & 1 & 0 & 0 & 0 & 0 & 0\\ & & & I_\ell & 0 & 0 & 0 & 0\\ & & & & I_\ell & 0 & 0 & a'\\ & & & & & 1 & 0 & 0\\ & & & & & & I_{n''} & 0\\ & & & & & & & I_{r''}\end{pmatrix}\begin{pmatrix} I_{r''} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ & I_{n''} & 0 & 0 & 0 & 0 & 0 & 0\\ & & 1 & c'' & 0 & 0 & 0 & 0\\ & & & I_\ell & 0 & 0 & 0 & 0\\ & & & & I_\ell & a'' & 0 & 0\\ & & & & & 1 & 0 & 0\\ & & & & & & I_{n''} & 0\\ & & & & & & & I_{r''}\end{pmatrix}=b'b'',$$

where

$$c' = -x_2''C' + x_2''y_1''C_1'', \quad c'' = -y_1''C', \quad a' = A_2''x_1'' - A'x_2''y_1'', \quad a'' = A'x_2''$$

so that

$$\bar{u}\begin{pmatrix} I_r & -Z \\ & I_r \end{pmatrix} = \begin{pmatrix} I_r & -Z \\ & I_r \end{pmatrix} b'b''\bar{u}''.$$

Since u' commutes with $\left(\begin{smallmatrix} I_r & -Z \\ & I_r \end{smallmatrix}\right)$ and

$$w^{-1}\begin{pmatrix} I_r & -Z \\ & I_r \end{pmatrix} w = \begin{pmatrix} I_{n''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & y_1'' & 0 & 0 & 0 \\ & & I_\ell & 0 & 0 & 0 & 0 & 0 \\ & & & I_{r''} & -x_2''y_2'' & 0 & -x_2'' & 0 \\ & & & & I_{r''} & 0 & 0 & 0 \\ & & & & & I_\ell & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & I_{n''} \end{pmatrix}$$

we find that

$$\phi_s\left(w^{-1}u'\bar{u}\begin{pmatrix} I_r & -Z \\ & I_r \end{pmatrix} J^{n,r}(zg), I_{2r''}, I_{M_P}\right) = \phi_s(w^{-1}u'b'b''\bar{u}''J^{n,r}(zg), I_{2r''}, I_{M_P}).$$

As $b' \in \hat{N}'_{r'',r'}$, we can change the variables $\bar{u}'' \mapsto \bar{u}$ and $u'b' \mapsto u'$ in the integral (7-11) to obtain

$$\text{(7-12)} \quad \int_{V_{\mathrm{GL}_{2n+1}}\backslash \mathrm{GL}_{2n+1}} W_v(zg) \int_{\bar{X}^{n,r}} \psi^{-1}_{\bar{X}^{n,r}}(\bar{u}) \\ \times \int_{\hat{N}'_{r'',r'}} \int_{\mathrm{Mat}_{r'',n''}} \int_{\mathrm{Mat}_{n'',r''}} \int_{\mathrm{Mat}_{r'',1}} \int_{\mathrm{Mat}_{1,r''}} \phi_s(w^{-1}u'b''\bar{u}J^{n,r}(zg);\ I_{2r''};\ I_{M_P}) \\ \times \psi^{-1}(A_2'x_2'' + y_1''C_1')\, dy_1''\, dx_2''\, dy_2''\, dx_1''\, du'\, d\bar{u}\, dg.$$

7.3.3. To accomplish the first step, observe that u' also commutes with b'', and

$$w^{-1}b''w = \begin{pmatrix} I_{n''} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & -y_1''C' & 0 & 0 & 0 & 0 & 0 \\ & & I_\ell & 0 & 0 & 0 & 0 & 0 \\ & & & I_{r''} & 0 & 0 & 0 & 0 \\ & & & & I_{r''} & 0 & 0 & 0 \\ & & & & & I_\ell & A'x_2'' & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & I_{n''} \end{pmatrix}.$$

These imply

$$\phi_s(w^{-1}u'b''\bar{u}J^{n,r}(zg);\ I_{2r''};\ I_{M_P})\psi^{-1}(A_2'x_2'' + y_1''C_1') \\ = \phi_s(w^{-1}u'\bar{u}J^{n,r}(zg);\ I_{2r''};\ I_{M_P}).$$

Therefore, (7-12) can be further changed into (7-6). This finishes the first step.

Step 2. The second step is show that the integral (7-6) can be further written as

$$\int_{V'_{\mathrm{GL}_{2n+1}} V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}} \overline{X}_{n,r''} \backslash \mathrm{GL}_{2n+1}} \int_{\overline{X}_{n,r''}} W_v(\bar{x}g) \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar{u}) \\ \times \int_{\hat{N}'_{r'',r'}} \phi_s(w^{-1}u'\bar{u}\jmath^{n,r}(g), I_{2r''}, I_{M_P})\, du'\, d\bar{u}\, d\bar{x}\, dg. \tag{7-13}$$

Here the dg-integration is understood in the sense of the Iwasawa decomposition, as previously explained.

We first factor the dg-integration in (7-6) through $\overline{X}_{n,r''}$ to obtain

$$\int_{V'_{\mathrm{GL}_{2n+1}} V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}} \overline{X}_{n,r''} \backslash \mathrm{GL}_{2n+1}} \int_{\overline{X}_{n,r''}} W_v(\bar{x}g) \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar{u}) \\ \times \int_{\hat{N}'_{r'',r'}} \phi_s(w^{-1}u'\bar{u}\jmath^{n,r}(\bar{x})\jmath^{n,r}(g), I_{2r''}, I_{M_P})\, du'\, d\bar{u}\, d\bar{x}\, dg. \tag{7-14}$$

At this point, we claim that $\bar{u}' := \jmath^{n,r}(\bar{x})^{-1}\bar{u}\jmath^{n,r}(\bar{x}) \in \overline{X}^{n,r}$ and $\psi_{\overline{X}^{n,r}}(\bar{u}') = \psi_{\overline{X}^{n,r}}(\bar{u})$. Indeed, if

$$\bar{x} = \begin{pmatrix} I_{r''} & & & & \\ x_1 & I_{n''} & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & I_{n''} & \\ 0 & 0 & 0 & x_2 & I_{r''} \end{pmatrix} \in \overline{X}_{n,r''}$$

then

$$\jmath^{n,r}(\bar{x}) = \begin{pmatrix} X_1 & & & \\ & I_\ell & & \\ & & I_\ell & \\ & & & X_2 \end{pmatrix}, \quad \text{where } X_1 = \begin{pmatrix} I_{r''} & & \\ x_1 & I_{n''} & \\ 0 & 0 & 1 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 1 & & \\ 0 & I_{n''} & \\ 0 & x_2 & 1_{r''} \end{pmatrix}.$$

Since

$$\bar{u} = \begin{pmatrix} I_{n+1} & & & \\ 0 & I_\ell & & \\ A & B & I_\ell & \\ 0 & C & 0 & I_{n+1} \end{pmatrix}$$

with $\psi_{\overline{X}^{n,r}}(\bar{u}) = \psi(A_{\ell,n+1} - C_{1,1})$, a simple computation shows

$$\bar{u}' = \begin{pmatrix} I_{n+1} & & & \\ 0 & I_\ell & & \\ AX_1 & B & I_\ell & \\ 0 & X_2^{-1}C & 0 & I_{n+1} \end{pmatrix}$$

with $(AX_1)_{\ell,n+1} = A_{\ell,n+1}$ and $(X_2^{-1}C)_{1,1} = C_{1,1}$. These prove the claim.

Now the integral (7-14) can be altered to

$$\int_{V'_{\mathrm{GL}_{2n+1}}V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}}\overline{X}_{n,r''}\backslash\mathrm{GL}_{2n+1}}\int_{\overline{X}_{n,r''}}W_v(\bar{x}g)\int_{\overline{X}^{n,r}}\psi^{-1}_{\overline{X}^{n,r}}(\bar{u}) \times\int_{\hat{N}'_{r'',r'}}\phi_s(w^{-1}u'\jmath^{n,r}(\bar{x})\bar{u}\jmath^{n,r}(g),I_{2r''},I_{M_P})\,du'\,d\bar{u}\,d\bar{x}\,dg \tag{7-15}$$

after changing the variable $\bar{u}'\mapsto\bar{u}$, and employing the fact that $d\bar{u}'=d\bar{u}$.

7.3.4. To proceed, let $u'\in\hat{N}'_{r'',r'}$ be given by (7-7). We have

$$u'\jmath^{n,r}(\bar{x})=\jmath^{n,r}(\bar{x})\begin{pmatrix}z_1&\\&z_2\end{pmatrix}u',$$

where

$$z_1=\begin{pmatrix}I_{r''}&0&0&0\\&I_{n''}&0&-x_1x'\\&&1&0\\&&&I_\ell\end{pmatrix}\quad\text{and}\quad z_2=\begin{pmatrix}I_\ell&0&y'x_2&0\\&1&0&0\\&&I_{n''}&0\\&&&I_{r''}\end{pmatrix}.$$

Since

$$w^{-1}\jmath^{n,r}(\bar{x})w=\begin{pmatrix}I_{n''}&0&0&x_1&0&0&0&0\\&1&0&0&0&0&0&0\\&&I_\ell&0&0&0&0&0\\&&&I_{r''}&0&0&0&0\\&&&&I_{r''}&0&0&x_2\\&&&&&I_\ell&0&0\\&&&&&&1&0\\&&&&&&&I_{n''}\end{pmatrix}$$

and

$$w^{-1}\begin{pmatrix}z_1&\\&z_2\end{pmatrix}w=\begin{pmatrix}I_{n''}&0&-x_1x'&0&0&0&0&0\\&1&0&0&0&0&0&0\\&&I_\ell&0&0&0&0&0\\&&&I_{r''}&0&0&0&0\\&&&&I_{r''}&0&0&0\\&&&&&I_\ell&0&y'x_2\\&&&&&&1&0\\&&&&&&&I_{n''}\end{pmatrix}$$

we find that

$$\begin{aligned}\phi_s(w^{-1}u'\jmath^{n,r}(\bar{x})\bar{u}\jmath^{n,r}(g),I_{2r''},I_{M_P})&=\phi_s\left(w^{-1}\begin{pmatrix}z_1&\\&z_2\end{pmatrix}u'\bar{u}\,\jmath^{n,r}(g),I_{2r''},I_{M_P}\right)\\&=\phi_s(w^{-1}u'\bar{u}\jmath^{n,r}(g),I_{2r''},I_{M_P}).\end{aligned}$$

From these, we see that (7-15) becomes (7-13). This verifies the second step.

Step 3. The last step is to establish (7-1). We first factor the dg-integration in (7-13) through $\jmath_{n,r''}(\mathrm{GL}_{2r''})$ to obtain

$$\int_{V''_{\mathrm{GL}_{2n+1}} V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}} \overline{X}_{n,r''}\jmath_{n,r''}(\mathrm{GL}_{2r''})\backslash \mathrm{GL}_{2n+1}} \int_{V_{\mathrm{GL}_{2r''}}\backslash \mathrm{GL}_{2r''}} \int_{\overline{X}_{n,r''}} W_v(\bar{x}\jmath_{n,r''}(h)g) \times \int_{\overline{X}^{n,r}} \psi^{-1}_{\overline{X}^{n,r}}(\bar{u}) \int_{\hat{N}'_{r'',r'}} \phi_s(w^{-1}u'\bar{u}\jmath^{n,r}(\jmath_{n,r''}(h)g), I_{2r''}, I_{M_P})\, du'\, d\bar{u}\, d\bar{x}\, dh\, dg. \tag{7-16}$$

Here $V''_{\mathrm{GL}_{2n+1}}$ denotes the subgroup of $V'_{\mathrm{GL}_{2n+1}}$ consisting of matrices of the form

$$\begin{pmatrix} I_{r''} & 0 & 0 & * & 0 \\ & I_{n''} & 0 & 0 & * \\ & & 1 & 0 & 0 \\ & & & I_{n''} & 0 \\ & & & & I_{r''} \end{pmatrix}$$

and the dg-integration in (7-16) should be understood in the sense of the Iwasawa decomposition.

To continue, let u' and $\bar{u}$ be given by (7-7) and (7-8), respectively. We have

$$w^{-1}u'\bar{u}w = \begin{pmatrix} I_{n''+1} & & & & & \\ 0 & I_\ell & & & & \\ 0 & x' & I_{r''} & & & \\ 0 & C' & 0 & I_{r''} & & \\ A'' & B' & A' & y' & I_\ell & \\ 0 & C'' & 0 & 0 & 0 & I_{n''+1} \end{pmatrix}, \tag{7-17}$$

where $B' = B + y'C'$.

Let $\overline{Y}$ be the subgroup of GL_{2r} consisting of the matrices similar to those given by (7-17), but with B' replaced by arbitrary matrices $B \in \mathrm{Mat}_{\ell,\ell}$. We adapt $\psi_{\overline{X}^{n,r}}$ to be the character $\psi_{\overline{Y}}$ of $\overline{Y}$. More concretely, if $\bar{y} \in \overline{Y}$ is given by (7-17) (with B' replaced by B), then

$$\psi_{\overline{Y}}(\bar{y}) = \psi((A'')_{n''+1,\ell} - (C'')_{1,1}).$$

With these, we see that (7-16) becomes

$$\int_{V'_{\mathrm{GL}_{2n+1}} V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}} \overline{X}_{n,r''}\jmath_{n,r''}(\mathrm{GL}_{2r''})\backslash \mathrm{GL}_{2n+1}} \int_{\overline{Y}} \psi^{-1}_{\overline{Y}}(\bar{y}) \times \int_{V_{\mathrm{GL}_{2r''}}\backslash \mathrm{GL}_{2r''}} \int_{\overline{X}_{n,r''}} W_v(\bar{x}\jmath_{n,r''}(h)g)\phi_s\big(\bar{y}w^{-1}\jmath^{n,r}(\jmath_{n,r''}(h)g), I_{2r''}, I_{M_P}\big)\, d\bar{x}\, dh\, d\bar{y}\, dg. \tag{7-18}$$

7.3.5. To finish the proof of the case $r'' \le n < r$, let $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_{2r''}$ with $a, b, c, d \in \mathrm{Mat}_{r'',r''}$. We have

$$w^{-1}\jmath^{n,r}(\jmath_{n,r''}(h))w = \begin{pmatrix} I_{r'} & & \\ & h & \\ & & I_{r'} \end{pmatrix}.$$

Now the key observation is that the subgroup $\overline{Y}$, the character $\psi_{\overline{Y}}$ and the Haar measure $d\bar{y}$ are invariant under the conjugation of $w^{-1}\jmath^{n,r}(\jmath_{n,r''}(h))w$. It follows that

$$\phi_s\big(\bar{y}w^{-1}\jmath^{n,r}(\jmath_{n,r''}(h)g),\, I_{2r''},\, I_{M_P}\big)=\phi_s(\bar{y}w^{-1}\jmath^{n,r}(g),\, h,\, I_{M_P}).$$

Consequently, the integral (7-18) can be changed to

$$\int_{V'_{\mathrm{GL}_{2n+1}}V^{\diamondsuit}_{\mathrm{GL}_{2n''+1}}\overline{X}_{n,r''}\jmath_{n,r''}(\mathrm{GL}_{2r''})\backslash\mathrm{GL}_{2n+1}}\int_{\overline{Y}}\psi_{\overline{Y}}^{-1}(\bar{y}) \times\int_{V_{\mathrm{GL}_{2r''}}\backslash\mathrm{GL}_{2r''}}\int_{\overline{X}_{n,r''}}W_v(\bar{x}\jmath_{n,r''}(h)g)\phi_s(\bar{y}w^{-1}\jmath^{n,r}(g),h,I_{M_P})\,d\bar{x}\,dh\,d\bar{y}\,dg. \tag{7-19}$$

Since the inner integral in (7-19) represents the Rankin–Selberg integral attached to π and τ'' for every fixed $g\in\mathrm{GL}_{2n+1}$ and $\bar{y}\in\overline{Y}$, we conclude that (7-1) holds when $r''\le n<r$ by the same argument as in the proof for the case $n<r''$.

7.4. ***The case: $r\le n$.*** By (3-7) and (7-2), the Rankin–Selberg integral $\Psi_{n,r}(v\otimes\xi_s)$ is equal to

$$\int_{V_{\mathrm{GL}_{2r}}\backslash\mathrm{GL}_{2r}}\int_{\overline{X}_{n,r}}\int_{\hat{N}_{r'',r'}}W_v(\bar{x}\jmath_{n,r}(h))\phi_s(w^{-1}uh,I_{2r''},I_{M_P})\psi^{-1}_{\hat{N}_{r'',r'}}(u)\,du\,d\bar{x}\,dh. \tag{7-20}$$

Since $\jmath_{n,r}(u)\in V_{\mathrm{GL}_{2n+1}}$ for $u\in\hat{N}_{r'',r'}$ and $\psi_{\mathrm{U}_{2n+1}}(\jmath_{n,r}(u))=\psi^{-1}_{\hat{N}_{r'',r'}}(u)$, we find that

$$W_v(\jmath_{n,r}(u)h)=\psi_{\mathrm{U}_{2n+1}}(\jmath_{n,r}(u))W_v(h)=\psi^{-1}_{\hat{N}_{r'',r'}}(u)W_v(u).$$

As a result, the integral (7-20) can be written as

$$\int_{V_{\mathrm{GL}_{2r}}\backslash\mathrm{GL}_{2r}}\int_{\overline{X}_{n,r}}\int_{\hat{N}_{r'',r'}}W_v(\jmath_{n,r}(u)\bar{x}\jmath_{n,r}(h))\phi_s(w^{-1}uh,I_{2r''},I_{M_P})\,du\,d\bar{x}\,dh. \tag{7-21}$$

We claim that $\bar{x}':=\jmath_{n,r}(u)\bar{x}\jmath_{n,r}(u)^{-1}\in\overline{X}_{n,r}$ and $d\bar{x}'=d\bar{x}$. For this, define

$$\bar{x}=\begin{pmatrix} I_{r''} &&&&&& \\ 0 & I_{r'} &&&&& \\ A_1 & A_1 & I_{n-r} &&&& \\ 0&0&0&1&&& \\ 0&0&0&0&I_{n-r}&& \\ 0&0&0&0&B_2&I_{r'}& \\ 0&0&0&0&B_1&0&I_{r''}\end{pmatrix},\quad u=\begin{pmatrix} I_{r''} & x & 0&0&0&0&0 \\ & I_{r'} & 0&0&0&0&0 \\ && I_{n-r}&0&0&0&0 \\ &&&1&0&0&0 \\ &&&&I_{n-r}&0&0 \\ &&&&&I_{r'}&y \\ &&&&&&I_{r''}\end{pmatrix}. \tag{7-22}$$

Then

$$\bar{x}' = \begin{pmatrix} I_{r''} & & & & & & \\ 0 & I_{r'} & & & & & \\ A_1 & A_2' & I_{n-r} & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & I_{n-r} & & \\ 0 & 0 & 0 & 0 & B_2' & I_{r'} & \\ 0 & 0 & 0 & 0 & B_1 & 0 & I_{r''} \end{pmatrix}$$

with $A_2' = A_2 - A_1 x$ and $B_2' = B_2 + yB_1$. This verifies the claim.

From this, we see that the integral (7-21) can be altered to

$$\text{(7-23)} \quad \int_{V_{\mathrm{GL}_{2r}} \backslash \mathrm{GL}_{2r}} \int_{\bar{X}_{n,r}} \int_{\hat{N}_{r'',r'}} W_v(\bar{x} J_{n,r}(uh)) \phi_s(w^{-1}uh, I_{2r''}, I_{M_P}) \, du \, d\bar{x} \, dh$$
$$= \int_{V_{M_{P'}} N_{Q_{2r}} \backslash \mathrm{GL}_{2r}} \int_{\bar{X}_{n,r}} W_v(\bar{x} J_{n,r}(h)) \phi_s(w^{-1}h, I_{2r''}, I_{M_P}) \, d\bar{x} \, dh$$

after changing the variable $\bar{x}' \mapsto \bar{x}$.

Here $P' \subset \mathrm{GL}_{2r}$ stands for the parabolic subgroup of block upper triangular matrices whose Levi subgroup $M_{P'}$ is isomorphic to $\mathrm{GL}_{r''} \times \mathrm{GL}_{r'} \times \mathrm{GL}_{r'} \times \mathrm{GL}_{r''}$, and we put $V_{M_{P'}} = M_{P'} \cap V_{\mathrm{GL}_{2r}}$.

7.4.1. Let $\check{N}_{r'',r'}$ be the subgroup of GL_{2r} consisting of the matrices of the form

$$\begin{pmatrix} I_{r''} & & & \\ * & I_{r'} & & \\ 0 & 0 & I_{r'} & \\ 0 & 0 & * & I_{r''} \end{pmatrix}.$$

We factor the dh-integration in (7-23) through $\check{N}_{r'',r'}$ to obtain

$$\text{(7-24)} \quad \int_{V_{M_{P'}} N_{Q_{2r}} \check{N}_{r'',r'} \backslash \mathrm{GL}_{2r}} \int_{\bar{X}_{n,r}} \int_{\check{N}_{r'',r'}} W_v(\bar{x} J_{n,r}(\bar{y}h)) \phi_s(w^{-1}\bar{y}h, I_{2r''}, I_{M_P}) \, d\bar{y} \, d\bar{x} \, dh.$$

As before, the dh-integration should be understood in the sense of the Iwasawa decomposition.

Now if

$$\bar{y} = \begin{pmatrix} I_{r''} & & & \\ y_1 & I_{r'} & & \\ 0 & 0 & I_{r'} & \\ 0 & 0 & y_2 & I_{r''} \end{pmatrix} \in \check{N}_{r'',r'} \quad \text{then} \quad w^{-1}\bar{y}w = \begin{pmatrix} I_{r''} & y_1 & 0 & 0 \\ 0 & I_{r'} & 0 & 0 \\ & & I_{r'} & y_2 \\ & & & I_{r''} \end{pmatrix},$$

which implies

$$\phi_s(w^{-1}\bar{y}h, I_{2r''}, I_{M_P}) = \phi_s(w^{-1}h, I_{2r''}, I_{M_P}).$$

We thus conclude that (7-24) becomes

$$\int_{V_{M_{P'}}N_{Q_{2r}}\check{N}_{r'',r'}\backslash \mathrm{GL}_{2r}}\int_{\overline{X}_{n,r}}\int_{\check{N}_{r'',r'}} W_v(\bar{x}\,j_{n,r}(\bar{y}h))\phi_s(w^{-1}h, I_{2r''}, I_{M_P})\,d\bar{y}\,d\bar{x}\,dh. \tag{7-25}$$

7.4.2. We proceed to compute $\bar{x}\,j_{n,r}(\bar{y})$. Let $\bar{x}\in\overline{X}_{n,r}$ be as in (7-22) and $\bar{y}\in\check{N}_{r'',r'}$ be as above. We have

$$j_{n,r}(\bar{y})=\begin{pmatrix} I_{r''} \\ y_1 & I_{r'} \\ 0 & 0 & I_{n-r} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & I_{n-r} \\ 0 & 0 & 0 & 0 & 0 & I_{r'} \\ 0 & 0 & 0 & 0 & 0 & y_2 & I_{r''} \end{pmatrix}$$

so that

$$\bar{x}\,j_{n,r}(\bar{y})=\begin{pmatrix} I_{r''} \\ y_1 & I_{r'} \\ A_1 & A_2 & I_{n-r} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & I_{n-r} \\ 0 & 0 & 0 & 0 & B_2 & I_{r'} \\ 0 & 0 & 0 & 0 & B_1' & y_2 & I_{r''} \end{pmatrix},$$

where $B_1' = B_1 + y_2B_2$.

We can decompose $\bar{x}\,j_{n,r}(\bar{y})$ as

$$\begin{pmatrix} I_{r''} \\ y_1 & I_{r'} \\ A_1 & 0 & I_{n-r} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & I_{n-r} \\ 0 & 0 & 0 & 0 & 0 & I_{r'} \\ 0 & 0 & 0 & 0 & B_1 & y_2 & I_{r''} \end{pmatrix}\begin{pmatrix} I_{r''} \\ 0 & I_{r'} \\ 0 & A_2 & I_{n-r} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & I_{n-r} \\ 0 & 0 & 0 & 0 & B_2 & I_{r'} \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{r''} \end{pmatrix}=\bar{x}''\bar{y}'.$$

Observe that $\bar{x}''\in\overline{X}_{n,r''}$. On the other hand, let $\check{N}'_{r'',r'}$ be the subgroup of GL_{2n+1} consisting of the matrices $\bar{y}'$ (as A_2, B_2 vary).

Then the integral (7-25) can be written as

$$\int_{V_{M_{P'}}N_{Q_{2r}}\check{N}_{r'',r'}\backslash \mathrm{GL}_{2r}}\int_{\check{N}'_{r'',r'}}\int_{\overline{X}_{n,r''}} W_v(\bar{x}''\bar{y}'\,j_{n,r}(h))\phi_s(w^{-1}h, I_{2r''}, I_{M_P})\,d\bar{x}''\,d\bar{y}'\,dh.$$

7.4.3. The last step is to factor the dh-integration in the previous display through $\iota_{r,r''}(\mathrm{GL}_{2r''})$:

$$\int_{V'_{M_{P'}} N'_{Q_{2r}} \check{N}_{r'',r'} \jmath_{r,r''}(\mathrm{GL}_{2r''})\backslash \mathrm{GL}_{2r}} \int_{\check{N}'_{r'',r'}} \int_{V_{\mathrm{GL}_{2r''}}\backslash \mathrm{GL}_{2r''}} \times \int_{\bar{X}_{n,r''}} W_v\big(\bar{x}''\bar{y}'\jmath_{n,r}(\iota_{r,r''}(h')h)\big)\phi_s(w^{-1}\iota_{r,r''}(h')h, I_{2r''}, I_{M_P})\, d\bar{x}''\, dh'\, d\bar{y}'\, dh. \tag{7-26}$$

Here $V'_{M_{P'}}$ (resp. $N'_{Q_{2r}}$) is the subgroup of $V_{M_{P'}}$ (resp. $N_{Q_{2r}}$) consisting of the matrices of the form

$$\begin{pmatrix} I_{r''} & & & \\ & z & & \\ & & z' & \\ & & & I_{r''} \end{pmatrix} \quad \left(\text{resp.} \quad \begin{pmatrix} I_{r''} & 0 & * & 0 \\ & I_{r'} & * & * \\ & & I_{r'} & 0 \\ & & & I_{r''} \end{pmatrix}\right)$$

for $z, z' \in V_{\mathrm{GL}_{2r'}}$ and $\iota_{r,r''} : \mathrm{GL}_{2r''} \hookrightarrow \mathrm{GL}_{2r}$ is the embedding given by

$$h' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b \\ & I_{2r'} & \\ c & & d \end{pmatrix},$$

where $a, b, c, d \in \mathrm{Mat}_{r'',r''}$.

Since

$$w^{-1}\iota_{r,r''}(h)w = \begin{pmatrix} I_{r'} & & \\ & h' & \\ & & I_{r'} \end{pmatrix}$$

we see that

$$\phi_s(w^{-1}\iota_{r,r''}(h')h, I_{2r''}, I_{M_P}) = \phi_s(w^{-1}h, h', I_{M_P}).$$

Moreover, since $\jmath_{n,r}(\iota_{r,r''}(h')) = \jmath_{n,r''}(h')$ and $\bar{y}'\jmath_{n,r''}(h') = \jmath_{n,r''}(h')\bar{y}'$, the integral (7-26) becomes

$$\int_{V'_{M_{P'}} N'_{Q_{2r}} \check{N}_{r'',r'}\iota_{r,r''}(\mathrm{GL}_{2r''})\backslash \mathrm{GL}_{2r}} \int_{\check{N}'_{r'',r'}} \int_{V_{\mathrm{GL}_{2r''}}\backslash \mathrm{GL}_{2r''}} \times \int_{\bar{X}_{n,r''}} W_v(\bar{x}''\jmath_{n,r''}(h')\bar{y}'\jmath_{n,r}(h))\phi_s(w^{-1}h, h', I_{M_P})\, d\bar{x}''\, dh'\, d\bar{y}'\, dh. \tag{7-27}$$

As the inner integral in (7-27) represents the Rankin–Selberg integral attached to π and τ'' for every fixed $h \in \mathrm{GL}_{2r}$ and $\bar{y}' \in \check{N}'_{r'',r'}$, we conclude that (7-1) holds when $r \le n$ using the same argument as in the proof for the case $n < r''$. This conclude the proof of (7-1), and also the proof of Proposition 5.2. □

8. Minimal case

In this final section, we prove Theorem 4.1(6). We mention that when E is a field, Kaplan also did similar computations in [24, Section 6.1].

8.1. ***Preliminaries.*** Recall that $\delta \in E^\times$ is an element such that $\bar{\delta} = -\delta$, and we have put $\Delta = \delta^2 \in F^\times$. Moreover, when $E = F \times F$, $\delta = (\delta_0, -\delta_0)$ for some $\delta_0 \in F^\times$.

8.1.1. Suppose that E is a field. We have an isomorphism

$$\mathrm{SL}_2(F) \xrightarrow{\sim} \mathrm{SU}_2(F); \quad h_1 \mapsto \begin{pmatrix} \delta & \\ & 1 \end{pmatrix} h_1 \begin{pmatrix} \delta & \\ & 1 \end{pmatrix}^{-1}.$$

It then follows from the Hilbert's Satz 90 that every $h \in \mathrm{U}_2(F)$ can be written as

$$h = \begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} \delta & \\ & 1 \end{pmatrix} h_1 \begin{pmatrix} \delta & \\ & 1 \end{pmatrix}^{-1} \tag{8-1}$$

for some $a \in E^\times$ and $h_1 \in \mathrm{SL}_2(F)$. Note that $\det(h) = a\bar{a}^{-1}$.

8.1.2. Let τ be an irreducible representation of $E^\times$. Therefore, $\tau = \chi$ is a character of $E^\times$ if E is a field, and τ is of the form $\chi_1 \boxtimes \chi_2$ for some characters χ_1, χ_2 of $F^\times$ when $E = F \times F$. We denote by χ_0 the restriction of χ to $F^\times$. Note that $\tau^*(a) = \chi(\bar{a})^{-1}$ if E is a field, and $\tau^* = \chi_2^{-1} \boxtimes \chi_1^{-1}$ if $E = F \times F$.

Since $Q_2 = B_{\mathrm{U}_2}$, the intertwining map (see Section 3.3)

$$A_\psi(w_{1,1}, \tau, s) : \mathcal{V}_{B_{\mathrm{U}_2}}^{\mathrm{U}_2}(\tau_s) \to \mathcal{V}_{B_{\mathrm{U}_2}}^{\mathrm{U}_2}(\tau_{1-s}^*)$$

is given by

$$A_\psi(w_{1,1}, \tau, s)\xi_s(h) = \int_{V_{\mathrm{U}_2}(F)} \xi_s(w_{1,1}^{-1} uh)\, du.$$

Given that $V_{\mathrm{U}_2}(F) \cong F$, the Haar measure du is self-dual with respect to ψ_2 when $E = F \times F$. On the other hand, when E is a field and we identify F with E^0 through the mapping $x \mapsto \delta x$, the Haar measure becomes $|\delta|_E^{1/2}$ times the self-dual measure with respect to ψ_2.

The normalized one $\mathcal{A}_\psi(w_{1,1}, \tau, s)$ is defined to satisfy

$$\int_{V_{\mathrm{U}_2}(F)} \xi_s(w_{1,1} uh)\psi'(u_{1,2})\, du = \tau(\delta)|\delta|_E^{s-\frac{1}{2}} \int_{V_{\mathrm{U}_2}(F)} \mathcal{A}_\psi(w_{1,1}, \tau, s)\xi_s(w_{1,1} uh)\psi'(u_{1,2})\, du. \tag{8-2}$$

Here ψ' is the character of E given by $\psi'(x) = \psi_E(\delta x)$ when E is a field, and $\psi' = \psi_{2\delta_0}$ when $E = F \times F$. As pointed out before, the Haar measures du in both sides of the identity above can be chosen arbitrarily, provided they are consistence.

8.1.3. Let $\varphi \in \mathcal{S}(F^2)$ be a Bruhat–Schwartz function, $s \in \mathbb{C}$ and $h \in \mathrm{U}_2(F)$. We define the *Godement section* $f_s^{\varphi}(h; \tau)$ attached to φ and τ by

$$f_s^{\varphi}(h; \tau) = \chi(a)|a|_E^s \int_{F^\times} \varphi((0, t)h_1)\chi_0(t)|t|_F^{2s}\, d^\times t \tag{8-3}$$

if E is a field and h is written as (8-1), and

$$f_s^{\varphi}(h; \tau) = \chi_1(\det(h))|\det(h)|_F^s \int_{F^\times} \varphi((0, t)h)\chi_1\chi_2(t)|t|_F^{2s}\, d^\times t \tag{8-4}$$

if $E = F \times F$.

Since the integrals (8-3) and (8-4) are essentially the Tate integrals (see [39]), they converge absolutely for $\mathrm{Re}(s) \gg 0$ and admit the meromorphic continuation to the whole complex plane. Moreover, it is not hard to check that $f_s^{\varphi}(-; \tau)$ is well defined when E is a field, i.e., independent of the decomposition (8-1) (see [3, Lemma 2.5]), and that

$$f_s^{\varphi}(-; \tau) \in \mathcal{V}_{B_{\mathrm{U}_2}}^{\mathrm{U}_2}(\tau_s)$$

provided that the integral is defined at s (see [3; 21]).

To describe the action of the intertwining map $\mathcal{A}_\psi(w_{1,1}, \tau, s)$ on $f_s^{\varphi}(-; \tau)$, we define the Fourier transform $\hat{\varphi}$ of φ by

$$\hat{\varphi}(x, y) = \int_F \int_F \varphi(z, w)\psi_2(zy - wx)\, dz\, dw,$$

where dz, dw are the Haar measures on F that are self-dual with respect to ψ_2.

The following lemma plays a pivotal role in our computation.

Lemma 8.1. *We have*

$$\mathcal{A}_\psi(w_{1,1}, \tau, s) f_s^{\varphi}(-; \tau) = \begin{cases} \chi^{-1}(-\delta)|\delta|_E^{-s+\frac{1}{2}} f_{1-s}^{\hat{\varphi}}(-; \tau^*) & \text{if } E \text{ is a field,} \\ \chi_1(-1) f_{1-s}^{\hat{\varphi}}(-; \tau^*) & \text{if } E = F \times F. \end{cases}$$

Proof. Assume first that $E = F \times F$. Then by (5-2), (5-3) and [8, Proposition 4.5.9], we have

$$\int_{V_{\mathrm{U}_2}(F)} \xi_s(w_{1,1}uh)\psi'(u_{1,2})\, du$$
$$= \chi_2(-1)\, |\delta_0|_F^{\frac{1}{2}}\, \gamma(2s-1, \chi_1\chi_2, \psi_{2\delta_0}) \int_{V_{\mathrm{U}_2}(F)} A_\psi(w_{1,1}, \tau, s)\xi_s(w_{1,1}uh)\psi'(u_{1,2})\, du.$$

for $\xi_s \in \mathcal{V}_{B_{\mathrm{U}_2}}^{\mathrm{U}_2}(\tau_s)$. Since

$$\gamma(2s-1, \chi_1\chi_2, \psi_{2\delta_0}) = \chi_1\chi_2(\delta_0)\, |\delta_0|_F^{2s-\frac{3}{2}}\, \gamma(2s-1, \chi_1\chi_2, \psi_2)$$

we find that

$$\mathcal{A}_\psi(w_{1,1}, \tau, s) = \gamma(2s-1, \chi_1\chi_2, \psi_2) A_\psi(w_{1,1}, \tau, s)$$

by comparing with (8-2).

On the other hand, if $\xi_s = f_s^\varphi$, then

$$\chi_1(-1)\gamma(2s-1, \chi_1\chi_2, \psi_2) A_\psi(w_{1,1}, \tau, s) f_s^\varphi(h; \tau) = f_{1-s}^{\hat{\varphi}}(h; \tau^*) \tag{8-5}$$

by the results in [14, Section 4.B]. Together, we conclude that

$$\mathcal{A}_\psi(w_{1,1}, \tau, s) f_s^\varphi(-; \tau) = \chi_1(-1) f_{1-s}^{\hat{\varphi}}(-; \tau^*),$$

which verifies the case where $E = F \times F$.

Next, assume that E is a field. Let $\tau_0 = 1 \boxtimes \chi_0$ be an irreducible representation of $F^\times \times F^\times$, where 1 stands for the trivial character of $F^\times$. We have

$$f_s^\varphi(h; \tau) = \chi(a)|a|_E^s f_s^\varphi(h_1; \tau_0)$$

provided that h is of the form (8-1). In particular, we can reduce the computation to the previous case.

Since

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \delta x \\ & 1 \end{pmatrix} = \begin{pmatrix} \delta^{-1} & \\ & -\delta \end{pmatrix}\begin{pmatrix} \delta & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \delta & \\ & 1 \end{pmatrix}^{-1}, \tag{8-6}$$

where $x \in F$, we find that

$$\chi^{-1}(-\delta)|\delta|_E^{s-\frac{1}{2}} \gamma(2s-1, \chi_0, \psi_2) A_\psi(w_{1,1}, \tau, s) f_s^\varphi(I_2; \tau) = f_{1-s}^{\hat{\varphi}}(I_2; \tau^*)$$

by (8-5). It follows that

$$\chi^{-1}(-\delta)|\delta|_E^{s-\frac{1}{2}} \gamma(2s-1, \chi_0, \psi_2) A_\psi(w_{1,1}, \tau, s) f_s^\varphi(h; \tau) = f_{1-s}^{\hat{\varphi}}(h; \tau^*) \tag{8-7}$$

for every $h \in \mathrm{U}_2(F)$. Note that the term $|\delta|_E^{1/2}$ arises due to the choice of measures.

In the remaining computations, let dx be the measure on F that is self-dual with respect to ψ_2. We apply (8-6) to compute

$$\begin{aligned}
\int_{V_{\mathrm{U}_2}(F)} & f_s^\varphi(w_{1,1}u; \tau)\psi'^{-1}(u_{1,2})\, du \\
&= \int_F f_s^\varphi\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \delta x \\ & 1 \end{pmatrix}; \tau\right)\psi_2^{-1}(\Delta x)\, dx \\
&= \chi(\delta^{-1})|\delta^{-1}|_E^s \int_F f_s^\varphi\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}; \tau_0\right)\psi_2^{-1}(\Delta x)\, dx \\
&= \chi^{-1}(-\delta)|\delta|_E^{-s} \int_F \int_{F^\times} \varphi(t, x)\chi_0(t)|t|_F^{2s-1}\psi_2^{-1}(\Delta t^{-1}x)\, d^\times t\, dx.
\end{aligned}$$

It is clear that the above integral converges absolutely for every s.

By replacing φ, χ and s with $\hat{\varphi}$, χ^* and $1-s$, respectively, and by applying the Fourier inversion formula, we compute

$$\begin{aligned}\int_{V_{\mathrm{U}_2}(F)} & f^{\hat{\varphi}}_{1-s}(w_{1,1}u;\tau^*)\psi'^{-1}(u_{1,2})\,du \\ &= \chi(\delta)|\delta|_E^{s-1}\int_F\int_{F^\times}\hat{\varphi}(t,x)\chi_0^{-1}(t)|t|_F^{1-2s}\psi_2^{-1}(\Delta t^{-1}x)\,d^\times t\,dx \\ &= \chi(\delta)|\delta|_E^{s-1}\int_F\int_{F^\times}\varphi(\Delta t^{-1},x)\chi_0^{-1}(t)|t|_F^{1-2s}\psi_2^{-1}(tx)\,d^\times t\,dx \\ &= \chi^{-1}(\delta)|\delta|_E^{-s}\int_F\int_{F^\times}\varphi(t,x)\chi_0^{-1}(t)|t|_F^{1-2s}\psi_2^{-1}(\Delta t^{-1}x)\,d^\times t\,dx \\ &= \chi(-1)\int_{V_{\mathrm{U}_2}(F)} f^{\varphi}_{s}(w_{1,1}u;\tau)\psi'^{-1}(u_{1,2})\,du.\end{aligned}$$

From this and (8-7), we deduce

$$\mathcal{A}_\psi(w_{1,1},\tau,s)=\gamma(2s-1,\chi_0,\psi_2)A_\psi(w_{1,1},\tau,s)$$

by comparing with (8-2). Therefore,

$$\mathcal{A}_\psi(w_{1,1},\tau,s)f^{\varphi}_s(-;\tau)=\chi^{-1}(-\delta)|\delta|_E^{-s+\frac{1}{2}}f^{\hat{\varphi}}_{1-s}(-;\tau^*).$$

This finishes the proof. □

We record the following corollary from the proof of Lemma 8.1:

Corollary 8.2. *We have*

$$\mathcal{A}_\psi(w_{1,1},\tau,s)=\begin{cases}\gamma(2s-1,\chi_0,\psi_2)A_\psi(w_{1,1},\tau,s) & \text{if } E \text{ is a field},\\ \gamma(2s-1,\chi_1\chi_2,\psi_2)A_\psi(w_{1,1},\tau,s) & \text{if } E=F\times F.\end{cases}$$

8.2. ***Proof of the case where E is a field.*** Note that in this case,

$$\mathrm{U}_1(F)=E^1:=\{a\in E^\times \mid |a|_E=1\}.$$

On the other hand, the Hilbert's Satz 90 induces the exact sequence

$$1\to F^\times \xrightarrow{\mathrm{id}} E^\times \xrightarrow{\iota} E^1\to 1.$$

Here id stands for the identity map and $\iota(a)=a\bar{a}^{-1}$ for $a\in E^\times$. We can therefore identity $\mathrm{U}_1(F)$ with $E^\times/F^\times$.

The representation π of $\mathrm{U}_1(F)$ is now a character η_1 of E^1. By pulling back, we obtain a character η of $E^\times$ that is trivial on $F^\times$. Note that η is the standard base change of η_1 to $\mathrm{GL}_1(E)=E^\times$. The representation τ is also a character of $E^\times$, and we denote it by χ as before.

8.2.1. Our aim is to prove

$$\Gamma(s, \eta_1 \times \chi, \psi) = \eta_1(-1)\gamma(s, \eta\chi, \psi_E). \tag{8-8}$$

To achieve this, we establish a connection between the local Rankin–Selberg integrals and the local Tate integrals by invoking the Godement sections introduced in Section 8.1.3.

Let $v \in \mathcal{V}_{\eta_1}$ and $\xi_s \in \mathcal{V}_{B_{\mathrm{U}_2}}^{\mathrm{U}_2}(\chi_s)$. The local Rankin–Selberg integral $\Psi_{0,1}(v \otimes \xi_s)$ is of the form

$$\int_{\mathrm{U}_1(F)} \eta_1(\alpha)\xi_s(\jmath^{0,1}(\alpha))\, d\alpha = \int_{E^\times/F^\times} \eta(a)\xi_s(\jmath^{0,1}(a\bar{a}^{-1}))\, d^\times a, \tag{8-9}$$

where the embedding $\jmath^{0,1} : \mathrm{U}_1(F) \hookrightarrow \mathrm{U}_2(F)$ is given by

$$\jmath^{0,1}(\alpha) = \frac{1}{2}\begin{pmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{pmatrix}$$

for $\alpha \in \mathrm{U}_1(F)$.

By [12, Lemma 3.1], we may assume that $\xi_s = f_s^{\varphi}(-;\chi)$ for some $\varphi \in \mathcal{S}(F^2)$. Then by letting $\alpha = a\bar{a}^{-1}$ with $a \in E^\times$ and the decomposition

$$\jmath^{0,1}(\alpha) = \jmath^{0,1}(a\bar{a}^{-1}) = \begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix}\begin{pmatrix} \delta & \\ & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} a^{-1}+\bar{a}^{-1} & \delta^{-1}(a^{-1}-\bar{a}^{-1}) \\ \delta(\bar{a}-a) & \bar{a}+a \end{pmatrix}\begin{pmatrix} \delta & \\ & 1 \end{pmatrix}^{-1}$$

we get that (see (8-3))

$$\xi_s(\jmath^{0,1}(\alpha)) = f_s^{\varphi}(\jmath^{0,1}(a\bar{a}^{-1});\chi) = \chi(a)|a|_E^s \int_{F^\times} \Phi(ta)\chi_0(t)|t|_F^{2s}\, d^\times t. \tag{8-10}$$

Here for $a \in E$, we have put

$$\Phi(a) = \varphi\left(\frac{\delta(\bar{a}-a)}{2}, \frac{\bar{a}+a}{2}\right),$$

which is a Bruhat–Schwartz function on E.

Now (8-9) and (8-10) imply

$$\Psi_{0,1}(v \otimes \xi_s) = \int_{E^\times} \Phi(a)\eta\chi(a)|a|^s\, d^\times a, \tag{8-11}$$

which is exactly the local Tate integral attached to Φ and $\eta\chi$.

8.2.2. Next, we compute $\Psi_{0,1}(v \otimes \mathcal{A}_\psi(w_1, 1, \chi, s)\xi_s)$ with $\xi_s = f_s^{\varphi}$. By Lemma 8.1 and the above calculations (with χ, φ and s replaced by χ^*, $\hat{\varphi}$ and $1-s$, respectively), we find that

$$\Psi_{0,1}(v \otimes \mathcal{A}_\psi(w_{1,1}, \chi, s)\xi_s) = \chi^{-1}(-\delta)|\delta|_E^{-s+\frac{1}{2}} \int_{E^\times} \Phi'(a)\eta\chi^*(a)|a|_E^{1-s}\, d^\times a, \tag{8-12}$$

where

$$\Phi'(a) := \hat{\varphi}\Big(\frac{\delta(\bar{a}-a)}{2}, \frac{\bar{a}+a}{2}\Big)$$

for $a \in E$.

To complete the proof, we claim

$$\Phi'(a) = |\delta|_E^{\frac{1}{2}} \cdot \hat{\Phi}(-\delta\bar{a}), \tag{8-13}$$

where $\hat{\Phi}$ is for the Fourier transform of Φ with respect to ψ_E given by

$$\hat{\Phi}(a) = \int_E \Phi(b)\psi_E(ab)\, db$$

and the Haar measure db on E is chosen to be self-dual with respect to ψ_E.

Assuming the claim for a moment, we see from (8-11), (8-12) and the functional equations for the local Tate integrals (see [8, Section 3.1]) that

$$\begin{aligned}\Psi_{0,1}(v\otimes\mathcal{A}_\psi(w_{1,1},\chi,s)\xi_s) &= \chi^{-1}(-\delta)|\delta|_E^{-s+1}\int_{E^\times}\hat{\Phi}(-\delta\bar{a})\eta\chi^*(a)|a|_E^{1-s}\,d^\times a\\ &= \eta_1(-1)\int_{E^\times}\hat{\Phi}(a)(\eta\chi)^{-1}(a)|a|_E^{1-s}\,d^\times a\\ &= \eta_1(-1)\gamma(s,\eta\chi,\psi_E)\int_{E^\times}\Phi(a)\eta\chi(a)|a|^s\,d^\times a\\ &= \eta_1(-1)\gamma(s,\eta\chi,\psi_E)\Psi_{0,1}(v\otimes\xi_s).\end{aligned}$$

From this, (8-8) follows.

8.2.3. It remains to verify the claim. We note

$$\int_E f(b)\,db = |\delta|_E^{\frac{1}{2}}\int_F\int_F f(z+\delta w)\,dz\,dw$$

for every $f\in L^1(E)$. Here db is the Haar measure on E that is self-dual with respect to ψ_E, and du, dv are the Haar measures on F that are self-dual with respect to ψ_2.

Now, write $a = x+\delta y$ for some $x, y\in F$; we have

$$\begin{aligned}\hat{\Phi}(-\delta\bar{a}) &= |\delta|_E^{\frac{1}{2}}\int_F\int_F \Phi(z+\delta w)\psi_2(-\Delta xw+\Delta yz)\,dz\,dw\\ &= |\delta|_E^{\frac{1}{2}}\int_F\int_F \varphi(-\Delta w, z)\psi_2(-\Delta xw+\Delta yz)\,dz\,dw\\ &= |\delta|_E^{-\frac{1}{2}}\int_F\int_F \varphi(v,u)\psi_2(xw+\Delta yz)\,dz\,dw\\ &= |\delta|_E^{-\frac{1}{2}}\Phi'(a).\end{aligned}$$

This proves the claim, and thus completes the proof of the case where E is a field. □

8.3. ***Proof of the case where $E = F \times F$.*** In this case, the representation π of $\mathrm{GL}_1(F) = F^\times$ is a character η of $F^\times$, and the representation τ of $\mathrm{G}_1(F) = F^\times \times F^\times$ is of the form $\chi_1 \boxtimes \chi_2$ as before.

Our aim is to prove

$$\Gamma(s, \eta \times \tau, \psi) = \eta(-1)\gamma(s, \eta\chi_1, \psi)\gamma(s, \eta^{-1}\chi_2, \psi). \tag{8-14}$$

The idea is again to connect the local Rankin–Selberg integrals to the local Tate integrals by using the Godement sections.

Let $v \in \mathcal{V}_\eta$ and $\xi_s \in \mathcal{V}_{V_{\mathrm{GL}_2}}^{\mathrm{GL}_2}(\tau_s)$. Then the local Rankin–Selberg integral $\Psi_{0,1}(v \otimes \xi_s)$ is of the form

$$\Psi_{0,1}(v \otimes \xi_s) = \int_{F^\times} \eta(a)\xi_s(\jmath^{0,1}(a))\, d^\times a,$$

where the embedding $\jmath^{0,1} : \mathrm{GL}_1(F) \hookrightarrow \mathrm{GL}_2(F)$ is given by

$$\jmath^{0,1}(a) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

As before, we may assume that $\xi_s = f_s^\varphi(-; \tau)$ for some $\varphi \in \mathcal{S}(F^2)$. For the ease of notation, we put

$$h_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \varphi' = \rho(h_0)\varphi,$$

where ρ stands for the right translation of $\mathrm{GL}_2(F)$ on $\mathcal{S}(F^2)$. Note that $h_0^{-1} = 2h_0$.

At this point, we further assume that

$$\varphi'(x, y) = \varphi_1(x)\varphi_2(y)$$

for some Bruhat–Schwartz functions $\varphi_1, \varphi_2 \in \mathcal{S}(F)$.

8.3.1. Using (8-4), we compute

$$\begin{aligned} &\Psi_{0,1}(v \otimes \xi_s) \\ &= \int_{F^\times} \eta(a)\xi_s(\jmath^{0,1}(a))\, d^\times a \\ &= \int_{F^\times} \eta(a) f_s^\varphi\left(h_0^{-1}\begin{pmatrix} 1 & \\ & a \end{pmatrix} h_0; \tau\right) d^\times a \\ &= \int_{F^\times}\int_{F^\times} \eta\chi_1(a)|a|_F^s\, \varphi'(t, -ta)\chi_1\chi_2(t)|t|_F^{2s}\, d^\times t\, d^\times a \\ &= \eta\chi_1(-1)\left(\int_{F^\times} \varphi_2(a)\eta\chi_1(a)|a|_F^s\, d^\times a\right)\left(\int_{F^\times} \varphi_1(t)\eta^{-1}\chi_2(t)|t|_F^s\, d^\times t\right), \end{aligned} \tag{8-15}$$

which is essentially a product of two local Tate integrals.

8.3.2. We continue to deal with $\Psi_{0,1}(v\otimes\mathcal{A}_\psi(w_{1,1},\tau,s)\xi_s)$. By Lemma 8.1 and the above calculations (with τ, φ and s replaced by τ^*, $\hat\varphi$ and $1-s$, respectively), we obtain

$$\begin{aligned}\text{(8-16)}\quad &\Psi_{0,1}(v\otimes\mathcal{A}_\psi(w_1,1,\chi,s)\xi_s)\\ &=\eta\chi_1\chi_2(-1)\int_{F^\times}\int_{F^\times}\hat\varphi'(t,a)\,\eta\chi_2^{-1}(a)|a|_F^{1-s}\,\eta^{-1}\chi_1^{-1}(t)|t|_F^{1-s}\,d^\times t\,d^\times a,\end{aligned}$$

where we put $\hat\varphi'=\rho(h_0)\hat\varphi$.

Now, we claim

$$\hat\varphi'(x,y)=|2|_F^{-1}\,\widehat{\varphi'}\Big(-\frac{x}{2},-\frac{y}{2}\Big)=\hat\varphi_2(x)\hat\varphi_1(-y). \tag{8-17}$$

Here for $f\in\mathcal{S}(F)$, we define the Fourier transform $\hat f$ of f by

$$\hat f(x)=\int_F f(y)\psi(xy)\,dy,$$

where dy is the Haar measure on F that is self-dual with respect to ψ.

Assuming the claim for a moment, we derive from (8-15), (8-16) and the functional equations of the Tate integrals (see [8, Section 3.1]) that

$$\begin{aligned}&\Psi_{0,1}(v\otimes\mathcal{A}_\psi(w_{1,1},\chi,s)\xi_s)\\ &=\eta\chi_1\chi_2(-1)\int_{F^\times}\int_{F^\times}\hat\varphi'(t,a)\,\eta\chi_2^{-1}(a)|a|_F^{1-s}\,\eta^{-1}\chi_1^{-1}(t)|t|_F^{1-s}\,d^\times t\,d^\times a\\ &=\chi_1(-1)\Big(\int_{F^\times}\hat\varphi_2(t)\eta^{-1}\chi_1^{-1}(t)|t|_F^{1-s}\,d^\times t\Big)\Big(\int_{F^\times}\hat\varphi_1(a)\eta\chi_2^{-1}(a)|a|_F^{1-s}\,d^\times a\Big)\\ &=\eta(-1)\gamma(s,\eta\chi_1,\psi)\gamma(s,\eta^{-1}\chi_2,\psi)\Psi_{0,1}(v\otimes\xi_s).\end{aligned}$$

From this, (8-14) follows.

8.3.3. It remains to verify the claim. We begin with verifying the first equality in (8-17), which is a special case of

$$\widehat{\rho(h)\phi}=|\det(h)|_F^{-1}\rho(h')\hat\phi$$

for $h\in\mathrm{GL}_2(F)$ and $\phi\in\mathcal{S}(F^2)$, where

$$h'=\begin{pmatrix}&-1\\1&\end{pmatrix}{}^th^{-1}\begin{pmatrix}&1\\-1&\end{pmatrix}.$$

In fact, a simple calculation shows

$$\begin{aligned}\widehat{\rho(h)\phi}(x,y)&=\int_F\int_F\phi((z,w)h)\psi_2\Big((z,w)\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\Big)\,dz\,dw\\ &=|\det(h)|_F^{-1}\int_F\int_F\phi((z,w))\psi_2\Big((z,w)\begin{pmatrix}&1\\-1&\end{pmatrix}{}^th'\begin{pmatrix}x\\y\end{pmatrix}\Big)\,dz\,dw\\ &=|\det(h)|_F^{-1}\rho(h')\hat\phi(x,y).\end{aligned}$$

To establish the second equality in (8-17), we note that if d_2z (resp. dz) is the Haar measure on F that is self-dual with respect to ψ_2 (resp. ψ), then $d_2z = |2|_F^{1/2} dz$. Now, we compute

$$\begin{aligned}|2|_F^{-1}\,\widehat{\varphi'}\Big(-\frac{x}{2}, -\frac{y}{2}\Big) &= |2|_F^{-1}\int_F\int_F \varphi'(z,w)\psi_2\Big(-\frac{zy}{2}+\frac{wx}{2}\Big)\,d_2z\,d_2w \\ &= \int_F\int_F \varphi_1(z)\varphi_2(w)\psi(-zy+wx)\,dz\,dw \\ &= \hat{\varphi}_2(x)\hat{\varphi}_1(-y).\end{aligned}$$

This verifies the claim, and we have completed the proof of Theorem 4.1.

Acknowledgements

Part of this work was conducted during Cheng's visit to the Mathematics Department of Kyoto University in late July 2023. He thanks S.-Y. Chen and A. Ichino for the kind invitation and gracious hospitality. We also thank M.-L. Hsieh and K. Morimoto for their comments on the first draft. Finally, the authors are grateful to the referees for pointing out an error in the original manuscript and for providing many valuable suggestions that greatly improved it.

The work of Cheng is partially supported by the MOST, under grant number 112-2115-M-032-004-MY3. The work of Wang is partially supported by the MOST, under grant number 111-2115-M-032-001.

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Received March 28, 2024. Revised March 10, 2025.

YAO CHENG
DEPARTMENT OF APPLIED MATHEMATICS AND DATA SCIENCE
TAMKANG UNIVERSITY
NEW TAIPEI CITY
TAIWAN
briancheng@o365.tku.edu.tw

CHIAN-JEN WANG
DEPARTMENT OF APPLIED MATHEMATICS AND DATA SCIENCE
TAMKANG UNIVERSITY
NEW TAIPEI CITY
TAIWAN
142888@o365.tku.edu.tw

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Volume 337 No. 1 July 2025