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REPRODUCING KERNEL FUNCTIONS AND BOUNDARY LIMITS IN MODEL SPACES

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We investigate the boundary behavior of functions in model spaces, emphasizing their approach towards boundary points through arbitrary but fixed regions. We generalize classical results to encompass *any* approach region, thereby resolving an open question about reproducing kernel functions. The study connects the geometric nature of approach regions to the analytic properties of functions in the model space as they approach the unit circle. Key conditions are established to fully characterize boundedness, weak convergence, and the existence of limits within the approach region for functions in these spaces. Furthermore, precise criteria for norm convergence of reproducing kernel functions are provided, offering deeper insights into how boundary geometry influences analytic behavior. These results extend classical boundary value problems to an ultimately broader framework, highlighting the interplay between geometry and analysis.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$, and H^2 denote the Hardy space on $\mathbb{D}$. The space H^2 can also be identified with a closed subspace of $L^2(\mathbb{T})$ via nontangential limits. A bounded analytic function u on the unit disk is said to be inner if $|u(\xi)| = 1$ a.e. on $\mathbb{T}$. It is well known [2] that every inner function u admits the factorization

$$u(z) = cB(z)s(z)\Delta(z) \tag{1-1}$$

with

$$\begin{aligned} B(z) &= \prod_{n\geq 1} \frac{\bar{a}_n}{|a_n|}\frac{a_n - z}{1-\bar{a}_n z}, \\ s(z) &= \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\sigma(\theta)\right\}, \\ \Delta(z) &= \exp\left\{-\sum_{n\geq 1} \rho_n \frac{e^{i\zeta_n}+z}{e^{i\zeta_n}-z}\right\}, \end{aligned}$$

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where c is a unimodular constant, $\{a_n\}$ is a Blaschke sequence which means it satisfies $\sum(1-|a_n|)<\infty$, (we assume $a_n/|a_n|\equiv 1$ if $a_n=0$), σ is a finite positive continuous singular measure with respect to Lebesgue measure, and $\zeta_n\in[0,2\pi)$, $\rho_n>0$, $\sum\rho_n<\infty$.

For each nonconstant inner function u, define the model space $K_u=H^2\ominus uH^2$, which is a reproducing kernel Hilbert space whose kernel has the form

$$k_\lambda^u(z)=\frac{1-\overline{u(\lambda)}u(z)}{1-\bar{\lambda}z},\quad \lambda,z\in\mathbb{D}.$$

In particular, when $u=B$, let $B_1=1$, $B_n(z)=\prod_{1\le i\le n-1}(a_i-z)/(1-\bar{a}_iz)$, $n=2,3,\dots$, and

$$e_n(z)=\frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_nz}B_n(z),\quad n\ge 1. \tag{1-2}$$

Then $\{e_n\}$ is an orthonormal basis for K_B. Such a basis is the so-called Takenaka–Malmquist–Walsh basis (see [9, Proposition 5.25]).

For $\xi\in\mathbb{T}$ and $\alpha>1$, the Stolz domain at ξ is defined by

$$\Gamma_\alpha(\xi)=\bigl\{z\in\mathbb{D}:|z-\xi|<\alpha(1-|z|)\bigr\}.$$

A function $f:\mathbb{D}\to\mathbb{C}$ is said to have a nontangential limit at ξ if $\lim_{z\to\xi}f(z)$ exists whenever z converges to ξ in every fixed Stolz domain $\Gamma_\alpha(\xi)$. The function f is said to have angular derivative in the sense of Carathéodory (ADC) at the point $\xi\in\mathbb{T}$, denoted by $f'(\xi)$, if the difference quotient $(f(z)-\eta)/(z-\xi)$ has a nontangential limit for some $\eta\in\mathbb{T}$. This is the celebrated Julia–Carathéodory theorem as follows [5; 11].

Theorem A. *For $b:\mathbb{D}\to\mathbb{D}$ analytic and $\xi\in\mathbb{T}$, the following are equivalent*:

(1) $\liminf_{z\to\xi}\bigl(1-|b(z)|\bigr)/(1-|z|)<\infty$, *where the limit is taken as z tends to ξ unrestrictedly in $\mathbb{D}$.*

(2) *b has an ADC at ξ.*

(3) *Both b and b' have* (*finite*) *nontangential limits at ξ, with* $|\lim_{r\to 1}b(r\xi)|=1$.

There are many extensions for this result to higher dimensions for holomorphic mappings of several complex variables; see, e.g., [1; 4; 13; 14; 16]. In particular, if b is an inner function, due to Ahern and Clark, the Julia–Carathéodory theorem can be more accurately described as follows (see [3] or see [9, Theorem 7.24]).

Theorem B. *Let u be an inner function factorized according to* (1-1). *Then the following are equivalent*:

(i) *u has an ADC at ξ.*

(ii) $\liminf_{z\to\xi}\|k_z^u\|^2=\liminf_{z\to\xi}\bigl(1-|u(z)|\bigr)/(1-|z|)<\infty$.

(iii) *For every fixed Stolz domain $\Gamma_\alpha(\xi)$ at ξ*, $\sup\{\|k_z^u\|:z\in\Gamma_\alpha(\xi)\}<\infty$.

(iv) *Each* $f \in K_u$ *has a nontangential limit at* $\xi \in \mathbb{T}$.

(v)

$$\sum_{n\geq 1} \frac{1-|a_n|}{|\xi - a_n|^2} + \int_0^{2\pi} \frac{d\sigma(\theta)}{|\xi - e^{i\theta}|^2} + \sum_{n\geq 1} \frac{\rho_n}{|\xi - e^{i\zeta_n}|^2} < \infty. \tag{1-3}$$

Moreover, when these conditions hold, $k_\xi^u = (1 - \overline{u(\xi)}u)/(1 - \bar{\xi}z) \in K_u$ *and* $u'(\xi) = \bar{\xi}u(\xi)|u'(\xi)|$, *where* $u(\xi)$ *and* $u'(\xi)$ *are the nontangential limits of* $u(z)$ *and* $u'(z)$ *at* ξ, *respectively, and* $k_z^u \to k_\xi^u$ *in the norm of* K_u *as* z *tends to* ξ *nontangentially from* $\mathbb{D}$.

To extend Theorem B to a broader context of convergence, let Ω_ξ be an *approach region* with vertex at $\xi \in \mathbb{T}$. By the term "approach region", we mean an open connected subset of $\mathbb{D}$ whose closure intersects $\mathbb{T}$ only at ξ. For example, an approach region might be the interior of a triangle whose vertex is at ξ (a Stolz approach region), or the interior of an internally tangent circle with contact point ξ (an oricyclic approach region), or perhaps one might want to consider other tangential approach regions. In fact, Hartmann and Ross [10, Theorem 4.1] gave an alternate proof of (iii) $\Longleftrightarrow$ (iv) of Theorem B, which can be used to prove the following result: for any approach region Ω_ξ, every $f \in K_u$ has a limit as $z \to \xi$ within $z \in \Omega_\xi$ if and only if $\sup\{\|k_z^u\| : z \in \Omega_\xi\} < \infty$. Therefore, it is natural to explore the following question.

Question 1.1. Is there a condition, similar to Frostman's condition (1-3), that one could exploit to determine whether or not $\sup\{\|k_z^u\| : z \in \Omega_\xi\} < \infty$?

This question, posed in [9, pp. 168–169], serves as the foundation for our investigation in this work. The structure of the paper is as follows. In Section 2, we outline the origin of our approach and present the main results, Theorems 2.1 and 2.2. The first theorem extends Theorem B to the context of weak convergence, but crucially, it applies to arbitrary approach regions — addressing the central objective of this work. The second theorem provides a complete characterization of norm convergence, thereby fully resolving Question 1.1. Sections 3 and 4 include several technical lemmas that are noteworthy in their own right and also serve as essential components for the respective proofs of the main results. Finally, in Section 5, we explore deeper into the convergence results within two specific approach regions, further illustrating the robustness and applicability of the main theorems.

2. Our approach and the main results

For $c > 0$ and $\gamma \geq 1$, define

$$R(c,\gamma) := \{z \in \mathbb{D} \setminus \{0\} : 1 - |z| \geq c \lfloor \arg z \rfloor^\gamma\} \quad \text{(see Figure 1)},$$

$$\Gamma_{c,\gamma} := \{z \in \mathbb{D} \setminus \{0\} : 1 - |z| > c|1 - z|^\gamma\},$$

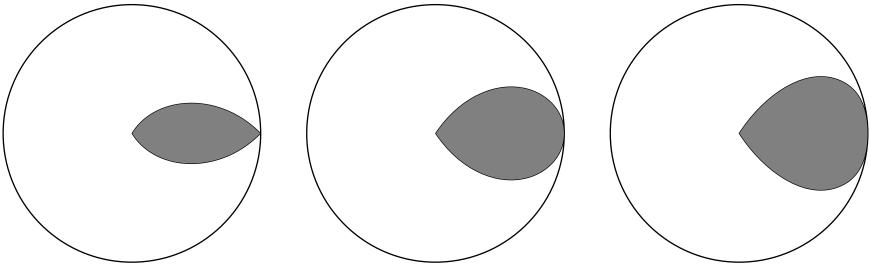

Figure 1. $R(1,1)$ (left), $R(1,2)$ (middle) and $R(1,3)$ (right).

where $\lfloor \arg z \rfloor$ means the length of the shorter one of the two arcs on $\mathbb{T}$ joining $z/|z|$ and 1. If $z \in \mathbb{D}$ with $\operatorname{Re} z > 0$ and $|z| > \frac{1}{2}$, then $\lfloor \arg z \rfloor \in [0, \pi/2]$ and hence $\sin \arg z \sim \lfloor \arg z \rfloor$, so that

$$|1-z|^{\gamma} \sim (1-|z|)^{\gamma} + \lfloor \arg z \rfloor^{\gamma},$$

which implies that there exist two positive numbers $c_1(\gamma)$ and $c_2(\gamma)$, such that

$$\Gamma_{c_1(\gamma),\gamma} \subset R(c,\gamma) \subset \Gamma_{c_2(\gamma),\gamma}.$$

A function f defined on $\mathbb{D}$ is said to have a T_γ-limit at 1 if f has a limit as $z \to 1$ within $R(c,\gamma)$ for every $c > 0$. In particular, f has a T_1-limit at 1 if and only if f has a nontangential limit at 1. For $\gamma > 1$, Protas [15, Theorem 3, Example 2] gave a sufficient condition that each $f \in K_u$ has a T_γ-limit at 1 and showed that this condition is not necessary for some model space K_B with a Blaschke product B. Leung [12, Theorem 4] proved the following result concerned with existence of T_γ-limits of functions in K_u.

Theorem C. *Let u be an inner function factorized according to* (1-1) *and let $\gamma \geq 1$. Then each $f \in K_u$ has a T_γ-limit at* 1 *if and only if* (1-3) *holds for $\xi = 1$ and*

$$\limsup_{t \to 0^+} F_{t,\gamma} < \infty,$$

where

$$F_{t,\gamma} = \left\{ \sum_{t/2 < \lfloor \arg a_n \rfloor < 2t} \frac{1-|a_n|}{t^{2\gamma} + (t - \lfloor \arg a_n \rfloor)^2} + \left(\int_{t/2}^{2t} + \int_{2\pi-2t}^{2\pi-t/2} \right) \frac{d\sigma(\lfloor \theta \rfloor)}{t^{2\gamma} + (t - \lfloor \theta \rfloor)^2} + \sum_{t/2 < \lfloor \zeta_n \rfloor < 2t} \frac{\rho_n}{t^{2\gamma} + (t - \lfloor \zeta_n \rfloor)^2} \right\}. \tag{2-1}$$

In this paper, we follow Leung's footsteps and generalize this result to *any approach region* and answer Question 1.1.

For a finite Blaschke product B, the dimension of the model space K_B is finite and any $f \in K_B$ is analytic on $c\mathbb{D}$ for some $|c| > 1$. So, in this paper, we always assume that B is an infinite Blaschke product. Throughout this paper, for $\omega \in \overline{\mathbb{D}}$, $\arg \omega$ is always restricted to the interval $[0, 2\pi)$ ($\arg w = 0$ if $w = 0$). For $z \in \mathbb{D}$, write

$$E_z = \begin{cases} \varnothing & \text{if } \arg z = 0, \\ ((\arg z)/2, 2\arg z) & \text{if } \arg z \in (0, \pi), \\ (2(\arg z - \pi), \pi + (\arg z)/2) & \text{if } \arg z \in [\pi, 2\pi), \end{cases}$$

and

$$F_z = \Biggl\{ \sum_{\arg a_n \in E_z} \frac{1 - |a_n|}{(1 - |z|)^2 + (\arg z - \arg a_n)^2} + \int_{E_z} \frac{d\sigma(\theta)}{(1 - |z|)^2 + (\arg z - \theta)^2} + \sum_{\zeta_n \in E_z} \frac{\rho_n}{(1 - |z|)^2 + (\arg z - \zeta_n)^2} \Biggr\}.$$

Intuitively, when $\arg z \in [\pi, 2\pi)$, $\{e^{i\theta} : \theta \in E_z\} = \overline{\{e^{i\theta} : \theta \in E_{\bar{z}}\}}$, i.e., they are symmetric about the x-axis.

We can now state the main results of this work.

Theorem 2.1. *Let u be an inner function factorized according to* (1-1). *Suppose Ω_1 is any approach region with vertex at* 1. *Then the following are equivalent.*

(i) *We have*

$$\sup_{z \in \Omega_1} \|k_z^u\| < \infty.$$

(ii) *k_z^u converges weakly as $z \to 1$ within Ω_1.*

(iii) *For each $f \in K_u$, f has a limit as $z \to 1$ within Ω_1.*

(iv) *For each $f \in K_u$, f is bounded on Ω_1.*

(v) (1-3) *holds for $\xi = 1$ and*

$$\limsup_{\substack{z \in \Omega_1 \\ z \to 1}} F_z < \infty. \tag{2-2}$$

Note that as z tends to 1 nontangentially from $\mathbb{D}$, k_z^u converges weakly if and only if k_z^u converges in the norm of K_u. So it is natural to consider for any approach region Ω_1, when k_z^u converges in the norm of K_u as $z \to 1$ within Ω_1. Fortunately, we get a complete characterization.

Theorem 2.2. *Let u be an inner function factorized according to* (1-1). *Suppose Ω_1 is an approach region with vertex at* 1. *Then $k_z^u \to k_1^u$ in the norm of K_u as $z \to 1$ within Ω_1 if and only if* (1-3) *holds for $\xi = 1$ and*

$$\lim_{\substack{z \in \Omega_1 \\ z \to 1}} F_z = 0. \tag{2-3}$$

3. Weak convergence

In this section, we establish some technical lemmas and then use them to give the proof of Theorem 2.1. First we show that the convergence of all functions in K_u is a stronger condition than the convergence of any divisor of u.

Lemma 3.1. *Let u be an inner function, and let Ω_1 be an approach region with vertex at 1. If each $f \in K_u$ has a limit as $z \to 1$ within Ω_1, then any divisor v of u has a limit of modulus 1 as $z \to 1$ within Ω_1.*

Proof. Since v is a divisor of u, $K_v \subseteq K_u$ and for any reproducing kernel k_z^v in K_v, we have

$$\|k_z^v\|^2 = k_z^v(z) = \frac{1-|v(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

The hypothesis together with the uniform boundedness principle yields

$$\sup_{z\in\Omega_1} \|k_z^v\| \le \sup_{z\in\Omega_1} \|k_z^u\| < \infty,$$

which implies $|v(z)| \to 1$ as $z \to 1$ within Ω_1.

Again by hypothesis, each $g \in K_v$ has a limit as $z \to 1$ within Ω_1. In particular, choose $z_0 \in \Omega_1$ such that $v(z_0) \neq 0$; then $k_{z_0}^v$ has a limit as $z \to 1$ within Ω_1 and hence v has a limit of modulus 1 as $z \to 1$ within Ω_1. □

Remark 3.2. From the proof of Lemma 3.1, it is easy to see that if we replace the approach region Ω_1 by $\partial\Omega_1 \setminus \{1\}$, $\overline{\Omega}_1 \setminus \{1\}$ or any arc lying in $\mathbb{D}$ and terminating at 1, Lemma 3.1 still holds.

Ahern and Clark [2] proved that K_u is unitarily equivalent to the direct sum of three L^2 spaces. In particular,

$$\text{(3-1)}\quad \|k_z^s\|^2 = 2\int_0^{2\pi} \frac{|s_\theta(z)|^2}{|1-e^{-i\theta}z|^2}\,d\sigma(\theta), \quad \|k_z^\Delta\|^2 = 2\sum_{n=1}^{\infty} \rho_n \frac{\int_{n-1}^{n} |\Delta_\lambda(z)|^2\,d\lambda}{|1-e^{-i\zeta_n}z|^2},$$

with

$$s_\theta(z) = \exp\left\{-\int_0^\theta \frac{e^{it}+z}{e^{it}-z}\,d\sigma(t)\right\}, \quad \theta \in [0, 2\pi),$$

and

$$\Delta_\lambda(z) = \exp\left\{-\sum_{j=1}^{n} \rho_j \frac{e^{i\zeta_j}+z}{e^{i\zeta_j}-z} - (\lambda-n)\rho_{n+1}\frac{e^{i\zeta_{n+1}}+z}{e^{i\zeta_{n+1}}-z}\right\},$$

where $\lambda \in (0, \infty)$ and n is the integral part of λ (see [2, pp. 194–195]). Moreover, recall that $\{e_n\}$ is an orthonormal basis for K_B (see (1-2)) and hence by Parseval's equality, we have

$$\|k_z^B\|^2=\sum_{n=1}^{\infty}|\langle k_z^B,e_n\rangle|^2=\sum_{n=1}^{\infty}|e_n(z)|^2=\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|1-\bar{a}_n z|^2}|B_n(z)|^2. \tag{3-2}$$

Moreover, [2, Lemma 4.1] showed that for any $f\in K_u$, f admits an orthogonal decomposition

$$f=P_Bf+BP_s\bar{B}f+BsP_\Delta\overline{Bs}f,$$

where P_B, P_s and P_Δ are the projections onto K_B, K_s and K_Δ. In particular, if $f=k_z^u$, then

$$\begin{aligned}k_z^u(w)&=\langle P_Bk_z^u+BP_s\bar{B}k_z^u+BsP_\Delta\overline{Bs}k_z^u,k_w^u\rangle\\&=\langle P_Bk_z^u,k_w^u\rangle+\langle BP_s\bar{B}k_z^u,k_w^u\rangle+\langle BsP_\Delta\overline{Bs}k_z^u,k_w^u\rangle\\&=\langle P_Bk_z^u,k_w^B\rangle+B(w)\langle P_s\bar{B}k_z^u,k_w^s\rangle+B(w)s(w)\langle P_\Delta\overline{Bs}k_z^u,k_w^\Delta\rangle\\&=\langle k_z^u,k_w^B\rangle+B(w)\langle k_z^u,Bk_w^s\rangle+B(w)s(w)\langle k_z^u,Bsk_w^\Delta\rangle\\&=k_z^B(w)+B(w)\overline{B(z)}k_z^s(w)+B(w)s(w)\overline{B(z)s(z)}k_z^\Delta(w).\end{aligned}$$

Therefore, we get the following lemma.

Lemma 3.3. *Let u be an inner function factorized according to* (1-1). *Then*

$$\begin{aligned}\|k_z^u\|^2&=\|k_z^B\|^2+|B(z)|^2\|k_z^s\|^2+|B(z)s(z)|^2\|k_z^\Delta\|^2\\&=\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|1-\bar{a}_n z|^2}|B_n(z)|^2+2|B(z)|^2\int_0^{2\pi}\frac{|s_\theta(z)|^2}{|1-e^{-i\theta}z|^2}\,d\sigma(\theta)\\&\qquad+2|B(z)s(z)|^2\sum_{n=1}^{\infty}\rho_n\frac{\int_{n-1}^{n}|\Delta_\lambda(z)|^2\,d\lambda}{|1-e^{-i\zeta_n}z|^2}.\end{aligned}$$

Lemma 3.4. *Let $a\in\overline{\mathbb{D}}$ and $z\in\mathbb{D}$ with $|z|>\frac{1}{2}$. Then the following hold*:

(i) *We have*

$$|1-\bar{a}z|^2\le 2\big((1-|z|)^2+(1-|a|)^2+(\arg z-\arg a)^2\big).$$

(ii) *If* $\arg a\notin E_z$, *then*

$$|1-\bar{a}z|^2\ge\frac{|1-a|^2}{8}.$$

(iii) *If* $\arg a\in E_z$ *and* $|a|>\frac{1}{2}$, *then*

$$(1-|z|)^2+(1-|a|)^2+(\arg z-\arg a)^2\le\pi^2|1-\bar{a}z|^2.$$

Proof. (i): This follows a direct calculation:

$$\begin{aligned}|1-\bar{a}z|^2 &= (1-|az|)^2+4|az|\sin^2\frac{\arg(\bar{a}z)}{2}\\ &= (1-|az|)^2+4|az|\sin^2\frac{\arg z-\arg a}{2}\\ &\le 2\big((1-|z|)^2+(1-|a|)^2\big)+(\arg z-\arg a)^2\\ &\le 2\big((1-|z|)^2+(1-|a|)^2+(\arg z-\arg a)^2\big).\end{aligned}\tag{3-3}$$

(ii): We divide this part into three cases.

Case 1: $\arg z=0$. In this case, combining (3-3) and $|z|>\frac{1}{2}$, we have

$$|1-\bar{a}z|^2>(1-|a|)^2+2|a|\sin^2\frac{\arg a}{2}\ge\frac{|1-a|^2}{2}.$$

Case 2: $\arg z\in(0,\pi)$ and $\arg a\notin E_z$. In this case either $0\le\arg a\le(\arg z)/2$ or $\arg a\ge 2\arg z$. In the first situation, we have $0\le\arg a\le\arg z-\arg a\le\pi$. In the second situation, we have

$$\begin{cases}(\arg a)/2\le\arg a-\arg z\le\pi & \text{if } 2\arg z\le\arg a\le\pi+\arg z,\\ 2\pi-\arg a\le 2\pi-\arg a+\arg z\le\pi & \text{if } \pi+\arg z\le\arg a<2\pi.\end{cases}$$

If $0\le\arg a\le\arg z-\arg a\le\pi$, then

$$\sin^2\frac{\arg z-\arg a}{2}\ge\sin^2\frac{\arg a}{2}\ge\sin^2\frac{\arg a}{4}.$$

If $2\arg z\le\arg a\le\pi+\arg z$, then

$$\sin^2\frac{\arg z-\arg a}{2}\ge\sin^2\frac{\arg a}{4}.$$

If $\pi+\arg z\le\arg a<2\pi$, then

$$\begin{aligned}\sin^2\frac{\arg z-\arg a}{2}&=\sin^2\frac{2\pi-\arg a+\arg z}{2}\ge\sin^2\frac{\arg a}{2}\\ &=4\sin^2\frac{\arg a}{4}\cos^2\frac{\arg a}{4}\ge 2\cos^2\frac{\arg a}{4}\\ &\ge\cos^2\frac{\arg a}{4}.\end{aligned}$$

So in both situations, combining (3-3) and $|z|>\frac{1}{2}$, we get

$$|1-\bar{a}z|^2\ge(1-|a|)^2+2|a|\min\left\{\sin^2\frac{\arg a}{4},\cos^2\frac{\arg a}{4}\right\}.$$

Again by $\sin^2\frac{\arg a}{2}=4\sin^2\frac{\arg a}{4}\cos^2\frac{\arg a}{4}$, we have

$$\min\left\{\sin^2\frac{\arg a}{4},\cos^2\frac{\arg a}{4}\right\}\ge\frac{1}{4}\sin^2\frac{\arg a}{2}.$$

This leads to

$$|1-\bar{a}z|^2 \geq (1-|a|)^2 + \frac{|a|}{2}\sin^2\frac{\arg a}{2} \geq \frac{|1-a|^2}{8}.$$

Case 3: $\arg z \in [\pi, 2\pi)$ and $\arg a \notin E_z$. By the symmetry, i.e., $\overline{\{e^{i\theta} : \theta \in E_z\}} = \{e^{i\theta} : \theta \in E_{\bar{z}}\}$, the desired result immediately follows from Case 2.

(iii): The condition $\arg a \in E_z$ implies $\arg z \in (0, 2\pi)$ and then

$$\begin{cases} |\arg z - \arg a| \leq \arg z \leq \pi & \text{if } \arg z \in (0,\pi), \\ |\arg z - \arg a| \leq 2\pi - \arg z \leq \pi & \text{if } \arg z \in [\pi, 2\pi). \end{cases} \tag{3-4}$$

Combining (3-3) and $|z| > \frac{1}{2}$, $|a| > \frac{1}{2}$, we have

$$|1-\bar{a}z|^2 \geq (1-|az|)^2 + \sin^2\frac{\arg z - \arg a}{2}.$$

Moreover, it follows from $(1-xy) \geq (2-x-y)/2$ for $0 \leq x, y \leq 1$ that

$$|1-\bar{a}z|^2 \geq \frac{(2-|a|-|z|)^2}{4} + \sin^2\frac{\arg z - \arg a}{2}.$$

This together with $(2-x-y)^2 = ((1-x)+(1-y))^2 \geq (1-x)^2 + (1-y)^2$ for $0 \leq x, y \leq 1$, $\sin x \geq 2x/\pi$ for $x \in [0, \pi/2]$ and (3-4) gives

$$\begin{aligned} |1-\bar{a}z|^2 &\geq \frac{(1-|a|+1-|z|)^2}{4} + \frac{(\arg z - \arg a)^2}{\pi^2} \\ &\geq \frac{(1-|a|)^2 + (1-|z|)^2 + (\arg z - \arg a)^2}{\pi^2}, \end{aligned}$$

as desired. □

Lemma 3.5. *Let B be a Blaschke product with zeros $\{a_n\}$ and Ω_1 be an approach region with vertex at* 1. *For $z \in \mathbb{D}$, write*

$$G_z = \sum_{\substack{\arg a_n \in E_z \\ |a_n| > \frac{1}{2}}} \frac{1-|a_n|}{(1-|z|)^2 + (1-|a_n|)^2 + (\arg z - \arg a_n)^2},$$

$$H_z = \sum_{\arg a_n \in E_z} \frac{1-|a_n|}{(1-|z|)^2 + (\arg z - \arg a_n)^2}.$$

If $\sum_{n\geq 1}(1-|a_n|)/|1-a_n|^2 < \infty$, then there exists a constant C, with $0 < C < 1$, such that

$$C \limsup_{\substack{z\in\Omega_1 \\ z\to 1}} H_z \leq \limsup_{\substack{z\in\Omega_1 \\ z\to 1}} G_z \leq \limsup_{\substack{z\in\Omega_1 \\ z\to 1}} H_z.$$

Proof. If $\arg z = 0$, then $E_z = \varnothing$ and the desired result is obvious. So we can assume that $\arg z \in (0, 2\pi)$. It is clear that $G_z \le H_z$ and hence it suffices to show that

$$C \limsup_{\substack{z\in\Omega_1\\ z\to 1}} H_z \le \limsup_{\substack{z\in\Omega_1\\ z\to 1}} G_z < \infty$$

for some $C \in (0, 1)$.

Now assume $M = \limsup_{z\in\Omega_1, z\to 1} G_z < \infty$, for each $\epsilon > 0$, there exist $r \in (0, 1)$ and $t_0 \in \big(0, \frac{1}{2}\big)$ such that

$$\frac{1 - |a_n|}{(1 - |z|)^2 + (1 - |a_n|)^2 + (\arg z - \arg a_n)^2} \le M + \epsilon,$$

when $z \in \Omega_1$, $|z| \in (r, 1)$, $\arg z \in (0, t_0) \cup (2\pi - t_0, 2\pi)$, $\arg a_n \in E_z$ and $|a_n| > \frac{1}{2}$.

If $\sum_{n\ge 1}(1 - |a_n|)/|1 - a_n|^2 < \infty$, then we conclude from [6, p. 336] that

$$\sum_{\arg a_n\in(0,\pi)} \frac{1 - |a_n|}{(\arg a_n)^2} + \sum_{\arg a_n\in[\pi,2\pi)} \frac{1 - |a_n|}{(2\pi - \arg a_n)^2} < \infty.$$

So there exists $t \in (0, t_0)$ such that

$$1 - |a_n| \le \frac{1}{M + \epsilon + 2},$$

and hence $|a_n| > \frac{1}{2}$ for $\arg a_n \in (0, t) \cup (2\pi - t, 2\pi)$. This implies

$$\begin{aligned}(1 - |a_n|)^2 &\le \frac{1 - |a_n|}{(M + \epsilon + 2)}\\ &\le \frac{M + \epsilon}{M + \epsilon + 2}\big((1 - |z|)^2 + (1 - |a_n|)^2 + (\arg z - \arg a_n)^2\big),\end{aligned}$$

when $z \in \Omega_1$, $|z| \in (r, 1)$, $\arg z \in (0, t/2) \cup (2\pi - t/2, 2\pi)$ and $\arg a_n \in E_z$. This can be rewritten as

$$(1 - |a_n|)^2 \le \frac{M + \epsilon}{2}\big((1 - |z|)^2 + (\arg z - \arg a_n)^2\big),$$

when $z \in \Omega_1$, $|z| \in (r, 1)$, $\arg z \in (0, t/2) \cup (2\pi - t/2, 2\pi)$ and $\arg a_n \in E_z$. Therefore,

$$\frac{1 - |a_n|}{(1 - |z|)^2 + (1 - |a_n|)^2 + (\arg z - \arg a_n)^2} \ge \frac{2}{M + \epsilon + 2} \frac{1 - |a_n|}{(1 - |z|)^2 + (\arg z - \arg a_n)^2},$$

when $z \in \Omega_1$, $|z| \in (r, 1)$, $\arg z \in (0, t/2) \cup (2\pi - t/2, 2\pi)$ and $\arg a_n \in E_z$. Choose $C = 2/(M + \epsilon + 2)$ and we complete the proof. □

Now we are at the position to prove Theorem 2.1.

The proof of Theorem 2.1. (ii) $\Longleftrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are obvious and (iv) $\Longrightarrow$ (i) follows from the uniform boundedness principle.

(i) $\Longrightarrow$ (iii): This follows from the proof of [10, Theorem 4.1].

(i) $\iff$ (v): This is the delicate part. It follows from $\sup_{z\in\Omega_1}\|k_z^u\|<\infty$ that $\liminf_{z\to 1}\|k_z^u\|<\infty$, which is equivalent to (1-3) for $\xi=1$ by Theorem B. Therefore, since in (v) we already assume (1-3) for $\xi=1$, to finish the proof, it suffices to show that if (1-3) holds for $\xi=1$, then $\sup_{z\in\Omega_1}\|k_z^u\|<\infty$ if and only if (2-2) holds.

If (1-3) holds for $\xi=1$ and $|z|>\frac{1}{2}$, then by Lemma 3.4(ii), we have

$$\text{(3-5)}\quad \sum_{\arg a_n\notin E_z}\frac{1-|a_n|^2}{|1-\bar{a}_n z|^2}|B_n(z)|^2+\int_{\theta\notin E_z}\frac{|s_\theta(z)|^2}{|1-e^{-i\theta}z|^2}\,d\sigma(\theta)+\sum_{\zeta_n\notin E_z}\rho_n\frac{\int_{n-1}^{n}|\Delta_\lambda(z)|^2\,d\lambda}{|1-e^{-i\zeta_n}z|^2}$$
$$\le 16\sum_{n=1}^{\infty}\frac{1-|a_n|}{|1-a_n|^2}+8\int_0^{2\pi}\frac{d\sigma(\theta)}{|1-e^{i\theta}|^2}+8\sum_{n=1}^{\infty}\frac{\rho_n}{|1-e^{i\zeta_n}|^2}<\infty.$$

Also it is clear that

$$\text{(3-6)}\quad \sum_{\substack{\arg a_n\in E_z\\ |a_n|\le\frac{1}{2}}}\frac{1-|a_n|^2}{|1-\bar{a}_n z|^2}\le 4\sum_{n=1}^{\infty}(1-|a_n|^2)<\infty.$$

It follows from Lemma 3.3 that

$$\|k_z^u\|^2=\|k_z^B\|^2+|B(z)|^2\|k_z^s\|^2+|B(z)s(z)|^2\|k_z^\Delta\|^2,$$

which together with $\sup_{z\in\Omega_1}\|k_z^u\|<\infty$ and Lemma 3.1 implies $|u(z)|\to 1$ as $z\to 1$ within Ω_1. Hence we can find $r\in\left(\frac{1}{2},1\right)$ such that $|u(z)|>\frac{1}{2}$ whenever $z\in\Omega_1$ and $|z|>r$. Recall that when $|u(z)|>\frac{1}{2}$, each of its factors has to be bounded from below by $\frac{1}{2}$. So $\sup_{z\in\Omega_1}\|k_z^u\|<\infty$ is equivalent to

$$\text{(3-7)}\quad \sup_{\substack{z\in\Omega_1\\ |z|>r}}\left(\|k_z^B\|^2+\|k_z^s\|^2+\|k_z^\Delta\|^2\right)<\infty.$$

Combining (3-5), (3-6) and Lemma 3.3 together, (3-7) in turn is equivalent to

$$\sum_{\substack{\arg a_n\in E_z\\ |a_n|>\frac{1}{2}}}\frac{1-|a_n|^2}{|1-\bar{a}_n z|^2}+\int_{\theta\in E_z}\frac{d\sigma(\theta)}{|1-e^{-i\theta}z|^2}+\sum_{\zeta_n\in E_z}\frac{\rho_n}{|1-e^{-i\zeta_n}z|^2}<\infty,$$

when $z\in\Omega_\xi$ and $|z|>r$. This is equivalent to (2-2) by the combination of Lemmas 3.4 and 3.5. This completes the proof. □

Remark 3.6. From the proof of Theorem 2.1, it is easy to see that if we replace the approach region Ω_1 by $\partial\Omega_1\setminus\{1\}$, $\overline{\Omega}_1\setminus\{1\}$ or any arc lying in $\mathbb{D}$ and terminating at 1, Theorem 2.1 still holds.

4. Norm convergence

We establish another set of technical lemmas which are used to prove Theorem 2.2.

By Theorem B and [7, Corollary 1.7.14, p. 29], we have the following lemma.

Lemma 4.1. *Let u be an inner function factorized according to* (1-1). *If* (1-3) *holds for* $\xi = 1$, *then*

$$\begin{aligned}\|k_1^u\|^2 &= \|k_1^B\|^2 + \|k_1^s\|^2 + \|k_1^\Delta\|^2 \\ &= \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-a_n|^2} + 2\int_0^{2\pi} \frac{d\sigma(\theta)}{|1-e^{i\theta}|^2} + 2\sum_{n=1}^{\infty} \frac{\rho_n}{|1-e^{i\zeta_n}|^2}.\end{aligned}$$

Lemma 4.2. *Let B be a Blaschke product with zeros $\{a_n\}$ such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-a_n|^2} < \infty.$$

Then

$$\|k_1^B\|^2 = \lim_{\substack{z\in\mathbb{D}\\ z\to 1}} \sum_{\arg a_n \notin E_z} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2.$$

Proof. By hypothesis, it is clear that $\{n \in \mathbb{N} : \arg a_n = 0\}$ is a finite set. Hence B admits a decomposition $B = f_0 f_1$, where f_0 and f_1 are two Blaschke products with zeros $\{a_n : \arg a_n = 0\}$ and $\{a_n : \arg a_n \in (0, 2\pi)\}$, respectively.

Note that $k_z^B = k_z^{f_0} + \overline{f_0(z)} f_0 k_z^{f_1}$ and hence

$$\|k_z^B\|^2 = \|k_z^{f_0}\|^2 + |f_0(z)|^2 \|k_z^{f_1}\|^2.$$

Observe that f_0 is a finite Blaschke product and for each a_n with $\arg a_n = 0$, we have $\arg a_n \notin E_z$ for any $z \in \mathbb{D}$, so by (3-3), we conclude that

$$\|k_1^{f_0}\|^2 = \lim_{\substack{z\in\mathbb{D}\\ z\to 1}} \|k_z^{f_0}\|^2 = \lim_{\substack{z\in\mathbb{D}\\ z\to 1}} \sum_{\arg a_n \notin E_z} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2.$$

Moreover, it is clear that $\lim_{z\in\mathbb{D}, z\to 1} |f_0(z)| = 1$. So to end the proof, it suffices to consider the case that the zeros $\{a_n\}$ of B satisfy $\arg a_n \in (0, 2\pi)$.

Now again by hypothesis, there exists a natural number N such that

$$\sum_{n=N+1}^{\infty} \frac{1-|a_n|^2}{|1-a_n|^2} < \epsilon.$$

There also exist two positive numbers $r \in \left(\frac{1}{2}, 1\right)$ and $t_0 \in (0, \pi/2)$ such that

$$\left| \sum_{n=1}^{N} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2 - \sum_{n=1}^{N} \frac{1-|a_n|^2}{|1-a_n|^2} \right| < \epsilon,$$

when $|z| \in (r, 1)$ and $\arg z \in [0, t_0) \cup (2\pi - t_0, 2\pi)$.

For $|z| \in (r, 1)$, Lemma 3.4(ii) shows that

$$\sum_{\substack{n=N+1 \\ \arg a_n \notin E_z}}^{\infty} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2 \leq 8 \sum_{n=N+1}^{\infty} \frac{1-|a_n|^2}{|1-a_n|^2} < 8\epsilon.$$

Put

$$t = \min\bigl\{t_0, \min\{(\arg a_n)/2 : 1 \leq n \leq N, \arg a_n \in (0,\pi)\}, \\ \min\{(2\pi - \arg a_n)/2 : 1 \leq n \leq N, \arg a_n \in [\pi, 2\pi)\}\bigr\}.$$

Here we claim that, for $1 \leq n \leq N$, if $\arg z \in [0,t) \cup (2\pi - t, 2\pi)$, then $\arg a_n \notin E_z$. To do this, three cases arise. If $\arg z = 0$, then $E_z = \varnothing$, so that $\arg a_n \notin E_z$ for every natural number n. But if $\arg z \in (0,t)$, then for $\arg a_n \in E_z$, we have $0 < (\arg a_n)/2 < \arg z < t$, which implies $n > N$. Finally, if $\arg z \in (2\pi - t, 2\pi)$, whenever $\arg a_n \in E_z$, we have $0 < (2\pi - \arg a_n)/2 < (2\pi - \arg z) < t$, which implies $n > N$. Thus the claim is proved.

Let $|z| \in (r, 1)$ and $\arg z \in [0,t) \cup (2\pi - t, 2\pi)$. Then the above discussion implies

$$\begin{aligned} &\Biggl| \sum_{\arg a_n \notin E_z} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2 - \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-a_n|^2} \Biggr| \\ &\quad \leq \Biggl| \sum_{\arg a_n \notin E_z} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2 - \sum_{n=1}^{N} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2 \Biggr| \\ &\quad\quad + \Biggl| \sum_{n=1}^{N} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} |B_n(z)|^2 - \sum_{n=1}^{N} \frac{1-|a_n|^2}{|1-a_n|^2} \Biggr| + \Biggl| \sum_{n=1}^{N} \frac{1-|a_n|^2}{|1-a_n|^2} - \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-a_n|^2} \Biggr| \\ &\quad < 10\epsilon, \end{aligned}$$

which together with Lemma 4.1 gives the desired result. ☐

Lemma 4.3. *Let Ω_1 be an approach region with vertex at* 1 *and let $u = s\Delta$ be a singular inner function factorized according to* (1-1). *If* $\sup_{z \in \Omega_1} \|k_z^{s\Delta}\| < \infty$, *then*

$$\|k_1^s\|^2 = 2 \lim_{\substack{z \in \Omega_1 \\ z \to 1}} \int_{\theta \notin E_z} \frac{|s_\theta(z)|^2}{|1-e^{-i\theta}z|^2}\, d\sigma(\theta) \tag{4-1}$$

and

$$\|k_1^\Delta\|^2 = 2 \lim_{\substack{z \in \Omega_1 \\ z \to 1}} \sum_{\zeta_n \notin E_z} \rho_n \frac{\int_{n-1}^{n} |\Delta_\lambda(z)|^2\, d\lambda}{|1-e^{-i\zeta_n}z|^2}. \tag{4-2}$$

Proof. To begin with, we first prove (4-1). It is clear that

$$\sup_{z \in \Omega_1} \|k_z^s\| \leq \sup_{z \in \Omega_1} \|k_z^u\| < \infty.$$

So Lemma 3.1 together with Theorem 2.1 implies that s has a limit of modulus 1 as $z \to 1$ within Ω_1 and (1-3) holds for $\xi = 1$. Consequently,

$$M = \int_0^{2\pi} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} < \infty. \tag{4-3}$$

Recall that σ is a finite positive continuous singular measure with respect to Lebesgue measure, so by (4-3), we conclude that for each $\epsilon > 0$, there exists $t_0 \in (0, \pi/2)$ such that

$$\int_{\theta \in B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} < \epsilon, \tag{4-4}$$

where $B_{t_0} = [0, t_0) \cup (2\pi - t_0, 2\pi)$. There also exist two positive numbers $r \in \left(\frac{1}{2}, 1\right)$ and $t \in (0, t_0/2)$ such that $1 - |s(z)|^2 < \epsilon$ and

$$\left| \int_{\theta \notin B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{-i\theta} z|^2} - \int_{\theta \notin B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} \right| < \epsilon, \tag{4-5}$$

when $z \in \Omega_1$, $|z| \in (r, 1)$ and $\arg z \in [0, t) \cup (2\pi - t, 2\pi)$.

Now let $z \in \Omega_1$, $|z| \in (r, 1)$ and $\arg z \in [0, t) \cup (2\pi - t, 2\pi)$. Then $E_z \subseteq B_{t_0}$ and hence Lemma 3.4(2) yields

$$\int_{\theta \in (B_{t_0} \setminus E_z)} \frac{|s_\theta(z)|^2}{|1 - e^{-i\theta} z|^2} \, d\sigma(\theta) \le 8 \int_{\theta \in B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} < 8\epsilon \tag{4-6}$$

and

$$\int_{\theta \notin B_{t_0}} \frac{1 - |s_\theta(z)|^2}{|1 - e^{-i\theta} z|^2} \, d\sigma(\theta) \le 8\big(1 - |s(z)|^2\big) \int_{\theta \notin E_z} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} \le 8M\epsilon. \tag{4-7}$$

Putting (4-4)–(4-7) together, it follows that

$$\begin{aligned}
&\left| \int_{\theta \notin E_z} \frac{|s_\theta(z)|^2}{|1 - e^{-i\theta} z|^2} \, d\sigma(\theta) - \int_0^{2\pi} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} \right| \\
&\quad \le \left| \int_{\theta \notin E_z} \frac{|s_\theta(z)|^2}{|1 - e^{-i\theta} z|^2} \, d\sigma(\theta) - \int_{\theta \notin B_{t_0}} \frac{|s_\theta(z)|^2}{|1 - e^{-i\theta} z|^2} \, d\sigma(\theta) \right| \\
&\qquad + \left| \int_{\theta \notin B_{t_0}} \frac{|s_\theta(z)|^2}{|1 - e^{-i\theta} z|^2} \, d\sigma(\theta) - \int_{\theta \notin B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{-i\theta} z|^2} \right| \\
&\qquad + \left| \int_{\theta \notin B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{-i\theta} z|^2} - \int_{\theta \notin B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} \right| \\
&\qquad + \left| \int_{\theta \notin B_{t_0}} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} - \int_0^{2\pi} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^2} \right| \\
&\quad \le (10 + 8M)\epsilon,
\end{aligned}$$

which together with Lemma 4.1 shows that (4-1) holds.

The proof of (4-2) is similar to the proof of (4-1) and hence it will be omitted. This completes the proof. □

The proof of Theorem 2.2. To begin with, we show that $k_z^u \to k_1^u$ in the norm of K_u as $z \to 1$ within Ω_1 if and only if $\|k_1^u\| = \lim_{z\in\Omega_1, z\to 1} \|k_z^u\|$. Indeed, if $\|k_1^u\| = \lim_{z\in\Omega_1, z\to 1} \|k_z^u\|$, then Theorem B implies that $u(1)$ exists and Lemma 3.1 implies that

$$u_{\Omega_1}(1) := \lim_{\substack{z\in\Omega_1\\ z\to 1}} u(z), \quad |u_{\Omega_1}(1)| = 1.$$

A theorem of Lindelöf [8, Theorem 2.3, p. 19] says that if $h \in H^\infty$ and $h(z) \to A$ as $z \to 1$ along a continuous path $G \subset \mathbb{D}$ terminating at 1, then $h(z) \to A$ as $z \to 1$ in any Stolz domain $\Gamma_\alpha(1)$. This implies $u_{\Omega_1}(1) = u(1)$. So for every $w \in \mathbb{D}$,

$$k_1^u(w) = \frac{1-\overline{u(1)}u(w)}{1-w} = \lim_{\substack{z\in\Omega_1\\ z\to 1}} \frac{1-\overline{u(z)}u(w)}{1-\bar{z}w} = \lim_{\substack{z\in\Omega_1\\ z\to 1}} k_z^u(w). \tag{4-8}$$

Moreover, by Theorem 2.1, it follows that k_z^u weakly as $z \to 1$ within Ω_1 and we denote this limit by k_{1,Ω_1}^u. Combining this and (4-8), we have $k_{1,\Omega_1}^u = k_1^u$.

Therefore,

$$\lim_{\substack{z\in\Omega_1\\ z\to 1}} \|k_z^u - k_1^u\|^2 = \lim_{\substack{z\in\Omega_1\\ z\to 1}} (\|k_z^u\|^2 - 2\operatorname{Re}\langle k_z^u, k_1^u\rangle + \|k_1^u\|^2) = 0.$$

Consequently, to end the proof, it suffices to show that if $\sup_{z\in\Omega_1} \|k_z^u\| < \infty$ and (1-3) holds for $\xi = 1$, then $\|k_1^u\| = \lim_{z\in\Omega_1, z\to 1} \|k_z^u\|$ if and only if (2-3) holds.

It is clear that $\{n \in \mathbb{N} : |a_n| \le \frac{1}{2}\}$ is a finite set. Hence we assume that $|a_j| \le \frac{1}{2}$ for $1 \le j \le N_1$ and $|a_j| > \frac{1}{2}$ for all $j > N_1$, where N_1 is some positive integer. Now write $B(z) = f_0(z) f_1(z)$, where f_0 is a finite Blaschke product with zeros $\{a_n : 1 \le n \le N_1\}$ and f_1 is a Blaschke product with zeros $\{a_n : n > N_1\}$. Note that $k_z^B = k_z^{f_0} + \overline{f_0(z)} f_0 k_z^{f_1}$ and hence

$$\|k_z^u\|^2 = \|k_z^{f_0}\|^2 + |f_0(z)|^2 \|k_z^{f_1}\|^2 + |B(z)|^2 \|k_z^s\|^2 + |B(z)s(z)|^2 \|k_z^\Delta\|^2.$$

Now Lemma 4.1 implies

$$\|k_1^u\|^2 = \|k_1^{f_0}\|^2 + \|k_1^{f_1}\|^2 + \|k_1^s\|^2 + \|k_1^\Delta\|^2.$$

Since f_0 is a finite Blaschke product, it is clear that

$$\|k_1^{f_0}\| = \lim_{\substack{z\in\mathbb{D}\\ z\to 1}} \|k_z^{f_0}\|.$$

Lemma 3.1 shows that both B and s have limits of modulus 1 as $z \to 1$ within Ω_1. This together with Lemmas 3.4, 3.5, 4.2 and 4.3 shows that $\|k_1^u\| = \lim_{z\in\Omega_1, z\to 1} \|k_z^u\|$ if and only if (2-3) holds. This completes the proof. □

5. Special approach regions

In this section, we consider two special approach regions:

(i) For $c > 0$, an oricyclic approach region is defined by

$$E(c,1)=\{z\in\mathbb{D}:|1-z|^2<c(1-|z|^2)\}.$$

A simple computation shows that $E(c,1)$ is an open disk internally tangent to the circle at 1 with center $1/(1+c)$ and radius $c/(1+c)$.

(ii) For $c > 0$ and $\gamma \geq 1$, let

$$R'(c,\gamma)=\{z\in\mathbb{D}\setminus\{0\}:1-|z|>c\lfloor\arg z\rfloor^{\gamma}\},\tag{5-1}$$

where $\lfloor\arg z\rfloor$ means the length of the shorter one of the two arcs on $\mathbb{T}$ joining $z/|z|$ and 1.

In what follows, we first show that f has a limit as $z \to 1$ within $R'(c,\gamma)$ if and only if f has a limit as $z \to 1$ within $R(c,\gamma)$. In fact, we can prove this equivalence in any approach region.

Lemma 5.1. *Let u be an inner function and Ω_1 be an approach region with vertex at* 1. *Then*

$$\sup_{z\in\Omega_1}\|k_z^u\|<\infty \iff \sup_{\omega\in\bar{\Omega}_1\setminus\{1\}}\|k_\omega^u\|<\infty.$$

Proof. It is clear that

$$\sup_{\omega\in\bar{\Omega}_1\setminus\{1\}}\|k_\omega^u\|<\infty \implies \sup_{z\in\Omega_1}\|k_z^u\|<\infty$$

and since every $f \in K_u$ is continuous at every point $\eta \in \partial\Omega_1 \setminus \{1\}$,

$$\sup_{\eta\in\partial\Omega_1\setminus\{1\}}|f(\eta)|\leq\sup_{z\in\Omega_1}|f(z)|,$$

so that

$$\sup_{\eta\in\Omega_1}\|k_\eta^u\|<\infty\implies\sup_{z\in\bar{\Omega}_1\setminus\{1\}}\|k_z^u\|<\infty$$

by Theorem 2.1 and Remark 3.6. □

In particular, when $\Omega_1 = R'(c,\gamma)$, we recover Theorem C [12, Theorem 4].

Corollary 5.2. *Let u be an inner function factorized according to* (1-1). *Then*

$$\sup_{z\in R(c,\gamma)}\|k_z^u\|<\infty$$

if and only if (1-3) *holds for $\xi = 1$ and*

$$\limsup_{t\to0^+}F_{t,\gamma}<\infty,$$

where $F_{t,\gamma}$ is seen in (2-1).

Proof. By Remark 3.6 for $\Omega_1 = R'(c, \gamma)$, to finish the proof, it suffices to show that

$$\limsup_{\substack{z\in R(c,\gamma)\\ z\to 1}} F_z < \infty \iff \limsup_{t\to 0^+} F_{t,\gamma} < \infty.$$

$\Longleftarrow$: For $z \in \mathbb{D}$ with $\lfloor \arg z \rfloor \in [0, \pi/4)$, write $t = \lfloor \arg z \rfloor$. If $\arg z \in [0, \pi/4)$, then $\theta \in E_z$ implies

$$\lfloor \theta \rfloor = \theta \in (t/2, 2t) \quad \text{and} \quad (\arg z - \theta)^2 = (t - \lfloor \theta \rfloor)^2. \tag{5-2}$$

If $\arg z \in (7\pi/4, 2\pi)$, then $\theta \in E_z$ implies

$$\begin{aligned} \lfloor \theta \rfloor = 2\pi - \theta \in \bigl(2\pi - (\pi + (\arg z)/2), 2\pi - 2(\arg z - \pi)\bigr) = (t/2, 2t),\\ (\arg z - \theta)^2 = ((2\pi - t) - (2\pi - \theta))^2 = (t - \lfloor \theta \rfloor)^2. \end{aligned} \tag{5-3}$$

For $z \in R(c, \gamma)$ with $\lfloor \arg z \rfloor \in [0, \pi/4)$, we have $1 - |z| \geq c \lfloor \arg z \rfloor^{\gamma}$ and then by (5-2) and (5-3), we have

$$F_z \leq \frac{1}{\min\{c^2, 1\}} F_{t,\gamma},$$

which implies

$$\limsup_{\substack{z\in R(c,\gamma)\\ z\to 1}} F_z \leq \frac{1}{\min\{c^2, 1\}} \limsup_{t\to 0^+} F_{t,\gamma} < \infty.$$

$\Longrightarrow$: It is easy to see that

$$\limsup_{\substack{z\in \partial R(c,\gamma)\\ z\to 1}} F_z \leq \limsup_{\substack{z\in R(c,\gamma)\\ z\to 1}} F_z < \infty.$$

For $t \in [0, t_0]$ for some $t_0 \in (0, \pi/4)$, there exist $z_1, z_2 \in \partial R(c, \gamma)$ such that $\arg z_1 = t$ and $\arg z_2 = 2\pi - t$. Thus $\lfloor \theta \rfloor \in (t/2, 2t)$ implies that either $\theta \in (t/2, 2t)$ if $\theta \in [0, \pi)$ or $2\pi - \theta \in (t/2, 2t)$ if $\theta \in [\pi, 2\pi)$. In the first situation,

$$\theta \in E_{z_1} \quad \text{and} \quad (t - \lfloor \theta \rfloor)^2 = (\arg z_1 - \theta)^2. \tag{5-4}$$

In the second situation,

$$\begin{aligned} \theta \in \left(2\pi - 2t, 2\pi - \frac{t}{2}\right) &= \left(2\pi - 2(2\pi - \arg z_2), 2\pi - \frac{2\pi - \arg z_2}{2}\right)\\ &= \left(2(\arg z_2 - \pi), \pi + \frac{\arg z_2}{2}\right)\\ &= E_{z_2},\\ (t - \lfloor \theta \rfloor)^2 = (2\pi - \arg z_2 &- (2\pi - \theta))^2 = (\arg z_2 - \theta)^2. \end{aligned} \tag{5-5}$$

For $z \in \partial R'(c,\gamma) \setminus \{1\}$, we have $1-|z| = c\lfloor \arg z \rfloor^{\gamma}$ and then by (5-4) and (5-5), we have

$$F_{t,\gamma} \le \frac{1}{\min\{1/c^2, 1\}} \max\{F_{z_1}, F_{z_2}\},$$

and if $t \to 0^+$, then both z_1 and z_2 tend to 1. Therefore,

$$\limsup_{t\to 0^+} F_{t,\gamma} \le \frac{1}{\min\{1/c^2, 1\}} \limsup_{\substack{z\in\partial R(c,\gamma)\\ z\to 1}} F_z < \infty. \qquad \square$$

Combining Corollary 5.2 and Lemma 5.1, we immediately get the following result.

Corollary 5.3. *Let u be an inner function factorized according to* (1-1). *Then* $\sup_{z\in R'(c,\gamma)} \|k_z^u\| < \infty$ *if and only if* $\sup_{z\in R(c,\gamma)} \|k_z^u\| < \infty$. *In particular,*

$$\sup_{z\in E(c,1)} \|k_z^u\| < \infty$$

if and only if (1-3) *holds for* $\xi = 1$ *and*

$$\limsup_{t\to 0^+} F_{t,2} < \infty,$$

where $F_{t,2}$ *is seen in* (2-1).

Note that (2-2) is not easy to calculate. So in what follows, for the approach region $R'(c,\gamma)$, we give a simple example showing that (2-2) can be greatly simplified.

Example 5.4. Let B be a Blaschke product with zeros $\{a_n = r_n e^{i/2^n} : n \in \mathbb{N}\}$ and $R'(c,\gamma)$ be defined by (5-1). In a moment, we will show that k_z^B converges weakly as $z \to 1$ within $R'(c,\gamma)$ if and only if

$$\sum_{n=1}^{\infty} 4^n(1-r_n) < \infty \quad \text{and} \quad \limsup_{n\to\infty} 4^{n\gamma}(1-r_n) < \infty.$$

It has been shown by Cargo [6, p. 336] that $\sum_{n=1}^{\infty}(1-r_n)/|1-r_n e^{i/2^n}|^2 < \infty$ is equivalent to $\sum_{n=1}^{\infty} 4^n(1-r_n) < \infty$. So by Corollary 5.2, it suffices to show that

$$\limsup_{\arg z\to 0} \sum_{\frac{1}{2^n}\in E_z} \frac{1-r_n}{(\arg z)^{2\gamma} + \left(\arg z - \frac{1}{2^n}\right)^2} < \infty \iff \limsup_{n\to\infty} 4^{n\gamma}(1-r_n) < \infty.$$

$\Longrightarrow$: This follows by choosing the subsequence $\{\arg z = 1/2^n\}$.

$\Longleftarrow$: For each $\arg z \in (0, \pi/4)$, $1/2^n \in E_z$ if and only if $-\log_2(\arg z) - 1 < n < -\log_2(\arg z) + 1$, which means that there are at most two natural numbers n_1, n_2

such that $1/2^{n_1}, 1/2^{n_2} \in E_z$, and hence $\arg z > 1/2^{n_i+1}$ for $i = 1, 2$. Therefore,

$$\limsup_{\arg z\to 0} \sum_{\frac{1}{2^n}\in E_z} \frac{1-r_n}{(\arg z)^{2\gamma} + \left(\arg z - \frac{1}{2^n}\right)^2} \leq \limsup_{n_1,n_2\to\infty} \{4^{(n_1+1)\gamma}(1-r_{n_1}) + 4^{(n_2+1)\gamma}(1-r_{n_2})\} < \infty,$$

as desired.

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