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Let (M, g, J) be a Riemannian manifold with a compatible integrable complex structure $J \in \mathrm{End}(T_{\mathbb{R}}M)$ and $\mathcal{A}_{g,J}$ be the space of real connections on $T_{\mathbb{R}}M$ preserving both g and J. We investigate the relationship between the geometry of real connections in $\mathcal{A}_{g,J}$ and that of Hermitian connections on $T^{1,0}M$. In particular, we study the geometry of the real Chern connection $\nabla^{\mathrm{Ch},\mathbb{R}}$ on (M, g, J), and obtain Kähler–Einstein metrics by using real Chern–Einstein metrics.

1. Introduction

Let (M, g, J) be a Riemannian manifold with a compatible integrable complex structure $J \in \mathrm{End}(T_{\mathbb{R}}M)$ and $\mathcal{A}_g$ be the space of real connections on $T_{\mathbb{R}}M$ compatible with g. Let h be the corresponding Hermitian metric of (g, J) and $\mathcal{B}_h$ be the space of affine connections on the holomorphic tangent bundle $T^{1,0}M$ compatible with h. For any $\nabla \in \mathcal{A}_g$, we can extend it to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. The restriction of the complexified connection ∇ to the holomorphic tangent bundle $T^{1,0}M$ is denoted by $\widehat{\nabla}$. It is obvious that $\widehat{\nabla} \in \mathcal{B}_h$. This gives a natural projection $\pi : \mathcal{A}_g \to \mathcal{B}_h$ and it is easy to see that

$$\mathcal{A}_g \cong \mathcal{B}_h \times \Gamma\big(M, \Omega^1(\mathrm{Hom}(T^{1,0}M, T^{0,1}M))\big). \tag{1-1}$$

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Let $\mathcal{A}_{g,J}$ be the space of real connections on $T_{\mathbb{R}}M$ compatible with both g and J. One can see that there is an isomorphism $\rho : \mathcal{A}_{g,J} \to \mathcal{B}_h$ and

$$\mathcal{A}_{g,J} \cong \mathcal{B}_h \cong \Gamma\big(M, \Omega^1(\mathrm{End}(T^{1,0}M))\big). \tag{1-2}$$

In the field of complex geometry, several classical connections in $\mathcal{B}_h$ are investigated extensively in the literature. For instance,

(1) the Chern connection ∇^{Ch}: the unique connection compatible with the Hermitian metric h and also the holomorphic structure $\bar\partial$;

(2) the Strominger–Bismut connection ∇^{SB} [Strominger 1986; Bismut 1989];

(3) $\hat\nabla^{\mathrm{LC}}$, the restriction of the complexified Levi-Civita connection ∇^{LC} to $T^{1,0}M$.

When (M, g, J) is Kähler, all these connections are the same.

It is well known that $\nabla^{\mathrm{LC}} \in \mathcal{A}_{g,J}$ if and only if $\nabla^{\mathrm{LC}} J = 0$, i.e., (M, g, J) is Kähler. Although $\rho : \mathcal{A}_{g,J} \to \mathcal{B}_h$ is an isomorphism, the relationship between the geometry of real connections in $\mathcal{A}_{g,J}$ and that of Hermitian connections in $\mathcal{B}_h$ is still mysterious. For instance, we set

$$\nabla^{\mathrm{Ch},\mathbb{R}} := \rho^{-1}(\nabla^{\mathrm{Ch}}) \in \mathcal{A}_{g,J} \tag{1-3}$$

and it is called the *real Chern connection*. It is obvious that $\nabla^{\mathrm{Ch},\mathbb{R}} \neq \nabla^{\mathrm{LC}}$ when (M, g, J) is not Kähler. For any $X, Y, Z, W \in T_{\mathbb{R}}M$, the curvature of $\nabla^{\mathrm{Ch},\mathbb{R}}$ is

$$R^{\mathrm{Ch},\mathbb{R}}(X, Y, Z, W) = g(\nabla^{\mathrm{Ch},\mathbb{R}}_X \nabla^{\mathrm{Ch},\mathbb{R}}_Y Z - \nabla^{\mathrm{Ch},\mathbb{R}}_Y \nabla^{\mathrm{Ch},\mathbb{R}}_X Z - \nabla^{\mathrm{Ch},\mathbb{R}}_{[X,Y]} Z, W).$$

The *real Chern–Ricci curvature* $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is defined by using the Riemannian metric g:

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)(X, Y) = \sum_{i=1}^{2n} R^{\mathrm{Ch},\mathbb{R}}(X, e_i, e_i, Y),$$

where $\{e_i\}_{i=1}^{2n}$ is an orthonormal frame with respect to g.

Definition 1.1. $(M, g, J, \nabla^{\mathrm{Ch},\mathbb{R}})$ is called *real Chern–Einstein* if

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) = \lambda g \quad \text{for some } \lambda \in \mathbb{R}. \tag{1-4}$$

If $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) = 0$, it is also called *real Chern–Ricci flat*. Moreover, $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ has positive real Chern–Ricci curvature if $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) \in \Gamma(M, \mathrm{Sym}^{\otimes 2} T_{\mathbb{R}}M)$ and it is positive definite. The negativity can be defined similarly.

There is a natural question concerning the relationship between real Chern–Einstein metrics and Kähler–Einstein metrics.

Question 1.2. Let (M, g, J) be a compact Hermitian manifold. If $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is real Chern–Einstein, then is (M, g, J) necessarily Kähler–Einstein? More generally, if $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is positive (resp. negative), is $c_1(M) > 0$ (resp. $c_1(M) < 0$)?

We can also ask similar questions for other connections in $\mathcal{A}_{g,J}$ or $\mathcal{A}_g$. One shall see that the answer to the above question is quite involved. Moreover, the existence of real Chern–Ricci flat metrics is significantly different from others.

Theorem 1.3. *Let (M, g, J) be a compact Hermitian manifold. Suppose $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is real Chern–Einstein with constant $\lambda \in \mathbb{R}$. If $\lambda \neq 0$, then (M, g, J) is Kähler–Einstein.*

Theorem 1.4. *Let (M, g, J) be a compact Hermitian manifold. If $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is positive (resp. negative), then M is projective and $c_1(M) > 0$ (resp. $c_1(M) < 0$).*

However, when $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is real Chern–Ricci flat, i.e., $\lambda = 0$, (M, g, J) is not necessarily Kähler–Ricci flat. We construct explicit real Chern–Ricci flat metrics on $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. On the contrary, it is well known that, there is *no* Chern–Ricci flat metric on $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ since $c_1^{\mathrm{BC}}(\mathbb{S}^{2n-1} \times \mathbb{S}^1)$ is not zero.

Theorem 1.5. *There exist real Chern–Ricci flat metrics on the Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$.*

We also obtain a classification for Hermitian surfaces with real Chern–Ricci flat metrics.

Theorem 1.6. *Let (M, g, J) be a compact Hermitian manifold. Suppose $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is real Chern–Ricci flat. Then (M, J) is one of the following:*

(1) *(M, J) is Kähler: $c_1(M, J) = 0$, i.e., (M, J) has a Kähler–Ricci flat metric.*

(2) *(M, J) is not Kähler:*

  (a) *$\kappa(M) = 0$ and $c_1^{\mathrm{BC}}(M) = 0$. Moreover, (M, J) has a balanced metric and K_M is a holomorphic torsion: $K_M^{\otimes m} = 0$ for some $m \in \mathbb{N}_+$.*

  (b) *$\kappa(M) = -\infty$ and $c_1^{\mathrm{AC}}(M) = 0$.*

Moreover, the only non-Kähler compact complex surface which can support a real Chern–Ricci metric is the Hopf surface.

Question 1.2 can also be proposed for arbitrary $\nabla \in \mathcal{A}_{g,J}$. Indeed, for any $\nabla \in \mathcal{A}_{g,J}$, there exists some $A \in \Gamma\big(M, \Omega^1(\mathrm{End}(T_{\mathbb{R}}M))\big)$ such that

$$\nabla = \nabla^{\mathrm{Ch},\mathbb{R}} + A,$$

and we also set $\hat{\nabla} = \rho(\nabla) \in \mathcal{B}_h$. One can deduce from (1-2) that ∇ is determined by some $\theta \in \Gamma\big(M, \Omega^{1,0}(\mathrm{End}(T^{1,0}M))\big)$. We write ∇^θ for $\nabla \in \mathcal{A}_{g,J}$, and $\hat{\nabla}^\theta$ for $\hat{\nabla} \in \mathcal{B}_h$. The curvature tensors of ∇^θ and $\hat{\nabla}^\theta$ are denoted by R^θ, $\mathfrak{R}^\theta$ respectively. In local holomorphic coordinates $\{z^i\}$ of M, $\hat{\nabla}^\theta = \rho(\nabla^\theta) \in \mathcal{B}_h$ is given by

$$\hat{\nabla}^\theta_{\partial/\partial z^i} \frac{\partial}{\partial z^j} = (\Gamma^k_{ij} + \theta^k_{ij}) \frac{\partial}{\partial z^k}, \quad \hat{\nabla}^\theta_{\partial/\partial \bar{z}^i} \frac{\partial}{\partial z^j} = -h_{j\bar{q}} h^{k\bar{p}} \bar{\theta}^q_{ip} \frac{\partial}{\partial z^k},$$

where $\theta = \theta^k_{ij}\, dz^i \otimes dz^j \otimes (\partial/\partial z^k)$ and $\Gamma^k_{ij} = h^{k\bar{\ell}}(\partial h_{j\bar{\ell}}/\partial z^i)$ is the Christoffel symbol of the Chern connection. We also use conventions $R^\theta_{i\bar{j}k\bar{\ell}}$ and $\mathfrak{R}^\theta_{i\bar{j}k\bar{\ell}}$ for the components of R^θ and $\mathfrak{R}^\theta$ respectively. The curvature of the Chern connection ∇^{Ch} is denoted by Θ. We set $\mathfrak{Ric}^{(1)}(\theta)=\sqrt{-1}\,(h^{k\bar\ell}\mathfrak{R}^\theta_{i\bar jk\bar\ell})\,dz^i\wedge d\bar z^j$ and similarly, we denote the first Chern–Ricci curvature of the Chern connection ∇^{Ch} by $\Theta^{(1)}$.

Theorem 1.7. *For any $\nabla^\theta\in\mathcal{A}_{g,J}$, curvature tensors R^θ and $\mathfrak{R}^\theta$ are determined by*

$$\begin{aligned} R^\theta_{i\bar jk\bar\ell} &= \mathfrak{R}^\theta_{i\bar jk\bar\ell}\\ &= \Theta_{i\bar jk\bar\ell}-\left(h_{k\bar p}\frac{\partial\bar\theta^p_{j\ell}}{\partial z^i}+h_{p\bar\ell}\frac{\partial\theta^p_{ik}}{\partial\bar z^j}\right)+(\theta^p_{ik}\bar\theta^q_{j\ell}h_{p\bar q}-h^{m\bar n}\theta^p_{im}\bar\theta^q_{jn}h_{p\bar\ell}h_{k\bar q}),\end{aligned}$$

$$R^\theta_{ijk\bar\ell}=\mathfrak{R}^\theta_{ijk\bar\ell}$$

$$=\left(\frac{\partial\theta^m_{jk}}{\partial z^i}-\frac{\partial\theta^m_{ik}}{\partial z^j}+\Gamma^s_{jk}\theta^\ell_{is}-\Gamma^\ell_{js}\theta^s_{ik}+\Gamma^\ell_{is}\theta^s_{jk}-\Gamma^s_{ik}\theta^\ell_{js}\right)h_{m\bar\ell}+(\theta^m_{ip}\theta^p_{jk}-\theta^p_{ik}\theta^m_{jp})h_{m\bar\ell}.$$

Moreover, we have $[\mathfrak{Ric}^{(1)}(\theta)]=[\Theta^{(1)}]\in H^{1,1}_{\text{AC}}(M,\mathbb{R})$ *and*

$$\mathfrak{Ric}^{(1)}(\theta)=\Theta^{(1)}-\sqrt{-1}\,(\partial\bar\theta_1-\bar\partial\theta_1)$$

where $\theta_1=\theta^k_{ik}\,dz^i$.

By using the isomorphism $\rho:\mathcal{A}_{g,J}\to\mathcal{B}_h$, there exists a *unique* linear family in $\{\nabla^t\}_{t\in\mathbb{R}}\subset\mathcal{A}_{g,J}$ which connects the real Chern connection $\nabla^{\text{Ch},\mathbb{R}}=\rho^{-1}(\nabla^{\text{Ch}})$ and the real Strominger–Bismut connection $\nabla^{\text{SB},\mathbb{R}}=\rho^{-1}(\nabla^{\text{SB}})$, and it is given by

$$\nabla^t=(1-t)\nabla^{\text{Ch},\mathbb{R}}+t\nabla^{\text{SB},\mathbb{R}}. \tag{1-5}$$

This family was introduced by Gauduchon [1997], and we call it *Gauduchon connections*. These connections were systematically investigated in recent papers [Fu and Yau 2008; Fu et al. 2009; Andreas and Garcia-Fernandez 2012; 2014; Biswas and Mukherjee 2013; Fei and Yau 2015; Otal et al. 2017; Angella et al. 2022; Fu and Zhou 2022; Yau et al. 2023; Zhao and Zheng 2023] on the construction of invariant solutions to the Strominger systems on complex Lie groups and their quotients. A straightforward computation shows

$$\rho(\nabla^0)=\nabla^{\text{Ch}},\quad \rho(\nabla^{1/2})=\hat\nabla^{\text{LC}},\quad \rho(\nabla^1)=\nabla^{\text{SB}}. \tag{1-6}$$

Hence, these classical connections are all in the Gauduchon family. The curvature relations for ∇^{LC}, ∇^{SB} and $\hat\nabla^{\text{LC}}$ were extensively investigated and have been formulated in differential notions; e.g., [Yau 1974; Gray 1976; Gauduchon 1977a; 1977b; 1984; Tricerri and Vanhecke 1981; Apostolov and Drăghici 1999; Fu 2012; Liu and Yang 2012; 2017; Wang et al. 2020; Yang 2017; Angella et al. 2022; He et al. 2020; Yang and Zheng 2018a; 2018b]. We shall formulate them in a uniform way for readers' convenience. As usual, the curvature tensor of $\rho(\nabla^t)\in\mathcal{B}_h$ is denoted by $\mathfrak{R}(\omega,t)$ and some notions can be found in [Liu and Yang 2017].

Corollary 1.8. *The curvature tensor of the Gauduchon connection $\rho(\nabla^t) \in \mathcal{B}_h$ is*

$$\begin{aligned}&\mathfrak{R}_{i\bar{j}k\bar{\ell}}(\omega,t)\\&\quad=\Theta_{i\bar{j}k\bar{\ell}}+t(\Theta_{i\bar{\ell}k\bar{j}}+\Theta_{k\bar{j}i\bar{\ell}}-2\Theta_{i\bar{j}k\bar{\ell}})+t^2(T_{ik}^{p}\overline{T}_{j\ell}^{q}h_{p\bar{q}}-h^{p\bar{q}}h_{m\bar{\ell}}h_{k\bar{n}}T_{ip}^{m}\overline{T}_{jq}^{n}),\end{aligned}$$

where $T_{ij}^k=\Gamma_{ij}^k-\Gamma_{ji}^k$ are the components of the torsion tensor of ∇^{Ch}. The Ricci curvatures are given by

$$\begin{aligned}\mathfrak{Ric}^{(1)}(\omega,t)&=\Theta^{(1)}-t(\partial\partial^*\omega+\bar\partial\bar\partial^*\omega),\\ \mathfrak{Ric}^{(2)}(\omega,t)&=\Theta^{(1)}-(1-2t)\sqrt{-1}\,\Lambda\partial\bar\partial\omega-(1-t)(\partial\partial^*\omega+\bar\partial\bar\partial^*\omega)\\&\qquad\qquad+(1-t)^2\sqrt{-1}\,T\boxdot T-t^2\sqrt{-1}\,T\circ T,\\ \mathfrak{Ric}^{(3)}(\omega,t)&=\Theta^{(1)}-t\sqrt{-1}\,\Lambda\partial\bar\partial\omega-(1-t)\partial\partial^*\omega-t\bar\partial\bar\partial^*\omega\\&\qquad\qquad+(t-t^2)\sqrt{-1}\,T\boxdot T+t^2T((\partial^*\omega)^{\#}),\\ \mathfrak{Ric}^{(4)}(\omega,t)&=\Theta^{(1)}-t\sqrt{-1}\,\Lambda\partial\bar\partial\omega-(1-t)\bar\partial\bar\partial^*\omega-t\partial\partial^*\omega\\&\qquad\qquad+(t-t^2)\sqrt{-1}\,T\boxdot T+t^2\overline{T}((\partial^*\omega)^{\#}).\end{aligned}\tag{1-7}$$

The scalar curvatures are related by

$$\begin{cases}s^{(1)}(\omega,t)=s_{\mathrm{C}}-2t\langle\partial\partial^*\omega,\omega\rangle,\\ s^{(2)}(\omega,t)=s_{\mathrm{C}}-(1-2t)\langle\partial\partial^*\omega,\omega\rangle-t^2(2|\partial\omega|^2+|\partial^*\omega|^2),\end{cases}\tag{1-8}$$

where we use for scalar curvatures the notations

$$s^{(1)}(\omega,t)=h^{i\bar{j}}h^{k\bar{\ell}}\mathfrak{R}_{i\bar{j}k\bar{\ell}}(\omega,t),\quad s^{(2)}(\omega,t)=h^{i\bar{\ell}}h^{k\bar{j}}\mathfrak{R}_{i\bar{j}k\bar{\ell}}(\omega,t).$$

Let s be the Riemannian scalar curvature of the background Riemannian manifold (M,g). One can deduce (for example from [Yang 2020, Corollary 3.7]) that

$$s=2s_{\mathrm{C}}-2\sqrt{-1}\,\partial^*\bar\partial^*\omega-\tfrac{1}{2}|T|^2.\tag{1-9}$$

By using this formula and (1-7), one can get scalar curvature relations for connections ∇^{Ch}, $\widehat{\nabla}^{\mathrm{LC}}$ and ∇^{SB} simultaneously.

Remark 1.9. We should point out that many parts of Corollary 1.8 are known in the literature. Indeed, J.-X. Fu and X.-C. Zhou [2022] established a variety of scalar curvature formulas for Gauduchon connections for general almost Hermitian manifolds, and (1-7), (1-8) are also obtained in their paper. For more discussions on scalar curvatures of almost Hermitian manifolds, we refer to [Apostolov and Drăghici 1999; Li 2010; Zhang 2012; Lejmi and Upmeier 2020; Chen and Zhang 2023; Fu and Zhou 2022]. One can also formulate curvature relations in the conformal setting; e.g., [Chiose et al. 2019; Angella et al. 2017; Lejmi and Maalaoui 2018; Chen et al. 2021].

As applications of curvature relations discussed above and methods developed in [Liu and Yang 2017; 2018; He et al. 2020; Yang 2016; 2019a; 2019b; 2020], we obtain:

Theorem 1.10. *Let (M,ω) be a compact Hermitian manifold and ∇^t be the Gauduchon connection on M. If $s^{(1)}(\omega,t)\ge 0$ for some $t>0$, then either*

(1) $\kappa(M)=-\infty$; *or*

(2) $\kappa(M)=0$ *and (M,ω) is conformally balanced and K_M is a holomorphic torsion*: $K_M^{\otimes m}=0$ *for some $m\in\mathbb{N}_+$.*

By using Theorem 1.10, we can classify t-Gauduchon–Ricci flat surfaces. Recall that a Hermitian manifold (M,ω) is called *t-Gauduchon–Ricci flat* if

$$\mathfrak{Ric}^{(1)}(\omega,t)=0 \quad \text{for some } t\in\mathbb{R}. \tag{1-10}$$

When $t=0,\frac{1}{2}$ and 1, it is also called *Chern–Ricci flat*, *Levi-Civita–Ricci flat* and *Strominger–Bismut–Ricci flat* respectively.

Theorem 1.11. *Let S be a compact complex surface. If it admits a t-Gauduchon–Ricci flat metric ω for some $t>0$, then S is a minimal surface lying in one of the following*:

(1) *an Enriques surface*;

(2) *a bielliptic surface*;

(3) *a $K3$-surface*;

(4) *a torus*;

(5) *a Hopf surface.*

Note that (1)–(4) are all Kähler Calabi–Yau surfaces. We also construct an explicit family of t-Gauduchon–Ricci flat metrics on diagonal Hopf surfaces.

Theorem 1.12. *Let ω_0 be the canonical metric on the standard Hopf manifold $\mathbb{S}^{2n-1}\times\mathbb{S}^1$ $(n\ge 2)$. Then the Hermitian metric*

$$\omega_t=\omega_0+4\left(\frac{2(n-1)t}{n}-1\right)\cdot\sqrt{-1}\,\partial\bar{\partial}\log|z|^2,\quad t>0 \tag{1-11}$$

is t-Gauduchon–Ricci flat. That is, $\mathfrak{Ric}^{(1)}(\omega_t,t)=0$ for each $t>0$.

When $t=\frac{1}{2}$, the Hermitian metric

$$\omega_{\mathrm{LC}}=\omega_0-\frac{4}{n}\cdot\sqrt{-1}\,\partial\bar{\partial}\log|z|^2 \tag{1-12}$$

is exactly the *Levi-Civita–Ricci flat* metric on $\mathbb{S}^{2n-1}\times\mathbb{S}^1$ constructed in [Liu and Yang 2017, Theorem 6.2]; see also [Liu and Yang 2018, Theorem 7.3]. When $t=1$, the Hermitian metric

$$\omega_{\mathrm{SB}} = \omega_0 - \frac{4(n-2)}{n} \cdot \sqrt{-1}\,\partial\bar{\partial}\log|z|^2 \tag{1-13}$$

is a *Strominger–Bismut–Ricci flat* metric on $\mathbb{S}^{2n-1} \times \mathbb{S}^1$.

The paper is organized as follows. In Section 2, we recall basic materials for readers' convenience and fix our notations. In Section 3, the spaces of connections are discussed. The curvatures of real connections in $\mathcal{A}_{g,J}$ are computed in Section 4 and Theorem 1.7 is obtained. In Section 5, we establish Theorem 1.3, Theorem 1.4, Theorem 1.5 and Theorem 1.6. In Section 6 we classify compact complex surfaces with t-Gauduchon–Ricci flat metrics and prove Theorem 1.10 and Theorem 1.11. We construct t-Gauduchon–Ricci flat metrics in Section 7, and in the Appendix, we include a detailed computation for Corollary 1.8.

2. Background materials

In this section, we give some background materials for readers' convenience.

2.1. ***Levi-Civita connection and its complexification.*** Let's recall some elementary settings. Let $(M, g, \nabla^{\mathrm{LC}})$ be a $2n$-dimensional Riemannian manifold with the Levi-Civita connection ∇^{LC}. The tangent bundle of M is also denoted by $T_{\mathbb{R}}M$. The Riemannian curvature tensor of $(M, g, \nabla^{\mathrm{LC}})$ is

$$R(X, Y, Z, W) = g(\nabla^{\mathrm{LC}}_X \nabla^{\mathrm{LC}}_Y Z - \nabla^{\mathrm{LC}}_Y \nabla^{\mathrm{LC}}_X Z - \nabla^{\mathrm{LC}}_{[X,Y]} Z, W)$$

for $X, Y, Z, W \in T_{\mathbb{R}}M$. Let $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C}$ be the complexification. One can extend the metric g and the Levi-Civita connection ∇^{LC} to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. Hence for any $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in T_{\mathbb{C}}M$,

$$R(aX, bY, cZ, dW) = abcd \cdot R(X, Y, Z, W).$$

Let (M, g, J) be an almost Hermitian manifold, i.e., $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$ with $J^2 = -1$, and for any $X, Y \in T_{\mathbb{R}}M$, $g(JX, JY) = g(X, Y)$. We define the Nijenhuis tensor $N_J : \Gamma(M, T_{\mathbb{R}}M) \times \Gamma(M, T_{\mathbb{R}}M) \to \Gamma(M, T_{\mathbb{R}}M)$ as

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

The almost complex structure J is *integrable* if $N_J \equiv 0$ and then (M, g, J) is a Hermitian manifold. We also extend J to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way: for any $X, Y \in T_{\mathbb{C}}M$, we still have $g(JX, JY) = g(X, Y)$. By Newlander–Nirenberg's theorem, there exist real coordinates $\{x^i, x^I\}$ such that $z^i = x^i + \sqrt{-1}x^I$ are local holomorphic coordinates on M. The Hermitian form $h : T_{\mathbb{C}}M \times T_{\mathbb{C}}M \to \mathbb{C}$ is given by

$$h(X, Y) := g(X, Y), \quad X, Y \in T_{\mathbb{C}}M. \tag{2-1}$$

By the J-invariant property of g,

$$h_{ij} := h\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\Big) = 0 \quad \text{and} \quad h_{\bar{i}\bar{j}} := h\Big(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\Big) = 0, \tag{2-2}$$

and

$$h_{i\bar{j}} := h\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\Big) = \tfrac{1}{2}(g_{ij} + \sqrt{-1}\, g_{iJ}). \tag{2-3}$$

It is obvious that $(h_{i\bar{j}})$ is a positive Hermitian matrix. Let ω be the fundamental 2-form associated to the J-invariant metric g, $\omega(X,Y) = g(JX,Y)$. In local complex coordinates, $\omega = \sqrt{-1}\, h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$. We shall use the components of the complexified curvature tensor R, for instance,

$$R_{i\bar{j}k\bar{\ell}} := R\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}\Big), \quad R_{ijk\ell} := R\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^\ell}\Big). \tag{2-4}$$

The components of complexified curvature tensor have the same properties as the components of the real curvature tensor. For instance, $R_{i\bar{j}k\ell} = -R_{\bar{j}ik\ell}$, $R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}}$, and in particular, the first Bianchi identity holds: $R_{i\bar{j}k\bar{\ell}} + R_{ik\bar{\ell}\bar{j}} + R_{i\bar{\ell}\bar{j}k} = 0$.

2.2. ***The induced Levi-Civita connection on $(T^{1,0}M, h)$.*** Since $T^{1,0}M$ is a subbundle of $T_{\mathbb{C}}M$, there is an induced connection $\widehat{\nabla}^{\mathrm{LC}}$ on $T^{1,0}M$ given by

$$\widehat{\nabla}^{\mathrm{LC}} = \pi \circ \nabla : \Gamma(M, T^{1,0}M) \xrightarrow{\nabla^{\mathrm{LC}}} \Gamma(M, T^*_{\mathbb{C}}M \otimes T_{\mathbb{C}}M) \xrightarrow{\pi} \Gamma(M, T^*_{\mathbb{C}}M \otimes T^{1,0}M).$$

Moreover, $\widehat{\nabla}^{\mathrm{LC}}$ is a metric connection on the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$ and it is determined by the relations

$$\widehat{\nabla}^{\mathrm{LC}}_{\partial/\partial z^i} \frac{\partial}{\partial z^k} := \widehat{\Gamma}^p_{ik} \frac{\partial}{\partial z^p} \quad \text{and} \quad \widehat{\nabla}^{\mathrm{LC}}_{\partial/\partial \bar{z}^j} \frac{\partial}{\partial z^k} := \widehat{\Gamma}^p_{\bar{j}k} \frac{\partial}{\partial z^p}, \tag{2-5}$$

where

$$\widehat{\Gamma}^k_{ij} = \tfrac{1}{2} h^{k\bar{\ell}} \Big(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j} \Big) \quad \text{and} \quad \widehat{\Gamma}^k_{\bar{i}j} = \tfrac{1}{2} h^{k\bar{\ell}} \Big(\frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^\ell} \Big). \tag{2-6}$$

The curvature tensor $\mathfrak{R} \in \Gamma(M, \Lambda^2 T^*_{\mathbb{C}}M \otimes T^{*1,0}M \otimes T^{1,0}M)$ of $\widehat{\nabla}^{\mathrm{LC}}$ is given by

$$\mathfrak{R}(X,Y)s = \widehat{\nabla}^{\mathrm{LC}}_X \widehat{\nabla}^{\mathrm{LC}}_Y s - \widehat{\nabla}^{\mathrm{LC}}_Y \widehat{\nabla}^{\mathrm{LC}}_X s - \widehat{\nabla}^{\mathrm{LC}}_{[X,Y]} s \tag{2-7}$$

for any $X, Y \in T_{\mathbb{C}}M$ and $s \in T^{1,0}M$. This curvature tensor has components

$$\begin{aligned}
\mathfrak{R}^\ell_{i\bar{j}k} &= -\Big(\frac{\partial \widehat{\Gamma}^\ell_{ik}}{\partial \bar{z}^j} - \frac{\partial \widehat{\Gamma}^\ell_{\bar{j}k}}{\partial z^i} + \widehat{\Gamma}^s_{ik} \widehat{\Gamma}^\ell_{\bar{j}s} - \widehat{\Gamma}^s_{\bar{j}k} \widehat{\Gamma}^\ell_{is} \Big), \\
\mathfrak{R}^\ell_{ijk} &= -\Big(\frac{\partial \widehat{\Gamma}^\ell_{ik}}{\partial z^j} - \frac{\partial \widehat{\Gamma}^\ell_{jk}}{\partial z^i} + \widehat{\Gamma}^s_{ik} \widehat{\Gamma}^\ell_{js} - \widehat{\Gamma}^s_{jk} \widehat{\Gamma}^\ell_{is} \Big), \\
\mathfrak{R}^\ell_{\bar{i}\bar{j}k} &= -\Big(\frac{\partial \widehat{\Gamma}^\ell_{\bar{i}k}}{\partial \bar{z}^j} - \frac{\partial \widehat{\Gamma}^\ell_{\bar{j}k}}{\partial \bar{z}^i} + \widehat{\Gamma}^s_{\bar{i}k} \widehat{\Gamma}^\ell_{\bar{j}s} - \widehat{\Gamma}^s_{\bar{j}k} \widehat{\Gamma}^\ell_{\bar{i}s} \Big).
\end{aligned} \tag{2-8}$$

With respect to the Hermitian metric h on $T^{1,0}M$, we use the convention

$$\mathfrak{R}_{\bullet\bullet k\bar{\ell}} := \sum_{s=1}^{n} \mathfrak{R}^{s}_{\bullet\bullet k} h_{s\bar{\ell}}. \tag{2-9}$$

Corollary 2.1. *We have the relations*

$$R^{\ell}_{ijk} = \mathfrak{R}^{\ell}_{ijk}, \quad R^{\ell}_{\bar{i}\bar{j}k} = \mathfrak{R}^{\ell}_{\bar{i}\bar{j}k},$$

and

$$R^{\ell}_{i\bar{j}k} = -\left(\frac{\partial \widehat{\Gamma}^{\ell}_{ik}}{\partial \bar{z}^{j}} - \frac{\partial \widehat{\Gamma}^{\ell}_{\bar{j}k}}{\partial z^{i}} + \widehat{\Gamma}^{s}_{ik}\widehat{\Gamma}^{\ell}_{\bar{j}s} - \widehat{\Gamma}^{s}_{\bar{j}k}\widehat{\Gamma}^{\ell}_{si} - \widehat{\Gamma}^{\ell}_{\bar{s}i}\cdot\overline{\widehat{\Gamma}^{s}_{\bar{k}j}}\right) = \mathfrak{R}^{\ell}_{i\bar{j}k} + \widehat{\Gamma}^{\ell}_{i\bar{s}}\cdot\overline{\widehat{\Gamma}^{s}_{j\bar{k}}}. \tag{2-10}$$

2.3. ***Curvature of the Chern connection on $(T^{1,0}M, h)$.*** On the Hermitian holomorphic tangent bundle $(T^{1,0}M, h)$, the Chern connection ∇^{Ch} is the unique connection which is compatible with the holomorphic structure and also the Hermitian metric. The curvature tensor of ∇^{Ch} is denoted by Θ and it has components

$$\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{p\bar{q}} \frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j} \frac{\partial h_{k\bar{q}}}{\partial z^i}. \tag{2-11}$$

It is well known that the (*first*) *Chern–Ricci curvature*

$$\Theta^{(1)} := \sqrt{-1}\,\Theta^{(1)}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \tag{2-12}$$

represents the first Chern class $c_1(M)$ of M where

$$\Theta^{(1)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log\det h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j}. \tag{2-13}$$

The *second Chern–Ricci curvature* $\Theta^{(2)} = \sqrt{-1}\,\Theta^{(2)}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ has components

$$\Theta^{(2)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{k\bar{\ell}i\bar{j}}. \tag{2-14}$$

The *Chern scalar curvature* s_{C} of the Chern curvature tensor Θ is defined by

$$s_{\mathrm{C}} = h^{i\bar{j}} h^{k\bar{\ell}} \Theta_{i\bar{j}k\bar{\ell}}. \tag{2-15}$$

Similarly, we can define

$$\Theta^{(3)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{i\bar{\ell}k\bar{j}}, \quad \Theta^{(4)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{k\bar{j}i\bar{\ell}} \quad \text{and} \quad s^{(2)}_{\mathrm{C}} = h^{i\bar{j}}\Theta^{(3)}_{i\bar{j}} = h^{i\bar{j}}\Theta^{(4)}_{i\bar{j}}. \tag{2-16}$$

The following lemma is well known; see for instance [Yang 2020, Lemma 3.6].

Lemma 2.2. *Let (X, ω) be a compact Hermitian manifold. Then*

$$\langle \bar{\partial}\bar{\partial}^*\omega, \omega\rangle = |\bar{\partial}^*\omega|^2 - \sqrt{-1}\,\partial^*\bar{\partial}^*\omega. \tag{2-17}$$

In particular, if ω is a Gauduchon metric, we have

$$\langle \bar\partial\bar\partial^*\omega, \omega\rangle = |\bar\partial^*\omega|^2. \tag{2-18}$$

According to [Liu and Yang 2017, Theorem 4.1], we have:

Theorem 2.3. *The Chern–Ricci curvatures are related by*

$$\begin{cases} \Theta^{(2)} = \Theta^{(1)} - \sqrt{-1}\,\Lambda(\partial\bar\partial\omega) - (\partial\partial^*\omega + \bar\partial\bar\partial^*\omega) + \sqrt{-1}\,T \boxdot \overline{T}, \\ \Theta^{(3)} = \Theta^{(1)} - \partial\partial^*\omega, \\ \Theta^{(4)} = \Theta^{(1)} - \bar\partial\bar\partial^*\omega, \end{cases} \tag{2-19}$$

where $T \boxdot \overline{T} := h^{p\bar q} h_{k\bar\ell} T^k_{ip} \cdot \overline{T}^\ell_{jq}\, dz^i \wedge d\bar z^j$. The scalar curvatures are related by

$$\begin{cases} s^{(2)}_{\mathrm{C}} = s_{\mathrm{C}} - \langle \partial\partial^*\omega, \omega\rangle, \\ s = 2s_{\mathrm{C}} - 2\sqrt{-1}\,\partial^*\bar\partial^*\omega - \frac{1}{2}|T|^2, \end{cases} \tag{2-20}$$

where s is the scalar curvature of the background Riemannian metric.

3. Space $\mathcal{A}_{g,J}$ of connections preserving metrics and complex structures

3.1. ***Spaces of real connections and Hermitian connections.*** Let (M, g, J) be a Riemannian manifold with a compatible integrable complex structure J. We consider two spaces of affine connections on the real tangent bundle $T_{\mathbb{R}}M$:

$$\mathcal{A}_g = \{\nabla \mid \nabla \text{ is an affine connection on } T_{\mathbb{R}}M \text{ satisfying } \nabla g = 0\},$$

$$\mathcal{A}_{g,J} = \{\nabla \mid \nabla \text{ is an affine connection on } T_{\mathbb{R}}M \text{ satisfying } \nabla g = 0 \text{ and } \nabla J = 0\}.$$

Clearly, $\iota : \mathcal{A}_{g,J} \hookrightarrow \mathcal{A}_g$. In the following, we shall give a characterization of connections in $\mathcal{A}_{g,J}$.

Lemma 3.1. *Let $\nabla^0 \in \mathcal{A}_{g,J}$. Suppose $\nabla = \nabla^0 + A$ with $A \in \Gamma\big(M, \Omega^1(\mathrm{End}(T_{\mathbb{R}}M))\big)$. Then $\nabla \in \mathcal{A}_{g,J}$ if and only if*

$$\begin{cases} [A, J] = 0, \\ g(AX, Y) + g(X, AY) = 0 \quad \text{for any } X, Y \in \Gamma(M, T_{\mathbb{R}}M). \end{cases}$$

Proof. It is easy to see that

$$\nabla J = \nabla^0 J + [A, J] = [A, J]$$

and

$$g(\nabla X, Y) + g(X, \nabla Y) - d(g(X, Y)) = g(AX, Y) + g(X, AY)$$

since $\nabla^0 J = 0$ and $\nabla^0 g = 0$. □

Let

$$A_{\mathbb{C}} \in \Gamma\big(M, \Omega^1(\mathrm{End}(T_{\mathbb{C}}M))\big)$$

be the complexification of $A \in \Gamma\big(M, \Omega^1(\mathrm{End}(T_{\mathbb{R}}M))\big)$. According to the decomposition $T_{\mathbb{C}}M = T^{1,0}M \bigoplus T^{0,1}M$, $A_{\mathbb{C}}$ has a matrix representation $\left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right]$, and $[A, J] = 0$ implies $A_{12} = A_{21} = 0$. Hence, we have $A_{\mathbb{C}} = A_{11} + A_{22}$ where $A_{11} \in \Gamma\big(M, \Omega^1(\mathrm{End}(T^{1,0}M))\big)$ and $A_{22} \in \Gamma\big(M, \Omega^1(\mathrm{End}(T^{0,1}M))\big)$. Let $A_{11} = \theta_1 + \theta_2$ and $A_{22} = \theta_3 + \theta_4$ where

$$\theta_1 \in \Gamma\big(M, \Omega^{1,0}(\mathrm{End}(T^{1,0}M))\big), \quad \theta_2 \in \Gamma\big(M, \Omega^{0,1}(\mathrm{End}(T^{1,0}M))\big),$$
$$\theta_3 \in \Gamma\big(M, \Omega^{1,0}(\mathrm{End}(T^{0,1}M))\big), \quad \theta_4 \in \Gamma\big(M, \Omega^{0,1}(\mathrm{End}(T^{0,1}M))\big).$$

Since A is real and $\nabla h = 0$, for any $X, Y \in \Gamma(M, T^{1,0}M)$, we have

$$\begin{cases} h(\theta_1 X, \overline{Y}) + h(X, \theta_3 \overline{Y}) = 0, \\ h(\theta_2 X, \overline{Y}) + h(X, \theta_4 \overline{Y}) = 0, \\ \bar{\theta_1} = \theta_4, \\ \bar{\theta_2} = \theta_3. \end{cases} \tag{3-1}$$

Hence, A is determined by θ_1. If we write $\theta_1 = \theta_{ij}^k \, dz^i \otimes dz^j \otimes (\partial/\partial z^k)$, then the complexification of $\nabla \in \mathcal{A}_{g,J}$ is given by

$$\begin{aligned} \nabla_{\partial/\partial z^i} \frac{\partial}{\partial z^j} &= \nabla^0_{\partial/\partial z^i} \frac{\partial}{\partial z^j} + \theta_{ij}^k \frac{\partial}{\partial z^k}, \\ \nabla_{\partial/\partial z^i} \frac{\partial}{\partial \bar{z}^j} &= \nabla^0_{\partial/\partial z^i} \frac{\partial}{\partial \bar{z}^j} - h_{q\bar{j}} h^{p\bar{k}} \theta_{ip}^q \frac{\partial}{\partial \bar{z}^k}, \end{aligned} \tag{3-2}$$

and their conjugations. Therefore, we have:

Proposition 3.2. $\mathcal{A}_{g,J} \cong \Gamma\big(M, \Omega^1(\mathrm{End}(T^{1,0}M))\big)$.

On the holomorphic tangent bundle $T^{1,0}M$, we consider the space

$$\mathcal{B}_h = \{\nabla \mid \nabla \text{ is an affine connection on } T^{1,0}M \text{ satisfying } \nabla h = 0\}. \tag{3-3}$$

It is easy to see that the Chern connection ∇^{Ch} is in $\mathcal{B}_h$. Moreover, for any $\nabla \in \mathcal{A}_{g,J}$, the restriction $\widehat{\nabla} : \Gamma(M, T^{1,0}M) \to \Gamma(M, \Omega^1(T^{1,0}M))$ of its complexification $\nabla : \Gamma(M, T_{\mathbb{C}}M) \to \Gamma(M, \Omega^1(T_{\mathbb{C}}M))$ is in $\mathcal{B}_h$. Hence, there is a natural map

$$\rho : \mathcal{A}_{g,J} \to \mathcal{B}_h. \tag{3-4}$$

Corollary 3.3. *$\rho : \mathcal{A}_{g,J} \to \mathcal{B}_h$ is an isomorphism.*

Indeed, for any $\nabla \in \mathcal{B}_h$, there exists some $B \in \Gamma\big(M, \Omega^1(\mathrm{End}(T^{1,0}M))\big)$ such that $\nabla = \nabla^{\mathrm{Ch}} + B$. We have the decomposition $B = B_1 + B_2$ where

$$B_1 = \theta_{ij}^k \, dz^i \otimes dz^j \otimes \frac{\partial}{\partial z^k} \in \Gamma\big(M, \Omega^{1,0}(\mathrm{End}(T^{1,0}M))\big)$$

and

$$B_2 = \eta_{\bar{i}j}^k \, d\bar{z}^i \otimes dz^j \otimes \frac{\partial}{\partial z^k} \in \Gamma\big(M, \Omega^{0,1}(\mathrm{End}(T^{1,0}M))\big).$$

Since $\nabla h = 0$, we deduce

$$\theta_{ij}^{k} h_{k\bar{\ell}} + \bar{\eta}_{\bar{i}\ell}^{p} h_{j\bar{p}} = 0. \tag{3-5}$$

Hence, the connection $\nabla \in \mathcal{B}_h$ is given by

$$\nabla_{\partial/\partial z^i} \frac{\partial}{\partial z^j} = \nabla^{\mathrm{Ch}}_{\partial/\partial z^i} \frac{\partial}{\partial z^j} + \theta_{ij}^{k} \frac{\partial}{\partial z^k}, \quad \nabla_{\partial/\partial \bar{z}^i} \frac{\partial}{\partial z^j} = -h_{j\bar{q}} h^{k\bar{p}} \bar{\theta}_{ip}^{q} \frac{\partial}{\partial z^k}. \tag{3-6}$$

That means $\nabla \in \mathcal{B}_h$ is determined by B_1, i.e., $\mathcal{B}_h \cong \Gamma\big(M, \Omega^{1,0}(\mathrm{End}(T^{1,0}M))\big)$. By using similar interpretations, one can show:

Corollary 3.4. $\mathcal{A}_g \cong \mathcal{B}_h \times \Gamma\big(M, \Omega^{1}(\mathrm{Hom}(T^{1,0}M, T^{0,1}M))\big)$.

3.2. ***Real correspondences for metric connections on $T^{1,0}M$.*** On a Hermitian manifold (M, g, J), we have the following diagram for spaces of connections:

$$\begin{array}{ccc} \mathcal{A}_{g,J} & \xrightarrow{\iota} & \mathcal{A}_g \\ & \searrow^{\rho} & \downarrow{\scriptstyle \pi} \\ & & \mathcal{B}_h \end{array}$$

As we discussed in the previous section, $\hat{\nabla}^{\mathrm{LC}}$, ∇^{Ch} and ∇^{SB} are all in $\mathcal{B}_h$. We shall consider the preimage of them under the isomorphism $\rho : \mathcal{A}_{g,J} \to \mathcal{B}_h$. By definition, $\nabla^{\mathrm{LC}} \in \mathcal{A}_{g,J}$ if and only if $\nabla^{\mathrm{LC}} J = 0$, that is if (M, g, J) is a Kähler manifold. It is a natural question to find the preimage when it is not Kähler. The following lemma is well known.

Lemma 3.5. *Let (M, g, J) be a Hermitian manifold. Then*:

- $\rho^{-1}(\nabla^{\mathrm{Ch}}) \in \mathcal{A}_{g,J}$ *is given by*

$$g(\nabla_X Y, Z) = g(\nabla^{\mathrm{LC}}_X Y, Z) - \tfrac{1}{2} d\omega(JX, Y, Z). \tag{3-7}$$

- $\rho^{-1}(\hat{\nabla}^{\mathrm{LC}}) \in \mathcal{A}_{g,J}$ *is given by*

$$g(\nabla_X Y, Z) = g(\nabla^{\mathrm{LC}}_X Y, Z) + \tfrac{1}{4} d\omega(JX, JY, JZ) - \tfrac{1}{4} d\omega(JX, Y, Z). \tag{3-8}$$

- $\rho^{-1}(\nabla^{\mathrm{SB}}) \in \mathcal{A}_{g,J}$ *is given by*

$$g(\nabla_X Y, Z) = g(\nabla^{\mathrm{LC}}_X Y, Z) + \tfrac{1}{2} d\omega(JX, JY, JZ). \tag{3-9}$$

Moreover, there exists a unique linear family $\{\nabla^t\}_{t\in\mathbb{R}} \subset \mathcal{A}_{g,J}$ such that

$$\rho(\nabla^0) = \nabla^{\mathrm{Ch}}, \quad \rho(\nabla^{\frac{1}{2}}) = \hat{\nabla}^{\mathrm{LC}} \quad \textit{and} \quad \rho(\nabla^1) = \nabla^{\mathrm{SB}},$$

and it is given by

$$g(\nabla^t_X Y, Z) = g(\nabla^{\mathrm{LC}}_X Y, Z) + \tfrac{1}{2} t\, d\omega(JX, JY, JZ) + \tfrac{1}{2}(t-1)\, d\omega(JX, Y, Z). \tag{3-10}$$

4. Curvatures of connections in $\mathcal{A}_{g,J}$

Recall that the real Chern connection $\nabla^{\mathrm{Ch},\mathbb{R}} = \rho^{-1}(\nabla^{\mathrm{Ch}}) \in \mathcal{A}_{g,J}$ is given in (3-7) and its complexification is determined by

$$\nabla^{\mathrm{Ch},\mathbb{R}}_{\partial/\partial z^i}\frac{\partial}{\partial z^j} = \Gamma^k_{ij}\frac{\partial}{\partial z^k}, \quad \nabla^{\mathrm{Ch},\mathbb{R}}_{\partial/\partial \bar z^i}\frac{\partial}{\partial z^j} = 0. \tag{4-1}$$

For any $\nabla \in \mathcal{A}_{g,J}$, there exists some $A \in \Gamma\big(M, \Omega^1(\mathrm{End}(T_{\mathbb{R}}M))\big)$ such that

$$\nabla = \nabla^{\mathrm{Ch},\mathbb{R}} + A$$

and its complexification $\nabla : \Gamma(M, T_{\mathbb{C}}M) \to \Gamma(M, \Omega^1(T_{\mathbb{C}}M))$ is given by

$$\begin{aligned} &\nabla_{\partial/\partial z^i}\frac{\partial}{\partial z^j} = (\Gamma^k_{ij} + \theta^k_{ij})\frac{\partial}{\partial z^k}, \qquad &&\nabla_{\partial/\partial \bar z^i}\frac{\partial}{\partial \bar z^j} = (\bar\Gamma^k_{ij} + \bar\theta^k_{ij})\frac{\partial}{\partial \bar z^k},\\ &\nabla_{\partial/\partial \bar z^i}\frac{\partial}{\partial z^j} = -h_{j\bar q}h^{k\bar p}\bar\theta^q_{ip}\frac{\partial}{\partial z^k}, &&\nabla_{\partial/\partial z^i}\frac{\partial}{\partial \bar z^j} = -h_{q\bar j}h^{p\bar k}\theta^q_{ip}\frac{\partial}{\partial \bar z^k}. \end{aligned} \tag{4-2}$$

Moreover, its restriction to $T^{1,0}M$, $\hat\nabla = \rho(\nabla) \in \mathcal{B}_h$ is

$$\hat\nabla_{\partial/\partial z^i}\frac{\partial}{\partial z^j} = (\Gamma^k_{ij} + \theta^k_{ij})\frac{\partial}{\partial z^k}, \quad \hat\nabla_{\partial/\partial \bar z^i}\frac{\partial}{\partial z^j} = -h_{j\bar q}h^{k\bar p}\bar\theta^q_{ip}\frac{\partial}{\partial z^k}. \tag{4-3}$$

To make this correspondence clearer for the readers, we write ∇^θ for $\nabla \in \mathcal{A}_{g,J}$ defined by (4-2), and $\hat\nabla^\theta$ for $\hat\nabla \in \mathcal{B}_h$ defined in (4-3). By using similar notations as in Section 2, the curvatures of ∇^θ and $\hat\nabla^\theta$ are denoted by $R^\theta, \mathfrak{R}^\theta$. More precisely,

$$R^\theta(X,Y,Z,W) = h(\nabla^\theta_X\nabla^\theta_Y Z - \nabla^\theta_Y\nabla^\theta_X Z - \nabla^\theta_{[X,Y]}Z, W) \tag{4-4}$$

for $X, Y, Z, W \in T_{\mathbb{C}}M$ and

$$\mathfrak{R}^\theta(X,Y,Z,W) = h(\hat\nabla^\theta_X\hat\nabla^\theta_Y Z - \hat\nabla^\theta_Y\hat\nabla^\theta_X Z - \hat\nabla^\theta_{[X,Y]}Z, W) \tag{4-5}$$

for $X, Y \in T_{\mathbb{C}}M$, $Z \in T^{1,0}M$, $W \in T^{0,1}M$. We also use conventions $R^\theta_{i\bar jk\bar\ell}$ and $\mathfrak{R}^\theta_{i\bar jk\bar\ell}$ for their components.

Proposition 4.1. *For any $\nabla^\theta \in \mathcal{A}_{g,J}$ with $\theta \in \Gamma\big(M, \Omega^{1,0}(\mathrm{End}(T^{1,0}M))\big)$, the curvature tensors R^θ and $\mathfrak{R}^\theta$ are determined by*

$$\begin{aligned} R^\theta_{i\bar jk\bar\ell} &= \mathfrak{R}^\theta_{i\bar jk\bar\ell}\\ &= \Theta_{i\bar jk\bar\ell} - \left(h_{k\bar p}\frac{\partial\bar\theta^p_{j\ell}}{\partial z^i} + h_{p\bar\ell}\frac{\partial\theta^p_{ik}}{\partial\bar z^j}\right) + (\theta^p_{ik}\bar\theta^q_{j\ell}h_{p\bar q} - h^{m\bar n}\theta^p_{im}\bar\theta^q_{jn}h_{p\bar\ell}h_{k\bar q}), \end{aligned}$$

$$\begin{aligned} &R^\theta_{ijk\bar\ell} = \mathfrak{R}^\theta_{ijk\bar\ell}\\ &= \left(\frac{\partial\theta^m_{jk}}{\partial z^i} - \frac{\partial\theta^m_{ik}}{\partial z^j} + \Gamma^s_{jk}\theta^\ell_{is} - \Gamma^\ell_{js}\theta^s_{ik} + \Gamma^\ell_{is}\theta^s_{jk} - \Gamma^s_{ik}\theta^\ell_{js}\right)h_{m\bar\ell} + (\theta^m_{ip}\theta^p_{jk} - \theta^p_{ik}\theta^m_{jp})h_{m\bar\ell}. \end{aligned}$$

Proof. Since $\nabla^\theta J = 0$, we have

$$R^\theta_{i\bar{j}k\bar{\ell}} = \mathfrak{R}^\theta_{i\bar{j}k\bar{\ell}}, \quad R^\theta_{ijk\bar{\ell}} = \mathfrak{R}^\theta_{ijk\bar{\ell}}.$$

Moreover,

$$\begin{aligned}\widehat{\nabla}^\theta_{\partial/\partial z^i}\widehat{\nabla}^\theta_{\partial/\partial z^j}\frac{\partial}{\partial z^k} &= \widehat{\nabla}^\theta_{\partial/\partial z^i}\Big(\Gamma^\ell_{jk}\frac{\partial}{\partial z^\ell} + \theta^\ell_{jk}\frac{\partial}{\partial z^\ell}\Big)\\ &= \bigg(\frac{\partial\Gamma^\ell_{jk}}{\partial z^i} + \Gamma^s_{jk}(\Gamma^\ell_{is} + \theta^\ell_{is}) + \frac{\partial\theta^\ell_{jk}}{\partial z^i} + \theta^s_{jk}(\Gamma^\ell_{is} + \theta^\ell_{is})\bigg)\frac{\partial}{\partial z^\ell},\end{aligned}$$

and

$$(\mathfrak{R}^\theta)^\ell_{ijk} = \bigg(\frac{\partial\theta^\ell_{jk}}{\partial z^i} - \frac{\partial\theta^\ell_{ik}}{\partial z^j} + \Gamma^s_{jk}\theta^\ell_{is} - \Gamma^\ell_{js}\theta^s_{ik} + \Gamma^\ell_{is}\theta^s_{jk} - \Gamma^s_{ik}\theta^\ell_{js}\bigg) + (\theta^s_{jk}\theta^\ell_{is} - \theta^s_{ik}\theta^\ell_{js}).$$

Similarly,

$$\begin{aligned}&h\Big(\widehat{\nabla}^\theta_{\partial/\partial z^i}\widehat{\nabla}^\theta_{\partial/\partial \bar{z}^j}\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\ &\qquad = \frac{\partial}{\partial z^i}h\Big(\widehat{\nabla}^\theta_{\partial/\partial \bar{z}^j}\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}\Big) - h\Big(\widehat{\nabla}^\theta_{\partial/\partial \bar{z}^j}\frac{\partial}{\partial z^k}, \widehat{\nabla}^\theta_{\partial/\partial z^i}\frac{\partial}{\partial \bar{z}^\ell}\Big)\\ &\qquad = \frac{\partial}{\partial z^i}h\Big(-h_{k\bar{q}}h^{s\bar{p}}\bar{\theta}^q_{jp}\frac{\partial}{\partial z^s}, \frac{\partial}{\partial \bar{z}^\ell}\Big) - h\Big(h_{k\bar{p}}h^{s\bar{q}}\bar{\theta}^p_{jq}\frac{\partial}{\partial z^s}, h_{p\bar{\ell}}h^{q\bar{t}}\theta^p_{iq}\frac{\partial}{\partial \bar{z}^t}\Big)\\ &\qquad = -\frac{\partial}{\partial z^i}(\bar{\theta}^s_{j\ell}h_{k\bar{s}}) - h^{q\bar{p}}\theta^s_{iq}\bar{\theta}^t_{jp}h_{s\bar{\ell}}h_{k\bar{t}}\\ &\qquad = -\frac{\partial}{\partial z^i}\bar{\theta}^s_{j\ell}h_{k\bar{s}} - \bar{\theta}^s_{j\ell}\Gamma^t_{ik}h_{t\bar{s}} - h^{q\bar{p}}\theta^s_{iq}\bar{\theta}^t_{jp}h_{s\bar{\ell}}h_{k\bar{t}},\end{aligned}$$

and

$$\begin{aligned}&h\Big(\widehat{\nabla}^\theta_{\partial/\partial \bar{z}^j}\widehat{\nabla}^\theta_{\partial/\partial z^i}\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\ &\quad = h\Big(\widehat{\nabla}^\theta_{\partial/\partial \bar{z}^j}\Big((\Gamma^s_{ik} + \theta^s_{ik})\frac{\partial}{\partial z^s}\Big), \frac{\partial}{\partial \bar{z}^\ell}\Big)\\ &\quad = h\Big(\frac{\partial}{\partial \bar{z}^j}(\Gamma^s_{ik} + \theta^s_{ik})\frac{\partial}{\partial z^s}, \frac{\partial}{\partial \bar{z}^\ell}\Big) + h\Big((\Gamma^s_{ik} + \theta^s_{ik})(-h_{s\bar{p}}h^{t\bar{q}}\bar{\theta}^p_{jq})\frac{\partial}{\partial z^t}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\ &\quad = h_{s\bar{\ell}}\frac{\partial\Gamma^s_{ik}}{\partial \bar{z}^j} + h_{s\bar{\ell}}\frac{\partial\theta^s_{ik}}{\partial \bar{z}^j} + (\Gamma^s_{ik} + \theta^s_{ik})(-h_{s\bar{p}}\bar{\theta}^p_{j\ell}).\end{aligned}$$

Hence, we obtain the curvature formulas in Proposition 4.1. □

By using Proposition 4.1, one has

Corollary 4.2. *The first Ricci curvature of* $\widehat{\nabla}^\theta$ *is*

$$\mathfrak{Ric}^{(1)}(\theta) = \Theta^{(1)} - \sqrt{-1}\,(\partial\bar{\theta}_1 - \bar{\partial}\theta_1),$$

where $\theta_1 = \theta^k_{ik}\, dz^i$.

There are two important linear families in $\mathcal{A}_{g,J}$. One is the Gauduchon family defined in (3-10) and in this case,

$$\theta_{ij}^k = t \cdot T_{ij}^k \tag{4-6}$$

and their curvatures are given in Corollary 1.8. The other family is $\theta = t\cdot\eta\otimes\mathrm{Id}_{T^{1,0}M}$ for some form $\eta = \eta_i\, dz^i \in \Gamma(M, \Omega_M^{1,0})$, and

$$\theta_{ij}^k = t \cdot \eta_i \delta_j^k. \tag{4-7}$$

Corollary 4.3. *The curvature formulas are*

$$\mathfrak{R}_{i\bar{j}k\bar{\ell}}(\theta) = \Theta_{i\bar{j}k\bar{\ell}} - t\left(\frac{\partial\bar{\eta}_j}{\partial z^i} + \frac{\partial\eta_i}{\partial\bar{z}^j}\right)h_{k\bar{\ell}}, \quad \mathfrak{R}_{ijk\bar{\ell}}(\theta) = t\left(\frac{\partial\eta_j}{\partial z^i} - \frac{\partial\eta_i}{\partial z^j}\right)h_{k\bar{\ell}},$$
$$\mathfrak{Ric}^{(1)}(\theta) = \Theta^{(1)} - nt\sqrt{-1}\,(\partial\bar{\eta} - \bar{\partial}\eta).$$

Remark 4.4. When $d\eta = 0$, one has

$$\mathfrak{R}_{i\bar{j}k\bar{\ell}}(\theta) = \Theta_{i\bar{j}k\bar{\ell}} \quad \text{and} \quad \mathfrak{R}_{ijk\bar{\ell}}(\theta) = 0$$

for any $t \in \mathbb{R}$.

5. Geometry of real Chern–Einstein metrics

In this section, we investigate real Chern–Einstein metrics and prove Theorem 1.3, Theorem 1.4, Theorem 1.5 and Theorem 1.6. Recall that the *real Chern–Ricci curvature* $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is defined by using the Riemannian metric g:

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)(X,Y) = \sum_{i=1}^{2n} R^{\mathrm{Ch},\mathbb{R}}(X, e_i, e_i, Y),$$

where $\{e_i\}_{i=1}^{2n}$ is an orthonormal frame with respect to g.

Proposition 5.1. *The complexification of* $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ *is given by*

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) = \Theta_{i\bar{j}}^{(3)}\, dz^i \otimes d\bar{z}^j + \Theta_{i\bar{j}}^{(4)}\, d\bar{z}^j \otimes dz^i \tag{5-1}$$

where $\Theta_{i\bar{j}}^{(3)}$ *and* $\Theta_{i\bar{j}}^{(4)}$ *are defined in* (2-16).

Proof. By using Theorem 1.7 for $\theta = 0$, we have

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial\bar{z}^j}\right) = h^{k\bar{\ell}} R_{ki\bar{j}\bar{\ell}} + h^{\bar{\ell}k} R_{\bar{\ell}i\bar{j}k} = h^{k\bar{\ell}} R_{i\bar{\ell}k\bar{j}} = h^{k\bar{\ell}}\Theta_{i\bar{\ell}k\bar{j}} = \Theta_{i\bar{j}}^{(3)},$$

where R stands for $R^{\mathrm{Ch},\mathbb{R}}$. Similarly, we have

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)\left(\frac{\partial}{\partial\bar{z}^j}, \frac{\partial}{\partial z^i}\right) = \Theta_{i\bar{j}}^{(4)}$$

and $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)(\partial/\partial z^i, \partial/\partial z^j) = \mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)(\partial/\partial\bar{z}^j, \partial/\partial\bar{z}^i) = 0.$ □

Definition 5.2. $(M, g, J, \nabla^{\mathrm{Ch},\mathbb{R}})$ is called *real Chern–Einstein* if

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) = \lambda g \quad \text{for some } \lambda \in \mathbb{R}. \tag{5-2}$$

If $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) = 0$, it is also called *real Chern–Ricci flat*. Moreover, $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ has positive real Chern–Ricci curvature if $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) \in \Gamma(M, \mathrm{Sym}^{\otimes 2} T_{\mathbb{R}} M)$ and it is positive definite. The negativity can be defined similarly.

Theorem 5.3. *Let* (M, g, J) *be a Hermitian manifold. Then* $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ *is positive if and only if*

$$\Theta^{(1)} - \partial\partial^*\omega \tag{5-3}$$

is a positive definite Hermitian $(1,1)$*-form. In particular,* $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ *is real Chern–Einstein with constant* $\lambda \in \mathbb{R}$ *if and only if*

$$\Theta^{(1)} - \partial\partial^*\omega = \lambda\omega, \tag{5-4}$$

where $\Theta^{(1)}$ *is the first Chern–Ricci curvature.*

Proof. By (5-1), if $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) \in \Gamma(M, \mathrm{Sym}^{\otimes 2} T_{\mathbb{R}} M)$, then $\Theta^{(3)}_{i\bar{j}} = \Theta^{(4)}_{i\bar{j}}$. Therefore,

$$\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) = \Theta^{(3)}_{i\bar{j}} (dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i).$$

Hence, the real Chern–Ricci curvature is positive definite if and only if $\Theta^{(3)}_{i\bar{j}} v^i \bar{v}^j > 0$ for every nonzero vector (v^i). By Theorem 2.3 or Corollary 1.8 (when $t = 0$), we know

$$\Theta^{(3)} = \sqrt{-1}\,\Theta^{(3)}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = \Theta^{(1)} - \partial\partial^*\omega$$

is a positive definite Hermitian $(1,1)$-form. In particular, $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ is real Chern–Einstein with constant $\lambda \in \mathbb{R}$ if and only if (5-4) holds. □

Proof of Theorem 1.3. By applying ∂ to (5-4), we have $\lambda\,\partial\omega = 0$. Hence if $\lambda \neq 0$, $d\omega = 0$ and (M, g, J) is Kähler–Einstein. □

Proof of Theorem 1.4. Suppose $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) > 0$. By Theorem 5.3 we deduce the Hermitian $(1,1)$-form

$$\Omega_0 := \Theta^{(1)} - \partial\partial^*\omega > 0$$

and $\partial\Omega_0 = 0$. Since Ω_0 is also real, we obtain $d\Omega_0 = 0$. Hence, Ω_0 is a Kähler form and (M, J) is a Kähler manifold. Moreover, the $(1,1)$-form $\partial\partial^*\omega$ is both d-closed and ∂-exact. By the $\partial\bar{\partial}$-lemma on the Kähler manifold (M, J), there exists some $f \in C^\infty(M, \mathbb{R})$ such that $\partial\partial^*\omega = -\sqrt{-1}\,\partial\bar{\partial} f$. Hence $\Theta^{(1)} = \Omega_0 + \sqrt{-1}\,\partial\bar{\partial} f$ and so $c_1(M, J) > 0$. The proof for $\mathrm{Ric}(\nabla^{\mathrm{Ch},\mathbb{R}}, g) < 0$ is similar. □

When $\lambda = 0$, we have the following result.

Corollary 5.4. *Let* (M, g, J) *be a Hermitian manifold. Then* $(\nabla^{\mathrm{Ch},\mathbb{R}}, g)$ *is real Chern–Ricci flat if and only if*

$$\Theta^{(1)} - \partial\partial^*\omega = \Theta^{(1)} - \bar{\partial}\bar{\partial}^*\omega = 0.$$

Proof of Theorem 1.5. On the standard Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ with canonical metric

$$\omega_0 = \sqrt{-1}\, h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = \frac{4\delta_{i\bar{j}}}{|z|^2} \sqrt{-1}\, dz^i \wedge d\bar{z}^j,$$

we know the metric

$$\omega = \omega_0 - \frac{4}{n} \cdot \sqrt{-1}\, \partial\bar{\partial} \log|z|^2 \tag{5-5}$$

is Levi-Civita–Ricci flat; see [Liu and Yang 2017, Theorem 6.2] or Theorem 1.12 with $t = \frac{1}{2}$. On the other hand, one can show directly (see also [Liu and Yang 2017, Theorem 6.2]) that

$$\partial\partial^*\omega = n\sqrt{-1}\, \partial\bar{\partial} \log|z|^2$$

where ∂^* is taken with respect to ω. By Corollary 5.4, we deduce that $(\nabla^{\mathrm{Ch},\mathbb{R}}, \omega)$ is real Chern–Ricci flat. □

Proof of Theorem 1.6. By Theorem 5.3, we have $\Theta^{(1)} = \partial\partial^*\omega$. Hence, the real $(1,1)$-form $\partial\partial^*\omega$ is d-closed and ∂-exact.

If (M, J) is Kähler, by the $\partial\bar{\partial}$-lemma, there exists some $f \in C^\infty(M, \mathbb{R})$ such that $\Theta^{(1)} = \partial\partial^*\omega = \sqrt{-1}\, \partial\bar{\partial} f$. In particular, $c_1(M) = 0$ on the Kähler manifold (M, J). By the Calabi–Yau theorem, there exists a Kähler–Ricci flat metric $\tilde{\omega}$ which is possibly different from ω.

When (M, J) is not Kähler, by (1-8), we have

$$s^{(1)}\big(\omega, \tfrac{1}{2}\big) = \mathrm{tr}_\omega \Theta^{(1)} - \langle \partial\partial^*\omega, \omega \rangle = 0.$$

By using Theorem 6.1 with $t = \frac{1}{2}$, we deduce that either

(1) $\kappa(M) = 0$ and $c_1^{\mathrm{BC}}(M) = 0$, and furthermore, (M, J) has a balanced metric and K_M is a holomorphic torsion: $K_M^{\otimes m} = 0$ for some $m \in \mathbb{N}_+$, or

(2) $\kappa(M) = -\infty$ and $c_1^{\mathrm{AC}}(M) = 0$.

When $\dim_{\mathbb{C}} M = 2$, by using Theorem 6.4 with $t = \frac{1}{2}$, we know the Hopf surface is the only non-Kähler surface which can support real Chern–Ricci flat metrics. □

Remark 5.5. On a Kähler Calabi–Yau manifold M, there exist non-Kähler metrics which are real Chern–Ricci flat. Indeed, let ω_{CY} be a Calabi–Yau Kähler metric on M. Then for any nonconstant smooth function $f \in C^\infty(M, \mathbb{R})$, by Yau's theorem [1978], there exists a Kähler metric ω_0 such that

$$\omega_0^n = e^{-f}\omega_{\mathrm{CY}}^n.$$

Let $\omega_f = e^f \omega_0$. We have

$$\bar\partial_f^* \omega_f = \bar\partial_0^* \omega_0 + (n-1)\sqrt{-1}\,\partial f = (n-1)\sqrt{-1}\,\partial f.$$

Hence $\bar\partial \bar\partial_f^* \omega_f = \partial \partial_f^* \omega_f = -\sqrt{-1}\,(n-1)\partial\bar\partial f$. Moreover, we have

$$\omega_f^n = e^{(n-1)f} \omega_{\mathrm{CY}}^n,$$

which implies $\Theta^{(1)}(\omega_f) = \Theta^{(1)}(\omega_{\mathrm{CY}}) - (n-1)\sqrt{-1}\,\partial\bar\partial f = \partial\partial_f^* \omega_f$. By Corollary 5.4, ω_f is a real Chern–Ricci-flat metric, and it is a non-Kähler metric.

6. Classification of compact complex surfaces with t-Gauduchon–Ricci flat metrics

In this section, we classify compact complex surfaces with t-Gauduchon–Ricci flat metrics. One of the key ingredients is understanding the geometry of scalar curvatures of Gauduchon connections. The following theorem generalizes results in [Liu and Yang 2017; 2018; Yang 2019b; He et al. 2020] to the Gauduchon family.

Theorem 6.1. *Let M be a compact complex manifold. Suppose ω is a Hermitian metric and ∇^t is the Gauduchon connection of M. If $s^{(1)}(\omega, t) \geq 0$ for some $t > 0$, then either*

(1) $\kappa(M) = -\infty$, *or*

(2) $\kappa(M) = 0$ *and (M, ω) is conformally balanced and K_M is a holomorphic torsion*: $K_M^{\otimes m} = 0$ *for some $m \in \mathbb{N}_+$.*

For $t < 0$, we have a similar result:

Theorem 6.2. *Let M be a compact complex manifold. Suppose ω is a Hermitian metric and ∇^t is the Gauduchon connection of M. If $s^{(1)}(\omega, t) \leq 0$ for some $t < 0$ and $s^{(1)}(\omega, t)$ is strictly negative at some point, then K_M^{-1} is not pseudoeffective.*

To prove Theorem 6.1 and Theorem 6.2, we first calculate the total scalar curvature of the Gauduchon metric in the conformal class of ω. It is well known that there exists a smooth function f on M such that $\omega_f = e^f \omega$ is Gauduchon: $\partial\bar\partial \omega_f^{n-1} = 0$.

Lemma 6.3. *Let s_f be the Chern scalar curvature of $\omega_f = e^f \omega$. Then we have*

$$\int_M s_f \frac{\omega_f^n}{n!} = \int_M f^{n-1} \cdot s^{(1)}(\omega, t) \cdot \frac{\omega^n}{n!} + t \int_M (|\bar\partial_f^* \omega_f|_f^2 + |\partial_f^* \omega_f|_f^2) \frac{\omega_f^n}{n!}. \tag{6-1}$$

Proof. From the relation $\Theta^{(1)}(\omega_f) = \Theta^{(1)}(\omega) - \sqrt{-1}\,n\partial\bar\partial f$, it follows that

$$\int_M s_f \frac{\omega_f^n}{n!} = \int_M \Theta^{(1)}(\omega_f) \wedge \frac{\omega_f^{n-1}}{(n-1)!} = \int_M (\Theta^{(1)}(\omega) - n\sqrt{-1}\,\partial\bar\partial f) \wedge \frac{\omega_f^{n-1}}{(n-1)!}$$
$$= \int_M \Theta^{(1)}(\omega) \wedge \frac{\omega_f^{n-1}}{(n-1)!}.$$

By using Corollary 1.8 and $\mathfrak{Ric}^{(1)}(\omega, t) = \Theta^{(1)}(\omega) - t(\bar\partial\bar\partial^*\omega + \partial\partial^*\omega)$, we get

$$\begin{aligned}\int_M s_f \frac{\omega_f^n}{n!} &= \int_M \Theta^{(1)}(\omega) \wedge \frac{\omega_f^{n-1}}{(n-1)!}\\ &= \int_M (\mathfrak{Ric}^{(1)}(\omega, t) + t\bar\partial\bar\partial^*\omega + t\partial\partial^*\omega) \wedge \frac{\omega_f^{n-1}}{(n-1)!}\\ &= \int_M e^{(n-1)f} \cdot s^{(1)}(\omega, t) \cdot \frac{\omega^n}{n!} + t\int_M (\bar\partial\bar\partial^*\omega + \partial\partial^*\omega) \wedge \frac{\omega_f^{n-1}}{(n-1)!}.\end{aligned}$$

By using the formula

$$\bar\partial_f^*\omega_f = \bar\partial^*\omega + (n-1)\sqrt{-1}\,\partial f,$$

see [Yang 2020, Lemma 3.4], we get

$$\bar\partial\bar\partial_f^*\omega_f = \bar\partial\bar\partial^*\omega - (n-1)\sqrt{-1}\,\partial\bar\partial f, \quad \partial\partial_f^*\omega_f = \partial\partial^*\omega - (n-1)\sqrt{-1}\,\partial\bar\partial f.$$

Therefore,

$$\begin{aligned}\int_M (\bar\partial\bar\partial^*\omega + \partial\partial^*\omega) \wedge \frac{\omega_f^{n-1}}{(n-1)!} &= \int_M (\bar\partial_f\bar\partial_f^*\omega + \partial_f\partial_f^*\omega) \wedge \frac{\omega_f^{n-1}}{(n-1)!}\\ &= \int_M (|\bar\partial_f^*\omega_f|_f^2 + |\partial_f^*\omega_f|_f^2)\frac{\omega_f^n}{n!}.\end{aligned}$$ □

As an application of Lemma 6.3, we can prove Theorems 6.1 and 6.2.

Proof of Theorem 6.1. If $s^{(1)}(\omega, t) \geq 0$ for some $t > 0$, by Lemma 6.3, we have

$$\int_M s_f \frac{\omega_f^n}{n!} \geq t\int_M (|\bar\partial_f^*\omega_f|_f^2 + |\partial_f^*\omega_f|_f^2)\frac{\omega_f^n}{n!} \geq 0.$$

If $\int_M s_f(\omega_f^n/n!) > 0$, by [Yang 2019b, Corollary 3.3], we have $\kappa(M) = -\infty$. If $\int_M s_f(\omega_f^n/n!) = 0$, then we must have $\bar\partial^*\omega_f = 0$, i.e., ω_f is a balanced metric. Hence ω is conformal balanced. In this case, by [Yang 2019b, Theorem 1.4], it follows that either $\kappa(M) = -\infty$ or $\kappa(M) = 0$ with K_M a holomorphic torsion. □

Proof of Theorem 6.2. By using Lemma 6.3 again, we deduce

$$\int_M s_f \frac{\omega_f^n}{n!} \leq \int_M e^{(n-1)f} \cdot s^{(1)}(\omega, t)\frac{\omega^n}{n!} < 0.$$

By [Yang 2019b, Theorem 1.3], K_M^{-1} is not pseudoeffective. □

Now we are ready to establish the classification.

Theorem 6.4. *Let S be a compact complex surface. If it admits a t-Gauduchon–Ricci flat metric ω for some $t > 0$, then S is a minimal surface lying in one of the following*:

(1) *an Enriques surface*;

(2) *a bielliptic surface*;

(3) *a $K3$-surface*;

(4) *a 2-torus*;

(5) *a Hopf surface.*

We shall prove Theorem 6.4 following ideas in [He et al. 2020]. By Theorem 6.1, we have:

Corollary 6.5. *Suppose $t > 0$. If a complex surface S can admit a t-Gauduchon–Ricci flat metric*, *then either*

(1) $\kappa(S) = -\infty$, *or*

(2) $\kappa(S) = 0$. *In this case*, ω *is conformal Kähler and* K_S *is a holomorphic torsion*: $K_S^{\otimes m} = 0$ *for some integer* $m \in \mathbb{Z}$.

In both cases, *we have* $c_1^2(S) = 0$.

We need two more lemmas with proofs similar to those in [He et al. 2020].

Lemma 6.6 [He et al. 2020, Theorem 4.3]. *Let S be a complex surface with $\kappa(S) = -\infty$. If S admits a t-Gauduchon–Ricci flat metric*, *then S must be non-Kähler.*

Lemma 6.7 [He et al. 2020, Theorem 5.1]. *Let S be a non-Kähler complex surface with $\kappa(S) = -\infty$. If $c_1^2(S) = 0$, then S must be minimal.*

As an application of Corollary 6.5, Lemma 6.6 and Lemma 6.7, one has:

Corollary 6.8. *If a compact complex surface S admits a t-Gauduchon–Ricci flat metric ω, then S must be minimal.*

Proof of Theorem 6.4. Suppose S supports a t-Gauduchon–Ricci flat metric, then by Corollary 6.5, $\kappa(S) \le 0$ and by Corollary 6.8, S is also minimal.

(A) $\kappa(S) = 0$. By Kodaira–Enriques's classification, S is exactly one of the following:

(1) an Enrique surface;

(2) a bielliptic surface;

(3) a K3 surface;

(4) a torus.

They are all Kähler Calabi–Yau.

(B) $\kappa(S) = -\infty$. By using Kodaira–Enriques's classification again, S can only be one of the following:

(1) a minimal rational surface;

(2) a ruled surface with $g > 0$;

(3) a surface of class VII_0.

By Lemma 6.6, S is non-Kähler and so S can only be a surface of class VII_0:

(1) a class VII_0 surface with type $b_2 > 0$;

(2) an Inoue surface;

(3) a Hopf surface.

By using similar strategies as in the proof of [He et al. 2020, Theorem 5.1], one can show S can only be a Hopf surface. □

Remark 6.9. An explicit t-Gauduchon–Ricci flat metric on a diagonal Hopf surface is constructed in Theorem 7.1.

7. Explicit construction of t-Gauduchon–Ricci flat metrics on Hopf manifolds

Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional ($n \geq 2$) Hopf manifold. It is diffeomorphic to $(\mathbb{C}^n - \{0\})/G$ where G is a cyclic group generated by the transformation $z \mapsto \frac{1}{2}z$. It has an induced complex structure from $\mathbb{C}^n - \{0\}$. On M, there is a natural induced metric ω_0 given by

$$\omega_0 = \sqrt{-1}\, h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = \sqrt{-1}\, \frac{4\delta_{i\bar{j}}}{|z|^2}\, dz^i \wedge d\bar{z}^j. \tag{7-1}$$

The main result of this section is the following:

Theorem 7.1. *The Hermitian metric*

$$\Omega_t = \omega_0 + 4\left(\frac{2(n-1)t}{n} - 1\right) \cdot \sqrt{-1}\,\partial\bar{\partial}\log|z|^2, \quad t > 0 \tag{7-2}$$

is t-Gauduchon–Ricci flat: $\mathfrak{Ric}^{(1)}(\Omega_t, t) = 0$.

Proof. We shall use similar constructions as in [Liu and Yang 2017, Section 6]. More precisely, we consider the perturbed Hermitian metric

$$\omega_\lambda = \omega_0 + 4\lambda\sqrt{-1}\,\partial\bar{\partial}\log|z|^2 \quad \text{with } \lambda > -1. \tag{7-3}$$

It is shown in [Liu and Yang 2017, Theorem 6.2] that

$$\Theta^{(1)}(\omega_\lambda) = n \cdot \sqrt{-1}\,\partial\bar{\partial}\log|z|^2 \tag{7-4}$$

and

$$\tfrac{1}{2}(\partial\partial^*\omega_\lambda + \bar{\partial}\bar{\partial}^*\omega_\lambda) = \frac{n-1}{1+\lambda} \cdot \sqrt{-1}\,\partial\bar{\partial}\log|z|^2. \tag{7-5}$$

By using Corollary 1.8, we deduce

$$\mathfrak{Ric}^{(1)}(\omega_\lambda, t) = \Theta^{(1)} - t(\partial\partial^*\omega_\lambda + \bar\partial\bar\partial^*\omega_\lambda) = \left(n - \frac{2(n-1)t}{1+\lambda}\right)\sqrt{-1}\,\partial\bar\partial\log|z|^2.$$

Therefore, $\mathrm{Ric}^{(1)}(\omega_\lambda, t) = 0$ if and only if $\lambda = (2(n-1)t/n) - 1$. □

Remark 7.2. Note that when $t = 0$, $\Omega_0 = \omega_0 - 4\sqrt{-1}\,\partial\bar\partial\log|z|^2$ is not a Hermitian metric since the corresponding matrix is not positive definite. It is also well known that there is no Chern–Ricci flat metric on $\mathbb{S}^{2n-1}\times\mathbb{S}^1$, although there are Levi-Civita–Ricci flat metrics $\left(t = \frac{1}{2}\right)$ and Strominger–Bismut–Ricci flat metrics $(t = 1)$. On the Hopf surface $\mathbb{S}^3\times\mathbb{S}^1$, the canonical metric $\Omega_1 = \omega_0$ is Strominger–Bismut–Ricci flat.

Appendix: Curvatures of Gauduchon connections

In this section, we give a detailed proof of Lemma 3.5 (which is definitely well known to experts) and Corollary 1.8.

Let $\{\nabla^{\lambda,\mu}\}_{\lambda,\mu\in\mathbb{R}} \subset \mathcal{A}_g$ be a family of affine connections on $T_{\mathbb{R}}M$ defined by

$$g(\nabla^{\lambda,\mu}_X Y, Z) := g(\nabla^{\mathrm{LC}}_X Y, Z) + \lambda\, d\omega(JX, JY, JZ) + \mu\, d\omega(JX, Y, Z), \tag{A-1}$$

for $X, Y, Z \in \Gamma(M, T_{\mathbb{R}}M)$. Let $\{z^i\}$ be the local holomorphic coordinates on M. We consider the complexification of $\nabla^{\lambda,\mu}$ by setting

$$\begin{aligned} \nabla^{\lambda,\mu}_{\partial/\partial z^i}\frac{\partial}{\partial z^j} &= \Gamma^k_{ij}(\lambda,\mu)\frac{\partial}{\partial z^k} + \Gamma^{\bar k}_{ij}(\lambda,\mu)\frac{\partial}{\partial \bar z^k},\\ \nabla^{\lambda,\mu}_{\partial/\partial \bar z^i}\frac{\partial}{\partial z^j} &= \Gamma^k_{\bar ij}(\lambda,\mu)\frac{\partial}{\partial z^k} + \Gamma^{\bar k}_{\bar ij}(\lambda,\mu)\frac{\partial}{\partial \bar z^k}. \end{aligned} \tag{A-2}$$

Lemma A.1. *We have the relations*

$$\Gamma^k_{ij}(\lambda,\mu) = \Gamma^k_{ij} - \left(\lambda + \mu + \tfrac{1}{2}\right)T^k_{ij}, \quad \Gamma^{\bar k}_{ij}(\lambda,\mu) = 0,$$

and

$$\Gamma^k_{\bar ij}(\lambda,\mu) = \left(\lambda + \mu + \tfrac{1}{2}\right)h^{k\bar m}h_{j\bar n}\overline{T^n_{im}}, \quad \Gamma^{\bar k}_{\bar ij}(\lambda,\mu) = \left(-\lambda + \mu + \tfrac{1}{2}\right)h^{m\bar k}h_{n\bar i}T^n_{jm},$$

where $\Gamma^k_{ij} = h^{k\bar\ell}(\partial h_{j\bar\ell}/\partial z^i)$ *and* $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$.

Proof. It follows from standard computations, and for readers' convenience we include a straightforward proof here. At first, we have

$$\nabla^{\mathrm{LC}}_{\partial/\partial z^i}\frac{\partial}{\partial z^j} = \tfrac{1}{2}h^{k\bar\ell}\left(\frac{\partial h_{j\bar\ell}}{\partial z^i} + \frac{\partial h_{i\bar\ell}}{\partial z^j}\right)\frac{\partial}{\partial z^k} \tag{A-3}$$

and

$$\nabla^{\mathrm{LC}}_{\partial/\partial \bar z^i}\frac{\partial}{\partial z^j} = \tfrac{1}{2}h^{k\bar\ell}\left(\frac{\partial h_{j\bar\ell}}{\partial \bar z^i} - \frac{\partial h_{j\bar i}}{\partial \bar z^\ell}\right)\frac{\partial}{\partial z^k} + \tfrac{1}{2}h^{k\bar q}\left(\frac{\partial h_{k\bar i}}{\partial z^j} - \frac{\partial h_{j\bar i}}{\partial z^k}\right)\frac{\partial}{\partial \bar z^q}. \tag{A-4}$$

Note also that $d\omega(\partial/\partial z^i, \partial/\partial z^j, \partial/\partial \bar{z}^\ell) = \sqrt{-1}\,(\partial h_{j\bar{\ell}}/\partial z^i - \partial h_{i\bar{\ell}}/\partial z^j)$. Hence, we have

$$\begin{aligned}
&h\Big(\nabla^{\lambda,\mu}_{\partial/\partial z^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\
&= h\Big(\nabla^{\mathrm{LC}}_{\partial/\partial z^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big) + \lambda\, d\omega\Big(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}, J\frac{\partial}{\partial \bar{z}^\ell}\Big) + \mu\, d\omega\Big(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\
&= \tfrac{1}{2}\Big(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j}\Big) + \lambda\sqrt{-1}\, d\omega\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big) + \sqrt{-1}\,\mu\, d\omega\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\
&= \frac{\partial h_{j\bar{\ell}}}{\partial z^i} - \big(\lambda + \mu + \tfrac{1}{2}\big)\Big(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} - \frac{\partial h_{i\bar{\ell}}}{\partial z^j}\Big).
\end{aligned}$$

Therefore,

$$\Gamma^k_{ij}(\lambda,\mu) = \Gamma^k_{ij} - \big(\lambda + \mu + \tfrac{1}{2}\big)T^k_{ij}.$$

Similarly,

$$\begin{aligned}
&h\Big(\nabla^{\lambda,\mu}_{\partial/\partial z^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big)\\
&= h\Big(\nabla^{\mathrm{LC}}_{\partial/\partial z^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big) + \lambda\, d\omega\Big(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}, J\frac{\partial}{\partial z^k}\Big) + \mu\, d\omega\Big(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big) = 0
\end{aligned}$$

since the metric is J-invariant. Therefore, $\Gamma^{\bar{k}}_{ij}(\lambda,\mu) = 0$.

For the second part, $d\omega(\partial/\partial \bar{z}^i, \partial/\partial z^j, \partial/\partial \bar{z}^\ell) = \sqrt{-1}\,(\partial h_{j\bar{\ell}}/\partial \bar{z}^i - \partial h_{j\bar{\imath}}/\partial \bar{z}^\ell)$ and

$$\begin{aligned}
&h\Big(\nabla^{\lambda,\mu}_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\
&= h\Big(\nabla^{\mathrm{LC}}_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big) + \lambda\, d\omega\Big(J\frac{\partial}{\partial \bar{z}^i}, J\frac{\partial}{\partial z^j}, J\frac{\partial}{\partial \bar{z}^\ell}\Big) + \mu\, d\omega\Big(J\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\
&= \tfrac{1}{2}\Big(\frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{\imath}}}{\partial \bar{z}^\ell}\Big) - \lambda\sqrt{-1}\, d\omega\Big(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big) - \sqrt{-1}\,\mu\, d\omega\Big(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^\ell}\Big)\\
&= \tfrac{1}{2}\Big(\frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{\imath}}}{\partial \bar{z}^\ell}\Big) + (\lambda+\mu)\Big(\frac{\partial h_{\ell\bar{j}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{\imath}}}{\partial \bar{z}^\ell}\Big) = \big(\lambda + \mu + \tfrac{1}{2}\big)\Big(\frac{\partial h_{\ell\bar{j}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{\imath}}}{\partial \bar{z}^\ell}\Big).
\end{aligned}$$

Therefore,

$$\Gamma^k_{\bar{\imath}j}(\lambda,\mu) = \big(\lambda + \mu + \tfrac{1}{2}\big)h^{k\bar{m}}h_{j\bar{n}}\overline{T}^n_{im}.$$

Similarly,

$$\begin{aligned}
&h\Big(\nabla^{\lambda,\mu}_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big)\\
&= h\Big(\nabla^{\mathrm{LC}}_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big) + \lambda\, d\omega\Big(J\frac{\partial}{\partial \bar{z}^i}, J\frac{\partial}{\partial z^j}, J\frac{\partial}{\partial z^k}\Big) + \mu\, d\omega\Big(J\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big)\\
&= \tfrac{1}{2}\Big(\frac{\partial h_{k\bar{\imath}}}{\partial z^j} - \frac{\partial h_{j\bar{\imath}}}{\partial z^k}\Big) + \sqrt{-1}\,(\lambda-\mu)\, d\omega\Big(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big)\\
&= \big(-\lambda + \mu + \tfrac{1}{2}\big)\Big(\frac{\partial h_{k\bar{\imath}}}{\partial z^j} - \frac{\partial h_{j\bar{\imath}}}{\partial z^k}\Big),
\end{aligned}$$

and we deduce

$$\Gamma_{\bar{i}j}^{\bar{k}}(\lambda,\mu)=\big(-\lambda+\mu+\tfrac{1}{2}\big)h^{m\bar{k}}h_{n\bar{i}}T_{jm}^{n}. \qquad \square$$

Proof of Lemma 3.5. By Lemma A.1 and (3-2), we deduced that $\nabla^{\lambda,\mu}\in\mathcal{A}_{g,J}$ if and only if $\Gamma_{\bar{i}j}^{\bar{k}}(\lambda,\mu)=0$:

$$\text{(A-5)}\qquad \nabla^{\lambda,\mu}\in\mathcal{A}_{g,J} \iff \big(-\lambda+\mu+\tfrac{1}{2}\big)\,d\omega=0.$$

On the other hand, the restricted connection $\widehat{\nabla}^{\lambda,\mu}=\pi(\nabla^{\lambda,\mu})$ on the holomorphic tangent bundle $T^{1,0}M$ is determined by

$$\text{(A-6)}\qquad \widehat{\nabla}^{\lambda,\mu}_{\partial/\partial z^i}\frac{\partial}{\partial z^j}=\Gamma_{ij}^{k}(\lambda,\mu)\frac{\partial}{\partial z^k},\quad \widehat{\nabla}^{\lambda,\mu}_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}=\Gamma_{\bar{i}j}^{k}(\lambda,\mu)\frac{\partial}{\partial z^k},$$

where

$$\Gamma_{ij}^{k}(\lambda,\mu)=\Gamma_{ij}^{k}-\big(\lambda+\mu+\tfrac{1}{2}\big)T_{ij}^{k},\quad \Gamma_{\bar{i}j}^{k}(\lambda,\mu)=\big(\lambda+\mu+\tfrac{1}{2}\big)h^{k\bar{m}}h_{j\bar{n}}\overline{T_{im}^{n}}.$$

Recall that the Chern connection ∇^{Ch} of $T^{1,0}M$ is characterized by

$$\nabla^{\mathrm{Ch}}_{\partial/\partial z^i}\frac{\partial}{\partial z^j}=\Gamma_{ij}^{k}\frac{\partial}{\partial z^k},\quad \nabla^{\mathrm{Ch}}_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}=0.$$

Hence,

$$\text{(A-7)}\qquad \pi(\nabla^{\lambda,\mu})=\nabla^{\mathrm{Ch}} \iff \big(\lambda+\mu+\tfrac{1}{2}\big)\,d\omega=0.$$

By using (A-5) and (A-7), we deduce

$$\text{(A-8)}\qquad \rho(\nabla^{\lambda,\mu})=\nabla^{\mathrm{Ch}} \iff d\omega=0 \ \text{ or } \ (\lambda,\mu)=\big(0,-\tfrac{1}{2}\big).$$

Thus, we obtain (3-7). Similarly, one can show (3-8) and (3-9). The uniqueness of the family (3-10) follows from the linear property. $\square$

A.1. ***Curvature formulas of Gauduchon connections.*** Recall that there is a linear family of connections defined by

$$g(\nabla_X^t Y,Z)=g(\nabla_X^{\mathrm{LC}}Y,Z)+\tfrac{1}{2}t\,d\omega(JX,JY,JZ)+\tfrac{1}{2}(t-1)\,d\omega(JX,Y,Z).$$

By using Lemma A.1, the Gauduchon connection is determined by

$$\text{(A-9)}\qquad \nabla^t_{\partial/\partial z^i}\frac{\partial}{\partial z^j}=\Gamma_{ij}^{k}(t)\frac{\partial}{\partial z^k}\quad\text{and}\quad \nabla^t_{\partial/\partial \bar{z}^i}\frac{\partial}{\partial z^j}=\Gamma_{\bar{i}j}^{k}(t)\frac{\partial}{\partial z^k},$$

where the coefficients Γ_{ij}^{k} and $\Gamma_{\bar{i}j}^{k}$ are given by

$$\Gamma_{ij}^{k}(t)=\Gamma_{ij}^{k}-tT_{ij}^{k}\quad\text{and}\quad \Gamma_{\bar{i}j}^{k}(t)=t\cdot h^{k\bar{m}}h_{j\bar{n}}\overline{T_{im}^{n}}.$$

Theorem A.2. *The curvature tensor of Gauduchon connection ∇^t is given by*

$$R_{i\bar{j}k\bar{\ell}}(t)$$
$$=\Theta_{i\bar{j}k\bar{\ell}}+t(\Theta_{i\bar{\ell}k\bar{j}}+\Theta_{k\bar{j}i\bar{\ell}}-2\Theta_{i\bar{j}k\bar{\ell}})+t^2(T_{ik}^{p}\overline{T_{j\ell}^{q}}h_{p\bar{q}}-h^{p\bar{q}}h_{m\bar{\ell}}h_{k\bar{n}}T_{ip}^{m}\overline{T_{jq}^{n}}).$$

Proof. In the setting of Proposition 4.1, $\theta_{ij}^k = -tT_{ij}^k$. Hence this last equation follows from Proposition 4.1 and the relation $\partial T_{ik}^\ell/\partial\bar{z}^j = -\Theta_{i\bar{j}k}^\ell + \Theta_{k\bar{j}i}^\ell$. □

Proof of Corollary 1.8. It follows from Theorem 2.3 and Theorem A.2. □

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