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Matlis duality for modules over commutative rings gives rise to the notion of Matlis reflexivity. It is shown that Matlis reflexive modules form a Krull–Schmidt category. For noetherian rings the absence of infinite direct sums is a characteristic feature of Matlis reflexivity. This leads to a discussion of objects that are extensions of artinian by noetherian objects. Classifications of Matlis reflexive modules are provided for some small examples.

1. Introduction

Matlis duality is one of the corner stones of commutative algebra. It is also used in noncommutative algebra, because it provides for any ring a relation between left and right modules. In this note we revisit this classical theory which is based upon the seminal work of Matlis [12]. Of particular interest are Matlis reflexive modules and we show that they form a Krull–Schmidt category. This class of modules admits nice descriptions when the ring is noetherian local and complete. For instance, they are precisely the modules that are extensions of artinian by noetherian modules [5; 17; 18]. Another characteristic feature of Matlis reflexivity is the absence of infinite direct sums. For any abelian category this leads us to an analysis of objects that are extensions of artinian by noetherian objects. The study of this finiteness condition goes back to Baer who called abelian groups with this property *minimax* [2].

Let A be a commutative ring and let Ω denote the set of maximal ideals of A. We write

$$E := E\Big(\coprod_{\mathfrak{m}\in\Omega} A/\mathfrak{m}\Big)$$

for the injective envelope of a coproduct of a representative set of simple A-modules; it is a minimal injective cogenerator for the category of A-modules. *Matlis duality* for A-modules is given by the assignment

$$M \mapsto M^\vee := \operatorname{Hom}_A(M, E).$$

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For any pair of A-modules M, N there is an isomorphism

$$\mathrm{Hom}_A(M, N^\vee) \cong \mathrm{Hom}_A(N, M^\vee)$$

and the evaluation map $M \to (M^\vee)^\vee$ is the unit of the adjunction. The module M is *Matlis reflexive* if the evaluation map is an isomorphism. Examples of Matlis reflexive modules are the modules of finite length.

2. The Krull–Schmidt property

Every finite length module admits an essentially unique decomposition into indecomposables. This generalises to Matlis reflexive modules.

Theorem 2.1. *The Matlis reflexive modules form a Serre subcategory which is Krull–Schmidt. Thus for any exact sequence* $0 \to M' \to M \to M'' \to 0$ *the module* M *is reflexive if and only if* M' *and* M'' *are reflexive. Moreover, any reflexive module admits an essentially unique decomposition into finitely many indecomposable modules with local endomorphism rings.*

The proof uses an intrinsic description of Matlis dual modules.

Lemma 2.2. *An A-module is pure-injective if and only if it is a direct summand of a Matlis dual module.*

Proof. It its well known that a Matlis dual module is pure-injective; see [1, Proposition I.10.1] or [10, Proposition 12.4.7]. For the converse note that the canonical map $M \to (M^\vee)^\vee$ is a pure monomorphism, so it splits when M is pure-injective. □

Proof of Theorem 2.1. The first statement follows from an application of the snake lemma, since Matlis duality is an exact functor and the canonical map $M \to (M^\vee)^\vee$ is a monomorphism. For the second assertion we use the fact that a reflexive module is pure-injective, by Lemma 2.2. A pure-injective module M admits a decomposition $M = M' \oplus M''$ such that M' is *discrete*, so a pure-injective envelope of a direct sum $\bigoplus_{i \in I} M_i$ of indecomposable direct summands of M, and M'' is *continuous*, so having no indecomposable direct summand [8, Theorem 8.25]. From Lemma 2.3 below it follows that $M'' = 0$ and that the set I is finite, so $M = \bigoplus_{i \in I} M_i$. This yields the Krull–Schmidt property since each indecomposable pure-injective module has a local endomorphism ring. □

Lemma 2.3. *Let* $M = \bigoplus_{i \in I} M_i$ *be a Matlis reflexive module. Then* $M_i = 0$ *for almost all i. Moreover, any nonzero Matlis reflexive module has an indecomposable direct summand.*

Proof. We have an isomorphism $M^\vee \cong \prod_{i \in I} M_i^\vee$, and applying the duality to the canonical inclusion

$$\bigoplus_{i \in I} M_i^\vee \to \prod_{i \in I} M_i^\vee$$

yields an epimorphism

$$(M^\vee)^\vee \cong \Big(\prod_{i\in I} M_i^\vee\Big)^\vee \to \Big(\bigoplus_{i\in I} M_i^\vee\Big)^\vee \cong \prod_{i\in I} (M_i^\vee)^\vee.$$

If M is reflexive, this identifies with the canonical inclusion $M \to \prod_{i\in I} M_i$, so it is an isomorphism. Thus $M_i = 0$ for almost all i.

Suppose $M \neq 0$ has no indecomposable direct summand. Then we get a decomposition $M = M_1 \oplus M^1$ such that both M_1 and M^1 are nonzero and admit no indecomposable direct summands. We continue and decompose $M^i = M_{i+1} \oplus M^{i+1}$ as before, for all $i \geq 1$. This yields an inclusion $\bigoplus_{i\geq 1} M_i \to M$. If M is reflexive, then the same holds for $\bigoplus_{i\geq 1} M_i$. But this is impossible by the first assertion. Thus M admits an indecomposable direct summand. □

Remark 2.4. The isomorphism classes of reflexive modules form a set, because the isomorphism classes of indecomposable pure-injective modules form a set.

Remark 2.5. Morita duality provides another context where the notion of a reflexive module has been studied before; see [13; 14]. This context is one way more general (as two rings are involved, not necessarily commutative), but also more restrictive (as the rings are necessarily semiperfect). Nonetheless, there are parallels. The first part of Lemma 2.3 is the assertion of [14, Lemma 2.13], though the proofs are different.

Let Λ be an associative A-algebra. Then Matlis duality $\operatorname{Hom}_A(-, E)$ induces an adjoint pair of functors between the categories of left and right Λ-modules, respectively. The above Theorem 2.1 remains true for Λ-modules, with same proof. For an intrinsic description of Matlis reflexive Λ-modules, see [7; 9].

Corollary 2.6. *For any associative algebra over a commutative ring, the Matlis reflexive modules form a Serre subcategory which is Krull–Schmidt.* □

Recall from [16] that a module M is *linearly compact* if for any codirected family of submodules $(M_i)_{i\in I}$ of M and any family of cosets $(x_i + M_i)_{i\in I}$ we have $\bigcap_{i\in I} x_i + M_i \neq \varnothing$ provided that $\bigcap_{i\in J} x_i + M_i \neq \varnothing$ for all finite $J \subseteq I$. An equivalent condition is that the canonical map $M \to \varprojlim M_i$ is an epimorphism. This condition is the categorical dual of the fact that for any directed family of submodules $M_i \subseteq M$ the canonical map $\varinjlim M_i \to M$ is a monomorphism. Thus any Matlis reflexive module is linearly compact (see Proposition 6.1).

The linearly compact modules form a Serre subcategory (see Propositions 2, 3, and 9 of [16]), and any decomposition $M = \bigoplus_{i\in I} M_i$ of a linearly compact module implies $M_i = 0$ for almost all i (see Proposition 6.1). Moreover, if M is linearly compact when viewed as a module over its endomorphism ring, then M is pure-injective [8, Corollary 7.4]. Thus any linearly compact module over a commutative ring is pure-injective. Combining these facts, it follows that the assertion of Theorem 2.1 remains true for the class of linearly compact modules.

Corollary 2.7. *For any commutative ring, the linearly compact modules form a Serre subcategory which is Krull–Schmidt.* □

Note that commutativity is needed. For instance, any artinian module is linearly compact, but the Krull–Schmidt property fails for artinian modules [11].

3. Matlis duality

Now assume that the commutative ring A is noetherian. Then we have the isomorphism

$$\coprod_{\mathfrak{m}\in\Omega} E(A/\mathfrak{m}) \cong E\Big(\coprod_{\mathfrak{m}\in\Omega} A/\mathfrak{m}\Big)$$

since any coproduct of injective modules is again injective. For any ideal $\mathfrak{a}$ we write $A^\wedge_{\mathfrak{a}} = \varprojlim A/\mathfrak{a}^i$ for the completion with respect to $\mathfrak{a}$. We set

$$\hat{A} := \prod_{\mathfrak{m}\in\Omega} A^\wedge_{\mathfrak{m}}.$$

Lemma 3.1. *There is an isomorphism of rings* $\operatorname{End}_A(E) \cong \hat{A}$.

Proof. For $\mathfrak{m}\in\Omega$ we have

$$E(A/\mathfrak{m}) \cong \varinjlim \operatorname{Hom}_A(A/\mathfrak{m}^i, E(A/\mathfrak{m})) = \varinjlim \operatorname{Hom}_A(A/\mathfrak{m}^i, E)$$

since $E(A/\mathfrak{m})$ is $\mathfrak{m}$-torsion. Thus

$$\begin{aligned}
\operatorname{End}_A(E) &\cong \prod_{\mathfrak{m}\in\Omega} \operatorname{Hom}_A(E(A/\mathfrak{m}), E)\\
&\cong \prod_{\mathfrak{m}\in\Omega} \operatorname{Hom}_A(\varinjlim \operatorname{Hom}_A(A/\mathfrak{m}^i, E), E)\\
&\cong \prod_{\mathfrak{m}\in\Omega} \varprojlim \operatorname{Hom}_A(\operatorname{Hom}_A(A/\mathfrak{m}^i, E), E)\\
&\cong \prod_{\mathfrak{m}\in\Omega} \varprojlim A/\mathfrak{m}^i\\
&= \hat{A}.
\end{aligned}$$

The first three isomorphisms are clear since $\operatorname{Hom}_A(-, E)$ sends colimits to limits. The last isomorphism uses that finite length modules are Matlis reflexive. □

For an $\hat{A}$-module M and $\mathfrak{m}\in\Omega$ we set $M(\mathfrak{m}) := MA^\wedge_{\mathfrak{m}}$. We say that M has *finite support* if $M(\mathfrak{m}) = 0$ for almost all $\mathfrak{m}\in\Omega$, and M is *locally finitely supported* if the following equivalent conditions hold:

(LF1) The canonical map $\bigoplus_{\mathfrak{m}\in\Omega} M(\mathfrak{m}) \to M$ is an isomorphism.

(LF2) M is a filtered colimit of noetherian modules with finite support.

From now on we restrict ourselves to the study of locally finitely supported $\hat{A}$-modules. These modules form a locally noetherian Grothendieck category, since $\{A^\wedge_\mathfrak{m} \mid \mathfrak{m} \in \Omega\}$ is a set of noetherian generators.

The following is essentially due to Matlis. In [12] he considered the case that A is local. We write art A for the category of artinian A-modules and noeth A for the category of noetherian A-modules.

Proposition 3.2. *The functors*

$$\operatorname{art} A = \operatorname{art} \hat{A} \underset{\operatorname{Hom}_{\hat{A}}(-,E)}{\overset{\operatorname{Hom}_A(-,E)=\operatorname{Hom}_{\hat{A}}(-,E)}{\rightleftarrows}} \operatorname{noeth} \hat{A}$$

are mutually quasi-inverse contravariant equivalences.

Proof. Locally finitely supported modules that are artinian or noetherian have finite support. Thus we may restrict to modules that are supported on a finite subset $\Psi \subseteq \Omega$. Let A_Ψ denote the localisation of A with respect to $\bigcap_{\mathfrak{m}\in\Psi}(A \smallsetminus \mathfrak{m})$. This ring is semilocal and its maximal ideals identify with the ones in Ψ. Thus the completion of A_Ψ with respect to the Jacobson radical $J(A_\Psi)$ is isomorphic to $\prod_{\mathfrak{m}\in\Psi} A^\wedge_\mathfrak{m}$. We conclude that the assertion reduces to the case that A is semilocal. Then the A-module E is artinian [10, Lemma 2.4.19], and therefore $\hat{A} \cong \operatorname{End}_A(E)$ is noetherian.

Now suppose that A is semilocal. Then the artinian A-modules are precisely the kernels of maps $E^p \to E^q$ for positive integers p, q, while the noetherian $\hat{A}$-modules are precisely the cokernels of maps $\hat{A}^p \to \hat{A}^q$ for positive integers p, q. It remains to observe that $\operatorname{Hom}_A(-, E)$ is an exact functor that induces a contravariant equivalence $\operatorname{add} E \xrightarrow{\sim} \operatorname{add} \hat{A}$, with quasi-inverse given by $\operatorname{Hom}_{\hat{A}}(-, E)$. Here, $\operatorname{add} M$ denotes the full subcategory of all direct summands of finite direct sums of copies of M, and the equivalence is clear from the isomorphism

$$\operatorname{End}_A(E) \xrightarrow{\sim} \operatorname{End}_{\hat{A}}(\hat{A}) = \hat{A}$$

from Lemma 3.1 which is induced by $\operatorname{Hom}_A(-, E)$. □

4. Categories of extensions

Let $\mathcal{A}$ be an abelian category. We fix a pair of Serre subcategories $\mathcal{C}, \mathcal{D}$ of $\mathcal{A}$ and write $\mathcal{C} * \mathcal{D}$ for the full subcategory consisting of objects $X \in \mathcal{A}$ that fit into an exact sequence

$$0 \to X' \to X \to X'' \to 0 \quad \text{with } X' \in \mathcal{C},\ X'' \in \mathcal{D}.$$

We say that $\mathcal{C} \subseteq \mathcal{A}$ is *right cofinal* if for every epimorphism $X \to Y$ in $\mathcal{A}$ with $Y \in \mathcal{C}$ there exists an epimorphism $\bar{X} \to Y$ in $\mathcal{C}$ that factors through $X \to Y$.

Lemma 4.1. *The subcategory $\mathcal{C} * \mathcal{D}$ of $\mathcal{A}$ is closed under subobjects and quotient objects. In particular, $\mathcal{C} * \mathcal{D}$ is an abelian category. Moreover, $\mathcal{C} * \mathcal{D}$ is closed under extensions provided that $\mathcal{C} \subseteq \mathcal{A}$ is right cofinal.*

Proof. The first assertion is clear. Now let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$ with subobjects $X' \subseteq X$ and $Z' \subseteq Z$ in $\mathcal{C}$ such that $X'' = X/X'$ and $Z'' = Z/Z'$ are in $\mathcal{D}$. We consider the pullback $Y \times_Z Z'$ and get an epimorphism $\pi : Y \times_Z Z' \to Z'$. The cofinality condition yields an epimorphism $\bar{Y} \to Z'$ in $\mathcal{C}$ that factors through π; in particular the composite with $Z' \to Z$ factors through $Y \to Z$ via a morphism $\bar{Y} \to Y$. We set $Y' := \operatorname{Im}(X' \oplus \bar{Y} \to Y)$ and $Y'' := Y/Y'$. Then we have $Y' \in \mathcal{C}$ and an exact sequence $X'' \to Y'' \to Z'' \to 0$ which shows that $Y'' \in \mathcal{D}$. Thus Y lies in $\mathcal{C} * \mathcal{D}$. □

Lemma 4.2. *The inclusions $\mathcal{C} \to \mathcal{C} * \mathcal{D}$ and $\mathcal{D} \to \mathcal{C} * \mathcal{D}$ induce equivalences*

$$\frac{\mathcal{C}}{\mathcal{C} \cap \mathcal{D}} \xrightarrow{\sim} \frac{\mathcal{C} * \mathcal{D}}{\mathcal{D}} \quad \textit{and} \quad \frac{\mathcal{D}}{\mathcal{C} \cap \mathcal{D}} \xrightarrow{\sim} \frac{\mathcal{C} * \mathcal{D}}{\mathcal{C}}.$$

Proof. We may assume $\mathcal{A} = \mathcal{C} * \mathcal{D}$. The inclusion $\mathcal{C} \to \mathcal{A}$ is exact and the kernel of the composite with the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{D}$ equals $\mathcal{C} \cap \mathcal{D}$. It is clear that the composite is essentially surjective, because for any $X \in \mathcal{A}$ the monomorphism $X' \to X$ is an isomorphism in $\mathcal{A}/\mathcal{D}$. In order to show that the induced functor $\mathcal{C}/(\mathcal{C} \cap \mathcal{D}) \to \mathcal{A}/\mathcal{D}$ is fully faithful we need to check the following cofinality condition: for each map $\phi \colon X \to Y$ such that $Y \in \mathcal{C}$ and $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ are in $\mathcal{D}$, there is a morphism $\psi \colon X' \to X$ such that $X' \in \mathcal{C}$ and $\operatorname{Ker}(\phi\psi)$ and $\operatorname{Coker}(\phi\psi)$ are in $\mathcal{D}$. But this is clear, since there is such a monomorphism ψ with $\operatorname{Coker}\psi \in \mathcal{D}$. The proof of the other equivalence is dual. □

Writing an object as an extension of objects in $\mathcal{C}$ and $\mathcal{D}$ is unique up to objects from $\mathcal{C} \cap \mathcal{D}$. To make this precise, fix a pair of exact sequences $0 \to X_i \to Y \to Z_i \to 0$ $(i = 1, 2)$ with $X_i \in \mathcal{C}$ and $Z_i \in \mathcal{D}$. We set $\bar{X} := X_1 \cap X_2$ and this yields an exact sequence $0 \to \bar{X} \to Y \to \bar{Z} \to 0$ such that $\bar{X} \in \mathcal{C}$ and $\bar{Z} \in \mathcal{D}$. Moreover, the induced morphisms $\bar{X} \to X_i$ and $\bar{Z} \to Z_i$ have kernels and cokernels in $\mathcal{C} \cap \mathcal{D}$. In the language of quotient categories we see that $\mathcal{C}$ and $\mathcal{D}$ form a torsion pair, modulo the subcategory $\mathcal{C} \cap \mathcal{D}$.

Lemma 4.3. *The categories $\mathcal{C}$ and $\mathcal{D}$ yield a pair of full subcategories*

$$\left(\frac{\mathcal{C}}{\mathcal{C} \cap \mathcal{D}}, \frac{\mathcal{D}}{\mathcal{C} \cap \mathcal{D}}\right)$$

of $(\mathcal{C} * \mathcal{D})/(\mathcal{C} \cap \mathcal{D})$ which form a torsion pair.

Proof. We may assume $\mathcal{A} = \mathcal{C} * \mathcal{D}$ and set $\bar{\mathcal{A}} = \mathcal{A}/(\mathcal{C} \cap \mathcal{D})$. The cofinality argument in the proof of Lemma 4.2 shows that the inclusions $\mathcal{C} \to \mathcal{A}$ and $\mathcal{D} \to \mathcal{A}$ induce fully faithful functors $\bar{\mathcal{C}} = \mathcal{C}/(\mathcal{C} \cap \mathcal{D}) \to \bar{\mathcal{A}}$ and $\bar{\mathcal{D}} = \mathcal{D}/(\mathcal{C} \cap \mathcal{D}) \to \bar{\mathcal{A}}$, respectively.

For a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of an abelian category to be a torsion pair, it suffices that $\operatorname{Hom}(X, Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$, and that every object X fits into an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

Any morphism in $\bar{\mathcal{A}}$ between objects in $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ is isomorphic to a morphism coming from $\mathcal{A}$, and we may assume it is a morphism between objects in $\mathcal{C}$ and $\mathcal{D}$. The image of such a morphism is in $\mathcal{C} \cap \mathcal{D}$, so is zero in $\bar{\mathcal{A}}$. For any $X \in \mathcal{A}$ the sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{C}$ and $X'' \in \mathcal{D}$ induces the required exact sequence in $\bar{\mathcal{A}}$. □

5. Artinian-by-noetherian objects

Let $\mathcal{A}$ be an abelian category. We call an object X in $\mathcal{A}$ *artinian-by-noetherian* if it fits into an exact sequence $0 \to X' \to X \to X'' \to 0$ such that X' is noetherian and X'' is artinian.

Lemma 5.1. *Let $\mathcal{A}$ be an abelian category and $0 \to X \to Y \to Z \to 0$ an exact sequence such that X is noetherian and Z is artinian. Then Y has no subquotient that is an infinite coproduct of nonzero objects.*

Proof. Any subquotient Y' of Y fits into an exact sequence $0 \to X' \to Y' \to Z' \to 0$ such that X' is noetherian and Z' is artinian. Thus we may assume that $Y' = Y = \coprod_{i \in \mathbb{N}} Y_i$ and need to show that $Y_i = 0$ for almost all i. For each $n \in \mathbb{N}$ let X_n denote the pullback of $X \to Y \leftarrow \coprod_{i \le n} Y_i$ and Z_n the pushout of $\coprod_{i \le n} Y_i \leftarrow Y \to Z$. We obtain chains

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \cdots \rightarrowtail X \quad \text{and} \quad Z \twoheadrightarrow \cdots \twoheadrightarrow Z_2 \twoheadrightarrow Z_1 \twoheadrightarrow Z_0$$

which stabilise, say for $n \ge n_0$, so $X_{n_0} = X$ and $Z = Z_{n_0}$. This yields an induced exact sequence

$$0 \to X \to \coprod_{i \le n_0} Y_i \to Z \to 0,$$

and therefore $Y_i = 0$ for all $i > n_0$. □

The converse of the preceding lemma requires an assumption. Recall that a Grothendieck category is *locally noetherian* if there is a generating set of noetherian objects. This means in particular that finitely generated and noetherian objects coincide. In addition we assume that the injective envelope of every simple object is artinian. Examples are the category of modules over a commutative noetherian ring, or more generally the category of quasicoherent sheaves on a noetherian scheme. Also, modules over a noetherian algebra have this property.

Lemma 5.2. *Let Λ be an A-algebra such that Λ is noetherian as an A-module. Then the injective envelope of each simple Λ-module is artinian.*

Proof. We may assume that the ring A is noetherian and then $E(A/\mathfrak{m})$ is artinian for every $\mathfrak{m} \in \Omega$; see [10, Lemma 2.4.19]. It follows that $\operatorname{Hom}_A(\Lambda, E(A/\mathfrak{m}))$ is an artinian Λ-module for each $\mathfrak{m}$. Let S be a simple Λ-module. The Λ-module

$$\operatorname{Hom}_A(\Lambda, E) \cong \coprod_{\mathfrak{m}\in\Omega} \operatorname{Hom}_A(\Lambda, E(A/\mathfrak{m}))$$

is an injective cogenerator. Thus for some $\mathfrak{m} \in \Omega$ the injective envelope $E(S)$ is a direct summand of $\operatorname{Hom}_A(\Lambda, E(A/\mathfrak{m}))$ and therefore artinian. □

Here is the converse of Lemma 5.1.

Lemma 5.3. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and suppose that injective envelopes of simple objects are artinian. Let X be an object having no subquotient that is an infinite coproduct of nonzero objects. Then X fits into an exact sequence $0 \to X' \to X \to X'' \to 0$ such that X' is noetherian and X'' is artinian.*

Proof. The assumption on $\mathcal{A}$ implies that an object is artinian if and only if its socle is an essential subobject and has finite length.

We construct a noetherian subobject $X' \subseteq X$ as follows. Suppose that X is not artinian. From our first observation it follows that $\operatorname{soc} X \subseteq X$ is not essential, so there is a finitely generated subobject $0 \neq V \subseteq X$ with $V \cap \operatorname{soc} X = 0$. Choose a maximal subobject $U \subseteq V$. Then the canonical map $X \to X/U$ induces a proper inclusion $\operatorname{soc} X \to \operatorname{soc}(X/U)$. If X/U is not artinian, we proceed and obtain a sequence of epimorphisms $X = X_0 \to X_1 \to X_2 \to \cdots$ such that $\operatorname{Ker}(X_i \to X_{i+1})$ is finitely generated and the induced map $\operatorname{soc} X_i \to \operatorname{soc} X_{i+1}$ is a proper inclusion for each $i \geq 0$. This must terminate after finitely many steps since $\varinjlim \operatorname{soc} X_i \subseteq \operatorname{soc}(\varinjlim X_i)$ has finite length, again by our assumption on X. Thus for some $i \geq 0$ the quotient $X'' = X_i$ is artinian, and $X' = \operatorname{Ker}(X \to X'')$ is finitely generated. □

Proposition 5.4. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and suppose that injective envelopes of simple objects are artinian. Then the artinian-by-noetherian objects form a Serre subcategory in $\mathcal{A}$; it is the smallest Serre subcategory containing both the artinian and the noetherian objects. Moreover, it consists of the objects having no subquotients that are infinite coproducts of nonzero objects.*

Proof. It follows from Lemma 4.1 that the artinian-by-noetherian objects form a Serre subcategory, because the noetherian objects are right cofinal in $\mathcal{A}$. The other description of the artinian-by-noetherian objects follows from Lemmas 5.1 and 5.3. □

Writing an object as an extension of an artinian and a noetherian object is unique up to objects of finite length. We make this more precise, using the language of quotient categories.

We write $\operatorname{arno}\mathcal{A}$ for the full subcategory consisting of all artinian-by-noetherian objects in $\mathcal{A}$.[1] As usual, $\operatorname{art}\mathcal{A}$, $\operatorname{noeth}\mathcal{A}$, and $\operatorname{fl}\mathcal{A}$ denote the full subcategories of objects in $\mathcal{A}$ that are artinian, noetherian, and finite length, respectively.

Let us record the basic properties of $\operatorname{arno}\mathcal{A}$; they are immediate consequences of Lemmas 4.2 and 4.3.

Proposition 5.5. *The categories* $\operatorname{noeth}\mathcal{A}$ *and* $\operatorname{art}\mathcal{A}$ *yield a pair of full subcategories*

$$\left(\frac{\operatorname{noeth}\mathcal{A}}{\operatorname{fl}\mathcal{A}}, \frac{\operatorname{art}\mathcal{A}}{\operatorname{fl}\mathcal{A}}\right)$$

of $(\operatorname{arno}\mathcal{A})/(\operatorname{fl}\mathcal{A})$ *which form a torsion pair.* □

Proposition 5.6. *The inclusions* $\operatorname{art}\mathcal{A} \to \operatorname{arno}\mathcal{A}$ *and* $\operatorname{noeth}\mathcal{A} \to \operatorname{arno}\mathcal{A}$ *induce equivalences*

$$\frac{\operatorname{art}\mathcal{A}}{\operatorname{fl}\mathcal{A}} \xrightarrow{\sim} \frac{\operatorname{arno}\mathcal{A}}{\operatorname{noeth}\mathcal{A}} \quad \textit{and} \quad \frac{\operatorname{noeth}\mathcal{A}}{\operatorname{fl}\mathcal{A}} \xrightarrow{\sim} \frac{\operatorname{arno}\mathcal{A}}{\operatorname{art}\mathcal{A}}.$$ □

6. Matlis reflexive modules

We keep the assumption that the commutative ring A is noetherian. It is convenient to call an A-module *complete* if the A-action factors through the canonical map $A \to \hat{A}$. Clearly, any Matlis dual module is complete. For that reason we restrict ourselves to the study of complete A-modules, which identify with $\hat{A}$-modules. This leads to a characterisation of Matlis reflexive modules as extensions of artinian by noetherian modules, which is well known when the ring A is local [5; 17; 18].

Proposition 6.1. *For a locally finitely supported $\hat{A}$-module M the following are equivalent*:

(1) *M is Matlis reflexive.*

(2) *M is linearly compact.*

(3) *M has no subquotient that is an infinite direct sum of nonzero modules.*

(4) *M has a noetherian submodule U such that M/U is artinian.*

Proof. (1) ⇒ (2): Given a codirected family of submodules $M_i \subseteq M$, the canonical map $\varinjlim (M/M_i)^\vee \to M^\vee$ is a monomorphism. Thus

$$M \cong (M^\vee)^\vee \to (\varinjlim (M/M_i)^\vee)^\vee \cong \varprojlim ((M/M_i)^\vee)^\vee \cong \varprojlim M/M_i$$

is an epimorphism.

[1] This is an abuse of notation, as Auslander introduces in [1] the same notation for modules that are extensions of noetherian-by-artinian modules. For instance, almost split sequences over noetherian algebras provide such extensions.

(2) ⇒ (3): Any subquotient of a linearly compact module is again linearly compact. Thus we may assume that there is a decomposition $M=\bigoplus_{i\in I} M_i$. The cofinite subsets $J\subseteq I$ yield a codirected family of submodules $M_J=\bigoplus_{i\in J} M_i$ of M. Choose $x_i\in M_i$ for each $i\in I$ and set $x_J=\sum_{i\notin J} x_i$. For any finite set of cofinite subsets $J_\alpha\subseteq I$ we have $x_J\in\bigcap_\alpha x_{J_\alpha}+M_{J_\alpha}$ for $J=\bigcap_\alpha J_\alpha$. Thus there exists $x\in\bigcap_J x_J+M_J$ when J runs through all cofinite subsets. This satisfies $x+\bigoplus_{j\neq i} M_j=x_i+\bigoplus_{j\neq i} M_j$ for all $i\in I$, and therefore $x_i=0$ for almost all i.

(3) ⇒ (4): Suppose M has no subquotient that is an infinite direct sum of nonzero modules. Then M has finite support and therefore it may be viewed as a module over a factor of $\hat{A}$ which is noetherian. Thus we can apply Lemma 5.3.

(4) ⇒ (1): Artinian and noetherian $\hat{A}$-modules are reflexive, by Proposition 3.2. As reflexive modules are closed under extensions, by Theorem 2.1, it follows that M is reflexive. □

We write refl $\hat{A}$ for the category of Matlis reflexive $\hat{A}$-modules and fl $\hat{A}$ for the category of finite length $\hat{A}$-modules. Then we obtain from Proposition 5.5 the torsion pair

$$\left(\frac{\operatorname{noeth}\hat{A}}{\operatorname{fl}\hat{A}},\frac{\operatorname{art}\hat{A}}{\operatorname{fl}\hat{A}}\right)$$

in $(\operatorname{refl}\hat{A})/(\operatorname{fl}\hat{A})$ and from Proposition 5.6 the pair of equivalences

$$\frac{\operatorname{art}\hat{A}}{\operatorname{fl}\hat{A}}\xrightarrow{\sim}\frac{\operatorname{refl}\hat{A}}{\operatorname{noeth}\hat{A}}\quad\text{and}\quad\frac{\operatorname{noeth}\hat{A}}{\operatorname{fl}\hat{A}}\xrightarrow{\sim}\frac{\operatorname{refl}\hat{A}}{\operatorname{art}\hat{A}}.$$

7. Examples

We provide descriptions of Matlis reflexive modules for some small examples. The first example illustrates why one passes from a noetherian ring A to the completion $\hat{A}$. Because of the Krull–Schmidt property, it suffices to list the indecomposable modules. Moreover, any indecomposable $\hat{A}$-module is the restriction of a module over the completion $A^\wedge_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of A. Thus it suffices to consider rings that are local and complete.

Example 7.1. Let $A=\mathbb{Z}$. It is easily checked that a $\mathbb{Z}$-module is Matlis reflexive if and only if it has finite length.

Example 7.2. Let A be a Dedekind domain. Passing from A to $\hat{A}$, we may assume that A is a complete discrete valuation ring, so local and complete of Krull dimension one. By inspection it is easily checked that an indecomposable A-module is Matlis reflexive if and only if it is pure-injective. The following is the complete list of indecomposable Matlis reflexive modules, where $\mathfrak{m}$ denotes the maximal ideal:

$$A=\varprojlim A/\mathfrak{m}^i,\quad E(A/\mathfrak{m})=\varinjlim A/\mathfrak{m}^i,\quad Q(A),\quad A/\mathfrak{m}^i\qquad(i\in\mathbb{N}).$$

The category $\operatorname{refl} A = \operatorname{arno} A$ is an abelian category with enough projective and enough injective objects. Note that the quotient ring $Q = Q(A)$ is the unique indecomposable object that is projective and injective. Set

$$\bar{A} = \begin{bmatrix} A & Q \\ 0 & Q \end{bmatrix} \quad \text{and} \quad T_2(Q) = \begin{bmatrix} Q & Q \\ 0 & Q \end{bmatrix}.$$

Then $\operatorname{Hom}_A(A \oplus Q, -)$ induces an equivalence $\operatorname{refl} A \xrightarrow{\sim} \operatorname{mod} \bar{A}$, and passing to the quotient modulo $\operatorname{fl} A$ yields equivalences

$$\frac{\operatorname{refl} A}{\operatorname{fl} A} \simeq \operatorname{mod} T_2(Q) \quad \text{and} \quad \frac{\operatorname{noeth} A}{\operatorname{fl} A} \simeq \operatorname{mod} Q \simeq \frac{\operatorname{art} A}{\operatorname{fl} A}.$$

Example 7.3. Let k be a field and $A = k[x, y]/(xy)$. There are two minimal prime ideals $\mathfrak{x} = (x)/(xy)$ and $\mathfrak{y} = (y)/(xy)$, with $A/\mathfrak{x} \cong k[y]$ and $A/\mathfrak{y} \cong k[x]$. For the maximal ideal $\mathfrak{n} = (x, y)/(xy)$ we have $A/\mathfrak{n} \cong k$. Thus the ring A fits into a pullback diagram:

$$\begin{array}{ccccc} & & k[x] & & \\ & \nearrow & & \searrow & \\ A & & & & k \\ & \searrow & & \nearrow & \\ & & k[y] & & \end{array}$$

In particular, $\operatorname{Spec} A$ is given by two copies of an affine line, glued via the distinguished maximal ideal $\mathfrak{n}$:

$$\operatorname{Spec} A = \operatorname{Spec} k[x] \amalg_{\mathfrak{n}} \operatorname{Spec} k[y].$$

For the set of maximal ideals we have

$$\Omega_A = \big(\Omega_{k[x]} \smallsetminus \{(x)\}\big) \cup \{\mathfrak{n}\} \cup \big(\Omega_{k[y]} \smallsetminus \{(y)\}\big).$$

For $\mathfrak{m} \in \Omega_A$ and $\mathfrak{m} \neq \mathfrak{n}$, the ring $A^\wedge_\mathfrak{m} \cong \operatorname{End}_A(E(A/\mathfrak{m}))$ is a complete discrete valuation ring, because $E(A/\mathfrak{m})$ is annihilated either by $\mathfrak{x}$ or by $\mathfrak{y}$ and identifies therefore with an indecomposable injective module over a polynomial ring. In particular, the description of the Matlis reflexive modules over $A^\wedge_\mathfrak{m}$ follows from the preceding example.

For $\mathfrak{m} = \mathfrak{n}$ we have $A^\wedge_\mathfrak{m} \cong k[\![x, y]\!]/(xy)$ and we refer to the next example for a description of its Matlis reflexive modules.

The algebra A is a string algebra. Crawley-Boevey offers in [3] a detailed discussion of modules over string algebras, including a classification of the artinian modules, and using the functorial filtration method, which goes back to Gelfand and Ponomarev [6]. Note that $\operatorname{art} A = \operatorname{art} \hat{A}$; so one may compare our description with the one given in [3, Theorem 1.3].

Example 7.4. Let k be a field and $A = k[\![x, y]\!]/(xy)$. This ring is a completion of $k[x, y]/(xy)$ and we have therefore an analogous pullback diagram:

$$\begin{array}{ccccc} & & k[\![x]\!] & & \\ & \nearrow & & \searrow & \\ A & & & & k \\ & \searrow & & \nearrow & \\ & & k[\![y]\!] & & \end{array}$$

The following is the complete list of indecomposable Matlis reflexive A-modules:

(1) string modules $M(C)$ given by finite or stabilising strings C,

(2) band modules $M(C, V)$ given by periodic strings C and indecomposable $k[t, t^{-1}]$-modules V of finite length.

This list can be deduced from work of Ebrahimi Atani [4], though our notation and terminology follows more closely that of [3]. For string modules given by infinite strings, see also Ringel [15]. The isomorphism type of a string or band module is essentially determined by the corresponding string C and the isomorphism type of the $k[t, t^{-1}]$-module V; see [3] for details.

A *string* is a possibly infinite word in the alphabet $\{x, y, x^-, y^-\}$ having no subword of the form xy, yx, x^-y^-, y^-x^-. We identify strings with diagrams where the letters are represented by arrows. Given a string C, we write C^- for the string which is obtained from C by changing each letter a in C to a^-, where $(a^-)^- = a$. A string is either finite or infinite, and for infinite strings we distinguish between type $\mathbb{N}$ and type $\mathbb{Z}$. A string C can be written as a sequence $(C_i)_{i\in I}$ indexed by an interval $I \subseteq \mathbb{Z}$. We say that an infinite string C is *stabilising* if $C_i = C_{i+1}$ for $|i| \gg 0$. We say that C is *periodic* if there exists some $n > 1$ such that $C_i = C_{i+n}$ for all $i \in \mathbb{Z}$. The minimal such n is called the *period*.

Let us define the *string module* $M(C)$ over A for a string C that is finite or stabilising. For finite C the module $M(C)$ has a k-basis that is indexed by the vertices of C and the arrows show the actions of x and y. For an infinite string we use truncations and note that they are again stabilising or finite. So for C of the form

$$\cdots \xrightarrow{a} \circ_2 \xrightarrow{a} \circ_1 \xrightarrow{a} \circ_0 \,\text{——}\, \circ \,\text{——}\, \cdots$$

(with $a = x$ or $a = y$) one takes the direct limit of the string modules given by the truncations

$$\circ_i \xrightarrow{a} \cdots \xrightarrow{a} \circ_1 \xrightarrow{a} \circ_0 \,\text{——}\, \circ \,\text{——}\, \cdots$$

and for a string of the form

$$\cdots \xleftarrow{a} \circ_2 \xleftarrow{a} \circ_1 \xleftarrow{a} \circ_0 \,\text{——}\, \circ \,\text{——}\, \cdots$$

one takes the inverse limit of the string modules given by the truncations

$$\circ_i \xleftarrow{a} \cdots \xleftarrow{a} \circ_1 \xleftarrow{a} \circ_0 \text{ --- } \circ \text{ --- } \cdots .$$

The definition of $M(C)$ depends on choices, but it can be shown that the isomorphism type of the A-module $M(C)$ is independent of any choices. For this one uses the observation that any concatenation $C = C'C''$ via an arrow from C'' to C' yields an exact sequence

$$0 \to M(C') \to M(C) \to M(C'') \to 0,$$

and there is an obvious swap when the connecting arrow goes in the other direction.

There are two distinguished strings:

$$C_\infty\colon \quad \cdots \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \cdots$$

and

$$D_\infty\colon \quad \cdots \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \xrightarrow{y} \circ \xrightarrow{y} \circ \xrightarrow{y} \cdots .$$

Let $\mathfrak{m}$ denote the maximal ideal of A. We claim that $M(C_\infty) \cong E(A/\mathfrak{m})$. In fact, we have, for $i \geq 1$,

$$\operatorname{soc}^i E(A/\mathfrak{m}) = \operatorname{Hom}_A(A/\mathfrak{m}^i, E(A/\mathfrak{m}) \cong M(C_{i-1})$$

with

$$C_i\colon \qquad \circ_i \xrightarrow{x} \cdots \xrightarrow{x} \circ_1 \xrightarrow{x} \circ_0 \xleftarrow{y} \circ_1 \xleftarrow{y} \cdots \xleftarrow{y} \circ_i,$$

and therefore

$$M(C_\infty) = \varinjlim M(C_i) \cong \varinjlim \operatorname{soc}^i E(A/\mathfrak{m}) = E(A/\mathfrak{m}).$$

For a finite string C we have $M(C)^\vee \cong M(C^-)$, and therefore

$$M(D_\infty) = \varprojlim M(C_i^-) \cong E(A/\mathfrak{m})^\vee = \operatorname{Hom}_A(E(A/\mathfrak{m}), E(A/\mathfrak{m})) \cong A.$$

For any finite or stabilising string C, the A-module $M(C)$ is an extension of an artinian by a noetherian module; see Lemma 7.5 below. In particular, $M(C)$ is Matlis reflexive. Note that $M(C)$ is artinian if and only if C contains no infinite substring of the form

$$\cdots \xleftarrow{x} \circ \xleftarrow{x} \circ \xleftarrow{x} \circ \quad \text{or} \quad \circ \xrightarrow{y} \circ \xrightarrow{y} \circ \xrightarrow{y} \cdots .$$

Analogously, $M(C)$ is noetherian if and only if C contains no infinite substring of the form

$$\cdots \xrightarrow{x} \circ \xrightarrow{x} \circ \xrightarrow{x} \circ \quad \text{or} \quad \circ \xleftarrow{y} \circ \xleftarrow{y} \circ \xleftarrow{y} \cdots .$$

Now consider an infinite string C of period n and set $R = k[t, t^{-1}]$. The string module $M(C)$ has a k-basis $\{b_i \mid i \in \mathbb{Z}\}$ that is indexed by the vertices of the diagram representing C, and the action of x and y is given by the arrows of this diagram. Then $M(C)$ is a free R-module of rank n with the action of t given by $tb_i = b_{i+n}$. For any R-module V one defines the *band module* $M(C, V) = V \otimes_R M(C)$.

For the quotient category $(\operatorname{refl} A)/(\operatorname{fl} A)$ one observes that each indecomposable object is either annihilated by x or by y. Thus the description reduces to modules over $k[\![x]\!]$ or $k[\![y]\!]$, and we get an equivalence

$$\frac{\operatorname{refl} A}{\operatorname{fl} A} \simeq \operatorname{mod} T_2\big(k((x))\big) \times \operatorname{mod} T_2\big(k((y))\big).$$

The following lemma has been used in the preceding example.

Lemma 7.5. *Let $A = k[\![x, y]\!]/(xy)$. For a finite or stabilising string C the module $M(C)$ is artinian-by-noetherian over A. Moreover, $M(C)^\vee \cong M(C^-)$.*

Proof. Clearly, $M(C)$ is of finite length if and only if C is finite.

Next consider the following type $\mathbb{N}$ strings, where $a = x$ or $a = y$:

$$C_a\colon\ \circ \xleftarrow{a} \circ \xleftarrow{a} \circ \xleftarrow{a} \cdots \qquad \text{and} \qquad D_a\colon\ \circ \xrightarrow{a} \circ \xrightarrow{a} \circ \xrightarrow{a} \cdots .$$

The module $M(C_a)$ equals the image of $M(C_\infty) \xrightarrow{a} M(C_\infty)$. Analogously, $M(D_a)$ equals the image of $M(D_\infty) \xrightarrow{a} M(D_\infty)$. Thus $M(C_a)$ is artinian and $M(D_a)$ is noetherian.

If C is of type $\mathbb{N}$, then C can be written as concatenation of two strings C' and C'' such that $C' \in \{C_x, C_y, D_x, D_y\}$ and C'' is finite. This gives rise to an extension of the form

$$0 \to M(C') \to M(C) \to M(C'') \to 0$$

or

$$0 \to M(C'') \to M(C) \to M(C') \to 0.$$

It follows that $M(C)$ is artinian or noetherian, since $M(C')$ is artinian or noetherian by our previous observation.

If C is of type $\mathbb{Z}$, then one can write it as concatenation of two type-$\mathbb{N}$ strings C' and C'', connected by an arrow from C'' to C'. This yields an exact sequence

$$0 \to M(C') \to M(C) \to M(C'') \to 0,$$

and therefore $M(C)$ is artinian-by-noetherian; see Proposition 5.4.

With the arguments from the first part of the proof, the assertion about Matlis duality reduces to the case that C is finite or of the form C_a or D_a. Here, one uses the exactness of Matlis duality and that $(\varinjlim M_i)^\vee \cong \varprojlim M_i^\vee$. □

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