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We determine those $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology that admit free circle actions up to almost diffeomorphism, provided that $n \equiv 5, 7 \bmod 8$.

1. Introduction

In this paper, unless otherwise stated, all manifolds under consideration are smooth, closed and oriented, and an oriented principal S^1-bundle is called an S^1*-bundle* for short. We are interested in the problem *when does a manifold M admit a free smooth action by the unit circle S^1* (free circle action for short in the sequel). There are studies of this problem for certain families of manifolds by various authors. For instance, when M is a homotopy sphere, this problem was studied in [Hsiang 1966; Lee 1968; Montgomery and Yang 1968; Schultz 1971]; when M is an $(n-1)$-connected $(2n+1)$-manifold, this problem has been solved for $n = 2$ [Duan and Liang 2005] and $n = 3$ [Jiang 2014]. For further examples see [Goldstein and Lininger 1972; Lininger 1972; Duan 2022; Galaz-García and Reiser 2025].

In this note we consider this problem for $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology when $n \geq 3$. The classification of $(n-1)$-connected $(2n+1)$-manifolds up to almost diffeomorphism was obtained in [Wall 1967; Wilkens 1972; Senger and Zhang 2023; Crowley 2002]. Recall that two n-manifolds M_1 and M_2 are *almost diffeomorphic* if there is a homotopy n-sphere Σ such that the connected sum $M_1 \# \Sigma$ is diffeomorphic to M_2 (see [Crowley and Wraith 2017, p. 223]). Carrying on [Wall 1967, Theorem 7; Wilkens 1972, Theorem 3], the following results will be proved in the Appendix.

Theorem 1.1. *Let M be an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology. When $n \equiv 5 \bmod 8$, M is almost diffeomorphic to the connected sum $\#_r(S^n \times S^{n+1})$ of r copies of $S^n \times S^{n+1}$ with $r \geq 0$. When $n \equiv 3 \bmod 4$, M is almost diffeomorphic to the sphere S^{2n+1} or $\#_r(S^n \times S^{n+1}) \# X_{l(M)}$ with $r \geq 0$.*

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Here we explain the manifold $X_{l(M)}$ in the above theorem. For $n \equiv 3 \bmod 4$, let $l(M)$ be the divisibility of the $\frac{n+1}{4}$-th Pontryagin class $p_{\frac{n+1}{4}}(M) \in H^{n+1}(M) = \mathbb{Z}$, i.e., $p_{\frac{n+1}{4}}(M)$ is an $l(M)$-fold multiple of a primitive element with $l(M) \geq 0$. In fact, it is well known (see Proposition A.2) that $l(M)$ is divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$ where $a_k = 1$ for k even and $a_k = 2$ for k odd, and $b_n = 2$ for $n = 3, 7$ and $b_n = 1$ for $n \neq 3, 7$. On the other hand, for any natural number l divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$, there exists a unique $(n+1)$-dimensional real vector bundle ξ_l over S^{n+1} whose Euler class and $\frac{n+1}{4}$-th Pontryagin class are

$$e(\xi_l) = 0, \quad p_{\frac{n+1}{4}}(\xi_l) = l \cdot \omega,$$

respectively, where ω is the preferred generator of $H^{n+1}(S^{n+1})$ (see Proposition A.3). Let X_l be the total space of the sphere bundle associated to ξ_l; then X_l is an $(n-1)$-connected $(2n+1)$-manifold with $H^{n+1}(X_l) \cong \mathbb{Z}$ and $l(X_l) = l$. See the Appendix for more details and a proof of Theorem 1.1.

As an application of this classification, we determine those $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology that admit free circle actions up to almost diffeomorphism, provided that $n \equiv 5, 7 \bmod 8$.

Theorem 1.2. *Let M be an $(n-1)$-connected $(2n+1)$-manifold where $H_n(M)$ is free and $n \equiv 5 \bmod 8$. Then there exists a homotopy sphere Σ such that $M \# \Sigma$ admits a free circle action.*

Recall that the Bernoulli numbers $B_1, B_2, \ldots$ are the coefficients in the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \frac{B_3}{6!}z^6 - \cdots,$$

(see [Milnor and Stasheff 1974, Appendix B]). For any rational number r let $\mathrm{den}(r)$ denote the denominator of r expressed as a fraction in lowest terms.

Theorem 1.3. *Let M be an $(n-1)$-connected $(2n+1)$-manifold where $H_n(M)$ is free and $n \equiv 7 \bmod 8$. Then there exists a homotopy sphere Σ such that $M \# \Sigma$ admits a free circle action if and only if one of the following conditions hold*:

(1) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1})$ for some integer $r \geq 0$;*

(2) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1}) \# X_l$ for some integer $r \geq 0$ and some nonnegative integer l divisible by $\left(\frac{n-1}{2}\right)! \, \mathrm{den}\left(\frac{B_{(n+1)/4}}{n+1}\right)$.*

Remark 1.4. In the above theorem, condition (1) is equivalent to that the n-th Betti number $b_n(M)$ of M is even and $l(M) = 0$; condition (2) is equivalent to that $b_n(M)$ is odd and $l(M)$ is divisible by $\left(\frac{n-1}{2}\right)! \, \mathrm{den}\left(\frac{B_{(n+1)/4}}{n+1}\right)$.

In the case $n \equiv 3 \bmod 8$, we obtain a partial version of Theorem 1.3 as follows.

Theorem 1.5. *Let M be an $(n-1)$-connected $(2n+1)$-manifold where $H_n(M)$ is free and $n \equiv 3 \bmod 8$. If there is a homotopy sphere Σ such that $M \# \Sigma$ admits a free circle action, then one of the following conditions hold:*

(1) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1})$ for some integer $r \geq 0$;*

(2) *M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1}) \# X_l$ for some integer $r \geq 0$ and some nonnegative integer l divisible by $\big(\frac{n-1}{2}\big)!\,\mathrm{den}\big(\frac{B_{(n+1)/4}}{n+1}\big)$.*

Conversely, for any integer $r \geq 0$, the connected sum $\#_{2r}(S^n \times S^{n+1})$ admits a free circle action; for any nonnegative integer r and nonnegative integer l divisible by $2 \cdot \big(\frac{n-1}{2}\big)!\,\mathrm{den}\big(\frac{B_{(n+1)/4}}{n+1}\big)$, there exists homotopy sphere Σ such that the connected sum $\#_{2r}(S^n \times S^{n+1}) \# X_l \# \Sigma$ admits a free circle action.

In Section 2 we prove Theorem 1.2 by a direct geometric construction. To prove Theorems 1.3 and 1.5, one observes that a manifold M admits a free circle action if and only if it is the total space of an S^1-bundle over a manifold N, and N is the orbit space of the action. The key ingredient in the proof is the analysis of the topological invariants of the orbit space. In Sections 3 and 4 we study the cohomology ring and the Pontryagin class of the orbit space, respectively. Finally we prove Theorems 1.3 and 1.5 in Section 5.

2. A construction of free circle actions

In this section we recall a construction of free circle actions and prove Theorem 1.2.

Given two m-manifolds N_1 and N_2, the connected sum $N_1 \# N_2$ is constructed as follows (see [Kervaire and Milnor 1963, Section 2]). Take embeddings

$$i_k : D^m \to N_k, \quad k = 1, 2,$$

such that i_1 preserves the orientation and i_2 reverses the orientation. We obtain $N_1 \# N_2$ from the disjoint sum $N_1 \setminus i_1(0) + N_2 \setminus i_2(0)$ by identifying $i_1(tu)$ with $i_2((1-t)u)$ where $u \in S^{m-1}$, $0 < t < 1$. Now given two oriented S^1-bundles $E_k \to N_k$, $k = 1, 2$, a new S^1-bundle over $N_1 \# N_2$ is constructed as follows. Take embeddings $j_k : S^1 \times D^m \to E_k$, $k = 1, 2$, such that j_k covers i_k, i.e., local trivializations of E_k over an embedded disk $D^m \subset N_k$. The *S^1-connected sum* $E_1 \#_{S^1} E_2$ is an $(m+1)$-manifold obtained from the disjoint sum $E_1 \setminus j_1(S^1 \times 0) + E_2 \setminus j_2(S^1 \times 0)$ by identifying $j_1(x, tu)$ with $j_2(x, (1-t)u)$ where $(x, u) \in S^1 \times S^{m-1}$, $0 < t < 1$. This construction does not depend on the choice of the embedded disk D^m by Palais' disk theorem [1960, Theorem B], and does not depend on the trivializations since the structure group of our fiber bundle is connected. The bundle projections $E_k \to N_k$, $k = 1, 2$, induce an S^1-bundle projection $E_1 \#_{S^1} E_2 \to N_1 \# N_2$ (see [Hambleton and Su 2013, Section 3]).

Let $m \geq 3$ and let $\alpha : S^1 \to \mathrm{SO}(m)$ represent the generator of $\pi_1 \mathrm{SO}(m) = \mathbb{Z}_2$. Let $\tau : S^1 \times D^m \to S^1 \times D^m$ be the homeomorphism given by $\tau(t, x) = (t, \alpha(t)x)$. Take an embedding $f : D^m \to N$ for an m-manifold N. Let $\Sigma_0 N$ and $\Sigma_1 N$ be the manifolds obtained by surgery along the embeddings

$$f_1 := \mathrm{id}_{S^1} \times f : S^1 \times D^m \to S^1 \times N,$$
$$f_2 := (\mathrm{id}_{S^1} \times f) \circ \tau : S^1 \times D^m \to S^1 \times N,$$

respectively, namely

$$\Sigma_i N := (S^1 \times N \setminus f_i(S^1 \times \mathring{D}^m)) \cup_{f_i|S^1 \times S^{m-1}} (D^2 \times S^{m-1}), \quad i = 1, 2,$$

where $\mathring{D}^m :=$ Interior D^m.

Lemma 2.1 [Duan 2022, Theorem B and Proposition 3.2]. *Let $E \to B$ be an S^1-bundle over an m-manifold with E simply connected and $m \geq 4$. Then for any simply connected m-manifold N, the S^1-connected sum $E \#_{S^1} (S^1 \times N)$ is diffeomorphic to $E \# \Sigma_0 N$ if B is nonspin and to $E \# \Sigma_1 N$ if B is spin.*

There are diffeomorphisms

$$\Sigma_0(S^p \times S^q) \cong \Sigma_1(S^p \times S^q) \cong (S^p \times S^{q+1}) \# (S^{p+1} \times S^q)$$

for any $p \leq q$ with $q \geq 3$

Proposition 2.2. *Let E be a simply connected $(2n+1)$-manifold and $n \geq 3$. If E admits a free circle action, then $\#_2(S^n \times S^{n+1}) \# E$ admits a free circle action.*

Proof. Let B be the orbit space of a free circle action on E. Then E is the total space of an S^1-bundle over B, and $E \#_{S^1} (S^1 \times S^n \times S^n)$ is the total space of an S^1-bundle over $B \# (S^n \times S^n)$. By Lemma 2.1, $E \#_{S^1} (S^1 \times S^n \times S^n)$ is diffeomorphic to either $E \# \Sigma_0(S^n \times S^n)$ or $E \# \Sigma_1(S^n \times S^n)$, and hence is diffeomorphic to $\#_2(S^n \times S^{n+1}) \# E$ by the second paragraph of Lemma 2.1. This proves the proposition. □

Corollary 2.3. *Let r, n be positive integers and $n \geq 3$. If Σ is a homotopy $(2n+1)$-sphere admitting a free circle action, then $\#_{2r}(S^n \times S^{n+1}) \# \Sigma$ admits a free circle action.*

Note that there are many exotic $(2n+1)$-spheres admitting free circle actions. In fact, the set of homotopy $(2n+1)$-spheres admitting free circle actions has been determined for $n = 3, 4, 5, 6$ in [Montgomery and Yang 1968; Brumfiel 1971, p. 402, Theorem I.10(i)].

Corollary 2.4. *Let r, n be nonnegative integers and $n \geq 2$. Then $\#_r(S^n \times S^{n+1})$ admits a free circle action.*

Proof. The case $n = 2$ has been proved in [Duan and Liang 2005, Corollary 2]. Note that S^{2n+1} admits a free circle action with orbit space $\mathbb{C}P^n$ and $S^n \times S^{n+1}$ admits a free circle action with orbit space $\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1}$ if n is odd or $S^n \times \mathbb{C}P^{\frac{n}{2}}$ if n is even. It follows that the case $n \geq 3$ is a direct consequence of Proposition 2.2. □

Proof of Theorem 1.2. By Theorem 1.1 there is a homotopy sphere Σ such that $M \# \Sigma$ is diffeomorphic to $\#_r(S^n \times S^{n+1})$ where r is the rank of $H_n(M)$. Hence the theorem follows directly from Corollary 2.4. □

3. Cohomology of the orbit space

A $(2n+1)$-manifold M admits a free circle action if and only if it is the total space of an S^1-bundle over a $2n$-manifold N. In this section we analyze the cohomology ring of N when M is $(n-1)$-connected with torsion free homology. In the next section we study the Pontryagin classes of N.

Lemma 3.1. *Let $n > 1$ be an odd integer. Let $S^1 \times M \to M$ be a free circle action on an $(n-1)$-connected $(2n+1)$-manifold M with torsion free homology. Then the orbit space N is a simply connected $2n$-manifold whose cohomology ring $H^*(N)$ is isomorphic to $H^*(\#_r(S^n \times S^n) \# \mathbb{C}P^n)$ or $H^*(\#_r(S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1}))$ for $r \geq 0$.*

Conversely, every simply connected $2n$-manifold N with

$$H^*(N) \cong H^*\Big(\mathop{\#}_r (S^n \times S^n) \# \mathbb{C}P^n\Big) \quad \textit{or} \quad H^*\Big(\mathop{\#}_r (S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\Big)$$

for $r \geq 0$ can be realized as the orbit space of some free circle action on an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology.

Proof. To prove the lemma, it suffices to verify that for an S^1-bundle $S^1 \to M \to N$, the manifolds M and N satisfy the conditions stated in the lemma. The fundamental groups of M and N are related by the homotopy exact sequence of the S^1-bundle

$$\pi_1(S^1) \to \pi_1(M) \to \pi_1(N) \to 0. \tag{1}$$

The cohomology groups of M and N are related by the Gysin sequence

$$H^{j-2}(N) \xrightarrow{-\cup t} H^j(N) \to H^j(M) \to H^{j-1}(N) \xrightarrow{-\cup t} H^{j+1}(N), \tag{2}$$

where $t \in H^2(N)$ is the Euler class of the S^1-bundle.

Now assume that M is an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology. The exact sequence (1) implies that N is simply connected. The cohomology ring $H^*(N)$ is computed from the Gysin sequence

$$H^{i-1}(M) \to H^{i-2}(N) \xrightarrow{-\cup t} H^i(N) \to H^i(M). \tag{3}$$

Since M is $(n-1)$-connected, the exact sequence (3) implies that for $0 \le i \le n-1$,

$$H^i(N) = \begin{cases} \mathbb{Z}t^{\frac{i}{2}}, & \text{if } i \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

and for $n+3 \le i \le 2n$, the homomorphism $-\cup t : H^{i-2}(N) \to H^i(N)$ is an isomorphism. Moreover, the Gysin sequence implies that the map $H^n(N) \to H^n(M)$ is a monomorphism. Since M has torsion free homology, this implies $H^n(M)$ is free and hence $H^n(N)$ is free. Note that the rank of $H^n(N)$ must be even, because the intersection form of N is nondegenerate and skew-symmetric since n is odd. Moreover, the homomorphism $-\cup t : H^n(N) \to H^{n+2}(N)$ is trivial since $H^{n+2}(N) \cong H_{n-2}(N) = 0$. Now since $H^{n-1}(N) = \mathbb{Z}t^{\frac{n-1}{2}}$ and $H^{n+1}(N) \cong H_{n-1}(N) \cong \mathbb{Z}$, it remains to show the homomorphism $-\cup t : H^{n-1}(N) \to H^{n+1}(N)$ must be either an isomorphism or a trivial map. This is easily obtained by the Gysin sequence

$$H^{n-1}(N) \xrightarrow{-\cup t} H^{n+1}(N) \to H^{n+1}(M)$$

and the fact that $H^{n+1}(M) \cong H_n(M)$ is free.

Conversely, assume N is simply connected, the cohomology ring $H^*(N)$ is isomorphic to either $H^*\big(\#_r(S^n \times S^n)\#\mathbb{C}P^n\big)$ or $H^*\big(\#_r(S^n \times S^n)\#(\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\big)$ for some $r \ge 0$, and the Euler class t is a generator of $H^2(N) \cong \mathbb{Z}$. The exact sequence (1) implies that $\pi_1 M$ is abelian and hence $\pi_1 M \cong H_1(M)$. To prove that M is $(n-1)$-connected with torsion free homology, it suffices to show that $H_i(M)$ is trivial for $1 \le i \le n-1$ and $H_n(M)$ is torsion free.

This follows from the sequence (2). When $j = 1$, the sequence (2) implies that $H^1(M) = 0$ since $H^1(N) = 0$ and $-\cup t : H^0(N) \to H^2(N)$ is an isomorphism. When $2 \le j \le n-1$, the sequence (2) implies that $H^j(M) = 0$ because $H^{n-2}(N) = 0$ and $-\cup t : H^{i-2}(N) \to H^i(N)$ is an isomorphism for $2 \le i \le n-1$. When $j = n, n+1$, the sequence (2) implies that $H^j(M)$ is free because the cohomology groups of N are free, the homomorphism $-\cup t : H^i(N) \to H^{i+2}(N)$ is trivial for $i = n-2, n$ and $-\cup t : H^{n-1}(N) \to H^{n+1}(N)$ is either an isomorphism or a trivial map. This shows that $H_i(M)$ is trivial for $1 \le i \le n-1$ and $H_n(M) \cong H^{n+1}(M)$ is torsion free. □

4. Pontryagin classes of the orbit spaces

We have observed in the previous section that the cohomology ring of the orbit space N of a free circle action on M is isomorphic to $H^*\big(\#_r(S^n \times S^n)\#\mathbb{C}P^n\big)$ or $H^*\big(\#_r(S^n \times S^n)\#(\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\big)$. It will be shown in the proof of Theorems 1.3 and 1.5 that in the first case, the Pontryagin class $p_{\frac{n+1}{4}}(M)$ vanishes. In order to determine the Pontryagin class $p_{\frac{n+1}{4}}(M)$ in the second case, we analyze the Pontryagin class of N in this section. Since we are in the case $n \equiv 3 \bmod 4$, we assume $n = 4k-1$.

Lemma 4.1. *Let N be a simply connected $(8k-2)$-manifold whose cohomology ring is isomorphic to $H^*\big(\#_r(S^{4k-1}\times S^{4k-1})\,\#\,(\mathbb{C}P^{2k-1}\times S^{4k})\big)$ with $r\geq 0$ and $k\geq 1$. If the Pontryagin class $p_k(N)$ is a $d(N)$-fold multiple of a primitive element, then $d(N)$ is divisible by $(2k-1)!\cdot \mathrm{den}\big(\frac{B_k}{4k}\big)$.*

Proof. Let $\{\hat{A}_k(p_1,\ldots,p_k)\}$ be the multiplicative sequence of polynomials with

$$\frac{\sqrt{t}/2}{\sinh\big(\sqrt{t}/2\big)}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2^{2n}-2)B_n}{2^{2n}((2n)!)}t^n$$

as characteristic power series (see [Hirzebruch 1995, §1]). Let α_k be the coefficient of p_k in $\hat{A}_k(p_1,\ldots,p_k)$. We first show the following two facts:

(1) $\alpha_k\cdot d(N)$ is an integer.

(2) The integer $d(N)$ is divisible by $a_k\cdot(2k-1)!$, where $a_k=1$ if k is even, $a_k=2$ if k is odd.

The proof of (1) is based on the integrality of twisted $\hat{A}$-genera. Let $\mathrm{ch}(\eta)\in H^*(N;\mathbb{Q})$ be the Chern character of a virtual complex vector bundle η over N, $d\in H^2(N)$ be a cohomology class whose mod 2 reduction is the second Stiefel–Whitney class of N, $\hat{A}(N)=\sum_{i=1}^{\infty}\hat{A}_i(p_1(N),\ldots,p_i(N))$ be the $\hat{A}$-class of N, and $[N]\in H_{2n}(N)$ be the fundamental class of N. Then by [Hirzebruch 1995, Theorem 26.1.1], $\langle \mathrm{ch}(\eta)\cdot e^{d/2}\cdot\hat{A}(N),[N]\rangle$ is an integer. Now let L be the complex line bundle over N with first Chern class $c_1(L)=t$, a generator of $H^2(N)$. Take $\eta=(L-1)^{2k-1}$. By the assumption of $H^*(N)$,

$$\begin{aligned}\langle \mathrm{ch}(\eta)\cdot e^{d/2}\cdot\hat{A}(N),[N]\rangle &=\langle (e^t-1)^{2k-1}e^{d/2}\hat{A}(N),[N]\rangle\\ &=\langle t^{2k-1}e^{d/2}\hat{A}(N),[N]\rangle\\ &=\langle t^{2k-1}\hat{A}(N),[N]\rangle\\ &=\big\langle t^{2k-1}(\alpha_k p_k(N)+\cdots),[N]\big\rangle\\ &=\pm\alpha_k d(N),\end{aligned}$$

where the second and third equality signs hold because $t^{2k}=0$ and $t^{2k-1}d=0$. Moreover, the last equality sign follows from the fact that the bilinear forms

$$H^{4i}(N)\times H^{4k-4i}(N)\to H^{4k}(N)$$

induced by cup products vanish for $1\leq i\leq k-1$. This proves (1).

To prove (2), let $\pi:N_t\to N$ be the S^1-bundle over N with Euler class t, a generator of $H^2(N)$. In the segment of the Gysin sequence

$$H^{4k-2}(N)\xrightarrow{-\cup t}H^{4k}(N)\xrightarrow{\pi^*}H^{4k}(N_t)\to H^{4k-1}(N)\xrightarrow{-\cup t}H^{4k+1}(N),$$

the two homomorphisms $-\cup t$ are trivial, therefore we have a split short exact sequence

$$0 \to H^{4k}(N) \xrightarrow{\pi^*} H^{4k}(N_t) \to H^{4k-1}(N) \to 0. \tag{4}$$

Furthermore, the tangent bundle TN_t of N_t is isomorphic to the Whitney sum $\pi^*TN \oplus V$, where $V = \ker d\pi$ is the vertical bundle of the S^1-bundle. The orientable line bundle V is trivial, therefore $p_k(N_t) = \pi^* p_k(N)$ and hence $l(N_t) = d(N)$ by the sequence (4). By Lemma 3.1 N_t is a $(4k-2)$-connected $(8k-1)$-manifold. Equip N_t with a CW complex structure. Since $H^i(N_t; \pi_{i-1}\,\mathrm{SO}) = 0$ for $1 \le i \le 4k-1$, it follows that the stable tangent bundle of N_t has a cross section over the $(4k-1)$-skeleton of N_t and hence $l(N_t)$ is divisible by $a_k(2k-1)!$ according to [Kervaire 1959, Lemma 1.1]. This proves (2).

Now by (2), there is an integer β such that $d(N) = a_k(2k-1)!\beta$. Since $\alpha_k = -\frac{B_k}{2\cdot(2k)!}$(see [Borel and Hirzebruch 1960, 3.4]), we have

$$\alpha_k d(N) = -\frac{B_k}{2\cdot(2k)!} a_k(2k-1)!\beta = -\frac{B_k a_k \beta}{4k}$$

is an integer. It follows that β must be divisible by $\mathrm{den}\big(\frac{B_k a_k}{4k}\big)$ and hence $d(N) = a_k(2k-1)!\beta$ must be a multiple of $a_k(2k-1)!\,\mathrm{den}\big(\frac{B_k a_k}{4k}\big)$. Now it suffices to show that $\mathrm{den}\big(\frac{a_k B_k}{4k}\big) = \mathrm{den}\big(\frac{B_k}{4k}\big)/a_k$. If k is even, then $a_k = 1$ and this is clearly true. When k is odd, $a_k = 2$. It is known that $\mathrm{den}(B_k)$ is even (see [Milnor and Stasheff 1974, Appendix B, Theorem B.3]), hence the numerator of B_k is odd. From this it is easy to see $\mathrm{den}\big(\frac{2B_k}{4k}\big) = \mathrm{den}\big(\frac{B_k}{4k}\big)/2$. □

Lemma 4.2. *Let $a_k = 1$ when k is even, and $a_k = 2$ when k is odd. For any integer d divisible by $a_k \cdot (2k-1)! \cdot \mathrm{den}\big(\frac{B_k}{4k}\big)$, there exists an $(8k-2)$-manifold N homotopy equivalent to $\#_r(S^{4k-1} \times S^{4k-1})\#(\mathbb{C}P^{2k-1} \times S^{4k})$ for $r \ge 0$, such that the Pontryagin class $p_k(N)$ is a d-fold multiple of a primitive element.*

The proof of this lemma uses surgery theory. We recall some elementary notions here (see [Wall 1999, p. 109; Browder 1972, pp. 45–46]). Let BO_k be the classifying space of the orthogonal group O_k. Let G_k be the topological monoid of self-homotopy equivalences of S^{k-1} and let BG_k be its classifying space. The natural maps $G_k \to G_{k+1}$ and $O_k \to O_{k+1}$ induce maps $\mathrm{BG}_k \to \mathrm{BG}_{k+1}$ and $\mathrm{BO}_k \to \mathrm{BG}_{k+1}$, respectively. Then $\mathrm{BO} = \lim_{k\to\infty} \mathrm{BO}_k$ is the classifying space of stable vector bundles and $\mathrm{BG} = \lim_{k\to\infty} \mathrm{BG}_k$ is the classifying space of stable spherical fibrations. By taking the sphere bundle associated to a real vector bundle one has a forgetful map $\mathrm{BO} \to \mathrm{BG}$. The induced homomorphism $J : \pi_i(\mathrm{BO}) \to \pi_i(\mathrm{BG})$ is the J-homomorphism.

Proof. Let ξ be a stable vector bundle over S^{4k}, whose sphere bundle is trivial as a stable spherical fibration. This means that ξ is in the kernel of the J-homomorphism

$J:\pi_{4k}(\mathrm{BO})\to\pi_{4k}(\mathrm{BG})$. The group $\pi_{4k}(\mathrm{BO})$ is isomorphic to $\mathbb{Z}$ and $\ker J$ is a subgroup of index $\mathrm{den}\big(\frac{B_k}{4k}\big)$ (first proved by Adams [1965, Theorem 3.7] assuming the Adams conjecture [1963, Conjecture 1.2], which was later proved by Quillen [1971]). It is shown by Kervaire [1959, Lemma 1.1] that the image of the homomorphism $p_k:\pi_{4k}(\mathrm{BO})\to\mathbb{Z}$, $\xi\mapsto\langle p_k(\xi),[S^{4k}]\rangle$, is a subgroup of index $a_k\cdot(2k-1)!$.

For an integer d divisible by $a_k\cdot(2k-1)!\cdot\mathrm{den}\big(\frac{B_k}{4k}\big)$, we may choose a vector bundle ξ such that $\langle p_k(\xi),[S^{4k}]\rangle=d$. Since ξ is a vector bundle reduction of the Spivak normal fibration of S^{4k}, we may consider the surgery problem

$$\begin{array}{ccc} \nu X & \xrightarrow{\bar f} & \xi \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^{4k} \end{array}$$

where X is a closed $4k$-manifold, and f is a map of degree 1, covered by a bundle map $\bar f$ from the stable normal bundle νX to ξ. The surgery obstruction, that is, the obstruction to doing surgery on (X,f) to get a homotopy equivalence $f':X'\to S^{4k}$, is $\theta(\bar f,f)\in L_{4k}(\mathbb{Z})\cong\mathbb{Z}$, where $L_{4k}(\mathbb{Z})$ is the Wall surgery obstruction group.

Taking the product with $\mathbb{C}P^{2k-1}$ we have a surgery problem

$$\begin{array}{ccc} \nu\mathbb{C}P^{2k-1}\times\nu X & \xrightarrow{\mathrm{id}\times\bar f} & \nu\mathbb{C}P^{2k-1}\times\xi \\ \downarrow & & \downarrow \\ \mathbb{C}P^{2k-1}\times X & \xrightarrow{\mathrm{id}\times f} & \mathbb{C}P^{2k-1}\times S^{4k} \end{array}$$

By the product formula of surgery obstructions (see [Browder 1972, p. 33; Wall 1999, Lemma 13B.4]), the surgery obstruction of this surgery problem is

$$\theta(\mathrm{id}\times\bar f,\mathrm{id}\times f)=\chi(\mathbb{C}P^{2k-1})\cdot\theta(\bar f,f)\in L_{8k-2}(\mathbb{Z}),$$

where $\chi(\mathbb{C}P^{2k-1})$ is the Euler characteristic of $\mathbb{C}P^{2k-1}$. Since $\chi(\mathbb{C}P^{2k-1})=2k$ and $L_{8k-2}(\mathbb{Z})\cong\mathbb{Z}/2$, the surgery obstruction $\theta(\mathrm{id}\times\bar f,\mathrm{id}\times f)$ vanishes. Therefore by surgery we get a homotopy equivalence $g:Y\to\mathbb{C}P^{2k-1}\times S^{4k}$, which is covered by a bundle map $\bar g:\nu Y\to\nu\mathbb{C}P^{2k-1}\times\xi$. The existence of $\bar g$ implies that $\tau Y=g^*(\tau\mathbb{C}P^{2k-1}\times\xi^{-1})$ where τY and $\tau\mathbb{C}P^{2k-1}$ are the stable tangent bundles of Y and $\mathbb{C}P^{2k-1}$, respectively, and ξ^{-1} is the stable inverse of ξ. Thus

$$p_k(Y)=p_k(\tau Y)=g^*p_k(\tau\mathbb{C}P^{2k-1}\times\xi^{-1}).$$

Since $p_k(\tau\mathbb{C}P^{2k-1}\times\xi^{-1})=-\pi_2^*(p_k(\xi))$, where $\pi_2:\mathbb{C}P^{2k-1}\times S^{4k}\to S^{4k}$ is the projection, the divisibility of $p_k(Y)$ equals the divisibility of $p_k(\xi)$. Taking connected sum with r copies of $S^{4k-1}\times S^{4k-1}$ we get N. □

5. Proof of Theorems 1.3 and 1.5

Proof of Theorems 1.3 and 1.5. Let N be the orbit space of a free circle action on M. Then M is the total space of an S^1-bundle η over N, $\pi : M \to N$ with Euler class t, a generator of $H^2(N)$. The tangent bundle TM is isomorphic to the Whitney sum $\pi^*TN \oplus V$, where $V = \ker d\pi$ is the vertical bundle of η and is trivial as an orientable line bundle. Therefore

$$p_{\frac{n+1}{4}}(M) = \pi^* p_{\frac{n+1}{4}}(N).$$

By Lemma 3.1, the cohomology ring $H^*(N)$ is isomorphic to

$$H^*\Big(\mathop{\#}_r (S^n \times S^n) \# \mathbb{C}P^n\Big) \quad \text{or} \quad H^*\Big(\mathop{\#}_r (S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\Big).$$

If $H^*(N)$ is isomorphic to $H^*\big(\#_r (S^n \times S^n) \# \mathbb{C}P^n\big)$, then in the Gysin sequence

$$H^{n-1}(N) \xrightarrow{-\cup t} H^{n+1}(N) \xrightarrow{\pi^*} H^{n+1}(M) \to H^n(N) \to 0$$

the homomorphism $-\cup t$ is an isomorphism, therefore the homomorphism π^* is trivial and hence

$$H_n(M) \cong H^{n+1}(M) \cong H^n(N) \cong \mathbb{Z}^{2r}, \quad p_{\frac{n+1}{4}}(M) = \pi^* p_{\frac{n+1}{4}}(N) = 0.$$

If $H^*(N)$ is isomorphic to $H^*\big(\#_r (S^n \times S^n) \# (\mathbb{C}P^{\frac{n-1}{2}} \times S^{n+1})\big)$, then in the Gysin sequence

$$H^{n-1}(N) \xrightarrow{-\cup t} H^{n+1}(N) \xrightarrow{\pi^*} H^{n+1}(M) \to H^n(N) \to 0$$

the homomorphism $-\cup t$ is trivial. Therefore we have a split short exact sequence

$$0 \to H^{n+1}(N) \xrightarrow{\pi^*} H^{n+1}(M) \to H^n(N) \to 0.$$

This implies that $H^{n+1}(M) \cong \mathbb{Z}^{2r+1}$ and the divisibility $l(M)$ of $p_{\frac{n+1}{4}}(M)$ is equal to the divisibility $d(N)$ of $p_{\frac{n+1}{4}}(N)$. By Lemma 4.1, the integer $l(M)$ must satisfy the conditions in Theorems 1.3 and 1.5.

Conversely, the total space of the S^1-bundle over $\#_r (S^n \times S^n) \# \mathbb{C}P^n$ with Euler class a generator of $H^2\big(\#_r (S^n \times S^n) \# \mathbb{C}P^n\big)$ is diffeomorphic to $\#_{2r}(S^n \times S^{n+1})$, as we have seen in the proof of Proposition 2.2. On the other hand, for an integer l divisible by $a_{\frac{n+1}{4}} \cdot \big(\frac{n-1}{2}\big)!\,\mathrm{den}\big(\frac{B_{(n+1)/4}}{n+1}\big)$, let N be a manifold given in Lemma 4.2 with $d(N) = l$, and take the total space M of the S^1-bundle over N with Euler class a generator of $H^2(N)$. Then by the above discussion $H_n(M)$ is isomorphic to $\mathbb{Z}^{2r+1}$ and $l(M)$ equals $d(N)$. By Theorem 1.1 M is almost diffeomorphic to $\#_{2r}(S^n \times S^{n+1}) \# X_l$. □

Appendix

In this section we prove Theorem 1.1. This is based on a sequence of propositions. We first describe the structure of $(n-1)$-connected $(2n+1)$-manifolds with torsion free homology and $n \equiv 3 \bmod 4$ as a twisted double of $\natural_r(S^n \times D^{n+1})$, where $\natural_r(S^n \times D^{n+1})$ denotes the boundary connected sum of r copies of $S^n \times D^{n+1}$, whose boundary is $\#_r(S^n \times S^n)$.

Proposition A.1. *Let M be an $(n-1)$-connected $(2n+1)$-manifold with torsion free homology, $n \equiv 3 \bmod 4$. If $r := \operatorname{rank} H_n(M) \geq 1$, then M is diffeomorphic to $\big(\natural_r(S^n \times D^{n+1})\big) \cup_f \big(\natural_r(S^n \times D^{n+1})\big)$, where $f : \#_r(S^n \times S^n) \to \#_r(S^n \times S^n)$ is a diffeomorphism which induces the identity on homology.*

Proof. Since M is $(n-1)$-connected, by the Hurewicz theorem, the Hurewicz map $\pi_n(M) \to H_n(M)$ is an isomorphism. By Whitney embedding theorem and transversality, a basis $\{x_1, \dots, x_r\}$ of $\pi_n(M)$ can be represented by disjoint embedded spheres $S^n_1, \dots, S^n_r \subset M$. For $n \equiv 3 \bmod 4$, $\pi_{n-1}\,\mathrm{SO}(n+1) = 0$, therefore the normal bundles of these embedded spheres are trivial. Choose a framing of the normal bundle, we have submanifolds $S^n_i \times D^{n+1} \subset M$ for $i = 1, \dots r$. Connecting these submanifolds by tubes $D^{2n} \times [0, 1]$, we have a submanifold $V_1 = \natural_r(S^n \times D^{n+1}) \subset M$. From the exact sequence

$$\cdots \to H_{k+1}(M, M - \mathring{V}_1) \to H_k(M - \mathring{V}_1) \to H_k(M) \to H_k(M, M - \mathring{V}_1) \to \cdots$$

and the isomorphisms $H_k(M, M - \mathring{V}_1) \cong H_k(V_1, \partial V_1) \cong H^{2n+1-k}(V_1)$ (excision and Poincaré duality), one may deduce that $H_k(M - \mathring{V}_1) = 0$ for $k \neq 0, n$, and the homomorphism $H_n(M - \mathring{V}_1) \to H_n(M)$ induced by the inclusion map is an isomorphism since $H_{n+1}(M) \to H_{n+1}(M, M - \mathring{V}_1)$ coincides with the homomorphism $H_{n+1}(M) \to \mathbb{Z}^r$, $u \mapsto (u \cdot x_1, \dots, u \cdot x_r)$, where $u \cdot x_k$ denotes the intersection number. Let $\bar{x}_1, \dots, \bar{x}_r \in H_n(M - \mathring{V}_1)$ be the preimages of $x_1, \dots, x_r$, which form a basis of $H_n(M - \mathring{V}_1)$. By the same process, we have a submanifold $V_2 = \natural_r(S^n \times D^{n+1}) \subset M - \mathring{V}_1$. By construction, the inclusion $V_2 \to M - \mathring{V}_1$ induces an isomorphism $H_k(V_2) \cong H_k(M - \mathring{V}_1)$ for all k. Therefore $H_k(M - (\mathring{V}_1 \cup \mathring{V}_2), \partial V_2) \cong H_k(M - \mathring{V}_1, V_2) = 0$ for all k. Hence $M - (\mathring{V}_1 \cup \mathring{V}_2)$ is an h-cobordism and there is a diffeomorphism $F : M - (\mathring{V}_1 \cup \mathring{V}_2) \to \big(\#_r(S^n \times S^n)\big) \times [0, 1]$, such that the restriction of F on $\partial V_2 = \#_r(S^n \times S^n)$ is the identity. Define $f = F|_{\partial V_1} : \#_r(S^n \times S^n) \to \#_r(S^n \times S^n)$; then M is diffeomorphic to the twisted double $\big(\natural_r(S^n \times D^{n+1})\big) \cup_f \big(\natural_r(S^n \times D^{n+1})\big)$.

Now we look at the isomorphism $f_* : H_n\big(\#_r(S^n \times S^n)\big) \to H_n\big(\#_r(S^n \times S^n)\big)$ induced by f. Let $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ be the standard symplectic basis of $H_n\big(\#_r(S^n \times S^n)\big)$ represented by $S^n_i \times \{*\}$ and $\{*\} \times S^n_i$. Let $A_i \subset M - (\mathring{V}_1 \cup \mathring{V}_2) \subset M$ be an embedded n-sphere representing the homology class x_i for $i = 1, \dots, r$. Then by the constructions of V_1 and V_2, under the inclusions $\partial V_j \to M - (\mathring{V}_1 \cup \mathring{V}_2)$,

$j = 1, 2$, the image of e_i is $[A_i]$. Since $F|_{\partial V_2}$ is the identity, we get $f_*(e_i) = e_i$. Also note that f_* preserves the intersection form, which is a standard symplectic form. Therefore the isomorphism $f_* : H_n\big(\#_r(S^n \times S^n)\big) \to H_n\big(\#_r(S^n \times S^n)\big)$ is represented by the symplectic matrix

$$\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \in \mathrm{Sp}(2r, \mathbb{Z}).$$

In the segment of the Mayer–Vietoris sequence

$$H_n\Big(\mathop{\#}_r (S^n \times S^n)\Big) \xrightarrow{(i_{1*}, i_{2*})} H_n(V_1) \oplus H_n(V_2) \to H_n(M) \to 0$$

under the basis $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ of $H_n\big(\#_r(S^n \times S^n)\big)$ and under the basis $\{x_1, \dots, x_r, \bar{x}_1, \dots \bar{x}_r\}$ of $H_n(V_1) \oplus H_n(V_2)$ the homomorphism (i_{1*}, i_{2*}) is represented by the matrix

$$\begin{pmatrix} I & I \\ 0 & A \end{pmatrix}.$$

Since $H_n(M)$ is a free abelian group of rank r, we have $A = 0$. Therefore the induced isomorphism f_* is the identity. □

Proposition A.2. *Assume that* $n \equiv 3 \bmod 4$. *Let M be an* $(n-1)$*-connected* $(2n+1)$*-manifold with torsion free homology and let $l(M)$ be the divisibility of $p_{\frac{n+1}{4}}(M)$. Then $l(M)$ is divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$ where*

$$a_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad b_n = \begin{cases} 2 & \text{if } n = 3, 7, \\ 1 & \text{if } n \neq 3, 7. \end{cases}$$

Proof. In Proposition A.1 we have seen that the manifold M is a twisted double $\big(\natural_r(S^n \times D^{n+1})\big) \cup_f \big(\natural_r(S^n \times D^{n+1})\big)$, where $f : \#_r(S^n \times S^n) \to \#_r(S^n \times S^n)$ is a diffeomorphism which induces the identity on homology. By the embedding theorem of Haefliger [1961], homotopic embeddings of S^n into $\#_r(S^n \times S^n)$ are isotopic. Since f_* is the identity, after an isotopy we may assume $f|_{\{*\} \times S_i^n}$ is the identity. Then a basis of $H_{n+1}(M)$ is represented by $S_i^{n+1} = D_i^{n+1} \cup D_i^{n+1}$, $i = 1, \dots, r$. The restriction of the stable tangent bundle of M on S_i^{n+1}, $\tau M|_{S_i^{n+1}}$, is the direct sum of the normal bundle of S_i^{n+1} in M and the stable tangent bundle of S_i^{n+1}. The former is an n-dimensional vector bundle, and the latter is stably trivial. Therefore, the clutching function of $\tau M|_{S_i^{n+1}}$ is in the image of $\pi_n(\mathrm{SO}(n)) \to \pi_n(\mathrm{SO})$, which is the full group $\pi_n(\mathrm{SO})$ when $n > 7$, or is a subgroup of index 2 when $n = 3, 7$. It is shown in [Kervaire 1959, Lemma 1.1] that for a stable vector bundle ξ over S^{n+1}, $p_{\frac{n+1}{4}}(\xi)$ is an $a_{\frac{n+1}{4}}\left(\frac{n-1}{2}\right)!$-fold multiple of the clutching function of ξ under the identification $H^{n+1}(S^{n+1}; \pi_n(\mathrm{SO})) = \pi_n(\mathrm{SO})$, where $a_m = 2$ when m is odd, and $a_m = 1$ when m is even. This finishes the proof of the proposition. □

Proposition A.3. *Assume that* $n \equiv 3 \bmod 4$. *For any natural number* l *divisible by* $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$, *there exists a unique* $(n+1)$*-dimensional real vector bundle* ξ_l *over* S^{n+1} *whose Euler class and* $\frac{n+1}{4}$*-th Pontryagin class are*

$$e(\xi_l) = 0, \quad p_{\frac{n+1}{4}}(\xi_l) = l \cdot \omega,$$

respectively, where ω *is the preferred generator of* $H^{n+1}(S^{n+1})$.

Proof. For any natural number l divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$, according to [Kervaire 1959, Lemma 1.1] there exists an $(n+2)$-dimensional real vector bundle η over S^{n+1} such that $p_{\frac{n+1}{4}}(\eta) = l \cdot \omega$. Let $\alpha \in \pi_n \operatorname{SO}(n+2)$ correspond to the bundle η (that is, α is represented by the clutching function of η). Since $\pi_{n+1} \operatorname{SO}(n+2)$ is torsion for $n \equiv 3 \bmod 4$ by [Kervaire 1960], it follows that $\pi_n \operatorname{SO}(n+1)$ is computed from the homotopy exact sequence for the fiber bundle $\operatorname{SO}(n+1) \to \operatorname{SO}(n+2) \to \operatorname{SO}(n+2)/\operatorname{SO}(n+1) = S^{n+1}$:

$$0 \to \pi_{n+1} S^{n+1} \xrightarrow{\partial} \pi_n \operatorname{SO}(n+1) \xrightarrow{i_*} \pi_n \operatorname{SO}(n+2) \to 0. \tag{5}$$

Since $\pi_n \operatorname{SO}(n+2) = \pi_n \operatorname{SO} = \mathbb{Z}$ for $n \equiv 3 \bmod 4$, it follows that the exact sequence (5) splits and hence there is a homomorphism $s : \pi_n \operatorname{SO}(n+2) \to \pi_n \operatorname{SO}(n+1)$ such that the composition $i_* s$ is the identity map.

Let $s(\eta)$ be the $(n+1)$-dimensional real vector bundle over S^{n+1} corresponding to $s(\alpha) \in \pi_n \operatorname{SO}(n+1)$. Note first that η is the stabilization of $s(\eta)$ since $i_* s$ is the identity map. This implies that $p_{\frac{n+1}{4}}(s(\eta)) = p_{\frac{n+1}{4}}(\eta) = l \cdot \omega$.

Next we claim that the Euler class $e(s(\eta))$ is even. The reason is as follows. When $n = 3, 7$, one has $b_n = 2$ and hence $p_{\frac{n+1}{4}}(\eta) = l \cdot \omega$ is divisible by $a_{\frac{n+1}{4}} \cdot 2 \cdot \left(\frac{n-1}{2}\right)!$. This implies that the class α of the clutching function of the bundle η is even in the homotopy group $\pi_n \operatorname{SO}(n+2) = \mathbb{Z}$ [Kervaire 1959, Lemma 1.1]. Thus $s(\alpha) \in \pi_n \operatorname{SO}(n+1)$ is divisible by 2. Since taking Euler class induces a homomorphism $e : \pi_n \operatorname{SO}(n+1) \to H^{n+1}(S^{n+1})$ (see [Milnor and Stasheff 1974, Lemma 20.10]), this implies that $e(s(\eta)) = e(s(\alpha))$ is even. When $n \neq 3, 7$, since S^n is not parallelizable, it follows from [Kervaire 1959, Lemma 6.4] that $w_{n+1}(\eta) = 0$. Hence $w_{n+1}(s(\eta)) = w_{n+1}(\eta) = 0$. In terms of the equation $e(s(\eta)) \bmod 2 = w_{n+1}(s(\eta))$ (see [Bredon 1993, p. 421, 17.2]), one has $e(s(\eta)) \bmod 2 = 0$ and hence $e(s(\eta))$ is even.

Let ι be the generator of $\pi_{n+1} S^{n+1}$ represented by the identity map. Since $\partial(\iota) \in \pi_n \operatorname{SO}(n+1)$ corresponds to the tangent bundle of S^{n+1}, it follows from $\chi(S^{n+1}) = 2$ that there is an integer m such that $e(s(\alpha) + m\partial\iota) = 0$. Let ξ_l denote the $(n+1)$-dimensional real vector bundle corresponding to $s(\alpha) + m\partial\iota$. Then $e(\xi_l) = 0$ and

$$p_{\frac{n+1}{4}}(\xi_l) = p(s(\alpha)) = l,$$

where $p : \pi_n \operatorname{SO}(n+1) \to H^{n+1}(S^{n+1})$ is the homomorphism induced by taking the $\frac{n+1}{4}$-th Pontryagin class (see [Milnor and Stasheff 1974, Lemma 20.10]). This shows the existence of ξ_l.

To show the uniqueness of ξ_l, let ξ' be any $(n+1)$-dimensional real vector bundle over S^{n+1} such that

$$e(\xi') = 0, \quad p_{\frac{n+1}{4}}(\xi') = l \cdot \omega. \tag{6}$$

Since there are integers x, y such that ξ' corresponds to the class $xs(\alpha_0) + y\partial\iota$ where α_0 is a generator of $\pi_n \operatorname{SO}(n+2)$, (6) implies that x, y must satisfy the system of equations

$$xe(s(\alpha_0)) + ye(\partial\iota) = 0, \quad xp(s(\alpha_0)) + yp(\partial\iota) = l \cdot \omega.$$

This system of equations has at most one solution because $p(\partial\iota) = 0$ and both of $p(s(\alpha_0))$ and $e(\partial\iota)$ are nonzero. This shows the uniqueness of ξ_l. □

When $n > 3$ and $n \neq 7$, Wall [1967, Theorem 7] determined the diffeomorphism classes of $(n-1)$-connected almost closed $(2n+1)$-manifolds in terms of certain systems of invariants including homology, characteristic classes, linking form and so on. Thus this gives a system of complete invariants of $(n-1)$-connected $(2n+1)$-manifolds up to almost diffeomorphisms. On the other hand, all possible values of the invariants can be realized by these $(2n+1)$-manifolds except for $n = 4, 8, 9$ according to the work of Senger and Zhang [2023, Theorem 1.10]. In fact, they determined precisely which exotic $2n$-spheres are the boundaries of $(n-1)$-connected almost closed $(2n+1)$-manifolds when $n \geq 3$. As a consequence, they deduce that every $(n-1)$-connected almost closed $(2n+1)$-manifold may be filled in to obtain a closed manifold when $n \geq 3$ and $n \neq 4, 8, 9$.

Moreover, when $n = 3, 7$, Wilkens [1971, Theorem 3; 1972] proved that the almost diffeomorphism classes of $(n-1)$-connected $(2n+1)$-manifolds M are classified by the invariants including homology, characteristic classes, linking form, modulo a finite ambiguity if the torsion part of $H_n(M)$ is of even order. Crowley [2002, Theorem B] completed the almost diffeomorphism classification by defining a family of quadratic refinement of the linking form. For more background on the classification of $(n-1)$-connected $(2n+1)$-manifold, see [Crowley and Nordström 2019; Crowley 2002]. To prove Theorem 1.1, we only need the special cases of the results obtained by Wall and Wilkens as follows.

Theorem A.4 [Wall 1967, Theorem 7]. *Assume that $n \equiv 5 \bmod 8$. Then the homology group $H_n(P)$ of an $(n-1)$-connected $(2n+1)$-manifold P with free homology is a complete invariant of almost diffeomorphisms. That is, if P_1 and P_2 are two $(n-1)$-connected $(2n+1)$-manifolds with free homology, then P_1 is almost diffeomorphic to P_2 if and only if $H_n(P_1)$ is isomorphic to $H_n(P_2)$.*

Proposition A.5 [Wall 1967, Theorem 7; Wilkens 1972, Theorem 3]. *Assume that $n \equiv 3 \bmod 4$. Then $\{H_n(P), p_{\frac{n+1}{4}}(P)\}$ of an $(n-1)$-connected $(2n+1)$-manifold P with free homology is a set of complete invariants of almost diffeomorphisms. That is,* *if P_1 and P_2 are two $(n-1)$-connected $(2n+1)$-manifolds with free homology, then there exists an almost diffeomorphism $f : P_1 \to P_2$ if and only if there is an isomorphism $F : H^{n+1}(P_2) \to H^{n+1}(P_1)$ such that $F(p_{\frac{n+1}{4}}(P_2)) = p_{\frac{n+1}{4}}(P_1)$.*

Proof of Theorem 1.1. When $n \equiv 5 \bmod 8$, by Theorem A.4, it suffices to show that there is an nonnegative integer r such that $\#_r\, S^n \times S^{n+1}$ and M have isomorphic n-homology groups. This follows directly from the assumption that $H_n(M)$ is free and by taking r to be rank $H_n(M)$. Now assume that $n \equiv 3 \bmod 4$. If $H_n(M) = 0$, then M is a homotopy sphere and hence is almost diffeomorphic to S^{2n+1}. If $H_n(M)$ is not trivial, take $r = \operatorname{rank} H_n(M) - 1$. It follows from Proposition A.2 that the natural number $l(M)$ is divisible by $a_{\frac{n+1}{4}} \cdot b_n \cdot \left(\frac{n-1}{2}\right)!$. By Proposition A.3, let $X_{l(M)}$ denote the total space of the sphere bundle associated to $\xi_{l(M)}$ and let $M' = \#_r\, S^n \times S^{n+1} \# X_{l(M)}$. Then there is an isomorphism $F : H^{n+1}(M') \to H^{n+1}(M)$ such that $F(p_{\frac{n+1}{4}}(M')) = p_{\frac{n+1}{4}}(M)$. Thus it follows from Proposition A.5 that M is almost diffeomorphic to $\#_r\, S^n \times S^{n+1} \# X_{l(M)}$. □

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