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THE RAMIFICATION TREE AND ALMOST DEDEKIND DOMAINS OF PRESCRIBED SP-RANK

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Given a valuation v on a field K and a chain $\mathcal{K} : K = K_0 \subseteq K_1 \subseteq \cdots$ of finite extensions of K, we construct a weighted tree $\mathcal{T}(v, \mathcal{K})$ encoding information about the ramification of v in the extensions K_i; conversely, when v is a discrete valuation, we show that a weighted tree $\mathcal{T}$ can be expressed as $\mathcal{T}(v, \mathcal{K})$ under some mild hypotheses on v or on $\mathcal{T}$. We use this correspondence to construct, for every countable successor ordinal number α and every discrete valuation ring V, an almost Dedekind domain D integral over V whose SP-rank is α. Subsequently, we extend this result to countable limit ordinal numbers by considering integral extensions of Dedekind domains with countably many maximal ideals.

1. Introduction

Let D be a Dedekind domain. Then, D has *prime factorization*, that is, every nonzero ideal of D can be written (uniquely) as a product of prime ideals, and indeed this property characterizes Dedekind domains. A weaker property is *radical factorization*: an ideal I has radical factorization if it can be written as a finite product of radical ideals, and a domain D is an *SP-domain* if every ideal has radical factorization. Every Dedekind domain is an SP-domain, but there are integral domains that are SP-domains without being Dedekind domains; nevertheless, every SP-domain that is not a field is an *almost Dedekind domain*, meaning that such a domain D is locally Dedekind or, equivalently, locally a discrete valuation ring.

Yet, not every almost Dedekind domain is an SP-domain. The *SP-rank* of an almost Dedekind domain, introduced in [11], is a measure of how far the domain is from having radical factorization: the SP-rank of D is an ordinal number that is 1 when D is an SP-domain. The SP-rank is defined by associating to each almost Dedekind domain a chain $\{\Delta_\alpha\}_\alpha$ of subsets of the maximal space $\mathrm{Max}(D)$, which also defines a chain $\{T_\alpha\}_\alpha$ of overrings of D; the definition of the chain $\{\Delta_\alpha\}_\alpha$ is formally very similar to the definition of the derived sequence of a topological

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space. In this analogy, the SP-rank of an almost Dedekind domain corresponds to the Cantor–Bendixson rank of a topological space.

So far, the examples of almost Dedekind domains that are not SP-domains had a single maximal ideal that is not critical (see below for the definition), or equivalently the first set Δ_1 of the chain $\{\Delta_\alpha\}$ is a singleton [4, Examples 3.4.1 and 3.4.2]. In particular, the SP-rank of such a domain is 2. In this paper, we construct almost Dedekind domains whose SP-rank is any arbitrary countable ordinal number α: in particular, we show that if α is a successor ordinal then the domain can be constructed as the integral closure of an (arbitrary) discrete valuation ring in an algebraic extension of its quotient field.

The main idea of the construction is to translate our problem in terms of trees. More precisely, given a discrete valuation v associated to a ring V and an infinite chain $\mathcal{K}$ of finite extensions of the quotient field of V, we define a weighted tree $\mathcal{T}(v, \mathcal{K})$ (called the *ramification tree* of v with respect to $\mathcal{K}$) whose elements are the extensions of v at the members of $\mathcal{K}$, and, given two extensions v_1, v_2, we have $v_1 \leq v_2$ if and only if v_2 is an extension of v_1; the weight of each edge is the ramification index of the extension. Using the possibility to construct extensions of valuations with prescribed ramification and inertia, we show that a tree $\mathcal{T}$ can be constructed as a ramification tree under some mild hypotheses either on $\mathcal{T}$ or on v (Propositions 3.4 and 3.5). Next, we define an SP-rank for trees, and we link the properties of $\mathcal{T}(v, \mathcal{K})$ (and, in particular, of its maximal paths) with the properties of the integral closure A of V in the union K_∞ of the elements of $\mathcal{K}$; thus, the problem of constructing an almost Dedekind domain of SP-rank α reduces to the construction of a tree with SP-rank α (Corollary 4.6). For successor ordinals, this is accomplished by an inductive construction (Theorems 5.5 and 5.7), while for limit ordinals it is necessary to consider a similar construction starting from a Dedekind domain with countably many maximal ideals (Theorem 6.3). In Section 7, we also analyze this construction from the point of view of the (inverse) topology on the maximal space of A.

An alternate construction is given in [8], where we construct almost Dedekind domains of arbitrary (not necessarily countable) SP-rank. However, the method in [8] only works for a specific construction, while the one in this paper can be applied to find almost Dedekind domains in much smaller fields (see, e.g., Corollary 5.8).

2. Preliminaries

2.1. ***Valuations.*** Let K be a field. A *valuation* v on K is a map $v : K \to \Gamma \cup \{\infty\}$, where $(\Gamma, +)$ is a totally ordered abelian group, such that, for every $x, y \in K$,

- $v(x) = \infty$ if and only if $x = 0$;
- $v(xy) = v(x) + v(y)$;
- $v(x + y) \geq \min\{v(x), v(y)\}$.

If the map v is surjective, Γ is said to be the *value group* of v, and is sometimes denoted by Γ_v.

The *valuation ring* associated to v is the ring $V := \{x \in K \mid v(x) \geq 0\}$; the quotient field of V is K. Conversely, if V is a ring with quotient field K such that, for every nonzero $x \in K$, at least one of x and x^{-1} is in V, then V is called a valuation ring, and there is a valuation v associated to V. A valuation ring V is always local; we denote its maximal ideal by $\mathfrak{m}_V$. The residue field $V/\mathfrak{m}_V$ of V is also called the residue field of v.

If $\Gamma \simeq \mathbb{Z}$, then v is said to be *discrete* and V is called a *discrete valuation ring* (DVR).

Let L be a field extending K. An *extension* of v to L is a valuation w on L such that $w|_K = v$; we also write $v \subseteq w$. If W is the valuation ring associated to w, then $W \cap K = V$, and we also say that W is an extension of V. Every extension w of v to L defines a map of residue fields $V/\mathfrak{m}_V \to W/\mathfrak{m}_W$ and an injective map $\Gamma_v \to \Gamma_w$. The degree $[W/\mathfrak{m}_W : V/\mathfrak{m}_V]$ is called the *inertial degree* of the extension (and is denoted by $f(w/v)$) while the index $(\Gamma_w : \Gamma_v)$ is called the *ramification degree*, and is denoted by $e(w/v)$. These degrees are multiplicative, i.e., if $v \subseteq v' \subseteq v''$ is a chain of extensions, then $e(v''/v) = e(v''/v')e(v'/v)$ and $f(v''/v) = f(v''/v')f(v'/v)$.

If $[L : K] < \infty$, v is a valuation on K and $w_1, \dots, w_g$ are extensions of v to L, then their inertial and ramification degrees are linked by the *fundamental inequality* [3, Theorem 3.3.4]

$$\sum_{i=1}^{g} e(w_i/v) f(w_i/v) \leq [L : K]. \tag{1}$$

If v is discrete, L is separable over K and $w_1, \dots, w_g$ are all the extensions of v to L, then the fundamental inequality becomes an equality [3, Theorem 3.3.5]; if L is also Galois over K, then $e(w_i/v) = e(w_j/v)$ and $f(w_i/v) = f(w_j/v)$ for all i, j (see [3, Proposition 3.2.16] or [9, Chapter 6, J]).

Conversely, we can always construct a field extension where a valuation extends with given inertia and ramification; see, for example, [9, Chapter 6, Theorem 4].

Theorem 2.1. *Let $v_1, \dots, v_k$ be discrete valuations on a field K. Let $g_1, \dots, g_k, e_{ij}, f_{ij}$ ($i = 1, \dots, k$, $j = 1, \dots, g_i$) be integers such that $\sum_{j=1}^{g_i} e_{ij} f_{ij} = n$ for each i, for some fixed n. Suppose that the residue field K_i of v_i has extensions L_{ij} such that $[L_{ij} : K_i] = f_{ij}$. Then, there is a finite separable extension L of K of degree n such that each valuation v_i has exactly g_i extensions, namely $w_{i1}, \dots, w_{ig_i}$, with ramification indices $(\Gamma_{w_{ij}}(L) : \Gamma_{v_i}(K)) = e_{ij}$ and such that the residue field of w_{ij} is L_{ij}.*

2.2. *Almost Dedekind domains.* Let D be an integral domain that is not a field. Then D is said to be an *almost Dedekind domain* if D_M is a discrete valuation ring for every $M \in \mathrm{Max}(D)$; an almost Dedekind domain is always a one-dimensional

Prüfer domain. An overring of an almost Dedekind domain D (i.e., a ring between D and its quotient field) is either the quotient field of D or an almost Dedekind domain.

Let D be an almost Dedekind domain with quotient field K. Any nonzero ideal I of D defines an ideal function

$$\nu_I : \mathrm{Max}(D) \to \mathbb{Z}, \quad M \to v_M(ID_M),$$

where $v_M : K \to \mathbb{Z} \cup \{\infty\}$ is the valuation relative to D_M and $v_M(ID_M)$ equals $\inf\{v(x) \mid x \in ID_M\}$.

A maximal ideal M of D is said to be *critical* if it does not contain any nonzero radical finitely generated ideal, or equivalently if $\sup \nu_I(\mathrm{Max}(D)) \geq 2$ for every nonzero finitely generated ideal $I \subseteq M$. We denote by $\mathrm{Crit}(D)$ the set of critical maximal ideals of D; this set is empty if and only if D is an SP-domain (see [7, Theorem 2.1] or [4, Theorem 3.1.2]).

We define recursively a chain $\{\mathrm{Crit}^\alpha(D)\}_\alpha$ of subsets of $\mathrm{Max}(D)$ and a chain $\{T_\alpha\}_\alpha$ of overrings of D in the following way, where α is an ordinal number:

- $T_0 := D$ and $\mathrm{Crit}^0(D) := \mathrm{Max}(D)$.
- If $\alpha = \beta + 1$ is a successor ordinal, then
$$\mathrm{Crit}^\alpha(D) := \{P \in \mathrm{Max}(D) \mid PT_\beta \in \mathrm{Crit}(T_\beta)\}.$$
- If α is a limit ordinal, then
$$\mathrm{Crit}^\alpha(D) := \bigcap_{\beta<\alpha} \mathrm{Crit}^\beta(D).$$
- $T_\alpha := \bigcap \{D_M \mid M \in \mathrm{Crit}^\alpha(D)\}$.

The set $\{\mathrm{Crit}^\alpha(D)\}_\alpha$ is a descending chain of subsets of $\mathrm{Max}(D)$, while $\{T_\alpha\}_\alpha$ is an ascending chain of subrings of K; we call the latter the *SP-derived sequence* of D. Moreover, if $T_\alpha \neq K$, the maximal ideals of T_α are exactly the extensions of the maximal ideals in $\mathrm{Crit}^\alpha(D)$ [11, Lemma 5.3].

The *SP-height* $\mathrm{SPh}(M)$ of a maximal ideal M is the smallest ordinal number α such that $MT_\alpha = T_\alpha$, or equivalently the smallest ordinal such that $M \notin \mathrm{Crit}^\alpha(D)$. The *SP-rank* of D is the smallest ordinal number α such that $\mathrm{Crit}^\alpha(D) = \emptyset$; this rank always exists [12, Theorem 5.1], and it is equal to the supremum of the SP-height of the maximal ideals of D. Equivalently, it is the smallest ordinal number α such that $T_\alpha = K$.

2.3. ***Topology.*** Let D be an integral domain. The *inverse topology* on the spectrum $\mathrm{Spec}(D)$ of D is the topology having, as a basis of open sets, the sets $V(I) := \{P \in \mathrm{Spec}(D) \mid I \subseteq P\}$, as I ranges among the finitely generated ideals of D.

We denote by $\mathrm{Max}(D)^{\mathrm{inv}}$ the maximal space of D, endowed with the inverse topology (i.e., with the subspace topology of the inverse topology of $\mathrm{Spec}(D)$).

The space $\mathrm{Max}(D)^{\mathrm{inv}}$ is always Hausdorff [2, Corollary 4.4.9], while the Zariski and the inverse topology agree on $\mathrm{Max}(D)$ if and only if $\mathrm{Max}(D)^{\mathrm{inv}}$ is compact [2, Corollary 4.4.17]. When D is one-dimensional (in particular, when D is an almost Dedekind domain) this happens if and only if the Jacobson radical of D is nonzero [6, Lemma 6.3(4)] (note that this result is stated in the reference only for one-dimensional Prüfer domains, but its proof does not need the Prüfer hypothesis).

The sets $\mathrm{Crit}^{\alpha}(D)$, where D is an almost Dedekind domain, are always closed in $\mathrm{Max}(D)^{\mathrm{inv}}$.

2.4. ***Trees.*** A *tree* is a partially ordered set $(\mathcal{T}, \leq)$ with a unique minimal element r, such that, for every $t \in \mathcal{T}$, the set $\{s \in \mathcal{T} \mid s < t\}$ is well ordered; the order type of this set is called the *height* of t. The element r is called the *root* of $\mathcal{T}$ and is the unique element of height 0; we also call the elements of $\mathcal{T}$ its *vertexes*. We denote by $\mathcal{T}(\alpha)$ the set of elements with height α. Throughout the paper, we shall assume that $\mathcal{T}$ is an ω-*tree*, that is, *every vertex has finite height*, and that $\mathcal{T}$ *has no maximal elements*.

An *edge* of $\mathcal{T}$ is a pair (a, b) of vertexes such that $a < b$ and there are no elements strictly between a and b; we denote the set of edges of $\mathcal{T}$ by $E(\mathcal{T})$. A *path* is a sequence $(a_i)_{i \in I}$ (where I is finite or $\mathbb{N}$) such that (a_i, a_{i+1}) is an edge for every $i \in I$; we call a_0 the *starting point* of the path. If $\pi' \subseteq \pi$ are paths, we say that π' is a subpath of π and that π is an extension of π'; if π has no proper extensions, we say that π is *maximal*. It is easy to see that, when $\mathcal{T}$ is an ω-tree, every infinite path is contained in a unique maximal path, and that a path is maximal if and only if it is infinite and its starting point is the root of $\mathcal{T}$. We denote by $\mathrm{MaxPath}(\mathcal{T})$ the set of maximal paths of $\mathcal{T}$. If π is a path and $t \in \pi$, we also say that π *passes through* t.

By a *weight* on $\mathcal{T}$ we mean a function $w : \mathcal{T} \times \mathcal{T} \to \mathbb{N}$ such that $w(a, b) > 0$ if and only if (a, b) is an edge of $\mathcal{T}$; in particular, $w(a, a) = 0$ for all a. The function w can also be considered as a function from $E(\mathcal{T})$ to $\mathbb{N}^+$. The (*outbound*) *weight* of $a \in \mathcal{T}$ is

$$w(a) := \sum_{b \in \mathcal{T}} w(a, b) = \sum_{(a,b) \in E(\mathcal{T})} w(a, b).$$

We say that $\mathcal{T}$ is

- *locally bounded* if, for every $n \in \mathbb{N}$, the set $\{w(a) \mid a \in \mathcal{T}(n)\}$ is bounded;
- *balanced* if, for every $n \in \mathbb{N}$, $w(a) = w(b)$ for every $a, b \in \mathcal{T}(n)$;
- *totally balanced* if $w(a) = w(b)$ for every $a, b \in \mathcal{T}$; in this case, this value is called the *weight of* $\mathcal{T}$.

If $\pi = (a_i)_{i \in \mathbb{N}}$ is a path, the *weight* of π is

$$w(\pi) := \prod_{i=0}^{\infty} w(a_i, a_{i+1}) \in \mathbb{N}^+ \cup \{\infty\}.$$

We say that π is

- *finitely ramified* if $w(\pi) < \infty$;
- *unramified* if $w(\pi) = 1$.

If π is finitely ramified, then $w(a_i, a_{i+1}) = 1$ for all large i; we call the largest infinite subpath of π that is unramified the *unramified subpath* of π, and we denote it by $u(\pi)$. We say that a tree $\mathcal{T}$ is *finitely ramified* if every maximal path is finitely ramified.

If $a \in \mathcal{T}$, we set $\{a\}^\uparrow := \{b \in \mathcal{T} \mid a \leq b\}$. In particular, $\{a\}^\uparrow$ is a tree with root a.

3. The correspondence

Let v be a valuation on a field K, with corresponding valuation ring V, and consider a chain

$$\mathcal{K} : K = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_n \subsetneq \cdots$$

of field extensions of K with $[K_{i+1} : K_i] < \infty$ for every i. The *ramification tree* of v with respect to $\mathcal{K}$, which we denote by $\mathcal{T}(v, \mathcal{K})$ or $\mathcal{T}(V, \mathcal{K})$, is defined in the following way:

- The elements of $\mathcal{T}(v, \mathcal{K})$ are the extensions of v at K_i, for each i.
- If $v_1, v_2 \in \mathcal{T}(v, \mathcal{K})$, we set $v_1 \leq v_2$ if v_2 is an extension of v_1.
- If (v_1, v_2) is an edge, $w(v_1, v_2)$ is the ramification index of the extension $v_1 \subseteq v_2$, i.e.,
$$w(v_1, v_2) = e(v_2/v_1) := [\Gamma_{v_2} : \Gamma_{v_1}].$$

We can also interpret the vertexes of $\mathcal{T}(v, \mathcal{K})$ in a different way: since the extensions of v to K_i are in bijective correspondence with the set $\mathrm{Max}(A_i)$ of the maximal ideals of A_i (the integral closure of V in K_i), the set $\mathcal{T}(v, \mathcal{K})$ corresponds to the union $\bigcup_{i \in \mathbb{N}} \mathrm{Max}(A_i)$ and, if M, N are two ideals in the union, then $M \leq N$ if and only if $M \subseteq N$. In particular, if $M \leq N$ then either $M = N$ or M, N are maximal ideals of distinct A_i's.

We use the same terminology also if the chain $\mathcal{K}$ is finite, that is, if we only have a finite chain $K = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_n$. In this case, the tree $\mathcal{T}(v, \mathcal{K})$ is finite.

Proposition 3.1. *Preserve the notation above and let $\mathcal{T} := \mathcal{T}(v, \mathcal{K})$.*

(a) *If (v_1, v_2) is an edge and v_1 is a valuation on K_i, then v_2 is a valuation on K_{i+1}.*

(b) *$\mathcal{T}$ is locally bounded.*

Proof. (a) If (v_1, v_2) is an edge, then v_2 is a proper extension of v_1, and thus v_2 is a valuation on K_j for some $j > i$. If $j > i + 1$, then we have $v_1 < v_2|_{K_{i+1}} < v_2$, against the fact that (v_1, v_2) is an edge. Thus $j = i + 1$, as claimed.

(b) Fix $i \in \mathbb{N}$ and let $v_1 \in \mathcal{T}(v, \mathcal{K})$ be a valuation on K_i. Then,

$$w(v_1) = \sum_{(v_1,v_2)\in E(\mathcal{T})} w(v_1, v_2) = \sum_{(v_1,v_2)\in E(\mathcal{T})} e(v_2/v_1).$$

By the previous point, if (v_1, v_2) is an edge then v_2 is a valuation on K_{i+1}; by the fundamental inequality (1), it follows that $w(v_1) \leq [K_{i+1} : K_i]$. Since this quantity does not depend on v_1 but only on i, the tree $\mathcal{T}$ is locally bounded. □

Every K_i is a finite extension of K; therefore, if v is discrete, all the integral closures A_i of V in K_i are Dedekind domains. The main object of interest of this paper is their union, or equivalently the integral closure of V in the union of all K_i.

Proposition 3.2. *Preserve the notation above, let $K_\infty := \bigcup_{i\in\mathbb{N}} K_i$, and let A be the integral closure of V in K_∞. Let $\mathcal{T} := \mathcal{T}(v, \mathcal{K})$ and let $\mathcal{E}(v, K_\infty)$ be the set of extensions of v to K_∞. For every $M \in \mathrm{Max}(A)$, let v_M be the valuation relative to A_M. Then, there are natural bijective correspondences between $\mathcal{E}(v, K_\infty)$, $\mathrm{Max}(A)$ and $\mathrm{MaxPath}(\mathcal{T})$, given by*

$$\begin{gathered}\mathrm{Max}(A) \leftrightarrow \mathcal{E}(v, K_\infty),\\ M \to v_M,\\ \mathfrak{m}_{v_\infty} \cap A \leftarrow v_\infty,\end{gathered}$$

and

$$\begin{gathered}\mathcal{E}(v, K_\infty) \leftrightarrow \mathrm{MaxPath}(\mathcal{T}),\\ v_\infty \to (v_\infty|_{K_i})_{i\in\mathbb{N}},\\ \bigcup_{i\in\mathbb{N}} v_i \leftarrow (v_i)_{i\in\mathbb{N}},\end{gathered}$$

where $\big(\bigcup_{i\in\mathbb{N}} v_i\big)(x) = v_j(x)$ if $j \in \mathbb{N}$ and $x \in K_j$.

Proof. The bijective correspondence between $\mathcal{E}(v, K_\infty)$ and $\mathrm{Max}(A)$ is a consequence of [5, Theorem 20.1].

If $v_\infty \in \mathcal{E}(v, K_\infty)$, then $v_i := v_\infty|_{K_i}$ is an extension of v, and v_j is an extension of v_i if $j > i$; therefore, $\pi := (v_i)_{i\in\mathbb{N}}$ is a maximal path in $\mathcal{T}$. Conversely, if $\pi = (v_i)_{i\in\mathbb{N}}$ is a maximal path in $\mathcal{T}$, then the union $v_\infty := \bigcup_{i\in\mathbb{N}} v_i$ is well-defined since v_j extends v_i when $j > i$. It is straightforward to see that these two correspondences are inverses of each other. □

We now concentrate on the main case that is of interest for this paper, namely the case in which v is a discrete valuation. In this case, we have the following.

Proposition 3.3. *Preserve the notation above, and suppose that v is discrete. Then, A is an almost Dedekind domain if and only if $\mathcal{T}$ is finitely ramified.*

Proof. If π is the maximal path corresponding to a maximal ideal M of A, then the ramification index of M over $M \cap V$ is, by construction, equal to the ramification index of π. The result now follows from [1, Corollary 3.6]. □

Let now $v' \in \mathcal{T}(v, \mathcal{K})$. By what we have seen, we can associate to v' a maximal ideal M of some A_i; thus, we can also associate to v' a finitely generated ideal of A, namely MA, and we have a natural map

$$\Psi : \mathcal{T}(v, \mathcal{K}) \to \mathcal{I}_f(A), \quad v' \mapsto MA,$$

where $\mathcal{I}_f(A)$ is the set of finitely generated ideals of A. The map Ψ is in general not surjective (for example, a necessary condition for I to be in the image is that $I \cap A_i$ is radical for some i). However, if I is a finitely generated ideal of A, we can find an index i such that K_i contains all elements of a family of generators of I; then, $I = JA$ is the extension of J, an ideal of the Dedekind domain A_i, and thus J is a product of ideals of A_i associated to elements of $\mathcal{T}$. In particular, I can be written as a product of ideals of the form $\Psi(v_i)$, with $v_i \in \mathcal{T}$.

So far, we have started from a chain of extensions and constructed a tree from it. However, under mild hypotheses we can actually go back, and construct a chain of extensions from a tree. We give two such constructions, one with a stronger hypothesis on the tree and one with a stronger hypothesis on the valuation.

Proposition 3.4. *Let $\mathcal{T}$ be a balanced tree, and let v be a discrete valuation on K. Then, there is a chain $\mathcal{K}$ of finite separable extensions of K such that $\mathcal{T} \simeq \mathcal{T}(v, \mathcal{K})$.*

Proof. We construct the fields K_i by induction. If $i = 0$, we take $K = K_0$. Suppose that we have constructed extensions up to K_i such that the ramification tree of the chain $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_i$ coincides with the set of elements of $\mathcal{T}$ of height at most i. Let $v_1, \ldots, v_k$ be the extensions of v to K_i, corresponding respectively to $a_1, \ldots, a_k \in \mathcal{T}$. For each i, let $(a_i, b_{i,1}), \ldots, (a_i, b_{i,t_i})$ be the edges starting from a_i. By the extension theorem (Theorem 2.1), since the tree is balanced, we can find an extension K_{i+1} of K_i of degree $w(a_1) = \cdots = w(a_k)$ such that each v_i has t_i extensions of ramification degree $w(a_i, b_{i,1}), \ldots, w(a_i, b_{i,t_i})$ and such that every extension of residue fields is trivial. The ramification tree of $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{i+1}$ now is isomorphic to the subset of $\mathcal{T}$ of the elements of height at most $i + 1$, and the claim follows by induction. □

Proposition 3.5. *Let $\mathcal{T}$ be a locally bounded tree, and let v be a discrete valuation on K with finite residue field. Then, there is a chain $\mathcal{K}$ of finite separable extensions of K such that $\mathcal{T} \simeq \mathcal{T}(v, \mathcal{K})$.*

Proof. We construct the fields K_i by induction. If $i = 0$, we take $K = K_0$. Suppose that we have constructed extensions up to K_i such that the ramification tree of the chain $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_i$ coincides with the set of elements of $\mathcal{T}$ of height at most i. Let $v_1, \dots, v_k$ be the extensions of v to K_i, corresponding respectively to $a_1, \dots, a_k \in \mathcal{T}$. Note that each K_i is a finite extension of K, and thus the residue field of v_i is finite: in particular, the residue field has finite extensions of any degree. Let δ be the least common multiple of $w(a_1), \dots, w(a_k)$. For each i, let $(a_i, b_{i,1}), \dots, (a_i, b_{i,t_i})$ be the edges starting from a_i. By Theorem 2.1, we can find an extension K_{i+1} of K_i of degree δ such that each v_i has t_i extensions, say $v_{i,1}, \dots, v_{i,t_i}$, such that the extension $v_i \subseteq v_{i,r}$ has ramification degree $e_r := w(a_i, b_{i,r})$ and inertial degree $f_r := \delta/w(a_i)$: this is possible since

$$\sum_{r=1}^{t_i} e_r f_r = \sum_{r=1}^{t_i} w(a_i, b_{i,r}) \frac{\delta}{w(a_i)} = w(a_i) \cdot \frac{\delta}{w(a_i)} = \delta$$

for every i.

The ramification tree of $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{i+1}$ now is isomorphic to the subset of $\mathcal{T}$ of the elements of height at most $i+1$, and the claim follows by induction. □

4. Criticality

In this section, we see how we can use the ramification tree to detect the SP-height of a maximal ideal. We begin with a purely ring-theoretic characterization.

Proposition 4.1. *Let A be an almost Dedekind domain, and let $M \in \mathrm{Max}(A)$. Let α be an ordinal number. Then, $M \in \mathrm{Crit}^\alpha(A)$ if and only if for every nonzero finitely generated ideal $I \subseteq M$ and every $\beta < \alpha$ there is a maximal ideal $N \in \mathrm{Crit}^\beta(A)$ such that $v_I(N) \geq 2$.*

Proof. Let $\{T_\alpha\}$ be the SP-derived sequence of A.

Suppose $M \in \mathrm{Crit}^\alpha(A)$, let $I \subseteq M$ be a nonzero and finitely generated ideal of A and let $\beta < \alpha$. Then, $IT_\beta \neq T_\beta$ since $MT_\beta \neq T_\beta$. If IT_β is radical, then MT_β is a noncritical maximal ideal of T_β, and thus $M \notin \mathrm{Crit}^{\beta+1}(A) \supseteq \mathrm{Crit}^\alpha(A)$, a contradiction. Hence IT_β is not radical, i.e., there is an $N' \in \mathrm{Max}(T_\beta)$ such that $v_{IT_\beta}(N') \geq 2$. By construction, $N := N' \cap A \in \mathrm{Crit}^\beta(A)$; hence, $D_N = (T_\beta)_{N'}$ by [5, Theorem 26.1], and $v_I(N) = v_{IT_\beta}(N') \geq 2$. The claim is proved.

Suppose that $M \notin \mathrm{Crit}^\alpha(A)$, and let β be an ordinal number such that the SP-height of M is $\beta + 1$. Then, $\beta < \alpha$, and thus MT_β is not a critical maximal ideal. Hence it contains a nonzero finitely generated radical ideal J; let I be a finitely generated ideal of A such that $J = IT_\beta$. If $N \in \mathrm{Crit}^\beta(A)$ contains I, then $v_I(N) = v_J(NT_\beta) = 1$, and thus the condition of the statement does not hold. □

The following lemma represents the translation from ideal functions to the ramification tree.

Lemma 4.2. *Fix an $i \in \mathbb{N}$, and let $V, \mathcal{K}, K_\infty, A_i, A$ be as in Section 3; suppose that V is a DVR and A is almost Dedekind. Let $M \in \mathrm{Max}(A)$ and let $N := M \cap A_i$;*

let v_i be the element of $\mathcal{T}$ corresponding to N. Let π be the path corresponding to M in $\mathcal{T}(v,\mathcal{K})$ and let π' be the infinite subpath of π starting from v_i. Then,

$$\nu_{NA}(M)=w(\pi').$$

Proof. Let $N_j := M\cap A_{i+j}$; then, $N=N_0$ and $N_0, N_1,\dots$ is the sequence of prime ideals corresponding to π'; let v_i be the valuation corresponding to N_i. Consider the ideal function $\nu_{N_jA_{i+j+1}}$ on the maximal ideals of A_{i+j+1} (which is a Dedekind domain, hence almost Dedekind): by definition of the ramification index,

$$\nu_{N_jA_{i+j+1}}(N_{j+1})=e(v_{j+1}/v_j)=w(v_j,v_{j+1}).$$

Since M is the union of the N_j and the ramification indices multiply, we obtain that

$$\nu_{NA}(M)=\prod_{j\geq 0}e(v_{j+1}/v_j)=\prod_{j\geq 0}w(v_j,v_{j+1})=w(\pi'),$$

as claimed. □

Proposition 4.3. *Let V, $\mathcal{K}$, K_∞, A_i, A be as in Section 3; suppose that V is a DVR and A is almost Dedekind. Let M be a maximal ideal of A and let π be the corresponding maximal path in $\mathcal{T}(v,\mathcal{K})$. Then, the following are equivalent:*

(i) $M\in\mathrm{Crit}(A)$.

(ii) *For all $v_i\in u(\pi)$ there is a path π' starting from v_i such that $w(\pi')>1$.*

(iii) *There is an infinite unramified subpath π_0 of π such that, for all $v_i\in\pi_0$, there is a path π' starting from v_i with $w(\pi')>1$.*

Proof. (i) $\Longrightarrow$ (ii): Suppose $M\in\mathrm{Crit}(A)$ and let $v_i\in u(\pi)$. Then, v_i corresponds to a maximal ideal N of A_i, the integral closure of V in K_i; let $I:=NA$. Then, I is finitely generated; if π_0 is the subpath of π starting from N, then $\nu_I(M)=w(\pi_0)=1$ because π_0 is a subpath of $u(\pi)$, which is unramified. Since M is critical, I is not radical, and thus there is a maximal ideal M' of A such that $\nu_I(M')\geq 2$; by construction, we must have $M'\cap A_i=N$. Let π' be the subpath of the path corresponding to M' that starts from N. Then, by Lemma 4.2, $w(\pi')=\nu_I(M')\geq 2>1$, as claimed.

(ii) $\Longrightarrow$ (iii): This is obvious (just take $\pi_0=u(\pi)$).

(iii) $\Longrightarrow$ (i): Suppose that M is not critical, let $I=(x_1,\dots,x_k)A$ be a nonzero finitely generated radical ideal contained in M, and fix $v_1\in\pi_0$ of height λ. Let t be an integer such that $t\geq\lambda$ and such that $x_1,\dots,x_k\in A_t$: then, there is a prime ideal N of A_t such that $(x_1,\dots,x_k)A_t\subseteq N\subseteq M$. Let v_2 be the vertex of $\mathcal{T}$ associated to N; then, $v_2\in\pi$ and $v_2\geq v_1$, so that $v_2\in\pi_0$. Applying the hypothesis on v_2, we can find a path π' starting from v_2 such that $w(\pi')>1$. Let M' be the maximal ideal of A corresponding to (the maximal extension of) π': then,

$I \subseteq NA \subseteq M'$, and thus since I is radical we must have $\nu_I(M') = 1$. However, using Lemma 4.2,

$$\nu_I(M') \geq \nu_{NA}(M') = w(\pi') \geq 2,$$

a contradiction. Therefore M must be critical, as claimed. □

Let now $\mathcal{T}$ be a tree such that all its paths are finitely ramified. We define recursively:

- $\mathrm{Crit}^0(\mathcal{T}) := \mathrm{MaxPath}(\mathcal{T})$.
- If $\alpha = \beta + 1$, then

$$\mathrm{Crit}^\alpha(\mathcal{T}) := \{\pi \in \mathrm{MaxPath}(\mathcal{T}) \mid \forall v \in u(\pi) \text{ there is a path } \pi' \text{ starting from } v \text{ with } w(\pi') > 1 \text{ and such that the maximal extension of } \pi' \text{ is in } \mathrm{Crit}^\beta(\mathcal{T})\}.$$

- If α is a limit ordinal, then $\mathrm{Crit}^\alpha(\mathcal{T}) := \bigcap \{\mathrm{Crit}^\beta(\mathcal{T}) \mid \beta < \alpha\}$.

We define the *SP-height* $\mathrm{SPh}(\pi)$ of $\pi \in \mathrm{MaxPath}(\mathcal{T})$ as the smallest ordinal β such that $\pi \notin \mathrm{Crit}^\beta(\mathcal{T})$, and the *SP-rank* SP-rank$(\mathcal{T})$ of $\mathcal{T}$ as the smallest ordinal β such that $\mathrm{Crit}^\beta(\mathcal{T}) = \emptyset$.

Lemma 4.4. *Preserve the notation above. The SP-rank of $\mathcal{T}$ is the supremum of the SP-heights of the maximal paths of $\mathcal{T}$.*

Proof. This follows directly from the definitions. □

Putting together the two propositions above we get:

Theorem 4.5. *Preserve the notation above. Let $M \in \mathrm{Max}(A)$ and let π be the corresponding maximal path in $\mathcal{T}(v, \mathcal{K})$. Let α be an ordinal number. Then, $M \in \mathrm{Crit}^\alpha(A)$ if and only if $\pi \in \mathrm{Crit}^\alpha(\mathcal{T}(v, \mathcal{K}))$.*

Proof. Let $\mathcal{T} := \mathcal{T}(v, \mathcal{K})$. Proposition 4.3 shows that $M \in \mathrm{Crit}^1(A)$ (that is, M is critical) if and only if $\pi \in \mathrm{Crit}^1(\mathcal{T})$.

Suppose that, for every $\beta < \alpha$, we have that $M \in \mathrm{Crit}^\beta(A)$ if and only if $\pi \in \mathrm{Crit}^\beta(\mathcal{T})$. If α is a limit ordinal, this immediately implies that the same holds for α in place of β. Suppose that $\alpha = \beta + 1$ is a successor ordinal and that $M \in \mathrm{Crit}^\alpha(A)$. If $v' \in u(\pi)$ and N is the maximal ideal associated to v', then NA is a nonzero finitely generated ideal contained in M. Then by Proposition 4.1, there is $M' \in \mathrm{Crit}^\beta(A)$ such that $\nu_{NA}(M') \geq 2$. In particular, if π_0 denotes the maximal path associated to M', then $\pi_0 \in \mathrm{Crit}^\beta(\mathcal{T}(v, \mathcal{K}))$ by the induction hypothesis and π_0 passes through v'. Let π' denote the subpath of π_0 starting from v'. Then by Lemma 4.2, we have $w(\pi') = \nu_{NA}(M') \geq 2$. Hence $\pi \in \mathrm{Crit}^\alpha(\mathcal{T}(v, \mathcal{K}))$.

Conversely, suppose that $\pi \in \mathrm{Crit}^\alpha(\mathcal{T}(v, \mathcal{K}))$ and let $I = (x_1, \ldots, x_k) \subseteq M$ be a nonzero finitely generated ideal. Then there is an integer t such that $x_1, \ldots, x_k \in A_t$

and such that $u(\pi)$ contains a vertex of height t. Let $J = (x_1, \dots, x_k)A_t$, let N be a maximal ideal of A_t such that $J \subseteq N \subseteq M$ and let v' be the vertex associated to N. Note that we have $v' \in u(\pi)$. Then $I \subseteq NA \subseteq M$ and hence there is a path π' starting in v' with $w(\pi') \geq 2$ such that the maximal extension of π' is contained in $\mathrm{Crit}^\beta(\mathcal{T}(v, \mathcal{K}))$. Let M' be the maximal ideal associated to the maximal extension of π'. Then by Lemma 4.2, we have $\nu_I(M') \geq \nu_{NA}(M') = w(\pi') = 2$. Since $M' \in \mathrm{Crit}^\beta(A)$ by the induction hypothesis, we obtain by Proposition 4.1 that $M \in \mathrm{Crit}^\alpha(A)$. □

Corollary 4.6. *Preserve the notation above. Then*, SP-rank$(A) =$ SP-rank$(\mathcal{T})$.

5. The construction

The results in the previous sections show that the problem of determining an almost Dedekind domain with given SP-rank α can be fully translated to the case of trees, and that it is enough to solve it in this context. In this section, we introduce a construction that allows to build trees with higher and higher SP-rank inductively. We first show that the SP-height of a path only depends on its tail.

Proposition 5.1. *Let $\mathcal{T}$ be a locally bounded finitely ramified tree. Let π be a maximal path and let $a \in \pi$. Then, the SP-height of π is equal to the SP-height of $\pi \cap \{a\}^\uparrow$ in $\{a\}^\uparrow$.*

Proof. By definition, $\pi \notin \mathrm{Crit}^1(\mathcal{T})$ if and only if there is a $b \in u(\pi)$ such that $w(\pi') = 1$ for every path π' starting at b. This is the case if and only if every $b' \in u(\pi)$ larger than b satisfies this property, which in turn is equivalent to $\pi \cap \{a\}^\uparrow \notin \mathrm{Crit}^1(\{a\}^\uparrow)$. Hence $\mathrm{SPh}(\pi) = 1$ if and only if $\mathrm{SPh}(\pi \cap \{a\}^\uparrow) = 1$. We proceed by induction.

Suppose that for every ordinal number $\gamma < \alpha$, we have $\mathrm{SPh}(\pi) = \gamma$ if and only if $\mathrm{SPh}(\pi \cap \{a\}^\uparrow) = \gamma$. Suppose that $\mathrm{SPh}(\pi) = \alpha$: then, α must be a successor ordinal, say $\alpha = \beta + 1$, and $\mathrm{SPh}(\pi \cap \{a\}^\uparrow) \geq \alpha$. By definition, there is a $b \in u(\pi)$ such that, for all paths π' starting from b, we have $w(\pi') = 1$ or the maximal extension of π' is not in $\mathrm{Crit}^\beta(\mathcal{T})$. The same holds for all elements larger than b, and thus in particular for $c := \sup\{a, b\}$. Since by the induction hypothesis, the restrictions of the paths in $\mathrm{Crit}^\beta(\mathcal{T})$ are in $\mathrm{Crit}^\beta(\mathcal{T} \cap \{a\}^\uparrow)$, we have $\pi \cap \{a\}^\uparrow \notin \mathrm{Crit}^\alpha(\mathcal{T} \cap \{a\}^\uparrow)$, and thus $\mathrm{SPh}(\pi \cap \{a\}^\uparrow) = \alpha$. Conversely, if $\mathrm{SPh}(\pi \cap \{a\}^\uparrow) = \alpha$, then we can find a $b \in \pi \cap \{a\}^\uparrow$ with the same property, and the same b works also for the whole path π. Hence $\mathrm{SPh}(\pi) = \alpha$.

By induction, it follows that, for every ordinal number α, we have $\mathrm{SPh}(\pi) = \alpha$ if and only if $\mathrm{SPh}(\pi \cap \{a\}^\uparrow) = \alpha$. In particular, the SP-heights are equal, as claimed. □

Let $\mathcal{T}_1, \dots, \mathcal{T}_n, \dots$ be a sequence of trees, and let r_i be the root of $\mathcal{T}_i$. We construct a new weighted tree $\mathcal{T} := \Lambda(\mathcal{T}_1, \dots, \mathcal{T}_n, \dots)$ in the following way:

- As a set, $\mathcal{T}$ is the disjoint union of $\mathcal{T}_i$ (for each i), and of a countable sequence $(x_i)_{i=0}^{\infty}$.
- The edges of $\mathcal{T}$ are:
  - (a, b), where $a, b \in \mathcal{T}_i$ for some $i \in \mathbb{N}$ and (a, b) is an edge in $\mathcal{T}_i$.
  - (x_i, x_{i+1}) for all $i \geq 0$.
  - (x_i, r_{i+1}) for all $i \geq 0$.
- Their weights are:
  - If $a, b \in \mathcal{T}_i$, then $w(a, b)$ is the weight of (a, b) in $\mathcal{T}_i$.
  - $w(x_i, x_{i+1}) = 1$.
  - $w(x_i, r_{i+1}) = 2$.

It is not hard to see that these conditions really define a tree with root x_0; indeed, the unique path connecting x_0 to x_n is $(x_0, x_1, \ldots, x_n)$ (as the only edge terminating in x_n is (x_{n-1}, x_n)), while if $v \in \mathcal{T}_i$ for some i, we can construct a path from x_0 to v by joining $(x_0, \ldots, x_{i-1})$ with the path from r_i to v, and this is unique since if $(x_0, v_1, \ldots, v_n = v)$ is a path connecting x_0 to v then there must be an edge (v_j, v_{j+1}) with $v_j \notin \mathcal{T}_i$ and $v_{j+1} \in \mathcal{T}_i$, which is only possible if $v_j = x_{i-1}$ and $v_{j+1} = r_i$.

An alternative way to construct $\mathcal{T} = \Lambda(\mathcal{T}_1, \ldots, \mathcal{T}_n, \ldots)$ is by recursion:

- We take x_0 as the root.
- x_0 has two direct successors, x_1 and r_1, with $w(x_0, x_1) = 1$ and $w(x_0, r_1) = 2$.
- $\{r_1\}^{\uparrow} \simeq \mathcal{T}_1$.
- x_1 has two direct successors, x_2 and r_2, with weight 1 and 2, respectively, and $\{r_2\}^{\uparrow} = \mathcal{T}_2$.
- x_2 has two direct successors, and so on.

Proposition 5.2. *Preserve the notation above. Let $i \in \mathbb{N}$, let π be a maximal path in $\mathcal{T}_i$, and let $\widetilde{\pi}$ be the maximal path in $\mathcal{T}$ extending π. Then,*

$$\widetilde{\pi} = (x_0, \ldots, x_{i-1}) \cup \{\pi\} \quad \textit{and} \quad w(\widetilde{\pi}) = 2w(\pi).$$

In particular, if each $\mathcal{T}_i$ is finitely ramified then so is $\mathcal{T} = \Lambda(\mathcal{T}_1, \ldots, \mathcal{T}_n, \ldots)$.

Proof. As x_0 is the root of $\mathcal{T}$, $(x_0, \ldots, x_{i-1}) \cup \{\pi\}$ is a maximal path, and it clearly extends π. Moreover,

$$w(\widetilde{\pi}) = w(x_0, x_1) \cdots w(x_{i-2}, x_{i-1}) \cdot w(x_{i-1}, r_i) \cdot w(\pi) = 2w(\pi)$$

by construction.

To prove the last claim, we note that a maximal path ρ is either equal to $\widetilde{\pi}$ for some maximal path π of $\mathcal{T}_i$ or $\rho = (x_i)_{i=0}^{\infty}$. In the former case, $w(\rho) = 2w(\pi) < \infty$

by our hypothesis and the previous part of the proof, while in the latter case $w(\rho)=1$ by construction. Hence $\mathcal{T}$ is finitely ramified. □

Proposition 5.3. *Preserve the notation above and suppose that* SP-rank$(\mathcal{T}_i) \leq$ SP-rank$(\mathcal{T}_j)$ *for all* $i \leq j$. *Then*

$$\text{SP-rank}(\Lambda(\mathcal{T}_1, \dots, \mathcal{T}_n, \dots)) = (\sup_n \text{SP-rank}(\mathcal{T}_n)) + 1.$$

Proof. Let π be a maximal path in $\mathcal{T} := \Lambda(\mathcal{T}_1, \dots, \mathcal{T}_n, \dots)$. If π contains r_n for some n, then the tail of π is in $\mathcal{T}_n$, and thus its SP-height is the same as the SP-height of $\pi \cap \{r_n\}^\uparrow$ in $\{r_n\}^\uparrow \simeq \mathcal{T}_n$ by Proposition 5.1, and in particular is at most SP-rank$(\mathcal{T}_n)$.

Suppose that π does not contain any r_i: then $\pi = (x_i)_{i=0}^\infty$ is unramified. For every i, any path $\pi' \neq \pi$ starting from x_i must contain the edge (x_j, r_{j+1}) for some $j \geq 1$ and since $w(x_j, r_{j+1}) = 2$, π' is not unramified. Let now $\alpha := \sup_n$ SP-rank$(\mathcal{T}_n)$. Since $\{\text{SP-rank}(\mathcal{T}_n)\}_n$ is a nondecreasing sequence, we find for all $\beta < \alpha$ and x_i a path π', starting at x_i, whose maximal extension is contained in $\text{Crit}^\beta(\mathcal{T})$ and is not unramified. Therefore, $\pi \in \text{Crit}^\alpha(\mathcal{T})$ and by the previous paragraph, we have $\text{Crit}^\alpha(\mathcal{T}) = \{\pi\}$. Therefore, $\text{Crit}^{\alpha+1}(\mathcal{T}) = \varnothing$ and SP-rank$(\mathcal{T}) = \alpha + 1$, proving our claim. □

Fix now a tree $\mathcal{T}$ of SP-rank 1. We want to construct a sequence $\mathcal{T}_\alpha$ of trees recursively, indexed by the countable successor ordinals α, such that SP-rank$(\mathcal{T}_\alpha) = \alpha$. We thus define:

(1) If $\alpha = 1$, then $\mathcal{T}_1 := \mathcal{T}$.

(2) If $\alpha = \beta + 1$ and β is a successor ordinal, then

$$\mathcal{T}_\alpha := \Lambda(\mathcal{T}_\beta, \mathcal{T}_\beta, \dots).$$

(3) If $\alpha = \beta + 1$ and β is a limit ordinal, then β has countable cofinality, and thus we can find an increasing sequence $\{\gamma_n\}_{n=0}^\infty$ of ordinals whose supremum is β, and we define

$$\mathcal{T}_\alpha := \Lambda(\mathcal{T}_{\gamma_0+1}, \mathcal{T}_{\gamma_1+1}, \dots, \mathcal{T}_{\gamma_n+1}, \dots).$$

Note that in the last construction there are many possible sequences $\{\gamma_n\}$, so the sequence of trees $\{\mathcal{T}_\alpha\}$ is not uniquely determined by $\mathcal{T}_1$.

Theorem 5.4. *Preserve the notation above and suppose that* $\mathcal{T}$ *is finitely ramified. Then, for every countable successor ordinal number* α, $\mathcal{T}_\alpha$ *is finitely ramified and* SP-rank$(\mathcal{T}_\alpha) = \alpha$.

Proof. We proceed by induction on α. If $\alpha = 1$ then $\mathcal{T}$ is finitely ramified and SP-rank$(\mathcal{T}_1) = 1$ by our hypothesis.

The fact that $\mathcal{T}_\alpha$ is finitely ramified follows from Proposition 5.2.

If $\alpha = \beta + 1$ and β is a successor ordinal, then $\mathcal{T}_\beta$ is defined and SP-rank$(\mathcal{T}_\beta) = \beta$ by the inductive hypothesis. Since $\{\text{SP-rank}(\mathcal{T}_\beta)\}_n$ is a constant sequence, we can apply Proposition 5.3 to obtain

$$\text{SP-rank}(\mathcal{T}_\alpha) = \text{SP-rank}(\mathcal{T}_\beta) + 1 = \beta + 1 = \alpha.$$

Suppose $\alpha = \beta + 1$ and β is a limit ordinal. Then $\lim_n(\gamma_n + 1) = \lim_n \gamma_n = \beta$ and $\{\text{SP-rank}(\mathcal{T}_{\gamma_n+1})\}_n$ is a nondecreasing sequence by the induction hypothesis. Thus, Proposition 5.3 yields

$$\text{SP-rank}(\mathcal{T}_\alpha) = (\sup_n \text{SP-rank}(\mathcal{T}_{\gamma_n+1})) + 1 = (\sup_n(\gamma_n + 1)) + 1 = \beta + 1 = \alpha.$$

Hence SP-rank$(\mathcal{T}_\alpha) = \alpha$ for every α, as claimed. □

We want to use this construction in two different ways, corresponding to Propositions 3.4 and 3.5.

Theorem 5.5. *Let α be a countable successor ordinal.*

(a) *There is a totally balanced finitely ramified tree $\mathcal{T}$ with* SP-rank$(\mathcal{T}) = \alpha$.

(b) *For every discrete valuation ring V, there is an almost Dedekind domain A such that $V \subseteq A$ is integral and* SP-rank$(A) = \alpha$.

Proof. (a) Let $\mathcal{T}_1$ be the ω-tree where each element has three direct successors and each edge has weight 1. Then, SP-rank$(\mathcal{T}_1) = 1$ since all paths are unramified (in particular, $\mathcal{T}_1$ is finitely ramified). Construct a sequence $\{\mathcal{T}_\alpha\}$ as above; we claim that each $\mathcal{T}_\alpha$ is finitely ramified and totally balanced with weight 3, and we proceed by induction.

The claim is trivially true for $\alpha = 1$. Theorem 5.4 shows that every $\mathcal{T}_\alpha$ is finitely ramified. We claim that, if $\mathcal{T}' := \Lambda(\mathcal{T}'_1, \dots, \mathcal{T}'_n, \dots)$ and each $\mathcal{T}'_i$ is totally balanced and has weight 3, then the same holds for $\mathcal{T}'$. Let thus $v \in \mathcal{T}'$: if $v \in \mathcal{T}'_i$ for some i then the claim is true by induction. If $v \notin \mathcal{T}'_i$, then (in the notation of the beginning of this section) $v = x_i$ for some i, and thus v has two direct successors, x_{i+1} and r_{i+1}, with $w(v, x_{i+1}) = 1$ and $w(v, r_{i+1}) = 2$. Thus also $w(v) = 3$, and $\mathcal{T}'$ is totally balanced.

Since all $\mathcal{T}_\alpha$ are built with the construction Λ, it follows by induction that each $\mathcal{T}_\alpha$ is balanced with weight 3.

(b) Let $\mathcal{T}$ be a finitely ramified totally balanced tree with SP-rank$(\mathcal{T}) = \alpha$. By Proposition 3.4, we can find a chain $\mathcal{K}$ of algebraic field extensions of the quotient field K of V such that $\mathcal{T}(V, \mathcal{K}) \simeq \mathcal{T}$, and by Proposition 3.3, the integral closure A of V in $K_\infty := \bigcup_{L \in \mathcal{K}} L$ is an almost Dedekind domain. Moreover, by the correspondence between critical sets, A has SP-rank α. The claim is proved. □

Remark 5.6. In the first part of the previous theorem, we constructed a totally balanced tree $\mathcal{T}$ with weight 3. To construct a sequence of trees with SP-rank α of weight $n > 3$, it is enough to take as $\mathcal{T}_1$ the tree where each element has n direct successors, and slightly change the construction Λ, so that $w(x_i, r_{i+1}) = n - 1$ (instead of 2) for all i.

Theorem 5.7. *Let α be a countable successor ordinal.*

(a) *There is a finitely ramified tree $\mathcal{T}$ such that* $\operatorname{MaxPath}(\mathcal{T})$ *is countable and* $\text{SP-rank}(\mathcal{T}) = \alpha$.

(b) *For every discrete valuation ring V with finite residue field, there is an almost Dedekind domain A such that $V \subseteq A$ is integral,* $\text{SP-rank}(A) = \alpha$ *and* $\operatorname{Max}(A)$ *is countable.*

Proof. (a) Let $\mathcal{T}_1$ be a totally ordered ω-tree, and let the weight of all its edges be 1. Construct a sequence $\{\mathcal{T}_\alpha\}$ as above; we claim that $\operatorname{MaxPath}(\mathcal{T}_\alpha)$ is countable for every α. Clearly $\mathcal{T}_1$ is finitely ramified and thus every $\mathcal{T}_\alpha$ is finitely ramified by Theorem 5.4.

Let $\mathcal{T}' := \Lambda(\mathcal{T}'_1, \dots, \mathcal{T}'_n, \dots)$. We claim that if each $\operatorname{MaxPath}(\mathcal{T}'_i)$ is countable, then also $\operatorname{MaxPath}(\mathcal{T}')$ is countable. Indeed, if π is a maximal path in $\mathcal{T}'$, then either the tail of π is in some $\{r_i\}^\uparrow \simeq \mathcal{T}'_i$ or $\pi = (x_i)$ (where r_i and x_i are as in the definition of Λ); thus, $\operatorname{MaxPath}(\mathcal{T}')$ is equal to a countable union of a countable family (the $\operatorname{MaxPath}(\mathcal{T}'_i)$) plus another element (the path $(x_i)_{i\in\mathbb{N}}$). Hence $\operatorname{MaxPath}(\mathcal{T}')$ is countable. Since all the $\mathcal{T}_\alpha$ are constructed with the construction Λ and $\operatorname{MaxPath}(\mathcal{T}_1)$ is a singleton, it follows by induction that each $\operatorname{MaxPath}(\mathcal{T}_\alpha)$ is countable.

(b) Apply Proposition 3.5 to the family found in the previous part of the proof. □

These two constructions are very general; we give an example in the next corollary.

Corollary 5.8. *Let p be a prime number, and let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. For every countable successor ordinal number α there is a field $F \subseteq \overline{\mathbb{Q}}$ such that the integral closure of $\mathbb{Z}_{(p)}$ in F is an almost Dedekind domain of SP-rank α with countable maximal space.*

Proof. The valuation ring $\mathbb{Z}_{(p)}$ has finite residue field. Hence we can apply Theorem 5.7. □

Theorems 5.5 and 5.7 were given only for successor ordinals. This is unavoidable in the current setting, as we show next.

Proposition 5.9. *Let V be a discrete valuation ring and let A be an almost Dedekind domain that is an integral extension of V. Then, the SP-rank of A is a successor ordinal.*

Proof. The maximal ideal $\mathfrak{m}$ of V is contained in every maximal ideal of A, and thus the Jacobson radical of A is nonzero. Therefore, $\mathrm{Max}(A)^{\mathrm{inv}}$ is a compact space. If $\alpha = \text{SP-rank}(A)$ is a limit ordinal, then

$$\varnothing = \bigcap_{\beta<\alpha} \mathrm{Crit}^{\beta}(A),$$

and no finite subintersection is empty. However, since every $\mathrm{Crit}^{\beta}(A)$ is closed in the inverse topology, this contradicts the compactness of the maximal space. Thus α must be a successor ordinal. □

6. Limit ordinals

As shown in Proposition 5.9, the previous construction cannot give us almost Dedekind domains with a limit ordinal as its SP-rank. To do so, we must start with a Dedekind domain instead of a DVR. We consider this problem in a slightly more general way. See [4, Section 6.3; 10] for the definition and properties of Jaffard families.

Proposition 6.1. *Let A be an almost Dedekind domain and let Θ be a Jaffard family on A.*

(a) *For every $P \in \mathrm{Max}(A)$, we have $\mathrm{SPh}(P) = \mathrm{SPh}(PT)$, where T is the only element of Θ such that $PT \neq T$.*

(b) $\text{SP-rank}(A) = \sup\{\text{SP-rank}(T) \mid T \in \Theta\}$.

Proof. Clearly the second statement is a direct consequence of the first one.

We first show that P is critical if and only if PT is critical. If P is not critical, there is a nonzero finitely generated radical ideal $I \subseteq P$; then, IT is a nonzero finitely generated radical ideal of T contained in PT, and thus PT is not critical. Conversely, if PT is not critical, then there is a nonzero finitely generated radical ideal $J \subseteq PT$. Then, $J' := J \cap A$ is radical, and since T is a Jaffard overring of A, J' is finitely generated too [10, Lemma 5.9]; since $J' \subseteq PT \cap A = P$ we have that P is not critical.

Suppose now that $\mathrm{SPh}(Q) = \mathrm{SPh}(QT)$ whenever $Q \neq QT$ and either $\mathrm{SPh}(Q) < \alpha$ or $\mathrm{SPh}(QT) < \alpha$. Suppose that $\mathrm{SPh}(P) \geq \alpha$ and $PT \neq T$. Let $\{A_\alpha\}$ and $\{T_\alpha\}$ be the SP-derived sequences of A and T, respectively; the inductive hypothesis implies that $T_\alpha = TA_\alpha$. In particular, $\mathrm{SPh}(P) = \alpha$ if and only if P is not critical in A_α, if and only if P is not critical in $TA_\alpha = T_\alpha$, if and only if $\mathrm{SPh}(PT) = \alpha$. By induction, $\mathrm{SPh}(P) = \mathrm{SPh}(PT)$ for all P. □

Corollary 6.2. *Let D be a Dedekind domain with quotient field K, $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ be a chain of finite field extensions of K, and let A be the integral closure of D in $K_\infty := \bigcup_{n\in\mathbb{N}} K_n$. Suppose that A is almost Dedekind. Then,*

$$\text{SP-rank}(A) = \sup_{M\in\mathrm{Max}(D)} \text{SP-rank}(\mathcal{T}(D_M, \mathcal{K})).$$

Proof. Let $\Theta := \{A_M \mid M \in \mathrm{Max}(D)\}$, where $A_M := (D \setminus M)^{-1}A$; then, A_M is the integral closure of D_M in K_∞. Then, Θ is a Jaffard family on A, and thus by Proposition 6.1 we have $\text{SP-rank}(A) = \sup\{\text{SP-rank}(A_M) \mid M \in \mathrm{Max}(D)\}$. By Corollary 4.6, $\text{SP-rank}(A_M) = \text{SP-rank}(\mathcal{T}(D_M, \mathcal{K}))$, and the claim follows. □

Unlike in the DVR case, when D is not semilocal, we cannot fully control the extension of all valuations induced by localizations of D, as there are infinitely many of them. However, we can modify our construction to consider only finitely many valuations at a time.

Theorem 6.3. *Let D be a Dedekind domain with countably infinite many maximal ideals and let α be a nonzero countable ordinal. Then, there is an almost Dedekind domain A such that $D \subseteq A$ is integral and* $\text{SP-rank}(D) = \alpha$.

Proof. We proceed by induction on α. If $\alpha = 1$, it is enough to take $A = D$.

Let now $\mathcal{T}_1, \ldots, \mathcal{T}_n, \ldots$ be a countable sequence of finitely ramified totally balanced trees with the same weight d and such that every maximal path is finitely ramified. We use $\mathcal{T}_i^{(k)}$ to denote the subtree of $\mathcal{T}_i$ containing the vertexes of height at most k. Let $\mathrm{Max}(D) = \{M_1, \ldots, M_n, \ldots\}$ and let v_i be the valuation relative to M_i. Using the extension theorem, we can construct a chain $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ of extensions in the following way.

We set K_0 as the quotient field of D.

We take K_1 as a finite extension of K_0 such that $\mathcal{T}(v_1, \{K_i\}_{i=0}^1) \simeq \mathcal{T}_1^{(1)}$ (that is, v_1 has as many extensions to K_1 as the number of elements of $\mathcal{T}_1$ of height 1, with the same ramification index as the edges in $\mathcal{T}_1$) and each inertial degree is 1. Note that each v_i with $i > 1$ has finitely many extensions to K_1, but neither do we know how many there are nor which inertial degree they have.

We construct K_2 as a finite extension of K_1 such that:

- $\mathcal{T}(v_1, \{K_i\}_{i=0}^2) \simeq \mathcal{T}_1^{(2)}$, and each inertial degree is 1.
- For every extension w of v_2 to K_1, we have $\mathcal{T}(w, \{K_i\}_{i=1}^2) \simeq \mathcal{T}_2^{(1)}$, and each extension has inertial degree 1 over w.

We construct K_3 in the same way as a finite extension of K_2, also considering the extensions of v_3 to K_2:

- $\mathcal{T}(v_1, \{K_i\}_{i=0}^3) \simeq \mathcal{T}_1^{(3)}$, and each inertial degree is 1.
- For every extension w of v_2 to K_1, we have $\mathcal{T}(w, \{K_i\}_{i=1}^3) \simeq \mathcal{T}_2^{(2)}$, and each extension has inertial degree 1 over w.
- For every extension w' of v_3 to K_2, we have $\mathcal{T}(w', \{K_i\}_{i=2}^3) \simeq \mathcal{T}_3^{(1)}$, and each extension has inertial degree 1 over w'.

More generally, we take K_n to be an extension of K_{n-1} such that, for every $k = 1, \ldots, n$ and every extension w of v_k to K_{k-1}, we have $\mathcal{T}(w, \{K_i\}_{i=k-1}^n) \simeq$

$\mathcal{T}_k^{(n+1-k)}$, and each extension has inertial degree 1 over w. (For example, if $k=1$ we control the extensions of v_1; if $k=2$ we control each extension of v_2 to K_1 separately and if $k=n$, we control the properties of the extensions of v_n to K_{n-1}.) In this way, while we don't know what happens to v_k between K_0 and K_{k-1}, we do control what happens above K_{k-1}.

Let $K_\infty := \bigcup_{n\in\mathbb{N}} K_n$ and let A be the integral closure of D in K_∞.

Fix now a j. Then, v_j has finitely many extensions to K_{j-1}, say $v_{j,1},\dots,v_{j,t}$, and by construction $\mathcal{T}(v_{j,i},\{K_{j+k}\}_{k\in\mathbb{N}})\simeq\mathcal{T}_j$. In particular, every maximal path π of $\mathcal{T}(v_j,\mathcal{K})$ ends up in a tree isomorphic to $\mathcal{T}_j$; it follows that π is finitely ramified (so $\mathcal{T}(v_j,\mathcal{K})$ is finitely ramified) and SP-rank$(\mathcal{T}(v_j,\mathcal{K}))=$ SP-rank$(\mathcal{T}_j)$. In particular, A is almost Dedekind by Proposition 3.3 (applied to each localization D_M).

Since SP-rank$(\mathcal{T}(v_j,\mathcal{K}))=$ SP-rank$(\mathcal{T}(D_{M_j},\mathcal{K}))$, by Corollary 6.2 we have

$$\text{SP-rank}(A)=\sup_{j\in\mathbb{N}}\text{SP-rank}(\mathcal{T}(D_{M_j},\mathcal{K}))=\sup_{j\in\mathbb{N}}\text{SP-rank}(\mathcal{T}_j).$$

If α is a successor ordinal, we can find a tree $\mathcal{T}$ with SP-rank α by Theorem 5.5, and take $\mathcal{T}_n:=\mathcal{T}$ for every n; if α is a limit ordinal, take a sequence γ_n of successor ordinals with limit α (notice that α has countable cofinality), and use Theorem 5.5 to find trees with SP-rank$(\mathcal{T}_n)=\gamma_n$. In both cases, the claim follows by the previous equality. □

7. Topology

Let $\mathcal{T}$ be an ω-tree, and consider the set MaxPath$(\mathcal{T})$ of maximal paths of $\mathcal{T}$. For each $a\in\mathcal{T}$, let

$$V(a):=\{\pi\in\text{MaxPath}(\mathcal{T})\mid a\in\pi\}.$$

Lemma 7.1. *The family $\{V(a)\mid a\in\mathcal{T}\}\cup\{\varnothing\}$ is closed under taking finite intersections, and thus is a basis of open sets for a topology on* MaxPath$(\mathcal{T})$.

Proof. Let $a,b\in\mathcal{T}$. If $V(a)\cap V(b)$ is nonempty, there is a maximal path π containing both a and b; since a path is totally ordered, we have $a\le b$ or $b\le a$. In the former case $V(a)\supseteq V(b)$, while in the latter $V(a)\subseteq V(b)$; in both cases, $V(a)\cap V(b)$ belongs to the family. □

We call this topology the *order topology* on MaxPath$(\mathcal{T})$.

Lemma 7.2. *Suppose that each element of $\mathcal{T}$ has only finitely many direct successors. Then, each $V(a)$ is closed in the order topology.*

Proof. Suppose that a is of height i: then the hypothesis guarantees that there are only finitely many elements of height i, say $a,b_1,\dots,b_k$. Then, every maximal path must pass either through a or through some b_t; therefore, $V(a)$ is the complement of $V(b_1)\cup\cdots\cup V(b_k)$, which is open. Hence $V(a)$ is also closed. □

Proposition 7.3. *Let W be a discrete valuation ring on K and let $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ be a chain of finite extensions of K. Let A be the integral closure of W in $K_\infty := \bigcup_{n\in\mathbb{N}} K_n$. Then, the natural correspondence $\Phi : \mathrm{Max}(A) \to \mathrm{MaxPath}(\mathcal{T}(W, \mathcal{K}))$ of Proposition 3.2 is a homeomorphism when $\mathrm{Max}(A)$ is endowed with the inverse topology and $\mathrm{MaxPath}(\mathcal{T}(W, \mathcal{K}))$ with the order topology.*

Proof. Let $a \in \mathcal{T}$. Then, a corresponds to a prime ideal P of A_n, the integral closure of W in K_n (for some n), and a path π contains a if and only if the corresponding maximal ideal $M := \Phi^{-1}(\pi)$ contains P. Hence, $\Phi^{-1}(V(a)) = V(PA)$. Moreover, P is finitely generated (since it is an ideal of a Dedekind domain) and thus $V(PA)$ is open in $\mathrm{Max}(A)^{\mathrm{inv}}$. Hence Φ is continuous.

To prove that Φ is open, let $I = (x_1, \dots, x_k)$ be a finitely generated ideal of A, let n be an integer such that $x_1, \dots, x_k \in K_n$, and let A_n be the integral closure of W in K_n; then, $J := (x_1, \dots, x_k)A_n$ is an ideal of A_n, and thus $J = P_1^{e_1} \cdots P_t^{e_t}$ for some prime ideals $P_1, \dots, P_t$ of A_n; hence,

$$\Phi(V(I)) = \Phi(V(JA)) = \Phi\Big(\bigcup_{i=1}^{t} V(P_i A)\Big) = \bigcup_{i=1}^{t} \Phi(V(P_i A)).$$

Let b_i be the element of $\mathcal{T}$ corresponding to P_i; by the previous part of the proof, $\Phi(V(P_i A)) = V(b_i)$; hence, $\Phi(V(I))$ is a union of open sets, and thus it is itself open. Thus, Φ is a homeomorphism. □

Let X be a topological space. A point $x \in X$ is *isolated* if $\{x\}$ is an open set; if x is not isolated, x is called a *limit point*. The set of limit points of X is called the *derived set* of X, and is denoted by $\mathcal{D}(X)$. More generally, if α is an ordinal number, we set

- $\mathcal{D}^0(X) := X$;
- if $\alpha = \beta + 1$ is a successor ordinal, $\mathcal{D}^\alpha(X) := \mathcal{D}(\mathcal{D}^\beta(X))$;
- if α is a limit ordinal, $\mathcal{D}^\alpha(X) := \bigcap_{\beta<\alpha} \mathcal{D}^\beta(X)$.

The smallest ordinal α such that $\mathcal{D}^\alpha(X) = \mathcal{D}^{\alpha+1}(X)$ is called the *Cantor–Bendixson rank* of X. If $X = \mathcal{D}(X)$ (i.e., if X has no isolated points) then X is said to be *perfect*.

Lemma 7.4. *Let $\mathcal{T}$ be a tree and $\pi \in \mathrm{MaxPath}(\mathcal{T})$. Then, π is isolated if and only if there is a $v \in \mathcal{T}$ such that π is the only maximal path containing v.*

Proof. If there is such a v, then $V(v) = \{\pi\}$ and π is isolated. If π is isolated, then $\{\pi\}$ is an open set. Since the family $\{V(a) \mid a \in \mathcal{T}\}$ is a basis that is closed by finite intersections, it must be $\{\pi\} = V(v)$ for some $v \in \mathcal{T}$; hence $v \in \pi$ and π is the only maximal path through v. □

Proposition 7.5. *Let α be a countable successor ordinal, and let W be a discrete valuation ring with finite residue field. Then, there is an almost Dedekind domain A that is integral over W such that $\text{SP-rank}(A) = \alpha$ and $\mathrm{Crit}^\beta(A) = \mathcal{D}^\beta(\mathrm{Max}(A)^{\mathrm{inv}})$ for every ordinal number $\beta \leq \alpha$.*

Proof. Let $\mathcal{T}$ be a totally ordered ω-tree such that all the edges have weight 1. Using the construction before Theorem 5.4, we can construct a finitely ramified locally bounded tree $\mathcal{T}_\alpha$ for every countable successor ordinal α that, by the proof of Theorem 5.7, has SP-rank equal to α and such that $\mathrm{MaxPath}(\mathcal{T}_\alpha)$ is countable. We claim that $\mathrm{Crit}^\beta(\mathcal{T}_\alpha) = \mathcal{D}^\beta(\mathrm{MaxPath}(\mathcal{T}_\alpha))$ for every $\beta \leq \alpha$.

We proceed by induction on α.

If $\alpha = 1$ then $\mathrm{MaxPath}(\mathcal{T}_1) = \{\mathcal{T}_1\}$ and the claim is obvious.

Suppose that $\alpha = \gamma + 1$, so that $\mathcal{T} := \mathcal{T}_\alpha := \Lambda(\mathcal{T}_{\gamma_1}, \dots, \mathcal{T}_{\gamma_n}, \dots)$ for a sequence $\{\gamma_n\}$ with limit γ (where, if γ is a successor ordinal, we have $\gamma_i = \gamma$ for all i). Set $X := \mathrm{MaxPath}(\mathcal{T})$. For each i, consider the set $X_i := \{\pi \in X \mid \pi \cap \mathcal{T}_{\gamma_i} \neq \varnothing\}$ and let r_i be the root of $\mathcal{T}_{\gamma_i}$. Then, $X_i = V(r_i)$ is closed by Lemma 7.2. Moreover, X_i is naturally homeomorphic to $\mathrm{MaxPath}(\mathcal{T}_{\gamma_i})$. Hence, the Cantor–Bendixson rank of $\pi \in X_i$ (as an element of X) is the same as the Cantor–Bendixson rank of its restriction to $\mathcal{T}_{\gamma_i}$; since the SP-rank too only depends on the tail of the path, it follows that for any $\pi \in X_i$ we have $\pi \in \mathcal{D}^\beta(X)$ if and only if $\pi \in \mathrm{Crit}^\beta(\mathcal{T})$.

Suppose now that $\pi \notin X_i$ for all i; therefore, $\pi = (x_i)_{i=1}^\infty$ and its SP-height is α. We claim that $\pi \in \mathcal{D}^\alpha(X)$. Let $\alpha_0 = \beta_0 + 1$ be the Cantor–Bendixson height of π. Then, π is an isolated point of $\mathcal{D}^{\beta_0}(X)$, and thus there is a basic open set $V(x)$ such that $V(x) \cap \mathcal{D}^{\beta_0}(X) = \{\pi\}$; thus, $x = x_i$ for some i. By construction, $V(x_i)$ contains all paths containing x_i, and thus $X_i \subseteq V(x_i)$. Suppose $\beta_0 < \beta$, i.e., $\alpha_0 < \alpha$. If γ (the ordinal just before α) is a successor, then $\gamma_i = \gamma$ and thus X_i contains elements of Cantor–Bendixson height γ, a contradiction. If γ is a limit ordinal, then there is a j such that $\gamma_j > \beta_0$; since $V(x_j) \subseteq V(x_i)$, we have $V(x_j) \cap \mathcal{D}^{\beta_0}(X) = \{\pi\}$, while $V(x_j) \cap \mathcal{D}^{\beta_0}(X)$ should contain some element of X_j, again a contradiction. Therefore, we must have $\alpha_0 = \alpha$, and $\pi \in \mathcal{D}^\alpha(X)$. By induction, the claim is proved.

Fix now an α and let $\mathcal{T} := \mathcal{T}_\alpha$. By Proposition 3.5 we can find a chain $\mathcal{K} : K = K_0 \subseteq K_1 \subseteq \cdots$ of finite separable extensions of K (where K is the quotient field of W) such that $\mathcal{T} \simeq \mathcal{T}(W, \mathcal{K})$. Let A be the integral closure of W in $K_\infty := \bigcup_{n \in \mathbb{N}} K_n$. Let $X := \mathrm{MaxPath}(\mathcal{T}(W, \mathcal{K}))$. The bijective correspondence $\Phi : X \to \mathrm{Max}(A)$ given by Proposition 3.2 preserves the sets Crit^β (by Theorem 4.5) and is a homeomorphism when $\mathrm{Max}(A)$ is endowed with the inverse topology and X with the order topology (Proposition 7.3). Hence,

$$\Phi(\mathrm{Crit}^\beta(X)) = \mathrm{Crit}^\beta(A) \quad \text{and} \quad \Phi(\mathcal{D}^\beta(X)) = \mathcal{D}^\beta(\mathrm{Max}(A)^{\mathrm{inv}}).$$

Since $\mathrm{Crit}^\beta(X) = \mathcal{D}^\beta(X)$ by the previous part of the proof, we also have

$$\mathrm{Crit}^\beta(A) = \mathcal{D}^\beta(\mathrm{Max}(A)^{\mathrm{inv}}). \qquad \square$$

The situation for the construction carried on in Theorem 5.5 is, on the other hand, completely different.

Lemma 7.6. *Let $\mathcal{T}$ be an ω-tree such that every element has only finitely many direct successors. Then*, $\mathrm{MaxPath}(\mathcal{T})$ *is compact.*

Proof. We can consider $\mathcal{T}$ as a weighted tree by prescribing to each edge a weight of 1; since there are only finitely many elements of height n, for every n, $\mathcal{T}$ is locally bounded. Let v be a discrete valuation on a field K with finite residue field; by Proposition 3.5, we can construct a chain $\mathcal{K}$ of extensions of K such that $\mathcal{T}(v, \mathcal{K}) \simeq \mathcal{T}$. By Proposition 7.3, $\mathrm{MaxPath}(\mathcal{T}) \simeq \mathrm{Max}(A)^{\mathrm{inv}}$, where A is the integral closure of V in $\bigcup_{n\in\mathbb{N}} K_n$. However, since the Jacobson radical of A is nonzero, the inverse topology on $\mathrm{Max}(A)$ coincides with the Zariski topology, and in particular $\mathrm{Max}(A)^{\mathrm{inv}}$ is compact. Thus also $\mathrm{MaxPath}(\mathcal{T})$ is compact. □

We denote by 2^ω the space of all countable $\{0, 1\}$-sequences, i.e., the product of countably many copies of $\{0, 1\}$; the space 2^ω is also homeomorphic to the Cantor set inside $\mathbb{R}$.

Proposition 7.7. *Let $\mathcal{T}$ be an ω-tree such that every element has at least two but only finitely many direct successors. Then*, $\mathrm{MaxPath}(\mathcal{T}) \simeq 2^\omega$.

Proof. We show that $\mathrm{MaxPath}(\mathcal{T})$ is nonempty, perfect, compact, totally disconnected, and metrizable; these properties characterize 2^ω [13, Chapter 30].

Clearly $\mathrm{MaxPath}(\mathcal{T})$ is nonempty, and it is compact by Lemma 7.6. To show that it is perfect, we must show that it has no isolated points; however, if π is an isolated point, there should be an a such that $V(a) = \{\pi\}$, that is, such that there is a unique maximal path containing a; this contradicts the fact that a has at least two direct successors.

By Lemma 7.2, $\mathrm{MaxPath}(\mathcal{T})$ has a basis of clopen subsets; since it is also T_1 (for each π we have $\{\pi\} = \bigcap \{V(a) \mid a \in \pi\}$), by [13, Theorem 29.5] $\mathrm{MaxPath}(\mathcal{T})$ is totally disconnected.

By [13, Lemma 29.6], $\mathrm{MaxPath}(\mathcal{T})$ is also Hausdorff, and being compact it is also normal (hence regular). Furthermore, it is second countable since $\mathcal{T}$ is countable and thus the set of all the $V(a)$ is countable. By Urysohn's Metrization Theorem (see, e.g., [13, Theorem 23.1]) $\mathrm{MaxPath}(\mathcal{T})$ is also metrizable.

Therefore, $\mathrm{MaxPath}(\mathcal{T})$ is nonempty, perfect, compact, totally disconnected, and metrizable, and thus homeomorphic to 2^ω. □

Corollary 7.8. *Let $\mathcal{T}$ be one of the trees $\mathcal{T}_\alpha$ constructed in the proof of Theorem 5.5. Then*, $\mathrm{MaxPath}(\mathcal{T}) \simeq 2^\omega$.

Proof. By construction, every element of $\mathcal{T}$ has at least two direct successors. Hence, we can apply Proposition 7.7 to all the trees $\mathcal{T}_\alpha$, and thus $\mathrm{MaxPath}(\mathcal{T}) \simeq 2^\omega$. □

In particular, if A is an almost Dedekind domain constructed in Theorem 5.5, then $\mathrm{Max}(A)^{\mathrm{inv}} \simeq 2^\omega$.

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