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MIRKO RÖSNER AND RAINER WEISSAUER

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For irreducible smooth representations Π of GSp(4, k) over a nonarchimedean local field k, Piatetski-Shapiro and Soudry (1981) have constructed an L-factor depending on the choice of a Bessel model. It factorizes into a regular part and an exceptional part. We determine the regular part for the case of split Bessel models.

1. Introduction

For infinite-dimensional irreducible smooth representations Π of $G = \mathrm{GSp}(4, k)$ with central character ω, where k is a local nonarchimedean field, and a smooth character μ of k^*, Piatetski-Shapiro [1997] constructed local L-factors

$$L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$$

attached to a choice of a Bessel model (Λ, ψ). To be precise, fix the standard Siegel parabolic subgroup $P = MN$ in G with Levi M and unipotent radical N. For a nondegenerate linear form ψ of N, the connected component $\widetilde{T}$ of the stabilizer of ψ in M is isomorphic to the unit group $L^\times$ for a quadratic extension L/k. A Bessel character is a pair (Λ, ψ) where Λ is a character of $\widetilde{T}$. The coinvariant space $(\Pi_\Lambda)_\psi$ with respect to the action of $\widetilde{T}N$ by (Λ, ψ) is at most one-dimensional [Piatetski-Shapiro 1997, Theorem 3.1; Roberts and Schmidt 2016, Theorem 6.3.2]. We give an independent proof in Theorem 5.1. If it is nonzero, we say Π has a

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Bessel model. Bessel models have been classified by Roberts and Schmidt [2016]. They are called anisotropic or split, depending on whether L is a field or not.

It was expected by Piatetski-Shapiro and Soudry [1981] that the L-factor does not depend on the choice of the Bessel model. They proved this for unitary fully Borel induced Π and unitary Λ [Piatetski-Shapiro and Soudry 1981, Theorem 2.4]. Novodvorsky made the claim [Novodvorsky 1979, Theorem 5] that for generic Π the L-factor $L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$ coincides with the L-factor $L^{\mathrm{Nov}}(s, \Pi, \mu)$, constructed by him in a completely different way using Whittaker models, which of course would imply that $L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$ is independent of the choice of the underlying Bessel model. This expectation holds true in the case of anisotropic Bessel models by results of Danişman [2014; 2015a; 2015b; 2017] and Rösner and Weissauer [2023].

To every Bessel model (Λ, ψ) we attach a Bessel module M that is a TS-module, where TS is isomorphic to the affine linear group $\mathrm{Gl}_a(1)$. The local L-factor of Piatetski-Shapiro is a product of three L-factors:

$$L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda) = L^{\mathrm{PS}}_{\mathrm{ex}}(s, \Pi, \mu, \Lambda)\, L^{\mathrm{PS}}_{\mathrm{sreg}}(s, \Pi, \mu, \Lambda)\, L(s, \mu \otimes M),$$

the *exceptional* part $L^{\mathrm{PS}}_{\mathrm{ex}}(s, \Pi, \mu, \Lambda)$, the *subregular* part $L^{\mathrm{PS}}_{\mathrm{sreg}}(s, \Pi, \mu, \Lambda)$ and an L-factor $L(s, \mu \otimes M)$ attached to a twist $\mu \otimes M$ of the Bessel module M. The product of the last two L-factors is the regular part $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$ in the sense of Piatetski-Shapiro [1997]. The subregular L-factors were computed[1] in [Rösner and Weissauer 2020], and the exceptional L-factors were computed in [Weissauer 2023] and [Rösner and Weissauer 2023].

To determine $L(s, \mu \otimes M)$ one has to compute M. Up to a character twist by $\nu^{-3/2}$ the module M is the quotient of the Bessel module $\widetilde{\Pi}$ by its subspace $\widetilde{\Pi}^S$ of S-invariants. Here $\widetilde{\Pi} = \Pi_\Lambda$ denotes the space of coinvariants with respect to the character Λ of $\widetilde{R}$. Notice that the Bessel module $\widetilde{\Pi}$ depends on the Bessel model (Λ, ψ).

Our main result is the determination of Bessel modules $\widetilde{\Pi} = \beta_\rho(\Pi)$ and $\beta^\rho(\Pi)$ for irreducible representations Π of G and split Bessel models of Π, see Table 5. This includes the computation of $\widetilde{\Pi}^S$ in every case. In fact, the L-factor $L(s, M)$ only depends on the T-eigenvalues of the torus T on the finite-dimensional quotient $\pi_0(\widetilde{\Pi})$ respectively the finite-dimensional subspace $\widetilde{\Pi}^S$ of $\widetilde{\Pi}$. Notice that $\pi_0(\widetilde{\Pi})$ can be computed from the Siegel–Jacquet module $J_P(\Pi)$ of Π by results of Tunnell [1983] and Waldspurger [1985, Lemma 8]. On the other hand, the subspace $\widetilde{\Pi}^S$ of S-invariants is a more delicate invariant that a priori cannot be extracted from the Siegel–Jacquet module $J_P(\Pi)$ alone. For details we refer to Section 5.4.

[1]In an earlier version of this paper and also in [Rösner and Weissauer 2017, Table 1] the L-factor $L(s, \mu \otimes M)$ was called the regular L-factor, not following the notation of [Piatetski-Shapiro 1997]. This fact does not affect the argument of [Rösner and Weissauer 2017].

It turns out that the regular factor $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$ does not depend on the choice of the split Bessel model; see Theorem 5.14. The list of the regular factors $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$ attached to split Bessel models can be found in Table 4. The exceptional factor does not depend on the choice of the split Bessel model either [Weissauer 2023]. Summarizing, we show:

(1) $L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$ does not depend on the choice of a split Bessel model.

(2) For generic Π, the L-factor $L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$ coincides with the L-factor of Novodvorsky [1979], explicitly calculated by Takloo-Bighash [2000].

(3) For noncuspidal Π, the L-factor $L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$ coincides with the L-factor of the Langlands parameter attached to Π by Roberts and Schmidt [2007, Table A.8].

For spherical representations Π, the Piatetski-Shapiro zeta integral has been computed explicitly by Lemma [2017, Section 5]. For Iwahori-spherical Π, the Bessel module has recently been studied by Pitale and Schmidt [2014] in terms of test vectors. For certain applications we refer to [Rösner and Weissauer 2017].

For the computations in the split case we use methods from homological algebra applied in the context of P_3-theory, as already considered in [Bernstein and Zelevinskii 1976] and [Roberts and Schmidt 2007]. This is explained in the Sections 3.1 and 3.2 and in Section 4.6, where we exploit further information from the Klingen–Jacquet module $J_Q(\Pi)$.

We then combine this with results obtained from a detailed study of the action of tori on the Siegel–Jacquet module $J_P(\Pi)$, and from the analysis of extensions that are defined by filtrations on Siegel-induced representations for which Π is an irreducible quotient; see Section 4.1. Here Theorem 4.20 and the combinatorial Lemma 4.22 are important steps. A specific complication, arising for split Bessel models, comes from the Bessel filtration on $\beta_\rho(I)$ for Siegel induced representations $I = \mathrm{Ind}_P^G(\sigma_\Pi)$. The fact that in the split case, in contrast to the anisotropic case, the functor β_ρ is not exact makes these filtrations rather involved (see Section 4.1). Concerning this, see also Lemma A.4 in the Appendix and the remark thereafter.

The last ingredient for the computation of $\widetilde{\Pi}$ in the split case comes from an underlying duality for Bessel models. This duality (Lemma 3.17) relates Bessel models for (Λ, ψ) and (Λ^*, ψ). If we describe Λ as in Section 1.1 by a character ρ, then Λ^* corresponds to the character $\rho^* = \omega\rho^{-1}$ for the central character ω of Π. With the combinatorial data obtained from Lemma 4.22 for the Bessel models attached to Λ and Λ^*, we obtain significantly stronger information. This finally leads to the data collected in Table 3 (multisets) of the Appendix. The content of this table is reminiscent of combinatorics of root data, yet in an obscure way. To exploit these data, we have to make use of a categorial Mellin functor that describes the composition of a Mellin transform and its inverse (see Lemma 3.25 and Theorem 3.27).

The explicit description of Bessel modules has applications for the construction of Euler systems for GSp(4) [Loeffler et al. 2022].

1.1. *Notation.* Fix a local nonarchimedean field k of characteristic $\mathrm{char}(k) \neq 2$ with finite residue field $\mathfrak{o}_k/\varpi\mathfrak{o}_k$ of cardinality q. The group of symplectic similitudes in four variables is

$$G = \mathrm{GSp}(4, k) = \{g \in \mathrm{Gl}(4, k) \mid g' J g = \lambda(g) \cdot J\}, \quad J = \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2, \end{pmatrix}$$

with the symplectic similitude factor $\lambda(g) \in k^\times$ and center $Z \cong k^\times$. Fix the standard Siegel parabolic subgroup $P = MN$ of G with Levi component M and unipotent radical N. We identify elements in N with vectors $(a, b, c) \in k^3$ and elements in M with pairs $(A, \lambda) \in \mathrm{Gl}(2) \times \mathrm{Gl}(1)$ via the embeddings

$$x_\lambda \cdot m_A = \begin{pmatrix} \lambda \cdot A & 0_2 \\ 0_2 & (A')^{-1} \end{pmatrix}, \quad s_{a,b,c} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We write $\tilde{t} = \mathrm{diag}(t_1, t_2, t_2, t_1)$, $x_\lambda = \mathrm{diag}(\lambda E, E)$ and $t_\lambda = \mathrm{diag}(E_2, \lambda E_2)$ in M for $t_1, t_2, \lambda \in k^\times$ and $s_b = s_{0,b,0}$. Notice $x_\lambda s_b x_\lambda^{-1} = s_{\lambda b}$. The Weyl group of G has order eight and is generated by

$$\mathbf{s}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

From Section 3 onwards, we use the following notation:

δ_P is the modulus character of P where $\delta_P(m_A x_\lambda s_{a,b,c}) = |\lambda \det(A)|^3$.

ω is a fixed smooth character of $Z \cong k^\times$.

ψ is a nontrivial character $N \to \mathbb{C}$ that vanishes on $\widetilde{N} = \{s_{*,0,*} \in N\}$.

$\widetilde{T} = \{\tilde{t} \mid t_1, t_2 \in k^\times\} \cong k^\times \times k^\times$ is a fixed split torus in M.

$T = \{x_\lambda \in M \mid \lambda \in k^\times\} \cong k^\times$.

$R = \widetilde{T}N \subseteq \ker(\delta_P)$ is a connected component of the centralizer of ψ in P.

$\widetilde{R} = \widetilde{T}\widetilde{N} \subseteq R$ is the Bessel group.

$S = \{s_b \in N, b \in k\} \cong k$ is the center of R.

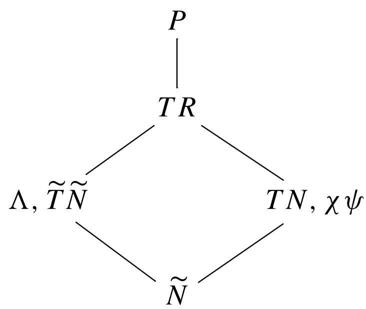
χ, μ, ρ are smooth characters of $k^\times$.

$\chi_{\mathrm{norm}} = \nu^{-3/2}\chi$ is a normalized character.

$\rho^{*} = \omega\rho^{-1}$ for fixed ω.

Λ is a smooth character of $\widetilde{R}$, which is trivial on $\widetilde{N}$ with $\Lambda|_Z = \omega$, and hence $\Lambda(\tilde{t}) = \rho(t_1)\rho^{*}(t_2)$ for some ρ.

There is an exact sequence $0 \to \widetilde{R} \to TR \to TS \to 0$ and direct product decompositions $N \cong S \times \widetilde{N}$ and $R \cong S \times \widetilde{R}$. The character (Λ, ψ) of R defines a *split* Bessel datum [Piatetski-Shapiro 1997].



For a totally disconnected locally compact group Γ let $\mathcal{C}_\Gamma$ be the category of complex vector spaces with a smooth action of Γ and let $\mathcal{C}_\Gamma^{\mathrm{fin}}$ be its full subcategory of representations with finite length. Objects of $\mathcal{C}_\Gamma$ are called Γ-*modules* or *representations of* Γ. For every subgroup $H \subseteq \Gamma$ with a smooth character $\chi : H \to \mathbb{C}^\times$ and every $V \in \mathcal{C}_\Gamma$ the χ-coinvariants, i.e., the maximal quotient of V on which H acts by the eigencharacter χ, is denoted by

$$V_\chi = V_{H,\chi} = V/V(H,\chi) \quad \text{for } V(H,\chi) = \langle hv - \chi(h)v \mid h \in H, v \in V\rangle.$$

We also consider the χ-invariants $V^\chi = \{v \in V \mid \forall h \in H : h \cdot v = \chi(h)v\}$, and the generalized χ-invariants $V^{(\chi)} = \{v \in V \mid \exists n\, \forall h \in H : (h - \chi(h))^n \cdot v = 0\}$. This defines the following functors:

κ is the left exact functor $\mathcal{C}_{TS} \to \mathcal{C}_{TS}$ that attaches to $V \in \mathcal{C}_{TS}$ its submodule $\kappa(V) = V^S$ of S-invariant vectors. Notice $\kappa(V/\kappa(V)) = 0$ by Lemma A.3.

k_ρ is the right exact functor $\mathcal{C}_n \to \mathcal{C}_{n-1}$ defined in Section 3.1 with its left derived functor k^ρ.

η is the exact functor $\mathcal{C}_G \to \mathcal{C}_{\mathrm{Gl}_a(2)}$ that attaches to $V \in \mathcal{C}_G$ its quotient $\overline{V} = V_{S_A}$ with the action of $\mathrm{Gl}_a(2)$ defined in Section 3.2.

β_ρ is the Bessel functor $k_\rho \circ \eta : \mathcal{C}_{TR} \to \mathcal{C}_{TS}$ which assigns to $V \in \mathcal{C}_{TR}$ the maximal quotient space $\widetilde{V} = V_{\widetilde{R},\Lambda}$ on which $\widetilde{R}$ acts by Λ, see Section 3.3.

β^ρ is the left exact functor $k^\rho \circ \eta : \mathcal{C}_{TR} \to \mathcal{C}_{TS}$ which assigns to $V \in \mathcal{C}_{TR}$ the Λ-eigenspace $(V_{\widetilde{N}})^{\widetilde{T},\Lambda}$ of the $\widetilde{N}$-coinvariants of V.

The twist by a smooth character μ of $k^\times$ is the exact functor $\mu:\mathcal{C}_G(\omega)\to\mathcal{C}_G(\mu^2\omega)$ given on objects by $V\mapsto \mu V=(\mu\circ\lambda)\otimes V$ with the obvious action on morphisms. The twist functor $\mathcal{C}_n\to\mathcal{C}_n,\ V\mapsto i_*(\mu\circ\det)\otimes V$ is also denoted by μ. Then there are natural equivalences of functors $\mathcal{C}_G\to\mathcal{C}_{TS}$:

$$\beta_{\mu\rho}\circ\mu\cong\mu\circ\beta_\rho,\quad \beta^{\mu\rho}\circ\mu\cong\mu\circ\beta^\rho.$$

2. Representations of $\mathrm{Gl}_a(1)$

In this section we consider the subgroup $\mathrm{Gl}_a(1)$ of $\mathrm{Gl}(2)$ generated by the affine linear transformations $[\lambda\,|\,b]:=\left(\begin{smallmatrix}\lambda & b\\ 0 & 1\end{smallmatrix}\right)\in\mathrm{Gl}(2,k)$ for $\lambda\in k^\times$ and $b\in k$; see also Section 3. Sending x_λ to $[\lambda\,|\,0]$ and s_b to $[1\,|\,b]$ defines an isomorphism $TS\cong\mathrm{Gl}_a(1)$ and we identify T and S with their images in $\mathrm{Gl}_a(1)$. Fix the category $\mathcal{C}_1=\mathcal{C}_{\mathrm{Gl}_a(1)}$ of smooth $\mathrm{Gl}_a(1)$-modules. Recall the well-known classification of $\mathrm{Gl}(1)$*-modules*:

Every $X\in\mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(1)}$ is finite-dimensional; the irreducible X are exactly the one-dimensional characters $\chi:\mathrm{Gl}(1)\to\mathbb{C}^\times$. The theory of Jordan normal forms implies for smooth $\mathrm{Gl}(1)$-characters χ_1,χ_2:

$$\dim\mathrm{Ext}^1_{\mathrm{Gl}(1)}(\chi_1,\chi_2)=\dim\mathrm{Hom}^1_{\mathrm{Gl}(1)}(\chi_1,\chi_2)=\begin{cases}1, & \chi_1=\chi_2,\\ 0, & \text{else.}\end{cases}$$

The higher $\mathrm{Ext}^n_{\mathrm{Gl}(1)}$-functors vanish for $n\geq 2$, since the cohomological dimension of $\mathcal{C}_{\mathrm{Gl}(1)}$ is one. Especially, every X decomposes as a direct sum $X=\bigoplus_\chi X^{(\chi)}$ of its generalized eigenspaces $X^{(\chi)}$ with respect to the smooth characters χ of $\mathrm{Gl}(1)$. The Jordan block $\chi^{(m)}$ is the unique indecomposable $\mathrm{Gl}(1)$-module of length m attached to the character χ. On $X^{(\chi)}$ the monodromy operator $\tau_\chi=\varpi-\chi(\varpi)\cdot\mathrm{id}$ is defined and nilpotent, so $X^{(\chi)}$ decomposes as a direct sum of Jordan blocks under the action of ϖ, unique up to ordering. We say that X is *cyclic* if every $X^{(\chi)}$ is indecomposable, which is equivalent to $\dim X^\chi=\dim X_\chi\leq 1$ for every χ.

2.1. ***Examples.*** All the examples listed below do frequently occur later. We introduce them here to precisely fix certain conventions and notations. Most statements are well known. For the convenience of the reader we give proofs.

Example 2.1. Fix a nontrivial smooth additive character $\psi:k\to\mathbb{C}^\times$ and let $\psi_b(x)=\psi(xb)$ for $b\in k$. The function spaces $C_c^\infty(k^\times)$, $C_c^\infty(k)$, $C_b^\infty(k^\times)$ of smooth complex valued functions g with compact (resp. bounded) support are $\mathrm{Gl}_a(1)$-modules by $(s_bg)(x)=\psi_b(x)g(x)$ and $(x_\lambda g)(x)=g(x\lambda)$. There are canonical $\mathrm{Gl}_a(1)$-equivariant embeddings $C_c^\infty(k^\times)\hookrightarrow C_c^\infty(k)\hookrightarrow C_b^\infty(k^\times)$. The Schwartz space $\mathbb{S}=C_c^\infty(k^\times)$ is an irreducible $\mathrm{Gl}_a(1)$-module that is invariant under twists by smooth characters χ of $k^\times$ by the isomorphism $i_*(\chi)\otimes\mathbb{S}\cong\mathbb{S},\ g\mapsto\chi g$.

Lemma 2.2. *For every submodule M of $C_b^\infty(k^\times)$ and every smooth character χ of T, the (T,χ)-invariant space $M^{T,\chi}=0$ and the S-invariant space $M^S=0$ vanish.*

Proof. $M^\chi = 0$ is implied by boundedness of the support. For every nonzero $\xi \in k^\times$, evaluation $g \mapsto g(\xi)$ defines an S-linear morphism $M^S \to (\mathbb{C}, \psi_\xi)$, so $g(\xi) = 0$ for every $g \in M^S$. □

Lemma 2.3. *Every nonzero* $\mathrm{Gl}_a(1)$*-submodule M of $C_b^\infty(k^\times)$ contains $\mathbb{S}$ and the action of S on $M/\mathbb{S}$ is trivial. The quotient $C_b^\infty(k^\times)/\mathbb{S}$ has a one-dimensional χ-eigenspace for every smooth T-character χ.*

Proof. For every $g \in M$ and every $s_b \in S$, the difference $s_b g - g$ has compact support by smoothness of ψ. If there is $x_0 \in k^\times$ with $g(x_0) \neq 0$, then there is $b \in K$ with $\psi(bx_0) \neq 1$, so $s_b g - g$ is a nontrivial element of $M \cap \mathbb{S}$ which generates $\mathbb{S}$ as an irreducible $\mathrm{Gl}_a(1)$-module. Especially, $M/\mathbb{S}$ is the S-coinvariant quotient M_S. Every χ-eigenvector $g + \mathbb{S}$ in $C_b^\infty(k^\times)/\mathbb{S}$ satisfies $g(x\lambda) = \chi(\lambda) g(x)$ for every $\lambda \in k^\times$ and sufficiently small x. Hence there is $C \in \mathbb{C}$ with $g(x) = C\chi(x)$ for small x and this g spans a one-dimensional χ-eigenspace in $C_b^\times(k^\times)/\mathbb{S}$. □

Example 2.4. Let $\mathbb{E} = C_c^\infty(S)$ denote the space of smooth complex valued functions with compact support in S. The $\mathrm{Gl}_a(1)$-module $\mathbb{E}$ defined by $(s_b f)(s) = f(s + s_b)$ and $(x_\lambda f)(s) = |\lambda|^{-1} f(\lambda^{-1}s)$ is naturally isomorphic to the compactly induced representation $\mathrm{ind}_T^{\mathrm{Gl}_a(1)}(\nu^{-1})$. Integration $f \mapsto \int_S f(s)\,\mathrm{d}s$ defines a $\mathrm{Gl}_a(1)$-equivariant map from $\mathbb{E}$ to the trivial $\mathrm{Gl}_a(1)$-module with kernel $\mathbb{E}^0$ and thus an exact sequence

$$0 \to \mathbb{E}^0 \to \mathbb{E} \to \mathbb{C} \to 0.$$

The Fourier transform $\mathcal{F} : \mathbb{E} \to C_c^\infty(k),\ \mathcal{F}(f)(x) = \int_S f(s)\psi(-xs)\,\mathrm{d}s$ yields a $\mathrm{Gl}_a(1)$-equivariant isomorphism with the $\mathrm{Gl}_a(1)$-module $C_c^\infty(k)$ in Example 2.1 such that $\mathcal{F}(\mathbb{E}^0) = \mathbb{S}$.

Example 2.5. Let $V \in \mathcal{C}_{\mathrm{Gl}(2)}$ be a smooth $\mathrm{Gl}(2)$-module. Pullback along the canonical embedding $\mathrm{Gl}_a(1) \hookrightarrow \mathrm{Gl}(2)$ of the mirabolic subgroup $\mathrm{Gl}_a(1)$ defines a $\mathrm{Gl}_a(1)$-module structure on V in a natural way. Every TS-module defines a $\mathrm{Gl}_a(1)$-module by pullback along the isomorphism $TS \cong \mathrm{Gl}_a(1),\ s_b x_\lambda \mapsto [\lambda\,|\,b] = \left(\begin{smallmatrix} \lambda & b \\ 0 & 1 \end{smallmatrix}\right)$. This defines an equivalence $\mathcal{C}_{TS} \cong \mathcal{C}_{Gl_a(1)}$.

2.2. *Modules of finite length.* Let $\mathcal{C} = \mathcal{C}_1^{\mathrm{fin}}$ be the full subcategory of $\mathcal{C}_1$ of $\mathrm{Gl}_a(1)$-modules of finite length.

Lemma 2.6. *An irreducible $M \in \mathcal{C}$ is either isomorphic to $\mathbb{S}$ or to a character*

$$i_*(\chi) : [\lambda\,|\,b] \mapsto \chi(\lambda)$$

for a smooth $\mathrm{Gl}(1)$*-character χ, see Lemma 3.1.*

Proof. If $M = M^S$, then M is inflated from an irreducible $\mathrm{Gl}(1)$-module χ. Assume $M \neq M^S$, then $M_\psi \neq 0$ by Lemma 3.1. By Frobenius reciprocity there is an embedding $M \hookrightarrow \mathrm{Ind}_S^{\mathrm{Gl}_a(1)}(\psi) \subseteq C^\infty(T)$. Every $f \in \mathrm{Ind}_S^{\mathrm{Gl}_a(1)}(\psi)$ is smooth, so $s_b f = f$

for sufficiently small $b \in k$. Thus $\psi(b\lambda) f(x_\lambda) = f(x_\lambda)$ for these b implies that $f(x_\lambda)$ vanishes for large values of $\lambda \in k^\times$. We obtain a $\mathrm{Gl}_a(1)$-embedding $\mathrm{Ind}_S^{\mathrm{Gl}_a(1)}(\psi) \hookrightarrow C_b^\infty(k^\times)$. Identify M with its image in $C_b^\infty(k^\times)$, then $M \cong \mathbb{S}$ by Lemma 2.3. □

We write $\chi \in \mathcal{C}$ instead of $i_*(\chi)$ when the meaning is clear. For $M \in \mathcal{C}$, the degree $\deg(M) = \dim(M_\psi)$ is the number of Jordan–Hölder constituents isomorphic to $\mathbb{S}$. Every nonzero submodule $M \subseteq C_b^\infty(k^\times)$ of finite length has degree one by Lemma 2.3. Here ψ is an arbitrary nontrivial S-character, defining the exact functor $j^!(M) = M_{S,\psi}$ of (S, ψ)-coinvariants. The functor of S-coinvariants $i^* : \mathcal{C} \to \mathcal{C}_T^{\mathrm{fin}}$, $M \mapsto M_S$ is exact and left adjoint to the inclusion functor $i_* : \mathcal{C}_T^{\mathrm{fin}} \hookrightarrow \mathcal{C}$. Their composition $\pi_0 = i_* i^*$ preserves characters and sends $\mathbb{S}$ to zero. There is an exact sequence

$$0 \to M^0 \to M \to \pi_0(M) \to 0$$

by Lemma 3.1, where $M^0 = j_! j^!(M)$ is isomorphic to $\mathbb{S}^{\oplus \deg(M)}$. The canonical morphism $M \to \pi_0(M)$ is also denoted π_0. The functors i^*, i_*, $j^!$, $j_!$ are introduced in Section 3 in greater generality.

Lemma 2.7. *For every* $\mathrm{Gl}_a(1)$*-module* $M \in \mathcal{C}$, *the* S*-invariant subspace* M^S *is the maximal finite-dimensional* $\mathrm{Gl}_a(1)$*-submodule of* M.

Proof. Every finite-dimensional $\mathrm{Gl}_a(1)$-submodule F of M is a trivial S-module and thus contained in M^S. Lemma 3.1 yields that $0 \to j_! j^!(F) \to F \to \pi_0(F) \to 0$ is an exact sequence in $\mathcal{C}_1$. The submodule $j_! j^!(F)$ is either zero or infinite-dimensional, but F is finite-dimensional, so $F \cong \pi_0(F)$ is a trivial S-module.

It remains to be shown that M^S itself is finite-dimensional. By Lemma A.1(5), it is sufficient to show that $i_* i^*(M)$ is finite-dimensional. But $i^*(M)$ is a T-module of finite length and thus finite-dimensional. □

Lemma 2.8. *For smooth* $M \in \mathcal{C}_1$ *and smooth* T*-characters* χ, *the subspace* $M^{(T,\chi)}$ *of* (T, χ)*-invariants is contained in the* S*-invariant subspace* M^S. *Especially, every finite-dimensional* T*-submodule of* M *is contained in* M^S.

Proof. For every $v \in M^{(T,\chi)}$, by smoothness there is a nontrivial $s_b \in S$ that fixes $v = s_b v$. This implies $s_{\lambda b} v = x_\lambda s_b x_\lambda^{-1} v = x_\lambda s_b \chi(\lambda)^{-1} v = v$ for every $\lambda \in k^\times$ and thus $v \in M^S$. □

Lemma 2.9. $M^S = \bigcap_{b \in k^\times} \ker(M \twoheadrightarrow M_{S,\psi_b})$ *for every* $M \in \mathcal{C}_1$.

Proof. The action of $[\lambda|*] \in \mathrm{Gl}_a(1)$ sends $M(S, \psi_b) = \ker(M \to M_{S,\psi_b})$ to $M(S, \psi_{\lambda^{-1}b})$, so $M' := \bigcap_{b \in k^\times} M(S, \psi_b)$ is a $\mathrm{Gl}_a(1)$-submodule of M. The functor $j^!$ of (S, ψ)-coinvariants is exact, so there is a commutative square of vector spaces

$$\begin{array}{ccc} M' & \hookrightarrow & M \\ \downarrow & & \downarrow \\ j^!(M') & \hookrightarrow & j^!(M) \end{array}$$

with surjective vertical and injective horizontal arrows. The composition morphism $M' \to j^!(M)$ vanishes by construction, so $j^!(M') = 0$. Lemma 3.1 applied to M' yields an isomorphism $M' \cong i_* i^*(M')$, and hence M' is a trivial S-module. The converse inclusion $M^S \subseteq M'$ is obvious. □

Lemma 2.10. *$\mathbb{S}$ is projective in $\mathcal{C}$, i.e.,* $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{S}, -) = 0$.

Proof. It suffices to show that $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{S}, M)$ vanishes for irreducible M. For finite-dimensional M and an exact sequence in $\mathcal{C}$ as in the upper row of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & \mathbb{S} & \longrightarrow & 0 \\ & & \cong \big\downarrow \pi_0 & & \big\downarrow \pi_0 & & \big\downarrow \pi_0 & & \\ 0 & \longrightarrow & M & \longrightarrow & E_S & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

the left vertical arrow is an isomorphism by Lemma 2.7 and exactness of π_0. The snake lemma provides a splitting $E(S) = \ker(E \to \pi_0(E)) \cong \mathbb{S}$ and thus shows the assertion for finite-dimensional M. For $M = \mathbb{S}$ note that by Lemma 3.1 the full subcategory of $\mathcal{C}$ of modules annihilated by i^* is equivalent to the category of finite dimensional complex vector spaces and thus semisimple. □

For M in $\mathcal{C}$ the lemma implies that $M^0 = \ker(M \to \pi_0(M))$ is the unique maximal subgroup of M generated by $\deg(M)$ copies of $\mathbb{S}$. However, $\mathbb{S}$ is not injective, as $\mathrm{Ext}^1_{\mathcal{C}}(-, \mathbb{S})$ is nontrivial by Lemma 2.21.

Lemma 2.11. *For every $M \in \mathcal{C}$ and every T-character χ the following holds*:

$$\dim M_{T,\chi} - \dim M^{T,\chi} = \deg M.$$

Proof. By Lemma A.2 there is a long exact sequence:

$$0 \to (\mathbb{S}^{\deg(M)})^\chi \to M^\chi \to \pi_0(M)^\chi \to \mathbb{S}_\chi^{\deg(M)} \to M_\chi \to \pi_0(M)_\chi \to 0.$$

Note that $\dim \pi_0(M)_\chi = \dim \pi_0(M)^\chi$ since $\pi_0(M)$ is finite-dimensional. We have $\mathbb{S}^\chi = 0$ by Lemma 2.2 and $\dim \mathbb{S}_\chi = 1$ by Proposition 4.3.2 of [Bump 1998], so counting dimensions implies the statement of Lemma 2.11. This is a special case of Proposition 3.6. □

2.3. ***Perfect modules.*** $M \in \mathcal{C}$ is said to be *perfect* if $M^{T,\chi} = 0$ holds for all smooth characters χ of T.

Example 2.12. Submodules of perfect modules and extensions of perfect modules are perfect. $\mathbb{S}$ is perfect by Lemma 2.2. Nonzero finite-dimensional modules $M \in \mathcal{C}$ are not perfect. The $\mathrm{Gl}_a(1)$-module $\mathbb{E} = C_c^\infty(S)$ of Example 2.4 is perfect of degree one by Lemma 2.2. Since $\mathbb{S} \cong \chi \otimes \mathbb{S}$, the twist $\mathbb{E}[\chi] = \chi \otimes \mathbb{E}$ is also perfect of degree one for every T-character χ and there is a nonsplit exact sequence $0 \to \mathbb{S} \to \mathbb{E}[\chi] \to \chi \to 0$.

Lemma 2.13. *For $M \in \mathcal{C}$ the following assertions are equivalent*:

(1) *M is perfect.*

(2) $M^S = 0$.

(3) $\dim(M_\chi) = \deg(M)$ *for all smooth T-characters χ.*

Proof. If the finite-dimensional T-module M^S is nonzero, each of its constituents χ yields $0 \neq (M^S)^\chi \subseteq M^\chi$, so M is not perfect. On the other hand, M^χ is a trivial S-module for every χ by Lemma 2.8, and hence $M^\chi \subseteq M^S$. Finally, equivalence between assertions (1) and (3) follows by Lemma 2.11. □

Lemma 2.14. *For $M \in \mathcal{C}$ of degree one and a T-character χ when* $\dim \pi_0(M)_{T,\chi} \leq 1$, *there can either be an embedding $i_*(\chi) \hookrightarrow M$, or an embedding $\mathbb{E}[\chi] \hookrightarrow M$, but not both.*

Proof. Suppose both embeddings exist. The first embedding $i_*(\chi) \hookrightarrow M$ has image in M^S, so both embeddings have zero intersection in M by perfectness of $\mathbb{E}[\chi]$. This yields an embedding $\mathbb{E}[\chi] \oplus \chi \hookrightarrow M$. Exactness of π_0 implies $\dim X^\chi \geq 2$ for $X = \pi_0(M)$, thus contradicts the assumption. □

Lemma 2.15. *For $M \in \mathcal{C}$ of degree one the following assertions are equivalent*:

(1) *M is perfect.*

(2) *M admits an embedding into $C_b^\infty(k^\times) \in \mathcal{C}_1$ (see Example 2.1).*

(3) *$i^*(M) = M_S$ is cyclic as a T-module and for every T-character χ with $\pi_0(M)_\chi \neq 0$ there is an embedding $\mathbb{E}[\chi] \hookrightarrow M$.*

Proof. (1) ⇒ (2): The degree of M is one, so $\dim M_\psi = 1$ for every nontrivial character ψ of S. Then the argument of Lemma 2.6 shows the existence of a nontrivial $\mathrm{Gl}_a(1)$-linear map $\ell : M \to C_b^\infty(k^\times)$. By Lemma 2.3 the image of ℓ has degree one. Therefore the kernel of ℓ is finite-dimensional and thus contained in M^S by Lemma 2.7. $M^S = 0$ vanishes by Lemma 2.13, so ℓ is injective.

(2) ⇒ (3): By Lemma 2.2 and exactness of π_0, as a submodule of $C_b^\infty(k^\times)$ of finite length, M satisfies $\dim(\pi_0(M)_\chi) = \dim(\pi_0(M)^\chi) \leq 1$. If $\dim(\pi_0(M)^\chi) = 1$, then there is an embedding $\chi \hookrightarrow \pi_0(M)$. The preimage of χ under the projection $M \to M/\mathbb{S} \cong \pi_0(M)$ has length two with constituents $\mathbb{S}$ and χ. By the uniqueness statement of Lemma 2.19, this preimage is isomorphic to $\chi \otimes \mathbb{E}$ as a $\mathrm{Gl}_a(1)$-module.

(3) ⇒ (1): If M is not perfect, then there is a character χ that embeds into $\kappa(M)$ by Lemma 2.13. By Lemma 2.14 there is no embedding $\mathbb{E}[\chi] \hookrightarrow M$, in contradiction to the assumption. □

Example 2.16 (the Kirillov model). An infinite-dimensional irreducible Gl(2)-module π, considered as a $\mathrm{Gl}_a(1)$-module as in Example 2.5, is perfect. Indeed, if there is a χ-eigenspace $\pi^\chi \neq 0$ for a T-character χ, then this eigenspace is a Gl(2)-submodule by Lemma 2.18 and this contradicts the irreducibility. By uniqueness of Whittaker models, π has degree one. Lemma 2.15 provides an embedding $\pi \hookrightarrow C_b^\infty(k^\times)$ of $\mathrm{Gl}_a(1)$-modules and the image contains $\mathbb{S} = C_c^\infty(k^\times)$. The quotient of π by $\mathbb{S}$ is the finite-dimensional unnormalized Jacquet-quotient π_S as a T-module.

Corollary 2.17 [Waldspurger 1985]. *For every infinite-dimensional irreducible smooth* $\pi \in \mathcal{C}_{\mathrm{Gl}(2)}$ *and every smooth character* ρ *of* $T = \{\mathrm{diag}(*, 1)\}$, *we have* $\dim \pi_{T,\rho} = 1$.

Proof. Consider π as a $\mathrm{Gl}_a(1)$-module as in Example 2.5. Then $\pi^{T,\rho} = 0$ by Lemma 2.18 and $\deg(\pi) = 1$ by the uniqueness of Whittaker models. Lemma 2.11 implies the assertion. □

Lemma 2.18. *Let* π *be a smooth* Gl(2)*-module. If a split maximal torus of* Gl(2) *acts on* $v \in \pi$ *by a character* χ, *then* v *is an eigenvector of* Gl(2). *Especially*, $\chi = \mu \circ \det$ *holds for a character* μ *of* $k^\times$.

Proof. By smoothness it is sufficient to show the assertion for a dense subset of Gl(2). By conjugation, v is an eigenvector of the standard torus. For the standard embedding $\mathrm{Gl}_a(1) \hookrightarrow \mathrm{Gl}(2)$, Lemma 2.7 implies that v is invariant under S, so it is an eigenvector of the standard Borel B. By conjugation, v is also an eigenvector of the opposite Borel $\bar{B}$. By Bruhat decomposition, $B\bar{B}$ is dense in Gl(2) and this implies the statement. □

Lemma 2.19. *For every cyclic* T*-module* $X \in \mathcal{C}_T^{\mathrm{fin}}$, *up to isomorphism there is a unique perfect* $\mathrm{Gl}_a(1)$*-module* $M \in \mathcal{C}$ *of degree one with* $\pi_0(M) \cong i_*(X)$.

Proof. Without loss of generality we can assume that $X = X^{(\chi)} = \chi^{(m)}$ is a Jordan block of length $m = \dim X$ attached to a character χ. By a twist we can assume $\chi = 1$. For every M that satisfies the requirements, there is a commutative diagram in $\mathcal{C}_1$ with exact rows and injective vertical arrows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{S} & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & \mathbb{S} & \longrightarrow & C_b^\infty(k^\times) & \longrightarrow & C_b^\infty(k^\times)/\mathbb{S} & \longrightarrow & 0
\end{array}$$

where $\varphi(X) \cong M/\mathbb{S}$ is in the kernel of the monodromy operator τ_χ^m on $C_b^\infty(k^\times)/\mathbb{S}$. Further, $(C_b^\infty(k^\times)/\mathbb{S})^\chi$ is one-dimensional by Lemma 2.2, so by Jordan normal forms $\dim\ker(\tau_\chi^m) \leq m$. By assumption $\varphi(X)$ has dimension m, so it coincides with this kernel and therefore M is unique up to isomorphism. It remains to show the existence. The generalized χ-eigenspace of $C_b^\infty(k^\times)/\mathbb{S}$ consists of functions locally constant modulo $\mathfrak{o}^\times$, thus uniquely determined on each $\mathfrak{o}^\times$-coset by the value $a(n) = f(\varpi^n)$. Using $k^\times = \varpi^{\mathbb{Z}} \times \mathfrak{o}^\times$ this generalized eigenspace can be identified with the subspace of $C(\mathbb{N},\mathbb{C})/C_c(\mathbb{N},\mathbb{C})$ represented by polynomial functions $a(n)$. Up to a sign, $\tau_\varpi = (x_\varpi - 1)$ acts on $C(\mathbb{N},\mathbb{C})/C_c(\mathbb{N},\mathbb{C})$ by the difference operator $a(n) \mapsto a(n) - a(n-1)$ and thus is nilpotent on X of the correct order. Then X corresponds the kernel of τ_ϖ^m and thus to polynomials of degree $< m$. Finally, define M as the fiber product of $C_b^\infty(k^\times)$ and X in the above diagram. Then M is perfect as a submodule of a perfect module. □

Universal extensions. Perfect modules $M \in \mathcal{C}$ of degree one are uniquely determined by $X = \pi_0(M) \in \mathcal{C}_T^{\mathrm{fin}}$ up to isomorphism by Lemma 2.19. Thus the perfect extensions

$$0 \to \mathbb{S} \to M \to X \to 0.$$

are universal and we write $\mathbb{E}[X] = M$. They exist if and only if X is cyclic. By construction, each isomorphism class $\mathbb{E}[X]$ admits a unique representative as a submodule of $C_b^\infty(k^\times)$. Note that $\mathbb{E}[0] = \mathbb{S}$ and $\mathbb{E}[\chi] = \chi \otimes \mathbb{E}$ for T-characters χ, see Example 2.12.

Lemma 2.20. *A perfect nonzero $M \in \mathcal{C}$ has degree at least one and admits a filtration of perfect modules in $\mathcal{C}$*

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_{\deg(M)} = M$$

such that the graded components $E_i = F_i/F_{i-1}$ are perfect of degree one.

Proof. For every $i \leq \deg(M)$, by Lemma 2.10 there always exists a quotient E_i of a perfect F_i of degree one and minimal dimension of $\pi_0(E_i)$. This E_i is perfect since otherwise $\pi_0(E_i)$ would not be minimal. The kernel $F_{i-1} = \ker(F_i \to E_i)$ is perfect as the submodule of the perfect F_i. Repeating this process gives the required filtration. □

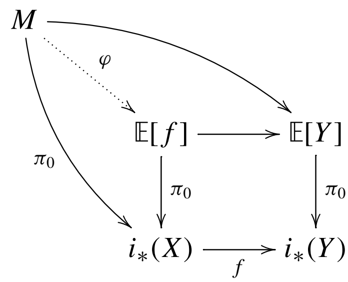
Remark. If M is not perfect, then the above construction applied to the perfect module M/M^S provides a filtration whose first term is finite-dimensional.

2.4. ***Modules of degree one.*** We classify the $\mathrm{Gl}_a(1)$-modules of degree one and their morphisms.

Lemma 2.21. *Every extension $0 \to \mathbb{S} \to M \to i_*(\chi) \to 0$ in $\mathcal{C}$ of a character $i_*(\chi) \in \mathcal{C}$ is either split $M \cong \mathbb{S} \oplus i_*(\chi)$ or isomorphic to $\mathbb{E}[\chi]$. In other words, $\mathrm{Ext}^1_{\mathcal{C}}(i_*(\chi), \mathbb{S})$ is one-dimensional.*

Proof. If $M^S \in \mathcal{C}$ is nonzero, then it is isomorphic to $i_*(\chi)$ because $\mathbb{S}$ is perfect. In that case the embedding $M^S \to M$ splits the sequence. Otherwise M is perfect by Lemma 2.13, so it admits an embedding $M \hookrightarrow C_b^\infty(k^\times)$ by Lemma 2.15. As a submodule of $C_b^\infty(k^\times)$, it is uniquely determined by $\pi_0(M) \cong X$ because of Lemma 2.19. □

For finite-dimensional T-modules $X, Y \in \mathcal{C}_T^{\mathrm{fin}}$ with cyclic Y and a T-morphism $f : X \to Y$, let $\mathbb{E}[f]$ denote the fiber product in $\mathcal{C}$ of $i_*(X)$ and $\mathbb{E}[Y]$ over $i_*(Y)$. It is explicitly given by $\mathbb{E}[f] = \{(x, m) \in i_*(X) \times \mathbb{E}[Y] \mid f(x) = \pi_0(m)\}$. We always assume that f is surjective by possibly replacing Y by the image of f. By the universal property of fiber products, for every M in $\mathcal{C}$ making the diagram of solid arrows commute, there is a unique $\varphi : M \to \mathbb{E}[f]$ making the whole diagram commute:



It turns out that every object of $\mathcal{C}$ of degree one is isomorphic to some $\mathbb{E}[f]$.

Lemma 2.22. *Every fiber product $\mathbb{E}[f]$ has degree one. Every $M \in \mathcal{C}$ of degree one is isomorphic to $\mathbb{E}[f]$ for the natural projection $f : X \to Y$ from $X = \pi_0(M)$ to the quotient $Y = X/\kappa(M)$. Especially, Y is always cyclic.*

Proof. $\mathbb{E}[f]$ modulo $\{(0, m) \in X \times \mathbb{E}[Y] \mid m \in \mathbb{S}\} \cong \mathbb{S}$ is finite-dimensional, so $\mathbb{E}[f]$ has degree one and $\pi_0(\mathbb{E}[f]) \cong \{(x, y) \in X \times Y \mid f(x) = y\}$ is isomorphic to X. By Lemmas 2.7 and 2.13, the maximal finite-dimensional submodule of $\mathbb{E}[f]$ is the kernel of the projection $\mathbb{E}[f] \to \mathbb{E}[Y]$, so $\kappa(\mathbb{E}[f]) \cong \ker(f)$. For the converse statement, let Q be a quotient of M of minimal length and degree one. The kernel of the projection $K = \ker(M \twoheadrightarrow Q)$ has degree zero and is thus a finite-dimensional submodule $K \subseteq \kappa(M)$. In fact $K = \kappa(M)$, since otherwise the length of $M/\kappa(M)$ would be smaller than that of Q. Further, $Q \cong M/\kappa(M)$ is perfect of degree one, since if it was not perfect, then it would admit a nontrivial finite-dimensional submodule which contradicts the minimality assumption. By exactness of the π_0-functor, $Q \cong \mathbb{E}[Y]$ is the universal extension attached to the T-module $Y = \pi_0(M)/\kappa(M)$. By Lemma 2.15, $Y \cong \pi_0(Q)$ is cyclic. The projection f from $X = \pi_0(M)$ to Y defines the fiber product $\mathbb{E}[f]$. The universal property gives rise to a unique morphism $\varphi : M \to \mathbb{E}[f]$. Since $\pi_0(\varphi) = \mathrm{id}_X$ and since the kernel of φ has degree zero, φ is an isomorphism by the five-lemma. $\square$

Lemma 2.23. *For a perfect $\mathbb{E}[X]$ of degree one in $\mathcal{C}$, a morphism $\varphi : \mathbb{E}[X] \to M$ in $\mathcal{C}$ either has finite-dimensional image in $\kappa(M)$ or is injective. If φ is injective, then $\pi_0(\varphi)$ is injective and* $\mathrm{coker}(\varphi) \cong \mathrm{coker}(\pi_0(\varphi))$ *in $\mathcal{C}$.*

Proof. If the image of φ is not finite-dimensional, then it has degree one. Then the kernel is a finite-dimensional submodule of $\mathbb{E}[X]$ and therefore zero. This shows the proposed dichotomy. For injective φ, the snake lemma applied to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{S} & \longrightarrow & \mathbb{E}[X] & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \varphi & & \downarrow \pi_0(\varphi) & & \\ 0 & \longrightarrow & \mathbb{S} & \longrightarrow & M & \longrightarrow & \pi_0(M) & \longrightarrow & 0 \end{array}$$

implies the final statement. □

Lemma 2.24. *For $M \in \mathcal{C}$ of degree one the following assertions are equivalent*:

(1) *M is indecomposable.*

(2) *Every Jordan block in $\pi_0(M)$ has nontrivial image in $\pi_0(M)/\kappa(M)$.*

(3) *$\pi_0(M)$ is cyclic and if $\pi_0(M)^\chi \neq 0$, then an embedding $\mathbb{E}[\chi] \hookrightarrow M/\kappa(M)$ exists.*

Proof. (1) ⇒ (2): If there is a Jordan block X of $\pi_0(M)$ contained in the submodule $\kappa(M)$, then we obtain decompositions of T-modules $\pi_0(M) \cong X \oplus \pi_0(M)/X$ and $\kappa(M) \cong X \oplus \kappa(M)/X$ as direct sums. By the uniqueness statement implicit in Lemma 2.22, M is isomorphic to $X \oplus M/X$.

(2) ⇒ (3): Fix a character χ that occurs in $\pi_0(M)$. If $\dim \pi_0(M)^\chi \geq 2$, then there are two χ-Jordan blocks and both have nontrivial image in $\pi_0(M)/\kappa(M)$. But $\pi_0(M)/\kappa(M)$ satisfies the monodromy property by Lemma 2.15, this gives a contradiction. Since χ occurs in $\pi_0(M)/\kappa(M) \cong \pi_0(M/\kappa(M))$, the embedding exists by Lemma 2.15.

(3) ⇒ (1): Assume that $M = M_1 \oplus M_2$ splits as a direct sum where M_2 is finite-dimensional. Fix a character χ that occurs in $M_2 = \pi_0(M_2)$. If there is an embedding $\mathbb{E}[\chi] \hookrightarrow M/\kappa(M) \cong M_1/\kappa(M_1)$, then $\pi_0(M_1)^\chi \neq 0$. Hence $\dim \pi_0(M)^\chi \geq 2$ implies a contradiction. □

Corollary 2.25. *Every perfect module $M \in \mathcal{C}$ of degree one is indecomposable.*

Proof. If M is perfect, then $\kappa(M) = 0$ by Lemma 2.13. The second assertion of Lemma 2.24 is obviously satisfied. □

2.5. ***L-functions and zeta integrals.*** Recall that $M \in \mathcal{C}$ is perfect of degree one if and only if $\dim M_{T,\chi} = 1$ holds for all smooth characters χ of $k^\times$. Then we have $\dim(M_\psi) = 1$ and every nonzero $\ell \in \mathrm{Hom}_{\mathbb{C}}(M_\psi, \mathbb{C})$ defines an embedding $p : M \hookrightarrow C_b^\infty(k^\times)$ by $p(m)(\lambda) = \ell(x_\lambda m)$, see Lemma 2.15. Conversely, if $0 \neq M \in \mathcal{C}$ has finite length, every such embedding p defines a functional $\ell(m) = p(m)(1)$.

Definition 2.26. Consider a perfect $\mathrm{Gl}_a(1)$-module $M \in \mathcal{C}$ of degree one. For a smooth T-character χ and $f \in p(M)$ define the *zeta integral*

$$Z(f, \chi, s) = \int_T f(x)\chi(x)|x|^s \, \mathrm{d}^\times x,$$

with Haar measure on $T \cong k^\times$ normalized by $\mathrm{vol}(\mathfrak{o}^\times) = 1$. The decomposition of the semisimplified T-module $\pi_0(M)^{ss} \cong \bigoplus_\mu a_\mu(M) \cdot \mu$ defines the product of Tate L-factors:

$$L(M, s) = \prod_\mu L(\mu, s)^{a_\mu(M)}.$$

Caveat. For an infinite-dimensional irreducible smooth Gl(2)-module, restriction to the mirabolic subgroup $\mathrm{Gl}_a(1) \subseteq \mathrm{Gl}(2)$ defines a perfect $\mathrm{Gl}_a(1)$-module by Kirillov theory. But in contrast to the L-factor defined by Jacquet and Langlands, our L-factor is offset by $s \mapsto s + \frac{1}{2}$.

Lemma 2.27. *For smooth χ and $f \in p(M)$ the quotients $\frac{Z(f,\chi,s)}{L(\chi\otimes M,s)}$ are entire.*

Proof. By a twist, we may assume $\chi = 1$. Every $f \in \mathbb{S}$ has compact support, so $Z(f, \chi, s)$ is entire. By linearity in f we can assume that $\pi_0(M)$ is a generalized eigenspace for a smooth character μ of T. By integration over the maximal compact subgroup $K_T \cong \mathfrak{o}^\times$ of T we can assume that μ is unramified, so $f \in M^{K_T}$ is fixed by K_T. Especially, $f(x) = \sum_n b_n 1_{\varpi^n\mathfrak{o}^\times}(x)$ with the characteristic functions $1_{\varpi^n\mathfrak{o}^\times}$ and $b_n \in \mathbb{C}$, so

$$Z(f, 1, s) = \sum_n b_n Z(1_{\varpi^n\mathfrak{o}^\times}, 1, s) = \sum_n b_n q^{-ns}, \quad q = |\varpi_k|^{-1}.$$

We can assume $b_n = 0$ for $n \le n_0$ because finite sums of q^{-ns} are entire. By Lemma 2.19, for sufficiently large n_0 there exists a polynomial $P_{f,\mu}(t)$ of degree $< a_\mu(M)$, such that $b_n = \mu(\varpi)^n P_{f,\mu}(n)$ for $n \ge n_0$.

Let the Laurent ring $\Lambda = \mathbb{C}[t, t^{-1}]$ in the indeterminate t act on $C_b^\infty(k^\times)$ by $t = x_\varpi^{-1} \in T$. Using power series expansion of $(1 - \mu(\varpi)t)^{-1}$ and $t^n \cdot 1_{\mathfrak{o}^\times} = 1_{\varpi^n\mathfrak{o}^\times}$ one defines a Λ-equivariant embedding

$$\tilde{\Lambda} := \Lambda\left[\frac{1}{1 - \mu(\varpi)t}\right] \hookrightarrow \mathbb{C}[\![t]\!][t^{-1}] \hookrightarrow C_b^\infty(k^\times), \quad P(t) \mapsto P(t) \cdot 1_{\mathfrak{o}^\times}.$$

The subspace $W = p(M)^{K_T}$ is finitely generated as a Λ-submodule of the localization $\tilde{\Lambda}$ of the Dedekind ring Λ, and hence defines a fractional Λ-ideal. On $\tilde{\Lambda}$ the zeta integral $Z(f, 1, s)$ is obtained from the evaluation $t \mapsto q^{-s}$. Since

$$\frac{d}{ds} P(q^{-s}) = -\log(q)\, t \frac{d}{dt} P(t),$$

the submodule $(1 - \mu(\varpi)t)^{a_\mu(M)} W$ is equal to Λ. Since $1 - \mu(\varpi)t$ specializes to $L(\mu, s)^{-1} = 1 - \mu(\varpi)q^{-s}$, the quotient $L(\mu, s)^{-\deg(P_{f,\mu})} Z(f, \chi, s)$ is entire. □

Lemma 2.28. *For perfect $M \in \mathcal{C}$ of degree one and positive integers n, we have* $\dim \mathrm{Hom}_T(\chi \otimes M, 1^{(n)}) = n$. *The exact sequence* $0 \to 1^{(n-1)} \to 1^{(n)} \to \mathbb{C} \to 0$ *of T-modules induces an injection* $\mathrm{Hom}_T(\chi \otimes M, 1^{(n-1)}) \hookrightarrow \mathrm{Hom}_T(\chi \otimes M, 1^{(n)})$. *Let $(\chi \otimes M)^{(n-1)} \subseteq \chi \otimes M$ be the T-submodule annihilated by the functionals in*

$\mathrm{Hom}_T(\chi \otimes M, 1^{(n-1)})$. *Then* $\mathrm{Hom}_T((\chi \otimes M)^{(n-1)}, \mathbb{C})$ *is spanned as a complex vector space by the functional* $I_\chi^{(n)} : \chi \otimes M \to \mathbb{C}$ *that is defined by*

$$I_\chi^{(n)}(f) = \frac{d^{n-1}}{ds^{n-1}}\left(\frac{Z(f, \chi, s)}{L(M \otimes \chi, s)}\right)\Big|_{s=0}.$$

Proof. Without restriction of generality $\chi = 1$. The subspace $W = M^{T(\mathfrak{o})} \subseteq M$ in $\tilde{\Lambda}$ is a fractional Λ-ideal such that $\prod_\mu (1-t)^{a_\mu(M)} = \Lambda$. Notice $(t-1) = -t\tau$ holds for the monodromy operator τ and $-t \in \Lambda^\times$ is a unit. Then $\tau^n = 0$ on $1^{(n)}$ implies $\mathrm{Hom}_T(M, 1^{(n)}) = \mathrm{Hom}_\Lambda(W, 1^{(n)}) = \mathrm{Hom}_\Lambda(W/(t-1)^n W, 1^{(n)})$. Since $\varphi : W/(t-1)^n W \cong \Lambda/(t-1)^n$, using the identification by the isomorphism $\varphi(P(t)) = \prod_\mu (1-\mu(\varpi)t)^{a_\mu(M)} P(t)$ we obtain

$$\mathrm{Hom}_\Lambda\left(\frac{W}{(t-1)^n W}, 1^{(n)}\right) \cong \mathrm{Hom}_\Lambda\left(\frac{\Lambda}{(t-1)^n}, 1^{(n)}\right) = \mathrm{Hom}_\Lambda(1^{(n)}, 1^{(n)}),$$

which is $\Lambda/(t-1)^n \cong 1^{(n)}$ as vector space, and hence $\dim(\mathrm{Hom}_T(M, \chi^{(n)})) = n$. On $\prod_\mu (1-\mu(\varpi)t)^{a_\mu(M)} W = \Lambda$ the zeta integral $Z(f, 1, s)$ is the evaluation $t \mapsto q^{-s}$, and $\frac{d}{ds} P(q^{-s}) = -\log(q) t \frac{d}{dt} P(t)$ holds. Since the higher derivations $\left(t\frac{d}{dt}\right)^i$ at $t = 1$ of order $i \le n-1$ are linear independent on $\Lambda/(t-1)^n$, the lemma now easily follows. □

Lemma 2.29. *For nonzero* $M \subseteq C_b^\infty(k^\times)$ *and every smooth character* χ, *the functional* $I_\chi : M \to \mathbb{C}$,

$$I_\chi(f) = \lim_{s\to 0} \frac{Z(f, \chi, s)}{L(\chi \otimes M, s)}, \quad f \in M$$

is nonzero and generates the one-dimensional space $\mathrm{Hom}_T(\chi \otimes M, \mathbb{C})$. *In other words,* $L(\chi \otimes M, s)$ *is the regularizing* L*-factor for all zeta integrals* $Z(f, \chi, s)$, *where* f *runs over the functions in* $M \subseteq C_b^\infty(k^\times)$.

Proof. Note that $Z(x_\lambda f, \chi, s) = \chi^{-1}(\lambda)|\lambda|^{-s} Z(f, \chi, s)$ and that the limit $s \to 0$ is defined by Lemma 2.27. By a twist we can assume $\chi = 1$. If $d = a_\chi(M) - 1 \ge 0$, nonvanishing follows from $I_\chi(f) = (-1)^d d!$ for $P_{f,\chi}(t) = t^d$ as in Lemma 2.27. For $a_\chi(M) = 0$, use $I_\chi(1_{\mathfrak{o}^\times}) \ne 0$ and $1_{\mathfrak{o}^\times} \in C_c^\infty(k^\times) \subseteq M$ by Lemma 2.3. □

3. Representations of affine linear groups in general

We review certain results of Bernstein and Zelevinskii [1976, Section 5] on the representation theory of affine linear groups. For a finite dimensional vector space V over k the affine linear group $\mathrm{Gl}_a(V) = \mathrm{Gl}(V) \ltimes V$ is the semidirect product of the general linear group $\mathrm{Gl}(V)$ with the group of translations by V. Elements of $\mathrm{Gl}_a(V)$ are denoted $[g\,|\,v]$ or $g.v$ for $g \in \mathrm{Gl}(V)$ and $v \in V$; and the group law is

$[g_1\,|\,v_1][g_2\,|\,v_2] = [g_1g_2\,|\,v_1 + g_1v_2]$. There are two pairs of adjoint exact functors

$$\mathcal{C}_{\widetilde{\mathrm{Gl}}_a(V')} \underset{j^!}{\overset{j_!}{\rightleftarrows}} \mathcal{C}_{\mathrm{Gl}_a(V)} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{C}_{\mathrm{Gl}(V)}$$

such that

$$\mathrm{Hom}_{\mathrm{Gl}_a(V)}(X, i_*Y) \cong \mathrm{Hom}_{\mathrm{Gl}(V)}(i^*X, Y)$$

and

$$\mathrm{Hom}_{\mathrm{Gl}_a(V)}(j_!X, Y) \cong \mathrm{Hom}_{\widetilde{\mathrm{Gl}}_a(V')}(X, j^!Y).$$

These functors determine the structure of $\mathcal{C}_{\mathrm{Gl}_a(V)}$:

Lemma 3.1 [Bernstein and Zelevinskii 1976, Section 5.12]. *There is a functorial exact sequence in $\mathcal{C}_{\mathrm{Gl}_a(V)}$:*

$$0 \longrightarrow j_!\,j^! \longrightarrow \mathrm{id} \longrightarrow i_*i^* \longrightarrow 0.$$

Functors $j_!$ and i_* induce equivalences $\mathcal{C}_{\widetilde{\mathrm{Gl}}_a(V')} \cong j_!(\mathcal{C}_{\widetilde{\mathrm{Gl}}_a(V')})$ and $\mathcal{C}_{\mathrm{Gl}(V)} \cong i_*(\mathcal{C}_{\mathrm{Gl}(V)})$ of abelian categories. The compositions $j^!i_* = 0$ and $i^*j_! = 0$ vanish, so these abelian subcategories are closed under extensions in $\mathcal{C}_{\mathrm{Gl}_a(V)}$. The compositions $j^!j_!$ and i^*i_* are naturally equivalent to the identity functor.

To be precise, fix a nontrivial k-linear form $\ell : V \to k$ and a nontrivial smooth character $\psi : k \to \mathbb{C}$. The kernel $V' = \ker(\ell)$ is a subspace of codimension one. This defines a nontrivial character $\psi_V : V \to \mathbb{C}$, by $\psi_V = \psi \circ \ell$. The group $\mathrm{Gl}_a(V)$ acts on its normal subgroup V by conjugation. The little group, the stabilizer of ℓ in $\mathrm{Gl}_a(V)$, is the subgroup $\widetilde{\mathrm{Gl}}_a(V') \ltimes V$ of $\mathrm{Gl}_a(V) = \mathrm{Gl}(V) \ltimes V$. Here the *mirabolic* subgroup $\widetilde{\mathrm{Gl}}_a(V')$ is the image of the natural embedding

$$\mathrm{Gl}_a(V') \ni [g\,|\,v] \mapsto \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \in \mathrm{Gl}(V)$$

with respect to the decomposition $V \cong V' \oplus k$. This defines the following functors:

- i^* is the coinvariants under the normal subgroup $V \hookrightarrow \mathrm{Gl}_a(V)$.
- i_* is the pullback under the projection $\mathrm{Gl}_a(V) \twoheadrightarrow \mathrm{Gl}(V)$.
- $j^!$ is the (V, ψ_V)-coinvariants under the normal subgroup $V \hookrightarrow \mathrm{Gl}_a(V)$.
- $j_! = \mathrm{ind}_{\widetilde{\mathrm{Gl}}_a(V') \ltimes V}^{\mathrm{Gl}_a(V)}(- \boxtimes \psi_V)$ is the unnormalized compact induction.

Write $\mathcal{C}_n$ for $\mathcal{C}_{\mathrm{Gl}_a(V)}$ if $n = \dim(V)$ and let $\mathcal{C}_n^{\mathrm{fin}} \subseteq \mathcal{C}_n$ be its full subcategory of representation with finite length. For $n = 1$ this is the category $\mathcal{C}$ discussed in the previous section. It is easy to see inductively that every irreducible object of $\mathcal{C}_n$ is of the form $(j_!)^k i_*(M)$ for some irreducible $M \in \mathcal{C}_{\mathrm{Gl}(n-k)}$, generalizing Lemma 2.6. We identify $\mathcal{C}_0$ with the category of complex vector spaces, then the irreducible $\mathrm{Gl}_a(V)$-module $\mathbb{S}_n = (j_!)^n(\mathbb{C})$ for $n = \dim V$ is defined for a choice of nontrivial characters $\psi_V, \psi_{V'}, \ldots$. We can identify $\mathbb{S}$ from the previous section with $\mathbb{S}_1$. One proves that $\mathbb{S}_n$ is projective in the category $\mathcal{C}_n$ in the same way as in Lemma 2.10.

Twists. Characters of $\mathrm{Gl}_a(V)$ factorize over $\det_a[g\,|\,v] = \det(g)$. For a smooth character $\chi : \mathrm{Gl}(1) \to \mathbb{C}^\times$, the twist functor

$$\mathcal{C}_n \to \mathcal{C}_n, \quad M \mapsto \chi \otimes M = (\chi \circ \det_a) \otimes M$$

is denoted by χ again, if no confusion is possible. The natural equivalences

$$i_* \circ \chi \cong \chi \circ i_*, \quad i^* \circ \chi \cong \chi \circ i^*, \quad j^! \circ \chi \cong \chi \circ j^!, \quad j_! \circ \chi \cong \chi \circ j_!$$

are obvious except for the last one which follows by partial induction and the formula $\mathrm{ind}_{\{1\}}^{k^\times}(\chi) \cong \chi \otimes \mathrm{ind}_{\{1\}}^{k^\times}(1) \cong \mathrm{ind}_{\{1\}}^{k^\times}(1)$. In the following we relax notation and identify $\mathrm{Gl}(V)$ and V with the subgroups $\mathrm{Gl}(V) \ltimes 0$ and $\{\mathrm{id}\} \ltimes V$, respectively.

3.1. ***The functors k_ρ and k^ρ.*** Fix a finite-dimensional k-vector space V and a subspace V' of codimension one as before. We write elements of $\mathrm{Gl}_a(V)$ as block matrices with respect to a fixed decomposition $V \cong V' \oplus k$. We identify $H = \mathrm{Gl}_a(V') \times \mathrm{Gl}(1)$ with its image under the embedding

$$H \hookrightarrow \mathrm{Gl}_a(V) \quad ([g\,|\,v'], x) \mapsto \left[\begin{matrix} g & 0 \\ 0 & x \end{matrix}\,\middle|\,\begin{matrix} v' \\ 0 \end{matrix}\right],$$

where H normalizes the subgroup $\widetilde{V}' = \left\{\left[\begin{smallmatrix} \mathrm{id} & * \\ 0 & 1 \end{smallmatrix}\,\middle|\,\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right\}$. For every $M \in \mathcal{C}_{\mathrm{Gl}_a(V)}$ the $\widetilde{V}'$-coinvariant quotient is a smooth H-module $M_{\widetilde{V}'} \in \mathcal{C}_H$. The corresponding functor $\mathcal{C}_{\mathrm{Gl}_a(V)} \to \mathcal{C}_H$ is exact because $\widetilde{V}'$ is compactly generated. A smooth character ρ of $\mathrm{Gl}(1)$ defines a character of $\{\mathrm{id}\} \times \mathrm{Gl}(1) \subseteq H$, explicitly given by $([\mathrm{id}\,|\,0], x) \mapsto \rho(x)$. By abuse of notation we denote this character by ρ again. The ρ-(co)invariant spaces

$$k_\rho(M) = (M_{\widetilde{V}'})_\rho, \quad k^\rho(M) = (M_{\widetilde{V}'})^\rho$$

are preserved by the subgroup $\mathrm{Gl}_a(V') \times \{1\}$ of H and thus define smooth modules $k_\rho(M)$ and $k^\rho(M)$ of $\mathrm{Gl}_a(V')$. We obtain right exact functors k_ρ and left exact functors k^ρ:

$$\mathcal{C}_{\mathrm{Gl}_a(V)} \underset{k_\rho}{\overset{k^\rho}{\rightrightarrows}} \mathcal{C}_{\mathrm{Gl}_a(V')}.$$

For twists by characters χ of $k^\times$ there are natural equivalences

$$k_\rho \circ \chi \cong \chi \circ k_{\rho\chi^{-1}}, \quad k^\rho \circ \chi \cong \chi \circ k^{\rho\chi^{-1}}.$$

Lemma A.2 implies:

Lemma 3.2. *A short exact sequence* $0 \to E \to F \to G \to 0$ *in* $\mathcal{C}_{\mathrm{Gl}_a(V)}$ *gives rise to a long exact sequence in* $\mathcal{C}_{\mathrm{Gl}_a(V')}$:

$$0 \to k^\rho(E) \to k^\rho(F) \to k^\rho(G) \to k_\rho(E) \to k_\rho(F) \to k_\rho(G) \to 0.$$

We say an object $M \in \mathcal{C}_n$ for $n \geq 1$ is *perfect* if it has finite length and $k^\rho(M) = 0$ holds for all characters ρ. The functors $j_!$, $j^!$ send perfect objects to perfect objects by Lemma 3.4. However, objects need not remain perfect after applying the functor k_ρ, see Theorem 3.24.

Lemma 3.3. *For every smooth ρ there is a natural equivalence*

$$k_\rho \circ i_* \cong k^\rho \circ i_*$$

of functors from $\mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(V)}$ to the full subcategory $i_*(\mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(V')})$ of $\mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}_a(V')}$.

Proof. For a smooth $\mathrm{Gl}(V)$-module π the isomorphisms $k_\rho(i_*(\pi)) \cong i_*((\pi_{\widetilde{V}'})_\rho)$ and $k^\rho(i_*(\pi)) \cong i_*((\pi_{\widetilde{V}'})^\rho)$ are obviously functorial. It remains to be shown that $(\pi_{\widetilde{V}'})_\rho$ and $(\pi_{\widetilde{V}'})^\rho$ are naturally equivalent as $\mathrm{Gl}(V')$-modules. The unnormalized Jacquet quotient $M = \pi_{\widetilde{V}'}$ is a module of finite length under the action of $\mathrm{Gl}(V') \times \mathrm{Gl}(1) \cong H \cap \mathrm{Gl}(V)$. Therefore M is a finite direct sum of products $X \boxtimes \chi^{(m)}$ where X is a $\mathrm{Gl}(V')$-module and $\chi^{(m)}$ is a $\mathrm{Gl}(1)$-Jordan block of length m. The ρ-(co)invariant space in M is then the direct sum of those X where $\chi = \rho$, so M^ρ and M_ρ are isomorphic. □

Lemma 3.4. *For smooth ρ there is a functorial exact sequence in $\mathcal{C}_n$, $n \geq 1$,*

$$0 \to j_! \circ k_{\nu\rho} \to k_\rho \circ j_! \to i_* \circ i^* \to 0$$

and a natural equivalence $j_! \circ k^{\nu\rho} \cong k^\rho \circ j_!$. Here the functors $k_{\nu\rho}$ and $k^{\nu\rho}$ are defined with respect to the twisted character $\nu\rho$.

Proof. Fix a nontrivial additive character ψ on $V = k^{n+1}$ and a subspace $V' \subseteq V$ of codimension one in the kernel of ψ. As before we write elements of $\mathrm{Gl}_a(V)$ as block matrices with respect to a fixed decomposition $V \cong V' \oplus k$. Fix subgroups $P = M \ltimes V$ and $Q = H \ltimes \widetilde{V}'$ of $\mathrm{Gl}_a(V)$ with the mirabolic group $M = \widetilde{\mathrm{Gl}}_a(V')$ and $H \cong \mathrm{Gl}_a(V') \times \mathrm{Gl}(1)$ as above. Bruhat decomposition implies that the double coset space $P \setminus \mathrm{Gl}_a(V)/Q$ is generated by the identity and a Weyl reflection $w \in \mathrm{Gl}(V)$ that does not preserve $\mathrm{Gl}(V')$. Note that $V' = H \cap wVw^{-1} = H \cap V$ and that $\psi^w = \psi \circ w^{-1}$ is nontrivial on V'. A dimension estimate implies that PwQ is open in $\mathrm{Gl}_a(V)$. Theorem 5.2 of Bernstein and Zelevinskii [1977] gives a short exact sequence of normalized functors $\mathcal{C}_M \to \mathcal{C}_H$:

$$0 \longrightarrow i_{V',\psi^w} \circ w \circ r_{M \cap w^{-1}\widetilde{V}'w} \longrightarrow r_{\widetilde{V}'} \circ i_{V,\psi} \longrightarrow i_{V',\psi} \circ r_{M \cap \widetilde{V}'} \longrightarrow 0.$$

These functors are explicitly given by

$$\begin{aligned} i_{V,\psi} &= j_! \circ \nu^{1/2}, & \mathcal{C}_M &\to \mathcal{C}_{\mathrm{Gl}_a(V)}, \\ r_{\widetilde{V}'} &= (\nu \boxtimes \nu^n)^{-1/2} \circ (-)_{\widetilde{V}'}, & \mathcal{C}_{\mathrm{Gl}_a(V)} &\to \mathcal{C}_H, \\ i_{V',\psi^w} \circ w &= (j_! \boxtimes \mathrm{id}) \circ (\nu \boxtimes 1)^{1/2}, & \mathcal{C}_{\widetilde{H}} &\to \mathcal{C}_H, \end{aligned}$$

$$\begin{aligned} r_{M\cap w^{-1}\widetilde{V}'w} &= (\nu\boxtimes\nu^{1-n})^{-1/2}\circ(-)_{M\cap w^{-1}\widetilde{V}'w}, &\quad \mathcal{C}_M\to\mathcal{C}_{\widetilde{H}},\\ i_{V',\psi} &= (i_*\boxtimes\mathcal{S})\circ\nu^{1/2}, &\quad \mathcal{C}_{\mathrm{Gl}(V')}\to\mathcal{C}_H,\\ r_{M\cap\widetilde{V}'} &= \nu^{-1/2}\circ i^*, &\quad \mathcal{C}_M\to\mathcal{C}_{\mathrm{Gl}(V')}, \end{aligned}$$

with $\widetilde{H} = M\cap w^{-1}H \cong \mathrm{Gl}_a(V'\cap wV')\times\mathrm{Gl}(1)$ and the regular Gl(1)-module $\mathcal{S}=\mathrm{ind}_{\{\mathrm{id}\}}^{\mathrm{Gl}(1)}(1)\cong\mathbb{S}|_{\mathrm{Gl}(1)}$. Twist the above sequence by the Gl(1)-character $\nu^{-(n-1)/2}$ and note that $\mathcal{S}$ is invariant under twists. This yields a short exact sequence of functors $\mathcal{C}_M\to\mathcal{C}_H$:

$$0\longrightarrow(j_!\boxtimes\nu^{-1})\circ(-)_{M\cap w^{-1}\widetilde{V}'w}\longrightarrow(-)_{\widetilde{V}'}\circ j_!\longrightarrow(i_*\circ i^*)\boxtimes\mathcal{S}\longrightarrow 0.$$

Finally, Lemma A.2 gives a long exact sequence of functors $\mathcal{C}_M\to\mathcal{C}_{\mathrm{Gl}_a(V')}$:

$$0\longrightarrow j_!k^{\nu\rho}\longrightarrow k^\rho j_!\longrightarrow i_*i^*\otimes\mathcal{S}^\rho\xrightarrow{\ \delta\ } j_!k_{\nu\rho}\longrightarrow k_\rho j_!\longrightarrow i_*i^*\otimes\mathcal{S}_\rho\longrightarrow 0.$$

Since $\mathcal{S}^\rho=0$ and $\mathcal{S}_\rho\cong\mathbb{C}$ by Proposition 4.3.2 of [Bump 1998], the assertion follows. □

Corollary 3.5. *The functors k_ρ and k^ρ send $\mathcal{C}_n^{\mathrm{fin}}$ to $\mathcal{C}_{n-1}^{\mathrm{fin}}$. For $X\in\mathcal{C}_{\mathrm{Gl}(1)}^{\mathrm{fin}}$ there is an exact sequence in $\mathcal{C}=\mathcal{C}_1^{\mathrm{fin}}$:*

$$0\to\mathbb{S}\otimes(\nu^{-1}X)_\rho\to k_\rho j_! i_*(X)\to i_*(X)\to 0$$

and an isomorphism $\mathbb{S}\otimes(\nu^{-1}X)^\rho\cong k^\rho j_! i_*(X)$. For $n\geq 1$ there are isomorphisms $k_\rho(\mathbb{S}_n)\cong\mathbb{S}_{n-1}$ and $k^\rho(\mathbb{S}_n)=0$ where $\mathbb{S}_n=(j_!)^n(\mathbb{C})\in\mathcal{C}_n^{\mathrm{fin}}$.

Proof. We proof the last assertion by induction over n. Indeed, $\mathbb{S}_1$ is perfect by Lemma 2.2 and $k_\rho(\mathbb{S}_1)=\mathbb{S}_\rho$ is one-dimensional by Lemma 2.11. Use Lemma 3.4 for the induction step and note that $i^*(\mathbb{S}_n)=0$ by Lemma 3.1. □

Proposition 3.6. *For $M\in\mathcal{C}_n^{\mathrm{fin}}$ with $n\geq 1$, in the Grothendieck group holds*

$$[k_\rho(M)]-[k^\rho(M)]=[j^!(M)]\quad\textit{in}\ K_0(\mathcal{C}_{n-1}^{\mathrm{fin}}).$$

Proof. The left-hand side is well defined and additive in M by Lemma 3.2, so we can assume that M is irreducible. Then $M\cong j_!^m i_*(\rho)$ for irreducible $\rho\in\mathcal{C}_{\mathrm{Gl}(n-m)}^{\mathrm{fin}}$ with $0\leq m\leq n$, see [Bernstein and Zelevinskii 1976, Section 5.13]. For $m=0$ the claim follows from Lemma 3.3 and the assertion $j^!i_*=0$. The general case follows by induction over m using Lemma 3.4, the assertions $i^*j_!=0$ and $j^!j_!\cong\mathrm{id}$ and the functorial exact sequence in Lemma 3.1. □

Corollary 3.7. *An M in $\mathcal{C}_n^{\mathrm{fin}}$ is perfect if and only if $[k_\rho(M)]\in K_0(\mathcal{C}_{n-1}^{\mathrm{fin}})$ does not depend on ρ.*

Proof. By Lemma 3.3 and Corollary 3.5, $k^\rho(M)$ is zero for almost all ρ. □

3.2. ***The functor η.*** The Klingen parabolic subgroup $Q = L_Q N_Q$ consists of all elements $g \in G$ that stabilize the line $\mathbb{C} \cdot e_1$, where e_1 denotes the first vector of the fixed symplectic basis of k^4. The unipotent radical N_Q of Q is nonabelian. The center of N_Q is the subgroup $S_A = \{s_{a,0,0} \mid a \in k\}$, it is normal in Q. The Levi subgroup $L_Q \subseteq Q$ is chosen to consist of all $g \in Q$ that stabilize the line $\mathbb{C} \cdot e_3$. There is an exact sequence

$$0 \longrightarrow \mathrm{Gl}_a(2) \xrightarrow{\;q\;} Q/S_A \xrightarrow{\;\mu\;} k^\times \longrightarrow 0.$$

Indeed, every $g \in Q$ is a unique product $g = z(g)\, l(g)\, n(g)$ of some $z(g)$ in the center $Z \cong k^\times$ of G, some $l(g) \in L_Q$ with $l(g) e_3 = e_3$, and $n(g) \in N_Q$ of the form

$$l(g) = \begin{pmatrix} \alpha\delta - \beta\gamma & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \in L_Q, \quad n(g) = \begin{pmatrix} 1 & -y & * & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y & 1 \end{pmatrix} \in N_Q.$$

The homomorphism q is well defined by

$$q \left[\begin{array}{cc|c} \alpha & \beta & 0 \\ \gamma & \delta & 0 \end{array}\right] = l(g) \quad \text{and} \quad q \left[\begin{array}{cc|c} 1 & 0 & b \\ 0 & 1 & y \end{array}\right] = n(g).$$

For every $I \in \mathcal{C}_G(\omega)$, the factor group Q/S_A acts naturally on the coinvariant quotient I_{S_A}. Restriction along q defines a smooth representation $\bar{I} = I_{S_A} \circ q$ of $\mathrm{Gl}_a(2)$. This defines an exact functor

$$\eta : \mathcal{C}_G(\omega) \to \mathcal{C}_2, \quad I \mapsto \bar{I}.$$

The constituents of $\bar{I}$ are described in Lemma 3.9. For more information, see [Roberts and Schmidt 2007, Section 2.5].

Twist. Recall that the twist by a smooth character μ of $k^\times$ sends the central character ω to $\omega\mu^2$. There is a natural equivalence of functors $\mathcal{C}_G(\omega) \to \mathcal{C}_2$:

$$\eta \circ \mu \cong \mu \circ \eta.$$

For a nontrivial smooth character $\psi : U \to \mathbb{C}$ of $U = N_Q \cap M$, let $J_P(I)_\psi$ be the twisted coinvariant space of the Siegel–Jacquet module of I. Fix equivalences of categories $\mathcal{C}_T \cong \mathcal{C}_{\mathrm{Gl}(1)}$ and $\mathcal{C}_{L_Q}(\omega) \cong \mathcal{C}_{\mathrm{Gl}(2)}$ by pullback along $\mathrm{Gl}(1) \to T,\ \lambda \mapsto x_\lambda$ and $q : \mathrm{Gl}(2) \to L_Q$, respectively.

Lemma 3.8. *There are natural equivalences of functors*

$$J_P(-)_\psi \cong i^* j^! \eta, \quad \mathcal{C}_G(\omega) \to \mathcal{C}_T \cong \mathcal{C}_{\mathrm{Gl}(1)},$$
$$J_Q(-) \cong i^* \eta, \quad \mathcal{C}_G(\omega) \to \mathcal{C}_{L_Q}(\omega) \cong \mathcal{C}_{\mathrm{Gl}(2)}.$$

In other words, the following diagrams commute up to natural equivalence:

$$
\begin{array}{ccc}
\mathcal{C}_G(\omega) & \xrightarrow{\ \eta\ } & \mathcal{C}_2 \\
{\scriptstyle J_P(-)_\psi}\downarrow & & \downarrow{\scriptstyle i^* j^!} \\
\mathcal{C}_T & \xrightarrow{\ \simeq\ } & \mathcal{C}_{\mathrm{Gl}(1)}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_G(\omega) & \xrightarrow{\ \eta\ } & \mathcal{C}_2 \\
{\scriptstyle J_Q(-)}\downarrow & & \downarrow{\scriptstyle i^*} \\
\mathcal{C}_{L_Q}(\omega) & \xrightarrow{\ \simeq\ } & \mathcal{C}_{\mathrm{Gl}(2)}
\end{array}
$$

Proof. This is clear by transitivity of the involved coinvariant functors. ☐

Lemma 3.9. *For $I \in \mathcal{C}_G(\omega)$ there are functorial exact sequences*:

$$0 \to j_! j^!(\bar{I}) \to \bar{I} \to i_*(J_Q(I)|_{\mathrm{Gl}(2)}) \to 0 \quad \textit{in } \mathcal{C}_2,$$
$$0 \to \mathbb{S}_1^{m_I} \to j^!(\bar{I}) \to i_*(J_P(I)_\psi) \to 0 \qquad \textit{in } \mathcal{C}_1,$$

where m_I denotes the dimension of the space of Whittaker functionals of I.

Proof. Apply the short exact sequence of Lemma 3.1 to $\bar{I}$ and to $j^!(\bar{I})$. The terms on the right-hand side are given by Lemma 3.8. Finally, note that by construction $\dim j^! j^!(\bar{I}) = m_I$. ☐

Corollary 3.10. *If $I \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ has finite length, then $\bar{I} \in \mathcal{C}_2^{\mathrm{fin}}$ has finite length.*

Proof. Lemma 3.9 gives an exact sequence in $\mathcal{C}_2$:

$$0 \to j_! i_*(J_P(I)_\psi) \to \bar{I}/\mathbb{S}_2^{m_I} \to i_*(J_Q(I)|_{\mathrm{Gl}(2)}) \to 0.$$

It is well known that the Jacquet modules have finite length. Recall that i_* and $j_!$ are exact and send irreducible modules to irreducible ones. ☐

Lemma 3.11. *For every $I \in \mathcal{C}_G(\omega)$ there is a long exact sequence in $\mathcal{C}_1$*:

$$0 \to k^\rho j_! j^!(\bar{I}) \to k^\rho(\bar{I}) \to k^\rho i_*(J_Q(I)) \to k_\rho j_! j^!(\bar{I}) \to k_\rho(\bar{I}) \to k_\rho i_*(J_Q(I)) \to 0.$$

If I has finite length, then $k^\rho i_*(J_Q(I))$ and $k_\rho i_*(J_Q(I))$ are finite-dimensional, so $\deg(k_\rho(\bar{I})) = \deg(k_\rho j_! j^!(\bar{I}))$ *and* $\deg(k^\rho(\bar{I})) = \deg(k^\rho j_! j^!(\bar{I}))$.

Proof. Lemma 3.2 applied to the first exact sequence of Lemma 3.9 shows the first statement. For the second statement note that the functors k_ρ, k^ρ send objects in $i_*(\mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(2)})$ to the full subcategory $i_*(\mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(1)})$ of $\mathcal{C}_1^{\mathrm{fin}}$ by Lemma 3.3. Counting degrees shows the third statement. ☐

Lemma 3.12. *For irreducible generic representations $\Pi \in \mathcal{C}_G(\omega)$ the module $A = j^!(\overline{\Pi})$ in $\mathcal{C}$ is perfect of degree one.*

Proof. By the uniqueness of Whittaker models, $\deg(A) = \dim j^! j^!(\overline{\Pi}) = 1$, so $\mathbb{S}$ embeds into A. Since $\mathbb{S}$ does not contain nontrivial finite dimensional submodules, the natural map $\kappa(A) \oplus \mathbb{S} \to A$ is injective and by degree reasons its cokernel

$Y = \pi_0(A)/\kappa(A)$ is finite-dimensional. From Lemma 3.2 we obtain for every ρ a long exact sequence in $\mathcal{C}_1$:

$$\cdots \to k^\rho \circ j_!(Y) \to k_\rho \circ j_!(\kappa(A) \oplus \mathbb{S}) \to k_\rho \circ j_!(A) \to k_\rho \circ j_!(Y) \to 0.$$

Let $\kappa(A) \neq 0$, then there is a smooth Gl(1)-character ρ such that $(\nu^{-1}\kappa(A))_\rho$ is nonzero. Corollary 3.5 implies that $\deg(k_\rho \circ j_!(\kappa(A)) > 0$, so the degree of $k_\rho \circ j_!(\kappa(A) \oplus \mathbb{S})$ is at least two by Corollary 3.5. Since Y is finite-dimensional, Proposition 3.6 implies that $\deg(k^\rho \circ j_!(Y)) = \deg(k_\rho \circ j_!(Y))$. By counting degrees in the long exact sequence we obtain $\deg(k_\rho \circ j_!(A)) \geq 2$. However, Lemmas 3.11 and 3.14 imply that $\deg(k_\rho \circ j_!(A)) = \deg(\widetilde{\Pi})$ and this is one by Corollary 3.20. This is a contradiction, so $\kappa(A) = 0$ vanishes and A is perfect by Lemma 2.13. ☐

3.3. ***Bessel functors β_ρ and β^ρ.*** For a smooth Gl(1)-character ρ the Bessel functor $\beta_\rho : \mathcal{C}_G(\omega) \to \mathcal{C}_{TS}$ sends a G-module I to the coinvariant quotient $\tilde{I} = \beta_\rho(I) = I_{\widetilde{R},\Lambda}$ where $\Lambda = \rho \boxtimes \rho^*$. The functor $\beta^\rho : \mathcal{C}_G(\omega) \to \mathcal{C}_{TS}$ is the functor of $(\widetilde{T}, \Lambda)$-invariants applied to the $\widetilde{N}$-coinvariants.

Lemma 3.13. *An exact sequence* $0 \to E \to F \to G \to 0$ *in* $\mathcal{C}_G(\omega)$ *gives rise to a long exact sequence in* $\mathcal{C}_{TS}$:

$$0 \to \beta^\rho(E) \to \beta^\rho(F) \to \beta^\rho(G) \to \beta_\rho(E) \to \beta_\rho(F) \to \beta_\rho(G) \to 0.$$

In other words, β^ρ *is the first left derived functor of* β_ρ.

Proof. This is implied by Lemmas A.1 and A.2 because $\widetilde{N}$ is compactly generated and $\widetilde{T}/Z_G \cong k^\times$. Alternatively, use Lemma 3.14 and exactness of η. ☐

Lemma 3.14. *There are natural equivalences* $\beta_\rho \cong k_\rho \circ \eta$ *and* $\beta^\rho \cong k^\rho \circ \eta$ *of functors* $\mathcal{C}_G(\omega) \to \mathcal{C}_1$. *In other words, there are commutative diagrams*

$$\begin{array}{ccc} \mathcal{C}_G(\omega) & \xrightarrow{\eta} & \mathcal{C}_2 \\ \downarrow{\scriptstyle \beta_\rho} & & \downarrow{\scriptstyle k_\rho} \\ \mathcal{C}_{TS} & \xrightarrow{\simeq} & \mathcal{C}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{C}_G(\omega) & \xrightarrow{\eta} & \mathcal{C}_2 \\ \downarrow{\scriptstyle \beta^\rho} & & \downarrow{\scriptstyle k^\rho} \\ \mathcal{C}_{TS} & \xrightarrow{\simeq} & \mathcal{C}_1 \end{array}$$

with k_ρ *and* k^ρ *defined in Section 3.1 .*

Proof. Transitivity of coinvariant functors gives an isomorphism $I_{\widetilde{N}} \cong \bar{I}_{L_Q \cap N}$ of smooth $T\widetilde{T}S$-modules, functorial in $I \in \mathcal{C}_G(\omega)$. The rest is straightforward. ☐

Corollary 3.15. *The Bessel functors* β_ρ *and* β^ρ *send* $\mathcal{C}_G^{\mathrm{fin}}(\omega)$ *to* $\mathcal{C}_{TS}^{\mathrm{fin}}$.

Proof. The corresponding assertions hold for η by Corollary 3.10 and for k_ρ, k^ρ by Corollary 3.5. ☐

Remark 3.16. Recall that the twist $\mathcal{C}_G(\omega) \to \mathcal{C}_G(\mu^2\omega)$ by a smooth character μ of Gl(1) satisfies natural equivalences $\beta_{\mu\rho} \circ \mu \cong \mu \circ \beta_\rho$ and $\beta^{\rho\mu} \circ \mu \cong \mu \circ \beta^\rho$. Recall that $\rho \mapsto \rho^* = \omega/\rho$ defines an involution on characters of Gl(1).

Lemma 3.17 (duality). *There are natural equivalences $\beta_\rho \cong \beta_{\rho^*}$ and $\beta^\rho \cong \beta^{\rho^*}$. For every $I \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ there is a commutative diagram of TS-modules*:

$$\begin{array}{ccc} I & \longrightarrow & \beta_\rho(I) \\ {\scriptstyle I(s_1)}\downarrow & & \downarrow\cong \\ I & \longrightarrow & \beta_{\rho^*}(I) \end{array}$$

Proof. The Weyl group element $s_1 \in G$ normalizes $\widetilde{T}$ and $\widetilde{N}$ and centralizes TS. Conjugation with s_1 permutes t_1 and t_2, thus sends $\Lambda = \rho \boxtimes \rho^*$ to $\Lambda^{s_1} = \rho^* \boxtimes \rho$. □

Proposition 3.18. *For $I \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ let*

$$d(I, \rho) = \dim\big(\mathrm{Hom}_T(\delta_P^{-1/2} J_P(I)_\psi, \nu^{-1/2}\rho)\big).$$

Then the degree of $\tilde{I} = \beta_\rho(I)$ satisfies the estimate

$$d(I, \rho) \le \deg(\tilde{I}) \le m_I + d(I, \rho),$$

where m_I denotes the multiplicity of Whittaker models of I. If $J_P(I)_\psi = 0$, then $\deg(\tilde{I}) = m_I$. *If $J_P(I) = 0$ vanishes, then $\tilde{I}$ is the direct sum of m_I copies of $\mathbb{S}$.*

Proof. The T-module $X = J_P(I)_\psi$ is isomorphic to $i^* j^!(\bar{I})$ by Lemma 3.8. The second exact sequence of Lemma 3.9, right-exactness of the functor $k_\rho \circ j_!$ and the assertion $k_\rho \circ j_!(\mathbb{S}) = \mathbb{S}$ by Corollary 3.5 yield a long exact sequence

$$\cdots \to k^\rho j_! i_*(X) \to \mathbb{S}^{m_I} \to k_\rho j_! j^!(\bar{I}) \to k_\rho j_! i_*(X) \to 0.$$

Corollary 3.5 and Lemma 3.8 imply

$$\deg(k_\rho j_! i_*(X)) = \deg(j_! k_{\nu\rho} i_*(X)) = \dim(\mathrm{Hom}_T(\nu^{-1}X, \rho)) = d(I, \rho).$$

Counting degrees in the long exact sequence yields the estimate

$$d(I, \rho) \le \deg(k_\rho j_! j^!(\bar{I})) \le m_I + d(I, \rho).$$

Now Lemmas 3.11 and 3.14 imply the first statement because

$$\deg(k_\rho j_! j^!(\bar{I})) = \deg(k_\rho(\bar{I})) = \deg(\tilde{I}).$$

If $X = 0$ vanishes, then by the long exact sequence above, $\mathbb{S}^{m_I} \cong k_\rho j_! j^!(\bar{I})$ which has degree $m_I = \deg(\tilde{I})$. Finally, if $J_P(I) = 0$ vanishes then so does the Borel–Jacquet module $J_P(I)_U$. Lemma 3.3 shows that $k_\rho\big(i_*(J_Q(I))\big) = 0$ and $k^\rho\big(i_*(J_Q(I))\big) = 0$ also vanish. Lemmas 3.11 and 3.14 imply that $k_\rho(\bar{I}) \cong k_\rho j_! j^!(\bar{I})$. Clearly $X = 0$ vanishes, so $k_\rho j_! j^!(\bar{I}) \cong \mathbb{S}^{m_I}$. □

Proposition 3.19. *For every finitely generated $I \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$,*

$$[\beta_\rho(I)] - [\beta^\rho(I)] = [j^!(\bar{I})]$$

holds in the Grothendieck group $K_0(\mathcal{C})$. Especially, the difference of degrees $\deg(\beta_\rho(I)) - \deg(\beta^\rho(I)) = m_I$ *is the multiplicity of Whittaker models of I.*

Proof. The first statement follows from Proposition 3.6 and Lemma 3.14. For $\bar{I} = \eta(I) \in \mathcal{C}_2^{\mathrm{fin}}$, the degree of $\beta^\rho(I)$ is $\dim j^! k^\rho(\bar{I})$ and the degree of $\beta_\rho(I)$ is $\dim j^! k_\rho(\bar{I})$. By definition, $m_I = \dim(j^!)^2(\bar{I}) = \deg(j^!(\bar{I}))$. □

Corollary 3.20. *For irreducible generic $\Pi \in \mathcal{C}_G(\omega)$ and every smooth ρ, the Bessel modules have degree* $\deg(\beta_\rho(\Pi)) = 1$ *and* $\deg(\beta^\rho(\Pi)) = 0$.

Proof. For $\widetilde{\Pi} = \beta_\rho(\Pi)$, Proposition 3.18 implies that $\deg(\widetilde{\Pi}) \le m_\Pi + d(\Pi, \rho)$ where $m_\Pi = 1$ is the multiplicity of Whittaker models. Note that $d(\Pi, \rho)$ can be nonzero only if ρ is in the multiset $\Delta_+(\Pi)$ defined in Section A.3. Replacing ρ by ρ^* does not change $\beta_\rho(\Pi)$ up to isomorphism (Lemma 3.17), so without loss of generality we can assume that $\rho \notin \Delta_+(\Pi)$ because $\Delta_+(\Pi) \cap \Delta_+^*(\Pi)$ is empty by Lemma A.9. This implies $d(\Pi, \rho) = 0$ and therefore $\deg(\widetilde{\Pi}) \le m_\Pi = 1$. Proposition 3.19 implies the assertion. □

Lemma 3.21. *For $I \in \mathcal{C}_G(\omega)$ with Siegel–Jacquet module $J_P(I) = I_N$ and for every Bessel character ρ, there is a functorial isomorphism of T-modules*:

$$i^*(\beta_\rho(I)) \cong J_P(I)_{\widetilde{T},\rho}.$$

If I has finite length, the T-modules $i^*(\beta^\rho(I))$ and $J_P(I)^{\widetilde{T},\rho}$ have the same Jordan–Hölder constituents.

Proof. The first assertion is a consequence of the transitivity property for coinvariant functors. For the second assertion, consider $J_P(I)$ as a $\mathrm{Gl}_a(1) \times T$-module where $[\alpha \,|\, v] \in \mathrm{Gl}_a(1)$ acts by m_A for $A = \left(\begin{smallmatrix} \alpha & v \\ 0 & 1 \end{smallmatrix}\right)$ and T by its natural embedding $T \hookrightarrow M$. Proposition 3.6 implies in the Grothendieck group of T-modules:

$$[J_P(I)_{\widetilde{T},\rho}] - [J_P(I)^{\widetilde{T},\rho}] = [j^!(J_P(I))] = [J_P(I)_\psi] \quad \text{in } K_0(\mathcal{C}_T^{\mathrm{fin}}).$$

There is an isomorphism of T-modules $J_P(I)_\psi \cong i^* j^!(\bar{I})$ by Lemma 3.8. The exact functor i^* applied to the equation of Proposition 3.19 yields

$$[i^*(\beta^\rho(I))] = [i^*(\tilde{I})] - [i^* j^!(\bar{I})]$$

and this shows the statement. □

Proposition 3.22. *For $I \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ one has $L(\beta_\rho(I), s) = L(J_P(I)_{\widetilde{T},\rho}, s)$ and*

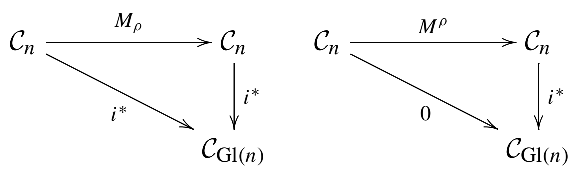
$$L(i_*(J_P(I)_\psi), s) = \frac{L(\beta_\rho(I), s)}{L(\beta^\rho(I), s)}.$$

The finite dimensional T-module $J_P(I)_\psi$ is viewed as a TS-module with the trivial S-action; *it is independent of ρ.*

Proof. This is a restatement of Lemma 3.21 and Proposition 3.19 in terms of Section 2.5. □

3.4. ***Mellin functors M_ρ and M^ρ.*** The Mellin functors attached to a smooth Gl(1)-character ρ are the additive endofunctors $M_\rho = k_\rho \circ j_!$ and $M^\rho = k^\rho \circ j_!$ of $\mathcal{C}_n$. Corollary 3.5 implies $M_\rho(\mathbb{S}_n) \cong \mathbb{S}_n$ and $M^\rho(\mathbb{S}_n) = 0$ for every ρ. In the Grothendieck group of $\mathcal{C}_n^{\mathrm{fin}}$, Proposition 3.6 implies $[X] = [M_\rho(X)] - [M^\rho(X)]$ for every X in $\mathcal{C}_n^{\mathrm{fin}}$. If X is perfect, $M^\rho(X) = 0$ vanishes for every ρ by Lemma 3.4.

Lemma 3.23. *There are natural equivalences $i^* \circ M_\rho \cong i^*$ and $i^* \circ M^\rho \cong 0$ of functors $\mathcal{C}_n \to \mathcal{C}_{\mathrm{Gl}(n)}$. In other words*, *the diagrams*



commute up to natural equivalence.

Proof. Apply the exact functor i^* to the short exact sequence and the natural isomorphism of Lemma 3.4. Note that $i^* j_! = 0$ and $i^* i_* = \mathrm{id}$. □

Theorem 3.24. *For smooth* Gl(1)*-characters μ and ρ there is an isomorphism of* $\mathrm{Gl}_a(1)$*-modules*

$$M_\rho(\mathbb{E}[\mu]) \cong \begin{cases} \mathbb{E}[\mu], & \mu \neq \nu^2\rho, \\ \mathbb{S} \oplus i_*(\mu), & \mu = \nu^2\rho. \end{cases}$$

Proof. Fix the subgroups

$$T = \left\{ \left[\begin{array}{cc|c} * & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \in \mathrm{Gl}_a(2) \right\}, \quad \widetilde{T} = \left\{ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & * & 0 \end{array}\right] \in \mathrm{Gl}_a(2) \right\},$$

$$S = \left\{ \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & 0 \end{array}\right] \in \mathrm{Gl}_a(2) \right\}, \quad U = \left\{ \left[\begin{array}{cc|c} 1 & * & 0 \\ 0 & 1 & 0 \end{array}\right] \in \mathrm{Gl}_a(2) \right\}$$

and their product

$$\Gamma = \left\{ \left[\begin{array}{cc|c} * & * & * \\ 0 & * & 0 \end{array}\right] \in \mathrm{Gl}_a(2) \right\}.$$

There is an exact sequence $0 \to U \to \Gamma \to TS \times \widetilde{T} \to 0$. By [Rösner and Weissauer 2020, Lemma 5.7] there is an isomorphism of $\mathrm{Gl}_a(2)$-modules

$$I := \mathrm{ind}_\Gamma^{\mathrm{Gl}_a(2)}(\sigma) \cong j_!(\mathbb{E}[\mu]),$$

where $\sigma \in \mathcal{C}_\Gamma$ is the $TS \times \widetilde{T}$-module $\mathbb{S}_1 \boxtimes \mu\nu^{-2}$. It only remains to determine $k_\rho(I)$. By Bruhat decomposition, the double coset space $\Gamma \setminus \mathrm{Gl}_a(2)/\Gamma$ is generated by id, the unipotent element $\tau = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and the Weyl group element $w = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. The orbit filtration of Bernstein and Zelevinskii [1977, Section 5.2] yields Γ-modules $0 \subseteq F_2 \subseteq F_1 \subseteq F_0 = I$ with quotients:

$I_{\mathrm{id}} := F_0/F_1 \cong \mathrm{ind}_\Gamma^\Gamma(\sigma) \cong \sigma$.

$I_\tau := F_1/F_2 \cong \mathrm{ind}_{\Gamma\cap\tau\Gamma\tau^{-1}}^\Gamma(\sigma^\tau)$, where $\Gamma \cap \tau\Gamma\tau^{-1} = T \ltimes SU$ and

$$\sigma^\tau\left(\left[\begin{matrix} a & b \\ 0 & 1 \end{matrix}\middle|\begin{matrix} s \\ 0 \end{matrix}\right]\right) = \sigma\left(\tau^{-1}\left[\begin{matrix} a & b \\ 0 & 1 \end{matrix}\middle|\begin{matrix} s \\ 0 \end{matrix}\right]\tau\right) = \sigma\left(\left[\begin{matrix} a & b \\ 0 & 1 \end{matrix}\middle|\begin{matrix} s+b \\ 0 \end{matrix}\right]\right).$$

$I_w := F_2 \cong \mathrm{ind}_{\Gamma\cap w\Gamma w^{-1}}^\Gamma(\sigma^w)$, with $\Gamma \cap w\Gamma w^{-1} = T \times \widetilde{T}$ and

$$\sigma^w\left(\left[\begin{matrix} a & 0 \\ 0 & d \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]\right) = \sigma\left(\left[\begin{matrix} d & 0 \\ 0 & a \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]\right).$$

Here the unipotent subgroups in [Bernstein and Zelevinskii 1977, Section 5.2] are considered to be trivial. The functor of coinvariants under U is exact and sends Γ-modules to $TS \times \widetilde{T}$-modules. Clearly, $(I_{\mathrm{id}})_U \cong \mathbb{S}_1 \boxtimes \mu\nu^{-2}$ as a representation of $TS \times \widetilde{T}$. Since U is normal in $\Gamma \cap \tau\Gamma\tau^{-1}$, there is an isomorphism

$$(I_\tau)_U \cong \mathrm{ind}_{T\ltimes SU}^\Gamma(\sigma^\tau)_U.$$

The integrand $(\sigma^\tau)_U = 0$ vanishes because $i^*(\mathbb{S}_1) = 0$, so $(I_\tau)_U = 0$. Finally, from [Bernstein and Zelevinskii 1977, Sectin 5.2] we get an isomorphism of $TS \times \widetilde{T}$-modules:

$$(I_w)_U \cong \mathrm{ind}_{T\times\widetilde{T}}^{TS\times\widetilde{T}}(\nu^{-1}\mu \boxtimes \mathcal{S}) \cong \mathbb{E}[\mu] \boxtimes \mathcal{S},$$

with $\mathcal{S} = \mathbb{S}|_{\mathrm{Gl}(1)}$ and the isomorphism $\mathrm{ind}_{\mathrm{Gl}(1)}^{\mathrm{Gl}_a(1)}(\nu^{-1}\mu) \cong \mathbb{E}[\mu]$ in Example 2.4. Summarily, there is an exact sequence of $TS \times \widetilde{T}$-modules:

$$0 \to \mathbb{E}[\mu] \boxtimes \mathcal{S} \to I_U \to \mathbb{S} \boxtimes \mu\nu^{-2} \to 0.$$

By Lemma A.2 this yields a long exact sequence of TS-modules:

$$0 \to \mathbb{S} \otimes (\mu\nu^{-2})^\rho \to \mathbb{E}[\mu] \otimes \mathcal{S}_\rho \to k_\rho(I) \to \mathbb{S} \otimes (\mu\nu^{-2})_\rho \to 0,$$

where $k^\rho(I) \cong M^\rho(\mathbb{E}[\mu]) = 0$ vanishes because $\mathbb{E}[\mu]$ is perfect. Since $\mathcal{S}_\rho \cong \mathbb{C}$, the statement follows from Lemma 2.10. □

Lemma 3.25. *For cyclic finite-dimensional smooth* $\mathrm{Gl}(1)$*-modules* $X \in \mathcal{C}_{\mathrm{Gl}(1)}^{\mathrm{fin}}$, *there are isomorphisms*

$$M_\rho i_*(X) \cong \begin{cases} \mathbb{E}[X \to \nu\rho], & \nu\rho \in X^{ss}, \\ i_*(X), & \nu\rho \notin X^{ss}, \end{cases} \qquad M^\rho i_*(X) \cong \begin{cases} \mathbb{S}, & \nu\rho \in X^{ss}, \\ 0, & \nu\rho \notin X^{ss}, \end{cases}$$

where X^{ss} *is the multiset of irreducible constituents of* X.

Proof. Since the Mellin functors are additive, it suffices to consider the case where $X = \mu^{(m)}$ is an indecomposable Jordan block of length $m \geq 1$ attached to a smooth character μ. For $\mu \neq \nu\rho$ the statement follows directly from Corollary 3.5, so we can assume $\mu = \nu\rho$. Corollary 3.5 shows that $M^{\rho} i_*(X) \cong \mathbb{S}$ and that there is an exact sequence $0 \to \mathbb{S} \to M_{\rho} i_*(X) \to i_*(X) \to 0$. We distinguish the cases $m = 1$ and $m > 1$.

For $m = 1$, the right-exact functor M_{ρ} applied to $0 \to \mathbb{S} \to \mathbb{E}[\mu] \to i_*(\mu) \to 0$ yields a long exact sequence in $\mathcal{C}_1$:

$$\cdots \to M_{\rho}(\mathbb{S}) \to M_{\rho}(\mathbb{E}[\mu]) \to M_{\rho} i_*(\mu) \to 0.$$

Theorem 3.24 implies that $M_{\rho}(\mathbb{E}[\mu]) \cong \mathbb{E}[\mu]$. Since $\mathbb{E}[\mu]$ and $M_{\rho} i_*(\mu)$ both have length two, the surjection $M_{\rho}(\mathbb{E}[\mu]) \to M_{\rho} i_*(\mu)$ is an isomorphism.

For $m > 1$ there is a Jordan block $Y \cong \mu^{(m-1)}$ with a short exact sequence

$$0 \to Y \to X \to \mu \to 0$$

and thus by Lemma 3.2 a long exact sequence in $\mathcal{C}_1$:

$$0 \longrightarrow M^{\rho} i_*(Y) \longrightarrow M^{\rho} i_*(X) \longrightarrow M^{\rho} i_*(\mu) \xrightarrow{\ \delta\ } M_{\rho} i_*(Y) \longrightarrow M_{\rho} i_*(X) \longrightarrow M_{\rho} i_*(\mu) \longrightarrow 0$$

By Corollary 3.5 every term in the long exact sequence has degree one. Since $M^{\rho} i_*(\mu) \cong \mathbb{S}$, counting degrees shows $\operatorname{coker}(\delta) \cong M_{\rho} i_*(Y)/\mathbb{S}$. Lemma 3.23 implies $M_{\rho} i_*(Y)/\mathbb{S} \cong i_*(Y)$. We obtain a short exact sequence in $\mathcal{C}_1^{\mathrm{fin}}$:

$$0 \longrightarrow i_*(Y) \longrightarrow M_{\rho} i_*(X) \longrightarrow M_{\rho} i_*(\mu) \longrightarrow 0.$$

Since $M_{\rho} i_*(\mu) \cong \mathbb{E}[\mu]$ is perfect, $i_*(Y)$ is the maximal finite-dimensional submodule of $M_{\rho} i_*(X)$. Lemma 3.23 yields $\pi_0 M_{\rho} i_*(X) \cong i_*(X)$, so Lemma 2.22 implies that $M_{\rho} i_*(X) \cong \mathbb{E}[X \to \mu]$. □

Lemma 3.26. *Let $\mu^{(2)} \in \mathcal{C}_{\mathrm{Gl}(1)}$ be an indecomposable Jordan block of length two for a smooth character μ of* Gl(1). *Then for every smooth character ρ of* Gl(1) *and $\rho^{*} = \mu^2 \nu^{-3} \rho^{-1}$ there is an isomorphism of* $\mathrm{Gl}_a(1)$*-modules*:

$$M_{\rho}(\mathbb{E}[\mu^{(2)}]) \cong M_{\rho^{*}}(\mathbb{E}[\mu^{(2)}]).$$

Proof. Fix a cuspidal irreducible $\pi \in \mathcal{C}_{\mathrm{Gl}(2)}(1)$ with trivial central character. The fully Siegel induced representation $\Pi = \pi \rtimes \mu\nu^{-3/2} \in \mathcal{C}_G$ is generic irreducible of type X with central character $\mu^2\nu^{-3}$. Note that $j^!(\overline{\Pi})$ is perfect of degree one by Lemma 3.12 and thus isomorphic to $\mathbb{E}[\mu^{(2)}]$ by Lemma 3.8. The Klingen–Jacquet module $J_Q(\Pi)$ vanishes for Π of type X, so Lemma 3.11 yields an isomorphism of $\mathrm{Gl}_a(2)$-modules $M_\rho(\mathbb{E}[\mu^{(2)}]) \cong \beta_\rho(\Pi)$ for every ρ. Lemma 3.17 implies the assertion. □

Theorem 3.27. *For every cyclic smooth* $\mathrm{Gl}(1)$*-module* $X \in \mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(1)}$ *of finite dimension and for every smooth character* ρ *of* $\mathrm{Gl}(1)$, *there are isomorphisms* $M^\rho(\mathbb{E}[X]) \cong 0$ *and*

$$M_\rho(\mathbb{E}[X]) \cong \begin{cases} \mathbb{E}[X], & \nu^2\rho \notin X^{ss},\ \nu\rho \notin X^{ss}, \\ \mathbb{E}[X \to \nu\rho], & \nu\rho \in X^{ss}, \\ \mathbb{E}[X \to X/\nu^2\rho], & \nu^2\rho \in X^{ss},\ \nu\rho \notin X^{ss}, \end{cases}$$

where X^{ss} *denotes the finite multiset of constituents of* X.

Proof. $M^\rho(\mathbb{E}[X])$ vanishes by Lemma 3.4. Proposition 3.6 implies that $M_\rho(\mathbb{E}[X])$ has the same constituents as $\mathbb{E}[X]$. Further, $i^*M_\rho(\mathbb{E}[X])$ is isomorphic to X by Lemma 3.23. By Lemma 2.22 it only remains to determine the maximal finite-dimensional submodule $\kappa(M_\rho(\mathbb{E}[X]))$ of $M_\rho(\mathbb{E}[X])$. Recall that for every submodule $Y \subseteq X$ there is an exact sequence of $\mathrm{Gl}_a(1)$-modules:

$$0 \to \mathbb{E}[Y] \to \mathbb{E}[X] \to i_*(X/Y) \to 0.$$

By Lemma 3.2, the Mellin functors yield a long exact sequence

$$(*) \qquad 0 \to M^\rho i_*(X/Y) \xrightarrow{\delta} M_\rho(\mathbb{E}[Y]) \to M_\rho(\mathbb{E}[X]) \to M_\rho i_*(X/Y) \to 0.$$

We distinguish four cases:

(1) If neither $\nu\rho$ nor $\nu^2\rho$ occur in X, then for every one-dimensional submodule $Y \subseteq X$, Theorems 3.24 and 3.25 applied to $(*)$ yield that $\mathbb{E}[Y] \hookrightarrow M_\rho(\mathbb{E}[X])$. By Lemma 2.15, $M_\rho(\mathbb{E}[X])$ is perfect.

(2) If $\nu\rho$ occurs in X, then by Lemma 3.25 the sequence $(*)$ with $Y = \ker(X \to \nu\rho)$ becomes

$$0 \to \mathbb{S} \xrightarrow{\delta} M_\rho(\mathbb{E}[Y]) \to M_\rho(\mathbb{E}[X]) \to \mathbb{E}[\mu] \to 0.$$

Lemma 3.23 implies that $\mathrm{coker}(\delta) \cong i_*(Y)$. This gives a short exact sequence $0 \to i_*(Y) \to M_\rho(\mathbb{E}[X]) \to \mathbb{E}[\mu] \to 0$, and hence $\kappa(M_\rho(\mathbb{E}[X])) \cong i_*(Y)$.

(3) Assume that $\nu^2\rho$ occurs exactly once in X but $\nu\rho$ does not occur in X. By Theorems 3.24 and 3.25, the sequence $(*)$ for $Y = \nu^2\rho$ becomes

$$0 \to \mathbb{S} \oplus i_*(\nu^2\rho) \to M_\rho(\mathbb{E}[X]) \to i_*(X/Y) \to 0.$$

That means $M_\rho(\mathbb{E}[X])$ admits a unique one-dimensional submodule isomorphic to $i_*(\nu^2\rho)$. We claim that $M_\rho(\mathbb{E}[X])/i_*(\nu^2\rho)$ is perfect. Indeed, for every constituent $\mu \neq \nu^2\rho$ of X, the sequence $(*)$ for $Y = \mu$ yields an embedding $\mathbb{E}[\mu] \hookrightarrow M_\rho(\mathbb{E}[X])$ and thus an embedding $\mathbb{E}[\mu] \hookrightarrow M_\rho(\mathbb{E}[X])/i_*(\nu^2\rho)$. This implies the claim by Lemma 2.15, so $\kappa(M_\rho(\mathbb{E}[X])) \cong i_*(\nu^2\rho)$.

(4) Suppose that $\nu^2\rho$ occurs in X more than once, but $\nu\rho$ does not occur in X. Lemma 3.26 and the argument for the second case yield isomorphisms

$$M_\rho(\mathbb{E}[(\nu^2\rho)^{(2)}]) \cong M_{\nu\rho}(\mathbb{E}[(\nu^2\rho)^{(2)}]) \cong \mathbb{E}[(\nu^2\rho)^{(2)} \to \nu^2\rho].$$

Then the exact sequence $(*)$ with $Y = (\nu^2\rho)^{(2)}$ yields an embedding

$$\mathbb{E}[(\nu^2\rho)^{(2)} \to \nu^2\rho] \hookrightarrow M_\rho(\mathbb{E}[X]).$$

Dividing out $i_*(\nu^2\rho)$ yields an embedding $\mathbb{E}[\nu^2\rho] \hookrightarrow M_\rho(\mathbb{E}[X])/i_*(\nu^2\rho)$. For every irreducible constituent $\mu \neq \nu^2\rho$ of X, an embedding $\mathbb{E}[\mu] \hookrightarrow M_\rho(\mathbb{E}[X])/i_*(\nu^2\rho)$ is constructed as in the previous case. By Lemma 2.15 $M_\rho(\mathbb{E}[X])/i_*(\nu^2\rho)$ is perfect. Finally, Lemma 2.22 implies the assertion. □

4. Siegel induced representations

We want to determine the TS-module $\pi_0(\widetilde{\Pi})$ for every irreducible $\Pi \in \mathcal{C}_G(\omega)$. By Lemma 3.21 it suffices to consider those Π whose Siegel–Jacquet module $J_P(\Pi) = \Pi_N$ does not vanish. By dual Frobenius reciprocity, Π is then isomorphic to a quotient of a Siegel induced representation $\mathrm{Ind}_P^G(\sigma_\Pi)$ from an irreducible $\sigma_\Pi \in \mathcal{C}_M$. This σ_Π is not uniquely determined by Π, we may choose it as in Table 1 and give the full list of possible σ_Π in Table 2. For simplicity, we fix a representative of each equivalence class under twisting.

For the standard Levi subgroup of the Siegel parabolic we use the decomposition $\mathrm{Gl}(2) \times \mathrm{Gl}(1) \cong M$ via $(A, \lambda) \mapsto m_A t_\lambda$ with notation as in Section 1.1. Every irreducible $\sigma_\Pi \in \mathcal{C}_M$ is of the form

$$\sigma_\Pi(m_A t_\lambda) = \pi(A)\chi_\Pi(\lambda)$$

for irreducible smooth representations $\pi = (\pi, V)$ of $\mathrm{Gl}(2)$ and χ_Π of $\mathrm{Gl}(1)$. The twist of $\sigma_\Pi = \pi \boxtimes \chi_\Pi$ by a $\mathrm{Gl}(1)$-character μ is $\mu \otimes \sigma_\Pi = \pi \boxtimes \chi_\Pi\mu$ and the contragredient is $\sigma_\Pi^\vee \cong \pi^\vee \boxtimes \chi_\Pi^{-1}$. Recall that the modulus factor of the Siegel parabolic P is given on M by $\delta_P(m_A t_\lambda) = |\det(A)/\lambda|^3$ with $\delta_P(x_\lambda) = |\lambda|^3$. The normalized Siegel induced representation

$$\pi \rtimes \chi_\Pi = \mathrm{Ind}_P^G(\sigma_\Pi)$$

is the right-regular action of G on the space of smooth functions $f : G \to V$ with compact support modulo P such that $f(msg) = \delta_P^{1/2}\sigma_\Pi(m) f(g)$ for $s \in N$, $m \in M$ and $g \in G$. The central character of $\pi \rtimes \chi_\Pi$ is $\omega = \omega_\pi \chi_\Pi^2$, where ω_π is the central character of π. There are natural isomorphisms

$$\mu \otimes \mathrm{Ind}_P^G(\sigma_\Pi) \cong \mathrm{Ind}_P^G(\mu \otimes \sigma_\Pi) \quad \text{and} \quad \mathrm{Ind}_P^G(\sigma_\Pi)^\vee \cong \mathrm{Ind}_P^G(\sigma_\Pi^\vee).$$

For details, see [Tadić 1994].

Theorem 4.1. *For every irreducible* $\Pi \in \mathcal{C}_G(\omega)$ *and* $\sigma_\Pi \in \mathcal{C}_M(\omega)$, *the following assertions are equivalent*:

(1) Π *is a quotient of* $\mathrm{Ind}_P^G(\sigma_\Pi)$.

(2) Π *is a submodule of* $\mathrm{Ind}_P^G(\omega \otimes \sigma_\Pi^\vee)$.

(3) $\omega \otimes \sigma_\Pi^\vee$ *is a quotient of* $\mathcal{J} = \delta_P^{-1/2} \otimes J_P(\Pi)$ *for the unnormalized Siegel–Jacquet module* $J_P(\Pi) = \Pi_N$.

For Π *normalized as in Table 1*, σ_Π *satisfies these assertions if and only if it appears in Table 2.*

Proof. The involution $I \mapsto \omega \otimes I^\vee$ of $\mathcal{C}_G^{\mathrm{fin}}(\omega)$ is a contravariant exact functor that preserves the irreducible constituents, so the first two assertions are equivalent. By dual Frobenius reciprocity, the second and third assertion are equivalent.

Since T is in the center of M, the decomposition $\mathcal{J} = \bigoplus_{\chi_{\mathrm{norm}}} \mathcal{J}^{(T,\chi_{\mathrm{norm}})}$ into generalized T-eigenspaces is preserved under the action of M. It is easy to see that for Π not of type VIa or VId, every constituent of $\mathcal{J}$ occurs as a quotient, see [Roberts and Schmidt 2007, Table A.3]. For type VIa, normalized as in Table 1, the constituents of $\mathcal{J}$ are $\mathrm{Sp}(\nu^{1/2}) \boxtimes \nu^{-1/2}$ (twice) and $(\nu^{1/2} \circ \det) \boxtimes \nu^{-1/2}$. However, the one-dimensional constituent of $\mathcal{J}$ is not a quotient because Π is not a constituent of $\mathrm{Ind}_P^G((\nu^{1/2} \circ \det) \boxtimes \nu^{-1/2})$, see [Roberts and Schmidt 2007, (2.11)]. For type VId the argument is analogous. □

Corollary 4.2. *For irreducible* $\Pi \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ *and for every* $\chi_{\mathrm{norm}} \in \Delta(\Pi)$, *there is an irreducible* $\sigma_\Pi = \pi \boxtimes \chi_\Pi \in \mathcal{C}_M^{\mathrm{fin}}(\omega)$ *such that* Π *is a quotient of* $\mathrm{Ind}_P^G(\sigma_\Pi)$ *and* $\chi_\Pi(t_\lambda) = \chi_{\mathrm{norm}}(x_\lambda)$.

Definition 4.3. For an admissible representation $\sigma_\Pi = \pi \boxtimes \chi_\Pi \in \mathcal{C}_M(\omega)$ and a smooth character ρ of $\mathrm{Gl}(1)$, the pair (σ_Π, ρ) will be called *ordinary* if either $\dim(\pi) = 1$ holds or if ρ is different from the two characters

$$\rho_+ = \nu^{-1/2}\chi_\Pi, \quad \rho_- = \rho_+^* = \omega/\rho_+.$$

If (σ_Π, ρ) is not ordinary and $\rho_+ \neq \rho_-$, it will be called *nonordinary*. The case where $\rho = \rho_+ = \rho_-$ is called *extraordinary*.

Notice that $\rho_+ = \rho_-$ if and only if $\nu^{-1}\chi_\Pi = \omega\chi_\Pi^{-1}$ if and only if $\omega_\pi = \nu^{-1}$. The case $\sigma_\Pi = (\chi_1 \times \nu) \boxtimes \nu^{-1/2}$ and $\rho = \chi_1 = \nu^{-1}$ is nonordinary by definition but behaves like an ordinary case. To be consistent we always assume for the family IIIa, IIIb that the character χ_1 satisfies $|\chi_1| = \nu^s$ with $s \geq 0$. This can be achieved by the Weyl reflection that switches χ_1 and χ_1^{-1}.

Lemma 4.4. *If* (σ_Π, ρ) *is nonordinary or extraordinary and* $\mathrm{Ind}_P^G(\sigma_\Pi)$ *is not irreducible, then* $(\omega \otimes \sigma_\Pi^\vee, \rho)$ *is ordinary.*

Proof. Note that $\omega \otimes \sigma_\Pi^\vee \cong \pi^\vee \boxtimes \chi_\Pi^*$. If $(\omega \otimes \sigma_\Pi^\vee, \rho)$ is not ordinary, then either $\rho_+(\chi_\Pi) = \nu^{-1/2}\chi_\Pi^*$ or $\rho_-(\chi_\Pi) = \nu^{-1/2}\chi_\Pi^*$. This amounts to $\chi_\Pi = \chi_\Pi^*$ and $\nu^{1/2}\chi_\Pi^* = \nu^{-1/2}\chi_\Pi^*$, respectively. The latter is clearly impossible, so $\chi_\Pi = \chi_\Pi^*$ implies $\chi_\Pi^2 = \omega$, and hence $\omega_\pi = 1$. This only occurs for the case $\pi = \nu \times \nu^{-1}$ that has already been excluded. □

4.1. ***Bessel filtration.*** Fix an admissible $(\pi, V) \in \mathcal{C}^{\mathrm{fin}}_{\mathrm{Gl}(2)}(\omega_\pi)$ of finite length with central character ω_π and a Gl(1)-character χ_Π. The Siegel induced representation $I = \mathrm{Ind}_P^G(\sigma_\Pi)$ for $\sigma_\Pi = \pi \boxtimes \chi_\Pi \in \mathcal{C}_M(\omega)$ carries a filtration of P-modules corresponding to Schubert cells on $P \setminus G$ defined by double cosets in $P \setminus G/P$, see [Roberts and Schmidt 2016, Section 5.2]. As modules of $RT \subseteq P$, this induces a finer filtration

$$0 \subseteq I_{\geq 3} \subseteq I_{\geq 2} \subseteq I_{\geq 1} \subseteq I_{\geq 0} = I,$$

whose associated graded components $I_i = I_{\geq i}/I_{\geq i+1}$ are described in Section A.2. Since the Bessel group $\widetilde{R}$ is a normal subgroup in RT, this allows to analyze the structure of $\beta_\rho(I) = \tilde{I}$ in terms of this filtration. We recall some notation and simple facts: identify TS with the group $\mathrm{Gl}_a(1)$ as in Example 2.5. The normalization factor for the Borel group B of upper triangular matrices in Gl(2) with unipotent radical U is $\delta_B^{1/2}(x_\lambda) = |\lambda|^{1/2}$. If π is irreducible, then the semisimplified Jacquet quotient as a T-module is

$$\delta_B^{-1/2}(\chi_\Pi \otimes \pi)_U^{ss} \cong \begin{cases} \chi_1\chi_\Pi \oplus \chi_2\chi_\Pi, & \pi = \chi_1 \times \chi_2, \\ \nu^{1/2}\chi\chi_\Pi, & \pi = \mathrm{Sp}(\chi), \\ \nu^{-1/2}\chi\chi_\Pi, & \pi = (\chi \circ \det), \\ 0, & \pi \text{ cuspidal.} \end{cases}$$

Fix smooth characters $\rho_+ = \nu^{-1/2}\chi_\Pi$ and $\rho_- = \rho_+^* = \omega\rho_+^{-1}$ of $F^\times$ as above. For the following lemma fix the filtration index $m = m(\sigma_\Pi, \Lambda) = \delta_{\rho\rho_+} + \delta_{\rho\rho_-}$ (Kronecker delta). It is equal to zero for $\rho \neq \rho_\pm$, to two for $\rho = \rho_+ = \rho_-$, and to one otherwise.

Lemma 4.5. *Assume that σ_Π is irreducible. For $I = \mathrm{Ind}_P^G(\sigma_\Pi)$ with filtration as above, the TS-modules $\delta_P^{-1/2} \otimes \tilde{I}_i$ are*

$$\delta_P^{-1/2} \otimes \tilde{I}_3 = \begin{cases} 0 & \textit{if } \pi = \mu \circ \det \textit{ and } \rho \neq \mu\chi_\Pi, \\ \chi_\Pi \otimes C_c^\infty(S) & \textit{otherwise,} \end{cases}$$

$$\delta_P^{-1/2} \otimes \tilde{I}_2 = \chi_\Pi \delta_B^{-1/2} \otimes \pi_U,$$

$$\delta_P^{-1/2} \otimes \tilde{I}_1 = m \cdot \chi_\Pi \delta_B^{-1/2} \otimes \pi,$$

$$\delta_P^{-1/2} \otimes \tilde{I}_0 = \begin{cases} 0 & \textit{if } \pi = \mu \circ \det \textit{ and } \rho \neq \mu\chi_\Pi, \\ \chi_\Pi^* & \textit{otherwise,} \end{cases}$$

and the TS-modules $\delta_P^{-1/2} \otimes \beta^\rho(I_i)$ are

$$\begin{aligned}
\delta_P^{-1/2} \otimes \beta^\rho(I_3) &= \begin{cases} \chi_\Pi \otimes C_c^\infty(S) & \textit{if } \pi = \mu \circ \det \textit{ and } \rho = \mu\chi_\Pi, \\ 0 & \textit{otherwise}, \end{cases} \\
\delta_P^{-1/2} \otimes \beta^\rho(I_2) &= 0, \\
\delta_P^{-1/2} \otimes \beta^\rho(I_1) &= m \cdot \chi_\Pi \delta_B^{-1/2} \otimes \pi, \\
\delta_P^{-1/2} \otimes \beta^\rho(I_0) &= \begin{cases} \chi_\Pi^{*} & \textit{if } \pi = \mu \circ \det \textit{ and } \rho = \mu\chi_\Pi, \\ 0 & \textit{otherwise}. \end{cases}
\end{aligned}$$

Concerning $\mathbb{E} = C_c^\infty(S)$ and π, π_U as TS-modules, see Examples 2.4 and 2.5.

Proof. By definition $\tilde{I}_i \cong ((I_i)_{\widetilde{N}})_{\widetilde{T},\rho}$ holds. The $\widetilde{N}$-coinvariant spaces $(I_i)_{\widetilde{N}}$ are given in Section A.2 and it remains to determine ρ-coinvariants. For $i = 3$ the result follows from Corollary 2.17 with the action on $C_c^\infty(S)$ given in Example 2.4. For $i = 2$ the ρ-coinvariant space is given by integrating $f \in C_c^\infty(k^\times, V_U)$ over $k^\times$. The action of $\widetilde{T}$ on $(I_1)_{\widetilde{N}} \cong V \oplus V$ is by multiplication with the character ρ_+ on one factor and ρ_- on the other factor. Thus $\tilde{I}_1$ is isomorphic to m copies of $\nu\chi_\Pi \otimes \pi$ as a $\mathrm{Gl}_a(1)$-module. The ρ-coinvariant space of $(I_0)_{\widetilde{N}}$ is given by Corollary 2.17. Finally, recall $\delta_P(x_\lambda s_b) = |\lambda|^3$. The proof for $\beta^\rho(I_i)$ is analogous. □

For $\dim(\pi) \neq 1$ and $i \neq 1$, all $\beta^\rho(I_i)$ above vanish. If (σ_Π, Λ) is ordinary, then all $\beta^\rho(I_i)$ with $i \neq 3$ are finite-dimensional.

The inclusions $I_{\geq i} \subseteq I$ induce monomorphisms $\beta^\rho(I_{\geq i}) \hookrightarrow \beta^\rho(I)$. However, the $\beta_\rho(I_{\geq i}) \to \beta_\rho(I)$ are not necessarily monomorphisms, since β_ρ is only right exact. Let $I_{\leq i} := I/I_{\geq i+1}$, then there is an exact sequence

$$0 \to I_{\geq 2} \to I \to I_{\leq 1} \to 0.$$

The Bessel functors applied to this sequence and to $0 \to I_1 \to I_{\leq 1} \to I_0 \to 0$ and $0 \to I_3 \to I_{\geq 2} \to I_2 \to 0$ yield a diagram

$$\begin{array}{ccccccccccccccc}
 & & 0 & & & & 0 & & 0 & & & & & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & & & \\
 & & \beta^\rho(I_3) & & & & \beta^\rho(I_1) & & \tilde{I}_3 & & & & \tilde{I}_1 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \beta^\rho(I_{\geq 2}) & \longrightarrow & \beta^\rho(I) & \longrightarrow & \beta^\rho(I_{\leq 1}) & \xrightarrow{\delta} & \tilde{I}_{\geq 2} & \longrightarrow & \tilde{I} & \longrightarrow & \tilde{I}_{\leq 1} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & \beta^\rho(I_0) & & \tilde{I}_2 & & & & \tilde{I}_0 & & \\
 & & & & & & & & \downarrow & & & & \downarrow & & \\
 & & & & & & & & 0 & & & & 0 & &
\end{array}$$

Lemma 4.6. *The sequences in the diagram are exact. If* $\pi \ncong (\nu^{-1/2} \circ \det)$, *the sequences* $0 \to \tilde{I}_1 \to \tilde{I}_{\leq 1} \to \tilde{I}_0 \to 0$ *and* $0 \to \beta^\rho(I_1) \to \beta^\rho(I_{\leq 1}) \to \beta^\rho(I_0) \to 0$ *are exact.*

Proof. The first assertion holds by Lemma 3.13. Note that $\beta^\rho(I_2) = 0$ by Lemma 4.5. It remains to be shown that the boundary morphism

$$\delta_2 : \beta^\rho(I_0) \to \tilde{I}_1$$

vanishes for $\pi \ncong (\nu^{-1/2} \circ \det)$. Indeed, Lemma 4.5 implies that $\beta^\rho(I_0) \neq 0$ is only possible if $\pi \cong (\mu \circ \det)$ and $\rho = \chi_\Pi \mu$ for a smooth character μ of $k^\times$. But then $\beta^\rho(I_0) \cong i_*(\nu^{3/2}\chi_\Pi^*)$ and

$$\tilde{I}_1 \cong m \cdot i_*(\delta_P^{1/2}\delta_B^{-1/2}\chi_\Pi \mu) = m \cdot i_*(\nu\rho).$$

Now $\delta_2 \neq 0$ implies $\rho = \nu^{1/2}\chi_\Pi^*$. The central character is $\omega = \mu^2\chi_\Pi^2 = \rho^2$, so this implies $\mu = \nu^{-1/2}$. □

If $\pi \cong (\nu^{-1/2} \circ \det)$, the unique quotient Π of I is of type VIb and does not admit a split Bessel model.

Lemma 4.7. *Fix* (σ_Π, ρ) *such that* π *is either infinite-dimensional or isomorphic to* $\rho\chi_\Pi^{-1} \circ \det$. *If* (σ_Π, ρ) *is ordinary, then* $\tilde{I}$ *has degree one and there is a monomorphism*

$$\tilde{I}_3 \cong \delta_P^{1/2} \otimes \mathbb{E}[\chi_\Pi] \hookrightarrow \tilde{I}.$$

Proof. For $i \neq 3$ the TS-modules $\beta^\rho(I_i)$ and $\tilde{I}_i$ are of finite dimension by Lemma 4.5. By Example 2.4, the TS-module $\tilde{I}_3$ is isomorphic to $\delta_P^{-1/2} \otimes \mathbb{E}[\chi_\Pi]$ and has degree one. By a diagram chase it is then easy to see that $\tilde{I}$ has degree one. In particular, the image of $\tilde{I}_3 \to \tilde{I}$ must have degree one. Since $\tilde{I}_3$ is perfect, the morphism $\tilde{I}_3 \to \tilde{I}$ is injective by Lemma 2.23. □

Lemma 4.8. *If* σ_Π *is not one-dimensional, then* $\beta^\rho(I)$ *is perfect for every* ρ. *Further,* $\beta^\rho(\Pi)$ *is perfect for every irreducible submodule or quotient* Π *of* I.

Proof. Lemma 4.5 implies $\beta^\rho(I_i) = 0$ for every $i \neq 1$ and $\kappa(\beta^\rho(I_1)) = 0$ by Kirillov theory. This implies $\kappa(\beta^\rho(I)) = 0$ by left-exactness of κ. By possibly replacing I with $\omega \otimes I^\vee$, we can assume that Π is a submodule of I by Theorem 4.1. Then $\beta^\rho(\Pi)$ is a submodule of $\beta^\rho(I)$ and thus perfect. □

Lemma 4.9. *If* $\Pi \in \mathcal{C}_G(\omega)$ *is generic irreducible, then* $\beta^\rho(\Pi) = 0$ *for all* ρ.

Proof. $\beta^\rho(\Pi)$ has degree zero by Corollary 3.20, so it is finite-dimensional. If $J_P(\Pi) \neq 0$, there is an irreducible $\sigma_\Pi \in \mathcal{C}_M$, not one-dimensional, such that Π is a quotient of $\mathrm{Ind}_P^G(\sigma)$, see Table 2. By Lemma 4.8, $\beta^\rho(\Pi)$ is perfect and thus zero. If $J_P(\Pi) = 0$, then Propositions 3.18 and 3.19 imply the assertion. □

Now Lemma 3.13 easily implies that:

Corollary 4.10. *An exact sequence* $0\to\Xi\to I\to\Pi\to 0$ *in* $\mathcal{C}_G^{\mathrm{fin}}(\omega)$ *with irreducible generic quotient* Π *gives an exact sequence* $0\to\widetilde{\Xi}\to\tilde{I}\to\widetilde{\Pi}\to 0$ *of* TS*-modules, and especially* $\deg(\tilde{I})=\deg(\widetilde{\Xi})+\deg(\widetilde{\Pi})$.

4.2. ***Siegel–Jacquet module*** $\mathcal{J}$. For irreducible $\Pi\in\mathcal{C}_G(\omega)$ the normalized Jacquet module with respect to the Siegel parabolic subgroup $P=MN$ is

$$\mathcal{J}=\delta_P^{-1/2}\otimes J_P(\Pi)=\delta_P^{-1/2}\otimes\Pi_N.$$

It is a smooth M-module of finite length. The Levi group M decomposes as

$$M=M^1\times\{t_\lambda=\mathrm{diag}(1,1,\lambda,\lambda)\mid\lambda\in k^\times\},$$

where $M^1=M\cap\mathrm{Sp}(4)$ is isomorphic to Gl(2) via $m_A\mapsto A$. In this section we describe $J_P(\Pi)_{T,\chi}$ as a Gl(2)-module for smooth characters χ of T. The irreducible Jordan–Hölder constituents of $\mathcal{J}$ are of the form $\sigma_\Pi=\pi\boxtimes\chi_\Pi$ for $\sigma_\Pi(t_\lambda m_A)=\pi(A)\chi_\Pi(\lambda)$ with central character $\omega=\omega_\pi\chi_\Pi^2$.

Remark 4.11. $\widetilde{T}$ embeds into M via $\tilde{t}=(t_1,t_2,t_2,t_1)=zt_\lambda m_A$ for $z=t_2\cdot\mathrm{id}$, $\lambda=t_1/t_2$ and $A=\mathrm{diag}(t_1/t_2,1)$. It acts on $\pi\boxtimes\chi_\Pi$ by

$$\tilde{t}\mapsto(\chi_\Pi\otimes\pi)(\mathrm{diag}(t_1,t_2))=\omega(t_2)\chi_\Pi(t_1/t_2)\,\pi(\mathrm{diag}(t_1/t_2,1)).$$

The torus T embeds into M via $x_\lambda=m_{\mathrm{diag}(\lambda,\lambda)}t_\lambda$, therefore on $\pi\boxtimes\chi_\Pi$ it acts by

$$\chi_{\mathrm{norm}}:x_\lambda\mapsto\omega_\pi\chi_\Pi(\lambda)=\chi_\Pi^{\circledast}(\lambda).$$

For the multiset $\Delta(\Pi)$ of smooth T-characters χ_{norm} arising from the irreducible constituents of $\mathcal{J}$, see Section A.3. For the unnormalized Jacquet modules $J_P(\Pi)$ the character χ_{norm} must be replaced by the corresponding unnormalized character $\chi=\nu^{3/2}\chi_{\mathrm{norm}}$. The operation of $\widetilde{T}$ does not depend on the normalization because $\widetilde{T}$ is in the kernel of the modulus character δ_P.

Lemma 4.12. *If a nongeneric irreducible* $\Pi\in\mathcal{C}_G(\omega)$ *admits a split Bessel model for a Bessel character* ρ, *then* $\rho=\mu\chi_\Pi$ *and* $\omega=(\mu\chi_\Pi)^2$ *holds for every one-dimensional Jordan–Hölder constituent* $\sigma=(\mu\circ\det)\boxtimes\chi_\Pi$ *of* $\mathcal{J}$.

Proof. By [Roberts and Schmidt 2007, Table A.3] and Theorem 4.1, Π is a quotient of the Siegel induced representation $I=\mathrm{Ind}_P^G((\mu\circ\det)\boxtimes\chi_\Pi)$. If $\rho\neq\mu\chi_\Pi$, then Lemmas 4.5 and 4.6 imply that $\tilde{I}$ and thus its quotient $\widetilde{\Pi}$ are finite-dimensional. This contradicts the assumption that ρ provides a split Bessel model for Π. The assertion $\omega=(\mu\chi_\Pi)^2$ is clear. ☐

Corollary 4.13. *Suppose* $\Pi\in\mathcal{C}_G(\omega)$ *is irreducible and not a twist of* VIa. *If* Π *has a split Bessel model with Bessel character* ρ, *the* ρ*-coinvariant space* σ_ρ *is one-dimensional for all Jordan–Hölder constituents* σ *of* $\mathcal{J}$.

Proof. Remark 4.11 implies that $\sigma_\rho = \sigma_{\widetilde{T},\rho}$ is the space of ρ-coinvariants under the $\widetilde{T}$-action defined by $\tilde{t} \mapsto \chi_\Pi \otimes \pi(\mathrm{diag}(t_1, t_2))$. Hence, by Corollary 2.17,

$$\dim(\sigma_\rho) = \begin{cases} 0, & \pi = \mu \circ \det \text{ and } \rho \neq \mu\chi_\Pi, \\ 1, & \text{otherwise.} \end{cases}$$

For nongeneric Π, Lemma 4.12 implies $\dim(\sigma_\rho) = 1$. The only generic Π with a one-dimensional constituent in $\mathcal{J}$ are the twists of type VIa. □

4.3. *Monodromy of $\mathcal{J}$.* T and M^1 commute, so $\mathcal{J} = \delta_P^{-1/2} \otimes J_P(\Pi) \in \mathcal{C}_M^{\mathrm{fin}}$ can be considered as an object in the abelian category $\mathcal{C}_{M^1}^{\mathrm{fin}}$ endowed with T-action. A smooth character χ of T defines χ-(co)invariant spaces $\mathcal{J}^\chi$, $\mathcal{J}_\chi \in \mathcal{C}_{M^1}^{\mathrm{fin}}$ and the generalized χ-eigenspace $\mathcal{J}^{(\chi)} \subseteq \mathcal{J}$ as the subspace annihilated by $(t - \chi(t))^N$ for all $t \in T$ and large N. Since T and M^1 commute, $\mathcal{J}$ decomposes as a direct sum

$$\mathcal{J} \cong \bigoplus_\chi \mathcal{J}^{(\chi)}.$$

Lemma 4.14. *For generic Π and every smooth T-character χ_{norm}, there is at most one generic constituent π in $\mathcal{J}_{T,\chi_{\mathrm{norm}}}$.*

Proof. Fix an additive character $\psi \neq 1$ of the unipotent radical U of $B_G \cap M$. Uniqueness of Whittaker models implies for all characters χ_{norm} of T that $\dim(\mathcal{J}_{\chi_{\mathrm{norm}}})_{U,\psi}$ equals the number of generic constituents π of $\mathcal{J}_{\chi_{\mathrm{norm}}}$. For generic Π we have shown in Lemma 3.12 that $A = j^!(\overline{\Pi}) \in \mathcal{C}$ is perfect. By Lemma 3.8, the T-module $i^*(A) \cong J_P(\Pi)_\psi$ is cyclic. Now Lemma 2.15 implies that the number of generic constituents is

$$\dim(\mathcal{J}_{\chi_{\mathrm{norm}}})_\psi = \dim(J_P(\Pi)_\psi)_\chi = \dim(\pi_0(A))_\chi \leq 1. \qquad \square$$

Monodromy filtration. Fix an E in $\mathcal{C}_{M^1 \times T}$ for some reductive group Γ and a T-character χ. On each $E^{(\chi)}$ the compact group $T(\mathfrak{o}) \cong \mathfrak{o}^\times$ acts by the character χ. Since $T \cong k^\times = \mathfrak{o}^\times \times \varpi^{\mathbb{Z}}$, it suffices to determine the action of the monodromy operator $\tau_\chi = \varpi - \chi(\varpi)$ on $E^{(\chi)}$. The morphism τ_χ is nilpotent of length bounded by the length $E^{(\chi)}$. There exists a unique finite increasing $\mathbb{Z}$-filtration $F_\bullet(V)$ of $V = E^{(\chi)}$ by M^1-submodules $F_i(V) \subseteq F_{i+1}(V)$ such that

$$\tau_\chi(F_i(V)) \subseteq F_{i-2}(V), \quad \tau_\chi^i : \mathrm{Gr}_i(V) \cong \mathrm{Gr}_{-i}(V) \qquad \text{for all } i \geq 0,$$

where $\mathrm{Gr}_i(V) = F_i(V)/F_{i-1}(V)$. Put $P_i(V) = \ker(\tau_\chi : \mathrm{Gr}_i(V) \to \mathrm{Gr}_{i-2}(V))$, then $P_i(V) = 0$ for $i > 0$ and $\mathrm{Gr}_i(V) \cong \bigoplus_{j \geq |i|, j \equiv i(2)} P_{-j}$ in $\mathcal{C}_{M^1}^{\mathrm{fin}}$.

For the induced filtration on $E^\chi = \ker(\tau_\chi)$ defined by $F_i(V) \cap E^\chi$ we have $\mathrm{Gr}_i(E^\chi) \cong P_i(V)$. For the induced filtration on the cokernel $E_\chi = \mathrm{coker}(\tau_\chi)$ defined by $\mathrm{im}(F_i(V) \to E_\chi)$ one has $\mathrm{Gr}_i(E_\chi) \cong P_{-i}(V)$. Notice that every constituent of $P_{-i}(V)$ has multiplicity $\geq i + 1$ in V.

For details, see [Kiehl and Weissauer 2001, p. 56], where the Tate twist can be omitted in our situation.

Lemma 4.15. *Fix $\Pi = \tau(S, \nu^{-1/2}\sigma)$ of type* VIa *and $\chi_{\mathrm{norm}} = \nu^{1/2}\sigma$. The Siegel–Jacquet module $\mathcal{J} = \mathcal{J}^{(\chi_{\mathrm{norm}})}$ has length three, with M^1-constituents $\nu^{1/2} \circ \det$ and two copies of $\mathrm{Sp}(\nu^{1/2})$. The coinvariant quotient $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is an indecomposable extension $0 \to (\nu^{1/2} \circ \det) \to \mathcal{J}_{\chi_{\mathrm{norm}}} \to \mathrm{Sp}(\nu^{1/2}) \to 0$, thus isomorphic to $1 \times \nu$ as a* Gl(2)*-module.*

Proof. The constituents of $\mathcal{J}$ are given in [Roberts and Schmidt 2007, Table A.3]. Since two of them are generic, Lemma 4.14 implies $\mathcal{J}_{\chi_{\mathrm{norm}}} \neq \mathcal{J}$. Thus the monodromy filtration is nontrivial in the sense that $\tau_{\chi_{\mathrm{norm}}}$ acts nontrivially, so $\mathcal{J} \neq \mathrm{Gr}_0(\mathcal{J})$. Now, since $\mathrm{Gr}_i(\mathcal{J}_{\chi_{\mathrm{norm}}}) \cong P_{-i}(\mathcal{J})$ and every constituent of $P_{-i}(\mathcal{J})$ has multiplicity $\geq i+1$ in $\mathcal{J}$, the only possibility is $P_0(\mathcal{J}) = \nu^{1/2} \circ \det$ and $P_{-1}(\mathcal{J}) = \mathrm{Sp}(\nu^{1/2})$. Since these primitive modules describe the induced increasing filtration on $\mathcal{J}_{\chi_{\mathrm{norm}}}$ defined by $F_i(\mathcal{J}_{\chi_{\mathrm{norm}}}) = \mathrm{im}(F_i(\mathcal{J}) \to \mathcal{J}_{\chi_{\mathrm{norm}}})$, we obtain $F_0(\mathcal{J}_{\chi_{\mathrm{norm}}}) = P_0(\mathcal{J}) \cong \nu^{1/2} \circ \det$ and $F_1(\mathcal{J}_{\chi_{\mathrm{norm}}})/F_0(\mathcal{J}_{\chi_{\mathrm{norm}}}) \cong P_1(\mathcal{J}) \cong \mathrm{Sp}(\nu^{1/2})$. Hence our claim follows that $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is an extension

$$0 \to (\nu^{1/2} \circ \det) \to \mathcal{J}_{\chi_{\mathrm{norm}}} \to \mathrm{Sp}(\nu^{1/2}) \to 0.$$

It remains to be shown that the sequence does not split. Indeed, if this were not true, then

$$\mathrm{Hom}_M(\mathcal{J}, (\nu^{1/2} \circ \det) \boxtimes \chi_\Pi) \cong \mathrm{Hom}_{M^1}(\mathcal{J}_{\chi_{\mathrm{norm}}}, (\nu^{1/2} \circ \det))$$

would not vanish for $\chi_\Pi = \chi_{\mathrm{norm}}^* = \nu^{-1/2}$. By dual Frobenius reciprocity, Π is then a submodule of the induced representation $I = \mathrm{Ind}_P^G((\nu^{\frac{1}{2}} \circ \det) \boxtimes \nu^{-1/2})$. However, the irreducible constituents of I are of type VId and VIb and thus not isomorphic to Π. This is a contradiction, so $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is indeed indecomposable. The unique indecomposable extension is the Gl(2)-module $1 \times \nu$. □

Lemma 4.16. *Fix an irreducible representation $\Pi \in \mathcal{C}_G(\omega)$ of type* VId *normalized as in Table 1 and the T-character $\chi_{\mathrm{norm}} = \nu^{-1/2}$. Then the Siegel–Jacquet module $\mathcal{J} = \delta_P^{-1/2} \otimes J_P(\Pi)$ has length three and its constituents are the $M^1 \times T$-modules $\mathrm{Sp}(\nu^{-1/2}) \boxtimes \chi_{\mathrm{norm}}$ and two copies of $(\nu^{-1/2} \circ \det) \boxtimes \chi_{\mathrm{norm}}$. The coinvariant quotient $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is a nonsplit extension of M^1-modules*

$$0 \to \mathrm{Sp}(\nu^{-1/2}) \to \mathcal{J}_{\chi_{\mathrm{norm}}} \to \nu^{-1/2} \circ \det \to 0$$

and is thus isomorphic to the Gl(2)*-module $\mathcal{J}_{\chi_{\mathrm{norm}}} \cong 1 \times \nu^{-1}$.*

Proof. For the constituents see [Roberts and Schmidt 2007, Table A.3], thus $\mathcal{J} = \mathcal{J}^{(\chi_{\mathrm{norm}})}$ is clear. We claim that the monodromy operator τ_χ acts nontrivially on $\mathcal{J}$ and the graded components of the monodromy filtration are the $M^1 \cong \mathrm{Gl}(2)$-modules

$$\mathrm{Gr}_1(\mathcal{J}) \cong (\nu^{-1/2} \circ \det), \quad \mathrm{Gr}_0(\mathcal{J}) \cong \mathrm{Sp}(\nu^{-1/2}), \quad \mathrm{Gr}_{-1}(\mathcal{J}) \cong (\nu^{-1/2} \circ \det).$$

This amounts to show that T does not act by an eigencharacter on $\mathcal{J}$, although its acts on each of its three constituents by the same character χ_{norm}. Indeed, $J_P(\Pi) \in \mathcal{C}_M^{\mathrm{fin}}$ admits the unnormalized Jacquet quotient $J_P(\Pi)_U$ with respect to the subgroup $U = \{m_A \in M \mid A = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$, which is isomorphic to the Borel–Jacquet module $J_{B_G}(\Pi)$. However, $J_{B_G}(\Pi)$ factorizes over the Klingen–Jacquet module $F = J_Q(\Pi) \cong i^*(\overline{\Pi}) \in \mathcal{C}_{\mathrm{Gl}(2)}$ and is isomorphic to $J_{B_G}(\Pi) \cong F_{U_Q}$ with respect to the maximal unipotent radical U_Q of $L_Q \cap B_G$ with the standard Klingen–Levi subgroup L_Q of Q as in Section 3.2. By [Roberts and Schmidt 2007, Table A.4], F contains the Jordan–Hölder factor $\nu^{1/2} \otimes (1 \times 1)$. Since $(1 \times 1)_{U_Q}$ is not semisimple, T does not act semisimply on $J_{B_G}(\Pi)$, see Remark 4.11. On the other hand, T is in the center of M and acts by the same character χ on the three constituents of $J_P(\Pi)$. This implies that the T-module $J_P(\Pi)$ is not semisimple and thus proves our claim. That the filtration of $J_P(\Pi)_\chi = \mathcal{J}_{\chi_{\mathrm{norm}}}$ and the graded components are of the required form is shown as in the case VIa. This implies that $\mathcal{J}_{\chi_{\mathrm{norm}}}$ sits in an exact sequence as required and it remains to be shown that this sequence does not split. But if $\mathcal{J}_{\chi_{\mathrm{norm}}}$ was semisimple, then Π would be a submodule of $\mathrm{Ind}_P^G(\mathrm{Sp}(\nu^{-1/2}) \boxtimes \nu^{1/2})$ by dual Frobenius reciprocity. This is a contradiction to Table 2, so $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is indecomposable as an $M^1 \cong \mathrm{Gl}(2)$-module and thus isomorphic to the unique indecomposable extension $(1 \times \nu^{-1})$. $\square$

Theorem 4.17. *For irreducible representations $\Pi \in \mathcal{C}_G(\omega)$ and characters χ_{norm} of T, the M^1-module $\mathcal{J}_{\chi_{\mathrm{norm}}} \in \mathcal{C}_{M^1}^{\mathrm{fin}}$ is irreducible or zero except for the following two cases where σ is a smooth character of* Gl(1):

> VIa $\Pi = \tau(S, \nu^{-1/2}\sigma)$ *and* $\chi_{\mathrm{norm}} = \nu^{1/2}\sigma$, *then* $\mathcal{J}_{\chi_{\mathrm{norm}}} \cong 1 \times \nu$,
>
> VId $\Pi = L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$ *and* $\chi_{\mathrm{norm}} = \nu^{-1/2}\sigma$, *then* $\mathcal{J}_{\chi_{\mathrm{norm}}} \cong 1 \times \nu^{-1}$.

Proof. By a twist we can assume that Π is normalized as in Table 1. For type VIa and VId see Lemmas 4.15 and 4.16. For generic Π not of type VIa, every constituent of $\mathcal{J}_{T,\chi_{\mathrm{norm}}}$ is generic, so the statement is implied by Lemma 4.14. For the remaining nongeneric Π the assertion follows from an inspection of Table A.3 of [Roberts and Schmidt 2007] and Table 3 except for case IIb.

For case IIb, Π is the fully induced representation $\Pi = \mathrm{Ind}_P^G((\chi_1 \circ \det) \boxtimes \chi_1^{-1})$. The constituents of $\mathcal{J} = \delta_P^{-1/2} \otimes J_P(\Pi)$ are $(\chi_1\nu^{-1/2} \times \chi_1^{-1}\nu^{-1/2}) \boxtimes \nu^{1/2}$ and $(\chi_1 \circ \det) \boxtimes \chi_1^{-1}$ and $(\chi_1^{-1} \circ \det) \boxtimes \chi_1$. Now assume $\mathcal{J}_{\chi_{\mathrm{norm}}}$ has length greater than one. Since $\chi_1^2 \neq \nu^{\pm 1}$ by definition, this implies $\chi_{\mathrm{norm}} = \chi_1 = \chi_1^{-1}$. Dual Frobenius reciprocity yields

$$\mathrm{Hom}_G\big(\Pi, \mathrm{Ind}_P^G((\chi_1 \circ \det) \boxtimes \chi_1^{-1})\big) \cong \mathrm{Hom}_{M^1}(\mathcal{J}_{\chi_{\mathrm{norm}}}, (\chi_1 \circ \det))$$

and this is one-dimensional, because Π is irreducible. Especially, $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is an indecomposable two-dimensional M^1-module. Every finite-dimensional smooth module of $M^1 \cong \mathrm{Gl}(2)$ factorizes over the determinant, so we obtain an isomorphism $\mathcal{J}_{\chi_{\mathrm{norm}}} \cong (\chi_1^{(2)} \circ \det)$. The center $\{\mathrm{diag}(\lambda I_2, \lambda^{-1} I_2) \mid \lambda \in k^\times\}$ of M^1 acts by $\lambda \mapsto \chi^{(2)}(\lambda^2)$, thus not semisimply. On the other hand, the center of G acts semisimply by multiplication with ω, so the action of T on $\mathcal{J}_{\chi_{\mathrm{norm}}}$ cannot be semisimple. This provides a contradiction. □

4.4. *Monodromy of the Bessel module.* The following monodromy theorem is one of the key inputs for Theorem 5.3. For fixed characters ρ of $\widetilde{T}$ and χ of T let $\widetilde{\Pi} = \beta_\rho(\Pi)$ and $\mathcal{J} = \delta_P^{-1/2} \otimes J_P(\Pi)$, then by transitivity of coinvariant functors there are isomorphisms

$$(\mathcal{J}_{\chi_{\mathrm{norm}}})_\rho \cong (J_P(\Pi)_\chi)_\rho \cong (\widetilde{\Pi}_S)_\chi \cong \pi_0(\widetilde{\Pi})_\chi.$$

We estimate the dimension in Theorem 4.20. More precisely, we show that $\pi_0(\widetilde{\Pi})$ is cyclic as a T-module except for the case where Π is of type VIa.

Lemma 4.18. *For irreducible $\Pi \in \mathcal{C}_G(\omega)$ of type* VIa *with normalization as in Table 1, all Bessel characters ρ and characters χ_{norm} of T:*

$$\dim(\pi_0(\widetilde{\Pi})_\chi) = \begin{cases} 0, & \chi_{\mathrm{norm}} \neq \nu^{1/2}, \\ 1, & \chi_{\mathrm{norm}} = \nu^{1/2},\ \rho \neq 1, \\ 2, & \chi_{\mathrm{norm}} = \nu^{1/2},\ \rho = 1. \end{cases}$$

Proof. Lemma 4.15 implies $\mathcal{J} = \mathcal{J}^{(\chi_{\mathrm{norm}})}$ for $\chi_{\mathrm{norm}} = \nu^{1/2}$ and yields a short exact sequence of $\mathrm{Gl}(2) \cong M^1$-modules

$$0 \to (\nu^{1/2} \circ \det) \to \mathcal{J}_{\chi_{\mathrm{norm}}} \to \mathrm{Sp}(\nu^{1/2}) \to 0.$$

By Remark 4.11, $\widetilde{T}$ acts on $\mathcal{J}_{\chi_{\mathrm{norm}}}$ by $\tilde{t} \mapsto \nu^{-1/2}(t_1 t_2)\mathcal{J}_{\chi_{\mathrm{norm}}}(m_{\mathrm{diag}(t_1,t_2)})$, therefore Lemma A.2 gives a long exact sequence

$$\cdots \to \mathrm{Sp}^\Lambda \to (1 \circ \det)_\Lambda \to (\nu^{-1/2} \otimes \mathcal{J}_{\chi_{\mathrm{norm}}})_\Lambda \to \mathrm{Sp}_\Lambda \to 0$$

of Λ-(co)invariant spaces with respect to the action of $\widetilde{T}$ by $\Lambda = \rho \boxtimes \rho^*$. Since $\mathrm{Sp}^\Lambda = 0$ and $\dim(\mathrm{Sp}_\Lambda) = 1$ by Corollary 2.17, counting dimensions yields the assertion. □

Lemma 4.19. *For irreducible $\Pi \in \mathcal{C}_G(\omega)$ of type* VId *with normalization as in Table 1, all Bessel characters ρ and characters χ of T, we have*

$$\dim(\pi_0(\widetilde{\Pi})_\chi) = \begin{cases} 0, & \chi_{\mathrm{norm}} \neq \nu^{-1/2}, \\ 1, & \chi_{\mathrm{norm}} = \nu^{-1/2}. \end{cases}$$

Proof. Lemma 4.16 implies $\mathcal{J} = \mathcal{J}^{(\chi_{\mathrm{norm}})}$ for $\chi_{\mathrm{norm}} = \nu^{-1/2}$ and yields an isomorphism of $M^1 \cong \mathrm{Gl}(2)$-modules $\mathcal{J}_{\chi_{\mathrm{norm}}} \cong (1 \times \nu^{-1})$. Now consider $\mathcal{J}_{\chi_{\mathrm{norm}}}$ as a $\mathrm{Gl}_a(1)$-module via $\mathrm{Gl}_a(1) \hookrightarrow \mathrm{Gl}(2) \cong M^1$ as in Example 2.5. By Kirillov theory, $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is perfect of degree one as a $\mathrm{Gl}_a(1)$-module since $1 \times \nu^{-1}$ does not admit a one-dimensional Gl(2)-subrepresentation. Lemma 2.11 yields the assertion. □

Theorem 4.20 (monodromy theorem). *For irreducible $\Pi \in \mathcal{C}_G(\omega)$, all smooth Bessel characters ρ and all smooth characters χ of T we have*

$$\dim(\pi_0(\widetilde{\Pi})_{T,\chi}) \le 1$$

except for the case $\Pi = \tau(S, \nu^{-1/2}\sigma)$ *of type* VIa *and* $(\rho, \chi_{\mathrm{norm}}) = (\sigma, \nu^{1/2}\sigma)$, *then* $\dim(\pi_0(\widetilde{\Pi})_{T,\chi}) = 2$.

Proof. By a twist we can assume that Π is normalized as in Table 1. Recall that $\pi_0(\widetilde{\Pi})_\chi \cong (\mathcal{J}_{\chi_{\mathrm{norm}}})_{\widetilde{T},\rho}$ as before. If the M^1-module $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is irreducible, Corollary 2.17 implies that $\dim(\mathcal{J}_{\chi_{\mathrm{norm}}})_{\widetilde{T},\rho} \le 1$. Indeed, $\mathcal{J}_{\chi_{\mathrm{norm}}}$ is irreducible by Theorem 4.17 except for the cases discussed in Lemmas 4.18 and 4.19. □

Corollary 4.21. $\dim k_\chi(\widetilde{\Pi}) \le 1$ *holds for irreducible nongeneric* $\Pi \in \mathcal{C}_G(\omega)$ *and all smooth characters* ρ *and* χ *of* Gl(1).

Proof. The degree of $\widetilde{\Pi}$ is either one or zero by Theorem 5.1. If the degree is one, $\widetilde{\Pi}$ is perfect by Corollary 5.7, so the assertion holds by Lemma 2.13. If $\widetilde{\Pi}$ has degree zero, then $\widetilde{\Pi}$ is isomorphic to $\pi_0(\widetilde{\Pi})$ and the assertion follows from Theorem 4.20. □

4.5. *Constituents of the Bessel module.* For an irreducible $\Pi \in \mathcal{C}_G(\omega)$ and a Bessel character ρ let $\widetilde{\Pi} = \beta_\rho(\Pi)$ be the corresponding Bessel module. Let $\mathcal{J} = \delta_P^{-1/2} \otimes J_P(\Pi)$ be the normalized Siegel–Jacquet module. We want to find out the multiset $\tilde{\Delta}(\Pi)$ of Jordan–Hölder constituents χ_{norm} of the T-module $\delta_P^{-1/2} \otimes \pi_0(\widetilde{\Pi}) \cong \mathcal{J}_{\widetilde{T},\rho}$.

Lemma 4.22. *Suppose* $\Pi \in \mathcal{C}_G(\omega)$ *is irreducible, normalized as in Table 1 and* ρ *is a smooth Bessel character. If* ρ *defines a Bessel model for* Π, *then*

$$\tilde{\Delta}(\Pi) = \begin{cases} \{\nu^{1/2}, \nu^{1/2}\} & \textit{for type } \mathsf{VIa}, \\ \{\nu^{-1/2}, \nu^{-1/2}\} & \textit{for type } \mathsf{VId}, \\ \Delta(\Pi) & \textit{otherwise}. \end{cases}$$

If ρ *does not define a Bessel model for* Π, *then*

$$\tilde{\Delta}(\Pi) = \begin{cases} \{\nu^{-3/2}\} & \textit{for type } \mathsf{IVd},\ \rho = 1, \\ \{\xi\nu^{-1/2}\} & \textit{for type } \mathsf{Vd},\ \rho = 1, \\ \{\nu^{-1/2}\} & \textit{for type } \mathsf{Vd},\ \rho = \xi, \\ \{\nu^{1/2}\} & \textit{for type } \mathsf{VIb},\ \rho = 1, \\ \Delta_0(\Pi) & \textit{otherwise}. \end{cases}$$

Proof. We can assume $J_P(\Pi) \neq 0$. If Π is generic, every ρ provides a Bessel model by Corollary 3.20. Lemma 4.9 implies $\beta^\rho(\Pi) = 0$, so Proposition 3.19 yields $[\widetilde{\Pi}] = [j^! \overline{\Pi}]$ in the Grothendieck group $K_0(\mathcal{C})$ and this means $\tilde{\Delta}(\Pi) = \Delta_0(\Pi)$ by exactness of π_0. The multiset $\Delta_0(\Pi)$ of constituents of the T-module $\mathcal{J}_\psi$ is given by the T-action on the generic constituents of $\mathcal{J}$, see [Roberts and Schmidt 2007, Table A.3]. Except for type VIa, every constituent of $\mathcal{J}$ is generic.

Now suppose Π is nongeneric, but not of type IIb or type VId. The action of M on $\mathcal{J}$ respects the decomposition into generalized T-eigenspaces

$$\mathcal{J} = \bigoplus_{\chi} \mathcal{J}^{(\chi_{\mathrm{norm}})}.$$

The generalized eigenspaces $\mathcal{J}^{(\chi_{\mathrm{norm}})}$ are irreducible M-modules, so M acts semisimply on $\mathcal{J}$. We treat the generic constituents of $\mathcal{J}$, indexed by $\Delta_0(\Pi)$, separately from the one-dimensional constituents. By Corollary 2.17 the generic constituents of $\mathcal{J}$ contribute a one-dimensional ρ-equivariant quotient for every ρ, so $\Delta_0(\Pi) \subseteq \tilde{\Delta}(\Pi)$. For the one-dimensional constituents $\sigma = (\mu \circ \det) \boxtimes \chi_\Pi$ of $\mathcal{J}$, Lemma 4.12 asserts that if a split Bessel model exists for some Bessel character ρ, then the torus $\widetilde{T}$ acts on σ by $\rho = \mu\chi_\Pi$. In that case the T-action on σ contributes the character $\chi_{\mathrm{norm}} = \chi_\Pi^{\circledast} = \mu^2\chi_\Pi$ to $\tilde{\Delta}$ if and only if ρ provides a Bessel model, see Remark 4.11. For types IVd, Vd and VIb there is no split Bessel model, but in these cases the Siegel–Jacquet module is finite-dimensional and $\mathcal{J}_{\widetilde{T},\rho}$ is easy to determine.

For case IIb, Π is the fully induced representation $I = \mathrm{Ind}_P^G((\chi_1 \circ \det) \boxtimes \chi_1^{-1})$ and we use the Bessel filtration from Section 4.1. Lemma 4.6 yields an exact sequence of TS-modules

$$\cdots \to \beta^\rho(I_{\leq 1}) \xrightarrow{\delta} \tilde{I}_{\geq 2} \to \tilde{I} \to \tilde{I}_{\leq 1} \to 0$$

and the corresponding vertical sequences. First assume $\rho = 1$, which gives a split Bessel model for I. This is an ordinary case, so $\tilde{I}_{\leq 1} \cong \tilde{I}_0$ and $\beta^\rho(I_{\leq 1}) \cong \beta^\rho(I_0)$. Lemma 4.5 gives $\beta^\rho(I_0) \cong \nu^{3/2}\chi_1$, implies that $\tilde{I}_3$ is perfect and that $\tilde{I}_2 \cong \nu \neq \nu^{3/2}\chi_1$, so the T-equivariant coboundary map $\delta : \beta^\rho(I_{\leq 1}) \to \tilde{I}_{\geq 2}$ is zero. If we apply the functor π_0, we obtain the T-characters $\pi_0(\tilde{I}_3) \cong \nu^{3/2}\chi_1$, $\pi_0(\tilde{I}_2) \cong \nu$ and $\pi_0(\tilde{I}_0) \cong \nu^{3/2}\chi_1^{-1}$. By exactness of π_0 this implies $\tilde{\Delta}(\Pi) = \{\chi_1, \chi_1^{-1}, \nu^{-1/2}\} = \Delta(\Pi)$. Now assume $\rho \neq 1$, which does not give a split Bessel model. $\beta^\rho(I)$ is perfect by Lemma 4.8 and finite-dimensional by Proposition 3.19, thus zero. Now Proposition 3.19 implies $\tilde{\Delta}(\Pi) = \Delta_0(\Pi)$.

For case VId, $\mathcal{J} = \mathcal{J}^{(\chi_{\mathrm{norm}})}$ for $\chi_{\mathrm{norm}} = \nu^{-1/2}$. By Lemma 4.16, $\mathcal{J}_{\chi_{\mathrm{norm}}} \cong (1 \times \nu^{-1})$ is indecomposable as a module of $M^1 \cong \mathrm{Gl}(2)$. By Lemma 2.18, $(\mathcal{J}_{\chi_{\mathrm{norm}}})^{\widetilde{T},\rho} = 0$ and by Lemma 4.19, $(\mathcal{J}_{\chi_{\mathrm{norm}}})_{\widetilde{T},\rho}$ is one-dimensional. Lemma A.2 implies an exact sequence of T-modules:

$$0 \to ((\nu^{-1/2} \circ \det) \boxtimes \nu^{1/2})_{\widetilde{T},\rho} \to \mathcal{J}_{\widetilde{T},\rho} \to (\mathcal{J}_{\chi_{\mathrm{norm}}})_{\widetilde{T},\rho} \to 0.$$

If $\rho = 1$, this gives $\dim(\mathcal{J}_{\widetilde{T},\rho}) = 2$, thus $\tilde{\Delta}(\Pi) = \{\nu^{-1/2}, \nu^{-1/2}\}$. If $\rho \neq 1$, then $((\nu^{-1/2} \circ \det) \boxtimes \nu^{1/2})_{\widetilde{T},\rho} = 0$ vanishes, therefore $\mathcal{J}_{\widetilde{T},\rho}$ is one-dimensional and $\tilde{\Delta}(\Pi) = \{\nu^{-1/2}\} = \Delta_0(\Pi)$. □

4.6. *Klingen–Jacquet module.* The Levi component L_Q of the standard Klingen parabolic Q is the direct product $\{\mathrm{diag}(t, 1, t^{-1}, 1) \mid t \in k^\times\} \times \mathrm{Gl}(2)$ where $\mathrm{Gl}(2)$ is embedded into L_Q as in Section 3.2. The irreducible representations of L_Q have the form

$$(\chi' \boxtimes \tau)(t, m) = \chi'(t \det(m))\, \tau(m) \quad \text{for } \tau \in \mathcal{C}_{\mathrm{Gl}(2)},\ \chi' \in \mathcal{C}_{\mathrm{Gl}(1)}.$$

The notation is chosen so that the upper left matrix entry of an element in L_Q acts with χ'. The square root of the modulus character of $m \in \mathrm{Gl}(2)$ embedded as above is $\delta_Q^{1/2}(m) = |\det(m)|$. For an irreducible $\Pi \in \mathcal{C}_G(\omega)$ let B denote the unnormalized Klingen–Jacquet module $J_Q(\Pi)$ as a $\mathrm{Gl}(2)$-module with respect to the above embedding. Thus every constituent π of B is of the form

$$\pi \cong \delta_Q^{1/2} \otimes (\chi' \circ \det) \otimes \tau) = (\nu\chi' \circ \det) \otimes \tau,$$

where $\chi' \boxtimes \tau$ is from [Roberts and Schmidt 2007, Table A.4]. Note that τ determines χ' because $\omega = \chi'\omega_\tau$ is the central character of Π.

Lemma 4.23. *For an irreducible $\Pi \in \mathcal{C}_G(\omega)$, every irreducible constituent of $k^\rho(i_*(B)) \cong k_\rho(i_*(B))$ in $\mathcal{C}$ is a T-character*

$$\chi = \nu^2\omega\chi'\rho^{-1}$$

for some χ' occurring in the Klingen–Jacquet module as above.

Proof. By right-exactness, every constituent of $k_\rho i_*(B)$ is of the form $k_\rho i_*(\pi)$ for some π in B. The central character of π is $\omega_\pi = \nu^2\chi'^2\omega_\tau = \chi\rho$. □

Lemma 4.24. *Let $\Pi \in \mathcal{C}_G(\omega)$ be irreducible generic and fix a smooth character ρ of* $\mathrm{Gl}(1)$ *with $\rho \notin \Delta_+(\Pi)$. Then*

$$k^\rho i_*(B) \cong k_\rho i_*(B) \cong \begin{cases} i_*(\nu^2\rho), & \text{type IIIa, IVa, VIa and } \rho \in \Delta_-(\Pi), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See Lemma 5.10 in [Rösner and Weissauer 2020]. □

5. Main results

In this section we prove our main results. First, we classify the split Bessel models for Π and determine the Bessel modules $\widetilde{\Pi} = \beta_\rho(\Pi)$ and $\beta^\rho(\Pi)$ for irreducible representations $\Pi \in \mathcal{C}_G(\omega)$ and every smooth $\mathrm{Gl}(1)$-character ρ. Then we describe the Bessel model as a TS-module and recall Piatetski-Shapiro's construction.

5.1. ***Split Bessel models.*** Bessel models for irreducible $\Pi \in \mathcal{C}_G(\omega)$ have been classified by Prasad and Takloo-Bighash [2011] and by Roberts and Schmidt [2016]. We give a new proof in the split case. Note that by Lemma 5.10, Π admits a split Bessel model for $\Lambda = \rho \boxtimes \rho^*$ if and only if the degree of $\beta_\rho(\Pi)$ is nonzero.

Theorem 5.1. *For an irreducible $\Pi \in \mathcal{C}_G(\omega)$ the degree of $\beta_\rho(\Pi) = \widetilde{\Pi}$ is*

$$\deg(\widetilde{\Pi}) = \begin{cases} 1 & \text{if } \Pi \text{ is generic or } \rho \in \Delta_+(\Pi), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For generic Π, see Corollary 3.20. For nongeneric Π, the inequality of Proposition 3.18 is sharp, so $\deg(\widetilde{\Pi}) = \dim(\mathrm{Hom}_T(\delta_P^{-1/2} J_P(I)_\psi, \nu^{-1/2}\rho))$. This is nonzero if and only if $\rho \in \Delta_+(\Pi)$. The degree cannot exceed one, because no T-character occurs in $\Delta_+(\Pi)$ more than once. $\square$

5.2. ***Bessel module $\beta^\rho(\Pi)$.***

Proposition 5.2. *For irreducible $\Pi \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ and smooth Bessel characters ρ, the Bessel module $\beta^\rho(\Pi)$ is given by Table 5.*

Proof. Assume that the Siegel–Jacquet module $J_P(\Pi)$ is nonzero. For generic Π, see Lemma 4.9. If Π is nongeneric and ρ provides a split Bessel model, then $\widetilde{\Pi}$ has degree one and $\beta^\rho(\Pi)$ also has degree one by Proposition 3.19. By Lemma 3.21, the constituents χ of the T-module $X \cong \pi_0(\beta^\rho(\Pi))$ are those of $J_P(\Pi)^{\widetilde{T},\rho}$, i.e., corresponding to χ_{norm} in the multiset $\tilde{\Delta}(\Pi) \backslash \Delta_0(\Pi) = \Delta_1(\Pi)$ given by Lemma 4.22. We have to show that $\beta^\rho(\Pi)$ is perfect in the cases where $\Delta_1(\Pi)$ is not empty. For types IIb, IVb and Vbc this follows from Lemma 4.8. It only remains to consider the case where Π is of type VId and $\rho = 1$. Then $\Delta_1(\Pi)$ contains only one element $\chi_{\mathrm{norm}} = \nu^{-1/2}$. By Lemma 2.15 it is sufficient to construct an embedding $\mathbb{E}[\nu] \hookrightarrow \beta^\rho(\Pi)$. Indeed, Π is a quotient of $I = \mathrm{Ind}_P^G(\sigma_\Pi)$ for the representation $\sigma_\Pi = (\nu^{1/2} \circ \det) \boxtimes \nu^{-1/2}$ of M. Lemma 4.5 for $\rho = 1$ provides an embedding $\mathbb{E}[\nu] \cong \beta^\rho(I_3) \hookrightarrow \beta^\rho(I)$. By degree reasons, this provides a nontrivial morphism $\mathbb{E}[\chi] \to \beta^\rho(\Pi)$, which is injective by Lemma 2.23. If Π is nongeneric and ρ does not give a split Bessel model, then $\deg(\widetilde{\Pi}) = 0$. Proposition 3.19 yields $\deg(\beta^\rho(\Pi)) = 0$, so $\beta^\rho(\Pi)$ is a finite-dimensional TS-module with constituents χ corresponding to χ_{norm} in $\Delta_1(\Pi) = \tilde{\Delta}(\Pi) \setminus \Delta_0(\Pi)$. The constituents of $\tilde{\Delta}(\Pi)$ have been determined in Lemma 4.22. Finally, if the Siegel–Jacquet module $J_P(\Pi)$ vanishes, then $\widetilde{\Pi} \cong \mathbb{S}^{m_\Pi}$, where m_Π is the multiplicity of Whittaker models, as shown in Proposition 3.18. Lemma 3.21 implies $\pi_0(\beta^\rho(\Pi)) = 0$. The degree $\deg(\beta^\rho(\Pi)) = \deg(\widetilde{\Pi}) - m_\Pi = 0$ vanishes by Proposition 3.19. $\square$

5.3. ***Bessel module $\beta_\rho(\Pi)$.*** We obtain a complete classification of split Bessel modules.

Theorem 5.3. *For every irreducible representation* $\Pi \in \mathcal{C}^{\mathrm{fin}}_{\mathrm{GSp}(4)}(\omega)$ *and every Bessel character* ρ, *the Bessel module* $\widetilde{\Pi} = \beta_\rho(\Pi)$ *is explicitly given in Table 5.*

Proof. For generic Π we can assume $\rho \notin \Delta_+(\Pi)$ by Lemma 3.17 and Corollary A.11. Then the assertion follows from Proposition 5.8.

If Π is not generic and $\rho \in \Delta_+(\Pi)$, i.e., if ρ provides a split Bessel model for Π, then the Bessel module is perfect by Corollary 5.7. It is explicitly given by Lemmas 2.22 and 4.22.

If Π is not generic and $\rho \notin \Delta_+(\Pi)$ does not yield a split Bessel model for Π, the action of S on $\widetilde{\Pi}$ is trivial by Lemma 2.7 and Theorem 5.1, so $\widetilde{\Pi}$ is isomorphic to $\pi_0(\widetilde{\Pi})$. The constituents of $\pi_0(\widetilde{\Pi})$ are given by Lemma 4.22. There are no nontrivial extensions between these constituents, so $\widetilde{\Pi}$ is semisimple. □

Corollary 5.4. *For spherical unitary irreducible* $\Pi \in \mathcal{C}_G(\omega)$ *and unitary Bessel characters* ρ *that provide a split Bessel model to* Π, *the module* $\widetilde{\Pi}$ *is perfect.*

Proof. If Π is generic and spherical unitary, then it is fully Borel induced $\chi_1 \times \chi_2 \rtimes \chi_\Pi$ from unramified characters χ_1, χ_2, χ_Π with $|\chi_\Pi \chi_i| = \nu^\beta$ for a real $-\frac{1}{2} < \beta < \frac{1}{2}$, see [Roberts and Schmidt 2007, Tables A.2, A.15]. For this case and for every nongeneric Π, Theorem 5.3 implies the statement. □

The rest of this section provides the details for the proof of Theorem 5.3. By Lemma 2.22 it suffices to determine $\pi_0(\widetilde{\Pi})$ and $\kappa(\widetilde{\Pi})$. The T-module $\pi_0(\widetilde{\Pi})$ is uniquely characterized by its monodromy properties and the list of its constituents, see Theorem 4.20 and Lemma 4.22. We distinguish the cases where Π is nongeneric and where Π is generic. In both cases our proof depends on the exact sequence of Lemma 3.11. For this reason we consider the Mellin functors $M_\rho = k_\rho \circ j_!$ and $M^\rho = k^\rho \circ j_!$ that were studied in Section 3.4.

For generic Π the submodule $A = j^!(\overline{\Pi}) \in \mathcal{C}_1$ is perfect of degree one, see Lemma 3.12. It turns out that the Mellin transform M_ρ applied to A catches an essential part of $\widetilde{\Pi}$ through the existence of a natural morphism $M_\rho(A) \to \widetilde{\Pi}$. This is an isomorphism in most cases, but not for certain cases of type IIIa, IVa, VIa with $\rho \in \Delta_-$ where the morphism $M_\rho(\Pi) \to \widetilde{\Pi}$ is not an isomorphism, so these cases have to be discussed separately. Then the key result used is the computation of the Mellin transform in Theorem 3.27. So the main task for the proof in the generic case is the determination of the Mellin transform $M_\rho(A)$. The combinatorial results of Section A.3 single out two series of exceptional cases, the *extraordinary exceptional cases* and the *fully induced nonordinary exceptional cases*, where the Bessel module $\widetilde{\Pi}$ is not perfect.

The case where Π is nongeneric is simpler. It turns out that $\kappa(\widetilde{\Pi})$ always vanishes and in order to show this we verify the conditions of the embedding criterion in Lemma 2.15. Hence our proof starts with the discussion of the nongeneric cases and we prove that $\widetilde{\Pi}$ is always perfect in these cases.

Lemma 5.5. *Suppose χ is a character of* Gl(1) *and $I \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$. An embedding $\mathbb{E}[\chi] \hookrightarrow M_\rho(j^!(\bar{I}))$ gives rise to an embedding $\mathbb{E}[\chi] \hookrightarrow \beta_\rho(I)$.*

Proof. Lemma 3.11 gives a long exact sequence in $\mathcal{C}$:

$$\cdots \to k^\rho i_*(B) \xrightarrow{\delta} M_\rho(A) \to k_\rho(\bar{I}) \to k_\rho i_*(B) \to 0$$

for $A = j^!(\bar{I})$ and $B = i^*(\bar{I}) \in \mathcal{C}_{\mathrm{Gl}(2)}$. The modules $k^\rho i_*(B)$ and $k_\rho i_*(B)$ are finite-dimensional. Hence the embedding $\mathbb{E}[\chi] \to M_\rho \alpha(I)$ remains nontrivial if composed with the morphism $f : M_\rho(\alpha(I)) \to k_\rho(\bar{I}) = \beta_\rho(I)$. The composition is injective by Lemma 2.23. $\square$

Proposition 5.6. *For nongeneric irreducible $\Pi \in \mathcal{C}_G(\omega)$ with Bessel character $\rho \in \Delta_+(\Pi)$ and every $\chi_{\mathrm{norm}} \in \tilde{\Delta}(\Pi)$ there is an embedding $\mathbb{E}[\chi] \hookrightarrow \widetilde{\Pi}$ in $\mathcal{C}$.*

Proof. By a twist we can assume that Π is normalized as in Table 1. The space of Whittaker functionals is isomorphic to $j^! A = 0$ for $A = j^! \overline{\Pi}$, so Lemma 3.1 implies $A = i_* i^*(A)$. Note that either $\chi_{\mathrm{norm}} \in \Delta_0(\Pi)$ or $\chi_{\mathrm{norm}} \in \Delta_1(\Pi)$.

First we consider the case where $\chi_{\mathrm{norm}} \in \Delta_0(\Pi)$, or in other words where $\nu^{1/2}\chi_{\mathrm{norm}} \in \Delta_+(\Pi)$. Since Π is nongeneric, by Theorem 5.1 only the finitely many characters $\rho \in \Delta_+(\Pi)$ define split Bessel models. By Lemma A.9, the involution $\rho \mapsto \rho^* = \omega\rho^{-1}$ acts transitively on $\Delta_+(\Pi) = \Delta_+^*(\Pi)$, so either there is a unique Bessel character $\rho = \rho^*$ or there are two characters ρ, ρ^*. By Lemma 3.17, it is sufficient to fix the Bessel character $\rho = \nu^{1/2}\chi_{\mathrm{norm}}$. The corresponding unnormalized T-character is $\chi = \nu\rho \in \delta_P^{1/2}\Delta_0(\Pi)$. By Lemma 3.8, there is an embedding of T-modules $\chi \hookrightarrow i^* j^!(\overline{\Pi}) = J_P(\Pi)_\psi$ with finite-dimensional cokernel E. Since every character χ_{norm} occurs in $\Delta_0(\Pi)$ at most once (Table 3), E does not contain χ as a constituent. For the exact sequence $0 \to \chi \to i^* A \to E \to 0$ in $\mathcal{C}_T^{\mathrm{fin}}$, Lemma 3.2 yields a long exact sequence of $\mathrm{Gl}_a(1)$-modules:

$$\cdots \to M^\rho(i_*(E)) \to M_\rho(i_*(\chi)) \to M_\rho(i_* i^*(A)) \to M_\rho(i_*(E)) \to 0.$$

In Lemma 3.25 we have shown that $M^\rho(i_*(E)) = 0$ and $M_\rho(i_*(\chi)) \cong \mathbb{E}[\chi]$ for our fixed choice of $\rho = \nu^{-1}\chi$. Since $A = i_* i^* A$ in $\mathcal{C}_1$, as explained above, we obtain an embedding $\mathbb{E}[\chi] \hookrightarrow M_\rho(A)$. By Lemma 5.5 this yields an embedding $\mathbb{E}[\chi] \hookrightarrow k_\rho(\overline{\Pi}) = \widetilde{\Pi}$.

Now we consider the second case where $\chi_{\mathrm{norm}} \in \Delta_1(\Pi)$. This case occurs if and only if Π is type IIb, IVb, Vb, Vc, VId, see Table 3. By Corollary 4.2 there is an irreducible one-dimensional M-module $\sigma_\Pi = \pi \boxtimes \chi_\Pi$ such that Π is a quotient of $I = \mathrm{Ind}_P^G(\sigma_\Pi)$ and $\chi_\Pi = \chi_{\mathrm{norm}}$ as Gl(1)-characters. Then (σ_Π, ρ) is an ordinary pair, so $\tilde{I}$ has degree one by Lemma 4.7 and there is an embedding $\mathbb{E}[\delta_P^{1/2}\chi_\Pi] = \mathbb{E}[\chi] \hookrightarrow \tilde{I}$. By uniqueness of Bessel models (Theorem 5.1), $\widetilde{\Pi}$ has degree one, so the kernel of the projection $\tilde{I} \to \widetilde{\Pi}$ has degree zero. The composition $\mathbb{E}[\chi] \to \widetilde{\Pi}$ is nonzero, so by Lemma 2.23 it is an embedding. $\square$

Corollary 5.7. *For nongeneric irreducible $\Pi \in \mathcal{C}_G^{\mathrm{fin}}(\omega)$ and every smooth character ρ providing a split Bessel model, the Bessel module $\widetilde{\Pi}$ is perfect.*

Proof. The degree of $\widetilde{\Pi}$ is one by Theorem 5.1. By Lemma 2.15, $\widetilde{\Pi}$ is perfect if and only if $\pi_0(\widetilde{\Pi})$ is cyclic and there is an embedding $\mathbb{E}[\chi] \hookrightarrow \widetilde{\Pi}$ for every $\chi_{\mathrm{norm}} \in \tilde{\Delta}(\Pi)$. The first condition is shown in Theorem 4.20 and the embedding is given by Proposition 5.6. □

Proposition 5.8. *Fix an irreducible generic $\Pi \in \mathcal{C}_G(\omega)$ and a Bessel character ρ with $\rho \notin \Delta_+(\Pi)$. Let $X = i^* j^!(\overline{\Pi})$ be the unique cyclic* Gl(1)*-module with constituents given by $\delta_P^{1/2}\Delta_0$. Then there is an isomorphism of* $\mathrm{Gl}_a(1)$*-modules*:

$$\beta_\rho(\Pi) \cong \begin{cases} i_*(\nu^2\rho) \oplus \mathbb{E}[\nu^2\rho], & \text{type VIa and } \rho \in \Delta_-(\Pi),\\ \mathbb{E}[X \to X/\nu^2\rho], & \text{type I, IIa, Va, Xa, XIa and } \rho \in \Delta_-(\Pi),\\ \mathbb{E}[X], & \text{otherwise}. \end{cases}$$

Proof. Fix $A = j^!(\overline{\Pi}) \in \mathcal{C}_1^{\mathrm{fin}}$ and $B = i^*(\overline{\Pi}) \in \mathcal{C}_{\mathrm{Gl}(2)}^{\mathrm{fin}}$ as in Section 3.2. It follows from Lemmas 3.12 and 3.8 that there is an isomorphism $A \cong \mathbb{E}[X]$. Recall that $\beta^\rho(\Pi) = 0$ vanishes by Lemma 4.9, so Lemma 3.11 yields a long exact sequence

$$(*) \qquad 0 \to k^\rho i_*(B) \xrightarrow{\delta} M_\rho(A) \to \beta_\rho(\Pi) \to k_\rho i_*(B) \to 0.$$

We distinguish four cases: If $\rho \notin \Delta_-(\Pi)$ or if Π is not of type IIIa, IVa, VIa, then $k^\rho(B) \cong k_\rho(B) = 0$ vanish by Lemma 4.24, so $(*)$ yields an isomorphism $M_\rho(A) \cong \beta_\rho(\Pi)$. The assertion follows from Theorem 3.27. If Π is of type IIIa with $\rho \in \Delta_-(\Pi)$, the assertion is shown in Lemma 5.9. If Π is of type IVa with $\rho \in \Delta_-(\Pi)$, then there is an isomorphism $\beta_\rho(\Pi) \cong \beta_{\rho^*}(\Pi) \cong \mathbb{E}[X]$ by Lemma 3.17 and because $\rho^* \notin \Delta_-(\Pi)$. Finally, assume Π is of type VIa with $\rho \in \Delta_-(\Pi)$. Lemma 4.24 implies that $(*)$ becomes an exact sequence

$$0 \to i_*(\nu^2\rho) \xrightarrow{\delta} M_\rho(A) \to \widetilde{\Pi} \to i_*(\nu^2\rho) \to 0.$$

It follows from Theorem 3.27 that $M_\rho(A) \cong \mathbb{E}[(\nu^2\rho)^{(2)} \to \nu^2\rho]$, so $\mathrm{coker}(\delta)$ is perfect and isomorphic to $\mathbb{E}[\nu^2\rho]$. That means the maximal finite-dimensional submodule $\kappa(\beta_\rho(\Pi))$ can be at most one-dimensional. It follows from Theorem 4.20 and Lemma 4.22 that $\pi_0(\beta_\rho(\Pi))$ is semisimple and isomorphic to $\nu^2\rho \oplus \nu^2\rho$. Since $\pi_0(\widetilde{\Pi})/\kappa(\widetilde{\Pi})$ is cyclic by Lemma 2.22, $\kappa(\widetilde{\Pi})$ must be at least one-dimensional and thus isomorphic to $i_*(\nu^2\rho)$. The assertion follows from Lemma 2.22. □

Lemma 5.9. *For irreducible $\Pi \in \mathcal{C}_G(\omega)$ of type* IIIa *and for $\rho \in \Delta_-(\Pi)$, the Bessel module $\widetilde{\Pi} = \beta_\rho(\Pi)$ is perfect and isomorphic to $\mathbb{E}[\nu^2\rho \oplus \nu^2\rho^*]$.*

Proof. Recall $\tilde{\Delta}(\Pi) = \Delta_0(\Pi) = \{\nu^{1/2}\rho, \nu^{1/2}\rho^*\}$ by Lemma 4.22 with $\rho \ncong \rho^*$. We claim there is an embedding $\mathbb{E}[\nu^2\rho^*] \hookrightarrow \beta_\rho(\Pi)$. Indeed, $A = j^!(\overline{\Pi})$ is perfect by

Lemma 3.12, so there is a short exact sequence

$$0 \to \mathbb{E}[\nu^2\rho^*] \to A \to i_*(\nu^2\rho) \to 0.$$

Lemma 3.2 yields a long exact sequence

$$\cdots \to M^\rho i_*(\nu^2\rho) \to M_\rho(\mathbb{E}[\nu^2\rho^*]) \to M_\rho(A) \to M_\rho i_*(\nu^2\rho) \to 0.$$

Theorem 3.24 and Lemma 3.25 give a short exact sequence

$$0 \to \mathbb{E}[\nu^2\rho^*] \to M_\rho(A) \to i_*(\nu^2\rho) \to 0.$$

Lemma 5.5 implies the claim. Switching ρ and ρ^* in the above argument yields an embedding $\mathbb{E}[\nu^2\rho] \hookrightarrow \beta_{\rho^*}(\Pi) \cong \beta_\rho(\Pi)$ by Lemma 3.17. Now the Bessel module $\widetilde{\Pi}$ is perfect by Lemma 2.15. Finally, the isomorphism $\beta_\rho(\Pi) \cong \mathbb{E}[\nu^2\rho \oplus \nu^2\rho^*]$ follows from Lemma 2.22. □

5.4. *Bessel functionals.* A *Bessel datum* (Λ, ψ) consists of a nondegenerate character $\psi : N \to \mathbb{C}$ and a character $\Lambda : D \to \mathbb{C}$ where D is the connected component of the centralizer of ψ in M and where Λ coincides with ω on the center Z of G. We only consider *split* Bessel data in the sense that $D \cong k^\times \times k^\times$ is a split torus. Without loss of generality we can assume that ψ factorizes over S, so $D = \widetilde{T}$ and $\Lambda(\tilde{t}) = \rho(t_1)\,\rho^*(t_2)$, compare [Piatetski-Shapiro and Soudry 1981]. For $I \in \mathcal{C}_G^{\text{fin}}(\omega)$, a *Bessel functional* attached to (Λ, ψ) with values in a $\mathbb{C}$-vector space X is a $\mathbb{C}$-linear map

$$\ell : I \to X$$

such that $\ell(I(\tilde{r}s)w) = \Lambda(\tilde{r})\,\psi(s)\,\ell(w)$ for all $w \in I$, all $\tilde{r} \in \widetilde{R}$ and $s \in S$, i.e.,

$$\ell \in \mathrm{Hom}_{S\widetilde{R}}(I, (\psi \boxtimes \Lambda) \otimes_{\mathbb{C}} X) \cong \mathrm{Hom}_{\mathbb{C}}((\tilde{I})_\psi, X)$$

for the TS-module $\tilde{I} = \beta_\rho(I)$. Every such Bessel functional ℓ factorizes over the *universal Bessel functional* $\ell_{\text{univ}} : I \to (\tilde{I})_\psi$ composed with a complex linear map $(\tilde{I})_\psi \to X$. This defines a TS-linear map

$$p_\ell : I \to C_b^\infty(k^\times, X), \quad v \mapsto \varphi_v$$

defined by $\varphi_v(\lambda) = \ell(I(x_\lambda)v)$. The image $\mathcal{K}_\ell(I)$ of p_ℓ is called a *Bessel model* if it is nonzero. For $\ell = \ell_{\text{univ}}$ this defines a nonfaithful functor $\mathcal{K} : \mathcal{C}_G^{\text{fin}}(\omega) \to \mathcal{C}$ sending I to the image of $p_{\ell_{\text{univ}}}$.

Lemma 5.10. *There is an exact sequence of functors* $\mathcal{C}_G^{\text{fin}}(\omega) \to \mathcal{C}$:

$$0 \longrightarrow \kappa \circ \beta_\rho \longrightarrow \beta_\rho \longrightarrow \mathcal{K} \longrightarrow 0.$$

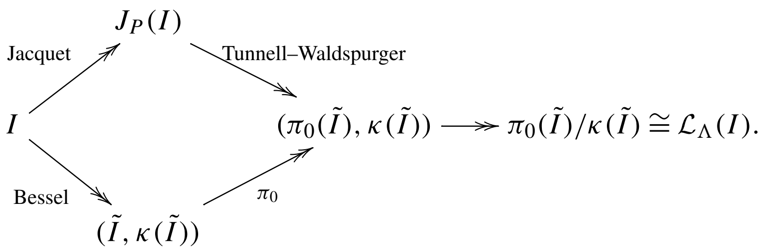
Especially, $\tilde{I}$ *and* $\mathcal{K}(I)$ *have the same degree in* $\mathcal{C}$ *for every* $I \in \mathcal{C}_G^{\text{fin}}(\omega)$.

Proof. p_ℓ for $\ell = \ell_{\text{univ}}$ factorizes over the projection $I \to \tilde{I}$ and defines a unique $\tilde{p}_\ell \in \text{Hom}_{TS}(\tilde{I}, \mathcal{K}(I))$. For $\varphi_v = \tilde{p}_\ell(v)$ and fixed $\lambda \in k^\times$ notice $\varphi_v(\lambda) = 0$ if and only if $v \in \ker(\tilde{I} \to \tilde{I}_{\psi_\lambda})$ for $\psi_\lambda(s_b) = \psi(s_{\lambda b})$. Since ψ_λ for $\lambda \in k^\times$ runs over all nontrivial characters of S, Lemma 2.9 implies that the kernel of the universal map $\tilde{p}_\ell$ is $\kappa(\tilde{I})$, compare [Danişman 2014, Lemmas 4.1, 4.2 and Proposition 4.7]. □

Corollary 5.11. *For* $I \in \mathcal{C}_G^{\text{fin}}(\omega)$ *with* $\deg(\tilde{I}) = 1$, *there are isomorphisms in* $\mathcal{C}$ $\mathcal{L}_\Lambda(I) := \pi_0(\mathcal{K}(I)) \cong \pi_0(\tilde{I})/\kappa(\tilde{I})$ *and* $\mathcal{K}(I) \cong \mathbb{E}[\mathcal{L}_\Lambda(I)]$.

Proof. The TS-module $\mathcal{K}(I)$ in $\mathcal{C}$ is perfect by Lemmas 2.3 and 2.22, compare [Danişman 2014, Lemma 4.1]. □

The L-factor of $\mathcal{K}(I)$ is thus determined by the constituents of $\mathcal{L}_\Lambda(I)$, (see Section 2.5) and these are obtained as follows: Lemma 3.21 yields a commutative diagram in $\mathcal{C}_T$, functorial in $I \in \mathcal{C}_G^{\text{fin}}(\omega)$, where the arrows are the obvious natural transformations. The upper left and the lower right diagonal arrows correspond to exact coinvariant functors. The functors corresponding to the upper right and lower left arrows are right-exact. They are exact in the analogous situation for anisotropic Bessel models.



The object on top of the diagram is the unnormalized Siegel–Jacquet module $J_P(I)$. The T-action on $J_P(I)$ commutes with the action of $\text{Gl}(2) \cong M^1$, but T does not act by a central character. Endowed with the trivial action of S, we view $J_P(I)$ as a $T\tilde{T}S$-module. The object on the bottom is the TS-module $\tilde{I} = \beta_\rho(I)$ with its maximal finite-dimensional submodule $\kappa(\tilde{I})$. In the quotient $\pi_0(\tilde{I}) \cong J_P(I)_{\tilde{T},\rho}$ there is inherited from $\tilde{I}$ the T-submodule $\kappa(\tilde{I}) \subseteq \pi_0(\tilde{I})$ by Lemma A.1(5). This object is not visible in the Jacquet module $J_P(I)$.

5.5. *Regular Poles of Piatetski-Shapiro L-functions.* For infinite-dimensional irreducible representations $\Pi \in \mathcal{C}_G(\omega)$ of $G = \text{GSp}(4)$ and a smooth character μ of $\text{Gl}(1)$, Piatetski-Shapiro and Soudry [Piatetski-Shapiro 1997; Piatetski-Shapiro and Soudry 1981] have defined local L-factors $L^{\text{PS}}(s, \Pi, \mu, \Lambda)$, attached to a choice of a Bessel model. We recall this construction in the case of split Bessel models.

Fix a nonzero split Bessel functional $\ell : \Pi \to \mathbb{C}$, equivariant with respect to a split Bessel datum (Λ, ψ) as in the previous section. Bessel functionals are unique up to scalars by Theorem 5.1. Every $v \in \Pi$ defines a Bessel function $W_v(g) := \ell(\Pi(g)v)$

whose restriction to TS coincides with φ_v. Attached to $v \in \Pi$ and Schwartz–Bruhat functions $\Phi \in \mathcal{S}(V)$ for $V = k^4$ is the zeta-integral

$$Z^{\mathrm{PS}}(s, v, \Phi, \mu, \Lambda) = \int_{\widetilde{N}\backslash H} W_v(h)\Phi((0, 0, 1, 1)h)\,\mu(\lambda_G(h))|\lambda_G(h)|^{s+\frac{1}{2}}\,\mathrm{d}h,$$

where H is the subgroup generated by $T\widetilde{T}$, $\widetilde{N}$ and $J = \left(\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right)$, i.e.,

$$H = \left\{ \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix} \in G \right\} \cong \mathrm{Gl}(2) \times_{\mathrm{Gl}(1)} \mathrm{Gl}(2).$$

The zeta-integral converges for sufficiently large $\mathrm{Re}(s)$ and admits a unique meromorphic continuation to $\mathbb{C}$.

Definition 5.12. The Piatetski-Shapiro L-factor $L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda)$ is the regularization L-factor of $Z^{\mathrm{PS}}(s, v, \Phi, \mu, \Lambda)$ varying over all $v \in \Pi$ and $\Phi \in \mathcal{S}(V)$.

It decomposes as a product

$$L^{\mathrm{PS}}(s, \Pi, \mu, \Lambda) = L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)\, L^{\mathrm{PS}}_{\mathrm{ex}}(s, \Pi, \mu, \Lambda),$$

where the regular part $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$ is the regularizing L-factor of the zeta functions subject to the condition $\Phi(0, 0, 0, 0) = 0$, see [Piatetski-Shapiro 1997]. The exceptional factor $L^{\mathrm{PS}}_{\mathrm{ex}}(s, \Pi, \mu, \Lambda)$ is described in [Weissauer 2023]. In the previous section we defined the perfect TS-module $\mathcal{K}(\Pi) \cong \widetilde{\Pi}/\widetilde{\Pi}^S$. The next propositions show that the L-factor of $\nu^{-3/2}\mu \otimes \mathcal{K}(\Pi)$ divides the regular factor $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$. In the analogous case of anisotropic Bessel models, these L-factors coincide as shown in [Danişman 2014, Proposition 2.5]. In the situation of split Bessel models however, these factors do not coincide due to the presence of so-called subregular poles; for details see [Rösner and Weissauer 2020].

Proposition 5.13. *Fix a smooth character μ of* Gl(1). *For an irreducible $\Pi \in \mathcal{C}_G(\omega)$ with a split Bessel model (Λ, ψ) for a smooth Bessel character $\Lambda = \rho \boxtimes \rho^*$ of $\widetilde{T}$, the L-factor $L(s, \nu^{-3/2}\mu \otimes \mathcal{K}_{\mathrm{univ}}(\Pi))$ equals the regularizing L-factor of the zeta functions $Z^{\mathrm{PS}}(s, v, \Lambda, \Phi, \mu)$ varying over every $v \in \Pi$ and $\Phi \in \mathcal{S}(V)$ subject to the conditions $\Phi(*, 0, *, 0) = 0$ and $\Phi(0, *, 0, *) = 0$.*

Proof. Fix a zeta-function $Z^{\mathrm{PS}}(s, v, \mu, \Phi, \Lambda)$ where $\Phi(x_1, y_1, x_2, y_2)$ vanishes on the hyperplanes $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$. Without loss of generality we can assume that $\mu = 1$, otherwise replace Π with $\mu \otimes \Pi$. Further, we can assume $\Phi(x_1, x_2, y_1, y_2) = \Phi_1(x_1, y_1)\Phi_2(x_2, y_2)$ for certain $\Phi_i \in \mathcal{S}(k^2 \setminus \{0\})$, because these Φ span the tensor product

$$\mathcal{S}(k^2 \setminus \{0\} \times k^2 \setminus \{0\}) \cong \mathcal{S}(k^2 \setminus \{0\}) \otimes \mathcal{S}(k^2 \setminus \{0\}).$$

By Iwasawa decomposition, every $h \in H$ is a product $h = \tilde{n} x_\lambda \tilde{t}(k_1, k_2) \in \widetilde{N} T \widetilde{T} H(\mathfrak{o}_k)$, where $H(\mathfrak{o}_k) = H \cap \mathrm{GSp}(4, \mathfrak{o}_k)$ and $(k_1, k_2) \in H(\mathfrak{o}_k)$ is the image of $k_1, k_2 \in \mathrm{Gl}(2, \mathfrak{o}_k)$ under the isomorphism $(\mathrm{Gl}(2) \times_{\mathrm{Gl}(1)} \mathrm{Gl}(2)) \cong H$. Up to a constant volume factor, the zeta-integral is

$$\begin{aligned} &Z^{\mathrm{PS}}(s, v, \mu, \Phi, \Lambda) \\ &\quad = \int_{\widetilde{N} \backslash H} W_v(h) \Phi((0, 0, 1, 1)h) |\lambda_G(h)|^{s+\frac{1}{2}} \, \mathrm{d}h \\ &\quad = Z^{\mathrm{PS}}_{\mathrm{reg}}(s, W_v) \int_{H(\mathfrak{o}_k)} \int_{\widetilde{T}} \Phi_1^{k_1}((0, t_2)) \Phi_2^{k_2}((0, t_1)) \Lambda(\tilde{t}) |t_1 t_2|^{s+\frac{1}{2}} \, \mathrm{d}\tilde{t} \, \mathrm{d}(k_1, k_2), \end{aligned}$$

with the regular zeta-integral $Z^{\mathrm{PS}}_{\mathrm{reg}}(s, W_v) = \int_T W_v(x_\lambda) |\lambda|^{s-\frac{3}{2}} \, \mathrm{d}x_\lambda$. The action of $H(\mathfrak{o}_k)$ replaces Φ_i by $\Phi_i^{k_i}(g) = \Phi_i(g k_i)$ and thus permutes $\mathcal{S}(k^2 \setminus \{0\})$. For each $k_i \in \mathrm{Gl}(2, \mathfrak{o}_k)$, the function $\Phi_i^{k_i}(0, *)$ has compact support in $k^\times$, therefore the integral over $\widetilde{T}$ is a finite sum. By smoothness of Φ_i, the integral over the compact group $H(\mathfrak{o}_k)$ is also a finite sum. For suitable choices of Φ_i one can arrange that the integral over $H(\mathfrak{o}_k)\widetilde{T}$ is nonzero. Hence for each fixed $v \in \Pi$, the poles of $Z^{\mathrm{PS}}(s, v, \mu, \Phi, \Lambda)$ varying over Φ as above are exactly the poles of the regular zeta-integral $Z^{\mathrm{PS}}_{\mathrm{reg}}(s, W_v)$. By definition, $L\big(s - \frac{3}{2}, \mathcal{K}(\Pi)\big)$ is the regularizing L-factor of the regular zeta integrals varying over $v \in \Pi$. □

Theorem 5.14. *For irreducible $\Pi \in \mathcal{C}_G(\omega)$ with split Bessel model (Λ, ψ) and for every smooth $\mathrm{Gl}(1)$-character μ, the regular part $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$ of the Piatetski-Shapiro L-factor is independent of the choice of the split Bessel model. The regular factor coincides with the product of Tate factors given by Table 4.*

Proof. Proposition 5.13 implies that $L(s, \nu^{-3/2}\mu \otimes \mathcal{K}(\Pi))$ divides the regular factor $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda)$ and thus yields a product

$$L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda) = L\big(s - \tfrac{3}{2}, \mu \otimes \mathcal{K}(\Pi)\big) \, L^{\mathrm{PS}}_{\mathrm{sreg}}(s, \Pi, \mu, \Lambda),$$

where the rightmost term is the subregular factor described in [Rösner and Weissauer 2020]. For the Bessel module $\widetilde{\Pi}$ with its S-invariant subspace $\kappa(\widetilde{\Pi}) = \widetilde{\Pi}^S$, see Theorem 5.3. By Corollary 5.11 this determines

$$L(s, \mu \otimes \mathcal{K}(\Pi)) = \frac{L(s, \mu \otimes \widetilde{\Pi})}{L(s, \mu \otimes \widetilde{\Pi}^S)}. \qquad \square$$

Appendix

A.1. *Functors.* Fix a totally disconnected locally compact group Γ. By definition, Γ is *compactly generated* if it is the union of its compact subgroups.

Lemma A.1. *If Γ is compactly generated, the following assertions hold for every smooth character $\chi : \Gamma \to \mathbb{C}^\times$.*

(1) *The functor* $\mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$ *of* (Γ, χ)*-coinvariants is exact.*

(2) *Smooth extensions of* (Γ, χ)*-eigenmodules are* (Γ, χ)*-eigenmodules.*

(3) $(M/M^{\Gamma,\chi})^{\Gamma,\chi} = 0$ *vanishes for every* $M \in \mathcal{C}_M$.

(4) *The functor of* (Γ, χ)*-invariants is exact if and only if* Γ *is compact.*

(5) *For every* $M \in \mathcal{C}_\Gamma$, *the canonical morphism* $M^{\Gamma,\chi} \to M_{\Gamma,\chi}$ *is injective.*

Proof. The first assertion is well known, see, e.g., [Bernstein and Zelevinskii 1976, 2.35]. The second assertion follows from the first assertion and the five-lemma applied to the projection onto (Γ, χ)-coinvariants. For the third assertion, the functor of (Γ, χ)-invariants applied to the short exact sequence

$$0 \to M^{\Gamma,\chi} \to M \to M/M^{\Gamma,\chi} \to 0$$

yields a long exact sequence

$$0 \to (M^{\Gamma,\chi})^{\Gamma,\chi} \to M^{\Gamma,\chi} \to (M/M^{\Gamma,\chi})^{\Gamma,\chi} \to \mathrm{Ext}^1_\Gamma(\chi, M^{\Gamma,\chi}) \to \dots.$$

By the second assertion $\mathrm{Ext}^1_\Gamma(\chi, M^{\Gamma,\chi}) = 0$ vanishes. Since $(M^{\Gamma,\chi})^{\Gamma,\chi} = M^{\Gamma,\chi}$, the third assertion follows. The fourth assertion is well known. For the last assertion note that by exactness of the coinvariant functor, the natural injection $M^{\Gamma,\chi} \hookrightarrow M$ yields an injection of coinvariants $M^{\Gamma,\chi} = (M^{\Gamma,\chi})_{\Gamma,\chi} \hookrightarrow M_{\Gamma,\chi}$. □

Lemma A.2. *If* Γ *admits a normal subgroup* $\widetilde{\Gamma}$ *isomorphic to* $k^\times$, *then every exact sequence* $0 \to E \to F \to G \to 0$ *in* $\mathcal{C}_\Gamma$ *yields an exact sequence in* $\mathcal{C}_\Gamma$:

$$0 \to E^{\widetilde{\Gamma}} \to F^{\widetilde{\Gamma}} \to G^{\widetilde{\Gamma}} \to E_{\widetilde{\Gamma}} \to F_{\widetilde{\Gamma}} \to G_{\widetilde{\Gamma}} \to 0.$$

Proof. By $k^\times \cong \varpi^{\mathbb{Z}} \times \mathfrak{o}^\times$ and the compactness of $\mathfrak{o}^\times$ this is immediately reduced to the corresponding well-known statement for $\Gamma = \mathbb{Z}$, i.e., the snake lemma for the monodromy operator $\tau = \varpi - \mathrm{id}$. □

Lemma A.3. *Let* Γ *be compactly generated modulo a central subgroup* Z.

(1) *For a representation* $M \in \mathcal{C}_\Gamma(\omega)$ *and a character* $\chi \in \mathcal{C}_\Gamma(\omega)$ *with central character* ω, *the coinvariant functor* $M \mapsto \widetilde{M} = M_{\Gamma,\chi}$ *is exact.*

(2) *The kernel of the natural projection* $M \to \widetilde{M}$ *is the union of the kernels of the pseudoprojectors* $P_{K,\chi} = \int_K \chi^{-1}(k)\, M(k)\, \mathrm{d}k$ *for all compact open subgroups* $K \subseteq \Gamma$.

Lemma A.4. *Fix a product* $R = S \times \widetilde{R}$ *of a totally disconnected group* $\widetilde{R}$ *compactly generated modulo the center* $Z \subseteq \widetilde{R}$ *and the compactly generated group* S. *Fix* $V \in \mathcal{C}_R(\omega)$ *on which* Z *acts by a central character* ω. *Suppose* $\Lambda : \widetilde{T} \to \mathbb{C}^\times$ *is a smooth character with* $\Lambda|_Z = \omega$, *and let* $\widetilde{V} = V_{\widetilde{R},\Lambda} \in \mathcal{C}_S$ *denote the space of coinvariants. Then the natural map*

$$\widetilde{V^S} \hookrightarrow (\widetilde{V})^S$$

is injective. If $\widetilde{R}$ *is compact modulo* Z, *then* $\widetilde{V^S} \cong (\widetilde{V})^S$.

Proof. The exact functor of $(\widetilde{R}, \Lambda)$-coinvariants (Lemma A.1) applied to the exact sequence $0 \to V^S \to V \to V/V^S \to 0$ for $W = V/V^S$ yields an exact sequence

$$0 \to \widetilde{V^S} \to \widetilde{V} \to \widetilde{W} \to 0.$$

The functor of S-invariants gives an injection $\widetilde{V^S} = (\widetilde{V^S})^S \hookrightarrow \widetilde{V}^S$. By Lemma A.1.3 notice $(V/V^S)^S = 0$. If $\widetilde{R}$ is compact modulo Z, then for $W \in C_{\widetilde{R}}(\omega)$ the quotient $\widetilde{W} = W_{\widetilde{R},\Lambda}$ is isomorph to the submodule $W^{\widetilde{R},\Lambda}$ of invariants in $W = V/V^S$, and hence $\widetilde{W}^S \subseteq (V/V^S)^S = 0$. This implies $\widetilde{V^S} \cong \widetilde{V}^S$. □

Remark. For anisotropic Bessel models of irreducible representations Π in $\mathcal{C}_G(\omega)$, the Bessel group $\widetilde{R}$ and $S \cong k$ obviously satisfies the conditions of Lemma A.4 and $\widetilde{R}$ is compact mod Z. Thus for anisotropic Bessel models Lemma A.4 reduces the key assertion $\kappa(\Pi) = 0$ to the assertion $\Pi^S = 0$. The latter is easy to prove with an argument similar to Example 2.16 for Gl(2). For the anisotropic cases this is the underlying argument of [Danişman 2014]. In contrast, for split Bessel models the proof of the assertion $\kappa(\Pi) = 0$ (perfectness) requires the full machinery developed in Sections 4 and 5.

Lemma A.5. *Let $\mathbb{E}[X] \in \mathcal{C}$ be perfect of degree one. Fix an exact sequence in $\mathcal{C}$ and a monomorphism j as in the diagram*

$$\begin{array}{ccccccccc} & & & & \mathbb{E}[X] & & & & \\ & & & & {\scriptstyle j}\downarrow & \searrow{\scriptstyle\varphi} & & & \\ 0 & \longrightarrow & P & \xrightarrow[i]{} & M & \xrightarrow[p]{} & Q & \longrightarrow & 0 \end{array}$$

Then one of the following holds:

(1) *$\varphi = p \circ j$ is injective and i induces an inclusion $P \hookrightarrow M/\mathrm{im}(j)$.*

(2) *φ factorizes over a morphism $\psi : X \to \kappa(Q)$ and there is an injection $\mathbb{E}[\ker(\psi)] \hookrightarrow P$.*

Proof. If φ is injective, the intersection of $\mathrm{im}(j)$ and $\ker(p) = \mathrm{im}(i)$ is trivial. Otherwise, $K = \ker(\varphi)$ has degree one by Lemma 2.23, so $\mathrm{im}(\varphi) \subseteq \kappa(Q)$. Thus φ factorizes over $\psi : X \to \kappa(Q)$. We obtain $K \cong \mathbb{E}[Y]$ for $Y = \ker(\psi)$. By construction K injects into $i(P) = \ker(p)$. □

A.2. *Filtrations.* Fix an admissible representation (π, V) of $M_1 \cong \mathrm{Gl}(2)$ of finite length with central character ω_π and a character χ_Π of the torus $\{t_\lambda \mid \lambda \in k^\times\}$. The irreducible representation $\sigma_\Pi = \pi \boxtimes \chi_\Pi$ of M determines the (normalized) Siegel induced representation $I = \mathrm{Ind}_P^G(\sigma_\Pi)$. Recall that I is isomorphic to the right regular action of G on the space of smooth functions $f : G \to V$ such that

$$f(mng) = \sigma_\Pi(m)\, \delta_P^{1/2}(m) f(g)$$

for $m \in M,\ n \in N$ and $g \in G$ with the modulus character $\delta_P(m_A t_\lambda) = \left|\frac{\det A}{\lambda}\right|^3$. By the Bruhat decomposition, the double coset space $P \setminus G/P$ is represented by the relative Weyl group $W_P = \{id, \boldsymbol{s}_2, \boldsymbol{w}\}$ with the long root $\boldsymbol{w} = \boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2$. This gives rise to a filtration $I^\bullet_{\text{cell}}$ of I by P-modules:

$$0 \subseteq I^2_{\text{cell}} \subseteq I^1_{\text{cell}} \subseteq I^0_{\text{cell}} = I,$$

with quotients $I^i_{\text{cell}}/I^{i+1}_{\text{cell}}$ isomorphic to certain induced representations. These are given explicitly in Lemma 5.2.1 of [Roberts and Schmidt 2016] where $I^\bullet_{\text{cell}}$ is denoted $I^\bullet$. The restriction to $T\widetilde{T}N$ gives rise to a filtration

$$0 \subseteq I_{\geq 3} \subseteq I_{\geq 2} \subseteq I_{\geq 1} \subseteq I_{\geq 0} = I,$$

with quotients $I_i = I_{\geq i}/I_{\geq i+1} \in \mathcal{C}_{T\widetilde{T}N}$ as follows:

(1) $I_3 = I_{\geq 3}$ is the restriction of the P-module I^2_{cell} to $T\widetilde{T}N$. It is isomorphic to the complex vector space $C^\infty_c(k^3, V)$ with the action of $T\widetilde{T}N$ on $f \in C_c(k^3, V)$:

$$\begin{aligned}(s_{a,b,c} f)(x, y, z) &= f(x+a, y+b, z+c),\\ (\tilde{t} f)(x, y, z) &= \pi(\operatorname{diag}(t_2, t_1))\chi_\Pi(t_1t_2) f\left(\frac{xt_2}{t_1}, y, \frac{zt_1}{t_2}\right),\\ (x_\lambda f)(x, y, z) &= |\lambda|^{-3/2}\chi_\Pi(\lambda) f(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-1}z).\end{aligned}$$

(2) The P-module $I^1_{\text{cell}}/I^2_{\text{cell}}$ admits a model as certain subspace W of $C^\infty(k^2, V)$, see [Roberts and Schmidt 2016, p. 517]. The action of $T\widetilde{T}N$ on $f \in W$ is

$$\begin{aligned}(s_{a,b,c} f)(u, w) &= \pi\begin{pmatrix}1 & b+ua\\ 0 & 1\end{pmatrix} f(u, w+au^2+2bu+c),\\ (\tilde{t} f)(u, w) &= \chi_\Pi(t_1t_2)\,\omega_\pi(t_1)|t_1/t_2|^{3/2} f\left(\frac{t_1}{t_2}u, \frac{t_1}{t_2}w\right),\\ (x_\lambda f)(u, w) &= \chi_\Pi(\lambda)\,\pi(\lambda, 1) f(u, \lambda^{-1}w).\end{aligned}$$

This action preserves the subspace $I_2 = C_c(k^\times \times k, V) \subseteq W$.

(3) $I_1 = W/I_2$ is isomorphic to the direct sum $C^\infty_c(k, V) \oplus C^\infty_c(k, V)$. An isomorphism is given by the pair of projections $p = (p_1, p_2)$ where $p_1(f)(w) = f(0, w)$ is evaluation at $u = 0$ and p_2 is the projection defined in [Roberts and Schmidt 2016, (102)]. Since $p_2(f)$ only depends on $f(u, w)$ for large values of u, we obtain an isomorphism. The action of $T\widetilde{T}N$ on $f \in C^\infty_c(k, V) \cong p_1(W/I_2)$ is given by the action on W for $u = 0$:

$$\begin{aligned}(s_{a,b,c} f)(w) &= \pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} f(w+c),\\ (\tilde{t} f)(w) &= \chi_\Pi(t_1t_2)\,\omega_\pi(t_1)|t_1/t_2|^{3/2} f\left(\frac{t_1}{t_2}w\right),\\ (x_\lambda f)(w) &= \chi_\Pi(\lambda)\,\pi(\operatorname{diag}(\lambda, 1)) f(\lambda^{-1}w).\end{aligned}$$

The second factor $p_2(W/I_2)$ is isomorphic to $p_1(W/I_2)$ as a TS-module via conjugation in I with the Weyl element s_1. The action of $\widetilde{T}$ and $\widetilde{N}$ is the same up to interchanging t_1, t_2, and a, c, respectively.

(4) I_0 is the restriction of the P-module $I^0_{\text{cell}}/I^1_{\text{cell}}$ to $T\widetilde{T}N$. It is isomorphic to V with the trivial action of N and $T\widetilde{T}$ acting on $v \in V$ by

$$\tilde{t}v = \chi_\Pi(t_1t_2)\,\pi(\mathrm{diag}(t_1, t_2))\,v, \quad x_\lambda v = |\lambda|^{3/2}\chi_\Pi(\lambda)\,\omega_\pi(\lambda)\,v.$$

$\widetilde{N}$-coinvariants. The functor of $\widetilde{N}$-coinvariants is exact, so the $T\widetilde{T}S$-module $I_{\widetilde{N}}$ admits a filtration with quotients isomorphic to $(I_i)_{\widetilde{N}}$. It is straightforward to show that these are explicitly given by:

(1) $(I_3)_{\widetilde{N}} \cong C_c^\infty(k, V)$ with action on $f \in C_c^\infty(k, V)$ by

$$(s_{0,b,0}f)(y) = f(y+b), \quad (\tilde{t}f)(y) = \pi(\mathrm{diag}(t_2, t_1))\chi_\Pi(t_1t_2)f(y),$$
$$(x_\lambda f)(y) = |\lambda|^{1/2}\chi_\Pi(\lambda)f(\lambda^{-1}y).$$

(2) $(I_2)_{\widetilde{N}} \cong C_c^\infty(k^\times, V_{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)})$ with action on $f \in C_c^\infty(k^\times, V_{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)})$ by

$$(s_{0,b,0}f)(u) = f(u), \quad (\tilde{t}f)(u) = \chi_\Pi(t_1t_2)\,\omega_\pi(t_1)|t_1/t_2|^{1/2} f\left(\frac{t_1}{t_2}u\right),$$
$$(x_\lambda f)(u) = |\lambda|\chi_\Pi(\lambda)\,\pi_{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)}(\lambda, 1)f(u).$$

(3) $(I_1)_{\widetilde{N}} \cong V \oplus V$ with action on $v \in V$ in the first factor by

$$(s_{0,b,0}v) = \pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}v, \quad (\tilde{t}v) = |t_1/t_2|^{1/2}\chi_\Pi(t_1t_2)\,\omega_\pi(t_1)\,v,$$
$$(x_\lambda f)(w) = |\lambda|\chi_\Pi(\lambda)\,\pi(\mathrm{diag}(\lambda, 1))\,v$$

and the same action on the second factor up to interchanging t_1 and t_2.

(4) $(I_0)_{\widetilde{N}} \cong I_0$ with the same action of $T\widetilde{T}S$ as on I_0.

A.3. *Combinatorics.* In this section we study Gl(1)-characters that occur as T-characters in the normalized Siegel–Jacquet module $\mathcal{J} = \delta_P^{1/2} \otimes J_P(\Pi)$ of an irreducible $\Pi \in \mathcal{C}_G(\omega)$. Let

$$\Delta_0(\Pi) \subseteq \tilde{\Delta}(\Pi) \subseteq \Delta(\Pi)$$

denote the finite multisets of Gl(1)-characters by which $x_\lambda \in T$ acts on the irreducible constituents of $\mathcal{J}_{U,\psi}$, of $\mathcal{J}_{\widetilde{T},\rho}$ and of $\mathcal{J}$, respectively, where $\tilde{\Delta}(\Pi)$ of course depends on the choice of the Bessel character ρ. We see in Lemma A.6 that it does not depend on ρ as long as ρ provides a split Bessel model. Let $\Delta_1(\Pi)$ denote the multiset of characters by which x_λ acts on the constituents of $\beta^\rho(\Pi)$ for the same

Bessel character. Proposition 3.19 asserts a union of multisets

$$\Delta_0(\Pi) \sqcup \Delta_1(\Pi) = \tilde{\Delta}(\Pi).$$

Fix the multisets

$$\Delta_\pm(\Pi) = \nu^{\pm 1/2}\Delta_0(\Pi).$$

We have shown in Lemma 4.22 that:

Lemma A.6. *For irreducible $\Pi \in \mathcal{C}_G(\omega)$, normalized as in Table 1, and $\tilde{\Delta}(\Pi)$ defined with respect to a Bessel character that provides a split Bessel model*

$$\tilde{\Delta}(\Pi) = \begin{cases} \{\nu^{1/2}, \nu^{1/2}\}, & \textit{case } \mathsf{VIa}, \\ \{\nu^{-1/2}, \nu^{-1/2}\}, & \textit{case } \mathsf{VId}, \\ \Delta(\Pi), & \textit{otherwise}. \end{cases}$$

For generic Π we have $\tilde{\Delta}(\Pi) = \Delta_0(\Pi)$.

Lemma A.7. *For irreducible representations $\Pi \in \mathcal{C}_G(\omega)$ with nonzero Siegel–Jacquet module the corresponding multisets are listed in Table 3.*

Proof. The constituents of $\Delta(\Pi)$ are given in [Roberts and Schmidt 2007, Table A.3] and $\tilde{\Delta}(\Pi)$ is given by Lemma A.6. The rest is straightforward. □

For the involution $\rho \mapsto \rho^* = \omega\rho^{-1}$ and multisets M as above define

$$M^* = \{\rho^* \mid \rho \in M\}.$$

Attached to every irreducible Π is the group $A(\Pi) = \{\chi \mid \chi \otimes \Pi \cong \Pi\}$ of T-characters that preserve Π under twisting. The involution $\rho \mapsto \rho^*$ generates orbits $\{\rho, \rho^*\}$ of cardinality one or two.

Lemma A.8. *For irreducible Π with $J_P(\Pi) \neq 0$ the group $A(\Pi)$ is trivial except for twists of the cases* Va *and* Vd *where it is $A(\Pi) = \{1, \xi\}$.*

Proof. For every $\chi \in A(\Pi)$ we have $\chi\Delta(\Pi) = \Delta(\Pi)$. This already implies $\chi = 1$ except for the cases Va, Vd where it implies $\chi \in \{1, \xi\}$. Up to semisimplification, the Borel induced representation $\xi \times \xi \rtimes \chi_\Pi$ is invariant under twists with the quadratic character ξ. Its unique essentially square-integrable constituent of type Va must be preserved under this twist. The constituents of type Vb and Vc are ξ-twists of each other, so Vd is also preserved. □

For irreducible representations $\Pi \in \mathcal{C}_G(\omega)$ Tables 1 and 3 imply that:

Lemma A.9. (1) *For generic irreducible Π the intersection $\Delta_+(\Pi) \cap \Delta_+^*(\Pi)$ is empty. For nongeneric irreducible Π we have $\Delta_+(\Pi) = \Delta_+^*(\Pi)$ and the involution $\rho \mapsto \rho^*$ acts transitively on $\Delta_+(\Pi)$.*

(2) *The intersection* $\Delta_-(\Pi)\cap\Delta_+(\Pi)$ *is empty except for twists of*:

type	IIIa	IIIa	IIIb	IIIb
with	$\chi_1=\nu$	$\chi_1=\nu^{-1}$	$\chi_1=\nu$	$\chi_1=\nu^{-1}$
$\Delta_-(\Pi)\cap\Delta_+(\Pi)$	$\{\nu\}$	$\{1\}$	$\{1\}$	$\{\nu^{-1}\}$

(3) *The intersection* $\Delta_-(\Pi)\cap\Delta_-^{*}(\Pi)$ *is empty except for twists of*:

type	IIa	IIIa	IVc	Va	VIa	XIa
$\Delta_-(\Pi)\cap\Delta_-^{*}(\Pi)$	$\{1\}$	$\{1,\chi_1\}$	$\{1\}$	$\{1,\xi\}$	$\{1,1\}$	$\{1\}$

In these cases, the intersection is an orbit with basepoint $\rho=1$ *under the joint action of* $A(\Pi)$ *and the involution* $\rho\mapsto\rho^{*}$.

(4) *If there is* $\rho\in\Delta_-(\Pi)\cap\Delta_+^{*}(\Pi)$ *with* $\rho\notin\Delta_-^{*}(\Pi)\cup\Delta_+(\Pi)$, *then* (Π,ρ) *belongs to a twist of one of the following cases*:

type	I	IIa	X
$\nu^{1/2}\rho\in$	$\{1,\chi_1,\chi_2,\chi_1\chi_2\}$	$\{\chi_1,\chi_1^{-1}\}$	$\{1,\omega_{\pi_c}\}$

(5) *If there exists* $\rho\in\Delta_-^{*}(\Pi)\cap\Delta_+(\Pi)\cap\Delta_-(\Pi)$, *then* (Π,ρ) *is a twist of case* IIIa *with* $(\chi_1,\rho)=(\nu,\nu)$ *or* $(\chi_1,\rho)=(\nu^{-1},1)$.

type	Π	σ_Π	ω	conditions
I	$\chi_1\times\chi_2\rtimes\sigma$	$(\chi_1\times\chi_2)\boxtimes\sigma$	$\chi_1\chi_2\sigma^2$	$\chi_1,\chi_2,\chi_1\chi_2,\chi_1\chi_2^{-1}\neq\nu^{\pm1}$
IIa	$\chi_1\mathrm{St}\rtimes\sigma$	$\mathrm{Sp}(\chi_1)\boxtimes\sigma$	$\chi_1^2\sigma^2$	$\chi_1^2\neq\nu^{\pm1}$ and $\chi_1\neq\nu^{\pm3/2}$
IIb	$\chi_1\mathbf{1}\rtimes\sigma$	$(\chi_1\circ\det)\boxtimes\sigma$	$\chi_1^2\sigma^2$	$\chi_1^2\neq\nu^{\pm1}$ and $\chi_1\neq\nu^{\pm3/2}$
IIIa	$\chi_1\rtimes\sigma\mathrm{St}$	$(\chi_1^{-1}\times\nu^{-1})\boxtimes\nu^{1/2}\chi_1\sigma$	$\chi_1\sigma^2$	$\chi_1\neq1,\nu^{\pm2},\nu^{-1}$
IIIb	$\chi_1\rtimes\sigma\mathbf{1}$	$(\chi_1\times\nu)\boxtimes\nu^{-1/2}\sigma$	$\chi_1\sigma^2$	$\chi_1\neq1,\nu^{\pm2},\nu^{-1}$
IVa	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\mathrm{Sp}(\nu^{-3/2})\boxtimes\nu^{3/2}\sigma$	σ^2	
IVb	$L(\nu^2,\nu^{-1}\sigma\mathrm{St})$	$(\nu^{-3/2}\circ\det)\boxtimes\nu^{3/2}\sigma$	σ^2	
IVc	$L(\nu^{3/2}\mathrm{St},\nu^{-3/2}\sigma)$	$\mathrm{Sp}(\nu^{3/2})\boxtimes\nu^{-3/2}\sigma$	σ^2	
IVd	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$(\nu^{3/2}\circ\det)\boxtimes\nu^{-3/2}\sigma$	σ^2	
Va	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\mathrm{Sp}(\nu^{-1/2}\xi)\boxtimes\nu^{1/2}\sigma$	σ^2	$\xi^2=1\neq\xi$
Vb	$L(\nu^{1/2}\xi\mathrm{St},\nu^{-1/2}\sigma)$	$\mathrm{Sp}(\nu^{1/2}\xi)\boxtimes\nu^{-1/2}\sigma$	σ^2	$\xi^2=1\neq\xi$
Vc	$L(\nu^{1/2}\xi\mathrm{St},\nu^{-1/2}\xi\sigma)$	$\mathrm{Sp}(\nu^{1/2}\xi)\boxtimes\xi\nu^{-1/2}\sigma$	σ^2	$\xi^2=1\neq\xi$
Vd	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$(\nu^{1/2}\xi\circ\det)\boxtimes\nu^{-1/2}\sigma$	σ^2	$\xi^2=1\neq\xi$
VIa	$\tau(S,\nu^{-1/2}\sigma)$	$\mathrm{Sp}(\nu^{-1/2})\boxtimes\nu^{1/2}\sigma$	σ^2	
VIb	$\tau(T,\nu^{-1/2}\sigma)$	$(\nu^{-1/2}\circ\det)\boxtimes\nu^{1/2}\sigma$	σ^2	
VIc	$L(\nu^{1/2}\mathrm{St},\nu^{-1/2}\sigma)$	$\mathrm{Sp}(\nu^{1/2})\boxtimes\nu^{-1/2}\sigma$	σ^2	
VId	$L(\nu,1\rtimes\nu^{-1/2}\sigma)$	$(\nu^{1/2}\circ\det)\boxtimes\nu^{-1/2}\sigma$	σ^2	
X	$\pi_c\rtimes\sigma$	$\pi_c\boxtimes\sigma$	$\omega_{\pi_c}\sigma^2$	$\omega_{\pi_c}\neq\nu^{\pm1}$
XIa	$\delta(\nu^{1/2}\pi_c,\nu^{-1/2}\sigma)$	$\nu^{-1/2}\pi_c\boxtimes\nu^{1/2}\sigma$	σ^2	$\omega_{\pi_c}=1$
XIb	$L(\nu^{1/2}\pi_c,\nu^{-1/2}\sigma)$	$\nu^{1/2}\pi_c\boxtimes\nu^{-1/2}\sigma$	σ^2	$\omega_{\pi_c}=1$

Table 1. Irreducible representations Π of $\mathrm{GSp}(4,k)$ with $J_P(\Pi)\neq0$.

Π	σ_Π	$\ker(I \to \Pi)^{ss}$	$\rho_+(\sigma_\Pi)$	$\rho_-(\sigma_\Pi)$
I	$(\chi_1 \times \chi_2) \boxtimes 1$	0	$\nu^{-1/2}$	$\nu^{1/2}\chi_1\chi_2$
	$(\chi_1^{-1} \times \chi_2) \boxtimes \chi_1$	0	$\nu^{-1/2}\chi_1$	$\nu^{1/2}\chi_2$
	$(\chi_1 \times \chi_2^{-1}) \boxtimes \chi_2$	0	$\nu^{-1/2}\chi_2$	$\nu^{1/2}\chi_1$
	$(\chi_1^{-1} \times \chi_2^{-1}) \boxtimes \chi_1\chi_2$	0	$\nu^{-1/2}\chi_1\chi_2$	$\nu^{1/2}$
IIa	$\mathrm{Sp}(\chi_1) \boxtimes \chi_1^{-1}$	0	$\nu^{-1/2}\chi_1^{-1}$	$\nu^{1/2}\chi_1$
	$\mathrm{Sp}(\chi_1^{-1}) \boxtimes \chi_1$	0	$\nu^{-1/2}\chi_1$	$\nu^{1/2}\chi_1^{-1}$
	$\nu^{-1/2}(\chi_1 \times \chi_1^{-1}) \boxtimes \nu^{1/2}$	IIb	1	1
IIb	$(\chi_1 \circ \det) \boxtimes \chi_1^{-1}$	0	$\nu^{-1/2}\chi_1^{-1}$	$\nu^{1/2}\chi_1$
	$(\chi_1^{-1} \circ \det) \boxtimes \chi_1$	0	$\nu^{-1/2}\chi_1$	$\nu^{1/2}\chi_1^{-1}$
	$\nu^{1/2}(\chi_1 \times \chi_1^{-1}) \boxtimes \nu^{-1/2}$	IIa	ν^{-1}	ν
IIIa	$(\chi_1^{-1} \times \nu^{-1}) \boxtimes \nu^{1/2}\chi_1$	IIIb	χ_1	1
	$(\chi_1 \times \nu^{-1}) \boxtimes \nu^{1/2}$	IIIb	1	χ_1
IIIb	$(\chi_1^{-1} \times \nu) \boxtimes \nu^{-1/2}\chi_1$	IIIa	$\nu^{-1}\chi_1$	ν
	$(\chi_1 \times \nu) \boxtimes \nu^{-1/2}$	IIIa	ν^{-1}	$\nu\chi_1$
IVa	$\mathrm{Sp}(\nu^{-3/2}) \boxtimes \nu^{3/2}$	IVc	ν	ν^{-1}
IVb	$(\nu^{-3/2} \circ \det) \boxtimes \nu^{3/2}$	IVd	ν	ν^{-1}
	$(\nu^2 \times \nu^{-1}) \boxtimes \nu^{-1/2}$	IVa, IVc, IVd	ν^{-1}	ν
IVc	$\mathrm{Sp}(\nu^{3/2}) \boxtimes \nu^{-3/2}$	IVa	ν^{-2}	ν^2
	$(\nu^{-2} \times \nu) \boxtimes \nu^{1/2}$	IVa, IVb, IVd	1	1
IVd	$(\nu^{3/2} \circ \det) \boxtimes \nu^{-3/2}$	IVb	ν^{-2}	ν^2
Va	$\mathrm{Sp}(\nu^{-1/2}\xi) \boxtimes \nu^{1/2}$	Vb	1	1
	$\mathrm{Sp}(\nu^{-1/2}\xi) \boxtimes \xi\nu^{1/2}$	Vc	ξ	ξ
Vb	$\mathrm{Sp}(\nu^{1/2}\xi) \boxtimes \nu^{-1/2}$	Va	ν^{-1}	ν
	$(\nu^{-1/2}\xi \circ \det) \boxtimes \nu^{1/2}\xi$	Vd	ξ	ξ
Vc	$\mathrm{Sp}(\nu^{1/2}\xi) \boxtimes \nu^{-1/2}\xi$	Va	$\nu^{-1}\xi$	$\nu\xi$
	$(\nu^{-1/2}\xi \circ \det) \boxtimes \nu^{1/2}$	Vd	1	1
Vd	$(\nu^{1/2}\xi \circ \det) \boxtimes \nu^{-1/2}\xi$	Vb	$\nu^{-1}\xi$	$\nu\xi$
	$(\nu^{1/2}\xi \circ \det) \boxtimes \nu^{-1/2}$	Vc	ν^{-1}	ν
VIa	$\mathrm{Sp}(\nu^{-1/2}) \boxtimes \nu^{1/2}$	VIc	1	1
VIb	$(\nu^{-1/2} \circ \det) \boxtimes \nu^{1/2}$	VId	1	1
VIc	$\mathrm{Sp}(\nu^{1/2}) \boxtimes \nu^{-1/2}$	VIa	ν^{-1}	ν
VId	$(\nu^{1/2} \circ \det) \boxtimes \nu^{-1/2}$	VIb	ν^{-1}	ν
X	$\pi_c \boxtimes 1$	0	$\nu^{-1/2}$	$\nu^{1/2}\omega_{\pi_c}$
	$\pi_c^\vee \boxtimes \omega_{\pi_c}$	0	$\nu^{-1/2}\omega_{\pi_c}$	$\nu^{1/2}$
XIa	$\nu^{-1/2}\pi_c \boxtimes \nu^{1/2}$	XIb	1	1
XIb	$\nu^{1/2}\pi_c \boxtimes \nu^{-1/2}$	XIa	ν^{-1}	ν

Table 2. Siegel induced representations.

Π	$\Delta(\Pi)$	$\tilde{\Delta}(\Pi)$	$\Delta_0(\Pi)$	$\Delta_1(\Pi)$	$\Delta_+(\Pi)$	ω
I	$\{\sigma, \chi_1\sigma, \chi_2\sigma, \chi_1\chi_2\sigma\}$	$\{\sigma, \chi_1\sigma, \chi_2\sigma, \chi_1\chi_2\sigma\}$	$\{\sigma, \chi_1\sigma, \chi_2\sigma, \chi_1\chi_2\sigma\}$	$\varnothing$	$\nu^{1/2}\Delta_0(\Pi)$	$\chi_1\chi_2\sigma^2$
IIa	$\{\chi_1^2\sigma, \chi_1\sigma\nu^{1/2}, \sigma\}$	$\{\chi_1^2\sigma, \chi_1\sigma\nu^{1/2}, \sigma\}$	$\{\chi_1^2\sigma, \chi_1\sigma\nu^{1/2}, \sigma\}$	$\varnothing$	$\nu^{1/2}\Delta_0(\Pi)$	$\chi_1^2\sigma^2$
IIb	$\{\chi_1^2\sigma, \chi_1\sigma\nu^{-1/2}, \sigma\}$	$\{\chi_1^2\sigma, \chi_1\sigma\nu^{-1/2}, \sigma\}$	$\{\chi_1\sigma\nu^{-1/2}\}$	$\{\chi_1^2\sigma, \sigma\}$	$\{\chi_1\sigma\}$	$\chi_1^2\sigma^2$
IIIa	$\{\sigma\nu^{1/2}, \chi_1\sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}, \chi_1\sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}, \chi_1\sigma\nu^{1/2}\}$	$\varnothing$	$\{\sigma\nu, \chi_1\sigma\nu\}$	$\chi_1\sigma^2$
IIIb	$\{\sigma\nu^{-1/2}, \chi_1\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}, \chi_1\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}, \chi_1\sigma\nu^{-1/2}\}$	$\varnothing$	$\{\sigma, \chi_1\sigma\}$	$\chi_1\sigma^2$
IVa	$\{\sigma\nu^{3/2}\}$	$\{\sigma\nu^{3/2}\}$	$\{\sigma\nu^{3/2}\}$	$\varnothing$	$\{\sigma\nu^2\}$	σ^2
IVb	$\{\sigma\nu^{3/2}, \sigma\nu^{-1/2}\}$	$\{\sigma\nu^{3/2}, \sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{3/2}\}$	$\{\sigma\}$	σ^2
IVc	$\{\sigma\nu^{-3/2}, \sigma\nu^{1/2}\}$	$\{\sigma\nu^{-3/2}, \sigma\nu^{1/2}\}$	$\{\sigma\nu^{-3/2}, \sigma\nu^{1/2}\}$	$\varnothing$	$\{\sigma\nu, \sigma\nu^{-1}\}$	σ^2
IVd	$\{\sigma\nu^{-3/2}\}$		$\varnothing$		$\varnothing$	σ^2
Va	$\{\sigma\nu^{1/2}, \xi\sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}, \xi\sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}, \xi\sigma\nu^{1/2}\}$	$\varnothing$	$\{\sigma\nu, \xi\sigma\nu\}$	σ^2
Vb	$\{\sigma\nu^{-1/2}, \xi\sigma\nu^{1/2}\}$	$\{\sigma\nu^{-1/2}, \xi\sigma\nu^{1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\{\xi\sigma\nu^{1/2}\}$	$\{\sigma\}$	σ^2
Vd	$\{\sigma\nu^{-1/2}, \xi\sigma\nu^{-1/2}\}$		$\varnothing$		$\varnothing$	σ^2
VIa	$\{\sigma\nu^{1/2}, \sigma\nu^{1/2}, \sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}, \sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}, \sigma\nu^{1/2}\}$	$\varnothing$	$\{\sigma\nu, \sigma\nu\}$	σ^2
VIb	$\{\sigma\nu^{1/2}\}$		$\varnothing$		$\varnothing$	σ^2
VIc	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\varnothing$	$\{\sigma\}$	σ^2
VId	$\{\sigma\nu^{-1/2}, \sigma\nu^{-1/2}, \sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}, \sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\}$	σ^2
X	$\{\sigma, \omega_{\pi_c}\sigma\}$	$\{\sigma, \omega_{\pi_c}\sigma\}$	$\{\sigma, \omega_{\pi_c}\sigma\}$	$\varnothing$	$\{\sigma\nu^{1/2}, \omega_{\pi_c}\sigma\nu^{1/2}\}$	$\omega_{\pi_c}\sigma^2$
XIa	$\{\sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}\}$	$\{\sigma\nu^{1/2}\}$	$\varnothing$	$\{\sigma\nu\}$	σ^2
XIb	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\{\sigma\nu^{-1/2}\}$	$\varnothing$	$\{\sigma\}$	σ^2

Table 3. Multisets.

Proposition A.10. *Fix a generic irreducible* Π. *In each orbit* $\{\rho, \rho^*\}$ *we can choose* ρ *such that exactly one of the following assertions holds*:

(1) ρ *is not in* $\Delta_+(\Pi) \cup \Delta_-(\Pi)$.

(2) ρ *is in* $\Delta_+^*(\Pi) \cap \Delta_-(\Pi)$ *and not in* $\Delta_-^*(\Pi) \cup \Delta_+(\Pi)$.

(3) ρ *is in* $\Delta_-(\Pi) \cap \Delta_-^*(\Pi)$ *and not in* $\Delta_+(\Pi)$.

Especially, $\rho \notin \Delta_+(\Pi)$ *holds in every case.*

Proof. Assume that assertion (1) cannot be satisfied, so ρ is both in $\Delta_+(\Pi) \cup \Delta_-(\Pi)$ and in $\Delta_+^*(\Pi) \cup \Delta_-^*(\Pi)$. If ρ is not in $\Delta_-(\Pi) \cap \Delta_-^*(\Pi)$, then ρ is in $\Delta_+(\Pi)$ or in $\Delta_+^*(\Pi)$. By choice of ρ we can assume that $\rho \in \Delta_+^*(\Pi)$. Lemma A.9 assures $\rho \notin \Delta_+(\Pi)$, and hence $\rho \in \Delta_-(\Pi)$. Especially, $\rho \notin \Delta_-^*(\Pi) \cup \Delta_+(\Pi)$, so (2) holds. Conversely, if ρ is in $\Delta_-(\Pi) \cap \Delta_-^*(\Pi)$, then the same holds for ρ^*. By possibly replacing ρ by ρ^* we can assume $\rho \notin \Delta_+(\Pi)$ by Lemma A.9, so (3) holds. ☐

We distinguish three series of exceptional cases for the pairs (Π, ρ).

Corollary A.11. *Suppose* $\Pi \in \mathcal{C}_G(\omega)$ *is generic and irreducible*, $J_P(\Pi) \neq 0$ *and* Π *is normalized as in Table 1. In each orbit* $\{\rho, \rho^*\}$ *one can choose* ρ *such that* $\rho \notin \Delta_+(\Pi)$ *and exactly one of the following assertions holds*:

(1) (Π, ρ) *belongs to the* ***nonexceptional*** *cases where* $\rho \notin \Delta_+(\Pi) \cup \Delta_-(\Pi)$.

(2) (Π, ρ) *belongs to the* ***fully induced nonordinary exceptional*** *cases*
 I *and* ρ *is in* $\{\nu^{-1/2}, \nu^{-1/2}\chi_1, \nu^{-1/2}\chi_2, \nu^{-1/2}\chi_1\chi_2\}$,
 IIa *and* ρ *is in* $\{\nu^{-1/2}\chi_1, \nu^{-1/2}\chi_1^{-1}\}$,
 X *and* ρ *is in* $\{\nu^{-1/2}, \nu^{-1/2}\omega_{\pi_c}\}$,

(3) (Π, ρ) *belongs to the* ***extraordinary exceptional*** *cases* IIa, Va, VIa, XIa *with* $\rho = \rho^* \in \Delta_-(\Pi)$.

(4) (Π, ρ) *belongs to the* ***exceptional*** *cases* IIIa *where* Π *is of type* IIIa *and* $\rho \in \Delta_-(\Pi) \cap \Delta_-^*(\Pi) = \{1, \chi_1\}$.

Proof. Use Proposition A.10. Assertion (2) is implied by Lemma A.9. Assertions (3) and (4) correspond to assertion (3) of Proposition A.10 by Lemma A.9. ☐

type	$\Pi \in \mathcal{C}_{\mathrm{GSp}(4)}(\omega)$	ρ	$L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, 1, \Lambda)$, $\Lambda = \rho \boxtimes \rho^*$ split
I	$\chi_1 \times \chi_2 \rtimes \sigma$	all	$L(s,\sigma)L(s,\chi_1\sigma)L(s,\chi_2\sigma)L(s,\chi_1\chi_2\sigma)$
IIa	$\chi\mathrm{St} \rtimes \sigma$	all	$L(s,\sigma)L(s,\chi^2\sigma)L(s,\nu^{1/2}\chi\sigma)$
IIb	$\chi\mathbf{1} \rtimes \sigma$	$\chi\sigma$	$L(s,\sigma)L(s,\chi^2\sigma)L(s,\nu^{-1/2}\chi\sigma)$
IIIa	$\chi \rtimes \sigma\mathrm{St}$	all	$L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{1/2}\sigma)$
IIIb	$\chi \rtimes \sigma\mathbf{1}$	σ, $\chi\sigma$	$L(s,\nu^{-1/2}\chi\sigma)L(s,\nu^{-1/2}\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{1/2}\sigma)$
IVa	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	all	$L(s,\nu^{3/2}\sigma)$
IVb	$L(\nu^2, \nu^{-1}\sigma\mathrm{St})$	σ	$L(s,\nu^{3/2}\sigma)L(s,\nu^{-1/2}\sigma)$
IVc	$L(\nu^{3/2}\mathrm{St}, \nu^{-3/2}\sigma)$	$\nu^{\pm1}\sigma$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{-3/2}\sigma)L(s,\nu^{3/2}\sigma)$
IVd	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	none	—
Va	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	all	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)$
Vb	$L(\nu^{1/2}\xi\mathrm{St}, \nu^{-1/2}\sigma)$	σ	$L(s,\nu^{-1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)$
Vc	$L(\nu^{1/2}\xi\mathrm{St}, \nu^{-1/2}\xi\sigma)$	$\xi\sigma$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\xi\sigma)$
Vd	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	none	—
VIa	$\tau(S, \nu^{-1/2}\sigma)$	all	$L(s,\nu^{1/2}\sigma)^2$
VIb	$\tau(T, \nu^{-1/2}\sigma)$	none	—
VIc	$L(\nu^{1/2}\mathrm{St}, \nu^{-1/2}\sigma)$	σ	$L(s,\nu^{-1/2}\sigma)L(s,\nu^{1/2}\sigma)$
VId	$L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$	σ	$L(s,\nu^{-1/2}\sigma)^2L(s,\nu^{1/2}\sigma)$
VII	$\chi \rtimes \pi$	all	1
VIIIa	$\tau(S, \pi)$	all	1
VIIIb	$\tau(T, \pi)$	none	—
IXa	$\delta(\nu\xi, \nu^{-1/2}\pi)$	all	1
IXb	$L(\nu\xi, \nu^{-1/2}\pi)$	none	—
X	$\pi \rtimes \sigma$	all	$L(s,\sigma)L(s,\omega_\pi\sigma)$
XIa	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	all	$L(s,\nu^{1/2}\sigma)$
XIb	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	σ	$L(s,\nu^{-1/2}\sigma)$
	cuspidal generic	all	1
	cuspidal nongeneric	none	—

Table 4. Spinor L-factors (regular part).

Recall that for an irreducible M-module $\sigma_\Pi = \pi \boxtimes \chi_\Pi$ as in Section 4 we have defined $\rho_+(\sigma_\Pi) = \nu^{-1/2}\chi_\Pi$ and $\rho_-(\sigma_\Pi) = \rho_+^*(\sigma_\Pi)$.

Lemma A.12. *Fix an infinite-dimensional irreducible $\sigma_\Pi \in \mathcal{C}_M(\omega)$ such that $\sigma_\Pi \ncong \omega \otimes \sigma_\Pi^\vee$ and that there is an exact sequence*

$$0 \to \Xi \to \mathrm{Ind}_P^G(\sigma_\Pi) \to \Pi \to 0,$$

with generic irreducible quotient Π and irreducible submodule Ξ. Then we have $\Delta_+(\Xi) = \{\rho_+(\sigma_\Pi), \rho_-(\sigma_\Pi)\}$ as sets, i.e., without counting multiplicity, and this is disjoint to $\{\rho_+(\omega \otimes \sigma_\Pi^\vee), \rho_-(\omega \otimes \sigma_\Pi^\vee)\}$.

Proof. See Tables 1, 2, 3. □

For irreducible $\Pi \in \mathcal{C}_G(\omega)$ and Bessel characters that provide a split Bessel model for Π, the multisets $\tilde{\Delta}(\Pi)$ and $\Delta_1(\Pi)$ are independent of the specific choice of the Bessel character by Lemma A.6.

Lemma A.13. *If $\sigma_\Pi \in \mathcal{C}_M(\omega)$ is one-dimensional with an exact sequence*

$$0 \to \Xi \to \mathrm{Ind}_P^G(\sigma_\Pi) \to \Pi \to 0,$$

where the irreducible quotient Π has a split Bessel model, then $\Delta_1(\Pi) \cap \Delta(\Xi)$ is empty.

Proof. By a twist we can assume that σ_Π is normalized as in Table 1. The constituents Π and Ξ are then given by Theorem 4.1 and Table 2. The statement follows from Table 3. □

A.4. *Tables.* Shown in Table 1 are the irreducible representations $\Pi \in \mathcal{C}_{\mathrm{GSp}(4)}(\omega)$ with nonzero Siegel–Jacquet module in the notation of [Sally and Tadić 1993] and [Roberts and Schmidt 2007]. By definition, the parameters $\chi_1, \chi_2, \sigma, \xi, \pi_c$ are subject to the conditions in the right column. For each Π we fix an irreducible $\sigma_\Pi \in \mathcal{C}_M(\omega)$ such that Π is the unique quotient of $\mathrm{Ind}_P^G(\sigma_\Pi)$. To be precise, π_c is a cuspidal irreducible smooth Gl(2)-module and $\mathrm{Sp}(\chi)$ is the unique quotient of $\nu^{-1/2}\chi \times \nu^{1/2}\chi \in \mathcal{C}_{\mathrm{Gl}(2)}$ for a Gl(1)-character χ. In Sections 4, A.3 and in Table 2, we frequently normalize Π by setting $\sigma = 1$ except for cases IIa, IIb where we set $\sigma = \chi_1^{-1}$.

For each irreducible $\Pi \in \mathcal{C}_G$ with $J_P(\Pi) \neq 0$, normalized as in Table 1, shown in Table 2 are the irreducible $\sigma_\Pi = \pi \boxtimes \chi_\Pi \in \mathcal{C}_M(\omega)$ such that Π is a quotient of the Siegel induced representation $I = \mathrm{Ind}_P^G(\sigma_\Pi)$. The third column lists the constituents of the kernel $\ker(I \to \Pi)$. The characters ρ_+ and ρ_- are defined in Section 4.1.

For irreducible $\Pi \in \mathcal{C}_G(\omega)$, Table 3 shows the multisets defined in Section A.3. The columns $\tilde{\Delta}(\Pi)$ and $\Delta_1(\Pi)$ refer only to the case where the Bessel character defines a split Bessel model for Π. There is no split Bessel model for blank cases.

type	Π	ρ	$\widetilde{\Pi} = \beta_\rho(\Pi)$	$\kappa = \widetilde{\Pi}^S$	$\beta^\rho(\Pi)$
I	$\chi_1 \times \chi_2 \rtimes \sigma$	$\rho \in \Delta_-(\Pi)$	$\mathbb{E}[X \to X/\kappa]$	$\nu^2\rho$	0
		$\rho \in \Delta_-^*(\Pi)$	$\mathbb{E}[X \to X/\kappa]$	$\nu^2\rho^*$	0
		every other ρ	$\mathbb{E}[X]$	0	0
IIa	$\chi \mathrm{St}_{\mathrm{Gl}(2)} \rtimes \sigma$	$\rho \in \Delta_-(\Pi)$	$\mathbb{E}[X \to X/\kappa]$	$\nu^2\rho$	0
		$\rho \in \Delta_-^*(\Pi)$	$\mathbb{E}[X \to X/\kappa]$	$\nu^2\rho^*$	0
		every other ρ	$\mathbb{E}[X]$	0	0
IIb	$\chi \mathbf{1}_{\mathrm{Gl}(2)} \rtimes \sigma$	$\rho = \chi\sigma$	$\mathbb{E}[X]$	0	$\mathbb{E}[X/\nu\chi\sigma]$
		$\rho \neq \chi\sigma$	$i_*(\nu\chi\sigma)$	$\nu\chi\sigma$	0
IIIa	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	all ρ	$\mathbb{E}[\nu^2\sigma \oplus \chi\nu^2\sigma]$	0	0
IIIb	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	$\rho \in \{\sigma, \chi\sigma\}$	$\mathbb{E}[\nu\sigma \oplus \chi_1\nu\sigma]$	0	$\mathbb{S}$
		$\rho \neq \sigma, \chi\sigma$	$i_*(\nu\sigma \oplus \chi_1\nu\sigma)$	$\nu\sigma \oplus \chi_1\nu\sigma$	0
IVa	$\sigma \mathrm{St}_G$	all ρ	$\mathbb{E}[\nu^3\sigma]$	0	0
IVb	$L(\nu^2, \nu^{-1}\mathrm{St})$	$\rho = \sigma$	$\mathbb{E}[\nu^3\sigma \oplus \nu\sigma]$	0	$\mathbb{E}[\nu^3\sigma]$
		$\rho \neq \sigma$	$i_*(\nu\sigma)$	$\nu\sigma$	0
IVc	$L(\nu^{3/2}\mathrm{St}, \nu^{-3/2}\sigma)$	$\rho = \nu^{\pm 1}\sigma$	$\mathbb{E}[\nu^2\sigma \oplus \sigma]$	0	$\mathbb{S}$
		$\rho \neq \nu^{\pm 1}\sigma$	$i_*(\nu^2\sigma \oplus \sigma)$	$\nu^2\sigma \oplus \sigma$	0
IVd	$\sigma \mathbf{1}_G$	$\rho = \sigma$	$i_*(\sigma)$	σ	$i_*(\sigma)$
		$\rho \neq \sigma$	0	0	0
Va	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\rho \in \{\sigma, \xi\sigma\}$	$i_*(\nu^2\rho) \oplus \mathbb{E}[\nu^2\xi\rho]$	$\nu^2\rho$	0
		$\rho \neq \sigma, \xi\sigma$	$\mathbb{E}[\nu^2\sigma \oplus \nu^2\xi\sigma]$	0	0
Vb	$L(\nu^{1/2}\xi\mathrm{St}, \nu^{-1/2}\sigma)$	$\rho = \sigma$	$\mathbb{E}[\nu\sigma \oplus \nu^2\xi\sigma]$	0	$\mathbb{E}[\nu^2\xi\sigma]$
		$\rho \neq \sigma$	$i_*(\nu\sigma)$	$\nu\sigma$	0
Vc	$L(\nu^{1/2}\xi\mathrm{St}, \nu^{-1/2}\xi\sigma)$	$\rho = \xi\sigma$	$\mathbb{E}[\nu^2\sigma \oplus \nu\xi\sigma]$	0	$\mathbb{E}[\nu^2\sigma]$
		$\rho \neq \xi\sigma$	$i_*(\nu\xi\sigma)$	$\nu\xi\sigma$	0
Vd	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$\rho \in \{\sigma, \xi\sigma\}$	$i_*(\nu\xi\rho)$	$\nu\xi\rho$	$i_*(\nu\xi\rho)$
		$\rho \neq \sigma, \xi\sigma$	0	0	0
VIa	$\tau(S, \nu^{-1/2}\sigma)$	$\rho = \sigma$	$i_*(\nu^2\sigma) \oplus \mathbb{E}[\nu^2\sigma]$	$\nu^2\rho$	0
		$\rho \neq \sigma$	$\mathbb{E}[(\nu^2\sigma)^{(2)}]$	0	0
VIb	$\tau(T, \nu^{-1/2}\sigma)$	$\rho = \sigma$	$i_*(\nu^2\sigma)$	$\nu^2\sigma$	$i_*(\nu^2\sigma)$
		$\rho \neq \sigma$	0	0	0
VIc	$L(\nu^{1/2}\mathrm{St}, \nu^{-1/2}\sigma)$	$\rho = \sigma$	$\mathbb{E}[\nu\sigma]$	0	$\mathbb{S}$
		$\rho \neq \sigma$	$i_*(\nu\sigma)$	$\nu\sigma$	0
VId	$L(\nu, 1 \rtimes \nu^{-1/2}\sigma)$	$\rho = \sigma$	$\mathbb{E}[(\nu\sigma)^{(2)}]$	0	$\mathbb{E}[\nu\sigma]$
		$\rho \neq \sigma$	$i_*(\nu\sigma)$	$\nu\sigma$	0
VII	$\chi \rtimes \pi$	all ρ	$\mathbb{S}$	0	0
VIIIa	$\tau(S, \pi)$	all ρ	$\mathbb{S}$	0	0
VIIIb	$\tau(T, \pi)$	all ρ	0	0	0
IXa	$\delta(\nu\xi, \nu^{-1/2}\pi)$	all ρ	$\mathbb{S}$	0	0
IXb	$L(\nu\xi, \nu^{-1/2}\pi)$	all ρ	0	0	0
X	$\pi \rtimes \sigma$	$\rho \in \Delta_-(\Pi)$	$\mathbb{E}[X \to X/\kappa]$	$\nu^2\rho$	0
		$\rho \in \Delta_-^*(\Pi)$	$\mathbb{E}[X \to X/\kappa]$	$\nu^2\rho^*$	0
		every other ρ	$\mathbb{E}[X]$	0	0
XIa	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\rho = \sigma$	$\mathbb{S} \oplus i_*(\nu^2\sigma)$	$\nu^2\rho$	0
		$\rho \neq \sigma$	$\mathbb{E}[\nu^2\sigma]$	0	0
XIb	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\rho = \sigma$	$\mathbb{E}[\nu\sigma]$	0	$\mathbb{S}$
		$\rho \neq \sigma$	$i_*(\nu\sigma)$	$\nu\sigma$	0
	cuspidal generic	all ρ	$\mathbb{S}$	0	0
	cuspidal nongeneric	all ρ	0	0	0

Table 5. Bessel modules.

For every irreducible smooth representation Π of GSp(4) over a local nonarchimedean field k, the third column in Table 4 lists the smooth characters ρ of $k^\times$ such that the character $\Lambda = \rho \boxtimes \omega\rho^{-1}$ of $\widetilde{T}$ yields a split Bessel model for Π, see Theorem 5.1. The last column gives the regular part of Piatetski-Shapiro's spinor L-factor attached to this split Bessel model. The notation follows [Sally and Tadić 1993] and [Roberts and Schmidt 2007]. For typographical reasons we set $\mu = 1$; the general case follows from the identity $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \Pi, \mu, \Lambda) = L^{\mathrm{PS}}_{\mathrm{reg}}(s, \mu \otimes \Pi, 1, \Lambda)$.

For irreducible smooth representations Π of GSp(4) and smooth Bessel characters ρ, Table 5 shows the Bessel modules $\widetilde{\Pi} = \beta_\rho(\Pi) \in \mathcal{C}_{TS}$ and $\beta^\rho(\Pi)$ and the S-invariant T-submodule $\kappa = \widetilde{\Pi}^S$ of $\widetilde{\Pi}$, see Theorem 5.3 and Proposition 5.2. Here X denotes the unique cyclic T-module with constituents in $\delta_P^{1/2}\tilde{\Delta}(\Pi)$. The notation follows [Sally and Tadić 1993] and [Roberts and Schmidt 2007].

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MIRKO RÖSNER
MATHEMATISCHES INSTITUT
RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG
HEIDELBERG
GERMANY
mroesner@mathi.uni-heidelberg.de

RAINER WEISSAUER
MATHEMATISCHES INSTITUT
RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG
HEIDELBERG
GERMANY
weissauer@mathi.uni-heidelberg.de

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