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If a reduced bivariate polynomial is quasihomogeneous, then its discriminant is a monomial. Over fields of characteristic 0, we show that if one adds another simple condition, this becomes an equivalence. We also give a third equivalent condition that is stated geometrically.

1. Introduction

Recall that a bivariate polynomial $f \in \mathbb{C}[x, y]$ is called *quasihomogeneous* if there are integers w, α, β with α, β not both equal to 0 such that every nonzero monomial $c_{ij}x^i y^j$ of f satisfies

$$w = \alpha i + \beta j.$$

In other words, f is quasihomogeneous if the nodes of its Newton polytope [2] lie on a line. One calls α and β the *weights* of x and y, respectively, and $(w; \alpha, \beta)$ the type of f. Note that we do admit negative weights, so, for example, the polynomial $1 + xy + x^2y^2$ is quasihomogeneous of type $(0; 1, -1)$. Be aware that quasihomogeneity with strictly positive weights is usually called weighted homogeneity; this has also other equivalent characterizations; see, e.g., [10].

The purpose of this note is to provide, for reduced bivariate polynomials f, a characterization of quasihomogeneity which is local geometric, in the sense that it suffices to verify given conditions at every point of the Zariski closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the variety defined by f. Here and also in the remainder of this note, $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is the one-dimensional projective space over $\mathbb{C}$, and we more generally identify varieties with their $\mathbb{C}$-valued points. Varieties do not need to be irreducible.

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Our main result also states yet another equivalent condition about f in terms of the discriminant of f with respect to one of the variables; this condition is an explicit algebraic reformulation of the geometric characterization.

Before stating the main result, let us recall the definition of discriminant (see, for example, [3, Chapter 12]).

Definition 1.1. Let $f(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x]$ be a polynomial. Set $n := \deg_y f$ and let $f_n \in \mathbb{C}[x]$ be the y^n-coefficient of f, considering the latter as a polynomial in y. The *discriminant* of f with respect to y is the polynomial $\mathrm{Disc}_y(f) \in \mathbb{C}[x]$ such that for every $a \in \mathbb{C}$,

$$\mathrm{Disc}_y(f)(a) := f_n^{2n-2}(a) \prod_{i<j} (b_i - b_j)^2,$$

where the b_i are the roots of $f(a, y)$. (For $n = 1$, one sets $\mathrm{Disc}_y(f) = 1$.)

If $V \subset \mathbb{C}^2$ is the variety defined by a polynomial $f = \sum_{i \le m, j \le n} a_{ij} x^i y^j \in \mathbb{C}[x, y]$, where $m = \deg_x f$ and $n = \deg_y f$, then its Zariski closure in $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by $\hat{f}(x, \tilde{x}, y, \tilde{y}) = \sum a_{ij} x^i \tilde{x}^{m-i} y^j \tilde{y}^{n-j}$. We call $\hat{f}$ the *multihomogenization of f* (see later Definition-Notation 2.5).

Here is the precise formulation of our main result.

Theorem 1.2. *Let $f(x, y) \in \mathbb{C}[x, y]$ be a complex bivariate polynomial written as $\sum_{i=0}^n f_i(x) y^i$ where $f_i \in \mathbb{C}[x]$, and f_0, f_n are nonzero polynomials. Suppose that f is reduced, i.e., no irreducible factor appears multiple times. We write $\hat{f} \in \mathbb{C}[x, \tilde{x}, y, \tilde{y}]$ for the multihomogenization of f. Then the following are equivalent:*

(A) *f is quasihomogeneous where the weight of x is nonzero.*

(B) *f_0, f_n and $\mathrm{Disc}_y(f)$ are (nonzero) monomials (in x).*

(C) *The subvarieties of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $\hat{f}$ and $y\hat{f}_y$ have no common point within $\mathbb{C}^\times \times \mathbb{P}^1$ (where $\hat{f}_y$ denotes the derivative of $\hat{f}$ with respect to the variable y).*

Note that our result is not symmetric in the variables x and y, and to be more precise, what we characterize is quasihomogeneity with the additional condition that the weight α of x is nonzero. To obtain a full characterization, one could combine our condition with the variant interchanging x and y, while the case $\alpha = 0$ is anyway simple to characterize.

Implications (A) $\Rightarrow$ (B) and (B) $\Leftrightarrow$ (C) are easy to show. (The proofs of (A) $\Rightarrow$ (B) $\Rightarrow$ (C) are given at the beginning of Section 3.) What is surprising is (C) $\Rightarrow$ (A) (respectively, (B) $\Rightarrow$ (A)), which is our main result.

While we have stated and first prove the result over $\mathbb{C}$, the conditions permit an easy generalization to polynomials over arbitrary fields of characteristic 0. The argument is given in Section 4, where we also show how to deduce a similar result about geometrically reduced polynomials over fields of sufficiently big positive characteristic (see Theorem 4.1). That argument for large positive characteristic requires some basic knowledge of model theory; for the convenience of the reader, we provide an informal introduction to this basic knowledge.

Our result is somewhat related to Bernstein–Kouchnirenko's theorem [5] (see also [2, Theorem 1] for a newer exposition), which expresses the number of common zeros in $(\mathbb{C}^\times)^2$ of two polynomials f, g under a certain genericity condition in terms of the mixed volume of their Newton polytopes. Indeed, suppose that (C) holds, so that $\hat{f}$ and $y\hat{f}_y$ have no common zero in $\mathbb{C}^\times \times \mathbb{P}^1$. This implies that f and $g := yf_y$ have no common zero in $(\mathbb{C}^\times)^2$. Provided f and yf_y satisfy the genericity condition, Bernstein–Kouchnirenko's theorem implies that the mixed volume of the Newton polytopes of f and yf_y is 0. Therefore, the nodes of the Newton polytope of f must lie on a line, that is, f is quasihomogeneous. While one can use this approach to prove Theorem 1.2 under an additional genericity assumption on f, for general f the polynomials f and yf_y need not satisfy the genericity condition needed by Bernstein–Kouchnirenko's theorem. The main point of our result is that the conclusion nevertheless holds without further assumptions on f and yf_y, whereas the genericity assumption in Bernstein–Kouchnirenko's theorem cannot simply be removed.

The idea of the proof of (C) ⇒ (A) is the following: Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the Zariski closure of the variety defined by f, and on X, consider the function

$$h\colon X \to \mathbb{P}^1, \quad (x, y) \mapsto \frac{yf_y(x, y)}{xf_x(x, y)}.$$

(It might not be well defined everywhere, but for the sake of this sketch, let us pretend it is.) Using a well-known criterion for quasihomogeneity (see Lemma 2.2), one finds that if f is not quasihomogeneous, then h is not constant on X. Since X is projective, this implies that there exists $(a, b) \in X$ with $h(a, b) = 0$. It turns out that $h(a, b)$ cannot be zero if $a \in \{0, \infty\}$, so $a \in \mathbb{C}^\times$, and from $h(a, b) = 0$, we deduce that (a, b) is a root of $yf_y(x, y)$, contradicting (C). To see that $h(0, b) \neq 0$ (the case $h(\infty, b)$ is similar), we express the branches of X near $x = 0$ as Puiseux series: assuming $y = \sum_{r\in\mathbb{Q}} b_r x^r$, one verifies that the limit $\lim_{x\to 0} h(x, y)$ essentially only depends on the minimal r with $b_r \neq 0$, and one in particular obtains that the limit is never 0.

2. Auxiliary results

We start with the following simple but useful feature of the discriminant followed by two lemmas on quasihomogeneity (for Lemma 2.2 see also, e.g., [6, Exercise 3 on p. 37]).

Fact 2.1. *For every* $a \in \mathbb{C}$, $\operatorname{Disc}_y(f)(a) = 0$ *if and only if* $f(a, y)$ *and* $f_y(a, y)$ *have a common zero or* $\deg_y f(a, y) < \deg_y f(x, y)$. □

Lemma 2.2. *Let $f \in \mathbb{C}[x, y]$ be a polynomial. Then f is quasihomogeneous if and only if there are $w, \alpha, \beta \in \mathbb{C}$, not all zero, such that*

$$wf = \alpha x f_x + \beta y f_y.$$

Proof. Write f as $\sum c_{ij} x^i y^j$ and let I be the support of f, that is, the set given by $I = \{(i, j) \in \mathbb{N} : c_{ij} \neq 0\}$. Then

$$x f_x = \sum_{(i,j)\in I} i c_{ij} x^i y^j \quad \text{and} \quad y f_y = \sum_{(i,j)\in I} j c_{ij} x^i y^j. \tag{2.3}$$

If f is quasihomogeneous of type $(w; \alpha, \beta)$, then using that we have $w = \alpha i + \beta j$ for $(i, j) \in I$, one easily deduces that $wf = \alpha x f_x + \beta y f_y$.

For the converse, suppose there are $w, \alpha, \beta \in \mathbb{C}$, not all zero, such that $wf = \alpha x f_x + \beta y f_y$. Then, by using (2.3) and comparing monomials, for every $(i, j) \in I$ we obtain $\alpha i + \beta j = w$, which means that f is quasihomogeneous. We note that w, α, β can be taken to be integers, since all $(i, j) \in I$ have integer coordinates. □

Lemma 2.4. *Let $f \in \mathbb{C}[x, y]$ be a quasihomogeneous polynomial of type $(w; \alpha, \beta)$ with α positive and α, β coprime. Then, there are integers $k, k', \ell, d \geqslant 0$ and $c, a_1, \ldots, a_d \in \mathbb{C}^\times$ such that f, considered as a Laurent polynomial, can be written as*

$$f = c x^k y^\ell \prod_{i=1}^{d} (a_i - x^{-\beta} y^\alpha) = c x^{k'} y^\ell \prod_{i=1}^{d} (a_i x^\beta - y^\alpha).$$

Note that at least one of those two expressions is a product of polynomials (depending on the sign of β).

Proof. Write $f = \sum_{i=0}^{n} b_i x^{k_i} y^{\ell_i}$ with $n \geqslant 0$, $b_i \in \mathbb{C}^\times$, $k_i, \ell_i \in \mathbb{N}$, where the numbering is chosen so that $\ell_0 < \cdots < \ell_n$. Since f is quasihomogeneous of type $(w; \alpha, \beta)$ (and using that α and β are coprime), we have

$$f = \sum_{i=0}^{n} b_i x^{k_0 - \beta m_i} y^{\ell_0 + \alpha m_i}$$

for some $m_i \in \mathbb{N}$. (That m_i is nonnegative follows from the assumption that $\alpha > 0$ and that $\ell_0 \leqslant \ell_i$ for all $0 \leqslant i \leqslant m$.) This can be written as

$$f = x^{k_0} y^{\ell_0} \sum_{i=0}^{n} b_i (x^{-\beta} y^\alpha)^{m_i} = x^{k_0} y^{\ell_0} \cdot g(x^{-\beta} y^\alpha)$$

for some polynomial $g \in \mathbb{C}[z]$ whose constant coefficient (which is equal to b_0) is nonzero, so we find $d \in \mathbb{N}$ and $c, a_1, \ldots, a_d \in \mathbb{C}^\times$ such that

$$f = c x^{k_0} y^{\ell_0} \prod_{i=1}^{d} (a_i - x^{-\beta} y^\alpha),$$

establishing the first expression for f. For the second one, we pull out $x^{-\beta}$ from each factor of the product to obtain

$$f = cx^{k_0 - d\beta} y^{\ell_0} \prod_{i=1}^{d} (a_i x^{\beta} - y^{\alpha}),$$

so it remains to verify that $k' := k_0 - d\beta$ is nonnegative. Indeed, this expression has a monomial of the form $cx^{k'} y^{\ell_0} \cdot (-y^{\alpha})^d$, so we must have $k' \geqslant 0$ since no negative power of x appears in f. □

Definition-Notation 2.5. (1) By a multihomogeneous polynomial of multidegree $(m_1, \dots, m_n)$ we mean a polynomial $f \in \mathbb{C}[x_1, \tilde{x}_1, \dots, x_n, \tilde{x}_n]$ such that every monomial of f has the form $ax_1^{i_1} \tilde{x}_1^{m_1 - i_1} \cdots x_n^{i_n} \tilde{x}_n^{m_n - i_n}$.

(2) Given a polynomial $f = \sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[x_1, \dots, x_n]$ of degree m_i in x_i, we define its *multihomogenization* $\hat{f} \in \mathbb{C}[x_1, \tilde{x}_1, \dots, x_n, \tilde{x}_n]$ as

$$\hat{f} := \sum a_{i_1 \cdots i_n} x_1^{i_1} \tilde{x}_1^{m_1 - i_1} \cdots x_n^{i_n} \tilde{x}_n^{m_n - i_n}.$$

Note that any multihomogeneous polynomial $g \in \mathbb{C}[x_1, \tilde{x}_1, \dots, x_n, \tilde{x}_n]$ defines a subvariety of $(\mathbb{P}^1)^n$. As mentioned before (in the case $n = 2$), if $\hat{f}$ is the multihomogenization of a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, the variety defined by $\hat{f}$ corresponds to the Zariski closure in $(\mathbb{P}^1)^n$ of the subvariety of $\mathbb{C}^n$ defined by f (via the natural embedding of $\mathbb{C}^n$ into $(\mathbb{P}^1)^n$).

Remark 2.6. If $\hat{f}, \hat{g}, \hat{h}$ are the multihomogenizations of polynomials $f, g, h \in \mathbb{C}[x_1, \dots, x_n]$, then we have $f = gh$ if and only if $\hat{f} = \hat{g}\hat{h}$. In particular, f is irreducible if and only if $\hat{f}$ is irreducible.

For the following lemmas, we use the following assumptions and notation (which will be relevant for the proof of (C) ⇒ (A)):

Assumption 2.7. We fix the following objects:

- $\hat{f} \in \mathbb{C}[x, \tilde{x}, y, \tilde{y}]$ is a multihomogeneous irreducible polynomial which is not a monomial (i.e., not equal to any of $x, \tilde{x}, y, \tilde{y}$).
- $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the irreducible projective variety defined by $\hat{f}$.
- $X_0 \subset X$ is the Zariski locally closed set given by

$$X_0 = \left\{([x : \tilde{x}], [y : \tilde{y}]) \in X : x\hat{f}_x(x, \tilde{x}, y, \tilde{y}) \neq 0\right\}.$$

- $V \subset (\mathbb{P}^1)^3$ is the projective variety defined by the multihomogeneous polynomials (in the variables $x, \tilde{x}, y, \tilde{y}, z, \tilde{z}$)

$$\hat{f}(x, \tilde{x}, y, \tilde{y}) \quad \text{and} \quad \tilde{z}y\hat{f}_y(x, \tilde{x}, y, \tilde{y}) - zx\hat{f}_x(x, \tilde{x}, y, \tilde{y}).$$

- $h\colon X_0 \to \mathbb{P}^1$ is the function sending each $([x:\tilde{x}],[y:\tilde{y}]) \in X_0$ to the unique $[z:\tilde{z}] \in \mathbb{P}^1$ such that $([x:\tilde{x}],[y:\tilde{y}],[z:\tilde{z}]) \in V$. More specifically,

$$h([x:\tilde{x}],[y:\tilde{y}]) = \frac{y\hat{f}_y(x,\tilde{x},y,\tilde{y})}{x\hat{f}_x(x,\tilde{x},y,\tilde{y})} \in \mathbb{C} \subset \mathbb{P}^1.$$

- $V' \subset V$ is the Zariski closure of the graph of h.

Remark 2.8. Note that those assumptions have the following symmetry: if we set $\hat{f}^{\#}(x,\tilde{x},y,\tilde{y}) := \hat{f}(\tilde{x},x,y,\tilde{y})$ and let $V'^{\#} \subset (\mathbb{P}^1)^3$ be obtained using $\hat{f}^{\#}$ instead of $\hat{f}$, then $([x:\tilde{x}],[y:\tilde{y}],[z:\tilde{z}]) \in V'^{\#}$ if and only if $([x:\tilde{x}],[y:\tilde{y}],[-z:\tilde{z}]) \in V'$. To see this, it suffices to verify that X_0, V and h (and hence also V') do not change if we replace $x\hat{f}_x(x,\tilde{x},y,\tilde{y})$ by $-\tilde{x}\hat{f}_{\tilde{x}}(x,\tilde{x},y,\tilde{y})$ in the definitions of X_0, V and h. Indeed, it is clear that X_0 does not change; to see that V and h do not change either, write $\hat{f} = \sum c_{ij}x^i\tilde{x}^{n-i}y^j\tilde{y}^{m-j}$. Then

$$x\hat{f}_x = \sum_{i,j} ic_{ij}x^i\tilde{x}^{n-i}y^j\tilde{y}^{m-j} \quad\text{and}\quad \tilde{x}\hat{f}_{\tilde{x}} = \sum_{i,j}(n-i)c_{ij}x^i\tilde{x}^{n-i}y^j\tilde{y}^{m-j}.$$

This implies that $x\hat{f}_x + \tilde{x}\hat{f}_{\tilde{x}} = n\hat{f}$. Therefore, for any $([x:\tilde{x}],[y:\tilde{y}]) \in X$, we have that $x\hat{f}_x = -\tilde{x}\hat{f}_{\tilde{x}}$.

Remark 2.9. Remark 2.8 holds analogously if one swaps y and $\tilde{y}$ instead of x and $\tilde{x}$ (and again changes the sign of z).

In the following, we write Y^{Zar} for the Zariski closure of a set $Y \subset (\mathbb{P}^1)^n$.

Lemma 2.10 (under Assumption 2.7). *We have $X_0^{\mathrm{Zar}} = X$ and $\pi_{12}(V') = X$, where $\pi_{12}\colon (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^2$ is the projection to the first two coordinates.*

Proof. Set

$$Y := \{([x:\tilde{x}],[y:\tilde{y}]) \in (\mathbb{P}^1)^2 \mid x\hat{f}_x(x,\tilde{x},y,\tilde{y}) = 0\}.$$

Since $\dim Y = \dim X = 1$ and X is irreducible, in order to conclude $X_0^{\mathrm{Zar}} = X$, it suffices to show that X is not contained in Y. If X is contained in Y, this implies that $\hat{f}$ divides $x\hat{f}_x$. For degree reasons, this would mean equality up to a factor from $\mathbb{C}^\times$, which implies that $\hat{f}$ is either a monomial (namely equal to x) or not irreducible and hence contradicts the assumptions.

For the second part, note that

$$\pi_{12}(V') = \pi_{12}(\mathrm{graph}(h)^{\mathrm{Zar}}) \overset{(\star)}{=} \pi_{12}(\mathrm{graph}(h))^{\mathrm{Zar}} = X_0^{\mathrm{Zar}} = X,$$

where the inclusion "$\supset$" in $(\star)$ uses that since π_{12} is proper, it sends the closed set $\mathrm{graph}(h)^{\mathrm{Zar}}$ to a closed set. □

Note that above any point of X, there are only finitely many points of V'.

Lemma 2.11 (under Assumption 2.7). *V' and $\big\{([0:1],[1:0])\big\}\times\mathbb{P}^1\times\{[0:1]\}$ are disjoint.*

Proof. By Remarks 2.8 and 2.9, it suffices to prove that V' and $\{[0:1]\}\times\mathbb{C}\times\{[0:1]\}$ are disjoint. Since this last set is a subset of $\mathbb{C}^3\subset(\mathbb{P}^1)^3$, we can (for simplicity) dehomogenize everything: setting $f(x,y):=\hat{f}(x,1,y,1)\in\mathbb{C}[x,y]$, we consider the restriction of h to $X_0\cap\mathbb{C}^2$, which is given by $h(x,y)=yf_y(x,y)/(xf_x(x,y))$, and what we need to show is that the Zariski closure of its graph and $\{0\}\times\mathbb{C}\times\{0\}$ are disjoint.

We first treat the point $(0,0,0)$. Afterwards, we will reduce the general case to this one.

Part 1: *proving that V' does not contain* $(0,0,0)$. By (a version of) Puiseux's theorem [1, Corollary 1.5.5], we can write f as

$$f=ux^r\prod_{i=1}^{k}(y-s_i),\tag{2.12}$$

where $r\in\mathbb{N}$, $u\in\mathbb{C}[\![x,y]\!]$ is an invertible power series in x, y and each s_i is a Puiseux series in x, that is, $s_i\in x^{1/N}\mathbb{C}[\![x^{1/N}]\!]$ for some $N\geq 1$. (Following the convention of [1], in a Puiseux series, we allow only strictly positive powers of x.) Without loss of generality, replacing x by t^N for some suitable large integer N, we may suppose that all exponents in the series are integers, and therefore, we can work with power series. Indeed, note that by setting $f^\#(t,y):=f(t^N,y)$, we obtain that the corresponding map $h^\#(t,y)=yf^\#_y/(tf^\#_t)$ satisfies $Nh^\#(t,y)=h(t^N,y)$. Therefore, the corresponding set $V'^\#$ contains $(0,0,0)$ if and only if V' does.

Next, note that in (2.12), we have $r=0$. Indeed, set $q=u\prod_{i=1}^k(y-s_i)\in\mathbb{C}[\![x,y]\!]\subset\mathbb{C}((y))((x))$ and let v_x denote the x-adic valuation on $\mathbb{C}((y))((x))$. Then, we have $v_x(q)=0$ (since u is invertible and $v_x(y-s_i)=0$). On the other hand, since $qx^r=f\in\mathbb{C}[x,y]$, we have $q\in\mathbb{C}[x,x^{-1},y]$. In particular, we can write q as $\sum_{i\in I}a_ix^i$ with $a\in\mathbb{C}[y]$ and I a finite subset of $\mathbb{Z}$. But since $v_x(q)=0$, we must have $a_i=0$ for all $i<0$. Therefore $q\in\mathbb{C}[x,y]$. If $r>0$, then $f=x^rq$ would not be irreducible, hence $r=0$.

Since u is invertible, there is an open neighborhood $U\subset\mathbb{C}^2$ of $(0,0)$ where u does not vanish. Hence $X\cap U$ is the union of the graphs $\{(x,y)\in U\mid y=s_i(x)\}$ of the power series s_i. Moreover, let us assume that U is small enough so that $X\cap U=X_0\cap U$.

Note that for each of those power series, if we take x in such a way that $(x,s_i(x))\in U$, then we have

$$-s_i'(x)=\frac{f_x(x,s_i(x))}{f_y(x,s_i(x))}\tag{2.13}$$

(where s_i' denotes the derivative of s_i). Indeed, composing the map $(\mathrm{id}, s_i)\colon \mathbb{C} \to \mathbb{C}^2$, $x \mapsto (x, s_i(x))$, with f gives the zero map (because f is zero on the graph of s_i). Thus, the derivative of the composed function is zero; expressed using the chain rule, this means $f_x(x, s_i(x)) + f_y(x, s_i(x)) \cdot s_i'(x) = 0$. Since $f_x(x, s_i(x)) \neq 0$ on X_0, this firstly implies that the above denominator $f_y(x, s_i(x))$ is nonzero and secondly, we obtain (2.13).

Note that s_i is not the 0 series, since otherwise, by irreducibility of X, X would consist only of $\mathbb{P}^1 \times \{0\}$, which contradicts $\hat{f}$ not being a monomial. Thus we can write $s_i(x) = \sum_{j \geqslant M_i} b_{i,j} x^j$ for $M_i \geqslant 1$ and $b_{i,M_i} \neq 0$. Then $s_i'(x) = \sum_{j \geqslant M_i} j b_{i,j} x^{j-1}$ and hence

$$\begin{aligned}\lim_{x\to 0} h(x, s_i(x)) &= \lim_{x\to 0} \frac{s_i(x) f_y(x, s_i(x))}{x f_x(x, s_i(x))}\\ &= \lim_{x\to 0} -\frac{s_i(x)}{x s_i'(x)} = \lim_{x\to 0} -\frac{\sum_{j\geqslant M_i} b_{i,j} x^j}{\sum_{j\geqslant M_i} j b_{i,j} x^j} = -\frac{1}{M_i} \neq 0,\end{aligned}$$

where the second equality is by (2.13). Applying this to each of the s_i shows that $(0, 0, 0)$ does not lie in the closure of $\mathrm{graph}(h)$ in the analytic topology. The closure of the graph in the Zariski topology is the same, so $(0, 0, 0) \notin V'$.

Part 2: *proving that V' does not contain* $(0, y_0, 0)$, *for* $y_0 \in \mathbb{C}^\times$. Consider the change of variables $y \mapsto y + y_0$, i.e., set $f^\#(x, y) = f(x, y + y_0)$. Note that

$$h^\#(x, y) = \frac{y f_y^\#(x, y)}{x f_x^\#(x, y)} = \frac{y f_y(x, y + y_0)}{x f_x(x, y + y_0)} = \frac{y}{y + y_0} h(x, y + y_0).$$

By part 1, $(0, 0, 0)$ does not belong to the closure of the graph of $h^\#$. Therefore $(0, y_0, 0)$ does not belong to the closure of the graph of h. Indeed, if (x_n, y_n) is a sequence of points in X_0 converging to $(0, y_0)$, then $\lim_{n\to\infty} h(x_n, y_n) = 0$ would in particular imply $\lim_{n\to\infty} \frac{y_n}{y_n + y_0} h(x_n, y_n) = 0$. □

3. Proof of Theorem 1.2

Suppose $f = \sum_{i=0}^n f_i(x) y^i$ is a reduced polynomial such that f_0 and f_n are nonzero polynomials. We show

(A) ⇒ (B) ⇒ (C) ⇒ (A).

(A) ⇒ (B): Suppose f is quasihomogeneous of type $(w; \alpha, \beta)$ with $\alpha \neq 0$. Without loss of generality we may assume that $\alpha > 0$ and that α and β are coprime. It is clear that f_0 and f_n are monomials. To see that $\mathrm{Disc}_y(f)$ is a monomial, by Lemma 2.4, we can write f both as

$$c x^k y^\ell \prod_{i=1}^d (a_i - x^{-\beta} y^\alpha) \quad \text{and} \quad c x^{k'} y^\ell \prod_{i=1}^d (a_i x^\beta - y^\alpha),$$

where one of the two expressions is a product of polynomials, and where the a_i are nonzero. Suppose the former is a product polynomials (so $\beta \leqslant 0$), the other case being similar. Since f is reduced we have that $0 \leqslant k, \ell \leqslant 1$ and all a_i must be different. Moreover, for every $e \in \mathbb{C}^\times$, the equation $a_i - e^{-\beta} y^\alpha = 0$ has no multiple roots. Therefore, also $f(e, y)$ has no multiple roots. Since f_n is a monomial, we also have $f_n(e) \neq 0$, so we obtain (by Fact 2.1) that $(\mathrm{Disc}_y(f))(e) \neq 0$ for all $e \in \mathbb{C}^\times$. In other words, the only possible root of $\mathrm{Disc}_y(f)$ is 0, meaning that it is a monomial.

(B) ⇒ (C): Suppose (B) holds but $\hat{f}$ and $y\hat{f}_y$ have a common zero $([a:1], [b:\tilde{b}]) \in \mathbb{C}^\times \times \mathbb{P}^1$. Since f_0 is a monomial, we must have $b \neq 0$, as otherwise, $\hat{f}(a, 1, 0, \tilde{b}) = f_0(a)\tilde{b}^n = 0$ would imply that $a = 0$. Similarly, $\tilde{b} \neq 0$, since f_n is a monomial. Hence, without loss of generality, $\tilde{b} = 1$. Therefore,

$$\hat{f}(a, 1, b, 1) = f(a, b) = 0 \quad \text{and} \quad b\hat{f}_y(a, 1, b, 1) = bf_y(a, b) = 0.$$

Since $b \neq 0$, the latter implies $f_y(a, b) = 0$. By Fact 2.1, we obtain $\mathrm{Disc}_y(f)(a) = 0$, which implies that $a = 0$ since $\mathrm{Disc}_y(f)$ is a monomial, a contradiction.

(C) ⇒ (A): We first reduce to the case of irreducible polynomials.

Claim 1. *It suffices to prove (C) ⇒ (A) when f is irreducible.*

Proof. Let f be a polynomial satisfying (C), that is, $\hat{f}$ and $y\hat{f}_y$ have no common root in $\mathbb{C}^\times \times \mathbb{P}^1$. Suppose further that $f = gh$ and that the implication (C) ⇒ (A) holds for g and h. We show that f is quasihomogeneous with nonzero weight of x. By Remark 2.6, we have that $\hat{f} = \hat{g}\hat{h}$. Moreover, the usual derivation rules imply

$$y\hat{f}_y = \hat{g}(y\hat{h}_y) + \hat{h}(y\hat{g}_y).$$

This shows that $\hat{g}$ and $y\hat{g}_y$ (resp. $\hat{h}$ and $y\hat{h}_y$) have no common root in $\mathbb{C}^\times \times \mathbb{P}^1$ as otherwise $\hat{f}$ and $y\hat{f}_y$ would have one. Therefore, by assumption, both g and h are quasihomogeneous with nonzero weight of x. If either g or h is a monomial, it is easy to see that f is quasihomogeneous, so we are done. So suppose g and h are not monomials. In order to deduce (A) for f, it suffices to verify that g and h have the same weights (up to some factor), so suppose otherwise. In that case, we will see that g and h have a common zero in $(x_0, y_0) \in \mathbb{C}^\times \times \mathbb{C}^\times$. This implies that $([x_0:1], [y_0:1])$ is a common root of $\hat{f}$ and $y\hat{f}_y$, contradicting the assumption.

As a referee pointed out, the existence of such a common zero (x_0, y_0) follows from Bernstein–Kouchnirenko's theorem, as, e.g., stated in [2, Theorem 1]. Indeed, from g and h being quasihomogeneous of different weight ratio, one deduces that they are Newton-nondegenerate (in the sense of [2, Definition 4]) and that the mixed volume $\mathrm{MV}(\Delta_g, \Delta_h)$ of their Newton polytopes is nonzero. The existence of (x_0, y_0) then follows from [2, Theorem 1].

We nevertheless give a self-contained argument for the existence of a common zero (x_0, y_0) of g and h. Using Lemma 2.4, write

$$g = cx^k y^\ell \prod_i (a_i x^\beta - y^\alpha) \quad \text{and} \quad h = dx^{k'} y^{\ell'} \prod_j (b_j x^\delta - y^\gamma)$$

for integers k, ℓ, k', ℓ', α, β, γ and δ with $\alpha \neq 0$, $\gamma \neq 0$ and $c, d, a_i, b_j \in \mathbb{C}^\times$, and where neither of the products over i and j are empty. The weight difference implies that $\beta/\alpha \neq \delta/\gamma$.

Let a be any of the a_i and let b be any of the b_j. We will find a common zero $(x_0, y_0) \in \mathbb{C}^\times \times \mathbb{C}^\times$ of the Laurent polynomials $ax^\beta - y^\alpha$ and $bx^\delta - y^\gamma$, which is hence a common zero of g and h.

In seeking a common root of the factors above, we may suppose that $(\alpha, \gamma) = 1$ (that is, they are coprime), if necessary via a change of variables $t = y^{(\alpha,\gamma)}$. Now let x_0 be any $(\delta\alpha - \beta\gamma)$-th root of a^γ / b^α. This implies

$$a^\gamma x_0^{\beta\gamma} = b^\alpha x_0^{\delta\alpha} =: w.$$

We need to find a y_0 such that $y_0^\alpha = ax_0^\beta$ (which is a γ-th root of w) and $y_0^\gamma = bx_0^\delta$ (which is an α-th root of w). Let z_0 be a fixed $(\alpha\gamma)$-th root of w and let ζ be a primitive $|\alpha\gamma|$-th root of unity. Then we have

$$ax_0^\beta = z_0^\alpha \zeta^{i\alpha} \quad \text{and} \quad bx_0^\delta = z_0^\gamma \zeta^{j\gamma}$$

for some integers i and j. If we set $y_0 = z_0 \zeta^k$ for some integer k, then our two conditions on y_0 become

$$z_0^\alpha \zeta^{k\alpha} = z_0^\alpha \zeta^{i\alpha} \quad \text{and} \quad z_0^\gamma \zeta^{k\gamma} = z_0^\gamma \zeta^{j\gamma}.$$

This corresponds to the modular equations

$$k\alpha \equiv i\alpha \mod \alpha\gamma \qquad \text{and} \qquad k\gamma \equiv j\gamma \mod \alpha\gamma,$$

which have a common solution since α and γ are coprime. □

To show (C) ⇒ (A) we will prove its contrapositive, so assume the negation of (A), that is, either f is not quasihomogeneous, or it is quasihomogeneous only using $\alpha = 0$ as the weight of x. The latter means that f is a polynomial in x only (since we assumed $f_0 \neq 0$) but not a monomial. Then f has a root $(a, 0)$ for some $a \in \mathbb{C}^\times$, and since $y\hat{f}_y$ is the zero-polynomial, $([a:1], [0:1])$ is a common root of $\hat{f}$ and $y\hat{f}_y$, contradicting (C).

We are left with the case where f is not quasihomogeneous. We let $\hat{f}$ be the multihomogenization of f and use all the notation from Assumption 2.7.

Claim 2. *The function $h\colon X_0 \to \mathbb{P}^1$ is not constant.*

Proof. Suppose h is constant. Then

$$y\hat{f}_y(x,\tilde{x},y,\tilde{y}) = x\hat{f}_x(x,\tilde{x},y,\tilde{y})\cdot c$$

for some constant $c \in \mathbb{C}$, for all $([x:\tilde{x}],[y:\tilde{y}]) \in X_0$. Since this polynomial equality holds on X_0, it also holds on the Zariski closure of X_0, which is X by Lemma 2.10. Thus, the polynomial $y\hat{f}_y - x\hat{f}_x \cdot c \in \mathbb{C}[x,\tilde{x},y,\tilde{y}]$ is a multiple of $\hat{f}$, that is,

$$y\hat{f}_y - x\hat{f}_x \cdot c = g\hat{f}$$

for some $g \in \mathbb{C}[x,\tilde{x},y,\tilde{y}]$. For degree reasons, g is constant equal to some $d \in \mathbb{C}$. Setting the variables $\tilde{x} = \tilde{y} = 1$, we obtain $yf_y - xf_x \cdot c = df$. Now Lemma 2.2 implies that f is quasihomogeneous, contradicting our assumption. ☐

Claim 3. *Let $\pi_3\colon (\mathbb{P}^1)^3 \to \mathbb{P}^1$ be the projection onto the third coordinate. Then $\pi_3|_{V'}$ is surjective.*

Proof. Since π_3 is a proper morphism, it is a closed map. Hence the image of V' is closed. But in $\mathbb{P}^1$, a closed set is either finite or the whole $\mathbb{P}^1$. If the image is finite, it is a singleton (by irreducibility of V'). But then h would be constant, contradicting Claim 2. ☐

By Claim 3, there exists a $([x_0:\tilde{x}_0],[y_0:\tilde{y}_0]) \in X$ such that

$$([x_0:\tilde{x}_0],[y_0:\tilde{y}_0],[0:1]) \in V'.$$

By Lemma 2.11, we have $[x_0:\tilde{x}_0] \notin \{[0:1],[1:0]\}$. Therefore $([x_0:\tilde{x}_0],[y_0:\tilde{y}_0]) \in \mathbb{C}^\times \times \mathbb{P}^1$ is a root of $\hat{f}$ (by definition of X) and a root of $y\hat{f}_y$ by definition of h, completing the proof.

4. Final remarks

We show in this section how Theorem 1.2 for $\mathbb{C}$ implies a corresponding result over arbitrary fields K of characteristic 0 and, for a fixed degree d, over fields K of characteristic p for large p depending on d. This is the content of Theorem 4.1 below. When K is not algebraically closed, Theorem 1.2(C) needs to be slightly adapted, for example, as follows:

(C′) Denote by K^{alg} the algebraic closure of K and write $\mathbb{P}^1(K^{\mathrm{alg}})$ for the projective space over K^{alg}. Then the subvarieties of $\mathbb{P}^1(K^{\mathrm{alg}}) \times \mathbb{P}^1(K^{\mathrm{alg}})$ defined by $\hat{f}$ and $y\hat{f}_y$ have no common point within $(K^{\mathrm{alg}})^\times \times \mathbb{P}^1(K^{\mathrm{alg}})$.

(Equivalently, one could also write that the subscheme of $\mathbb{G}_{m,K} \times \mathbb{P}^1_K$ defined by $\hat{f}$ and $y\hat{f}_y$ is empty.)

Theorem 4.1. *Let K be a field of characteristic $p \geqslant 0$.*

(1) *If $p = 0$, Theorem 1.2 holds over K, where* (C) *is replaced by the above* (C′).

(2) *For every* $d \in \mathbb{N}$ *there is* $N_d \in \mathbb{N}$ *such that if* $p > N_d$, *Theorem 1.2 holds over* K *for all geometrically reduced polynomials* $f \in K[x, y]$ *of* (*total*) *degree at most* d, *again using* (C′) *instead of* (C).

Note that the proof of (2) uses a model-theoretic argument and in particular requires familiarity with the notion of first-order sentences in the language of rings (which can be found in any model-theoretic text book such at [8] or [9]). The only result we use is the transfer principle for algebraically closed fields, which is stated explicitly, e.g., as [7, Theorem 2.4.4] or [4, Theorem 1.14]. For readers not familiar with model theory, below is a very informal introduction to the things we need.

A first-order sentence φ in the language of rings is a mathematical statement that can be written by starting with polynomial equations with coefficients in $\mathbb{Z}$ and combining them with boolean operators $\wedge$, $\vee$, $\neg$ and the quantifiers $\forall$ and $\exists$. The sentence φ doesn't specify which sets the quantifiers run over. Instead, one then says that φ holds in a field K if φ becomes true when one lets all quantifiers run over K. The transfer principle for algebraically closed fields asserts (in particular) that for each such φ, there exists an nonnegative integer N_φ such that φ holds in all algebraically closed fields of characteristic 0 if and only if φ holds in all algebraically closed fields of characteristic $p > N_\varphi$.

Proof of Theorem 4.1. For (1), let K be a field of characteristic 0 and let $f = \sum_{i=0}^{n} f_i(x)y^i \in K[x, y]$ be a reduced bivariate polynomial with $f_i \in K[x]$, where f_0 and f_n are nonzero polynomials. Let $c = (c_{i,j})$ be the coefficients of f. Then $\mathbb{Q}(c)$ is an extension of $\mathbb{Q}$ of finite transcendence degree. Let $\sigma : \mathbb{Q}(c) \to \mathbb{C}$ be an embedding and $f^\sigma \in \mathbb{C}[x, y]$ be the image of f under σ. Since we are in characteristic 0, f being reduced implies that it is geometrically reduced, and hence, f^σ is also reduced (when considered as a polynomial over $\mathbb{C}$). We can thus apply Theorem 1.2 to f^σ and obtain that the equivalences (A) $\Longleftrightarrow$ (B) $\Longleftrightarrow$ (C) hold for f^σ. Moreover, over $\mathbb{C}$, (C) is equivalent to (C′). Since each of (A), (B) and (C′) hold for f if and only if the corresponding condition holds for f^σ, we are done.

For (2), let us first show the statement when K is algebraically closed. To obtain this, it suffices to verify that for each fixed d, there exists a first-order sentence φ_d in the language of rings which expresses exactly that Theorem 1.2 holds for all polynomials f of degree at most d. Indeed, once we have such a φ_d, we know that φ_d holds in every algebraically closed field of characteristic 0, so by the transfer principle of algebraically closed fields (see, e.g., [7, Theorem 2.4.4]), there exists an $N_d \in \mathbb{N}$ such that φ_d holds for all algebraically closed fields K of characteristic $p > N_d$, meaning that Theorem 1.2 holds in K for all f of degree at most d.

While specifying φ_d is fairly standard for model theorists, we give some explanations for readers not so used to model theory:

- To quantify over all polynomials f of degree at most d, we use one universal quantifier for each of the (finitely many) coefficients of f.
- f_0 and f_n being nonzero is a finite boolean combination about certain coefficients of f being nonzero.
- f being reduced can be expressed as follows: there exist no $g, h \in \mathbb{C}[x, y]$ such that $f = g^2h$. Of course we may restrict to g and h of degree at most d, so that the quantifiers over g and h can be written as quantifiers over the (finitely many) coefficients of g and h. (And note that $f = g^2h$ can be expressed as a first-order formula since it is a polynomial expression in the coefficients of f, g, and h.)
- Theorem 1.2(A) is just a finite boolean combination of some of the coefficients of f being nonzero, and the same holds for f_0 and f_n being monomials.
- Expressing that the discriminant $\mathrm{Disc}_y(f)$ is a monomial works similarly, using that the coefficients of $\mathrm{Disc}_y(f)$ are polynomials in the coefficients of f (see [3, Chapter 12]).
- Finally, turning (C) into a first-order formula is straightforward, where a point $([x : \tilde{x}], [y : \tilde{y}]) \in \mathbb{P}^1 \times \mathbb{P}^1$ is represented by x, $\tilde{x}$, y, $\tilde{y}$ (and by quantifying over x, $\tilde{x}$, y, $\tilde{y}$).

Now, to conclude for all fields of characteristic $p > N_d$, fix first an algebraically closed field F of characteristic $p > N_d$ with infinite transcendence degree over $\mathbb{F}_p$. Let K be any field of characteristic p and $f \in K[x, y]$ be a bivariate polynomial which is geometrically reduced. Let $c = (c_{i,j})$ be the coefficients of f. Then $\mathbb{F}_p(c)$ is an extension of $\mathbb{F}_p$ of finite transcendence degree. Letting $\sigma : \mathbb{F}_p(c) \to F$ be an embedding and $f^\sigma \in F[x, y]$ be the image of f under σ we conclude as in case (1), noting that f^σ is reduced. ☐

We finish by asking the following questions:

Question 4.2. *Does Theorem 1.2 hold for geometrically reduced polynomials over all fields of positive characteristic?*

Question 4.3. *Is there a suitable analogue of Theorem 1.2 in higher dimension (i.e., for curves in $\mathbb{C}^n$ or for hypersurfaces in $\mathbb{C}^n$, or maybe for arbitrary varieties in $\mathbb{C}^n$)?*

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