Pacific Journal of Mathematics

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Volume 338 No. 1 September 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 338, No. 1, 2025
https://doi.org/10.2140/pjm.2025.338.1

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We determine those $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology that admit free circle actions up to almost diffeomorphism, provided that $n \equiv 5, 7 \bmod 8$.

1. Introduction

In this paper, unless otherwise stated, all manifolds under consideration are smooth, closed and oriented, and an oriented principal S^1-bundle is called an *S^1-bundle* for short. We are interested in the problem *when does a manifold M admit a free smooth action by the unit circle S^1* (free circle action for short in the sequel). There are studies of this problem for certain families of manifolds by various authors. For instance, when M is a homotopy sphere, this problem was studied in [Hsiang 1966; Lee 1968; Montgomery and Yang 1968; Schultz 1971]; when M is an $(n-1)$-connected $(2n+1)$-manifold, this problem has been solved for $n = 2$ [Duan and Liang 2005] and $n = 3$ [Jiang 2014]. For further examples see [Goldstein and Lininger 1972; Lininger 1972; Duan 2022; Galaz-García and Reiser 2025].

In this note we consider this problem for $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology when $n \geq 3$. The classification of $(n-1)$-connected $(2n+1)$-manifolds up to almost diffeomorphism was obtained in [Wall 1967; Wilkens 1972; Senger and Zhang 2023; Crowley 2002]. Recall that two n-manifolds M_1 and M_2 are *almost diffeomorphic* if there is a homotopy n-sphere Σ such that the connected sum $M_1 \# \Sigma$ is diffeomorphic to M_2 (see [Crowley and Wraith 2017, p. 223]). Carrying on [Wall 1967, Theorem 7; Wilkens 1972, Theorem 3], the following results will be proved in the Appendix.

Theorem 1.1. *Let M be an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology. When $n \equiv 5 \bmod 8$, M is almost diffeomorphic to the connected sum $\#_r(S^n \times S^{n+1})$ of r copies of $S^n \times S^{n+1}$ with $r \geq 0$. When $n \equiv 3 \bmod 4$, M is almost diffeomorphic to the sphere S^{2n+1} or $\#_r(S^n \times S^{n+1}) \# X_{l(M)}$ with $r \geq 0$.*

MSC2020: 57R19, 57S25.
Keywords: circle actions, highly connected manifolds, surgery.

Here we explain the manifold $X_{l(M)}$ in the above theorem. For $n \equiv 3 \bmod 4$, let $l(M)$ be the divisibility of the $\frac{n+1}{4}$-th Pontryagin class $p_{\frac{n+1}{4}}(M) \in H^{n+1}(M) = \mathbb{Z}$, i.e., $p_{\frac{n+1}{4}}(M)$ is an $l(M)$-fold multiple of a primitive element with $l(M) \geq 0$. In fact, it is well known (see Proposition A.2) that $l(M)$ is divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$ where $a_k = 1$ for k even and $a_k = 2$ for k odd, and $b_n = 2$ for $n = 3, 7$ and $b_n = 1$ for $n \neq 3, 7$. On the other hand, for any natural number l divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$, there exists a unique $(n+1)$-dimensional real vector bundle ξ_l over S^{n+1} whose Euler class and $\frac{n+1}{4}$-th Pontryagin class are

$$e(\xi_l) = 0, \quad p_{\frac{n+1}{4}}(\xi_l) = l \cdot \omega,$$

respectively, where ω is the preferred generator of $H^{n+1}(S^{n+1})$ (see Proposition A.3). Let X_l be the total space of the sphere bundle associated to ξ_l; then X_l is an $(n-1)$-connected $(2n+1)$-manifold with $H^{n+1}(X_l) \cong \mathbb{Z}$ and $l(X_l) = l$. See the Appendix for more details and a proof of Theorem 1.1.

As an application of this classification, we determine those $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology that admit free circle actions up to almost diffeomorphism, provided that $n \equiv 5, 7 \bmod 8$.

Theorem 1.2. *Let M be an $(n-1)$-connected $(2n+1)$-manifold where $H_n(M)$ is free and $n \equiv 5 \bmod 8$. Then there exists a homotopy sphere Σ such that $M \# \Sigma$ admits a free circle action.*

Recall that the Bernoulli numbers $B_1, B_2, \ldots$ are the coefficients in the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \frac{B_3}{6!}z^6 - \cdots,$$

(see [Milnor and Stasheff 1974, Appendix B]). For any rational number r let $\operatorname{den}(r)$ denote the denominator of r expressed as a fraction in lowest terms.

Theorem 1.3. *Let M be an $(n-1)$-connected $(2n+1)$-manifold where $H_n(M)$ is free and $n \equiv 7 \bmod 8$. Then there exists a homotopy sphere Σ such that $M \# \Sigma$ admits a free circle action if and only if one of the following conditions hold*:

(1) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1})$ for some integer $r \geq 0$;*

(2) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1}) \# X_l$ for some integer $r \geq 0$ and some nonnegative integer l divisible by $\left(\frac{n-1}{2}\right)! \operatorname{den}\left(\frac{B_{(n+1)/4}}{n+1}\right)$.*

Remark 1.4. In the above theorem, condition (1) is equivalent to that the n-th Betti number $b_n(M)$ of M is even and $l(M) = 0$; condition (2) is equivalent to that $b_n(M)$ is odd and $l(M)$ is divisible by $\left(\frac{n-1}{2}\right)! \operatorname{den}\left(\frac{B_{(n+1)/4}}{n+1}\right)$.

In the case $n \equiv 3 \bmod 8$, we obtain a partial version of Theorem 1.3 as follows.

Theorem 1.5. *Let M be an $(n-1)$-connected $(2n+1)$-manifold where $H_n(M)$ is free and $n \equiv 3 \bmod 8$. If there is a homotopy sphere Σ such that $M \# \Sigma$ admits a free circle action, then one of the following conditions hold:*

(1) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1})$ for some integer $r \geq 0$;*

(2) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1}) \# X_l$ for some integer $r \geq 0$ and some nonnegative integer l divisible by $\left(\frac{n-1}{2}\right)! \operatorname{den}\left(\frac{B_{(n+1)/4}}{n+1}\right)$.*

Conversely, for any integer $r \geq 0$, the connected sum $\#_{2r}(S^n \times S^{n+1})$ admits a free circle action; for any nonnegative integer r and nonnegative integer l divisible by $2 \cdot \left(\frac{n-1}{2}\right)! \operatorname{den}\left(\frac{B_{(n+1)/4}}{n+1}\right)$, there exists homotopy sphere Σ such that the connected sum $\#_{2r}(S^n \times S^{n+1}) \# X_l \# \Sigma$ admits a free circle action.

In Section 2 we prove Theorem 1.2 by a direct geometric construction. To prove Theorems 1.3 and 1.5, one observes that a manifold M admits a free circle action if and only if it is the total space of an S^1-bundle over a manifold N, and N is the orbit space of the action. The key ingredient in the proof is the analysis of the topological invariants of the orbit space. In Sections 3 and 4 we study the cohomology ring and the Pontryagin class of the orbit space, respectively. Finally we prove Theorems 1.3 and 1.5 in Section 5.

2. A construction of free circle actions

In this section we recall a construction of free circle actions and prove Theorem 1.2.

Given two m-manifolds N_1 and N_2, the connected sum $N_1 \# N_2$ is constructed as follows (see [Kervaire and Milnor 1963, Section 2]). Take embeddings

$$i_k : D^m \to N_k, \quad k = 1, 2,$$

such that i_1 preserves the orientation and i_2 reverses the orientation. We obtain $N_1 \# N_2$ from the disjoint sum $N_1 \setminus i_1(0) + N_2 \setminus i_2(0)$ by identifying $i_1(tu)$ with $i_2((1-t)u)$ where $u \in S^{m-1}$, $0 < t < 1$. Now given two oriented S^1-bundles $E_k \to N_k$, $k = 1, 2$, a new S^1-bundle over $N_1 \# N_2$ is constructed as follows. Take embeddings $j_k : S^1 \times D^m \to E_k$, $k = 1, 2$, such that j_k covers i_k, i.e., local trivializations of E_k over an embedded disk $D^m \subset N_k$. The S^1-*connected sum* $E_1 \#_{S^1} E_2$ is an $(m+1)$-manifold obtained from the disjoint sum $E_1 \setminus j_1(S^1 \times 0) + E_2 \setminus j_2(S^1 \times 0)$ by identifying $j_1(x, tu)$ with $j_2(x, (1-t)u)$ where $(x, u) \in S^1 \times S^{m-1}$, $0 < t < 1$. This construction does not depend on the choice of the embedded disk D^m by Palais' disk theorem [1960, Theorem B], and does not depend on the trivializations since the structure group of our fiber bundle is connected. The bundle projections $E_k \to N_k$, $k = 1, 2$, induce an S^1-bundle projection $E_1 \#_{S^1} E_2 \to N_1 \# N_2$ (see [Hambleton and Su 2013, Section 3]).

Let $m \geq 3$ and let $\alpha : S^1 \to \mathrm{SO}(m)$ represent the generator of $\pi_1 \mathrm{SO}(m) = \mathbb{Z}_2$. Let $\tau : S^1 \times D^m \to S^1 \times D^m$ be the homeomorphism given by $\tau(t, x) = (t, \alpha(t)x)$. Take an embedding $f : D^m \to N$ for an m-manifold N. Let $\Sigma_0 N$ and $\Sigma_1 N$ be the manifolds obtained by surgery along the embeddings

$$f_1 := \mathrm{id}_{S^1} \times f : S^1 \times D^m \to S^1 \times N,$$
$$f_2 := (\mathrm{id}_{S^1} \times f) \circ \tau : S^1 \times D^m \to S^1 \times N,$$

respectively, namely

$$\Sigma_i N := (S^1 \times N \setminus f_i(S^1 \times \mathring{D}^m)) \cup_{f_i|S^1 \times S^{m-1}} (D^2 \times S^{m-1}), \quad i = 1, 2,$$

where $\mathring{D}^m :=$ Interior D^m.

Lemma 2.1 [Duan 2022, Theorem B and Proposition 3.2]. *Let $E \to B$ be an S^1-bundle over an m-manifold with E simply connected and $m \geq 4$. Then for any simply connected m-manifold N, the S^1-connected sum $E \#_{S^1} (S^1 \times N)$ is diffeomorphic to $E \# \Sigma_0 N$ if B is nonspin and to $E \# \Sigma_1 N$ if B is spin.*

There are diffeomorphisms

$$\Sigma_0(S^p \times S^q) \cong \Sigma_1(S^p \times S^q) \cong (S^p \times S^{q+1}) \# (S^{p+1} \times S^q)$$

for any $p \leq q$ with $q \geq 3$

Proposition 2.2. *Let E be a simply connected $(2n+1)$-manifold and $n \geq 3$. If E admits a free circle action, then $\#_2(S^n \times S^{n+1}) \# E$ admits a free circle action.*

Proof. Let B be the orbit space of a free circle action on E. Then E is the total space of an S^1-bundle over B, and $E \#_{S^1} (S^1 \times S^n \times S^n)$ is the total space of an S^1-bundle over $B \# (S^n \times S^n)$. By Lemma 2.1, $E \#_{S^1} (S^1 \times S^n \times S^n)$ is diffeomorphic to either $E \# \Sigma_0(S^n \times S^n)$ or $E \# \Sigma_1(S^n \times S^n)$, and hence is diffeomorphic to $\#_2(S^n \times S^{n+1}) \# E$ by the second paragraph of Lemma 2.1. This proves the proposition. □

Corollary 2.3. *Let r, n be positive integers and $n \geq 3$. If Σ is a homotopy $(2n+1)$-sphere admitting a free circle action, then $\#_{2r}(S^n \times S^{n+1}) \# \Sigma$ admits a free circle action.*

Note that there are many exotic $(2n+1)$-spheres admitting free circle actions. In fact, the set of homotopy $(2n+1)$-spheres admitting free circle actions has been determined for $n = 3, 4, 5, 6$ in [Montgomery and Yang 1968; Brumfiel 1971, p. 402, Theorem I.10(i)].

Corollary 2.4. *Let r, n be nonnegative integers and $n \geq 2$. Then $\#_r(S^n \times S^{n+1})$ admits a free circle action.*

Proof. The case $n=2$ has been proved in [Duan and Liang 2005, Corollary 2]. Note that S^{2n+1} admits a free circle action with orbit space $\mathbb{C}P^n$ and $S^n \times S^{n+1}$ admits a free circle action with orbit space $\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1}$ if n is odd or $S^n \times \mathbb{C}P^{\frac{n}{2}}$ if n is even. It follows that the case $n \geq 3$ is a direct consequence of Proposition 2.2. □

Proof of Theorem 1.2. By Theorem 1.1 there is a homotopy sphere Σ such that $M \# \Sigma$ is diffeomorphic to $\#_r(S^n \times S^{n+1})$ where r is the rank of $H_n(M)$. Hence the theorem follows directly from Corollary 2.4. □

3. Cohomology of the orbit space

A $(2n+1)$-manifold M admits a free circle action if and only if it is the total space of an S^1-bundle over a $2n$-manifold N. In this section we analyze the cohomology ring of N when M is $(n-1)$-connected with torsion free homology. In the next section we study the Pontryagin classes of N.

Lemma 3.1. *Let $n > 1$ be an odd integer. Let $S^1 \times M \to M$ be a free circle action on an $(n-1)$-connected $(2n+1)$-manifold M with torsion free homology. Then the orbit space N is a simply connected $2n$-manifold whose cohomology ring $H^*(N)$ is isomorphic to $H^*(\#_r(S^n \times S^n) \# \mathbb{C}P^n)$ or $H^*(\#_r(S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1}))$ for $r \geq 0$.*

Conversely, every simply connected $2n$-manifold N with

$$H^*(N) \cong H^*\Big(\#_r (S^n \times S^n) \# \mathbb{C}P^n\Big) \quad \text{or} \quad H^*\Big(\#_r (S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\Big)$$

for $r \geq 0$ can be realized as the orbit space of some free circle action on an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology.

Proof. To prove the lemma, it suffices to verify that for an S^1-bundle $S^1 \to M \to N$, the manifolds M and N satisfy the conditions stated in the lemma. The fundamental groups of M and N are related by the homotopy exact sequence of the S^1-bundle

$$\pi_1(S^1) \to \pi_1(M) \to \pi_1(N) \to 0. \tag{1}$$

The cohomology groups of M and N are related by the Gysin sequence

$$H^{j-2}(N) \xrightarrow{-\cup t} H^j(N) \to H^j(M) \to H^{j-1}(N) \xrightarrow{-\cup t} H^{j+1}(N), \tag{2}$$

where $t \in H^2(N)$ is the Euler class of the S^1-bundle.

Now assume that M is an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology. The exact sequence (1) implies that N is simply connected. The cohomology ring $H^*(N)$ is computed from the Gysin sequence

$$H^{i-1}(M) \to H^{i-2}(N) \xrightarrow{-\cup t} H^i(N) \to H^i(M). \tag{3}$$

Since M is $(n-1)$-connected, the exact sequence (3) implies that for $0 \leq i \leq n-1$,

$$H^i(N) = \begin{cases} \mathbb{Z}t^{\frac{i}{2}}, & \text{if } i \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

and for $n+3 \leq i \leq 2n$, the homomorphism $-\cup t : H^{i-2}(N) \to H^i(N)$ is an isomorphism. Moreover, the Gysin sequence implies that the map $H^n(N) \to H^n(M)$ is a monomorphism. Since M has torsion free homology, this implies $H^n(M)$ is free and hence $H^n(N)$ is free. Note that the rank of $H^n(N)$ must be even, because the intersection form of N is nondegenerate and skew-symmetric since n is odd. Moreover, the homomorphism $-\cup t : H^n(N) \to H^{n+2}(N)$ is trivial since $H^{n+2}(N) \cong H_{n-2}(N) = 0$. Now since $H^{n-1}(N) = \mathbb{Z}t^{\frac{n-1}{2}}$ and $H^{n+1}(N) \cong H_{n-1}(N) \cong \mathbb{Z}$, it remains to show the homomorphism $-\cup t : H^{n-1}(N) \to H^{n+1}(N)$ must be either an isomorphism or a trivial map. This is easily obtained by the Gysin sequence

$$H^{n-1}(N) \xrightarrow{-\cup t} H^{n+1}(N) \to H^{n+1}(M)$$

and the fact that $H^{n+1}(M) \cong H_n(M)$ is free.

Conversely, assume N is simply connected, the cohomology ring $H^*(N)$ is isomorphic to either $H^*\big(\#_r(S^n \times S^n) \# \mathbb{C}P^n\big)$ or $H^*\big(\#_r(S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\big)$ for some $r \geq 0$, and the Euler class t is a generator of $H^2(N) \cong \mathbb{Z}$. The exact sequence (1) implies that $\pi_1 M$ is abelian and hence $\pi_1 M \cong H_1(M)$. To prove that M is $(n-1)$-connected with torsion free homology, it suffices to show that $H_i(M)$ is trivial for $1 \leq i \leq n-1$ and $H_n(M)$ is torsion free.

This follows from the sequence (2). When $j = 1$, the sequence (2) implies that $H^1(M) = 0$ since $H^1(N) = 0$ and $-\cup t : H^0(N) \to H^2(N)$ is an isomorphism. When $2 \leq j \leq n-1$, the sequence (2) implies that $H^j(M) = 0$ because $H^{n-2}(N) = 0$ and $-\cup t : H^{i-2}(N) \to H^i(N)$ is an isomorphism for $2 \leq i \leq n-1$. When $j = n, n+1$, the sequence (2) implies that $H^j(M)$ is free because the cohomology groups of N are free, the homomorphism $-\cup t : H^i(N) \to H^{i+2}(N)$ is trivial for $i = n-2, n$ and $-\cup t : H^{n-1}(N) \to H^{n+1}(N)$ is either an isomorphism or a trivial map. This shows that $H_i(M)$ is trivial for $1 \leq i \leq n-1$ and $H_n(M) \cong H^{n+1}(M)$ is torsion free. □

4. Pontryagin classes of the orbit spaces

We have observed in the previous section that the cohomology ring of the orbit space N of a free circle action on M is isomorphic to $H^*\big(\#_r(S^n \times S^n) \# \mathbb{C}P^n\big)$ or $H^*\big(\#_r(S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\big)$. It will be shown in the proof of Theorems 1.3 and 1.5 that in the first case, the Pontryagin class $p_{\frac{n+1}{4}}(M)$ vanishes. In order to determine the Pontryagin class $p_{\frac{n+1}{4}}(M)$ in the second case, we analyze the Pontryagin class of N in this section. Since we are in the case $n \equiv 3 \bmod 4$, we assume $n = 4k-1$.

Lemma 4.1. *Let N be a simply connected $(8k-2)$-manifold whose cohomology ring is isomorphic to $H^*\big(\#_r(S^{4k-1}\times S^{4k-1})\,\#\,(\mathbb{C}P^{2k-1}\times S^{4k})\big)$ with $r\geq 0$ and $k\geq 1$. If the Pontryagin class $p_k(N)$ is a $d(N)$-fold multiple of a primitive element, then $d(N)$ is divisible by $(2k-1)!\cdot\mathrm{den}\big(\frac{B_k}{4k}\big)$.*

Proof. Let $\{\hat{A}_k(p_1,\ldots,p_k)\}$ be the multiplicative sequence of polynomials with

$$\frac{\sqrt{t}/2}{\sinh\big(\sqrt{t}/2\big)}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2^{2n}-2)B_n}{2^{2n}((2n)!)}t^n$$

as characteristic power series (see [Hirzebruch 1995, §1]). Let α_k be the coefficient of p_k in $\hat{A}_k(p_1,\ldots,p_k)$. We first show the following two facts:

(1) $\alpha_k\cdot d(N)$ is an integer.

(2) The integer $d(N)$ is divisible by $a_k\cdot(2k-1)!$, where $a_k=1$ if k is even, $a_k=2$ if k is odd.

The proof of (1) is based on the integrality of twisted $\hat{A}$-genera. Let $\mathrm{ch}(\eta)\in H^*(N;\mathbb{Q})$ be the Chern character of a virtual complex vector bundle η over N, $d\in H^2(N)$ be a cohomology class whose mod 2 reduction is the second Stiefel–Whitney class of N, $\hat{A}(N)=\sum_{i=1}^{\infty}\hat{A}_i(p_1(N),\ldots,p_i(N))$ be the $\hat{A}$-class of N, and $[N]\in H_{2n}(N)$ be the fundamental class of N. Then by [Hirzebruch 1995, Theorem 26.1.1], $\langle\mathrm{ch}(\eta)\cdot e^{d/2}\cdot\hat{A}(N),[N]\rangle$ is an integer. Now let L be the complex line bundle over N with first Chern class $c_1(L)=t$, a generator of $H^2(N)$. Take $\eta=(L-1)^{2k-1}$. By the assumption of $H^*(N)$,

$$\begin{aligned}\langle\mathrm{ch}(\eta)\cdot e^{d/2}\cdot\hat{A}(N),[N]\rangle&=\langle(e^t-1)^{2k-1}e^{d/2}\hat{A}(N),[N]\rangle\\&=\langle t^{2k-1}e^{d/2}\hat{A}(N),[N]\rangle\\&=\langle t^{2k-1}\hat{A}(N),[N]\rangle\\&=\big\langle t^{2k-1}(\alpha_k p_k(N)+\cdots),[N]\big\rangle\\&=\pm\alpha_k d(N),\end{aligned}$$

where the second and third equality signs hold because $t^{2k}=0$ and $t^{2k-1}d=0$. Moreover, the last equality sign follows from the fact that the bilinear forms

$$H^{4i}(N)\times H^{4k-4i}(N)\to H^{4k}(N)$$

induced by cup products vanish for $1\leq i\leq k-1$. This proves (1).

To prove (2), let $\pi:N_t\to N$ be the S^1-bundle over N with Euler class t, a generator of $H^2(N)$. In the segment of the Gysin sequence

$$H^{4k-2}(N)\xrightarrow{-\cup t}H^{4k}(N)\xrightarrow{\pi^*}H^{4k}(N_t)\to H^{4k-1}(N)\xrightarrow{-\cup t}H^{4k+1}(N),$$

the two homomorphisms $-\cup t$ are trivial, therefore we have a split short exact sequence

$$0 \to H^{4k}(N) \xrightarrow{\pi^*} H^{4k}(N_t) \to H^{4k-1}(N) \to 0. \tag{4}$$

Furthermore, the tangent bundle TN_t of N_t is isomorphic to the Whitney sum $\pi^* TN \oplus V$, where $V = \ker d\pi$ is the vertical bundle of the S^1-bundle. The orientable line bundle V is trivial, therefore $p_k(N_t) = \pi^* p_k(N)$ and hence $l(N_t) = d(N)$ by the sequence (4). By Lemma 3.1 N_t is a $(4k-2)$-connected $(8k-1)$-manifold. Equip N_t with a CW complex structure. Since $H^i(N_t; \pi_{i-1}\,\mathrm{SO}) = 0$ for $1 \le i \le 4k-1$, it follows that the stable tangent bundle of N_t has a cross section over the $(4k-1)$-skeleton of N_t and hence $l(N_t)$ is divisible by $a_k(2k-1)!$ according to [Kervaire 1959, Lemma 1.1]. This proves (2).

Now by (2), there is an integer β such that $d(N) = a_k(2k-1)!\beta$. Since $\alpha_k = -\frac{B_k}{2\cdot(2k)!}$(see [Borel and Hirzebruch 1960, 3.4]), we have

$$\alpha_k d(N) = -\frac{B_k}{2\cdot(2k)!} a_k (2k-1)!\beta = -\frac{B_k a_k \beta}{4k}$$

is an integer. It follows that β must be divisible by $\mathrm{den}\big(\frac{B_k a_k}{4k}\big)$ and hence $d(N) = a_k(2k-1)!\beta$ must be a multiple of $a_k(2k-1)!\,\mathrm{den}\big(\frac{B_k a_k}{4k}\big)$. Now it suffices to show that $\mathrm{den}\big(\frac{a_k B_k}{4k}\big) = \mathrm{den}\big(\frac{B_k}{4k}\big)/a_k$. If k is even, then $a_k = 1$ and this is clearly true. When k is odd, $a_k = 2$. It is known that $\mathrm{den}(B_k)$ is even (see [Milnor and Stasheff 1974, Appendix B, Theorem B.3]), hence the numerator of B_k is odd. From this it is easy to see $\mathrm{den}\big(\frac{2B_k}{4k}\big) = \mathrm{den}\big(\frac{B_k}{4k}\big)/2$. □

Lemma 4.2. *Let $a_k = 1$ when k is even, and $a_k = 2$ when k is odd. For any integer d divisible by $a_k \cdot (2k-1)! \cdot \mathrm{den}\big(\frac{B_k}{4k}\big)$, there exists an $(8k-2)$-manifold N homotopy equivalent to $\#_r(S^{4k-1} \times S^{4k-1})\#(\mathbb{C}P^{2k-1} \times S^{4k})$ for $r \ge 0$, such that the Pontryagin class $p_k(N)$ is a d-fold multiple of a primitive element.*

The proof of this lemma uses surgery theory. We recall some elementary notions here (see [Wall 1999, p. 109; Browder 1972, pp. 45–46]). Let BO_k be the classifying space of the orthogonal group O_k. Let G_k be the topological monoid of self-homotopy equivalences of S^{k-1} and let BG_k be its classifying space. The natural maps $G_k \to G_{k+1}$ and $O_k \to O_{k+1}$ induce maps $\mathrm{BG}_k \to \mathrm{BG}_{k+1}$ and $\mathrm{BO}_k \to \mathrm{BG}_{k+1}$, respectively. Then $\mathrm{BO} = \lim_{k\to\infty} \mathrm{BO}_k$ is the classifying space of stable vector bundles and $\mathrm{BG} = \lim_{k\to\infty} \mathrm{BG}_k$ is the classifying space of stable spherical fibrations. By taking the sphere bundle associated to a real vector bundle one has a forgetful map $\mathrm{BO} \to \mathrm{BG}$. The induced homomorphism $J : \pi_i(\mathrm{BO}) \to \pi_i(\mathrm{BG})$ is the J-homomorphism.

Proof. Let ξ be a stable vector bundle over S^{4k}, whose sphere bundle is trivial as a stable spherical fibration. This means that ξ is in the kernel of the J-homomorphism

$J : \pi_{4k}(\mathrm{BO}) \to \pi_{4k}(\mathrm{BG})$. The group $\pi_{4k}(\mathrm{BO})$ is isomorphic to $\mathbb{Z}$ and $\ker J$ is a subgroup of index $\mathrm{den}\big(\frac{B_k}{4k}\big)$ (first proved by Adams [1965, Theorem 3.7] assuming the Adams conjecture [1963, Conjecture 1.2], which was later proved by Quillen [1971]). It is shown by Kervaire [1959, Lemma 1.1] that the image of the homomorphism $p_k : \pi_{4k}(\mathrm{BO}) \to \mathbb{Z}$, $\xi \mapsto \langle p_k(\xi), [S^{4k}]\rangle$, is a subgroup of index $a_k \cdot (2k-1)!$.

For an integer d divisible by $a_k \cdot (2k-1)! \cdot \mathrm{den}\big(\frac{B_k}{4k}\big)$, we may choose a vector bundle ξ such that $\langle p_k(\xi), [S^{4k}]\rangle = d$. Since ξ is a vector bundle reduction of the Spivak normal fibration of S^{4k}, we may consider the surgery problem

$$\begin{array}{ccc} \nu X & \xrightarrow{\bar{f}} & \xi \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{f} & S^{4k} \end{array}$$

where X is a closed $4k$-manifold, and f is a map of degree 1, covered by a bundle map $\bar{f}$ from the stable normal bundle νX to ξ. The surgery obstruction, that is, the obstruction to doing surgery on (X, f) to get a homotopy equivalence $f' : X' \to S^{4k}$, is $\theta(\bar{f}, f) \in L_{4k}(\mathbb{Z}) \cong \mathbb{Z}$, where $L_{4k}(\mathbb{Z})$ is the Wall surgery obstruction group.

Taking the product with $\mathbb{C}P^{2k-1}$ we have a surgery problem

$$\begin{array}{ccc} \nu\mathbb{C}P^{2k-1} \times \nu X & \xrightarrow{\mathrm{id}\times\bar{f}} & \nu\mathbb{C}P^{2k-1} \times \xi \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}P^{2k-1} \times X & \xrightarrow{\mathrm{id}\times f} & \mathbb{C}P^{2k-1} \times S^{4k} \end{array}$$

By the product formula of surgery obstructions (see [Browder 1972, p. 33; Wall 1999, Lemma 13B.4]), the surgery obstruction of this surgery problem is

$$\theta(\mathrm{id} \times \bar{f}, \mathrm{id} \times f) = \chi(\mathbb{C}P^{2k-1}) \cdot \theta(\bar{f}, f) \in L_{8k-2}(\mathbb{Z}),$$

where $\chi(\mathbb{C}P^{2k-1})$ is the Euler characteristic of $\mathbb{C}P^{2k-1}$. Since $\chi(\mathbb{C}P^{2k-1}) = 2k$ and $L_{8k-2}(\mathbb{Z}) \cong \mathbb{Z}/2$, the surgery obstruction $\theta(\mathrm{id} \times \bar{f}, \mathrm{id} \times f)$ vanishes. Therefore by surgery we get a homotopy equivalence $g : Y \to \mathbb{C}P^{2k-1} \times S^{4k}$, which is covered by a bundle map $\bar{g} : \nu Y \to \nu\mathbb{C}P^{2k-1} \times \xi$. The existence of $\bar{g}$ implies that $\tau Y = g^*(\tau\mathbb{C}P^{2k-1} \times \xi^{-1})$ where τY and $\tau\mathbb{C}P^{2k-1}$ are the stable tangent bundles of Y and $\mathbb{C}P^{2k-1}$, respectively, and ξ^{-1} is the stable inverse of ξ. Thus

$$p_k(Y) = p_k(\tau Y) = g^* p_k(\tau\mathbb{C}P^{2k-1} \times \xi^{-1}).$$

Since $p_k(\tau\mathbb{C}P^{2k-1} \times \xi^{-1}) = -\pi_2^*(p_k(\xi))$, where $\pi_2 : \mathbb{C}P^{2k-1} \times S^{4k} \to S^{4k}$ is the projection, the divisibility of $p_k(Y)$ equals the divisibility of $p_k(\xi)$. Taking connected sum with r copies of $S^{4k-1} \times S^{4k-1}$ we get N. □

5. Proof of Theorems 1.3 and 1.5

Proof of Theorems 1.3 and 1.5. Let N be the orbit space of a free circle action on M. Then M is the total space of an S^1-bundle η over N, $\pi : M \to N$ with Euler class t, a generator of $H^2(N)$. The tangent bundle TM is isomorphic to the Whitney sum $\pi^*TN \oplus V$, where $V = \ker d\pi$ is the vertical bundle of η and is trivial as an orientable line bundle. Therefore

$$p_{\frac{n+1}{4}}(M) = \pi^* p_{\frac{n+1}{4}}(N).$$

By Lemma 3.1, the cohomology ring $H^*(N)$ is isomorphic to

$$H^*\Big(\#_r (S^n \times S^n) \# \mathbb{C}P^n\Big) \quad \text{or} \quad H^*\Big(\#_r (S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\Big).$$

If $H^*(N)$ is isomorphic to $H^*\big(\#_r (S^n \times S^n) \# \mathbb{C}P^n\big)$, then in the Gysin sequence

$$H^{n-1}(N) \xrightarrow{-\cup t} H^{n+1}(N) \xrightarrow{\pi^*} H^{n+1}(M) \to H^n(N) \to 0$$

the homomorphism $-\cup t$ is an isomorphism, therefore the homomorphism π^* is trivial and hence

$$H_n(M) \cong H^{n+1}(M) \cong H^n(N) \cong \mathbb{Z}^{2r}, \qquad p_{\frac{n+1}{4}}(M) = \pi^* p_{\frac{n+1}{4}}(N) = 0.$$

If $H^*(N)$ is isomorphic to $H^*\big(\#_r (S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\big)$, then in the Gysin sequence

$$H^{n-1}(N) \xrightarrow{-\cup t} H^{n+1}(N) \xrightarrow{\pi^*} H^{n+1}(M) \to H^n(N) \to 0$$

the homomorphism $-\cup t$ is trivial. Therefore we have a split short exact sequence

$$0 \to H^{n+1}(N) \xrightarrow{\pi^*} H^{n+1}(M) \to H^n(N) \to 0.$$

This implies that $H^{n+1}(M) \cong \mathbb{Z}^{2r+1}$ and the divisibility $l(M)$ of $p_{\frac{n+1}{4}}(M)$ is equal to the divisibility $d(N)$ of $p_{\frac{n+1}{4}}(N)$. By Lemma 4.1, the integer $l(M)$ must satisfy the conditions in Theorems 1.3 and 1.5.

Conversely, the total space of the S^1-bundle over $\#_r (S^n \times S^n) \# \mathbb{C}P^n$ with Euler class a generator of $H^2\big(\#_r (S^n \times S^n) \# \mathbb{C}P^n\big)$ is diffeomorphic to $\#_{2r} (S^n \times S^{n+1})$, as we have seen in the proof of Proposition 2.2. On the other hand, for an integer l divisible by $a_{\frac{n+1}{4}} \cdot \big(\frac{n-1}{2}\big)! \,\mathrm{den}\big(\frac{B_{(n+1)/4}}{n+1}\big)$, let N be a manifold given in Lemma 4.2 with $d(N) = l$, and take the total space M of the S^1-bundle over N with Euler class a generator of $H^2(N)$. Then by the above discussion $H_n(M)$ is isomorphic to $\mathbb{Z}^{2r+1}$ and $l(M)$ equals $d(N)$. By Theorem 1.1 M is almost diffeomorphic to $\#_{2r} (S^n \times S^{n+1}) \# X_l$. □

Appendix

In this section we prove Theorem 1.1. This is based on a sequence of propositions. We first describe the structure of $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology and $n \equiv 3 \bmod 4$ as a twisted double of $\natural_r(S^n \times D^{n+1})$, where $\natural_r(S^n \times D^{n+1})$ denotes the boundary connected sum of r copies of $S^n \times D^{n+1}$, whose boundary is $\#_r(S^n \times S^n)$.

Proposition A.1. *Let M be an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology, $n \equiv 3 \bmod 4$. If $r := \operatorname{rank} H_n(M) \geq 1$, then M is diffeomorphic to $\big(\natural_r(S^n \times D^{n+1})\big) \cup_f \big(\natural_r(S^n \times D^{n+1})\big)$, where $f : \#_r(S^n \times S^n) \to \#_r(S^n \times S^n)$ is a diffeomorphism which induces the identity on homology.*

Proof. Since M is $(n-1)$-connected, by the Hurewicz theorem, the Hurewicz map $\pi_n(M) \to H_n(M)$ is an isomorphism. By Whitney embedding theorem and transversality, a basis $\{x_1, \dots, x_r\}$ of $\pi_n(M)$ can be represented by disjoint embedded spheres $S^n_1, \dots, S^n_r \subset M$. For $n \equiv 3 \bmod 4$, $\pi_{n-1} \mathrm{SO}(n+1) = 0$, therefore the normal bundles of these embedded spheres are trivial. Choose a framing of the normal bundle, we have submanifolds $S^n_i \times D^{n+1} \subset M$ for $i = 1, \dots r$. Connecting these submanifolds by tubes $D^{2n} \times [0, 1]$, we have a submanifold $V_1 = \natural_r(S^n \times D^{n+1}) \subset M$. From the exact sequence

$$\cdots \to H_{k+1}(M, M - \mathring{V}_1) \to H_k(M - \mathring{V}_1) \to H_k(M) \to H_k(M, M - \mathring{V}_1) \to \cdots$$

and the isomorphisms $H_k(M, M - \mathring{V}_1) \cong H_k(V_1, \partial V_1) \cong H^{2n+1-k}(V_1)$ (excision and Poincaré duality), one may deduce that $H_k(M - \mathring{V}_1) = 0$ for $k \neq 0, n$, and the homomorphism $H_n(M - \mathring{V}_1) \to H_n(M)$ induced by the inclusion map is an isomorphism since $H_{n+1}(M) \to H_{n+1}(M, M - \mathring{V}_1)$ coincides with the homomorphism $H_{n+1}(M) \to \mathbb{Z}^r$, $u \mapsto (u \cdot x_1, \dots, u \cdot x_r)$, where $u \cdot x_k$ denotes the intersection number. Let $\bar{x}_1, \dots, \bar{x}_r \in H_n(M - \mathring{V}_1)$ be the preimages of $x_1, \dots, x_r$, which form a basis of $H_n(M - \mathring{V}_1)$. By the same process, we have a submanifold $V_2 = \natural_r(S^n \times D^{n+1}) \subset M - \mathring{V}_1$. By construction, the inclusion $V_2 \to M - \mathring{V}_1$ induces an isomorphism $H_k(V_2) \cong H_k(M - \mathring{V}_1)$ for all k. Therefore $H_k(M - (\mathring{V}_1 \cup \mathring{V}_2), \partial V_2) \cong H_k(M - \mathring{V}_1, V_2) = 0$ for all k. Hence $M - (\mathring{V}_1 \cup \mathring{V}_2)$ is an h-cobordism and there is a diffeomorphism $F : M - (\mathring{V}_1 \cup \mathring{V}_2) \to \big(\#_r(S^n \times S^n)\big) \times [0, 1]$, such that the restriction of F on $\partial V_2 = \#_r(S^n \times S^n)$ is the identity. Define $f = F|_{\partial V_1} : \#_r(S^n \times S^n) \to \#_r(S^n \times S^n)$; then M is diffeomorphic to the twisted double $\big(\natural_r(S^n \times D^{n+1})\big) \cup_f \big(\natural_r(S^n \times D^{n+1})\big)$.

Now we look at the isomorphism $f_* : H_n\big(\#_r(S^n \times S^n)\big) \to H_n\big(\#_r(S^n \times S^n)\big)$ induced by f. Let $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ be the standard symplectic basis of $H_n\big(\#_r(S^n \times S^n)\big)$ represented by $S^n_i \times \{*\}$ and $\{*\} \times S^n_i$. Let $A_i \subset M - (\mathring{V}_1 \cup \mathring{V}_2) \subset M$ be an embedded n-sphere representing the homology class x_i for $i = 1, \dots, r$. Then by the constructions of V_1 and V_2, under the inclusions $\partial V_j \to M - (\mathring{V}_1 \cup \mathring{V}_2)$,

$j = 1, 2$, the image of e_i is $[A_i]$. Since $F|_{\partial V_2}$ is the identity, we get $f_*(e_i) = e_i$. Also note that f_* preserves the intersection form, which is a standard symplectic form. Therefore the isomorphism $f_* : H_n\big(\#_r(S^n \times S^n)\big) \to H_n\big(\#_r(S^n \times S^n)\big)$ is represented by the symplectic matrix

$$\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \in \mathrm{Sp}(2r, \mathbb{Z}).$$

In the segment of the Mayer–Vietoris sequence

$$H_n\Big(\mathop{\#}_r (S^n \times S^n)\Big) \xrightarrow{(i_{1*}, i_{2*})} H_n(V_1) \oplus H_n(V_2) \to H_n(M) \to 0$$

under the basis $\{e_1, \ldots, e_r, f_1, \ldots, f_r\}$ of $H_n\big(\#_r(S^n \times S^n)\big)$ and under the basis $\{x_1, \ldots, x_r, \bar{x}_1, \ldots \bar{x}_r\}$ of $H_n(V_1) \oplus H_n(V_2)$ the homomorphism (i_{1*}, i_{2*}) is represented by the matrix

$$\begin{pmatrix} I & I \\ 0 & A \end{pmatrix}.$$

Since $H_n(M)$ is a free abelian group of rank r, we have $A = 0$. Therefore the induced isomorphism f_* is the identity. □

Proposition A.2. *Assume that* $n \equiv 3 \bmod 4$. *Let* M *be an* $(n-1)$*-connected* $(2n+1)$*-manifold with torsion free homology and let* $l(M)$ *be the divisibility of* $p_{\frac{n+1}{4}}(M)$. *Then* $l(M)$ *is divisible by* $a_{\frac{n+1}{4}} \cdot b_n \cdot \big(\frac{n-1}{2}\big)!$ *where*

$$a_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad b_n = \begin{cases} 2 & \text{if } n = 3, 7, \\ 1 & \text{if } n \neq 3, 7. \end{cases}$$

Proof. In Proposition A.1 we have seen that the manifold M is a twisted double $\big(\natural_r(S^n \times D^{n+1})\big) \cup_f \big(\natural_r(S^n \times D^{n+1})\big)$, where $f : \#_r(S^n \times S^n) \to \#_r(S^n \times S^n)$ is a diffeomorphism which induces the identity on homology. By the embedding theorem of Haefliger [1961], homotopic embeddings of S^n into $\#_r(S^n \times S^n)$ are isotopic. Since f_* is the identity, after an isotopy we may assume $f|_{\{*\} \times S^n_i}$ is the identity. Then a basis of $H_{n+1}(M)$ is represented by $S^{n+1}_i = D^{n+1}_i \cup D^{n+1}_i$, $i = 1, \ldots, r$. The restriction of the stable tangent bundle of M on S^{n+1}_i, $\tau M|_{S^{n+1}_i}$, is the direct sum of the normal bundle of S^{n+1}_i in M and the stable tangent bundle of S^{n+1}_i. The former is an n-dimensional vector bundle, and the latter is stably trivial. Therefore, the clutching function of $\tau M|_{S^{n+1}_i}$ is in the image of $\pi_n(\mathrm{SO}(n)) \to \pi_n(\mathrm{SO})$, which is the full group $\pi_n(\mathrm{SO})$ when $n > 7$, or is a subgroup of index 2 when $n = 3, 7$. It is shown in [Kervaire 1959, Lemma 1.1] that for a stable vector bundle ξ over S^{n+1}, $p_{\frac{n+1}{4}}(\xi)$ is an $a_{\frac{n+1}{4}}\big(\frac{n-1}{2}\big)!$-fold multiple of the clutching function of ξ under the identification $H^{n+1}(S^{n+1}; \pi_n(\mathrm{SO})) = \pi_n(\mathrm{SO})$, where $a_m = 2$ when m is odd, and $a_m = 1$ when m is even. This finishes the proof of the proposition. □

Proposition A.3. *Assume that $n \equiv 3 \bmod 4$. For any natural number l divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$, there exists a unique $(n+1)$-dimensional real vector bundle ξ_l over S^{n+1} whose Euler class and $\frac{n+1}{4}$-th Pontryagin class are*

$$e(\xi_l) = 0, \quad p_{\frac{n+1}{4}}(\xi_l) = l \cdot \omega,$$

respectively, where ω is the preferred generator of $H^{n+1}(S^{n+1})$.

Proof. For any natural number l divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$, according to [Kervaire 1959, Lemma 1.1] there exists an $(n+2)$-dimensional real vector bundle η over S^{n+1} such that $p_{\frac{n+1}{4}}(\eta) = l \cdot \omega$. Let $\alpha \in \pi_n \operatorname{SO}(n+2)$ correspond to the bundle η (that is, α is represented by the clutching function of η). Since $\pi_{n+1} \operatorname{SO}(n+2)$ is torsion for $n \equiv 3 \bmod 4$ by [Kervaire 1960], it follows that $\pi_n \operatorname{SO}(n+1)$ is computed from the homotopy exact sequence for the fiber bundle $\operatorname{SO}(n+1) \to \operatorname{SO}(n+2) \to \operatorname{SO}(n+2)/\operatorname{SO}(n+1) = S^{n+1}$:

$$0 \to \pi_{n+1} S^{n+1} \xrightarrow{\partial} \pi_n \operatorname{SO}(n+1) \xrightarrow{i_*} \pi_n \operatorname{SO}(n+2) \to 0. \tag{5}$$

Since $\pi_n \operatorname{SO}(n+2) = \pi_n \operatorname{SO} = \mathbb{Z}$ for $n \equiv 3 \bmod 4$, it follows that the exact sequence (5) splits and hence there is a homomorphism $s : \pi_n \operatorname{SO}(n+2) \to \pi_n \operatorname{SO}(n+1)$ such that the composition $i_* s$ is the identity map.

Let $s(\eta)$ be the $(n+1)$-dimensional real vector bundle over S^{n+1} corresponding to $s(\alpha) \in \pi_n \operatorname{SO}(n+1)$. Note first that η is the stabilization of $s(\eta)$ since $i_* s$ is the identity map. This implies that $p_{\frac{n+1}{4}}(s(\eta)) = p_{\frac{n+1}{4}}(\eta) = l \cdot \omega$.

Next we claim that the Euler class $e(s(\eta))$ is even. The reason is as follows. When $n = 3, 7$, one has $b_n = 2$ and hence $p_{\frac{n+1}{4}}(\eta) = l \cdot \omega$ is divisible by $a_{\frac{n+1}{4}} \cdot 2 \cdot \left(\frac{n-1}{2}\right)!$. This implies that the class α of the clutching function of the bundle η is even in the homotopy group $\pi_n \operatorname{SO}(n+2) = \mathbb{Z}$ [Kervaire 1959, Lemma 1.1]. Thus $s(\alpha) \in \pi_n \operatorname{SO}(n+1)$ is divisible by 2. Since taking Euler class induces a homomorphism $e : \pi_n \operatorname{SO}(n+1) \to H^{n+1}(S^{n+1})$ (see [Milnor and Stasheff 1974, Lemma 20.10]), this implies that $e(s(\eta)) = e(s(\alpha))$ is even. When $n \neq 3, 7$, since S^n is not parallelizable, it follows from [Kervaire 1959, Lemma 6.4] that $w_{n+1}(\eta) = 0$. Hence $w_{n+1}(s(\eta)) = w_{n+1}(\eta) = 0$. In terms of the equation $e(s(\eta)) \bmod 2 = w_{n+1}(s(\eta))$ (see [Bredon 1993, p. 421, 17.2]), one has $e(s(\eta)) \bmod 2 = 0$ and hence $e(s(\eta))$ is even.

Let ι be the generator of $\pi_{n+1} S^{n+1}$ represented by the identity map. Since $\partial(\iota) \in \pi_n \operatorname{SO}(n+1)$ corresponds to the tangent bundle of S^{n+1}, it follows from $\chi(S^{n+1}) = 2$ that there is an integer m such that $e(s(\alpha) + m\partial\iota) = 0$. Let ξ_l denote the $(n+1)$-dimensional real vector bundle corresponding to $s(\alpha) + m\partial\iota$. Then $e(\xi_l) = 0$ and

$$p_{\frac{n+1}{4}}(\xi_l) = p(s(\alpha)) = l,$$

where $p : \pi_n \operatorname{SO}(n+1) \to H^{n+1}(S^{n+1})$ is the homomorphism induced by taking the $\frac{n+1}{4}$-th Pontryagin class (see [Milnor and Stasheff 1974, Lemma 20.10]). This shows the existence of ξ_l.

To show the uniqueness of ξ_l, let ξ' be any $(n+1)$-dimensional real vector bundle over S^{n+1} such that

$$e(\xi') = 0, \quad p_{\frac{n+1}{4}}(\xi') = l \cdot \omega. \tag{6}$$

Since there are integers x, y such that ξ' corresponds to the class $xs(\alpha_0) + y\partial\iota$ where α_0 is a generator of $\pi_n \operatorname{SO}(n+2)$, (6) implies that x, y must satisfy the system of equations

$$xe(s(\alpha_0)) + ye(\partial\iota) = 0, \quad xp(s(\alpha_0)) + yp(\partial\iota) = l \cdot \omega.$$

This system of equations has at most one solution because $p(\partial\iota) = 0$ and both of $p(s(\alpha_0))$ and $e(\partial\iota)$ are nonzero. This shows the uniqueness of ξ_l. □

When $n > 3$ and $n \neq 7$, Wall [1967, Theorem 7] determined the diffeomorphism classes of $(n-1)$-connected almost closed $(2n+1)$-manifolds in terms of certain systems of invariants including homology, characteristic classes, linking form and so on. Thus this gives a system of complete invariants of $(n-1)$-connected $(2n+1)$-manifolds up to almost diffeomorphisms. On the other hand, all possible values of the invariants can be realized by these $(2n+1)$-manifolds except for $n = 4, 8, 9$ according to the work of Senger and Zhang [2023, Theorem 1.10]. In fact, they determined precisely which exotic $2n$-spheres are the boundaries of $(n-1)$-connected almost closed $(2n+1)$-manifolds when $n \geq 3$. As a consequence, they deduce that every $(n-1)$-connected almost closed $(2n+1)$-manifold may be filled in to obtain a closed manifold when $n \geq 3$ and $n \neq 4, 8, 9$.

Moreover, when $n = 3, 7$, Wilkens [1971, Theorem 3; 1972] proved that the almost diffeomorphism classes of $(n-1)$-connected $(2n+1)$-manifolds M are classified by the invariants including homology, characteristic classes, linking form, modulo a finite ambiguity if the torsion part of $H_n(M)$ is of even order. Crowley [2002, Theorem B] completed the almost diffeomorphism classification by defining a family of quadratic refinement of the linking form. For more background on the classification of $(n-1)$-connected $(2n+1)$-manifold, see [Crowley and Nordström 2019; Crowley 2002]. To prove Theorem 1.1, we only need the special cases of the results obtained by Wall and Wilkens as follows.

Theorem A.4 [Wall 1967, Theorem 7]. *Assume that $n \equiv 5 \bmod 8$. Then the homology group $H_n(P)$ of an $(n-1)$-connected $(2n+1)$-manifold P with free homology is a complete invariant of almost diffeomorphisms. That is, if P_1 and P_2 are two $(n-1)$-connected $(2n+1)$-manifolds with free homology, then P_1 is almost diffeomorphic to P_2 if and only if $H_n(P_1)$ is isomorphic to $H_n(P_2)$.*

Proposition A.5 [Wall 1967, Theorem 7; Wilkens 1972, Theorem 3]. *Assume that $n \equiv 3 \bmod 4$. Then $\{H_n(P), p_{\frac{n+1}{4}}(P)\}$ of an $(n-1)$-connected $(2n+1)$-manifold P with free homology is a set of complete invariants of almost diffeomorphisms. That is,* *if P_1 and P_2 are two $(n-1)$-connected $(2n+1)$-manifolds with free homology, then there exists an almost diffeomorphism $f : P_1 \to P_2$ if and only if there is an isomorphism $F : H^{n+1}(P_2) \to H^{n+1}(P_1)$ such that $F(p_{\frac{n+1}{4}}(P_2)) = p_{\frac{n+1}{4}}(P_1)$.*

Proof of Theorem 1.1. When $n \equiv 5 \bmod 8$, by Theorem A.4, it suffices to show that there is an nonnegative integer r such that $\#_r\, S^n \times S^{n+1}$ and M have isomorphic n-homology groups. This follows directly from the assumption that $H_n(M)$ is free and by taking r to be rank $H_n(M)$. Now assume that $n \equiv 3 \bmod 4$. If $H_n(M) = 0$, then M is a homotopy sphere and hence is almost diffeomorphic to S^{2n+1}. If $H_n(M)$ is not trivial, take $r = \operatorname{rank} H_n(M) - 1$. It follows from Proposition A.2 that the natural number $l(M)$ is divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$. By Proposition A.3, let $X_{l(M)}$ denote the total space of the sphere bundle associated to $\xi_{l(M)}$ and let $M' = \#_r\, S^n \times S^{n+1} \# X_{l(M)}$. Then there is an isomorphism $F : H^{n+1}(M') \to H^{n+1}(M)$ such that $F(p_{\frac{n+1}{4}}(M')) = p_{\frac{n+1}{4}}(M)$. Thus it follows from Proposition A.5 that M is almost diffeomorphic to $\#_r\, S^n \times S^{n+1} \# X_{l(M)}$. □

Acknowledgements

The authors would like to thank Haibao Duan for bringing the topic to their attention, and thank Matthias Kreck for helpful conversations. We are grateful to the referees for valuable suggestions that improved the exposition of this paper. Y. Jiang is partially supported by NSFC 11801298 and NSFC 12371070. Y. Su is partially supported by NSFC 12071462.

References

[Adams 1963] J. F. Adams, "On the groups $J(X)$, I", *Topology* **2** (1963), 181–195. MR Zbl

[Adams 1965] J. F. Adams, "On the groups $J(X)$, II", *Topology* **3** (1965), 137–171. MR Zbl

[Borel and Hirzebruch 1960] A. Borel and F. Hirzebruch, "Characteristic classes and homogeneous spaces, III", *Amer. J. Math.* **82** (1960), 491–504. MR Zbl

[Bredon 1993] G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics **139**, Springer, 1993. MR Zbl

[Browder 1972] W. Browder, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete **65**, Springer, 1972. MR Zbl

[Brumfiel 1971] G. Brumfiel, "Homotopy equivalences of almost smooth manifolds", *Comment. Math. Helv.* **46** (1971), 381–407. MR Zbl

[Crowley 2002] D. J. Crowley, *The classification of highly connected manifolds in dimensions 7 and 15*, Ph.D. thesis, Indiana University, 2002, available at https://www.proquest.com/docview/305508775. Zbl

[Crowley and Nordström 2019] D. Crowley and J. Nordström, "The classification of 2-connected 7-manifolds", *Proc. Lond. Math. Soc.* (3) **119**:1 (2019), 1–54. MR Zbl

[Crowley and Wraith 2017] D. Crowley and D. J. Wraith, "Positive Ricci curvature on highly connected manifolds", *J. Differential Geom.* **106**:2 (2017), 187–243. MR Zbl

[Duan 2022] H. Duan, "Circle actions and Suspension operations on Smooth manifolds", preprint, 2022. arXiv 2202.06225v1

[Duan and Liang 2005] H. Duan and C. Liang, "Circle bundles over 4-manifolds", *Arch. Math.* (*Basel*) **85**:3 (2005), 278–282. MR Zbl

[Galaz-García and Reiser 2025] F. Galaz-García and P. Reiser, "Free torus actions and twisted suspensions", *Forum Math. Sigma* **13** (2025), art. id. e3. MR Zbl

[Goldstein and Lininger 1972] R. Z. Goldstein and L. Lininger, "A classification of 6-manifolds with free S^1 actions", pp. 316–323 in *Proceedings of the Second Conference on Compact Transformation Groups*, *I* (Amherst, MA, 1971), edited by H. T. Ku et al., Lecture Notes in Math. **298**, Springer, 1972. MR Zbl

[Haefliger 1961] A. Haefliger, "Differentiable imbeddings", *Bull. Amer. Math. Soc.* **67** (1961), 109–112. MR Zbl

[Hambleton and Su 2013] I. Hambleton and Y. Su, "On certain 5-manifolds with fundamental group of order 2", *Q. J. Math.* **64**:1 (2013), 149–175. MR Zbl

[Hirzebruch 1995] F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Springer, 1995. MR Zbl

[Hsiang 1966] W.-c. Hsiang, "A note on free differentiable actions of S^1 and S^3 on homotopy spheres", *Ann. of Math.* (2) **83** (1966), 266–272. MR Zbl

[Jiang 2014] Y. Jiang, "Regular circle actions on 2-connected 7-manifolds", *J. Lond. Math. Soc.* (2) **90**:2 (2014), 373–387. MR Zbl

[Kervaire 1959] M. A. Kervaire, "A note on obstructions and characteristic classes", *Amer. J. Math.* **81** (1959), 773–784. MR Zbl

[Kervaire 1960] M. A. Kervaire, "Some nonstable homotopy groups of Lie groups", *Illinois J. Math.* **4** (1960), 161–169. MR Zbl

[Kervaire and Milnor 1963] M. A. Kervaire and J. W. Milnor, "Groups of homotopy spheres, I", *Ann. of Math.* (2) **77** (1963), 504–537. MR Zbl

[Lee 1968] R. Lee, "Non-existence of free differentiable actions of S^1 and Z_2 on homotopy spheres", pp. 208–209 in *Proc. Conf. on Transformation Groups* (New Orleans, LA, 1967), edited by P. S. Mostert, Springer, 1968. MR Zbl

[Lininger 1972] L. Lininger, "S^1 actions on 6-manifolds", *Topology* **11** (1972), 73–78. MR Zbl

[Milnor and Stasheff 1974] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies **76**, Princeton Univ. Press, 1974. MR Zbl

[Montgomery and Yang 1968] D. Montgomery and C. T. Yang, "Differentiable actions on homotopy seven spheres, II", pp. 125–134 in *Proc. Conf. on Transformation Groups* (New Orleans, LA, 1967), edited by P. S. Mostert, Springer, 1968. MR Zbl

[Palais 1960] R. S. Palais, "Extending diffeomorphisms", *Proc. Amer. Math. Soc.* **11** (1960), 274–277. MR Zbl

[Quillen 1971] D. Quillen, "The Adams conjecture", *Topology* **10** (1971), 67–80. MR Zbl

[Schultz 1971] R. Schultz, "The nonexistence of free S^1 actions on some homotopy spheres", *Proc. Amer. Math. Soc.* **27** (1971), 595–597. MR Zbl

[Senger and Zhang 2023] A. Senger and A. Y. Zhang, "Inertia groups of $(n-1)$-connected $2n$-manifolds", preprint, 2023. arXiv 2211.00782v2

[Wall 1967] C. T. C. Wall, "Classification problems in differential topology, VI: Classification of $(s-1)$-connected $(2s+1)$-manifolds", *Topology* **6** (1967), 273–296. MR Zbl

[Wall 1999] C. T. C. Wall, *Surgery on compact manifolds*, 2nd ed., Mathematical Surveys and Monographs **69**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Wilkens 1971] D. L. Wilkens, *Closed* $(s-1)$*-connected* $(2s+1)$*-manifolds,* $s = 3, 7$, Ph.D. thesis, University of Liverpool, 1971, available at https://webhomes.maths.ed.ac.uk/~v1ranick/papers/wilkensphd.pdf.

[Wilkens 1972] D. L. Wilkens, "Closed $(s-1)$-connected $(2s+1)$-manifolds, $s = 3, 7$", *Bull. London Math. Soc.* **4** (1972), 27–31. MR Zbl

Received October 3, 2024. Revised April 23, 2025.

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PACIFIC JOURNAL OF MATHEMATICS

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 338 No. 1 September 2025