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SHA YAO, HAO-YUE LIU AND XIN-AN REN

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MATRIX LI–YAU–HAMILTON ESTIMATES FOR NONLINEAR HEAT EQUATIONS UNDER GEOMETRIC FLOWS

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We are concerned with the matrix Li–Yau–Hamilton estimates for nonlinear heat equations. Firstly, we derive such an estimate for positive solutions to the nonlinear heat equations under the Ricci flow. Then we consider the estimates for positive solutions to the nonlinear heat equations and the nonlinear backward conjugate heat equations under the Kähler–Ricci flow.

1. Introduction

The research of differential Harnack inequalities for parabolic equations on Riemannian manifolds originated in Li and Yau's celebrated paper [19], in which they got a differential Harnack inequality for positive solutions of the heat equation by using the maximum principle. They also derived a classical Harnack inequality of Moser [23] from their estimates. Later, the inequality was extended to a matrix version by Hamilton [18]. For curvature flows, Hamilton discovered a matrix differential Harnack inequality for Ricci flow [17]. This is often referred to as a matrix estimate. So these types of Harnack inequalities were called Li–Yau–Hamilton estimates in [27]. For Kähler–Ricci flow, Cao [1] obtained the matrix Li–Yau–Hamilton estimate. Similar Li–Yau–Hamilton estimates for Gauss curvature flow and Yamabe flow were proved by Chow in [4; 5]. If the Riemannian metric evolves by Ricci flow, Chow and Chu gave a nice geometric interpretation of Hamilton quantities in the Li–Yau–Hamilton estimate in [8]. In subsequent papers, Chow [6], as well as Chow and Chu [9], further deepened their understanding from this perspective. Using this geometric approach, Chow and Knopf gave a new Li–Yau–Hamilton estimate for Ricci flow in [10]. In [2], Cao and Ni extended the matrix Li–Yau–Hamilton estimate for heat equation on Riemannian manifolds to Kähler manifolds. Later, Chow and Ni [25] established a matrix Li–Yau–Hamilton estimate for the forward conjugate heat equation under the Kähler–Ricci flow. In [24], Ni more generally

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discussed the monotonicity and the related results of geometric interpretation of the linear trace Li–Yau–Hamilton on Kähler–Ricci flow. Analogous estimates were obtained by Ren, Yao, Shen and Zhang [30] in the constrained case.

Yau [34] considered the gradient and Hessian estimates for positive solutions of

$$u_t = \Delta u - Vu, \tag{1-1}$$

where $V = h(x) + k(u)$. By setting $v = \ln u$, one has

$$v_t = \Delta v + |\nabla v|^2 - k(e^v) - h(x). \tag{1-2}$$

Later, Ma, Zhao and Song [22] generalized the gradient estimates to degenerate parabolic equations

$$u_t = \Delta F(u) + H(u). \tag{1-3}$$

Xu [32] considered the gradient estimates of Hamilton type for degenerate parabolic equations

$$u_t = \Delta F(u). \tag{1-4}$$

In [3], the authors derived the gradient estimate for nonlinear heat equation

$$u_t = \Delta u + au \ln u. \tag{1-5}$$

This estimate is sharp if the Ricci curvature is nonnegative. The matrix version of this estimate on Riemannian manifolds was obtained by Wu [31] and on Kähler manifolds by Ren [29]. Moreover, the authors [33] extended the matrix Li–Yau–Hamilton estimates for semilinear parabolic equation

$$\frac{\partial}{\partial t} L = \Delta L + G(|\nabla L|^2) + F(L). \tag{1-6}$$

When the metrics are evolved by the Ricci flow [11; 15]

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{1-7}$$

Perelman [28] discovered a new differential Harnack estimate for the fundamental solution of the backward conjugate heat equation

$$\partial_t u + \Delta_{g(t)} u = Ru, \tag{1-8}$$

where R denotes the scalar curvature of $(M^n, g(t))$. For more discussions on Li–Yau–Hamilton estimate, we refer the reader to the surveys [7; 26], and the monographs [13, Chapters 15–16; 14, Chapters 23–26].

In [20], Li and Zhang proved new Li–Yau–Hamilton estimates for positive solutions to the heat equation and the backward conjugate heat equation under the Ricci flow. Their estimate does not require the condition of parallel Ricci curvature.

Theorem A (Li–Zhang). *Let $(M^n, g(t))$, $t \in [0, T]$, be a complete solution to the Ricci flow. Let $u : M^n \times [0, T] \to \mathbb{R}$ be a positive solution to the heat equation*

$$u_t - \Delta_{g(t)} u = 0. \tag{1-9}$$

Suppose that $(M^n, g(t))$ has nonnegative sectional curvature and $\mathrm{Ric} \le kg$ for some constant $k > 0$. Then

$$\nabla_i \nabla_j \ln u + \frac{k}{1 - e^{-2kt}} g_{ij} \ge 0 \tag{1-10}$$

for all $(x, t) \in M \times (0, T)$.

Recently, Li, Liu and Ren [21] proved matrix Li–Yau–Hamilton estimates for positive solutions to the heat equation and the backward conjugate heat equation under the Kähler–Ricci flow.

Theorem B (Li–Liu–Ren). *Let $(M^n, g(t))$, $t \in [0, T]$, be a complete solution to the ε-Kähler–Ricci flow*

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -\varepsilon R_{i\bar{j}} \tag{1-11}$$

with nonnegative bisectional curvature and $R_{i\bar{j}} \le k g_{i\bar{j}}$ for some constant $k > 0$. Let $u : M^n \times [0, T] \to \mathbb{R}$ be a positive solution to the heat equation

$$u_t - \Delta_{g(t)} u = 0. \tag{1-12}$$

Then

$$\nabla_i \nabla_{\bar{j}} \ln u + \frac{\varepsilon k}{1 - e^{-\varepsilon k t}} g_{i\bar{j}} \ge 0 \tag{1-13}$$

for all $(x, t) \in M \times (0, T)$.

Theorem C (Li–Liu–Ren). *Let $(M^n, g(t))$, $t \in [0, T]$, be a complete solution to the ε-Kähler–Ricci flow with nonnegative bisectional curvature and $R_{i\bar{j}} \le k g_{i\bar{j}}$ for some constant $k > 0$. Let $u : M^n \times [0, T] \to \mathbb{R}$ be a positive solution to the backward conjugate heat equation*

$$u_t + \Delta_{g(t)} u = \varepsilon R u. \tag{1-14}$$

Suppose that $\eta : (0, T) \to (0, \infty)$ is a smooth function satisfying the ordinary differential inequality

$$\eta' \le \eta^2 - \varepsilon k \eta - \frac{\varepsilon k}{t} \tag{1-15}$$

in $(0, T)$ and that $\eta(t) \to \infty$ as $t \to T$. Then

$$\varepsilon R_{i\bar{j}} - \nabla_i \nabla_{\bar{j}} \ln u - \eta g_{i\bar{j}} \le 0 \tag{1-16}$$

for all $(x, t) \in M \times (0, T)$. *In particular*, *we have*

$$\varepsilon R_{i\bar{j}} - \nabla_i \nabla_{\bar{j}} \ln u - \left(\frac{\varepsilon k}{1 - e^{-\varepsilon k(T-t)}} + \sqrt{\frac{k}{t}} \right) g_{i\bar{j}} \leq 0. \tag{1-17}$$

In this paper, we firstly consider the matrix Li–Yau–Hamilton estimate for the nonlinear heat equation under the Ricci flow and obtain the following.

Theorem 1.1. *Let* $(M^n, g(t))$, $t \in [0, T]$, *be a compact solution to the Ricci flow with nonnegative sectional curvature and* $\mathrm{Ric} \leq kg$ *for some constant* $k > 0$. *Let* $v : M^n \times [0, T] \to \mathbb{R}$ *be a solution to the nonlinear heat equation*

$$v_t = \Delta_{g(t)} v + |\nabla v|^2 + f(v) \tag{1-18}$$

with $f''(v) \geq 0$. *Suppose that* $c(t)$ *is a smooth function satisfying the ordinary differential inequality*

$$c'(t) + 2c^2(t) - (f'(v) + 2k)c(t) \geq 0 \tag{1-19}$$

in $(0, T)$ *and that* $c(t) \to \infty$ *as* $t \to 0^+$. *Then*

$$\nabla_i \nabla_j v + c(t) g_{ij} \geq 0. \tag{1-20}$$

If $f''(v) \geq 0$, and $f'(v) \leq 0$, then solving the ordinary differential equation

$$c'(t) + 2c^2(t) - 2kc(t) = 0 \tag{1-21}$$

leads to

$$c(t) = \frac{k}{1 - e^{-2kt}}.$$

Therefore, we can get the following result.

Corollary 1.2. *Let* $(M^n, g(t))$, $t \in [0, T]$, *be a compact solution to the Ricci flow. Let* $v : M^n \times [0, T] \to \mathbb{R}$ *be a solution to the nonlinear heat equation*

$$v_t = \Delta_{g(t)} v + |\nabla v|^2 + f(v), \tag{1-22}$$

where $f''(v) \geq 0$ *and* $f'(v) \leq 0$. *Suppose that* $(M^n, g(t))$ *has nonnegative sectional curvature and* $\mathrm{Ric} \leq kg$ *for some constant* $k > 0$. *Then*

$$\nabla_i \nabla_j v + \frac{k}{1 - e^{-2kt}} g_{ij} \geq 0. \tag{1-23}$$

Setting $f(v) = av$ and $v = \ln u$ in Theorem 1.1, the nonlinear heat equation turns out to be

$$u_t = \Delta_{g(t)} u + au \ln u. \tag{1-24}$$

By solving the ordinary differential equation

$$c'(t)+2c^2(t)-(a+2k)c(t)=0, \tag{1-25}$$

we have

$$c(t)=\frac{k+\frac{1}{2}a}{1-e^{-(2k+a)t}}.$$

Therefore, we have:

Corollary 1.3. *Let $(M^n, g(t))$, $t \in [0, T]$, be a compact solution to the Ricci flow. Let $u : M^n \times [0, T] \to \mathbb{R}$ be a positive solution to the nonlinear heat equation*

$$u_t=\Delta_{g(t)}u+au\ln u \tag{1-26}$$

for some constant $a \geq 0$. Suppose that $(M^n, g(t))$ has nonnegative sectional curvature and Ric $\leq kg$ *for some constant $k > 0$. Then*

$$\nabla_i\nabla_j \ln u+\frac{k+\frac{1}{2}a}{1-e^{-(2k+a)t}}g_{ij}\geq 0 \tag{1-27}$$

for all $(x, t) \in M \times (0, T)$.

We can also obtain the similar estimate for the nonlinear heat equations under the ε-Kähler–Ricci flow.

Theorem 1.4. *Let $(M^n, g(t))$, $t \in [0, T]$, be a compact solution to the ε-Kähler–Ricci flow with nonnegative bisectional curvature and $R_{i\bar{j}} \leq kg_{i\bar{j}}$ for some constant $k > 0$. Let $v : M^n \times [0, T] \to \mathbb{R}$ be a solution to the nonlinear heat equation*

$$v_t=\Delta_{g(t)}v+|\nabla v|^2+f(v) \tag{1-28}$$

with $f''(v) \geq 0$. Suppose that $c(t)$ is a smooth function satisfying the ordinary differential inequality

$$c'(t)+c^2(t)-(f'(v)+\varepsilon k)c(t)\geq 0 \tag{1-29}$$

in $(0, T)$ and that $c(t) \to +\infty$ as $t \to 0$. Then

$$\nabla_i\nabla_{\bar{j}}v+c(t)g_{i\bar{j}}\geq 0. \tag{1-30}$$

Setting $f(v) = av$ and $v = \ln u$ in Theorem 1.4, we have:

Corollary 1.5. *Let $(M^n, g(t))$, $t \in [0, T]$, be a compact solution to the ε-Kähler–Ricci flow. Let $u : M^n \times [0, T] \to \mathbb{R}$ be a positive solution to the nonlinear heat equation*

$$u_t=\Delta_{g(t)}u+au\ln u \tag{1-31}$$

for some constant $a \geq 0$. Suppose that $(M^n, g(t))$ has nonnegative bisectional curvature and $R_{i\bar{j}} \leq k g_{i\bar{j}}$ for some constant $k > 0$. Then

$$\nabla_i \nabla_{\bar{j}} \ln u + \frac{\varepsilon k + a}{1 - e^{-(\varepsilon k + a)t}} g_{i\bar{j}} \geq 0 \tag{1-32}$$

for all $(x, t) \in M \times (0, T)$.

Remark 1.6. If we take $a = 0$ in Corollaries 1.3 and 1.5, then we have Theorems A and B, respectively.

Finally, we will prove a matrix Li–Yau–Hamilton estimate for positive solutions to the nonlinear backward conjugate heat equation coupled with the ε-Kähler–Ricci flow.

Theorem 1.7. *Let $(M^n, g(t))$, $t \in [0, T]$, be a compact solution to the ε-Kähler–Ricci flow with nonnegative bisectional curvature and $R_{i\bar{j}} \leq k g_{i\bar{j}}$ for some constant $k > 0$. Let $v : M^n \times [0, T] \to \mathbb{R}$ be a solution to the nonlinear backward conjugate heat equation*

$$v_t + \Delta_{g(t)} v = \varepsilon R - |\nabla v|^2 + f(v) \tag{1-33}$$

with $f''(v) \leq 0$. Suppose that $\eta : (0, T) \to (0, \infty)$ is a smooth function satisfying the ordinary differential inequality

$$\eta' \leq \eta^2 - (\varepsilon k - f'(v))\eta - \varepsilon k \left(f'(v) + \frac{1}{t} \right) \tag{1-34}$$

in $(0, T)$, and that $\eta(t) \to +\infty$ as $t \to T$. Then

$$\varepsilon R_{i\bar{j}} - \nabla_i \nabla_{\bar{j}} \ln u - \eta g_{i\bar{j}} \leq 0 \tag{1-35}$$

for all $(x, t) \in M \times (0, T)$.

Setting $f(v) = av$ and $v = \ln u$ in Theorem 1.7, we have:

Corollary 1.8. *Let $(M^n, g(t))$, $t \in [0, T]$, be a compact solution to the ε-Kähler–Ricci flow with nonnegative bisectional curvature and $R_{i\bar{j}} \leq k g_{i\bar{j}}$ for some constant $k > 0$. Let $u : M^n \times [0, T] \to \mathbb{R}$ be a positive solution to the nonlinear backward conjugate heat equation*

$$u_t + \Delta_{g(t)} u = \varepsilon R u + a u \ln u. \tag{1-36}$$

Suppose that $\eta : (0, T) \to (0, \infty)$ is a smooth function satisfying the ordinary differential inequality

$$\eta' \leq \eta^2 - (\varepsilon k - a)\eta - \varepsilon k \left(a + \frac{1}{t} \right) \tag{1-37}$$

on $(0, T)$ *and that* $\eta(t) \to \infty$ *as* $t \to T$. *Then*

$$\varepsilon R_{i\bar{j}} - \nabla_i \nabla_{\bar{j}} \ln u - \eta g_{i\bar{j}} \le 0 \tag{1-38}$$

for all $(x, t) \in M \times (0, T)$.

The rest of the paper is organized as follows. We devote Section 2 to the notation and basic formulas in Riemannian geometry and Kähler geometry. In Section 3, we provide evolution equations of geometric quantities under the Ricci flow and the Kähler–Ricci flow, respectively. In Section 4, we prove Theorems 1.1 and 1.4. Section 5 presents the proof of Theorem 1.7.

2. Preliminaries

Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection of the metric g. Then the Riemann curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2-1}$$

In local coordinate, its components are determined by

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l_{ijk}\frac{\partial}{\partial x^l}. \tag{2-2}$$

We keep the notation

$$R_{ijkl} = g_{lm} R^m_{ijk} \tag{2-3}$$

as in [11]. The Ricci tensor is defined by

$$R_{ij} = R^k_{kij} \tag{2-4}$$

and scalar curvature is given by

$$R = g^{ij} R_{ij}. \tag{2-5}$$

We say that M has nonnegative sectional curvature if

$$R_{kijl} v^i v^j w^k w^l \ge 0 \tag{2-6}$$

for all $v, w \in T_x M$ at $x \in M$. We define Ricci curvature to be parallel if $\nabla_k R_{ij} = 0$ for all i, j, k. To communicate covariant differentiation, we need the following Ricci identities. If α is a 1-form, then

$$\nabla_i \nabla_j \alpha_k = \nabla_j \nabla_i \alpha_k - R^l_{ijk} \alpha_l. \tag{2-7}$$

Let M be a Kähler manifold with Kähler metric $g_{i\bar{j}}$. The Kähler form

$$\omega = \tfrac{1}{2}\sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \tag{2-8}$$

is a closed real (1, 1)-form, so we have

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}. \tag{2-9}$$

The Christoffel symbols of the metric $g_{i\bar{j}}$ are given by

$$\Gamma^k_{ij} = g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j}, \quad \Gamma^{\bar{l}}_{\bar{i}\bar{j}} = g^{k\bar{l}} \frac{\partial g_{k\bar{j}}}{\partial \bar{z}^i}, \tag{2-10}$$

where $g^{i\bar{j}} = (g_{i\bar{j}})^{-1}$. It is easy to see that Γ^k_{ij} is symmetric in i and j and $\Gamma^{\bar{l}}_{\bar{i}\bar{j}}$ is symmetric in $\bar{i}$ and $\bar{j}$.

The curvature tensor of the metric $g_{i\bar{j}}$ is defined as

$$R^j_{ik\bar{l}} = \frac{\partial \Gamma^j_{ik}}{\partial \bar{z}^l}, \quad R_{i\bar{j}k\bar{l}} = g_{p\bar{j}} R^p_{ik\bar{l}}. \tag{2-11}$$

It is easy to see that $R_{i\bar{j}k\bar{l}}$ is symmetric in i and k, in $\bar{j}$ and $\bar{l}$ and in pairs $i\bar{j}$ and $k\bar{l}$. The second Bianchi identity in the Kähler case reduces to

$$\nabla_p R_{i\bar{j}k\bar{l}} = \nabla_k R_{i\bar{j}p\bar{l}}, \quad \nabla_{\bar{q}} R_{i\bar{j}k\bar{l}} = \nabla_{\bar{l}} R_{i\bar{j}k\bar{q}}. \tag{2-12}$$

M is said to have nonnegative holomorphic bisectional curvature if

$$R_{i\bar{j}k\bar{l}} v^i v^{\bar{j}} w^k w^{\bar{l}} \geq 0 \tag{2-13}$$

for all nonzero vectors v and w in the holomorphic tangent space $T_x M$ at $x \in M$.

The Ricci tensor of the metric $g_{i\bar{j}}$ is obtained by taking the trace of the curvature tensor:

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}, \tag{2-14}$$

and the scalar curvature is given by

$$R = g^{i\bar{j}} R_{i\bar{j}}. \tag{2-15}$$

Finally, we give the commutation formulae for covariant differentiations in Kähler geometry. Covariant differentiations of the same type can be commuted freely, e.g.,

$$\nabla_k \nabla_j v_i = \nabla_j \nabla_k v_i, \quad \nabla_{\bar{k}} \nabla_{\bar{j}} v_i = \nabla_{\bar{j}} \nabla_{\bar{k}} v_i. \tag{2-16}$$

But we shall need following formulae when commuting covariant derivatives of different types

$$\nabla_k \nabla_{\bar{j}} v_i = \nabla_{\bar{j}} \nabla_k v_i - R_{k\bar{j}i\bar{l}} v_l. \tag{2-17}$$

3. Evolution equations

Let $(M^n, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow. In order to give the proof of Theorems 1.1 and 1.4, we give some lemma and propositions used in the proof. It should be pointed out that all the computations are taken in normal coordinate.

Lemma 3.1 [20]. *Under the Ricci flow, we have*

$$(\partial_t-\varepsilon\Delta_L)(\nabla_i\nabla_j f)=\nabla_i\nabla_j(\partial_t-\varepsilon\Delta)f+(1-\varepsilon)(\nabla_i R_{jk}+\nabla_j R_{ik}-\nabla_k R_{ij})\nabla_k f \tag{3-1}$$

for any smooth function $f(x,t)$.

Let $v : M^n \times [0, T] \to \mathbb{R}$ be a solution to the nonlinear heat-type equation

$$(\partial t-\varepsilon\Delta_{g(t)})v=\delta R+\varepsilon\,|\nabla v|^2+f(v). \tag{3-2}$$

We will derive the evolution equation of $H_{ij} := \nabla_i\nabla_j \ln u$. In the following, Δ_L denotes the Lichnerowicz Laplacian acting on symmetric two-tensors via

$$\Delta_L h_{ij}=\Delta h_{ij}+2R_{ikjl}h_{kl}-R_{ik}h_{jk}-R_{jk}h_{ik}. \tag{3-3}$$

We can get the following proposition.

Proposition 3.2. *Under the Ricci flow, we have*

$$\begin{aligned}(\partial_t-\varepsilon\Delta)H_{ij}&\\ =\delta\nabla_i\nabla_j R&+2\varepsilon(H_{ij}^2+R_{ikjl}\nabla_k v\nabla_l v+\nabla_k H_{ij}\nabla_k v)\\ &+(1-\varepsilon)(\nabla_i R_{jk}+\nabla_j R_{ik}-\nabla_k R_{ij})\nabla_k v\\ &+\varepsilon(2R_{ikjl}H_{kl}-R_{ik}H_{jk}-R_{jk}H_{ik})+f''(v)\nabla_i v\nabla_j v+f'(v)\nabla_i\nabla_j v,\end{aligned} \tag{3-4}$$

where $H_{ij}^2 := H_{ik}H_{jk}$.

Proof. Since v is a solution to (3-2), v satisfies

$$(\partial_t-\varepsilon\Delta)v=\delta R+\varepsilon|\nabla v|^2+f(v). \tag{3-5}$$

We calculate that

$$\begin{aligned}\nabla_i\nabla_j|\nabla v|^2&=\nabla_i\nabla_j(\nabla_k v\nabla_k v)\\ &=\nabla_i(\nabla_j\nabla_k v\nabla_k v+\nabla_k v\nabla_j\nabla_k v)\\ &=2\nabla_i(\nabla_j\nabla_k v\nabla_k v)\\ &=2\nabla_k(\nabla_i\nabla_j v)\nabla_k v+2R_{ikjl}\nabla_l v\nabla_k v+2H_{ij}^2\\ &=2H_{ij}^2+2R_{ikjl}\nabla_l v\nabla_k v+2\nabla_k H_{ij}\nabla_k v.\end{aligned} \tag{3-6}$$

Applying Lemma 3.1 to $f = v$ yields

$$(\partial_t - \varepsilon\Delta_L)H_{ij} = \nabla_i\nabla_j(\varepsilon|\nabla v|^2 + \delta R + f(v)) + (1-\varepsilon)(\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij})\nabla_k v$$
$$= \delta\nabla_i\nabla_j R + 2\varepsilon(H_{ij}^2 + R_{ikjl}\nabla_l v\nabla_k v + \nabla_k H_{ij}\nabla_k v) + (1-\varepsilon)(\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij})\nabla_k v + f''(v)\nabla_i v\nabla_j v + f'(v)\nabla_i\nabla_j v. \tag{3-7}$$

Combining with the identity

$$\Delta_L H_{ij} = \Delta H_{ij} + 2R_{ikjl}H_{kl} - R_{ik}H_{jk} - R_{jk}H_{ik}, \tag{3-8}$$

we obtain

$$(\partial_t - \varepsilon\Delta)H_{ij} = \delta\nabla_i\nabla_j R + 2\varepsilon(H_{ij}^2 + R_{ikjl}\nabla_l v\nabla_k v + \nabla_k H_{ij}\nabla_k v) + \varepsilon(2R_{ikjl}H_{kl} - R_{ik}H_{jk} - R_{jk}H_{ik}) + (1-\varepsilon)(\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij})\nabla_k v + f''(v)\nabla_i v\nabla_j v + f'(v)\nabla_i\nabla_j v. \tag{3-9}$$

The proof is complete. □

Let $(M^n, g(t))$, $t \in [0, T]$, be a solution to the ε-Kähler–Ricci flow. Let $v : M^n \times [0, T] \to \mathbb{R}$ be a solution to the nonlinear heat-type equation

$$(\partial t - \delta\Delta_{g(t)})v = \theta R + \delta|\nabla v|^2 + f(v). \tag{3-10}$$

We will derive the evolution equation of $H_{i\bar{j}} := \nabla_i\nabla_{\bar{j}}\ln u = \nabla_i\nabla_{\bar{j}}v$.

For simplicity of notation, we write $\Delta := \Delta_{g(t)}$, $v = \ln u$, $H_{ik} := \nabla_i\nabla_k v$ and $H_{\bar{j}\bar{k}} := \nabla_{\bar{j}}\nabla_{\bar{k}}v$. Moreover, Δ_L denotes the complex Lichnerowicz Laplacian acting on real (1,1)-tensors via

$$\Delta_L h_{i\bar{j}} := \Delta h_{i\bar{j}} + R_{i\bar{j}l\bar{k}}h_{k\bar{l}} - \tfrac{1}{2}R_{i\bar{l}}h_{l\bar{j}} - \tfrac{1}{2}R_{k\bar{j}}h_{i\bar{k}}. \tag{3-11}$$

In order to give the proof of Theorem 1.4, we need following proposition.

Proposition 3.3. *In the setting described above, we have*

$$(\partial_t - \delta\Delta_L)H_{i\bar{j}} = \theta\nabla_i\nabla_{\bar{j}}R + \delta(H_{i\bar{k}}H_{k\bar{j}} + H_{ik}H_{\bar{j}\bar{k}} + R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}}v) + \delta(\nabla_k H_{i\bar{j}}\nabla_{\bar{k}}v + \nabla_{\bar{k}}H_{i\bar{j}}\nabla_k v) + f''(v)\nabla_i v\nabla_{\bar{j}}v + f'(v)\nabla_i\nabla_{\bar{j}}v. \tag{3-12}$$

Proof. We calculate that

$$(\partial_t - \delta\Delta_L)H_{i\bar{j}} = \nabla_i\nabla_{\bar{j}}(\partial_t - \delta\Delta)v$$
$$= \nabla_i\nabla_{\bar{j}}(\theta R + \delta|\nabla v|^2 + f(v))$$
$$= \theta\nabla_i\nabla_{\bar{j}}R + \delta\nabla_i\nabla_{\bar{j}}|\nabla v|^2 + \nabla_i\nabla_{\bar{j}}f(v), \tag{3-13}$$

Commuting covariant derivatives produces

$$\nabla_i\nabla_{\bar j}|\nabla v|^2 = H_{i\bar k}H_{k\bar j}+H_{ik}H_{\bar j\bar k}+R_{i\bar k l\bar j}\nabla_k v\nabla_{\bar l}v+\nabla_k H_{i\bar j}\nabla_{\bar k}v+\nabla_{\bar k}H_{i\bar j}\nabla_k v. \tag{3-14}$$

Hence, we obtain

$$\begin{aligned}(\partial_t-\delta\Delta_L)H_{i\bar j}&=\theta\nabla_i\nabla_{\bar j}R+\delta(H_{i\bar k}H_{k\bar j}+H_{ik}H_{\bar j\bar k}+R_{i\bar k l\bar j}\nabla_k v\nabla_{\bar l}v)\\&\quad+\delta(\nabla_k H_{i\bar j}\nabla_{\bar k}v+\nabla_{\bar k}H_{i\bar j}\nabla_k v)+f''(v)\nabla_i v\nabla_{\bar j}v+f'(v)\nabla_i\nabla_{\bar j}v.\end{aligned}\tag{3-15}$$

The proof is complete. □

4. Matrix estimates for the nonlinear heat equation

In this section we will prove Theorems 1.1 and 1.4.

Proof of Theorem 1.1. Setting $\varepsilon=1$, $\delta=0$ in (3-4), we have that $H_{ij}:=\nabla_i\nabla_j\ln u$ satisfies

$$\begin{aligned}(\partial_t-\Delta)H_{ij}&=2H_{ij}^2+2R_{ikjl}\nabla_l v\nabla_k v+2\nabla_k H_{ij}\nabla_k v+2R_{ikjl}H_{kl}\\&\quad-R_{ik}H_{jk}-R_{jk}H_{ik}+f''(v)\nabla_i v\nabla_j v+f'(v)\nabla_i\nabla_j v,\end{aligned}\tag{4-1}$$

where $H_{ij}^2:=H_{ik}H_{jk}$. We define

$$Z_{ij}:=H_{ij}+c(t)g_{ij}. \tag{4-2}$$

Straightforward computation yields that

$$\begin{aligned}&2H_{ij}^2+2R_{ikjl}H_{kl}-R_{ik}H_{jk}-R_{jk}H_{ik}\\&\quad=2Z_{ij}^2-4cZ_{ij}+2c^2g_{ij}+2R_{ikjl}Z_{kl}-R_{ik}Z_{jk}-R_{jk}Z_{ik}.\end{aligned}\tag{4-3}$$

Combining with the identity (4-1), we get that

$$\begin{aligned}(\partial_t-\Delta)Z_{ij}&=(\partial_t-\Delta)(H_{ij}+c(t)g_{ij})\\&=(\partial_t-\Delta)H_{ij}+(\partial_t-\Delta)(c(t)g_{ij})\\&=2Z_{ij}^2-4cZ_{ij}+2c^2g_{ij}+2R_{ikjl}Z_{kl}-R_{ik}Z_{jk}-R_{jk}Z_{ik}+2R_{ikjl}\nabla_l v\nabla_k v\\&\qquad+2\nabla_k Z_{ij}\nabla_k v+f'(v)H_{ij}+f''(v)\nabla_i v\nabla_j v+c'g_{ij}-2cR_{ij}\\&=2Z_{ij}^2-4cZ_{ij}+2R_{ikjl}Z_{kl}-R_{ik}Z_{jk}-R_{jk}Z_{ik}+2R_{ikjl}\nabla_l v\nabla_k v\\&\qquad+2\nabla_k Z_{ij}\nabla_k v+f'(v)Z_{ij}+f''(v)\nabla_i v\nabla_j v\\&\qquad+(c'+2c^2-f'(v)c-2kc)g_{ij}+2c(kg_{ij}-R_{ij}).\end{aligned}$$

Using $2R_{ikjl}\nabla_l v\nabla_k v\geq 0$, $R_{ij}\leq kg_{ij}$, $f''(v)\geq 0$ and the condition that $c(t)$ satisfies

$$c'(t)+2c^2(t)-(f'(v)+2k)c(t)\geq 0, \tag{4-4}$$

we obtain that

$$(\partial_t - \Delta)Z_{ij} \geq 2Z_{ij}^2 - 4cZ_{ij} + 2R_{ikjl}Z_{kl} - R_{ik}Z_{jk} - R_{jk}Z_{ik} + f'(v)Z_{ij} + 2\nabla_k Z_{ij}\nabla_k v. \tag{4-5}$$

Since M is compact and $c(t) \to \infty$ as $t \to 0^+$, we have $Z_{ij} \geq 0$ as $t \to 0^+$. Then the tensor maximum principle of Hamilton implies that $Z_{ij} \geq 0$ for all $t \in [0, T]$, as it is obvious that

$$2Z_{ij}^2 - 4cZ_{ij} + 2R_{ikjl}Z_{kl} - R_{ik}Z_{jk} - R_{jk}Z_{ik} + f'(v)Z_{ij} \tag{4-6}$$

is nonnegative at a null-eigenvector of Z_{ij}. □

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Setting $\delta = 1$, $\theta = 0$ in (3-12), we have that $H_{i\bar{j}} := \nabla_i\nabla_{\bar{j}} \ln u$ satisfies

$$(\partial_t - \Delta_L)H_{i\bar{j}} = H_{i\bar{k}}H_{k\bar{j}} + H_{ik}H_{\bar{j}\bar{k}} + R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}} v + \nabla_k H_{i\bar{j}}\nabla_{\bar{k}} v + \nabla_{\bar{k}} H_{i\bar{j}}\nabla_k v + f''(v)\nabla_i v\nabla_{\bar{j}} v + f'(v)\nabla_i\nabla_{\bar{j}} v. \tag{4-7}$$

We define

$$Z_{i\bar{j}} := H_{i\bar{j}} + c(t)g_{i\bar{j}}. \tag{4-8}$$

It follows from

$$H_{i\bar{k}}H_{k\bar{j}} = Z_{i\bar{k}}Z_{k\bar{j}} - 2c(t)Z_{i\bar{j}} + c^2(t)g_{i\bar{j}} \tag{4-9}$$

that

$$\begin{aligned}
(\partial_t - \Delta_L)Z_{i\bar{j}} &= (\partial_t - \Delta_L)(H_{i\bar{j}} + c(t)g_{i\bar{j}}) \\
&= (\partial_t - \Delta_L)H_{i\bar{j}} + (\partial_t - \Delta_L)(c(t)g_{i\bar{j}}) \\
&= H_{i\bar{k}}H_{\bar{j}k} + H_{ik}H_{\bar{j}\bar{k}} + R_{i\bar{k}l\bar{j}}\nabla_{\bar{l}} v\nabla_k v + \nabla_k H_{i\bar{j}}\nabla_{\bar{k}} v + \nabla_{\bar{k}} H_{i\bar{j}}\nabla_k v \\
&\qquad + f'(v)H_{i\bar{j}} + f''(v)\nabla_i v\nabla_{\bar{j}} v + c'(t)g_{i\bar{j}} - \varepsilon c(t)R_{i\bar{j}} \\
&= H_{ik}H_{\bar{j}\bar{k}} + Z_{i\bar{k}}Z_{k\bar{j}} - 2c(t)Z_{i\bar{j}} + R_{i\bar{k}l\bar{j}}\nabla_{\bar{l}} v\nabla_k v + \nabla_k H_{i\bar{j}}\nabla_{\bar{k}} v \\
&\quad + \nabla_{\bar{k}} H_{i\bar{j}}\nabla_k v + f'(v)Z_{i\bar{j}} + f''(v)\nabla_i v\nabla_{\bar{j}} v \\
&\qquad + \big(c'(t) + c^2(t) - (\varepsilon k + f'(v))c(t)\big)g_{i\bar{j}} + \varepsilon c(t)(kg_{i\bar{j}} - R_{i\bar{j}}).
\end{aligned}$$

Using $R_{i\bar{k}l\bar{j}}\nabla_{\bar{l}} v\nabla_k v \geq 0$, $R_{i\bar{j}} \leq kg_{i\bar{j}}$, $f''(v) \geq 0$ and the condition that $c(t)$ satisfies

$$c'(t) + c^2(t) - (\varepsilon k + f'(v))c(t) \geq 0, \tag{4-10}$$

we obtain that

$$(\partial_t - \Delta_L)Z_{i\bar{j}} \geq Z_{i\bar{k}}Z_{k\bar{j}} - 2c(t)Z_{i\bar{j}} + \nabla_k Z_{i\bar{j}}\nabla_{\bar{k}} v + \nabla_{\bar{k}} Z_{i\bar{j}}\nabla_k v + f'(v)Z_{i\bar{j}}. \tag{4-11}$$

Noticing that $Z_{i\bar{k}}Z_{k\bar{j}}$, $-2c(t)Z_{i\bar{j}}$ and $f'(v)Z_{i\bar{j}}$ satisfy the null-eigenvector condition in Hamilton's tensor maximum principle [16] and $Z_{i\bar{j}}(x,t)\to\infty$ uniformly as $t\to 0$, we conclude that $Z_{i\bar{j}}(x,t)\geq 0$ on $M\times(0,T)$. □

5. Matrix estimates for the nonlinear backward conjugate heat equation

Proof of Theorem 1.7. By using Proposition 3.3 with $\delta=-1$ and $\theta=\varepsilon$, we have

$$-(\partial_t+\Delta_L)H_{i\bar{j}}=-\varepsilon\nabla_i\nabla_{\bar{j}}R+H_{i\bar{k}}H_{k\bar{j}}+H_{ik}H_{\bar{j}\bar{k}}+R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}}v +\nabla_k H_{i\bar{j}}\nabla_{\bar{k}}v+\nabla_{\bar{k}}H_{i\bar{j}}\nabla_k v-f''(v)\nabla_i v\nabla_{\bar{j}}v-f'(v)\nabla_i\nabla_{\bar{j}}v. \tag{5-1}$$

Under the ε-Kähler–Ricci flow, we have (see [12, page 123])

$$\partial_t R_{i\bar{j}}=\varepsilon\Delta_L R_{i\bar{j}}=\varepsilon\nabla_i\nabla_{\bar{j}}R, \tag{5-2}$$

which yields

$$(\partial_t+\Delta_L)(\varepsilon R_{i\bar{j}})=\varepsilon^2(\Delta R_{i\bar{j}}+R_{i\bar{j}l\bar{k}}R_{k\bar{l}}-R_{i\bar{k}}R_{k\bar{j}})+\varepsilon\nabla_i\nabla_{\bar{j}}R. \tag{5-3}$$

Also, we have

$$-(\partial_t+\Delta_L)(\eta g_{i\bar{j}})=-\eta' g_{i\bar{j}}+\varepsilon\eta R_{i\bar{j}}. \tag{5-4}$$

Set

$$Z_{i\bar{j}}:=\varepsilon R_{i\bar{j}}-H_{i\bar{j}}-\eta(t)g_{i\bar{j}}. \tag{5-5}$$

Combining the above evolution equations together, we derive that

$$\begin{aligned}
(\partial_t+\Delta_L)Z_{i\bar{j}} &= (\partial_t+\Delta_L)(\varepsilon R_{i\bar{j}}-H_{i\bar{j}}-\eta(t)g_{i\bar{j}})\\
&= (\partial_t+\Delta_L)(\varepsilon R_{i\bar{j}})-(\partial_t+\Delta_L)H_{i\bar{j}}-(\partial_t+\Delta_L)(\eta(t)g_{i\bar{j}})\\
&= H_{i\bar{k}}H_{k\bar{j}}+H_{ik}H_{\bar{j}\bar{k}}+R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}}v+\nabla_k H_{i\bar{j}}\nabla_{\bar{k}}v+\nabla_{\bar{k}}H_{i\bar{j}}\nabla_k v\\
&\qquad +\varepsilon^2(\Delta R_{i\bar{j}}+R_{i\bar{j}l\bar{k}}R_{k\bar{l}}-R_{i\bar{k}}R_{k\bar{j}})-f''(v)\nabla_i v\nabla_{\bar{j}}v\\
&\qquad -f'(v)\nabla_i\nabla_{\bar{j}}v-\eta' g_{i\bar{j}}+\varepsilon\eta R_{i\bar{j}}\\
&= H_{i\bar{k}}H_{k\bar{j}}+H_{ik}H_{\bar{j}\bar{k}}-\varepsilon^2 R_{i\bar{k}}R_{k\bar{j}}-\nabla_k Z_{i\bar{j}}\nabla_{\bar{k}}v-\nabla_{\bar{k}}Z_{i\bar{j}}\nabla_k v\\
&\quad +\varepsilon^2\left(\Delta R_{i\bar{j}}+R_{i\bar{j}l\bar{k}}R_{k\bar{l}}+\frac{1}{\varepsilon^2}R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}}v\right)\\
&\quad +\varepsilon^2\left(\frac{1}{\varepsilon}\nabla_k R_{i\bar{j}}\nabla_{\bar{k}}v+\frac{1}{\varepsilon}\nabla_k v\nabla_{\bar{k}}R_{i\bar{j}}+\frac{1}{\varepsilon t}R_{i\bar{j}}\right)+f'(v)Z_{i\bar{j}}-\eta' g_{i\bar{j}}\\
&\quad +\varepsilon\eta R_{i\bar{j}}-f''(v)\nabla_i v\nabla_{\bar{j}}v-\frac{\varepsilon}{t}R_{i\bar{j}}-f'(v)\varepsilon R_{i\bar{j}}+f'(v)\eta g_{i\bar{j}}.
\end{aligned}$$

Using the identity

$$-\tfrac{1}{2}Z_{i\bar{k}}(\varepsilon R_{k\bar{j}}+H_{k\bar{j}}-\eta g_{k\bar{j}})-\tfrac{1}{2}(\varepsilon R_{i\bar{k}}+H_{i\bar{k}}-\eta g_{i\bar{k}})Z_{k\bar{j}} = H_{i\bar{k}}H_{k\bar{j}}-\varepsilon^2 R_{i\bar{k}}R_{k\bar{j}}+2\varepsilon\eta R_{i\bar{j}}-\eta^2 g_{i\bar{j}}, \tag{5-6}$$

we get

$$\begin{aligned}(\partial_t+\Delta_L)Z_{i\bar{j}} &= H_{ik}H_{\bar{j}\bar{k}}+(\eta^2-\eta'+\eta f'(v))g_{i\bar{j}}-\left(\varepsilon\eta+\frac{\varepsilon}{t}+f'(v)\varepsilon\right)R_{i\bar{j}}\\ &\quad-\nabla_k Z_{i\bar{j}}\nabla_{\bar{k}}v-\nabla_{\bar{k}}Z_{i\bar{j}}\nabla_k v\\ &\quad+\varepsilon^2\left(\Delta R_{i\bar{j}}+R_{i\bar{j}l\bar{k}}R_{k\bar{l}}+\frac{1}{\varepsilon^2}R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}}v\right)\\ &\quad+\varepsilon^2\left(\frac{1}{\varepsilon}\nabla_k R_{i\bar{j}}\nabla_{\bar{k}}v+\frac{1}{\varepsilon}\nabla_k v\nabla_{\bar{k}}R_{i\bar{j}}+\frac{1}{\varepsilon t}R_{i\bar{j}}\right)\\ &\quad-\tfrac{1}{2}Z_{i\bar{k}}(\varepsilon R_{k\bar{j}}+H_{k\bar{j}}-\eta g_{k\bar{j}})-\tfrac{1}{2}(\varepsilon R_{i\bar{k}}+H_{i\bar{k}}-\eta g_{i\bar{k}})Z_{k\bar{j}}\\ &\quad+f'(v)Z_{i\bar{j}}-f''(v)\nabla_i v\nabla_{\bar{j}}v.\end{aligned} \tag{5-7}$$

Now for the ε-Kähler–Ricci flow, Cao's matrix Harnack estimate [1] with $X_i=\frac{1}{\varepsilon}\nabla_i v$ implies that

$$0\le \Delta R_{i\bar{j}}+R_{i\bar{j}l\bar{k}}R_{k\bar{l}}+\frac{1}{\varepsilon^2}R_{i\bar{k}l\bar{j}}\nabla_k v\nabla_{\bar{l}}v+\frac{1}{\varepsilon}\nabla_k R_{i\bar{j}}\nabla_{\bar{k}}v+\frac{1}{\varepsilon}\nabla_k v\nabla_{\bar{k}}R_{i\bar{j}}+\frac{1}{\varepsilon t}R_{i\bar{j}}. \tag{5-8}$$

Using $R_{i\bar{j}}\le k g_{i\bar{j}}$ and (1-34), we have

$$\begin{aligned}(\eta^2-\eta'+f'(v)\eta)g_{i\bar{j}}-\left(\varepsilon\eta+\frac{\varepsilon}{t}+f'(v)\varepsilon\right)R_{i\bar{j}} &\ge\left(\eta^2-\eta'+f'(v)\eta-\varepsilon k\eta-\frac{\varepsilon k}{t}-f'(v)k\varepsilon\right)g_{i\bar{j}}\\ &\ge 0.\end{aligned} \tag{5-9}$$

Therefore, we have

$$\begin{aligned}(\partial_t+\Delta_L)Z_{i\bar{j}}+\nabla_k Z_{i\bar{j}}\nabla_{\bar{k}}v+\nabla_{\bar{k}}Z_{i\bar{j}}\nabla_k v-f'(v)Z_{i\bar{j}}+f''(v)\nabla_i v\nabla_{\bar{j}}v\\ \ge-\tfrac{1}{2}Z_{i\bar{k}}(\varepsilon R_{k\bar{j}}+H_{k\bar{j}}-\eta g_{k\bar{j}})-\tfrac{1}{2}(\varepsilon R_{i\bar{k}}+H_{i\bar{k}}-\eta g_{i\bar{k}})Z_{k\bar{j}}.\end{aligned} \tag{5-10}$$

Observing that the right-hand side of the above inequality satisfies the null-eigenvector condition in Hamilton's tensor maximum principle [16] and $Z_{i\bar{j}}(x,t)\to-\infty$ uniformly as $t\to T$, we conclude that $Z_{i\bar{j}}(x,t)\le 0$ on $M\times(0,T)$. □

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SHA YAO
SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE
HENAN POLYTECHNIC UNIVERSITY
JIAOZUO
CHINA
yaosha@hpu.edu.cn

HAO-YUE LIU
SCHOOL OF SCIENCE
JIANGSU OCEAN UNIVERSITY
LIANYUNGANG
CHINA
lhyty7@163.com

XIN-AN REN
SCHOOL OF MATHEMATICS
CHINA UNIVERSITY OF MINING AND TECHNOLOGY
XUZHOU
CHINA
renx@cumt.edu.cn

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