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# A NOTE ON INFINITE PARTITIONS OF FREE PRODUCTS OF BOOLEAN ALGEBRAS

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**The cardinal invariants $\mathfrak{a}(n)$, for $1 \le n < \omega$, multidimensional generalizations of the mad family number $\mathfrak{a}$, were proved by Spinas (*Pacific J. Math.* 176:1 (1996), 249–262) to be greater than or equal to the bounding number $\mathfrak{b}$. The lack of knowledge of other lower bounds for these cardinal invariants was noted in the same article. We present a couple of more general results that give lower bounds to the cardinal $\mathfrak{a}(A \oplus B)$, where $A \oplus B$ is the free product of the Boolean algebras $A$ and $B$. One of them, when restricted to the free products of $\mathcal{P}(\omega)/\mathrm{fin}$, gives a new proof of the known result. The other has as a corollary a newer lower bound that adds relevant information to the open question of the consistency of $\mathfrak{a}(n+1) < \mathfrak{a}(n)$, for some $1 \le n < \omega$.**

## 1. Families mad in their own way

*Mad families* of infinite subsets of integers and their minimum size $\mathfrak{a}$ are a very important keystone and source of study in infinite combinatorics and cardinal characteristics of the continuum. The relation of $\mathfrak{a}$ to other cardinals has been extensively studied. Some of the cardinal characteristics of the continuum whose relation to $\mathfrak{a}$ has been of great importance are the *bounding* number $\mathfrak{b}$, the *splitting* number $\mathfrak{s}$ and the *dominating* number $\mathfrak{d}$. The relations provable in ZFC that involve these cardinal characteristics are $\mathfrak{b} \le \mathfrak{a}$, $\mathfrak{b} \le \mathfrak{d}$ and $\mathfrak{s} \le \mathfrak{d}$.

Since the consistency of $\omega_1 = \mathfrak{s} = \mathfrak{b} = \mathfrak{a} < \mathfrak{d} = \omega_2$ was established by the Cohen model, proving the consistency of some of the strict inequalities between any pair of these cardinals has gone hand in hand with the development of forcing techniques. Shelah [1984; 2004] proved the consistency of $\mathfrak{b} < \mathfrak{a}$ and $\mathfrak{d} < \mathfrak{a}$ developing, respectively, *creature* forcing and forcing iterations along a *template*. A *matrix iteration* was used in [Brendle and Fischer 2011] for proving the consistency of $\mathfrak{a} < \mathfrak{s}$. With the exception of $\mathfrak{d} < \mathfrak{a}$, all these inequalities are known to be consistent with the smaller cardinal equal to $\omega_1$ and the larger cardinal equal to $\omega_2$. Thus, the following important question remains open since the 1970's.

**Question 1.1** (Roitman). *Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?*

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*Keywords:* Boolean algebras, free products, mad families, cardinal invariants.

Since adding any reals destroys many ground model mad families, another way mad families have influenced research in forcing theory is the study of *indestructible* mad families. The standard proof of $\mathfrak{a} = \omega_1$ holding in the Cohen model consists of the construction, under the assumption of CH, of a Cohen-*indestructible* mad family, i.e., a mad family that remains a mad family after adding a Cohen real. Important results on the (consistent) existence of $\mathbb{P}$-indestructible mad families for large classes of forcing notions $\mathbb{P}$ can be found in [Raghavan 2009; Guzmán and Hrušák 2022; Brendle et al. 2022].

As a natural consequence of this broad research corpus, specializations and generalizations of the concept of mad families have arisen. Among the former, we can find mad families of functions (i.e., whose elements are subsets of $\omega \times \omega$ which are graphs of functions $f : \omega \to \omega$), which are studied in [Blass et al. 2005; Raghavan 2009], or mad families that are constructed by the union of definable subsets. If $\mathfrak{a}_{\text{closed}}$ is the least size of a family of closed subsets whose union is a mad family, in interesting contrast to what is known about $\mathfrak{a}$, in [Raghavan and Shelah 2012] it was proved that $\mathfrak{d} = \omega_1$ implies that $\mathfrak{a}_{\text{closed}} = \omega_1$, and in [Brendle and Raghavan 2014] the consistency of $\mathfrak{a}_{\text{closed}} < \mathfrak{b}$ was proved.

One generalization comes from the study of the quotients $\mathcal{P}(X)/\mathcal{I}$, where $X$ is a countable set and $\mathcal{I}$ an ideal on $X$, where a family $\mathcal{A} \subseteq \mathcal{P}(X) \setminus \mathcal{I}$ is called $\mathcal{I}$-mad if $X \cap Y \in \mathcal{I}$, for all different $X, Y \in \mathcal{A}$, and is maximal with that property. Defining $\bar{\mathfrak{a}}(\mathcal{I})$ as the least size of an uncountable $\mathcal{I}$-mad family, a parallel to mad families is the result that $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$, for all coanalytic $P$-ideals $\mathcal{I}$, which is proved in [Farkas 2011]. An interesting contrast to this is the consistency of $\bar{\mathfrak{a}}(\mathcal{ED}_{\text{fin}}) < \mathfrak{b}$, presented in [Brendle 2009], where $\mathcal{ED}_{\text{fin}}$ is an $F_\sigma$ ideal defined in [Hernández-Hernández and Hrušák 2007].

Also, since at least [Cummings and Shelah 1995], cardinal characteristics on uncountable cardinals have been studied. Recent advances in this topic, including the study of maximal $\kappa$-ad families, for uncountable cardinals $\kappa$, and its least possible size, denoted $\mathfrak{a}_\kappa$, can be found in [Blass et al. 2005; Raghavan and Shelah 2017; 2019; Brendle et al. 2018]. In contrast to the case $\kappa = \omega$, for which it is consistent that $\omega_1 = \mathfrak{b} < \mathfrak{a}$, for uncountable regular $\kappa$ we have that $\mathfrak{b}_\kappa = \kappa^+$ implies $\mathfrak{a}_\kappa = \kappa^+$.

Another natural path of generalization for mad families is the study of their *multi-dimensional* versions. A simple $n$-dimensional version of mad families, for $n < \omega$, would be the infinite partitions of the Boolean algebra $\bigoplus^n \mathcal{P}(\omega)/\text{fin}$, or the identical concept of maximal antichains of the poset $(\mathcal{P}(\omega)/\text{fin})^n$. The least sizes of these $n$-mad families, usually denoted $\mathfrak{a}(n)$, form a (not strictly) decreasing sequence of uncountable cardinals. This is the generalization mainly studied here.

To the author's knowledge, most of the advances in this topic have been the finding of some lower bounds for the number $\mathfrak{a}(\mathbb{P} \times \mathbb{Q})$, where $\mathbb{P}$ and $\mathbb{Q}$ are infinite posets with no minimum element, and for the numbers $\mathfrak{a}(n)$, for all $n < \omega$ (see [Kurilić

2017; Spinas 1996]). In particular, in the case of [Spinas 1996, Theorem 1.2], it was first proved that $\mathfrak{b} \leq \mathfrak{a}(n)$, for all $1 \leq n < \omega$. This result generalizes the known inequality $\mathfrak{b} \leq \mathfrak{a}$. In [Spinas 1996], both the existence of other lower bounds for $\mathfrak{a}(n)$ and the consistency of $\mathfrak{a}(n+1) < \mathfrak{a}(n)$, for some $n < \omega$, are left as the object of further research.

Here, the cardinal $\mathfrak{a}(A \oplus B)$, where $A$ and $B$ are infinite Boolean algebras, is studied. Since every partition $P$ of $A$ (and $B$) induces a partition of $A \oplus B$, namely $\{a \cdot 1 : a \in P\}$, it easily follows that $\mathfrak{a}(A \oplus B) \leq \mathfrak{a}(A), \mathfrak{a}(B)$, for all infinite Boolean algebras $A$ and $B$. This observation motivated the next question, asked in [Monk 2014, Problem 8].

**Question 1.2.** *Does*

$$\mathfrak{a}(A \oplus B) = \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$$

*hold for any pair of infinite Boolean algebras $A$ and $B$?*

As an advance for the solution of this question, the following lower bound was given to $\mathfrak{a}(A \oplus B)$ in [Santos 2023, Theorem 13].

**Theorem 1.3.** *If $A$ and $B$ are infinite Boolean algebras, then*

$$\min\Big\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{p}(A), \mathfrak{p}(B)\}\Big\} \leq \mathfrak{a}(A \oplus B).$$

Observe that from this theorem it follows that any instance of

$$\mathfrak{a}(A \oplus B) < \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$$

is one of $\mathfrak{p}(A), \mathfrak{p}(B) < \mathfrak{a}(A), \mathfrak{a}(B)$. Since $\mathfrak{a}(A) = \omega$ if and only if $\mathfrak{p}(A) = \omega$, for every infinite Boolean algebra $A$, to obtain such a counterexample we need $\omega_1 \leq \mathfrak{p}(A), \mathfrak{p}(B)$, and hence $\omega_1 \leq \mathfrak{a}(A \oplus B)$.

Theorem 3.2 gives another lower bound for $\mathfrak{a}(A \oplus B)$, generalizing [Spinas 1996, Theorem 1.2], with a simpler proof. In addition, in Theorem 3.1, another lower bound is given, which is related to the *splitting* number and improves that of Theorem 1.3, at least for *homogeneous* Boolean algebras.

Some notation, as well as the definitions and results on Boolean algebras, free products, and their cardinal invariants relevant for both theorems, are given in Section 2. For more basic concepts on Boolean algebras and cardinal characteristics of the continuum, the reader is referred to [Koppelberg 1989; Monk 2014; Blass 2010].

## 2. Boolean algebras and their cardinal invariants

Since every Boolean algebra is isomorphic to the family of clopen sets of some zero-dimensional compact Hausdorff space, equipped with the usual set-theoretic operations ($\cup$, $\cap$, $\setminus$) from now on $A$ and $B$ will be, respectively, the algebra of clopen sets of some zero-dimensional compact Hausdorff spaces $X$ and $Y$.[1]

Formally speaking, if $A$ and $B$ are two Boolean algebras, their free product, denoted $A \oplus B$, is an algebra $C$ such that there exist subalgebras $A'$, $B' \leq C$, such that $A \cong A'$, $B \cong B'$, such that $C$ is algebraically generated by $A' \cup B'$. Given two Boolean algebras $A$ and $B$, this algebra exists and is unique up to isomorphisms. In accordance with the use of topological duality settled at the beginning of this section, in what follows $A \oplus B$ will refer to the algebra of clopen sets of the product space $X \times Y$. Some basic kinds of infinite Boolean algebras are the following.

**Definition 2.1.** A Boolean algebra $A$ will be called homogeneous if for all $x \in A^+$, the Boolean algebra defined on $A \upharpoonright x := \{y \in A : y \subseteq x\}$ with the structure inherited from $A$ is isomorphic to $A$.

**Definition 2.2.** A Boolean algebra $A$ will be called atomless if for all $x \in A^+$ there exists $y \in A^+$ such that $y \subsetneq x$.

Now we have some definitions on the combinatorics of infinite Boolean algebras. By $A^+$ is denoted $A \setminus \{\varnothing\}$, the set of *positive* elements of $A$. We will say that *a splits b* if $b \cap a \neq \varnothing \neq b \setminus a$. If $P \subseteq A^+$, we will say that $P$ is a *centered family* if $\bigcap_{i<n} x_i \neq \varnothing$, for any nonempty finite collection of elements $x_0, \dots, x_{n-1} \in P$. It will be called a (*pairwise*) *disjoint family* if $a$ and $b$ are disjoint, for all distinct $a, b \in P$. Whenever $P$ is a centered family, if $x \in A^+$ and $x \subseteq a$, for all $a \in P$, $x$ will be called a *pseudointersection* of $P$. Two families $C$, $D \subseteq A^+$ will be called *orthogonal* if $x$ and $y$ are disjoint, for all $x \in C$ and all $y \in D$. Some special families arising from these simple concepts are the following.

**Definition 2.3.** Let $A$ be a Boolean algebra.

- A partition of $A$ is a disjoint family $P \subseteq A^+$ that is maximal with respect to this property.
- A splitting family of $A$ is a subset $P \subseteq A^+$ such that all positive elements of $A$ are split by some element of $P$.
- A Rothberger gap of $A$ consists of a pair $(C, D)$ of orthogonal families of $A^+$ such that $C$ is a countable disjoint family and there is no $x \in A^+$ disjoint to all elements of $C$ and an upper bound to all elements of $D$.

Each of these classes of subfamilies of a Boolean algebra $A$ is related to a *cardinal invariant*.

**Definition 2.4.** Let $A$ be an infinite Boolean algebra. Some of its combinatorial cardinal invariants are

[1] For topological duality, as well as other basic topics on Boolean algebras, the reader is referred to [Koppelberg 1989].

- $\mathfrak{a}(A) := \min\{|P| : P \subseteq A^+ \text{ is an infinite partition}\}$,
- $\mathfrak{p}(A) := \min\{|P| : P \subseteq A^+ \text{ is centered with no pseudointersection}\}$,
- $\mathfrak{s}(A) := \min\{|P| : P \subseteq A^+ \text{ is a splitting family}\}$,
- $\mathfrak{b}(A) := \min\{|D| : \exists C \in [A^+]^\omega\ (C, D) \text{ is a Rothberger gap}\}$.

Observe that $\mathfrak{s}(A)$ is well defined only if $A$ is atomless. For $\mathfrak{b}(A)$ to be defined, there must exist $C \in [A^+]^\omega$ with no least upper bound. This happens, for example, when $\omega_1 \leq \mathfrak{a}(A)$. The basic relations between these cardinal invariants are the following.

**Fact 2.5.** *Let $A$ be an infinite Boolean algebra. Then*

- $\mathfrak{p}(A) \leq \mathfrak{a}(A)$,
- $\mathfrak{p}(A) \leq \mathfrak{s}(A)$, *if $A$ is atomless*, *and*
- $\mathfrak{b}(A) \leq \mathfrak{a}(A)$, *if $\omega_1 \leq \mathfrak{a}(A)$.*

*Proof.* For the first inequality, observe that if $P \subseteq A^+$ is an infinite partition of $A$, then $\{X \setminus x : x \in P\}$. For the second one, observe that any maximal centered subfamily of a splitting family is a centered family with no pseudointersection.

Now suppose that $\mathfrak{a}(A) \geq \omega_1$. Let $\{a_\alpha : \alpha < \kappa\}$ be an infinite partition of size $\kappa = \mathfrak{a}(A)$. Observe that $\mathcal{A} := \{a_n : n < \omega\}$ and $\mathcal{B} := \{a_\alpha : \alpha \in \kappa \setminus \omega\}$ form a Rothberger gap. Otherwise, if there is $c \in A^+$ such that $a_n \cap c = \varnothing$, for all $n < \omega$ and $a_\alpha \subseteq c$, for all $\alpha \in \kappa \setminus \omega$, then $\mathcal{A} \cup \{c\}$ is an infinite partition of $A$, which is a contradiction. □

The behavior of the cardinal invariants $\mathfrak{b}(A)$ and $\mathfrak{s}(A)$ on free products is described in the following results. A proof of Theorem 2.7 can be found in [Santos 2023].

**Observation 2.6.** *If $A$ and $B$ are infinite Boolean algebras, then $\mathfrak{b}(A \oplus B) = \omega$.*

Let $\{a_n : n < \omega\}$ and $\{b_n : n < \omega\}$ be disjoint families of $A$ and $B$. Then $\{a_n \times b_n : n < \omega\}$ and $\{a_n \times b_m : m, n < \omega,\ n \neq m\}$ are both sides of a Rothberger gap.

**Theorem 2.7.** $\mathfrak{s}(A \oplus B) = \min\{\mathfrak{s}(A), \mathfrak{s}(B)\}$, *for all infinite atomless Boolean algebras $A$ and $B$.*

Now comes a word on some classic cardinal characteristics of the continuum, many of which are defined as cardinal invariants of the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$. This is the Boolean algebra defined on the equivalence classes of the relation $X \sim Y$, where $X \sim Y$ if and only if $(X \setminus Y) \cap (Y \setminus X)$ is finite, with the operations induced by the set-theoretic operations $\cup$, $\cap$ and $\setminus$.

The cardinal characteristic $\mathfrak{a}$, usually defined as the least size of an infinite *maximal almost disjoint* (mad) family, i.e., a family $\{A_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$, such that $|A_\alpha \cap A_\beta| < \omega$, for all $\alpha < \beta < \kappa$, and which is maximal with this property, can be defined as $\mathfrak{a}(\mathcal{P}(\omega)/\text{fin})$. Similarly, $\mathfrak{s} = \mathfrak{s}(\mathcal{P}(\omega)/\text{fin})$. The cardinal $\mathfrak{b}$ can be defined as $\mathfrak{b}(P(\omega)/\text{fin})$, although its usual definition is the smallest size of a family $\mathcal{F} \subseteq \omega^\omega$ such that for all $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $f(m) > g(m)$, for infinitely many $m < \omega$. The equivalence of both definitions was proved in [Rothberger 1941].

## 3. Lower bounds for $\mathfrak{a}(A \oplus B)$

By definition, if $c \in A \oplus B$, then there exist $\{a_i : i < k\} \subseteq A$ and $\{b_i : i < k\} \subseteq B$, for $k < \omega$, such that

$$c = \bigcup_{i<k} a_i \times b_i.$$

Since

$$c = \bigcup_{\varnothing \neq J \subseteq k} \Big( \bigcap_{i \in J} a_i \setminus \bigcup_{j \in k \setminus J} a_j \Big) \times \bigcup_{i \in J} b_i$$

we can always assume that either $\{a_i : i < k\}$ is a disjoint family or that $\{b_i : i < k\}$ is a disjoint family. Therefore, when dealing with infinite partitions (or disjoint families) of $A \oplus B$ we can always assume that they are of the form $\{a_\alpha \times b_\alpha : \alpha < \kappa\}$, where $a_\alpha \in A$ and $b_\alpha \in B$, for all $\alpha < \kappa$. Now we give the main results of the article, a couple of lower bounds for $\mathfrak{a}(A \oplus B)$. The similarity of the proof of Theorem 3.1 to that of [Raghavan and Steprāns 2023, Theorem 2.23], mainly in the use of $\kappa < \mathfrak{s}(A \restriction x)$, for any $x \in A^+$, which here is implied by homogeneity, has been observed by the author. More about this similarity will be given in Section 4.

**Theorem 3.1.** *Let $A$ and $B$ be homogeneous Boolean algebras. Then*

$$\min\big\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{s}(A), \mathfrak{s}(B)\}\big\} \leq \mathfrak{a}(A \oplus B).$$

*Proof.* Suppose that $\omega_1 \leq \kappa < \mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{s}(A), \mathfrak{s}(B)\}$ and that the set $P = \{a_\alpha \times b_\alpha : \alpha < \kappa\}$ is a disjoint subfamily of $A \oplus B$. Without loss of generality, we can suppose that $\kappa < \mathfrak{s}(A)$. We will prove two cases.

*Case* 1: There exists $E \in [\kappa]^\omega$ such that $\{a_\alpha : \alpha \in E\}$ is a centered family. Without loss of generality $E = \omega$. Since $\omega_1 \leq \mathfrak{p}(A)$, we can take $a' \in A^+$ such that $a' \subseteq a_n$, for all $n < \omega$. Furthermore, take $a \in A \restriction a'$ that witnesses that $\{a_\alpha \cap a' : \alpha < \kappa\}$ is not a splitting family of $A \restriction a'$, i.e., for all $\alpha < \kappa$, either $a \cap a_\alpha = \varnothing$ or $a \subseteq a_\alpha$. Since $E := \{\alpha < \kappa : a \subseteq a_\alpha\}$ is an infinite set, it follows that $\{b_\alpha : \alpha \in E\}$ is an infinite disjoint family of $B$. Also, since $\kappa < \mathfrak{a}(B)$, there exists $b \in B$ such that $b \cap b_\alpha = \varnothing$, for all $\alpha \in E$. Take $\alpha < \kappa$. If $\alpha \in E$, then $b \cap b_\alpha = \varnothing$. If $\alpha \notin E$, then $a \cap a_\alpha = \varnothing$. In either case $a \times b$ is disjoint from $a_\alpha \times b_\alpha$, which means that $P$ is not an infinite partition.

*Case* 2: The family $\{a_\alpha : \alpha \in E\}$ is not centered, for all $E \in [\kappa]^\omega$. Let $M \subseteq [\kappa]^{<\omega}$ be the collection of all subsets $E$ of $\kappa$ such that $\bigcap_{\alpha \in E} a_\alpha \neq \varnothing$, and maximal with that property. Observe that $M$ is infinite. In fact, if $E_0, \ldots, E_{n-1} \in M$, for $n < \omega$, taking $\alpha \in \kappa \setminus \bigcup_{i<n} E_i$ and extending $\{\alpha\}$ to $E$, maximal with the property that $\bigcap_{\alpha \in E} a_\alpha \neq \varnothing$, we obtain a new element of $M$.

For $E \in M$, define $d_E := \bigcap_{\alpha \in E} a_\alpha$. Observe that $\{d_E : E \in M\}$ is a disjoint family. Since $\kappa < \mathfrak{a}(A)$, there exists $x \in A^+$ such that $x \cap d_E = \varnothing$, for all $E \in M$. Since $A$ is homogeneous, the nonempty elements of $\{x \cap a_\alpha : \alpha < \kappa\}$ do not form a splitting family of $A \restriction x$. Take nonempty $c \subseteq x$ such that for all $\alpha < \kappa$ either $c \subseteq a_\alpha$ or $c \cap a_\alpha = \varnothing$.

Define $F := \{\alpha < \kappa : c \subseteq a_\alpha\}$. If $F = \varnothing$, then $c \times Y$ shows that $P$ is not a partition. Suppose $F$ is not empty. Extend $F$ to a family $E_0 \in M$. Since $c \cap d_{E_0} = \varnothing$, it follows that $F \subsetneq E_0$. Take $\beta \in E_0 \setminus F$. Since $\{a_\alpha : \alpha \in E_0\}$ is centered, then $b_\beta \cap b_\alpha = \varnothing$, for all $\alpha \in F$. Take $\alpha < \kappa$. If $\alpha \notin F$, then $c \cap a_\alpha = \varnothing$. If $\alpha \in F$, then $b_\beta \cap b_\alpha = \varnothing$. Either way, $c \times b_\beta$ witnesses that $P$ is not an infinite partition. □

Since $\max\{\mathfrak{p}(A), \mathfrak{p}(B)\} \leq \max\{\mathfrak{s}(A), \mathfrak{s}(B)\}$, for all homogeneous Boolean algebras $A$ and $B$, this more specific theorem gives an improvement to Theorem 1.3.

**Theorem 3.2.** *Let $A$ and $B$ be Boolean algebras such that $\omega_1 \leq \mathfrak{a}(A), \mathfrak{a}(B)$. Then*

$$\min\big\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{b}(A), \mathfrak{b}(B)\}\big\} \leq \mathfrak{a}(A \oplus B).$$

*Proof.* Suppose without loss of generality that $\kappa < \mathfrak{b}(A), \mathfrak{a}(B)$ and that the set $P = \{a_\alpha \times b_\alpha : \alpha < \kappa\}$ is a disjoint family of $A \oplus B$. As in the previous theorem, we have two cases.

*Case* 1: Suppose that $\{b_n : n < \omega\}$ is a centered family and, hence, that $\{a_n : n < \omega\}$ is a pairwise disjoint family. Take $b \in B^+$ such that $b \subseteq b_n$, for all $n < \omega$. Define $E := \{\alpha \in \kappa \setminus \omega : b_\alpha \cap b \neq \varnothing\}$. It follows that $a_n \cap a_\alpha = \varnothing$, for all $n < \omega$ and $\alpha \in E$. Since $|E| < \mathfrak{b}(A)$, there exists $c \in A$ such that $a_\alpha \subseteq c$, for all $\alpha \in E$, and $a_n \cap c = \varnothing$, for all $n < \omega$. Since $\{a_n : n < \omega\} \cup \{c\}$ is not an infinite partition of $A$, take $a \in A^+$ as a witness of this fact. Take $\alpha < \kappa$. If $\alpha \in \omega \cup E$, then $a \cap a_\alpha = \varnothing$. If $\alpha \notin \omega \cup E$, then $b \cap b_\alpha = \varnothing$. Either way, $a \times b$ witnesses that $P$ is not an infinite partition.

*Case* 2: Suppose that $\{b_\alpha : \alpha \in E\}$ is not centered, for all $E \in [\kappa]^\omega$. Let $\{F_\gamma : \gamma < \kappa\}$ be the family of all $F$ finite subsets of $\kappa$, maximal with the property that $\{b_\alpha : \alpha \in F\}$ is a centered family. Define $d_\gamma := \bigcap_{\alpha \in F_\gamma} b_\alpha$, for $\gamma < \kappa$. Clearly, the family $\{d_\gamma : \gamma < \kappa\}$ is pairwise disjoint. Since $\kappa < \mathfrak{a}(B)$, take $d \in B^+$ such that $d \cap d_\gamma = \varnothing$, for all $\gamma < \kappa$. Take $F \in [\kappa]^{<\omega}$, maximal with the property that $\{b_\alpha : \alpha \in F\} \cup \{d\}$ is a centered family. Take $\gamma < \kappa$ such that $F \subseteq F_\gamma$. Since $d \cap d_\gamma = \varnothing$, it follows that $F \neq F_\gamma$. Take $\beta \in F_\gamma \setminus F$ and define $b := \bigcap_{\alpha \in F} b_\alpha \cap d$. Take $\alpha < \kappa$. If $\alpha \in F$, then $a_\alpha \cap a_\beta = \varnothing$. If $\alpha \notin F$, then $b_\alpha \cap b = \varnothing$. Either way, $a_\beta \times b$ shows that $P$ is not an infinite partition. □

In Section 4, these results are applied to the Boolean algebras $\bigoplus_{i<n} \mathcal{P}(\omega)/\text{fin}$, that is, the free product of $n$ many copies of $\mathcal{P}(\omega)/\text{fin}$.

## 4. Mad families on $\omega^n$

Since this section is mainly set in $\omega^k$, for $2 \leq k < \omega$, some notation for its subsets is given here.

**Notation 4.1.** Take $2 \leq k < \omega$ and $A \subseteq \omega^k$. If $n < \omega$ define

$$A(n) := \{\bar{x} \in \omega^{k-1} : (n) \frown \bar{x} \in A\}.$$

**Notation 4.2.** For $X \subseteq \omega$ and $\{Y_n : n \in X\} \subseteq \mathcal{P}(\omega)$ define

$$\coprod_{n \in X} Y_n := \bigcup_{n \in X} \{n\} \times Y_n.$$

Before applying Theorems 3.1 and 3.2, we define a notion for $n$-dimensional mad family, for $1 \leq n < \omega$, where 1-mad family means simply mad family.

**Definition 4.3.** Take $2 \leq n < \omega$. An infinite family $\{(X_\alpha^0, \dots, X_\alpha^{n-1}) : \alpha < \kappa\} \subseteq ([\omega]^\omega)^n$ is called an $n$-ad family if for all $\alpha < \beta < \kappa$ there exists $i < n$ such that $|X_\alpha^i \cap X_\beta^i| < \omega$. It will be called an $n$-mad family if it is maximal with this property. Define $\mathfrak{a}(n)$ as the smallest size of an infinite $n$-mad family.

Observe that identifying the sequences $(X^0, \dots, X^{n-1}) \in ([\omega]^\omega)^n$ with the $n$-cube $\prod_{i<n} X_i \subseteq \omega^n$, and $X_i$ with an element of $\mathcal{P}(\omega)/\mathrm{fin}$, an $n$-mad family is just a partition of $\bigoplus_{i<n} \mathcal{P}(\omega)/\mathrm{fin}$, and $\mathfrak{a}(n) = \mathfrak{a}\big(\bigoplus_{i<n} \mathcal{P}(\omega)/\mathrm{fin}\big)$. Therefore, the theorems of the previous section give lower bounds to the numbers $\mathfrak{a}(n)$. However, this is not the only notion of "almost-disjointness", or even "cube" on $\omega^n$, and a word will be given on some of them for comparison.

Take, for example, the ideal

$$\mathrm{fin}^2 := \{X \subseteq \omega^2 : \forall^\infty n < \omega \ |X(n)| < \omega\}.$$

Elements of both Boolean algebras are of the form $\coprod_{n \in X} Y_n$, where $X, Y_n \in [\omega]^\omega$, for all $n \in X$. However, if elements of $\mathcal{P}(\omega^2)/\mathrm{fin}^2$ have no restriction on the family $\{Y_n : n \in X\}$, basic elements of $\mathcal{P}(\omega)/\mathrm{fin} \oplus \mathcal{P}(\omega)/\mathrm{fin}$ have the restriction of $Y_n = Y_m$, for all $n, m \in X$.

In general, for $2 < k < \omega$, we can recursively define

$$\mathrm{fin}^k := \{X \subseteq \omega^n : \forall^\infty n < \omega \ X(n) \in \mathrm{fin}^{k-1}\}$$

Although with a different notation, the numbers $\mathfrak{a}(\mathrm{fin}^k)$, for $2 \leq k < \omega$, are studied in [Raghavan and Steprāns 2023]. Their results will be compared with what is known about $\mathfrak{a}(n)$, for $n < \omega$, which is summarized here.

**Corollary 4.4.** (1) $\omega_1 \leq \mathfrak{a}(n+1) \leq \mathfrak{a}(n) \leq \mathfrak{a}$, *for all* $1 \leq n < \omega$.

(2) $\min\{\mathfrak{s}, \mathfrak{a}\} \leq \mathfrak{a}(n)$, *for all* $2 \leq n < \omega$.

(3) $\mathfrak{b} \leq \mathfrak{a}(n)$, *for all* $1 \leq n < \omega$.

*Proof.* (1) From the definition it follows that $\mathfrak{a}(2) \leq \mathfrak{a}(1) \leq \mathfrak{a}$. Taking this fact as the base case, suppose that $\mathfrak{a}(n) \leq \mathfrak{a}(n-1) \leq \mathfrak{a}$ has been proved for some $2 \leq n < \omega$. Since $\bigoplus_{i<n+1} \mathcal{P}(\omega)/\mathrm{fin} = \big(\bigoplus_{i<n} \mathcal{P}(\omega)/\mathrm{fin}\big) \oplus \mathcal{P}(\omega)/\mathrm{fin}$, it follows that $\mathfrak{a}(n+1) \leq \min\{\mathfrak{a}, \mathfrak{a}(n)\} = \mathfrak{a}(n)$. Since $\omega_1 \leq \mathfrak{p} \leq \mathfrak{a}(n)$, for all $1 \leq n < \omega$, item (1) is proved.

(2) Observe that $\mathfrak{s}\big(\bigoplus_{i\leq n}\mathcal{P}(\omega)/\mathrm{fin}\big)=\mathfrak{s}$, for all $1\leq n<\omega$. Trivially, $\min\{\mathfrak{a},\mathfrak{s}\}\leq\mathfrak{a}$. Suppose, as the induction hypothesis, $\min\{\mathfrak{a},\mathfrak{s}\}\leq\mathfrak{a}(n)$, for some $1\leq n<\omega$. By Theorem 3.1, it follows that $\min\{\mathfrak{a},\mathfrak{s}\}\leq\min\{\mathfrak{a},\mathfrak{a}(n),\mathfrak{s}\}\leq\mathfrak{a}(n+1)$.

(3) Taking $\mathfrak{b}\leq\mathfrak{a}$ as the base case, suppose that $\mathfrak{b}\leq\mathfrak{a}(n)$ has been proved for some $1\leq n<\omega$. From Theorem 3.2 and Observation 2.6, it follows that

$$\min\{\mathfrak{a},\mathfrak{a}(n),\max\{\mathfrak{b},\omega\}\}=\min\{\mathfrak{a}(n),\mathfrak{b}\}=\mathfrak{b}\leq\mathfrak{a}(n+1). \qquad \square$$

In parallel to these results, the following facts are proved in [Raghavan and Steprāns 2023, Corollaries 2.18 and 2.27].

**Proposition 4.5.** *Take* $2\leq k<\omega$. *Then*

- $\mathfrak{b}\leq\mathfrak{a}(\mathrm{fin}^k)$ *and*
- $\min\{\mathfrak{a},\mathfrak{s}\}\leq\mathfrak{a}(\mathrm{fin}^k)$.

To prove the first inequality, in [Raghavan and Steprāns 2023] the usual definition of $\mathfrak{b}$ was used. In contrast, the use of a nonsplitting family in Case 1 of the proof of Theorem 3.1 is very similar to the use of countably many nonsplitting families in the proof of [Raghavan and Steprāns 2023, Theorem 2.23]. Furthermore, it is easy to see that $\mathcal{P}(\omega)/\mathrm{fin}\oplus\mathcal{P}(\omega)/\mathrm{fin}$ is a subalgebra of $\mathcal{P}(\omega^2)/\mathrm{fin}^2$, and that all 2-ad families induce $\mathrm{fin}^2$-ad families. However, the similarity of these results does not seem to emerge from the structures themselves and their partitions. Indeed, "one in every two" 2-mad families is not a $\mathrm{fin}^2$-mad family. The next lemma, whose proof is included for completeness sake, will help to prove this.

**Lemma 4.6.** *Suppose that* $P=\{a_\alpha\times b_\alpha:\alpha<\kappa\}$ *is an infinite partition of* $A\oplus B$. *Then there exists* $\{\alpha_n:n<\omega\}\subseteq\kappa$ *such that* $\{a_{\alpha_n}:n<\omega\}$ *is a centered family or* $\{b_{\alpha_n}:n<\omega\}$ *is a centered family.*

*Proof.* Suppose that if $E\subseteq\kappa$ is such that $\{a_\alpha:\alpha\in E\}$ is a centered family, then $|E|<\omega$. Let $\{E_\beta:\beta<\kappa\}\subseteq[\kappa]^{<\omega}$ be the set of all $E\in[\kappa]^{<\omega}$ such that $\{a_\alpha:\alpha\in E\}$ is centered and maximal with this property. Observe that if $E$ is maximal with this property, then $\{b_\alpha:\alpha\in E\}$ is a disjoint family and

$$\bigcup_{\alpha\in E}b_\alpha=Y.$$

We will recursively construct a sequence $\{\alpha_n:n<\omega\}$, such that $\{b_{\alpha_n}:n<\omega\}$ is a centered family. Extend $\{0\}$ to a set $E_0$, maximal with the property that $\{a_\alpha:\alpha\in E_0\}$ is centered. Write $E_0=\{\alpha_0^0,\dots,\alpha_0^{k_0-1}\}$ and define

$$H_0^i:=\{\beta\in\kappa\setminus E_0:b_{\alpha_0^i}\cap b_\beta\neq\varnothing\},\quad\text{for all } i<k_0.$$

There exists $i_0<k_0$ such that $|H_0^{i_0}|=\kappa$. Define $\alpha_0:=\alpha_0^{i_0}$ and $H_0:=H_0^{i_0}$.

Suppose now that for some $n \geq 1$ we have constructed the sets $\{\alpha_l : l < n\} \subseteq \kappa$, $\{H_l : l < n\} \subseteq [\kappa]^\kappa$, and $\{E_l : l < n\} \subseteq [\kappa]^{<\omega}$ such that

- $\alpha_l \in E_l$,
- $\alpha_{l'} \neq \alpha_l$,
- $H_l \subseteq H_{l'}$, for all $l' < l < n$, and that
- $$b_\beta \cap \bigcap_{l<k} b_{\alpha_l} \neq \varnothing,$$
  if and only if $\beta \in H_{n-1}$, for all $\beta < \kappa$.

Define $b := \bigcap_{l<k} b_{\alpha_l}$ and take $\beta \in H_{n-1} \setminus \{\alpha_l : l < n\}$. Extend $\{\beta\}$ to a family $E_n$, maximal with the property that $\{a_\alpha : \alpha \in E_n\}$ is centered. Clearly

$$b = \bigcup_{\alpha \in E_n \cap H_{n-1}} b \cap b_\alpha.$$

Also $\alpha \neq \alpha_l$, for all $l < n$ and all $\alpha \in E_n$: otherwise we would have $b_{\alpha_l} \cap b_\beta \neq \varnothing \neq a_{\alpha_l} \cap a_\beta$, which is a contradiction. Write $E_n \cap H_{n-1} := \{\alpha_n^0, \ldots, \alpha_n^{k_n-1}\}$ and define

$$H_k^i := \{\beta \in H_{k-1} : b_{\alpha_n^i} \cap b \cap b_\beta \neq \varnothing\}.$$

Since there exists $i_n < k_n$ such that $|H_n^{i_n}| = \kappa$, define $\alpha_n := \alpha_n^{i_n}$ and $H_n := H_n^{i_n}$. Since we can continue this recursion, we get $\{\alpha_n : n < \omega\} \subseteq \kappa$ such that $\{b_{\alpha_n} : n < \omega\}$ is a centered family. □

Now, let $\{(X_\alpha^0, X_\alpha^1) : \alpha < \kappa\}$ be a 2-mad family. Without loss of generality, by the previous lemma, $\{X_n^0 : n < \omega\}$ is a centered family. Let $X$ be a pseudointersection of that family. It is easy to see that $\coprod_{n \in X} X_n^1$ witnesses that $\{X_\alpha^0 \times X_\alpha^1 : \alpha < \kappa\}$ is not a $\mathrm{fin}^2$-mad family. Therefore, the similarity on the lower bounds obtained for $\mathfrak{a}(k)$ and $\mathfrak{a}(\mathrm{fin}^k)$, for $2 \leq k < \omega$, as well as that of their proofs, could be pointing to some feature of multidimensional infinite combinatorics, which could also be reflected on the eventual constructions of models of $\mathfrak{a}(\mathrm{fin}^k) < \mathfrak{a}$ or of $\mathfrak{a}(k) < \mathfrak{a}$, for some $2 \leq k < \omega$, or on the not yet impossible ZFC proofs of the corresponding equalities.

We finish this article by listing the main open questions about the cardinals $\mathfrak{a}(n)$, for $n < \omega$.

**Question 4.7.** (1) *Is it consistent that* $\mathfrak{a}(n) < \mathfrak{a}(n-1)$, *for any* $2 \leq n < \omega$?

(2) *Is it consistent that* $\omega_1 = \mathfrak{b} = \mathfrak{s} = \mathfrak{a}(n) < \mathfrak{a}(n-1)$, *for any* $2 \leq n < \omega$?

(3) *Is it consistent that* $\omega_1 < \mathfrak{b} < \mathfrak{s} < \mathfrak{a}(n) = \mathfrak{a}(n-1)$, *for any* $n < \omega$?

The first and second questions also remain open for the cardinals $\mathfrak{a}(\mathrm{fin}^k)$, for $k < \omega$. The second question is emphasized because of its relation to the following question.

**Question 4.8** (Brendle and Raghavan). *Does* $\mathfrak{b} = \mathfrak{s} = \omega_1$ *imply that* $\mathfrak{a} = \omega_1$?

This open question, related to Question 1.1, as well as the history of separating $\mathfrak{a}$ from other cardinal characteristics by means of breakthroughs in forcing techniques, could point to the potential difficulties in answering Question 4.7. Related to the third question, in [Brendle and Fischer 2011] it was asked if it is consistent $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$. With current forcing techniques to make $\mathfrak{a}$ "large", such as those described in [Fischer and Mejia 2017], it seems that the same model for these inequalities would give a positive answer to the third question.

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