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A CONJECTURE ON MULTIPLICITIES FOR STRONGLY TEMPERED SPHERICAL VARIETIES

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We form a conjecture about the multiplicities for general strongly tempered spherical varieties without type N root. This generalizes the epsilon dichotomy conjectures of Gan et al. (2012) and Wan and Zhang (2022).

1. Introduction

Let F be a local field of characteristic 0, G be a connected reductive group defined over F and H be a closed connected subgroup of G. Assume that H is a spherical subgroup of G (i.e., H admits an open orbit in the flag variety of G). We say the spherical pair (G, H) is reductive if H is reductive.

We say that the spherical pair (G, H) is the Whittaker induction of a reductive spherical pair (G_0, H_0) if there exists a parabolic subgroup $P = MN$ of G and a generic character ξ of $N(F)$ such that $M \simeq G_0$, $H = H_0 N$, and H_0 is contained in the neutral component of the stabilizer of the character ξ in M under the adjoint action. Note that each character of $N(F)$ corresponds to an element in $\bar{\mathfrak{n}}(F)/[\bar{\mathfrak{n}}(F), \bar{\mathfrak{n}}(F)]$ where $\bar{P} = M\bar{N}$ is the parabolic subgroup opposite to P. We say a character of $N(F)$ is generic if the corresponding element in $\bar{\mathfrak{n}}(F)/[\bar{\mathfrak{n}}(F), \bar{\mathfrak{n}}(F)]$ belongs to an open orbit under the $M(F)$-adjoint action. If this is the case, we say (G, H) is the Whittaker induction of (G_0, H_0, ξ) (if H is already reductive, we can just let $(G_0, H_0, \xi) = (G, H, 1)$). In this paper, we will restrict ourselves to the case when (G, H) is the Whittaker induction of a reductive spherical pair (G_0, H_0, ξ). We can extend the character ξ to $H(F)$ by making it trivial on $H_0(F)$. For an irreducible smooth representation π of $G(F)$ whose central character is trivial on $Z_{G,H}(F) = Z_G(F) \cap H(F)$, we define the multiplicity

$$m(\pi, \xi) = \dim(\mathrm{Hom}_{H(F)}(\pi, \xi)).$$

To simplify the notation we will use $m(\pi)$ instead of $m(\pi, \xi)$ to denote the multiplicity if the choice of ξ is clear. We say that the representation is (H, ξ)-distinguished (or just H-distinguished if the choice of ξ is clear) if the multiplicity is nonzero.

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One of the fundamental problems in the *relative Langlands program* is to study the multiplicity $m(\pi, \xi)$. In general, one expects the multiplicity $m(\pi, \xi)$ to be finite and to detect some functorial structures of π. We refer the reader to [9] for a detailed discussion of these types of problems.

Among all spherical pairs, there is a special category called strongly tempered spherical pairs. More precisely, when H is reductive, we say the pair (G, H) is strongly tempered if all the matrix coefficients of tempered representations of $G(F)$ are integrable on $H(F)/Z_{G,H}(F)$ (here Z_G is the center of G and $Z_{G,H} = Z_G \cap H$). When H is not reductive and if the model (G, H) is the Whittaker induction of a reductive spherical pair (G_0, H_0, ξ), then we say that the pair (G, H) is strongly tempered if and only if (G_0, H_0) is strongly tempered.

According to the general conjecture of Sakellaridis and Venkatesh in Conjecture 16.5.1 of [9], for a strongly tempered spherical pair (G, H), if we assume that the spherical varieties $X = G/H$ do not have a type N spherical root (we refer the reader to Section 3.1 of [9] for the definition of spherical roots), then almost all the tempered local Vogan L-packets of $G(F)$ (with suitable central character) should contain at least one (H, ξ)-distinguished representation (i.e., almost all tempered local Vogan L-packets are (H, ξ)-distinguished). The key point is that in the strongly tempered case, conjecturally the L-group of the spherical variety $X = G/H$ is the L-group of $G/Z_{G,H}$, and hence one expects that almost all tempered local Vogan L-packet (with suitable central character) should be distinguished. Moreover, if the spherical variety only has one open Borel orbit over the local field F, then the general conjecture of Sakellaridis and Venkatesh predicts that almost all tempered local Vogan L-packets of $G(F)$ should contain exactly one (H, ξ)-distinguished representation (this is usually called a strong multiplicity one on L-packets). In general, we expect the multiplicity of each tempered local Vogan L-packet of $G(F)$ to be equal to the number of open Borel orbits of $X(F)$.

The most famous examples of strongly tempered spherical pairs without type N root are the orthogonal Gan–Gross–Prasad models $(\mathrm{SO}_{n+2k+1} \times \mathrm{SO}_n, \mathrm{SO}_n \ltimes N)$. Here N is some unipotent subgroup. For these cases, the local conjecture was formulated by Gan, Gross, and Prasad in Section 17 of [4]. They not only conjectured the property of strong multiplicity one on generic L-packets (i.e., each generic Vogan L-packet contains a unique distinguished element and its multiplicity is equal to one), but they also conjectured about the unique distinguished element in each L-packet. More precisely, for each local L-packet Π_ϕ ($\phi : W_F' \to {}^LG$ is a Langlands parameter where W_F' is the Weil-Deligne group), let Z_ϕ be the centralizer of the parameter and $S_\phi = Z_\phi / Z_\phi^\circ$ be its component group. The local Langlands conjecture states that for each choice of Whittaker datum, there is a natural bijection between the L-packet Π_ϕ and the set of irreducible representations of S_ϕ (denoted by $\hat{S}_\phi$). In Section 17 of [4], they defined a quadratic character of S_ϕ using some local epsilon factor and conjectured that the unique distinguished element in a generic L-packet is the one associated with this quadratic character. This is usually called the epsilon dichotomy conjecture or epsilon criterion. In [16] the authors formulated an analogue of this conjecture for 10 strongly tempered spherical varieties and proved the conjecture in many cases including all the archimedean cases.

In this paper, we make an analogue of this conjecture for the Whittaker induction of general strongly tempered spherical varieties without type N root, and in Section 3 we show that our conjecture recovers the conjectures in [4] and [16]. The most important advantage of our conjecture is that, unlike the conjectures in [4] and [16], our conjecture does not rely on specific knowledge of the component group S_ϕ and the centralizer Z_ϕ (in [4] the authors wrote down the component group S_ϕ explicitly and then defined the character on it, whereas in [16] the authors specifically wrote down elliptic elements in Z_ϕ and then defined the function on it explicitly). The reason we can do this is that based on all the existing examples of strongly tempered spherical varieties, we find that the L-function associated with strongly tempered spherical varieties should satisfy a property called anomaly free. While preparing this paper, we were very happy to learn that Ben-Zvi, Sakellaridis, and Venkatesh [1] have also found the same property which served as a key ingredient in their proposed relative Langlands duality (the name "anomaly free" comes from [1]). We refer the reader to Section 2 for more details.

Another advantage of our conjecture is that it applies to a general strongly tempered case, we do not even need to assume that the spherical variety $X = G/H$ has a unique rational open Borel orbit (in particular, it may not have strongly multiplicity one over the Vogan L-packet). In Section 4, we discuss some examples with more than one open Borel orbit and show that our conjecture holds for these models.

Remark 1.1. Prasad [7] gave a beautiful conjecture for the multiplicity of Galois model $(G, H) = (\mathrm{Res}_{E/F} H, H)$ where E/F is a quadratic extension. The Galois model case and the strongly tempered case are the two extreme cases in terms of the behavior of multiplicity. The Galois model case is purely related to functoriality, while the strongly tempered case is purely related to epsilon factors. We believe for general spherical variety without type N root, the behavior of the multiplicity should lie in between these two extreme cases. In other words, it should be a combination of functoriality and epsilon criterion. An example would be the Guo–Jacquet model $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(E))$ for which the multiplicity is related to both the functoriality and certain epsilon factor. We are trying to combine these two conjectures to make a conjecture of the multiplicity for general spherical variety without type N root.

The paper is organized as follows. In Section 2 we discuss the endoscopic datum, the local Langlands conjecture, and the anomaly free representation of L-groups. Then we state our conjecture. In Section 3 we show that our conjecture recovers the conjectures in [4] and [16]. In Section 4, we prove our conjecture for some cases with more than one open Borel orbit.

2. The conjecture

2.1. ***Extended endoscopic triple.*** Let G be a connected reductive group defined over F. Following Definition 2 of [5], we say $(G', s, {}^L\eta)$ is an extended endoscopic triple of G if G' is a quasisplit connected reductive group define over F, s is a semisimple element of $\hat{G}$, and ${}^L\eta$ is an L-embedding from ${}^LG'$ into ${}^LG$ such that the image of ${}^L\eta$ commutes with s and it induces an isomorphism between $\hat{G}'$ and $\hat{G}_s$ (here $\hat{G}_s$ is the neutral component of the centralizer of s in $\hat{G}$). Here $\hat{G}$ is the Langlands dual group of G and ${}^LG$ is the L-group of G (Section 2 of [2]).

We restrict ourselves to the case when each endoscopic datum $\mathcal{E} = (G', \mathcal{G}', s, {}^L\eta)$ of G (we refer the reader to Definition 1 of [5] for the definition of endoscopic datum) is also an extended endoscopic triple (this is equivalent to saying that $\mathcal{G}'$ in the endoscopic datum is an L-group) so that we only need to consider extended endoscopic triple instead of the more complicated endoscopic datum.

Remark 2.1. This assumption is true in many cases. For example, when G is a classical group, or when the derived group G_{der} of G is simply connected.

2.2. ***The local Langlands conjecture.*** We recall the local Langlands conjecture (see Conjecture E of [5]). Let G be a quasisplit reductive group defined over F and let $\{G_\alpha \mid \alpha \in H^1(F, G)\}$ be the set of pure inner forms of G. Let $\Pi_{\mathrm{irr,temp}}(G_\alpha)$ be the set of irreducible tempered representations of $G_\alpha(F)$. The local Langlands conjecture [11] states that

$$\bigcup_{\alpha\in H^1(F,G)} \Pi_{\mathrm{irr,temp}}(G_\alpha)$$

has a canonical partition into finite subsets (i.e., the local tempered Vogan L-packets) parametrized by Langlands parameters $\cup_\phi \Pi_\phi$, where ϕ runs over all the tempered L-parameters of G and

$$\Pi_\phi = \bigcup_{\alpha\in H^1(F,G)} \Pi_\phi(G_\alpha)$$

consists of a finite number of tempered representations with $\Pi_\phi(G_\alpha) \subset \Pi_{\mathrm{irr,temp}}(G_\alpha)$ such that the following conditions hold.

- There is a unique generic element in $\Pi_\phi(G)$ with respect to any Whittaker datum of G.
- For the given Whittaker datum, there is a bijection between $\hat{S}_\phi$, the set of irreducible representations of the component group $S_\phi = Z_\phi/Z_\phi^\circ$ of the Langlands parameter ϕ (Z_ϕ is the centralizer of $\mathrm{Im}(\phi)$ in $\hat{G}$), and Π_ϕ (denoted by $\pi \leftrightarrow \chi_\pi$) satisfying the following conditions.

⋄ The trivial character of S_ϕ corresponds to the unique generic element of $\Pi_\phi(G)$ with respect to the given Whittaker datum.

⋄ For $\alpha \in H^1(F, G)$, the distribution character

$$\theta_{\Pi_\phi(G_\alpha)} = \sum_{\pi \in \Pi_\phi(G_\alpha)} \dim(\chi_\pi)\theta_\pi$$

is stable. Moreover, $\iota(G_\alpha)\theta_{\Pi_\phi(G_\alpha)}$ is the endoscopic transfer of $\theta_{\Pi_\phi(G)}$ where $\iota(G_\alpha)$ is the Kottwitz sign.

⋄ For any $\alpha \in H^1(F, G)$ and $\pi \in \Pi_\phi(G_\alpha)$, the restriction of the central character of χ_π to $Z(\hat{G})^{\Gamma_F}$ is equal to χ_α. Here χ_α is the character of $Z(\hat{G})^{\Gamma_F}$ associated to α via the Kottwitz isomorphism $H^1(F, G) \simeq \pi_0(Z(\hat{G})^{\Gamma_F})^\vee$. Note that the representation χ_π of the component group can be viewed as a representation of the centralizer Z_ϕ of the image of ϕ, the group $Z(\hat{G})^{\Gamma_F}$ belongs to the center of Z_ϕ and hence it makes sense to talk about the restriction of the central character of χ_π to $Z(\hat{G})^{\Gamma_F}$.

⋄ For $s \in S_\phi$ and for an extended endoscopic triple $(G', s', {}^L\eta)$ of G such that $s' \in sZ_\phi^\circ$ and ϕ factors through ${}^L\eta$, let $\Pi_{\phi,s}(G')$ be the corresponding L-packet of G' and let $\theta_{\Pi_{\phi,s}(G')}$ be the distribution character of that packet (which is a stable character on $G'(F)$). Then for $\alpha \in H^1(F, G)$, the character

$$\theta_{\Pi_\phi,\alpha,s} = \sum_{\pi \in \Pi_\phi(G_\alpha)} \mathrm{tr}(\chi_\pi(s))\theta_\pi$$

is the endoscopic transfer of $\iota(G_\alpha)\theta_{\Pi_{\phi,s}(G')}$.

2.3. ***Anomaly free representation of L-groups.*** Given a symplectic representation $\rho_X : {}^LG \to \mathrm{GL}(V)$ of ${}^LG$, for an extended endoscopic triple $(G', s, {}^L\eta)$, let $V_{s,-}$ be the -1-eigenspace of $\rho_X(s)$. Then the extended endoscopic triple induces a symplectic representation of ${}^LG'$ on $V_{s,-}$ which will be denoted by $\rho_{X,s,{}^L\eta,-}$.

Definition 2.2 (see also Definition 5.1.2 and Proposition 5.1.5 of [1]). Assume that G is quasisplit. Let $T \subset G$ be the maximal quasisplit torus. We say that a symplectic representation $\rho_X : {}^LG \to \mathrm{GL}(V)$ of ${}^LG$ is anomaly free if it satisfies the following two conditions.

- The restriction of the representation (ρ_X, V) to ${}^LT$ can be decomposed into a direct sum of two representations of ${}^LT$ that are dual to each other, i.e.,

$$(\rho_X|_{{}^LT}, V) \simeq (\rho, W) \oplus (\rho^\vee, W).$$

- There exists a character χ of ${}^LT$ and a character η of ${}^LG$ such that

$$\det(\rho) = \chi^2 \cdot \eta|_{{}^LT}.$$

Remark 2.3. (1) If G is split (if and only if T is split), the first condition in the definition is always true. In this case the second condition is just the one in Proposition 5.1.5 of [1].

(2) The second condition in the definition does not depend on the decomposition $\rho_X = \rho \oplus \rho^\vee$ in the first condition.

(3) If $\rho_X = \rho_0 \oplus \rho_0^\vee$ where ρ_0 is a representation of ${}^L G$, then it is anomaly free.

(4) If $\rho_X = \rho_1 \oplus \rho_2$ with ρ_i being a representation of ${}^L G$ that is anomaly free, then ρ_X is anomaly free.

Definition 2.4. We say the symplectic representation ρ_X of ${}^L G$ is anomaly free under endoscopy if for every extended endoscopic triple $(G', s, {}^L\eta)$ of G, the symplectic representation $\rho_{X,s,{}^L\eta,-}$ of ${}^L G'$ is anomaly free.

Remark 2.5. If G is split adjoint, all its endoscopic groups are split. Then ρ_X is anomaly free under endoscopy if for any $s \in \hat{G}_{ss}$, the representation of $\hat{G}_s$ on $V_{s,-}$ is anomaly free. Here $\hat{G}_{ss}$ is the set of semisimple elements of $\hat{G}$.

2.4. ***Multiplicity for strongly tempered spherical varieties.*** Suppose that (G, H) is a strongly tempered spherical pair which is the Whittaker induction of (G_0, H_0, ξ) with no type N root. Assume that G has a quasisplit pure inner form and let G_{qs} be the quasisplit pure inner form of G. Also assume that $(\mathrm{Res}_{E/F}\, G_0, \mathrm{Res}_{E/F}\, H_0)$ is strongly tempered for any finite field extension E of F.[1]

The L-group of the spherical variety $X = H\backslash G$ is expected to take the form ${}^L G_X = {}^L G/Z_{G,H}$.[2] We would like to emphasize that, at present there is no general definition of the L-group for arbitrary spherical varieties. What is available is a definition of the dual group of spherical varieties, as developed in [3; 6; 9]. However, in the strongly tempered case, following the speculation in Section 17 of [9], it is expected that the L-group takes the form ${}^L G_X = {}^L G/Z_{G,H}$.

Conjecturally there is a representation $\rho_X : {}^L G_X \to \mathrm{GL}(V)$ of ${}^L G_X$ associated to (G, H, ξ) so that the square of the global period integral associated to X should be related to the central value of the automorphic L-function associated to ρ_X (in some cases the existence of ρ_X is known by the work of Sakellaridis and Wang [8; 10]). To continue our discussion, we make the following assumption (see a similar assumption in Section 5 of [1]).

Assumption 2.6. *The representation ρ_X is symplectic and anomaly free under endoscopy.*

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[1]This is to avoid those models that are only strongly tempered because G_0 is not split. For example, if G_0 is compact (say isomorphic to $\mathrm{SL}_1(D)$ for some division algebra D over F), then even the model (G_0, G_0) is strongly tempered but it is not strongly tempered after a suitable finite field extension.

[2]This is not true if we do not assume $(\mathrm{Res}_{E/F}\, G_0, \mathrm{Res}_{E/F}\, H_0)$ is strongly tempered for any finite field extension E of F.

Remark 2.7. We expect that for an extended endoscopic triple $(G', s, {}^L\eta)$ of G such that the split rank of G' is equal to the split rank of G (here split rank is the dimension of the maximal split tori), then the symplectic representation $\rho_{X,s,{}^L\eta,-}$ of ${}^LG'$ is anomaly free. In particular, we expect Assumption 2.6 to be true if the split rank of G' is equal to the split rank of G for all the extended endoscopic triple $(G', s, {}^L\eta)$ of G (e.g., when G is split adjoint).

However, we do have an example where this assumption fails. Among all the strongly tempered examples we know, the only case this assumption fails is the unitary Gan–Gross–Prasad model $(U_{n+2m+1} \times U_n, U_n \ltimes U)$. The epsilon dichotomy conjecture for this case is also slightly different from the rest strongly tempered cases. See Section 2.5 for details and for a generalization of our conjecture to the case when Assumption 2.6 fails.

Let $\phi' : W_F' \to {}^LG_X$ be a tempered Langlands parameter (recall that we have ${}^LG_X = {}^LG/Z_{G,H}$). We are going to define a function ω_{ϕ',ρ_X} on $Z_{\phi'}$. For $s \in Z_{\phi'}$, there exists an extended endoscopic triple $(G', s, {}^L\eta)$ of G (not necessarily unique) such that ϕ' factors through ${}^L\eta$ (i.e., there exists $\phi_0 : W_F' \to {}^LG'$ such that $\phi' = {}^L\eta \circ \phi_0$). Let T' be a maximal quasisplit torus of G' (that is equivalent to saying T' is a minimal Levi subgroup of G' since G' is quasisplit). Since ρ_X is anomaly free under endoscopy, the symplectic representation $\rho_{X,s,{}^L\eta,-}$ of ${}^LG'$ is anomaly free. Hence we can decompose the representation $\rho_{X,s,{}^L\eta,-}|_{{}^LT'}$ as $\rho \oplus \rho^\vee$ and there exists a character χ (resp. η) of ${}^LT'$ (resp. ${}^LG'$) such that

$$\det(\rho) = \chi^2 \cdot \eta|_{{}^LT}.$$

We define

$$\omega_{\phi',\rho_X}(s) = \eta \circ \phi_0(-1)\epsilon\left(\tfrac{1}{2}, \rho_{X,s,{}^L\eta,-} \circ \phi_0\right) \in \{\pm 1\}.$$

It is clear that this definition is independent of the choice of the decomposition

$$\rho_{X,s,{}^L\eta,-}|_{{}^LT'} = \rho \oplus \rho^\vee, \quad \det(\rho) = \chi^2\eta|_{{}^LT}.$$

However, it still depends on the choice of the extended endoscopic triple $(G', s, {}^L\eta)$ and the lifting ϕ_0. To continue our discussion, we assume the following conjecture.

Conjecture 2.8. *The function ω_{ϕ',ρ_X} is well defined* (*i.e.*, *it is independent of the choice of the extended endoscopic triple and the lifting*), *it induces a function of $S_{\phi'}$* (*i.e.*, *it is constant on each connected component of $Z_{\phi'}$*), *and it is a character of $S_{\phi'}$.*

At this moment we do not have a systematic way to study this conjecture. In the next section we will verify this conjecture for the orthogonal Gan–Gross–Prasad models and the models in [16].

Let $\phi : W_F' \to {}^LG$ be a tempered Langlands parameter of $G(F)$. We would like to define a set of irreducible representations of the component group S_ϕ. Let I' be the set of lifting of ϕ to ${}^LG_X$. For each lifting $\phi' : W_F' \to {}^LG_X$ of ϕ, the above discussion gives us a quadratic character ω_{ϕ',ρ_X} of $S_{\phi'}$ and we also have a map i from $S_{\phi'}$ to S_ϕ.[3] We use $i(S_{\phi'})$ to denote the image of the map i and let I be the subset of I' containing those ϕ' such that the character ω_{ϕ',ρ_X} is trivial on $\ker(i)$.

Definition 2.9. Let $I(\phi, \rho_X)$ be the multiset consisting of all irreducible components of $\mathrm{Ind}_{i(S_{\phi'})}^{S_\phi}(\omega_{\phi',\rho_X})$ as ϕ' runs through the set I.

Now we can formulate our conjecture for the multiplicity. Assume that there exists $\alpha \in \mathrm{Im}(H^1(F,H) \to H^1(F,G))$ such that the pure inner form G_α is quasisplit. We will denote such G_α by G_{qs}. The Whittaker datum of $G_{qs}(F)$ is a $\ker(H^1(F,Z_G) \to H^1(F,G))$-torsor. The below conjecture is about the multiplicity $m(\pi)$ of $G/H(F)$. To be specific, if H is reductive, then $m(\pi)$ is just given by

$$m(\pi) = \sum_{\alpha\in\ker(H^1(F,H)\to H^1(F,G))} \dim(\mathrm{Hom}_{H_\alpha(F)}(\pi, 1)).$$

In general, when (G, H) is the Whittaker induction of (G_0, H_0, ξ), let M_ξ be the stabilizer of the character ξ under the adjoint action of M. There is a natural bijection $\alpha \mapsto \xi_\alpha$ between $\ker(H^1(F, M_\xi) \to H^1(F,G))$ and the set of $M(F)$-conjugacy classes of the generic characters of $N(F)$. One example is when G is quasisplit and $P = MN$ is a Borel subgroup of G, in this case $M_\xi = Z_G$ and the set $\ker(H^1(F,Z_G) \to H^1(F,G))$ can be naturally identified with the set of Whittaker datum of G (up to conjugation). Since we have $H_0 \subset M_\xi$, each $\alpha \in \ker(H^1(F,H_0) \to H^1(F,G))$ induces a generic character ξ_α of $N(F)$ whose stabilizer in M contains $H_{0,\alpha}$. This gives us the model $(G, H_\alpha) = (G, H_{0,\alpha} \ltimes N)$ which is the Whittaker induction of $(G_0, H_{0,\alpha}, \xi_\alpha)$. Then we can define $m(\pi)$ to be

$$m(\pi) = \sum_{\alpha\in\ker(H^1(F,H_0)\to H^1(F,G))} \dim(\mathrm{Hom}_{H_\alpha(F)}(\pi, \xi_\alpha)).$$

Note that if the map $H^1(F,H_0) \to H^1(F,G)$ is injective, then $m(\pi)$ is just the multiplicity for $G(F)/H(F)$, i.e., $m(\pi) = \dim(\mathrm{Hom}_{H(F)}(\pi, \xi))$.

Conjecture 2.10. *Let π be an irreducible tempered representation of $G(F)$ whose central character is trivial on $Z_{G,H}(F)$, and let ϕ be the Langlands parameter of π. There exists a choice of Whittaker datum of G_{qs} (only depends on (G, H, ξ), in particular, independent of π) such that under this choice of Whittaker datum, the multiplicity $m(\pi)$ is equal to the number of irreducible representations in $I(\phi, \rho_X)$ that is equal to ω_π. Here ω_π is the irreducible representation of S_ϕ associated to π under the local Langlands correspondence (with respect to the choice of Whittaker datum).*

Moreover, the choice of the Whittaker datum is not necessarily unique. All the possible choices that make this conjecture hold form a torsor

$$\mathrm{Im}\big(\ker(H^1(F,Z_{G,H}) \to H^1(F,H_0)) \to \ker(H^1(F,Z_G) \to H^1(F,G))\big).$$

[3] This map is not necessarily injective/surjective.

Remark 2.11. The above conjecture is similar to the epsilon dichotomy conjecture for the Gan–Gross–Prasad models in [4] and for 10 strongly tempered models in [16]. But there are two important improvements (both in the definition of ω_{ϕ',ρ_X}). The definition of ω_{ϕ',ρ_X} in [16] only involves elements $s \in Z_\phi$ that belong to an elliptic extended endoscopic triple. For the cases considered in [16], we showed that each connected component of Z_ϕ contains an elliptic element, which justifies restricting attention to elliptic extended endoscopic triples. However, in the general setting, it is unclear whether every connected component of Z_ϕ contains an elliptic element. As a result, it is insufficient to consider only the elliptic case. In the present paper, our definition does not impose the elliptic condition.

Secondly, in [16] we explicitly write down s and define the term $\eta \circ \phi_0(-1)$ in the definition of ω_{ϕ',ρ_X} by an explicit formula. In [4], they explicitly write down a representative for each element of S_ϕ and then define the function ω_{ϕ',ρ_X} (in particular the term $\eta \circ \phi_0(-1)$) by an explicit formula. In this paper, we define the term $\eta \circ \phi_0(-1)$ in a conceptual way using the anomaly free property. This is a very important improvement because for general groups (e.g., E_7, E_8), it is very hard (at least for us) to explicitly write down the component group S_ϕ and representative of elements in S_ϕ for the general Langlands parameter (see Section 3.4 for an example involving E_7, at this moment this is the only strongly tempered example we known involving exceptional groups other than the Whittaker models).

Another important point in Conjecture 2.10 is that we can consider the case when there is more than one open Borel orbits (i.e., the multiplicity for the L-packet is not necessarily one).[4] This is the first time such a conjecture has been proposed (other than in some lower-rank cases). The key is to use the set of lifting and to consider the induced representation $\mathrm{Ind}_{i(S_{\phi'})}^{S_\phi}(\omega_{\phi',\rho_X})$.

However, one limitation compared to the Gan–Gross–Prasad models in [4] is that here we cannot pin down the choice of Whittaker datum in Conjecture 2.10.

2.5. *The cases when Assumption 2.6 fails.* As we mentioned in Assumption 2.6, among all the strongly tempered examples we know, the only case this assumption fails is the unitary Gan–Gross–Prasad model $(U_{n+2m+1} \times U_n, U_n \ltimes N)$. In fact we can see it from the case $(U_2 \times U_1, U_1)$. In this case the representation ρ_X is two-dimensional. However, if we consider the extended endoscopic triple $(G', s, {}^L\eta)$ of G with $G' = U_1 \times U_1 \times U_1$, then $\rho_{X,s,{}^L\eta,-}$ is one-dimensional which is not symplectic.

The epsilon dichotomy conjecture for this case (Conjecture 17.3 of [4]) is also different with the other cases. Namely, for all the other cases, like Conjecture 2.10, the character ω_{ϕ',ρ_X} of the component group $S_{\phi'}$ is defined purely using the representation ρ_X, while in the unitary Gan–Gross–Prasad model case [4], the definition

[4]Under our assumption this happens when the map $H^1(F, Z_G \cap H) \to H^1(F, H_0)$ is not injective.

of the character ω_{ϕ',ρ_X} depends on the choice of an additive character ψ_E of the quadratic extension E/F and this additive character also determines the choice of the Whittaker datum of G (i.e., in this case the definition of the character ω_{ϕ',ρ_X} is determined by the choice of Whittaker datum). For all the other cases, the character ω_{ϕ',ρ_X} does not depend on the choice of Whittaker datum and there are some natural choice of Whittaker datum (only depends on the model (G, H)) so that Conjecture 2.10 holds.

When Assumption 2.6 fails, we cannot define the function ω_{ϕ',ρ_X} as in Section 2.4. Instead, as mentioned in Remark 2.7, we expect that for those extended endoscopic triple $(G', s, {}^L\eta)$ of G where the split rank of G' is equal to the split rank of G, then the symplectic representation $\rho_{X,s,{}^L\eta,-}$ of ${}^LG'$ is anomaly free (we show in Section 3, this holds for the unitary Gan–Gross–Prasad models).

As a result, by using the same argument as in Section 2.4, we can define a function ω'_{ϕ',ρ_X} on the subset $Z'_{\phi'}$ of $Z_{\phi'}$ where $Z'_{\phi'}$ contains those $s \in Z_{\phi'}$ satisfying the following condition.

- There exists an extended endoscopic triple $(G', s, {}^L\eta)$ of G (not necessarily unique) such that ϕ' factors through ${}^L\eta$ and the split rank of G' is equal to the split rank of G.

Then we make the following conjecture which is an analogue of Conjecture 2.8.

Conjecture 2.12. *The function ω'_{ϕ',ρ_X} is well defined (i.e., it is independent of the choice of the extended endoscopic triple and the lifting). Moreover, there exists a character ω_{ϕ',ρ_X} of S_ϕ (not necessarily unique) such that ω_{ϕ',ρ_X} is equal to ω'_{ϕ',ρ_X} on $Z'_{\phi'}$.*

Then by using the same argument as in Section 2.4, we can also formulate Conjecture 2.10 in this case.

Conjecture 2.13. *For some suitable choice of Whittaker datum, there exists a character ω_{ϕ',ρ_X} of S_ϕ such that ω_{ϕ',ρ_X} is equal to ω'_{ϕ',ρ_X} on $Z'_{\phi'}$ and Conjecture 2.10 holds.*

In Section 3, we will show that Conjectures 2.12 and 2.13 hold for the unitary Gan–Gross–Prasad model.

2.6. ***How to prove Conjecture 2.10 and some open questions.*** Here we discuss some ideas about proving Conjecture 2.10 and some open questions. The first step is to prove a multiplicity formula $m(\pi) = m_{\mathrm{geom}}(\pi)$ for all tempered representations. Here $m_{\mathrm{geom}}(\pi)$ is defined in [15] and is called the geometric multiplicity. Such a multiplicity formula has been proved for many strongly tempered spherical varieties such as the Gan–Gross–Prasad models and the models in [16]. Moreover, for each given model, it seems that the current trace formula method (invented by Waldspurger in his proof of the orthogonal Gan–Gross–Prasad conjecture [12; 14]) can be used to prove the multiplicity formula. But it is still not clear how to write down the proof for the general case without using any feature pertaining to the specific model.

After proving the multiplicity formula, one can study the behavior of the geometric multiplicity under endoscopic. Together with some induction hypothesis (i.e., we assume the epsilon dichotomy conjecture holds for some models related to endoscopic groups of G), we can reduce the proof of Conjecture 2.10 to the computation of the sum of the multiplicity over the Vogan L-packet. This idea was invented by Waldspurger in his proof of the orthogonal Gan–Gross–Prasad conjecture [13]. As in the proof of the multiplicity formula, it seems that Waldspurger's method can be used for any given model, but it is not clear how to write it for a general case. In particular, if G' is an endoscopic group of G, it is not clear in general which models of G' should be related to (G, H). For a specific model, we know the model associated to G' by direct computation, but we do not have a general theory to explain this (i.e., we need a relative endoscopy theory for strongly tempered spherical varieties).

The last step, which is also the most difficult step, is to study the multiplicity of the L-packet. The goal is to relate it to the epsilon factor (under the language of [16], we call this the weak epsilon dichotomy conjecture, or just the weak conjecture). For this step, we do not have a systematic way to solve it at this moment. For the Gan–Gross–Prasad model, this was done by relating the multiplicity of the L-packet to the twisted multiplicity of the Gan–Gross–Prasad model of the general linear group. But this method does not work if the Langlands functoriality $\rho_X : {}^L G \to \mathrm{GL}(V)$ is not of twisted endoscopic type (in particular it does not apply to any of the cases in [16]). For all the models in [16] except the model $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, G(\mathrm{Sp}_4 \times \mathrm{Sp}_2))$, the authors in [18] proposed a method to prove the weak conjecture using the "dichotomy" behavior of certain degenerate principal series of GSp_6. The reason this method works is due to the fact that for all the models in [16] except the model $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, G(\mathrm{Sp}_4 \times \mathrm{Sp}_2))$, the epsilon factor can be defined using some local Rankin–Selberg integral involving the degenerate principal series of GSp_6 (in particular this method cannot be used to prove the weak conjecture of the Gan–Gross–Prasad model).[5] It is not clear at this moment how to prove the weak conjecture for the general case (although for all the strongly tempered models we know except the model $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, G(\mathrm{Sp}_4 \times \mathrm{Sp}_2))$, one of the two methods discussed here can be used to prove the weak conjecture).[6]

Another open question is regarding the case when $G/H(F) \neq G(F)/H(F)$. In this case, Conjecture 2.10 studies the multiplicity of $G/H(F)$, not the multiplicity

[5]The reason we exclude the model $(\mathrm{GSp}_6 \times \mathrm{GSp}_4, G(\mathrm{Sp}_4 \times \mathrm{Sp}_2))$ is that at this moment there is no Rankin–Selberg integral defining the epsilon factor associated to this model.

[6]An interesting point is that it seems these two methods are disjoint; we do not know any example where both methods can be used to prove the epsilon dichotomy conjecture.

of $G(F)/H(F)$. This is compatible with the philosophy of [9], but it would be nice to have a conjecture for the multiplicity of $G(F)/H(F)$.

The last open question is about the choice of Whittaker datum in Conjecture 2.10. In Conjecture 2.10, we were not able to specify the choice of Whittaker datum, we only conjectured that all the possible choices form a torsor

$$\mathrm{Im}\big(\ker(H^1(F, Z_{G,H}) \to H^1(F, H)) \to \ker(H^1(F, Z_G) \to H^1(F, G))\big).$$

Among all the known cases, the Whittaker datum is unique for all the models in [16] so this is not an issue; for the Gan–Gross–Prasad model, the Whittaker datum is not unique, and in Section 12 of [4] they gave a specific choice of the Whittaker model. But at this moment we do not know how to generalize it to general strongly tempered spherical varieties.

2.7. ***Why do we need anomaly free?*** Here we explain why we need the condition of anomaly free from our point of view and that of Ben-Zvi, Sakellaridis, and Venkatesh [1].

From our point of view, the anomaly free condition is used to define the term $\eta \circ \phi_0(-1)$ in the character

$$\omega_{\phi',\rho_X}(s) = \eta \circ \phi_0(-1)\epsilon\big(\tfrac{1}{2}, \rho_{X,s,{}^L\eta,-} \circ \phi_0\big) \in \{\pm 1\}.$$

In all the previous epsilon dichotomy conjectures [4; 16], this term was defined by an explicit computation. It was given so that for most unramified parameter ϕ_0, the value of $\omega_{\phi',\rho_X}(s)$ should be equal to 1 (this is because for most unramified parameters the component group is trivial and hence we need the character to also be trivial). With the assumption of anomaly free, we know that for most unramified parameter ϕ_0, the epsilon factor $\epsilon\big(\tfrac{1}{2}, \rho_{X,s,{}^L\eta,-} \circ \phi_0\big)$ is equal to $\eta \circ \phi_0(-1)$ and hence we can define the character ω_{ϕ',ρ_X} in this way.

From the point of view of [1], one of the goals is to equip each Hamiltonian G-space with the automorphic quantization, without passing the metaplectic cover of G. They introduce the notion of "anomaly free" to Hamiltonian G-spaces in [1, Definition 5.1.2] and conjecture such symplectic varieties admit an automorphic and spectral quantization. When M is a symplectic vector space, Definition 2.2 is equivalent to their definition (see [1, Proposition 5.1.5]). Moreover, Examples 5.1.7 and 5.1.9 in [1] give more hyperspecial vector spaces examples and elaborate more detailed connections with Table 1 of [16] and Remark 2.3.

3. Known examples with multiplicity one

We show that our conjecture recovers the epsilon dichotomy conjecture in [4] for the Gan–Gross–Prasad model and the epsilon dichotomy conjecture in [16] for 10 strongly tempered models.

3.1. ***The Whittaker model.*** Let G be a quasisplit reductive group defined over F, N be a maximal unipotent subgroup of G, and ξ be a generic character of $N(F)$. In this case ${}^LG_X = {}^LG$ and the representation ρ_X is zero-dimensional.

In this case, it is clear that Assumption 2.6 and 2.8 are satisfied. Moreover, the set $I(\phi, \rho_X)$ contains a unique element which is the trivial character of S_ϕ. Then Conjecture 2.10 follows from the local Langlands conjecture In this case the choice of Whittaker datum is unique (as $Z_{G,H}$ is trivial in this case) and it should be the one associated to ξ. It is clear that in this case if we do not make the correct choice of the Whittaker datum, then Conjecture 2.10 fails (this is because a generic L-packet may contain two representations, each of which is generic with respect to a different Whittaker datum).

3.2. ***The orthogonal Gan–Gross–Prasad model.*** We show that for the orthogonal Gan–Gross–Prasad models $(G, H)=(\mathrm{SO}_{a+2b+1}\times \mathrm{SO}_a, \mathrm{SO}_a \ltimes N)$, Conjecture 2.10 is the same as the epsilon dichotomy conjecture in [4]. In this case, we can identify ${}^LG_X$ with ${}^LG = \mathrm{Sp}_{2m}(\mathbb{C}) \times \mathrm{SO}_{2n}(\mathbb{C})$ or ${}^LG = \mathrm{Sp}_{2m}(\mathbb{C}) \times O_{2n}(\mathbb{C})$ where $\{2m+1, 2n\} = \{a+2b+1, a\}$ (depends on whether the even special orthogonal group has a split pure inner form). And the representation ρ_X is the $4mn$-dimensional tensor product representation of ${}^LG_X$. Moreover, $Z_{G,H}$ is trivial and the choice of Whittaker datum is unique (defined in Section 12 of [4]).

Let $\phi : W'_F \to {}^LG$ be a tempered L-parameter. The L-parameter ϕ can be identified with $M \otimes N$ where M (resp. N) is a homomorphism from W'_F into $\mathrm{Sp}_{2m}(\mathbb{C})$ (resp. $O_{2n}(\mathbb{C})$ or $\mathrm{SO}_{2n}(\mathbb{C})$). As in Section 4 of [4], we can decompose M and N as

$$M = \bigoplus_{i=1}^{a_1} m_{1i} M_{1i} + \bigoplus_{i=1}^{a_2} 2m_{2i} M_{2i} + \bigoplus_{i=1}^{a_3} m_{3i}(M_{3i} \oplus M_{3i}^\vee),$$
$$N = \bigoplus_{j=1}^{b_1} n_{1j} N_{1j} + \bigoplus_{j=1}^{b_2} 2n_{2j} N_{2j} + \bigoplus_{j=1}^{b_3} n_{3j}(N_{3j} \oplus N_{3j}^\vee),$$

where M_{1i}, N_{2j} are self-dual of symplectic type, M_{2i}, N_{1j} are self-dual of orthogonal type, and M_{3i}, N_{3j} are not self-dual. Then Z_ϕ and S_ϕ are given by

$$Z_\phi = \prod_{i=1}^{a_1} O(m_{1i}, \mathbb{C}) \times \prod_{i=1}^{a_2} \mathrm{Sp}(2m_{2i}, \mathbb{C}) \times \prod_{i=1}^{a_3} \mathrm{GL}(m_{3i}, \mathbb{C})$$
$$\times \prod_{j=1}^{b_1} O(n_{1j}, \mathbb{C}) \times \prod_{j=1}^{b_2} \mathrm{Sp}(2n_{2j}, \mathbb{C}) \times \prod_{j=1}^{b_3} \mathrm{GL}(n_{3j}, \mathbb{C}),$$
$$S_\phi = (\mathbb{Z}/2\mathbb{Z})^{a_1} \times (\mathbb{Z}/2\mathbb{Z})^{b_1}.$$

We just need to show that the function ω_{ϕ,ρ_X} defined in the previous section is the same as the character $\chi_N \times \chi_M$ defined in Section 6 of [4].

We first recall the definition of $\chi_N \times \chi_M$. For $a_M \in (\mathbb{Z}/2\mathbb{Z})^{a_1}$, let $M^{a_M} = \oplus_i M_{1i}$ where i runs over all the components of a_M with -1 coordinate. Similarly, we can also define N^{a_N} for $a_N \in (\mathbb{Z}/2\mathbb{Z})^{b_1}$. In Section 6 of [4], they define

$$\chi_N(a_M)\chi_M(a_N) = \epsilon(M^{a_M}\otimes N)\epsilon(M\otimes N^{a_N})\det(M^{a_M})(-1)^{\dim(N)/2}\det(N)(-1)^{\dim(M^{a_M})/2} \times \det(N^{a_N})(-1)^{\dim(M)/2}\det(M)(-1)^{\dim(N^{a_N})/2}.$$

Here to simplify the notation, for a symplectic representation V of W_F', we use $\epsilon(V)$ to denote $\epsilon\left(\frac{1}{2}, V\right)$.

Next, we show that ω_{ϕ,ρ_X} coincides with $\chi_N \times \chi_M$. Let

$$s = (g_{1i}, g_{2i}, g_{3i}, h_{1j}, h_{2j}, h_{3j})$$

be an element in Z_ϕ with

$$g_{1i} \in O(m_{1i},\mathbb{C}),\quad g_{2i} \in \mathrm{Sp}(2m_{2i},\mathbb{C}),\quad g_{3i} \in \mathrm{GL}(m_{3i},\mathbb{C}),$$
$$h_{1j} \in O(n_{1j},\mathbb{C}),\quad h_{2j} \in \mathrm{Sp}(2n_{2j},\mathbb{C}),\quad h_{3j} \in \mathrm{GL}(n_{3j},\mathbb{C}).$$

Let $a_M \times a_N$ be the corresponding element in S_ϕ. We let I_1 (resp. J_1) be the set of $1 \le i \le a_1$ (resp. $1 \le j \le b_1$) such that $g_{1i} \in O(m_{1i},\mathbb{C}) - \mathrm{SO}(m_{1i},\mathbb{C})$ (resp. $h_{1i} \in O(n_{1j},\mathbb{C}) - \mathrm{SO}(n_{1j},\mathbb{C})$) and let I_2 (resp. J_2) be the complement of I_1 (resp. J_1) in $\{1, 2, \cdots, a_1\}$ (resp. $\{1, 2, \cdots, b_1\}$). We let $I_{1,\mathrm{odd}}$ (resp. $I_{1,\mathrm{even}}$) be the set of $i \in I_1$ such that m_{1i} is odd (resp. even). Similarly, we can define $I_{2,\mathrm{odd}}$, $I_{2,\mathrm{even}}$, $J_{1,\mathrm{odd}}.J_{1,\mathrm{even}}$, $J_{2,\mathrm{odd}}$, $J_{2,\mathrm{even}}$. By Proposition 5.1 of [4], we have

$$\begin{aligned}\chi_N \times \chi_M(s) &= \Pi_{i\in I_1}\Pi_{1\le j\le b_1}\epsilon(M_{1i}\otimes N_{1j})^{n_{1j}}\det(N_{1j})^{n_{1j}\cdot\dim(M_{1j})/2} \\ &\qquad \times \Pi_{j\in J_1}\Pi_{1\le i\le a_1}\epsilon(M_{1i}\otimes N_{1j})^{m_{1i}}\det(N_{1j})^{m_{1i}\cdot\dim(M_{1j})/2} \\ &= \Pi_{(i,j)\in I_1\times J_{2,\mathrm{odd}}\cup I_{2,\mathrm{odd}}\times J_1\cup I_{1,\mathrm{even}}\times J_{1,\mathrm{odd}}\cup I_{1,\mathrm{odd}}\times J_{1,\mathrm{even}}}\epsilon(M_{1i}\otimes N_{1j})\det(N_{1j})^{\dim(M_{1j})/2}.\end{aligned}$$

Next we study the -1-eigenspace $V_{s,-}$ of $\rho_X(s)$, which is a direct sum of the -1-eigenspaces associated to $g_{ki}\times h_{lj}$ with $1 \le k, l \le 3$. We study each separately.

We first study the -1-eigenspace associated to $g_{1i}\times h_{2j}$. By using the tensor representation we can view $g_{1i}\times h_{2j}$ as an element in $\mathrm{GL}(m_{1i}n_{2j})$ and we let $2k$ be the dimension of the -1-eigenspace of this matrix (it is easy to see that this dimension is an even number). Then it is easy to see that the -1-eigenspace associated to $g_{1i}\times h_{2j}$ is $2k$-copy of $M_{1i}\otimes N_{2j}$. This representation is obviously anomaly free and we can choose the character η in the definition of anomaly free to be trivial for this representation. Moreover, by Proposition 5.1 of [4], the epsilon factor associated to it is also equal to 1. Hence the contribution of this -1-eigenspace to the character ω_{ϕ,ρ_X} is just 1.

Similarly, we can show that the contribution of the -1-eigenspaces coming from

$$g_{1i}\times h_{3j},\quad g_{2i}\times h_{1j},\quad g_{2i}\times h_{2j},\quad g_{2i}\times h_{3j},\quad g_{3i}\times h_{1j},\quad g_{3i}\times h_{2j},\quad g_{3i}\times h_{3j}$$

to the character ω_{ϕ,ρ_X} is also 1.

It remains to consider the -1-eigenspace associated to $g_{1i} \times h_{1j}$. By a similar argument as above we can show that the contribution of the -1-eigenspaces coming from $g_{1i} \times h_{1j}$ with

$$(i, j) \in I_1 \times J_{2,\text{even}} \cup I_{2,\text{even}} \times J_1 \cup I_{1,\text{odd}} \times J_{1,\text{odd}} \cup I_{1,\text{even}} \times J_{1,\text{even}} \cup I_2 \times J_2$$

to the character ω_{ϕ,ρ_X} is also 1.

Consider the -1-eigenspace associated to $g_{1i} \times h_{1j}$ with $(i, j) \in I_1 \times J_{2,\text{odd}}$. By using the tensor representation we can view $g_{1i} \times h_{1j}$ as an element in $\mathrm{GL}(m_{1i}n_{2j})$ and we let k be the dimension of the -1-eigenspace of this matrix (it is easy to see that this dimension is an odd number). Then it is easy to see that the -1-eigenspace associated to $g_{1i} \times h_{1j}$ is k-copy of $M_{1i} \otimes N_{1j}$. This representation is obviously anomaly free and we can choose the character η in the definition of anomaly free to be $\det(N_{1j})^{\dim(M_{1j})/2}$. Moreover, the epsilon factor associated to it is equal to $\epsilon(M_{1i} \otimes N_{1j})^k = \epsilon(M_{1i} \otimes N_{1j})$. Hence the contribution of this -1-eigenspace to ω_{ϕ,ρ_X} is $\epsilon(M_{1i} \otimes N_{1j}) \det(N_{1j})^{\dim(M_{1j})/2}$.

Similarly, we can show that the contribution of the -1-eigenspace associated to

$$g_{1i} \times h_{1j}, \quad (i, j) \in I_{2,\text{odd}} \times J_1 \cup I_{1,\text{even}} \times J_{1,\text{odd}} \cup I_{1,\text{odd}} \times J_{1,\text{even}}$$

to ω_{ϕ,ρ_X} is also $\epsilon(M_{1i} \otimes N_{1j}) \det(N_{1j})^{\dim(M_{1j})/2}$. This implies that ω_{ϕ,ρ_X} is the same as $\chi_N \times \chi_M$. In particular, we have proved that for the orthogonal Gan–Gross–Prasad model, Conjecture 2.10 is the same as the epsilon dichotomy conjecture in [4].

3.3. ***The unitary Gan–Gross–Prasad models.*** We show that for the unitary Gan–Gross–Prasad models $(G, H) = (U_{a+2b+1} \times U_a, U_a \ltimes N)$, Conjecture 2.10 is the same as the epsilon dichotomy conjecture in [4]. Let E/F be a quadratic extension. In this case, we have that ${}^L G_X = {}^L G = (\mathrm{GL}_{2m}(\mathbb{C}) \times \mathrm{GL}_{2n+1}(\mathbb{C})) \rtimes W_F'$ where $\{2m, 2n+1\} = \{a+2b+1, a\}$. And the representation ρ_X is the $8m(2n+1)$-dimensional tensor product representation of the base change.

Let $\phi : W_F' \to {}^L G$ be a tempered L-parameter. As in Section 8 of [4], we can identify ϕ with $M \otimes N$ where M (resp. N) is a conjugate-symplectic (resp. conjugate-orthogonal) representation of W_E' of dimension $2m$ (resp. $2n+1$). As in Section 4 of [4], we can decompose M and N as

$$M = \bigoplus_{i=1}^{a_1} m_{1i} M_{1i} + \bigoplus_{i=1}^{a_2} 2m_{2i} M_{2i} + \bigoplus_{i=1}^{a_3} m_{3i} (M_{3i} \oplus \overline{M}_{3i}^{\vee}),$$

$$N = \bigoplus_{j=1}^{b_1} n_{1j} N_{1j} + \bigoplus_{j=1}^{b_2} 2n_{2j} N_{2j} + \bigoplus_{j=1}^{b_3} n_{3j} (N_{3j} \oplus \overline{N}_{3j}^{\vee}),$$

where we say M_{1i}, N_{2j} are of conjugate-symplectic type, M_{2i}, N_{1j} are of conjugate-orthogonal type, and M_{3i}, N_{3j} are not conjugate self-dual. Then Z_ϕ and S_ϕ are given by

$$\begin{aligned} Z_\phi = &\prod_{i=1}^{a_1} O(m_{1i},\mathbb{C}) \times \prod_{i=1}^{a_2} \mathrm{Sp}(2m_{2i},\mathbb{C}) \times \prod_{i=1}^{a_3} \mathrm{GL}(m_{3i},\mathbb{C}) \\ &\times \prod_{j=1}^{b_1} O(n_{1j},\mathbb{C}) \times \prod_{j=1}^{b_2} \mathrm{Sp}(2n_{2j},\mathbb{C}) \times \prod_{j=1}^{b_3} \mathrm{GL}(n_{3j},\mathbb{C}) \\ S_\phi = &(\mathbb{Z}/2\mathbb{Z})^{a_1} \times (\mathbb{Z}/2\mathbb{Z})^{b_1}. \end{aligned}$$

In this case, the representation ρ_X is not anomaly free under endoscopy because the tensor product representation of the base change of $U_{n_1} \times U_{n_2}$ is not anomaly free when both n_1 and n_2 are odd (it is anomaly free when at least one of n_i is even). The endoscopic group of U_{2n+1} is always of the same split rank as U_{2n+1}, while the endoscopic group of U_{2m} is of the same rank only if it does not contains the product of two odd unitary groups. This in particular proves the assumption in Remark 2.7 (i.e., for every extended endoscopic triple $(G', s, {}^L\eta)$ of G, if the split rank of G' is equal to the split rank of G, then the symplectic representation $\rho_{X,s,{}^L\eta,-}$ of ${}^LG'$ is anomaly free). Moreover, in this case, Z'_ϕ is a normal subgroup of Z_ϕ and we have $|Z_\phi/Z'_\phi| = 1$ (resp. $|Z_\phi/Z'_\phi| = 2$) if the dimensional of M_{1i} are all even (resp. the dimension of M_{1i} is odd for some i). To be specific, Z'_ϕ contains those elements in Z_ϕ such that when projected to $\prod_{i=1}^{a_1} O(m_{1i,\mathbb{C}})$, the number of i such that the projection of the element to $O(m_{1i,\mathbb{C}})$ belongs to $O(m_{1i,\mathbb{C}}) - \mathrm{SO}(m_{1i,\mathbb{C}})$ and $\dim(M_{1i})$ being odd is even.

By a similar argument as in the orthogonal group case in Section 3.3 (the key is the formula of epsilon factor under induction in page 18, line 13 of [4]), we know that the function ω'_{ϕ,ρ_X} of Z'_ϕ defined in the previous section is the same as the restriction of the character $\chi_N \times \chi_M$ defined in Section 6 of [4] to Z'_ϕ.[7] In particular, this shows that our conjecture agrees with the conjecture in [4] for the unitary Gan–Gross–Prasad models.

3.4. ***The models in*** **[16].** We show that Conjecture 2.10 recovers the epsilon dichotomy conjecture of the 10 models considered in [16]. We only consider the most complicated model $(E_7, \mathrm{PGL}_2 \ltimes N)$. The other models in [16] follows from a similar and easier argument. We first prove Assumption 2.6. Then as in [16], by assuming the weak conjecture (Conjecture 1.6 of [16]) holds for the model $(E_7, \mathrm{PGL}_2 \ltimes N)$, we prove Conjecture 2.10.

We first prove Assumption 2.6 for this model. We write ${}^LG_X = {}^LG = \hat{G} \times W'_F$ where $\hat{G} = E_{7,sc}(\mathbb{C})$ is the simply connected form of E_7 and ρ_X is the 56-dimensional representation of $E_{7,sc}(\mathbb{C})$. To prove Assumption 2.6, we only need to prove the following proposition.

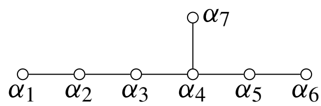
[7] It is worth mentioning that the character $\chi_N \times \chi_M$ defined in Section 6 of [4] depends on the choice of an additive character of E, but its value on Z'_ϕ does not depend on it by Theorem 6.1 of [4].

Proposition 3.1. *For $s \in \hat{G}_{ss}$, let $V_{s,-}$ be the -1-eigenspace of $\rho_X(s)$. The representation of $\hat{G}_s$ on $V_{s,-}$ is anomaly free.*

Proof. We use $\rho_{X,s,-}$ to denote the representation of $\hat{G}_s$ on $V_{s,-}$. If s is elliptic, the representation $\rho_{X,s,-}$ was described in Section 2.5 of [16]. From there it is easy to see that $\rho_{X,s,-}$ is anomaly free, we just need to use the fact that the following representations are anomaly free (which follows from an easy direct computation):

- The representation ρ_X of $\hat{G}$.
- The tensor representation of $\mathrm{Spin}_{12}(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/(\mathbb{Z}/2\mathbb{Z})$.
- The half-spin representation of $\mathrm{Spin}_{12}(\mathbb{C})$.
- The exterior cube representation of $\mathrm{SL}_6(\mathbb{C})/(\mathbb{Z}/3\mathbb{Z})$.
- The representation $\wedge^2 \otimes \mathrm{std}$ of $\mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/(\mathbb{Z}/4\mathbb{Z})$.

When s is not elliptic, consider the following Dynkin diagram of E_7:



Let L_i be the maximal Levi subgroup of $\hat{G}$ associated to the simple roots $\Delta - \{\alpha_i\}$ for $1 \le i \le 7$ with $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_7\}$. It is enough to show that $\rho_X|_{L_i}$ is anomaly free under endoscopy for all i. If $i = 1, 3, 7$, the restriction of ρ_X to L_i is of the form $\rho_X = \rho \oplus \rho^\vee$ for some representation ρ of L_i. Hence we know that $\rho_X|_{L_i}$ is anomaly free under endoscopy.

If $i = 6$ (resp. 5, 4, 2), then we have that the restriction ρ_X to L_i is of the form $\rho \oplus \rho^\vee \oplus \rho'$ where ρ is a representation of L_i and ρ' is the half-spin representation of $\mathrm{Spin}_{12}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})/(\mathbb{Z}/2Z)$ (resp. the exterior cube representation of $\mathrm{SL}_6(\mathbb{C})/(\mathbb{Z}/3\mathbb{Z})$, the representation $\wedge^2 \otimes \mathrm{std}$ of $\mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/(\mathbb{Z}/4\mathbb{Z})$, the tensor product representation of $\mathrm{Spin}_{10}(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/(\mathbb{Z}/4\mathbb{Z})$). It remains to show that ρ' is anomaly free under endoscopy. The argument is the same as above and we skip it here (i.e., we first consider the elliptic elements of L_i for which we can explicitly write down the representation $V_{s,-}$ and show that it is anomaly free, then we can further reduce to maximal Levi subgroup of L_i). This proves the proposition. □

Next, we assume the weak conjecture (Conjecture 1.6 of [16]) holds for the model $(E_7, \mathrm{PGL}_2 \ltimes N)$ and then prove Conjecture 2.10. Let (G, H) and (G_D, H_D) be as in Section 8 of [16]. Let $\phi : W_F' \to {}^L G$ be a tempered L-parameter of G and let ω_ϕ be the character of S_ϕ corresponds to the unique distinguished element in the L-packet $\Pi_\phi = \Pi_\phi(G) \cup \Pi_\phi(G_D)$. We need to show that $\omega_{\phi,\rho_X} = \omega_\phi$. In Section 2 of [16], we have defined a function $\omega_{\phi,H}$ on the elliptic elements of Z_ϕ and we have proved in Section 8 of [16] that $\omega_{\phi,H} = \omega_\phi$ on the elliptic elements of Z_ϕ.

It is clear from the definition that the functions $\omega_{\phi,H}$ and ω_{ϕ,ρ_X} are the same on all the elliptic elements formulas. Hence by using the result in Section 8 of [16] we proved that $\omega_{\phi,\rho_X}(s)=\omega_{\phi,H}(s)$ for all $s\in Z_\phi$ with s elliptic.

When s is not elliptic, the argument is the same as the elliptic case in Section 8 of [16]. Namely, let $(G', s, {}^L\eta)$ be the extended endoscopic triple such that ϕ factors through ${}^L\eta$ (i.e., there exists $\phi_0 : W_F' \to {}^LG'$ such that $\phi = {}^L\eta\circ\phi_0$). Then we study the behavior of the geometric multiplicity of the model under endoscopy between G and G'. When s is elliptic, this is done in Section 8.3 of [16]. If s is nonelliptic, the argument is the same. We can first pass from G to its Levi subgroup L_s (this step has already been done in Proposition 8.1 of [16]), then we can study the endoscopic transfer of the geometric multiplicity between L_s and G' (the argument is the same as the one in Section 8.3 of [16]). After we proved this identity, we get some formulas of distributions on G' which can be related to some models of G'. Then by using the weak conjecture, we can relate it to a certain epsilon factor and prove that $\omega_{\phi,\rho_X}(s)=\omega_{\phi,H}(s)$. Since the argument is the same as the elliptic case in Section 8 of [16], we only list the models related to G' and skip the remaining details.

- If L_s is associated to one of the following subsets of Δ:

$$\{\alpha_1,\alpha_3,\alpha_7\},\ \{\alpha_1,\alpha_3,\alpha_7,\alpha_i\},\quad i=2,4,5,6,$$
$$\{\alpha_1,\alpha_3,\alpha_7,\alpha_i,\alpha_j\},\quad \{i,j\}=\{2,4\},\{2,5\},\{2,6\},\{4,6\},\{5,6\},$$
$$\Delta\setminus\{\alpha_i\},\quad i=4,5,$$

then it is of type A and we must have $G'=L_s$. In this case, the model related to G' is the model $(G', G'\cap H)$ as in Proposition 8.1 of [16].

- If L_s is associated to $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_7\}$, then L_s is of type $D_4\times A_1$. In this case, G' is either equal to L_s or of the type $(A_1)^5$. If G' is of type $(A_1)^5$, the model related to G' is the trilinear GL_2 model. If $G'=L_s$, the model related to G' is the model $(G', G'\cap H)$ as in Proposition 8.1 of [16].

- If $L_s=L_2$, then L_s is of type $D_5\times A_1$. In this case, G' is either equal to L_s or of the type $A_3\times A_1\times A_1\times A_1$. If $G'=L_s$, the model related to G' is the model $(G', G'\cap H)$ as in Proposition 8.1 of [16]. If G' is of the type $A_3\times A_1\times A_1\times A_1$, the models related to G' are the model $(\mathrm{GL}_4\times\mathrm{GL}_2, \mathrm{GL}_2\times\mathrm{GL}_2)$ defined in [16] and the trilinear GL_2-model.

- If $L_s=L_6$, then L_s is of type D_6. In this case, we know that G' is equal to L_s, of type $D_4\times A_1\times A_1$ or of type $A_3\times A_3$. If $G'=L_s$, the model related to G' is the model $(G', G'\cap H)$ as in Proposition 8.1 of [16]. If G' is of the type $D_4\times A_1\times A_1$, the model related to G' is the model $(\mathrm{GSO}_8\times\mathrm{GL}_2, \mathrm{GL}_2\ltimes N)$ defined in [16]. If G' is of type $A_3\times A_3$, the model related to G' is the Whittaker model.

- For all the other cases, the model related to G' is the Whittaker model.

4. Some examples without multiplicity one

We discuss some models with more than one Borel orbits and show that our conjecture holds for these cases.

4.1. ***The model*** **(SL_2, GL_1).** We consider the model $(G, H) = (\mathrm{SL}_2, \mathrm{GL}_1)$ (i.e., H is a maximal split tori of G). In this case, ${}^LG = \mathrm{PGL}_2(\mathbb{C})$ and ${}^LG_X = \mathrm{SL}_2(\mathbb{C})$. For a tempered parameter ϕ of $G(F)$, the central character of the packet $\Pi_\phi(G)$ is trivial if and only if there is a lifting ϕ' of ϕ to ${}^LG_X$. In this case, the set of liftings I' contains $|F^\times/(F^\times)^2|/|S_\phi|$ many elements (any two different liftings are differed by a twist of quadratic character). The L-packet $\Pi_\phi(G)$ contains $|S_\phi|$ many representations and it is easy to see that each of them has multiplicity $|F^\times/(F^\times)^2|/|S_\phi| = |I'|$. Moreover, let J be the set of quadratic characters η such that $\phi' \simeq \phi' \otimes \eta$. Then $|I'| = |F^\times/(F^\times)^2|/|J|$ and $|S_\phi| = |J|$.

The representation ρ_X of ${}^LG = \mathrm{SL}_2(\mathbb{C})$ is just $\mathrm{std} \oplus \mathrm{std}$. Clearly Assumption 2.6 and Conjecture 2.8 hold and the character ω_{ϕ',ρ_X} of $S_{\phi'}$ is just the trivial character. As a result, the set I is equal to I' and the set

$$\{\chi_{\phi',\rho_X,j} \mid j \in J(\phi'),\ \phi' \in I\}$$

contains all the characters of S_ϕ, each of them appears exactly $|I| = |F^\times/(F^\times)^2|/|S_\phi|$ times. The choice of Whittaker datum does not matter in this case since the map

$$\ker(H^1(F, Z_{G,H}) \to H^1(F, H)) \to \ker(H^1(F, G) \to H^1(F, G))$$

is a bijection. This proves Conjecture 2.10.

4.2. ***The model*** **(SL_2, E^1).** We consider the model $(G, H) = (\mathrm{SL}_2, E^1)$ where E/F is a quadratic extension, $\eta_{E/F}$ is the quadratic character associated to E/F and $E^1 = \ker(\eta_{E/F})$ (i.e., H is a maximal elliptic tori of G). As in the previous case, we have ${}^LG = \mathrm{PGL}_2(\mathbb{C})$ and ${}^LG_X = \mathrm{SL}_2(\mathbb{C})$. For a tempered parameter ϕ of $G(F)$, the central character of the packet $\Pi_\phi(G)$ is trivial if and only if there is a lifting ϕ' of ϕ to ${}^LG_X$, and the set of liftings I' contains $|F^\times/(F^\times)^2|/|S_\phi|$ many elements and the L-packet $\Pi_\phi(G)$ contains $|S_\phi|$ many representations.

For this model, $X(F)$ is not equal to $G(F)/H(F)$ and it is equal to

$$X(F) = G(F)/H(F) \cup G(F)/H'(F),$$

where $H'(F)$ is another maximal elliptic tori of $G(F)$ that is isomorphic to E^1 (if $\eta_{E/F}(-1) = 1$ then H' is not conjugated to H; if $\eta_{E/F}(-1) = -1$ then we may just choose H' to be H). Hence Conjecture 2.10 studies the multiplicity

$$m(\pi) = \dim(\mathrm{Hom}_{H(F)}(\pi, 1)) + \dim(\mathrm{Hom}_{H'(F)}(\pi, 1)).$$

In this case, any maximal elliptic tori of $G(F)$ that is isomorphic to E^1 is either conjugated to H or H'. Since any two representations in the L-packet $\Pi_\phi(G)$ can be conjugated to each other by an element of $\mathrm{GL}_2(F)$, we know that the multiplicity $m(\pi)$ is constant among representations in the L-packet $\Pi_\phi(G)$.

If we view PGL_2 (resp. E^1) as SO_3 (resp. SO_2), then ρ_X is the restriction of the 4-dimensional tensor representation of ${}^L\mathrm{SO}_3 \times {}^L\mathrm{SO}_2$ to ${}^L\mathrm{SO}_3$. It is easy to see that Assumption 2.6 and Conjecture 2.8 hold in this case. Moreover, the component group $S_{\phi'}$ ($\phi' \in I'$ and $\pi_{\phi'}$ is the irreducible tempered representation of $\mathrm{PGL}_2(F)$ associated to ϕ') is either trivial or $\mathbb{Z}/2\mathbb{Z}$. If it is trivial, we have

$$\eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'}, \rho_X\big) = 1.$$

If it is equal to $\mathbb{Z}/2\mathbb{Z}$, then the character ω_{ϕ',ρ_X} is trivial (resp. the sign character) if $\eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'}, \rho_X\big) = 1$ (resp. $\eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'}, \rho_X\big) = -1$). This implies that

$$I = \big\{\phi' \in I' \mid \eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'}, \rho_X\big) = 1\big\}$$

and the set

$$\{\chi_{\phi',\rho_X,j} \mid j \in J(\phi'),\ \phi' \in I\}$$

contains all the characters of S_ϕ, each of them appears exactly $|I|$ times. Moreover, for any $\phi' \in I'$, if we let J' be the set of quadratic characters η such that

$$\eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'} \otimes \eta, \rho_X\big) = 1,$$

then $|I| = |J'|/|J|$. Here we recall from the previous subsection that J is the set of quadratic characters η such that $\phi' \simeq \phi' \otimes \eta$. Hence it remains to show that the multiplicity $m(\pi)$ is equal to $|I|$ for any π in the L-packet $\Pi_\phi(G)$. Since the $m(\pi)$ is constant among representations in the L-packet $\Pi_\phi(G)$, it is enough to show that

$$\begin{aligned}\dim(\mathrm{Hom}_{H(F)}(\Pi_\phi(G), 1)) &= \dim(\mathrm{Hom}_{H'(F)}(\Pi_\phi(G), 1)) \\ &= \tfrac{1}{2}(|I| \cdot |S_\phi|),\end{aligned}$$

where

$$\mathrm{Hom}_{H(F)}(\Pi_\phi(G), 1) = \bigoplus_{\pi \in \Pi_\phi(G)} \mathrm{Hom}_{H(F)}(\pi, 1),$$

$$\mathrm{Hom}_{H'(F)}(\Pi_\phi(G), 1) = \bigoplus_{\pi \in \Pi_\phi(G)} \mathrm{Hom}_{H'(F)}(\pi, 1).$$

Let ϕ' be an element in I'. We can view $\pi_{\phi'}$ as a tempered representation of $\mathrm{GL}_2(F)$ with trivial central character and we have $\Pi_\phi(G) = \pi_{\phi'}|_{\mathrm{SL}_2(F)}$. The model (PGL_2, E^1) is the famous Waldspurger model and we let $m'(\pi_{\phi'})$ be the multiplicity of $\pi_{\phi'}$ with respect to this model. The epsilon dichotomy conjecture for the Waldspurger model implies that

$$m'(\pi_{\phi'}) = 1 \iff \eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'}, \rho_X\big) = 1,$$
$$m'(\pi_{\phi'}) = 0 \iff \eta_{E/F}(-1)\epsilon\big(\tfrac{1}{2}, \pi_{\phi'}, \rho_X\big) = -1.$$

Hence (η runs over all the quadratic characters modulo the subgroup $\{1, \eta_{E/F}\}$),

$$\begin{aligned}\dim(\mathrm{Hom}_{H(F)}(\Pi_\phi(G), 1)) &= \dim(\mathrm{Hom}_{H'(F)}(\Pi_\phi(G), 1))\\ &= \sum_\eta m'(\pi_{\phi'} \otimes \eta)\\ &= \tfrac{1}{2}|J'| = \tfrac{1}{2}(|I| \cdot |S_\phi|).\end{aligned}$$

Here the identity

$$\dim(\mathrm{Hom}_{H(F)}(\Pi_\phi(G), 1)) = \dim(\mathrm{Hom}_{H'(F)}(\Pi_\phi(G), 1)) = \sum_\eta m'(\pi_{\phi'} \otimes \eta)$$

follows from the Frobenius reciprocity and the decomposition ($N_{E/F} : E^\times \to F^\times$ is the norm map)

$$\mathrm{Ind}_{E^1 \cdot F^\times}^{E^\times}(1) = \bigoplus_\eta \eta \circ N_{E/F}.$$

This proves Conjecture 2.10 for the model $(G, H) = (\mathrm{SL}_2, E^1)$.

Remark 4.1. By a similar argument we can also verify Conjecture 2.10 for the triple product model of SL_2 (that is, $(G, H) = ((\mathrm{SL}_2)^3, \mathrm{SL}_2)$) and U_2 (that is, $(G, H) = ((U_2)^3, U_2)$).

4.3. ***The model*** $(U_6, U_2 \ltimes N)$**.** We discuss the unitary Ginzburg–Rallis model $(U_6, U_2 \ltimes N)$ studied in [17]. In this case, we have ${}^LG = \mathrm{GL}_6(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$ and ${}^LG_X = \mathrm{SL}_6(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$. For a tempered parameter ϕ of $G(F)$, the central character of the packet $\Pi_\phi(G)$ is trivial if and only if there is a lifting ϕ' of ϕ to $G_X(F)$. In this case, the map $S_{\phi'} \to S_\phi$ is injective, and we have $S_\phi = S_{\phi'}$ (resp. $|S_\phi/S_{\phi'}| = 2$) if and only if ϕ' is not isomorphic to $\phi' \otimes \eta_{E/F}$ (resp. ϕ' is isomorphic to $\phi' \otimes \eta_{E/F}$). The set of liftings I' contains $2 \cdot |S_{\phi'}|/|S_\phi|$ many elements and we have $I = I'$.

The representation ρ_X of ${}^LG_X = \mathrm{SL}_6(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$ is the 20-dim exterior cube representation. By a similar but easier argument as the Gan–Gross–Prasad model case we can prove Assumption 2.6 and Conjecture 2.8 for this model. The set

$$\{\chi_{\phi', \rho_X, j} \mid j \in J(\phi'),\ \phi' \in I\}$$

contains two elements. If the lifting is unique, then $|S_\phi/S_{\phi'}| = 2$ and these two characters are the two characters of S_ϕ whose restriction to $S_{\phi'}$ is equal to ω_{ϕ', ρ_X}. If there are two liftings, then $S_\phi = S_{\phi'}$ and the two characters are just ω_{ϕ', ρ_X} for $\phi' \in I$ (this is equivalent to we only consider one lifting but we consider ρ_X and $\rho_X \otimes \eta_{E/F}$).

The choice of the Whittaker model does not matter since the map

$$\ker(H^1(F, Z_{G,H}) \to H^1(F, H)) \to \ker(H^1(F, G) \to H^1(F, G))$$

is a bijection.

We prove Conjecture 2.10 in this case by assuming that Conjecture 1.6 of [16] holds for the model $(GU_6, GU_2 \ltimes N)$.[8] Like the argument in Section 5 of [16], the key is to study the behavior of the multiplicity under endoscopy.

First, we consider the case when there are two liftings. In this case, the two choices of Whittaker data give the same parametrization of the L-packet. We use ϕ_i' $(i = 1, 2)$ to denote these two liftings. In this case $S_\phi = S_{\phi_i'}$. For $s \in S_\phi = S_{\phi_i'}$, as in Section 5.4 of [16], we can choose $s' \in sZ^\circ_{\phi_i'}$ so that s' is conjugated to $\pm I_6$ or $\pm\operatorname{diag}(I_4, -I_2)$. The value of $\omega_{\phi_i',\rho_X}(s)$ is the same as the one defined in Section 2.5 of [16]. Let

$$I_G = \bigl\{i \mid \eta_{E/F}(-1)\epsilon\bigl(\tfrac{1}{2}, \Pi_{\phi_i'}, \rho_X\bigr) = \varepsilon_G\bigr\},$$

where ε_G is equal to 1 (resp. -1) if G is quasisplit (resp. nonquasisplit). By the local Langlands correspondence discussed in Section 2.2, I_G is the set of i such that ω_{ϕ_i',ρ_X} corresponds to a representation in $\Pi_\phi(G)$.

By Conjecture 1.6 of [16] and Proposition 5.2 of [17], the multiplicity of the L-packet $\Pi_\phi(G)$ is equal to $|I_G|$. If $|I_G| = 0$, then the two characters ω_{ϕ_i',ρ_X} do not correspond to a representation in $\Pi_\phi(G)$ (both of them correspond to a representation of the pure inner form of G). This proves Conjecture 2.10. If $|I_G| = 2$, then the two characters ω_{ϕ_i',ρ_X} both correspond to a representation in $\Pi_\phi(G)$. Also in this case the L-packet $\Pi_\phi(G)$ has multiplicity two and we let $\omega_{\phi,i}$ $(i = 1, 2)$ be the two characters correspond to the distinguished elements in $\Pi_\phi(G)$ (these two characters may be the same). In this case, by the same argument as in Section 5.4 of [16], we can show that (note that in this case the multiplicity formula was proved in [17] and we can prove the endoscopic identity of the geometric multiplicity by the same argument as in Proposition 5.8 of [16])

$$\omega_{\phi,1}(s) + \omega_{\phi,2}(s) = \sum_{\pi\in\Pi_\phi(G)} \chi_\pi(s)m(\pi) = \omega_{\phi_1',\rho_X}(s) + \omega_{\phi_2',\rho_X}(s) \quad \text{for all } s \in S_\phi.$$

Here χ_π is defined in Section 2.2. This implies that $\{\omega_{\phi,1}, \omega_{\phi,2}\} = \{\omega_{\phi_1',\rho_X}, \omega_{\phi_2',\rho_X}\}$ and proves Conjecture 2.10 in this case. If $|I_G| = 1$, we may assume that $\phi_1' \in I_G$. In this case, the L-packet $\Pi_\phi(G)$ has multiplicity one and we let ω_ϕ be the character corresponding to the distinguished elements in $\Pi_\phi(G)$. In this case, by the same

[8] In [18] we have proposed a method to prove this conjecture, and we will prove it in our next paper.

argument as in Section 5.4 of [16], we can show that

$$\omega_\phi(s) = \sum_{\pi\in\Pi_\phi(G)} \chi_\pi(s)m(\pi) = \omega_{\phi_1',\rho_X}(s) \quad \text{for all } s \in S_\phi.$$

This implies that $\omega_\phi = \omega_{\phi_1',\rho_X}$ and proves Conjecture 2.10.

Next, we consider the case when the lifting is unique and we use ϕ' to denote this lifting. In this case, if $\eta_{E/F}(-1)\epsilon\left(\frac{1}{2}, \Pi_{\phi'}, \rho_X\right) = -\varepsilon_G$, the multiplicity of the L-packet $\Pi_\phi(G)$ is equal to 0 and the two characters in $I_{S_{\phi'}}^{S_\phi}(\omega_{\phi',\rho_X})$ does not correspond to a representation in $\Pi_\phi(G)$ (both of them corresponds to a representation of the pure inner form of G). This proves Conjecture 2.10. If $\eta_{E/F}(-1)\epsilon\left(\frac{1}{2}, \Pi_{\phi'}, \rho_X\right) = \varepsilon_G$, the multiplicity of the L-packet $\Pi_\phi(G)$ is equal to 2 and the two characters in $I_{S_{\phi'}}^{S_\phi}(\omega_{\phi',\rho_X})$ both correspond to a representation in $\Pi_\phi(G)$. We let χ_1, χ_2 be these two characters and we let $\omega_{\phi,i}$ $(i = 1, 2)$ be the two characters corresponding to the distinguished elements in $\Pi_\phi(G)$. In this case, the two representations that correspond to $\{\chi_1, \chi_2\}$ are independent of the choice of the Whittaker datum. In fact, if one choice of Whittaker datum $\omega_{\phi,i}$ corresponds to some representation π_i of the L-packet, then under the other choice of Whittaker datum $\omega_{\phi,1}$ (resp. $\omega_{\phi,2}$) corresponds to π_2 (resp. π_1).

By the same argument as in Section 5.4 of [16], we can show that

$$\omega_{\phi,1}(s) + \omega_{\phi,2}(s) = \sum_{\pi\in\Pi_\phi(G)} \chi_\pi(s)m(\pi) = \chi_1(s) + \chi_2(s) \quad \text{for all } s \in S_{\phi'}.$$

Since χ_1 and χ_2 are the only two characters of S_ϕ whose restriction to $S_{\phi'}$ is equal to ω_{ϕ',ρ_X}, we only need to show that $\omega_{\phi,1} \neq \omega_{\phi,2}$. Choose $s \in S_\phi - S_{\phi'}$ and we just need to show that $\omega_{\phi,1}(s) + \omega_{\phi,2}(s) = 0$. We can find $s' \in sZ_\phi^\circ$ such that s' belongs to an elliptic endoscopic triple $(G', s', {}^L\eta)$ of G with $G' = U_5 \times U_1$ or $G' = U_3 \times U_3$ and ϕ factors through ${}^LG'$. Then we need to study the behavior of the geometric multiplicity under the endoscopic transfer between G and G'. By a similar but easier argument as in Proposition 5.8 of [16], we can show that $m_{\mathrm{geom}}(\theta) = 0$ if θ is the endoscopic transfer of a stable distribution θ' of $G'(F)$. Here θ is a quasicharacter on $G(F)$ and the geometric multiplicity $m_{\mathrm{geom}}(\theta)$ is defined in Section 5.2 of [17]. This implies that

$$\omega_{\phi,1}(s) + \omega_{\phi,2}(s) = \sum_{\pi\in\Pi_\phi(G)} \chi_\pi(s)m(\pi) = 0.$$

Hence $\omega_{\phi,1} \neq \omega_{\phi,2}$ and this finishes the proof of Conjecture 2.10 for the model $(U_6, U_2 \ltimes N)$.

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