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We describe a construction procedure of infinite sets of 2-links in closed simply connected 4-manifolds that are topologically isotopic, smoothly inequivalent and componentwise topologically unknotted. These 2-links are the first such examples in the literature. The examples provided have surface and free groups as their 2-link groups, and display subtle exotic phenomena that are related to their linking. In particular, our examples are not parallel copies of exotic embeddings of 2-spheres that were previously known to exist nor their combinations with smoothly unknotted 2-spheres.

1. Introduction and main results

All embeddings and manifolds in this paper are smooth unless it is stated otherwise. The main contribution of this paper is to introduce a construction procedure of infinite sets of multiple components 2-links in closed 4-manifolds that are topologically isotopic, yet smoothly inequivalent, and whose components are topologically unknotted. The explicit examples that we produce have the property that the fundamental group of the complement can be chosen to be any free group or a surface group. We begin by making these notions precise in the following definition.

Definition 1. • A k-component 2-link

$$\Gamma = S_1 \sqcup \cdots \sqcup S_k \subset X \tag{1-1}$$

in a 4-manifold X is an unordered union of disjointly embedded 2-spheres S_i with trivial tubular neighborhood $\nu(S_i) = D^2 \times S^2$ for $i = 1, \ldots, k$ and $k \in \mathbb{N}$. We say that a component S_i is topologically unknotted if there is a locally flat embedded 3-ball $D_i \subset X$ such that $\partial D_i = S_i$.

• The fundamental group $\pi_1(X \setminus \Gamma)$ of the complement of (1-1) is called the 2-link group.

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• A 2-link (1-1) is symmetric if for any permutation σ of $\{1, \dots, k\}$ there is a diffeomorphism $\phi : X \to X$ such that $\phi(S_i) = S_{\sigma(i)}$ for $i = 1, \dots, k$.

• Two k-component 2-links Γ and Γ' embedded in a 4-manifold X are smoothly inequivalent if there is no diffeomorphism of pairs

$$(X, \Gamma) \to (X, \Gamma').$$

• A family of k-component 2-links is exotic if its elements are topologically isotopic and pairwise smoothly inequivalent.

There has been a flurry of research on exotic embeddings of surfaces in closed 4-manifolds ignited by Fintushel and Stern's foundational paper [14]. Finding such embeddings of nullhomotopic 2-spheres is particularly difficult: just notice that it is still unknown if there is an exotic S^2 in S^4. There is an implicit topologically unknotted 2-sphere that is smoothly knotted in a closed simply connected 4-manifold in work of Fintushel and Stern [13] as pointed out by Ray and Ruberman in [35], and infinite sets of such embeddings of 2-spheres were constructed by Torres in [42, Theorem A].

The construction procedure that we introduce in this paper extends these results to multiple components 2-links, where our key raw material is defined as follows.

Definition 2. An admissible 4-manifold M is a 4-manifold that satisfies the following properties:

• There is at least one basic class $k \in H^2(M; \mathbb{Z})$, i.e., the Seiberg–Witten invariant of M satisfies $\mathrm{SW}_M(k) \neq 0$.

• The 4-manifold M is simply connected.

• There is a pair of embedded disjoint 2-tori $T_1, T_2 \subset M$ such that

$$T_i \text{ has self-intersection } 0 \quad \text{for } i = 1, 2,$$

and

$$\pi_1(M) = \{1\} = \pi_1\big(M \setminus (\nu(T_1) \sqcup \nu(T_2))\big).$$

There is no shortage of admissible 4-manifolds [8; 13; 20; 28] where the quintessential examples are the elliptic surfaces $E(n)$ for $n \geq 2$. We now give a sample of the produce of our construction procedure with our main result. A closed oriented surface of genus g is denoted by Σ_g, and the free group of g generators by F_g.

Theorem A. *Let M be an admissible 4-manifold as in Definition 2 and fix $g \in \mathbb{N}$. Let G be a group isomorphic to either $\pi_1(\Sigma_g)$ or F_g, and let n be its rank. Let $\mathcal{K}$ be an infinite family of knots $K \subset S^3$ with pairwise distinct Alexander polynomials. There is an infinite set of smooth n-component 2-links*

$$\{\Gamma_K = S_{1,K} \sqcup \cdots \sqcup S_{n,K} \mid K \in \mathcal{K}\}, \tag{1-2}$$

which are smoothly embedded in $M \# n(S^2 \times S^2)$ and have the following properties:

(1) *Their* 2*-link group is* G*.*

(2) *The* 2*-links in* (1-2) *form an exotic family as in Definition 1.*

(3) *The components of* (1-2) *are topologically unknotted.*

(4) *Surgery on the components of* (1-2) *yields an infinite set of pairwise nondiffeomorphic closed* 4*-manifolds in a same homeomorphism class with fundamental group* G*.*

Theorem A unveils exotic phenomena of codimension-two embeddings that were not previously known to exist. A detailed description of the procedure is given in Section 2.2. Examples of exotic 2-links in closed 4-manifolds with homologically essential components are available. Auckly, Kim, Melvin and Ruberman [5, Theorem B] produced infinite sets of exotic 2-links with trivial 2-link group and whose components have self-intersection $+1$. Hayden, Kjuchukova, Krishna, Miller, Powell and Sunukjian provided pairs of exotic 2-component 2-links in [25, Theorem 8.2]; see [24] as well.

Much like the examples in [5; 25], linking is of significant importance regarding the subtlety of the exotic behavior of the 2-links of Theorem A. Less subtle phenomena can be readily obtained by taking parallel copies of the exotic 2-spheres in [42] and a combination of them with smoothly unknotted 2-spheres to produce exotic 2-links whose 2-link group is F_g; see Remark 22. In order to guarantee that our examples of Theorem A are not of this kind and to put in display the subtlety of their exotica, we introduce the following notion.

Definition 3. An exotic pair of smooth k-component 2-links

$$\Gamma = S_1 \sqcup \cdots \sqcup S_k \quad \text{and} \quad \Gamma' = S_1' \sqcup \cdots \sqcup S_k'$$

in the sense of Definition 1 is Brunnianly exotic if the 2-links

$$\Gamma \setminus S_i \quad \text{and} \quad \Gamma' \setminus S_j'$$

are smoothly equivalent for any $i, j = 1, \ldots, k$.

The Brunnianity property of Definition 3 rules out uninteresting constructions of exotic 2-links. Our second main result shows that the examples of Theorem A satisfy this property and that they stabilize by taking the connected sum with a single copy of $S^2 \times S^2$ at a point away from every element in (1-2).

Theorem B. • *Every* 2*-link in the infinite set* (1-2) *of Theorem A with free* 2*-link group is smoothly symmetric.*

• *There is an infinite subset of* (1-2) *made of pairwise Brunnianly exotic* 2*-links smoothly embedded in* $M \# g(S^2 \times S^2)$*. Moreover, elements in this infinite subset are pairwise smoothly equivalent in* $(M \# g(S^2 \times S^2)) \# (S^2 \times S^2)$*.*

Related notions of Brunnianity of 2-links in 4-manifolds are available in the literature, always in analogy with the properties of Brunnian links in S^3 introduced by Brunn in 1892 [10]. The notion of Brunnianity used by Auckly, Kim, Melvin and Ruberman in [5, Theorem B] is quite similar to ours albeit a bit stronger: their results address smooth isotopy by considering ordered links. We consider only equivalence and not isotopy between our surfaces, but do so irrespectively of their order. A stronger use of the adjective Brunnian is employed by Hayden, Kjuchukova, Krishna, Miller, Powell and Sunukjian in [25].

We finish the introduction with a pair of questions that arose during the production of this work, and a third question that was posed by a referee. The first question concerns a topological property of the 2-links constructed under our procedure. Notice that the 2-links (1-2) of Theorem A with F_g as 2-link group are smoothly linked: there are no g smoothly embedded disjoint 3-balls in $M \# g(S^2 \times S^2)$ such that each of them bounds a component of Γ_K. The locally flat embedded realm immediately comes to mind.

Question C. *Are the* 2*-links of Theorem A with free* 2*-link group topologically unlinked*? *That is*, *are there locally flat embedded disjoint* 3*-balls*

$$\Theta_g = D_1^3 \sqcup \cdots \sqcup D_g^3 \tag{1-3}$$

in $M \# g(S^2 \times S^2)$ *such that* $\partial\Theta_K = \Gamma_k$ *for any* $K \in \mathcal{K}$?

The second question regards a comparison of the Brunnianity property of Definition 3 and the Brunnian behavior of the examples of Hayden, Kjuchukova, Krishna, Miller, Powell and Sunukjian.

Question D. *Are the* 2*-links of Theorem A with free* 2*-link group Brunnian in the sense of* [25]? *That is*, *is* $\Gamma_K \setminus S_{i,K}$ *smoothly unlinked for every* $i = 1, \ldots, g$?

Bing doubling is the primary tool used by Hayden, Kjuchukova, Krishna, Miller, Powell and Sunukjian to construct their examples [25, Section 2]. It is canonical to wonder whether the examples of Theorem A can be obtained by Bing doubling a topologically unknotted but smoothly knotted 2-sphere in [13; 42] as an anonymous referee kindly pointed out to us.

Question E. *Let* S *be a topologically unknotted and smoothly knotted* 2*-sphere in a closed* 4*-manifold* M. *Is it possible to construct an example of an exotic* 2*-link in a closed* 4*-manifold by Bing doubling* S?

The paper is organized as follows. The construction procedure of infinite sets of exotic 2-links is laid down in Section 2.2. A topological restriction on admissible 4-manifolds is given in Section 2.1. The main building blocks are produced in Sections 3.1, 3.2 and 3.3. These sections contain results to pin down the diffeomorphism type of a 4-manifold constructed from surgeries that might be of independent interest, including a handlebody depiction of the Kodaira–Thurston manifold not previously available in the literature to the best of our knowledge. The smooth structures of Theorem A are constructed in Section 3.4. The diffeomorphism type of the ambient 4-manifolds are pinned down in Section 3.5. The 2-links are manufactured in Section 3.6, where we also show that they are pairwise topologically isotopic. The properties of Definition 3 are investigated in Section 3.8.

2. Recipe for exotic 2-links

2.1. ***Admissible 4-manifolds of Theorem A and their size.*** As already mentioned, there are many examples of admissible 4-manifolds in the literature [8; 13; 20; 28]. For example, in [8, Theorem 18], Baldridge and Kirk construct an admissible 4-manifold that is homeomorphic to $3\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$. The presence of the 2-tori T_1 and T_2 of Definition 2 imposes certain restrictions on the second Betti number of an admissible 4-manifold, which are recorded in the following inequality (see [17, Lemma 1]).

Lemma 4. *If M is an admissible 4-manifold in the sense of Definition 2, then*

$$b_2(M) \geq |\sigma(M)| + 4, \tag{2-1}$$

where $b_2(M)$ and $\sigma(M)$ are the second Betti number and the signature of M, respectively. In particular, M has an indefinite intersection form.

Proof. Since the complement $M \setminus (\nu(T_1) \sqcup \nu(T_2))$ is simply connected, each 2-torus T_i has a geometrically dual immersed 2-sphere S_i for $i = 1, 2$. The 2-torus T_i is disjoint from the 2-sphere S_j for $i \neq j$. We define the linear subspace

$$\mathcal{S} = \langle [T_1], [S_1], [T_2], [S_2] \rangle \subset H_2(M; \mathbb{R})$$

and compute the intersection form on its generators to be the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & * & 0 & * \\ 0 & 0 & 0 & 1 \\ 0 & * & 1 & * \end{bmatrix}.$$

This matrix is equivalent to $A = 2(+1) \oplus 2(-1)$ via a real change of basis. We can extend this new basis to a basis for $H_2(M; \mathbb{R})$, so that, after possibly changing again coordinates, the intersection form Q_M over $\mathbb{R}$ becomes

$$Q_M \cong \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

This yields the desired inequality

$$|\sigma(M)| = |\sigma(Q_M)| = |\sigma(B)| \leq \operatorname{rank}(B) = b_2(M) - 4. \qquad \square$$

Remark 5 (nonempty boundary and the size of admissible 4-manifolds). Hayden, Kjuchukova, Krishna, Miller, Powell and Sunukjian in [25] construct pairs of exotic properly embedded 2-disks in the 4-ball, where the adjective 'exotic' in this case means topologically isotopic rel. boundary and pairwise smoothly inequivalent (see [24] too). In order to distinguish the smooth structures of their complements, they make use of Stein structures and a well-known adjunction formula [21, Theorem 11.4.7]; see [3, Section 9.1]. In order to be able to distinguish infinite sets and not only pairs of exotic 2-links, we employ the Seiberg–Witten invariants instead and work with larger ambient 4-manifolds.

2.2. ***Strategy to construct exotic 2-links and prove Theorem A.***

(a) Start with an admissible 4-manifold of Definition 2 and perform Fintushel–Stern knot surgery [12; 15] along the 2-torus T_1 to produce an infinite set $\{M_K \mid K \in \mathcal{K}\}$ of pairwise nondiffeomorphic 4-manifolds that are homeomorphic to M. Notice that each one of these 4-manifolds contains a copy of the 2-torus T_2.

(b) Build the generalized fiber sum (see [20]) $Z_K = M_K \#_{T^2} B_G$ of M_K and a 4-manifold B_G. For an appropriate choice of the building block B_G, this yields an infinite set of pairwise nondiffeomorphic 4-manifolds in the homeomorphism type of Z_K with fundamental group G.

(c) Perform surgery along n loops $\mathcal{L}_G = \{\gamma_1, \dots, \gamma_n\}$ in Z_K whose homotopy classes are the generators of the group $\pi_1(Z_K) = G$, and produce a closed simply connected 4-manifold Z_K^* that is diffeomorphic to $M \# n(S^2 \times S^2)$ for every $K \in \mathcal{K}$. This step can be traced back to Wallace [44] and Milnor [29].

(d) The belt 2-spheres of the surgeries of (c) form a 2-link Γ_K of n nullhomotopic components smoothly embedded in $M \# n(S^2 \times S^2)$ whose 2-link group is isomorphic to G. A result of Sunukjian [39, Theorem 7.2] implies that each component of Γ_K is topologically unknotted.

(e) Any two 2-links Γ_K and $\Gamma_{K'}$ are smoothly inequivalent for any two different knots $K, K' \in \mathcal{K}$ since their complements

$$M \# n(S^2 \times S^2) \setminus \nu(\Gamma_K) \quad \text{and} \quad M \# n(S^2 \times S^2) \setminus \nu(\Gamma_{K'}) \tag{2-2}$$

are nondiffeomorphic. This is proven indirectly using gauge-theoretical invariants of the closed 4-manifolds Z_K and $Z_{K'}$ of (b).

(f) The explicit nature of our constructions yields a homeomorphism of the complements (2-2) that extends to a homeomorphism of pairs

$$(M \# n(S^2 \times S^2), \Gamma_K) \to (M \# n(S^2 \times S^2), \Gamma_{K'}), \tag{2-3}$$

which induces the identity map on homology. Results of Quinn and Perron allow us to conclude that the 2-links $\{\Gamma_K \mid K \in \mathcal{K}\}$ are topologically isotopic.

(g) Surgery on one loop in (c), instead of n loops, is already enough to produce the same diffeomorphism type regardless of the knot K. This is used to show that giving up one component in each 2-link stabilizes the family (1-2).

3. Building blocks and auxiliary results

We gather in this section the raw materials that are used in the proof of Theorem A and record several of their topological properties that are useful for our purposes. The following items are fixed in the sequel:

- a natural number $g \in \mathbb{N}$,
- a group $G \in \{F_g, \pi_1(\Sigma_g)\}$ of rank n,
- an infinite collection $\mathcal{K}$ of knots $K \subset S^3$ with pairwise distinct Alexander polynomials (for simplicity, we assume that the family $\mathcal{K}$ contains the unknot $U \subset S^3$),
- an admissible manifold M as in Definition 2.

3.1. ***$T^2 \times S^2 \# 2g(S^2 \times S^2)$ as a result of surgery to $T^2 \times \Sigma_g$.*** We begin this section with a description of the building block $B_{\pi_1(\Sigma_g)} = T^2 \times \Sigma_g$ (see (b) of Section 2.2), where T^2 is the 2-torus. To ease notation, we will call such building block B_g.

We pick on the surfaces T^2 and Σ_g simple closed curves x, y and a_i, b_i for $i = 1, \ldots, g$ as in Figure 1. In particular, writing $T^2 = S^1 \times S^1$, the loop x is $S^1 \times \{1\}$ and the loop y is $\{1\} \times S^1$. Moreover, we take as a parallel copy of x the loop $x' = S^1 \times \{-1\}$. Notice that the loops $\{a_i, b_i \mid i = 1, \ldots, g\}$ form a symplectic basis for the first homology group $H_1(\Sigma_g; \mathbb{Z})$.

On the other hand, the surfaces $T^2 \times \{pt\}$, $\{p\} \times \Sigma_g$ and the collections of 2-tori

$$\{x \times a_i \mid i = 1, \ldots, g\}, \quad \{x' \times b_i \mid i = 1, \ldots, g\}, \tag{3-1}$$

$$\{y \times a_i \mid i = 1, \ldots, g\} \quad \text{and} \quad \{y \times b_i \mid i = 1, \ldots, g\} \tag{3-2}$$

generate the second homology group $H_2(T^2 \times \Sigma_g; \mathbb{Z}) = \mathbb{Z}^{2+4g}$.

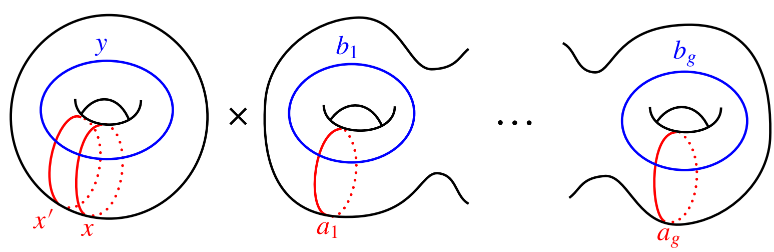

Figure 1. The 4-manifold $T^2 \times \Sigma_g$, the curves x, x', y in T^2 and a_i, b_i for $i = 1, \ldots, g$ in Σ_g.

The second building block is a 4-manifold B_g^* that is obtained from $T^2 \times \Sigma_g$ by doing loop surgery along the framed components of the 1-dimensional submanifold

$$\mathcal{L}_{\pi_1(\Sigma_g)} = \gamma_1 \sqcup \gamma_1' \sqcup \ldots \sqcup \gamma_g \sqcup \gamma_g' \subset T^2 \times \Sigma_g, \tag{3-3}$$

where γ_i is the loop $\{p\} \times a_i$ and γ_i' is $\{p'\} \times b_i$ for $p \neq p'$ and the framing is the one induced by the product structure of $T^2 \times \Sigma_g$.

In particular, we get B_g^* as

$$B_g^* = (T^2 \times \Sigma_g) \setminus \bigsqcup_{i=1}^{g} (\nu(\gamma_i) \sqcup \nu(\gamma_i')) \cup \bigsqcup_{i=1}^{2g} (D^2 \times S^2). \tag{3-4}$$

Notice that doing surgery on the loops (3-3) kills the subgroup

$$\{1\} \times \pi_1(\Sigma_g) < \pi_1(T^2 \times \Sigma_g) = \pi_1(T^2) \times \pi_1(\Sigma_g). \tag{3-5}$$

We also build an auxiliary 4-manifold $\widehat{B}_g$ as the result of a performing a total of $2g$ torus surgeries on the framed 2-tori (3-1). More explicitly, this 4-manifold is defined as

$$\widehat{B}_g = (T^2 \times \Sigma_g) \setminus \Big(\bigsqcup_{i=1}^{g} (\nu(x \times a_i) \sqcup \nu(x' \times b_i)) \cup_\phi \bigsqcup_{i=1}^{2g} (T^2 \times D^2) \Big), \tag{3-6}$$

where ϕ is a gluing diffeomorphism

$$\phi : \bigsqcup_{i=1}^{g} \partial\nu(x \times a_i) \cup \partial\nu(x' \times b_i) \to \bigsqcup_{i=1}^{2g} (T^2 \times \partial D^2)$$

that satisfies

$$(\phi|_{\partial\nu(x \times a_i)})_*^{-1}([\{p\} \times \partial D^2]) = [l_i] \in H_1(\partial\nu(x \times a_i); \mathbb{Z})$$

and a similar equation for the 2-tori of the form $x' \times b_i$ for any $i = 1, \ldots, g$. Here l_i is a Lagrangian push-off of the loop γ_i into $\partial\nu(x \times a_i)$.

This cut-and-paste construction is known as a multiplicity-zero log transform along the loops γ_i and γ_i' [21, p. 83]. We now briefly explain a trick due to Moishezon [30, Lemma 13], which will play a key role in Lemmas 6 and 10. In general, the transformation of $T^2 \times D^2$ under a multiplicity-zero log transform is depicted in Figure 2, left and middle, where we use the dotted circle notation for 1-handles [1; 21, Section 5.4; 36; 37]. Figure 2, left, depicts a copy of $T^2 \times D^2$ inside a smooth 4-manifold X, where all handles that go through the two dotted circles and the 0-framed circle are ignored. Figure 2, middle, depicts the changes in the handlebody of $T^2 \times D^2$ when the multiplicity-zero log transform is performed, yielding the manifold $\widehat{X}$ [3, Figure 6.9; 21, Figure 8.25]; all other handles are ignored. Figure 2, right, depicts the same area after performing a loop surgery on X, we call the result X^*. Notice that X^* can also be obtained by performing a

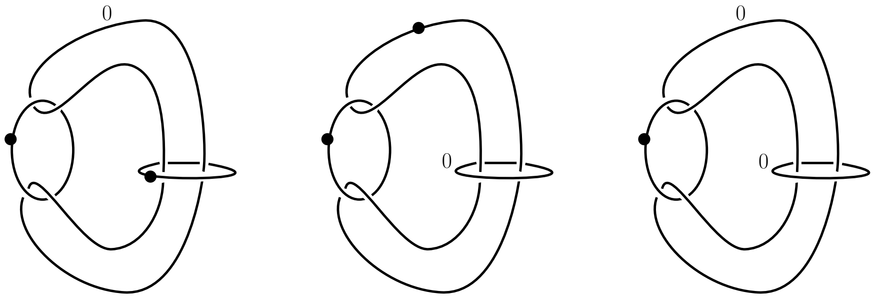

Figure 2. A trick due to Moishezon [30] as explained by Gompf in [19, proof of Lemma 3].

loop surgery on $\widehat{X}$. The trick is to split a loop surgery into a multiplicity-zero log transform followed by a different loop surgery.

The 2-torus $T^2 \times \{p\}$, equipped with the framing induced by the product structure in $T^2 \times \Sigma_g$, is disjoint from the families of 2-tori (3-1), and it defines a framed 2-torus embedded in $\widehat{B}_g$ and one in B_g^*. We denote both of them by T. We now identify the diffeomorphism type of (3-4).

Lemma 6. *Let $\widehat{B}_g$ be the* 4*-manifold in* (3-6). *There is a diffeomorphism*

$$\widehat{B}_g \approx T^2 \times S^2. \tag{3-7}$$

Moreover, *through this diffeomorphism the framed* 2*-torus $T \subset \widehat{B}_g$ is mapped to the framed* 2*-torus $T^2 \times \{p\} \subset T^2 \times S^2$ equipped with its standard product framing.*

Let B_g^* be the 4*-manifold defined in* (3-4). *There is a diffeomorphism*

$$B_g^* \approx (T^2 \times S^2) \# 2g(S^2 \times S^2). \tag{3-8}$$

Moreover, *through this diffeomorphism the framed* 2*-torus $T \subset B_g^*$ is mapped to the canonical* 2*-torus $T^2 \times \{p\} \subset (T^2 \times S^2) \# 2g(S^2 \times S^2)$ equipped with its product framing.*

Proof. We first prove the existence of the diffeomorphism (3-7). Write $T^2 \times \Sigma_g$ as $S^1 \times \partial(D^2 \times \Sigma_g)$, where the S^1 factor corresponds to the loop x. Use this splitting to view each 4-dimensional torus surgery as a (1+3)-dimensional surgery. In particular, the resulting manifold is $\widehat{B}_g = S^1 \times Y$, where the 3-manifold Y is obtained from $S^1 \times \Sigma_g$ by performing 0-Dehn surgeries around the $2g$ loops $\{1\} \times a_i$ and $\{-1\} \times b_i$ for $i = 1, \ldots, g$. This is equivalent to adding $2g$ 4-dimensional 2-handles with 0 framing along the same loops in $D^2 \times \Sigma_g$ to get a 4-manifold Z bounded by Y as in Figure 3. From Figure 4, after some handle slides, handle cancellations and isotopies, we conclude that Z is diffeomorphic to $D^2 \times S^2$. In particular, Y is diffeomorphic to $S^1 \times S^2$ and the existence of the diffeomorphism (3-7) follows. The torus T in $S^1 \times Y$ is $S^1 \times l$, where l is the meridian of the black 0-framed 2-handle in Figure 3. Notice that the handle slides in Figure 4 do not move the

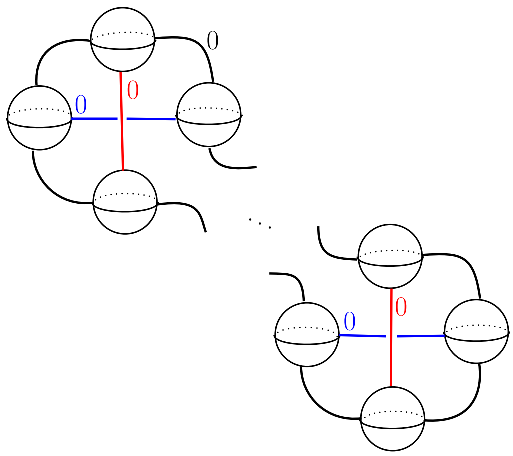

Figure 3. The 1-handles and the black 0-framed 2-handle represent the disk bundle $D^2 \times \Sigma_g$ over the orientable surface of genus g. The red loops are the loops $\{1\} \times a_i$ and the blue ones are the loops $\{-1\} \times b_i$. The manifold Z is obtained from $D^2 \times \Sigma_g$ by adding the 0-framed blue and red 2-handles. The boundary of the 4-manifold Z is the 3-manifold Y.

loop l. Under the diffeomorphism $Y \approx S^1 \times S^2$, the loop l corresponds to the loop $S^1 \times \{p\}$. Therefore, the 2-torus T is mapped to $S^1 \times (S^1 \times \{p\})$ and, after checking the framing of l, we conclude that the first part of the lemma holds.

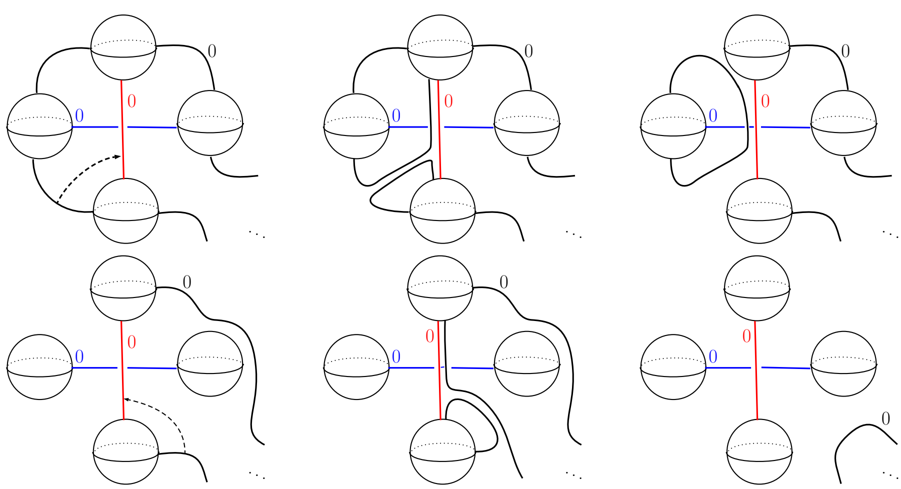

Figure 4. A handlebody of the manifold Z in the proof of Lemma 6. The dotted arrows indicate the 2-handle slide performed. If we apply g times this sequence of moves to the handlebody depicted in Figure 3, we see that the manifold Z has $2g$ canceling pairs of 1- and 2-handles, along with a 0-framed unknotted 2-handle. Thus, Z is diffeomorphic to $D^2 \times S^2$.

To show the existence of the diffeomorphism (3-8), we use the Moishezon trick, described above, to break down the surgery along the loops γ_i, γ_i' by first performing $2g$ multiplicity-zero log transforms along the tori $x \times a_i$ and $x' \times b_i$, and then $2g$ loop surgeries along some framed loops $c_1, \dots, c_{2g}$ which are nullhomotopic since $\pi_1(B_g^*) \cong \pi_1(\widehat{B}_g)$. The manifold B_g^* is the result of $2g$ surgeries along the loops c_i in $\widehat{B}_g = T^2 \times S^2$ and it is therefore diffeomorphic to either $\widehat{B}_g \# 2g(S^2 \times S^2)$ or $\widehat{B}_g \# 2g(S^2 \tilde{\times} S^2)$ [21, Section 5.2]. We conclude that B_g^* is diffeomorphic to $\widehat{B}_g \# 2g(S^2 \times S^2)$ since it admits a spin structure given that the surgeries preserve the spin structure. Another argument to see this, albeit more complicated, is to show that the intersection form of B_g^* is isomorphic to the intersection form of $T^2 \times \Sigma_g$ (which is even). To see this, consider a set of $4g+2$ surfaces representing a basis for $H_2(T^2 \times \Sigma_g; \mathbb{Z})$ and call Q the intersection matrix of this basis. Assume, in addition, that these surfaces are away from the loops (3-3); hence surgery on such loops leaves the surfaces intact. This yields a set of $4g+2$ surfaces in B_g^*, whose intersections give the matrix Q, which is unimodular. This, together with the fact that $b_2(B_g^*) = 4g+2$, implies that these surfaces must be a basis for $H_2(B_g^*; \mathbb{Z})$, on which the intersection matrix is again Q.

We now argue that the framed 2-torus $T \subset B_g^*$ is mapped to the canonical 2-torus inside $(T^2 \times S^2) \# 2g(S^2 \times S^2)$. The diffeomorphism (3-7) maps the framed loops $c_1, \dots, c_{2g} \subset \widehat{B}_g$ to framed loops $\bar{c}_1, \dots, \bar{c}_{2g} \subset T^2 \times S^2$. The loops $\bar{c}_i$ are nullhomotopic and disjoint from $T^2 \times \{p\}$, since the loops c_i are already nullhomotopic and disjoint from T inside $\widehat{B}_g$. Moreover, the loops $\bar{c}_i$ are nullhomotopic also in $T^2 \times (S^2 \setminus \{p\})$, given that the inclusion map $T^2 \times (S^2 \setminus \{p\}) \to T^2 \times S^2$ induces an isomorphism between fundamental groups. Therefore, each simple loop $\bar{c}_i$ bounds a smooth 2-disk disjoint from $T^2 \times \{p\}$. We can write $T^2 \times S^2$ as $(T^2 \times S^2) \# S^4$ and, by a smooth ambient isotopy of $T^2 \times S^2$, we can move one by one all the loops $\bar{c}_i$ into the S^4 factor keeping fixed a neighborhood of the 2-torus $T^2 \times \{p\}$. In this way the diffeomorphism of tuples

$$(\widehat{B}_g, T, c_1, \dots, c_{2g}) \approx \big((T^2 \times S^2) \# S^4, T^2 \times \{p\}, \bar{c}_1, \dots, \bar{c}_{2g}\big)$$

is inducing a diffeomorphism of pairs

$$(B_g^*, T) \approx \big((T^2 \times S^2) \# (S^4)^*, T^2 \times \{p\}\big), \tag{3-9}$$

where $(S^4)^*$ is the 4-manifold obtained as the result of $2g$ surgery operations along the framed loops $\bar{c}_1, \dots, \bar{c}_{2g}$ in S^4. By the previous paragraphs we know that $(S^4)^*$ is diffeomorphic to the connected sum of $2g$ copies of $S^2 \times S^2$, and we can conclude the proof of this lemma by composing the diffeomorphism (3-9) with a last diffeomorphism between $(T^2 \times S^2) \# (S^4)^*$ and $(T^2 \times S^2) \# 2g(S^2 \times S^2)$, which is the identity on a neighborhood of the 2-torus $T^2 \times \{p\}$. □

3.2. $(T^2 \times S^2)\#(S^2 \times S^2)$ ***as a result of surgery to the Kodaira–Thurston manifold.*** We now describe the Kodaira–Thurston manifold N, an essential building block for our constructions in the case $G = F_g$. It is the total space of a T^2-bundle over T^2 [7, Section 2.4; 8, Section 2] which can also be seen as a product $N = S^1 \times Y_{D_a}$ between the 1-sphere S^1 and the mapping torus Y_{D_a} of a Dehn twist $D_a : T^2 \to T^2$ around the loop $a := S^1 \times \{p\} \subset T^2 = S^1 \times S^1$. For convenience, we also define the auxiliary loop $b := \{p'\} \times S^1 \subset T^2$. In particular, the 3-manifold Y_{D_a} is the total space of a T^2-bundle over a circle y and its first homology group is $H_1(Y_{D_a}; \mathbb{Z}) = \mathbb{Z}b \oplus \mathbb{Z}y$. Notice that the loop a is homologically trivial in Y_{D_a}. Moreover, we write the T^2-bundle over T^2 structure of $N = S^1 \times Y_{D_a} = x \times Y_{D_a}$ as

$$a \times b \hookrightarrow N \to x \times y, \tag{3-10}$$

where our notation emphasizes the loops contained in the 2-torus fiber and the 2-torus section that generate the first homology group $H_1(N; \mathbb{Z}) = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}b$. Note that the fiber 2-torus $a \times b$ is geometrically dual to the 2-torus section $T := x \times y$. Moreover, there is another pair of geometrically dual 2-tori inside N, that we denote by $x \times b$ and $y \times a$, respectively, which are disjoint from both the fiber and the section; see [7, Section 2] for a detailed description of these submanifolds.

We equip N with a symplectic structure for which the 2-torus $x \times b \subset N$ is Lagrangian and use the unique Lagrangian framing when we perform surgery along this submanifold; this is explained in detail in [7, Section 2.1; 26]. Let m and l be the Lagrangian push-offs of x and b, respectively, and let μ be the meridian of $x \times b$, i.e., a curve isotopic to $\{p\} \times \partial D^2 \subset \partial\nu(x \times b)$. The triple $\{m, l, \mu\}$ is a basis for the group $H_1(\partial\nu(x \times b); \mathbb{Z})$; see [7, Section 2.1] for further details. We now define the 4-manifold

$$\widehat{N} = (N \setminus \nu(x \times b)) \cup_{\varphi_0} (T^2 \times D^2), \tag{3-11}$$

by using a diffeomorphism

$$\varphi_0 : \partial\nu(x \times b) \to T^2 \times \partial D^2 \tag{3-12}$$

satisfying the condition $\varphi_0(l)_* = [\{p\} \times \partial D^2] \in H_1(T^2 \times \partial D^2; \mathbb{Z})$. In other words, $\widehat{N}$ is obtained from N by applying a multiplicity-zero log transform.

We identify the diffeomorphism type of $\widehat{N}$ and of another useful 4-manifold obtained by the Kodaira–Thurston manifold via loop surgery in the following lemma.

Lemma 7. *Let $\widehat{N}$ be the* 4*-manifold in* (3-11)*. There is a diffeomorphism*

$$\hat{N} \approx T^2 \times S^2. \tag{3-13}$$

Let N^* be the 4*-manifold that is obtained from the Kodaira–Thurston manifold by performing surgery along the based loop b with respect to the framing induced by*

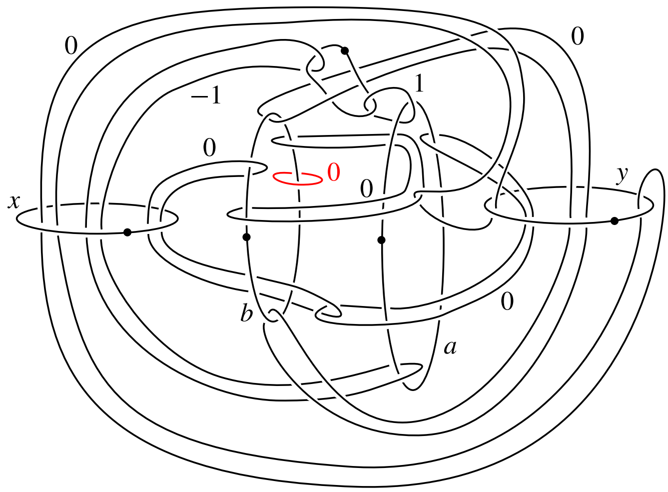

Figure 5. Handlebody depiction of the Kodaira–Thurston manifold N and the 4-manifold N^*.

the Lagrangian framing on $x \times b \subset N$. There is a diffeomorphism

$$N^* \approx (T^2 \times S^2) \# (S^2 \times S^2). \tag{3-14}$$

Moreover, the framed 2-torus $T \subset N$ is disjoint from the loop b, so it defines a framed 2-torus $T \subset N^*$ which is mapped through the diffeomorphism (3-14) *to the canonical 2-torus $T^2 \times \{p\} \subset (T^2 \times S^2) \# (S^2 \times S^2)$ equipped with its product framing.*

We present two proofs of the lemma for the convenience of the reader.

Proof. Given the relevance of the 4-manifold N^* in the proof of Theorem A, we begin with a proof of the existence of the diffeomorphism (3-14). The handlebody of the 4-manifold N^* is given in Figure 5, where its three 3-handles and its 4-handle are not drawn; its construction is explained in the next paragraph. The existence of the diffeomorphism (3-14) is seen by first using the 0-framed circle that links once the second dotted circle from left to right in Figure 5 to unlink all other attaching spheres of the 2-handles from this dotted circle. Cancel this 1-handle and 2-handle pair. Straightforward handle slides unlink the diagram even further, and two more 1- and 2-handle cancellations yield a handlebody of $(T^2 \times S^2) \# (S^2 \times S^2)$.

In order to draw this handlebody of N^*, we build heavily on work of Akbulut to first draw a Kirby diagram of the Kodaira–Thurston manifold. Equip the 4-torus $T^4 = x \times y \times a \times b$ with the product symplectic form. The Kodaira–Thurston manifold N is obtained from the 4-torus T^4 by applying one Luttinger surgery to the Lagrangian 2-torus $x \times a$ along the Lagrangian push-off of a [6]. We use this description of N in order to draw its handlebody by building on the handlebody of the 4-torus drawn in [3, Figure 4.5] and the depiction of Luttinger surgery [3, Section 6.4]. The handlebody of the Kodaira–Thurston manifold is given in Figure 5 without the 0-framed 2-handle that links the second dotted circle from left to right along with four 3-handles and a 4-handle; see [3, Figure 14.35].

The second argument to prove the lemma is as follows. The existence of the diffeomorphism (3-13) follows from a standard argument; see [8, Proof of Lemma 2]. The 4-manifold $\widehat{N}$ is diffeomorphic to the product $S^1 \times M^3$ of the circle with a 3-manifold M^3 that is obtained from the 3-torus $T^3 = y \times a \times b$ by applying a 1-Dehn surgery along a and a 0-Dehn surgery along b. The resulting 3-manifold M^3 has infinite cyclic fundamental group. By a similar Kirby diagram argument as for the diffeomorphism (3-7), we see that M^3 is diffeomorphic to $S^1 \times S^2$ and we conclude that $\widehat{N} = S^1 \times (S^1 \times S^2) = T^2 \times S^2$.

We now prove the existence of the diffeomorphism (3-14) using the Moishezon trick; it implies that the 4-manifold N^* is obtained by doing surgery along a nullhomotopic loop in $\widehat{N}$. Therefore, N^* is diffeomorphic to either $\widehat{N} \# (S^2 \times S^2)$ or $\widehat{N} \# (S^2 \tilde{\times} S^2)$. To check that the diffeomorphism type is the former, it is sufficient to proceed as in Lemma 6. Using (3-13), we conclude that $N^* \approx (T^2 \times S^2) \# (S^2 \times S^2)$. Moreover, this diffeomorphism can be chosen to send the framed 2-torus T to the canonical 2-torus with its product framing (see proof of Lemma 6). □

3.3. ***The 4-manifolds N_g for $g \in \mathbb{N}$.*** We now generalize the construction of the Kodaira–Thurston manifold N in the previous section to produce a 4-manifold N_g that will be the building block B_{F_g} mentioned in (b) of the strategy in Section 2 for any natural number g. The symplectic 4-manifold N_g is defined as

$$N_g := S^1 \times Y_g, \tag{3-15}$$

where Y_g is the mapping torus of

$$\varphi_g := D_{a_1} \circ \cdots \circ D_{a_g} : \Sigma_g \to \Sigma_g$$

and D_{a_i} is a Dehn twist around the loop $a_i \subset \Sigma_g$ defined in Figure 1.

Remark 8. It is possible to describe (3-15) as a symplectic sum

$$N_g = N_1 \#_{T^2} \cdots \#_{T^2} N_1 \tag{3-16}$$

of g copies of the Kodaira–Thurston manifold $N_1 = N$. As we already saw in Section 3.2, N is obtained by taking the product of S^1 with a mapping torus of a Dehn twist of T^2. This is generalized in the construction (3-15) of N_g given that a surface of genus g can itself be deconstructed as a connected sum of g copies of the 2-torus:

$$\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_{g \text{ times}}.$$

The 4-manifold N_g contains a symplectic 2-torus

$$T = x \times y \subset N_1 \setminus \nu(T^2) \subset N_g \tag{3-17}$$

as well as a Lagrangian 2-torus $x \times b$ for each copy of N_1 in (3-16). These 2-tori can also be seen as $x' \times b_i$ where x' is a parallel copy of x and b_i is the inclusion of b in N_g through the i-th copy of N_1 in (3-16). Let γ_i' be the framed loops

$$\gamma_i' = \{p'\} \times b_i \subset x' \times b_i \subset N_g \tag{3-18}$$

with framing induced by the Lagrangian framing on the 2-torus $x' \times b$. We gather together the framed loops (3-18) into a 1-dimensional submanifold

$$\mathcal{L}_{F_g} = \gamma_1' \sqcup \cdots \sqcup \gamma_g'. \tag{3-19}$$

The following homotopical properties of N_g are useful for our purposes.

Lemma 9. *Let $g \in \mathbb{N}$.*

- *The fundamental group of N_g is generated by the inclusion of the fundamental group of the torus $j_*\pi_1(T)$ and the loops $\tilde{\gamma}_i$ obtained by connecting the loops γ_i' to a base point.*
- *There is an isomorphism*

$$\frac{\pi_1(N_g)}{\langle j_*\pi_1(T)\rangle_N} \approx F_g, \tag{3-20}$$

where $\langle j_*\pi_1(T)\rangle_N$ is the normal subgroup generated by $j_*\pi_1(T)$ and the loops $\tilde{\gamma}_i$ correspond to generators.

Proof. We know that $\pi_1(Y_g)$ is given by an HNN extension, which yields

$$\pi_1(N_g) \cong \langle x\rangle \oplus \Big\langle y, a_1, b_1, \ldots, a_g, b_g \;\Big|\; [y, a_i] = [y, b_i]a^{-1} = \prod_{j=1}^{g} [a_j, b_j] = 1 \text{ for } i = 1, \cdots, g\Big\rangle$$

with $j_*(\pi_1(T)) = \langle x, y \mid [x, y]\rangle$. We conclude that

$$\frac{\pi_1(N_g)}{\langle j_*(\pi_1(T))\rangle_N} \cong \langle b_1, b_2, \ldots, b_g\rangle \cong F_g,$$

where $x \in \pi_1(T^2 \times \Sigma_g)$ is the homotopy class of the loop $x \times \{p\} \subset T^2 \times \Sigma_g$ connected to a base point, and similarly for y, a_i and b_i for $i = 1, \ldots, g$. □

Perform g surgeries to N_g along the framed loops (3-18) to produce a closed 4-manifold

$$N_g^* = \Big(N_g \setminus \bigsqcup_{i=1}^{g} \nu(\gamma_i')\Big) \cup \Big(\bigsqcup_{i=1}^{g} (D^2 \times S^2)\Big) \tag{3-21}$$

with fundamental group $\pi_1(N_g^*) = \mathbb{Z} \oplus \mathbb{Z}$. The 2-torus T is disjoint from the seam of the surgeries in (3-21) and it defines an embedded framed 2-torus T in N_g^*.

We now identify the diffeomorphism type of the 4-manifold N_g^*.

Lemma 10. *Let $g \in \mathbb{N}$ and let N_g^* be the manifold defined in* (3-21). *There is a diffeomorphism*

$$N_g^* \approx (T^2 \times S^2) \# g(S^2 \times S^2). \tag{3-22}$$

Moreover, through this diffeomorphism the framed 2-torus $T \subset N_g^*$ is mapped to the canonical 2-torus $T^2 \times \{p\} \subset (T^2 \times S^2) \# g(S^2 \times S^2)$ equipped with its product framing.

Proof. A proof of Lemma 10 by induction on g is obtained using Lemma 7 and the construction of N_g in terms of a generalized fiber sum of g copies of the Kodaira–Thurston manifold N_1 as it is described in Remark 8. The details are left to the reader. □

3.4. *Infinitely many inequivalent smooth structures.* The smooth 4-manifolds of the fourth clause of Theorem A are introduced in the following proposition.

Proposition 11. *Let M be an admissible 4-manifold in the sense of Definition 2. Let B_G be the building block in* (b) *of Section 2 that was defined in Sections 3.1 and 3.3 for the groups $G = \pi_1(\Sigma_g)$ and $G = F_g$, respectively.*

The generalized fiber sum

$$M \#_{T^2} B_G = (M \setminus \nu(T_2)) \cup (B_G \setminus \nu(T)) \tag{3-23}$$

admits infinitely many pairwise inequivalent smooth structures $\{Z_K \mid K \in \mathcal{K}\}$.

Moreover, the fundamental group of (3-23) *is isomorphic to G and it is generated by the homotopy classes of the loops obtained by connecting every component of $\mathcal{L}_G$,* (3-3) *and* (3-19), *to a common base point.*

Proof. For a knot $K \in \mathcal{K}$, perform Fintushel–Stern knot surgery to M along T_1 to obtain the 4-manifold M_K. Notice that we still have an inclusion

$$T_2 \subset (M \setminus \nu(T_1)) \xhookrightarrow{i} M_K,$$

since this cut-and-paste operation can be performed away from the 2-torus T_2. We abuse notation and denote $i(T_2)$ by T_2, and we make sure that the context resolves any confusion. The intersection forms of M and M_K are isomorphic as computed in [15; 22, V.4.1]. Moreover, there is an isomorphism of the second homology groups sending the homology class $[T_2] \in H_2(M; \mathbb{Z})$ onto $[T_2] \in H_2(M_K; \mathbb{Z})$ [22, Chapter V]. By Freedman's theorem [18] this isomorphism can be realized by a homeomorphism $f : M \to M_K$ such that the induced map in homology f_* sends the homology class $[T_2]$ onto itself:

$$\begin{array}{ccc} H_2(M;\mathbb{Z}) & \xrightarrow{f_*} & H_2(M_K;\mathbb{Z}) \\ \Cup & & \Cup \\ [T_2] & \longrightarrow & [T_2] \end{array}$$

By our hypothesis, the 2-tori $f(T_2)$, $T_2 \subset M_K$ have simply connected complement. A result of Sunukjian [39, Theorem 7.1] implies the existence of a homeomorphism of pairs

$$g : \big(M_K, \nu(f(T_2))\big) \to (M_K, \nu(T_2)) \tag{3-24}$$

and we obtain the desired homeomorphism of pairs

$$g \circ f : (M, \nu(T_2)) \to (M_K, \nu(T_2)). \tag{3-25}$$

We proceed to fiber sum both M and M_K with B_G. In particular, we perform the generalized fiber sum by identifying the 2-tori T_2 in M and M_K with the framed 2-torus $T \subset B_G$. In order to do so, we fix a smooth framing τ for the 2-torus T_2 in M and we endow the 2-torus $T_2 \subset M_K$ with a smooth framing τ_K as follows. Denote by $\tilde{\tau}_K$ the restriction of the map $\tau \circ (g \circ f)^{-1}$ to $\nu(T_2)$. Notice that $\tilde{\tau}_K : \nu(T_2) \to T^2 \times D^2$ is not necessarily a smooth map. However, any homeomorphism of a 3-manifold is isotopic to a diffeomorphism [11; 31, Theorem 6.3], and we can assume that $\tilde{\tau}_K$ is indeed smooth near the boundary. We have that the diffeomorphism

$$(\tilde{\tau}_K \circ \tau^{-1})_{|\partial} : T^2 \times S^1 \to T^2 \times S^1$$

induces a map $((\tilde{\tau}_K \circ \tau^{-1})_{|\partial})_* : H_1(T^2 \times S^1) \to H_1(T^2 \times S^1) \cong \mathbb{Z}^3$ that can be represented by a matrix of the form

$$A = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{C}} & 0 \\ & & 0 \\ p & q & 1 \end{pmatrix}$$

with $C \in \mathrm{SL}(2,\mathbb{Z})$. Let $\varphi_C : T^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the self-diffeomorphism of the 2-torus that induces an isomorphism $(\varphi_C)_* : H_1(T^2) \to H_1(T^2) = \mathbb{Z}^2$ given by the matrix C, and let $R_\theta : D^2 \to D^2$ be a rotation of angle θ of the 2-disk. Consider the diffeomorphism given by

$$\begin{aligned} \sigma : T^2 \times D^2 \approx \mathbb{R}^2/\mathbb{Z}^2 \times D^2 &\to T^2 \times D^2 \approx \mathbb{R}^2/\mathbb{Z}^2 \times D^2, \\ ((x_1, x_2), y) &\mapsto (\varphi_C(x_1, x_2), R_{2\pi(px_1+qx_2)}(y)), \end{aligned}$$

for $(x_1, x_2) \in T^2 = S^1 \times S^1$ and $y \in D^2$. The map induced by the diffeomorphism σ satisfies $(\sigma_{|\partial})_* = ((\tilde{\tau}_K \circ \tau^{-1})_{|\partial})_*$. We set $\tau_K = \sigma \circ \tau$ and point out that the diffeomorphisms $\tilde{\tau}_{K|\partial}$ and $\tau_{K|\partial}$ are smoothly isotopic by a result of Hatcher and Wahl [23, Proposition 2.1]. Without loss of generality, we assume that they are equal and pick τ_K to be the framing of $T_2 \subset M_K$. We define the smooth 4-manifolds

$$Z := M \#_{T^2} B_G \quad \text{and} \quad Z_K := M_K \#_{T^2} B_G. \tag{3-26}$$

Given our chosen framings, the homeomorphism (3-25) extends via the identity map to a homeomorphism $F : Z \to Z_K$ since τ_K defines the same gluing map as $\tilde{\tau}_K$. The claim regarding the fundamental group $\pi_1(M \#_{T_2} B_G) = G$ follows from Lemma 9 and the Seifert–van Kampen theorem.

To show that the set $\{Z_K \mid K \subset \mathcal{K}\}$ consists of pairwise nondiffeomorphic 4-manifolds, we look at the Seiberg–Witten invariant of each of its elements as a Laurent polynomial in the group ring $\mathbb{Z}[H_2(Z_K; \mathbb{Z})]$ for every $K \in \mathcal{K}$ [16, Lecture 2]. Recall that the Seiberg–Witten invariants $\{\mathrm{SW}_X(k) \mid k \in H_2(X)\}$ of a 4-manifold X can be combined into an element of $\mathbb{Z}[H_2(X; \mathbb{Z})]$ by associating a formal variable e^α to each homology class $\alpha \in H_2(X; \mathbb{Z})$ and by setting

$$\mathcal{SW}_X = \sum \mathrm{SW}_X(k) \cdot e^k, \tag{3-27}$$

where the sum is taken over all the basic classes $k \in H_2(X; \mathbb{Z})$; see [12, p. 200; 38] for further details. An admissible 4-manifold contains at least one basic class, and hence the invariant (3-27) is nonzero, and the same holds for the symplectic 4-manifold B_G by a result of Taubes [40].

We proceed to set up the use of a gluing formula due to Taubes [41] to compute (3-27) for Z_K; see [3, Section 13.9]. The boundary of $B_G \setminus \nu(T)$ is diffeomorphic to $S^1 \times S^1 \times \partial D^2$ by the diffeomorphism coming from the framing of $T \subset B_G$ and the rim 2-tori

$$S^1 \times \{p\} \times \partial D^2 \quad \text{and} \quad \{p\} \times S^1 \times \partial D^2$$

bound 3-manifolds in $B_G \setminus \nu(T)$. This fact combined with the Mayer–Vietoris sequence for $Z_K = (M_K \setminus \nu(T_2)) \cup (B_G \setminus \nu(T))$ implies that the inclusion induced homomorphism

$$j_* : H_2(B_G \setminus \nu(T); \mathbb{Z}) \to H_2(Z_K; \mathbb{Z})$$

is injective. The Mayer–Vietoris sequence for $M_K = (M_K \setminus \nu(T_2)) \cup \nu(T_2)$ yields that the inclusion induced homomorphisms

$$i_* : H_2(M_K \setminus \nu(T_2); \mathbb{Z}) \to H_2(Z_K; \mathbb{Z})$$

and

$$i'_* : H_2(M_K \setminus \nu(T_2); \mathbb{Z}) \to H_2(M_K; \mathbb{Z})$$

have the same kernel

$$\ker(i'_*) = \ker(i_*) = \langle [\tau_K^{-1}(\{p\} \times S^1 \times \partial D^2)], [\tau_K^{-1}(S^1 \times \{p\} \times \partial D^2)] \rangle.$$

On the other hand, $\mathcal{SW}_{M_K} = \mathcal{SW}_M \cdot \Delta_K(e^{2[T_1]}) \neq 0$ by [15] (see [3, Proposition 13.21; 12, p. 201]). Taubes' gluing formula [41, Theorem 1.1] implies that

$$\mathcal{SW}_{M_K} = i'_*(\mathcal{SW}_{M_K \setminus \nu(T_2)}) \cdot i''_*(\mathcal{SW}_{\nu(T_2)}).$$

It follows that $\mathcal{SW}_{M_K \setminus \nu(T^2)} \notin \ker(i'_*) = \ker(i_*)$. Moreover, $c := i_*([T'_1]) \neq 0$ where T'_1 is a push-off of T_1 inside $M_K \setminus \nu(T_2)$. We apply again Taubes' gluing formula [41, Theorem 1.1] and get

$$\begin{aligned} \mathcal{SW}_{Z_K} &= i_*(\mathcal{SW}_{M_K \setminus \nu(T_2)}) \cdot j_*(\mathcal{SW}_{B_G \setminus \nu(T)}) \\ &= i_*(\Delta_K(e^{2[T'_1]})) \cdot i_*(\mathcal{SW}_{M \setminus \nu(T_2)}) \cdot j_*(\mathcal{SW}_{(B_G \setminus \nu(T)}) \\ &= \Delta_K(e^{2c}) \cdot \mathcal{SW}_Z \\ &\not\cong \mathcal{SW}_Z \end{aligned}$$

for a knot K with nontrivial Alexander polynomial. The $\mathcal{SW}$-invariants, thus, are different [38], and we conclude that Z and Z_K are nondiffeomorphic. Moreover, for two knots K_1 and K_2 with Alexander polynomials $\Delta_{K_1} \neq \Delta_{K_2}$, we have that the 4-manifolds Z_{K_1} and Z_{K_2} have different $\mathcal{SW}$-invariants and, therefore, they are nondiffeomorphic. This concludes the proof of the proposition. $\square$

It is possible to give a more explicit computation of the value (3-27) for Z_K in the proof of Proposition 11 by using other gluing formulas [12, Section 1]. B. D. Park computed

$$\mathcal{SW}_{T^2 \times \Sigma_g} = (t^{-1} - t)^{2g-2} \quad \text{and} \quad \mathcal{SW}_{T^2 \times \Sigma_g^0} = (t^{-1} - t)^{2g-1}$$

for $t = [T^2 \times \{p\}]$ and $\Sigma_g^0 = \Sigma_g \setminus D^2$ in [32, Corollary 19]. Since the symplectic 4-manifold N_g is obtained by applying Luttinger surgeries to $T^2 \times \Sigma_g$, the invariants $\mathcal{SW}_{N_g}$ and $\mathcal{SW}_{N_g \setminus \nu(T)}$ can be computed using Park's work.

Remark 12. The diffeomorphism type of the generalized fiber sum $M \#_{T_2} (T^2 \times S^2)$ is the same regardless of the choice of framing τ or τ_K for T_2 and the product framing for $T^2 \times \{p\} \subset T^2 \times \Sigma_g$.

3.5. ***The ambient 4-manifolds of Theorem A.*** We now identify the diffeomorphism types of the ambient 4-manifolds of Theorem A. Since the codimension-three submanifold $\mathcal{L}_G \subset B_G$ is disjoint from T, it is embedded in the 4-manifold Z_K constructed in (3-26) and each of its components is framed. We define Z_K^* to be the 4-manifold obtained by doing loop surgery along each component of $\mathcal{L}_G \subset Z_K$. Moreover, we can define Γ to be the 2-link in Z_K^* given by the belt 2-spheres of the surgeries.

Proposition 13. *For every knot $K \in \mathcal{K}$, there is a diffeomorphism*

$$Z_K^* \approx M \# n(S^2 \times S^2). \tag{3-28}$$

Proof. We fix a knot $K \in \mathcal{K}$ and take a closer look at the assemblage of Z_K^* to prove the existence of the diffeomorphism (3-28). Recall that the manifold Z_K is defined as the generalized fiber sum of M_K and B_G along the 2-tori $T_2 \subset M_K$ with framing τ_K and $T \subset B_G$ with the Lagrangian framing. Since $\mathcal{L}_G \subset B_G$ is disjoint from T, the 4-manifold Z_K^* is the generalized fiber sum of M_K and B_G^* along $T_2 \subset M_K$ and $T \subset B_G^*$. Lemmas 6 and 10 say that there is a diffeomorphism

$$B_G^* \approx (T^2 \times S^2) \# n(S^2 \times S^2)$$

that sends the framed 2-torus T to the canonical 2-torus

$$T^2 \times \{p\} \subset (T^2 \times S^2) \# n(S^2 \times S^2). \tag{3-29}$$

Therefore, the 4-manifold Z_K^* is diffeomorphic to the generalized fiber sum of M_K and $(T^2 \times S^2) \# n(S^2 \times S^2)$ along the 2-tori $T_2 \subset M_K$ and (3-29). To sum up, using Remark 12, we have

$$Z_K^* = M_K \#_{T^2} B_G^* \approx M_K \#_{T^2} (T^2 \times S^2) \# n(S^2 \times S^2) \approx M_K \# n(S^2 \times S^2).$$

A result of Akbulut [2], Auckly [4] and Baykur [9] allows us to conclude that Z_K^* is diffeomorphic to the connected sum $M \# n(S^2 \times S^2)$ for any $K \subset S^3$. □

3.6. *The 2-links of Theorem A and topological isotopy.* In this section, we construct the 2-links of Theorem A and show that they are pairwise topologically isotopic and componentwise topologically unknotted.

Proposition 14. *For $G \in \{F_g, \pi_1(\Sigma_g)\}$ and $n = \mathrm{rk}(G)$, there is an infinite collection of n-component 2-links*

$$\{\Gamma_K \mid K \in \mathcal{K}\} \tag{3-30}$$

smoothly embedded in $M \# n(S^2 \times S^2)$ that are pairwise topologically isotopic and componentwise topologically unknotted.

The 4-manifold that is obtained from $M \# n(S^2 \times S^2)$ by doing surgeries along every component of (3-30) is diffeomorphic to the 4-manifold Z_K defined in (3-26) for any knot $K \in \mathcal{K}$.

Proof. Let $U \subset S^3$ be the unknot. For any knot $K \in \mathcal{K}$, there is a homeomorphism of pairs

$$f_K : (Z_K, \mathcal{L}_G) \to (Z_U, \mathcal{L}_G) \tag{3-31}$$

that restricts to the identity in a neighborhood of $\mathcal{L}_G$ by the proof of Proposition 11. The 4-manifolds Z_U^* and Z_K^* are the result of surgeries along the components of the submanifold $\mathcal{L}_G \subset B_G$ and we define $\Gamma \subset B_G^* \setminus \nu(T)$ to be the 2-link given by the disjoint union of the belt 2-spheres of such surgeries. In particular, there is an embedding of Γ inside Z_K^* for any knot $K \in \mathcal{K}$, and the number of components of Γ and of $\mathcal{L}_G$ are both equal to n. Moreover, the homeomorphism (3-31) induces a homeomorphism of pairs

$$g_K : (Z_K^*, \Gamma) \to (Z_U^*, \Gamma).$$

Proposition 13 allows us to fix a diffeomorphism $\phi_U : Z_U^* \to M \# n(S^2 \times S^2)$ and to define Γ_U as $\phi_U(\Gamma)$.

Notice that we also have a diffeomorphism

$$\phi_K : Z_K^* \to M \# n(S^2 \times S^2) \tag{3-32}$$

such that $(\phi_K)_* = (\phi_U \circ g_K)_*$ in homology for every $K \in \mathcal{K}$. Indeed, any diffeomorphism taken from Proposition 13 can be adjusted at the level of homology by composing it via a self-diffeomorphism of $M \# n(S^2 \times S^2)$ by a result of Wall [43, Theorem 2] since the intersection form of M is indefinite; see Lemma 4.

We define the 2-link (3-30) to be the image of Γ_K under the diffeomorphism (3-32), i.e.,

$$\Gamma_K := \phi_K(\Gamma). \tag{3-33}$$

We now argue that the 2-links Γ_K and Γ_U are topologically isotopic for any knot. The map

$$h_K = \phi_U \circ g_K \circ \phi_K^{-1} : (M \# n(S^2 \times S^2), \Gamma_K) \to (M \# n(S^2 \times S^2), \Gamma_U)$$

is a homeomorphism of pairs that induces the identity map in homology. Work of Perron [33] and Quinn [34] guarantees that h_K is isotopic to the identity map and, hence, we conclude that the 2-links Γ_K and Γ_U are topologically isotopic for any $K \in \mathcal{K}$. Each component of the 2-link (3-30) is topologically unknotted by either [42, Theorem B] or [39, Theorem 7.2]. This concludes the proof of the first clause of Proposition 14.

The second clause is straightforward from the construction of Z_K and $Z_K{}^*$. □

3.7. ***Smoothly inequivalent 2-links.*** In this section, we distinguish the smooth embeddings of our 2-links. The key idea is to tell them apart by looking at the smooth structures of their complements and use their Seiberg–Witten invariants indirectly. More precisely, we undo the surgeries and reconstruct Z_K from $M \# n(S^2 \times S^2)$ as

$$Z_K \approx \big(M \# n(S^2 \times S^2) \setminus (\nu(\Gamma_K)) \cup (\nu(\mathcal{L}_G))\big) \tag{3-34}$$

and use inequivalent smooth structures constructed in Proposition 11 in order to distinguish our 2-links.

Proposition 15. *Let Γ and Γ' be a pair of 2-links that are smoothly embedded in a smooth 4-manifold Z^*. Let Z and Z' be the 4-manifolds that are obtained from Z^* by doing surgery along every component of Γ and Γ', respectively. If there is no diffeomorphism $Z \to Z'$, then the 2-links Γ and Γ' are smoothly inequivalent.*

In particular, the 2-links (3-30) *constructed in Proposition 14 are pairwise smoothly inequivalent.*

The proof of Proposition 15 is immediate and, hence, omitted. Any distinct pair of 2-links in the collection (3-30) satisfies the assumptions of this proposition by Proposition 11.

3.8. ***Symmetric and Brunnianly exotic 2-links.*** In this section, we study some properties of the 2-links of Theorem A. We start with the following result regarding their symmetry.

Proposition 16. *Suppose $G = F_g$. Every 2-link $\Gamma_K \subset M \# g(S^2 \times S^2)$ belonging to the infinite collection* (3-30) *is smoothly symmetric.*

Proof. For any permutation σ of g elements, there is a self-diffeomorphism

$$f : N_g \to N_g \tag{3-35}$$

that is the identity on a neighborhood of the 2-torus (3-17) and such that it maps the framed loops (3-19) to $f(\gamma_i') = \gamma_{\sigma(i)}'$ for every $i = 1, \dots, g$. The map (3-35) yields a diffeomorphism of pairs

$$(Z_K, \gamma_1', \dots, \gamma_g') \to (Z_K, \gamma_{\sigma(1)}', \dots, \gamma_{\sigma(g)}'),$$

which allows us to define a diffeomorphism of pairs

$$(Z_K^*, S_1, \dots, S_g) \to (Z_K^*, S_{\sigma(1)}, \dots, S_{\sigma(g)}).$$

In particular, the 2-link $\Gamma := S_1 \sqcup \cdots \sqcup S_g \subset Z_K^*$ is smoothly symmetric and so is the 2-link (3-33) smoothly embedded in $M \# g(S^2 \times S^2)$, given that there is a diffeomorphism of pairs

$$\phi_K : (Z_K^*, \Gamma) \to (M \# g(S^2 \times S^2), \Gamma_K);$$

see (3-32). □

We will employ the following two results to address Brunnianity.

Lemma 17 (Mandelbaum [27], Gompf [19, Lemma 4]). *Let X and B be two oriented 4-manifolds (possibly with boundary) and let $T_X \subset X$ and $\alpha_B \times \beta_B = T_B \subset B$ be two smoothly embedded framed 2-tori. Suppose that X, B and $X \setminus \nu(T_X)$ are simply connected and that either X is spin, or $X \setminus \nu(T_X)$ is nonspin. Consider the generalized fiber sum*

$$F = X \#_{T^2} B$$

of X and B along T_X and T_B. Then $F \# S^2 \times S^2$ is diffeomorphic to $X \# B^*$, where B^* is the manifold obtained from B by doing surgery along the push-offs of the loops α_B and β_B. Moreover, we can assume the chosen diffeomorphism to be the identity on $\partial F = \partial X \sqcup \partial B$ if X or B have nonempty boundary.

Lemma 18. *Let M be an admissible 4-manifold. If M is nonspin, then, up to swapping the roles of the 2-tori T_1 and T_2, $M \setminus \nu(T_2)$ is nonspin.*

Proof. Since $M \setminus \nu(T_2)$ is simply connected, it is enough to show that there exists an element $\sigma \in H_2(M \setminus \nu(T_2); \mathbb{Z})$ with odd self-intersection. The assumption $\pi_1\big(M \setminus (\nu(T_1) \sqcup \nu(T_2))\big) = \{1\}$ implies the existence of immersed 2-spheres $S_1, S_2 \subset M$ such that $S_i \cdot T_i = 1$ for $i = 1, 2$ and $S_1 \cap T_2 = S_2 \cap T_1 = \varnothing$. If either of them has odd self-intersection, then the complement of one of the 2-tori contains a surface of odd self-intersection and we are done. Suppose that $S_2 \cdot S_2$ is even. Since M is nonspin, there exists an element $\alpha \in H_2(M; \mathbb{Z})$ with odd self-intersection. Define $\sigma := \alpha - (\alpha \cdot [T_2])[S_2]$ and notice that we have

$$\sigma \cdot \sigma = \alpha \cdot \alpha - 2(\alpha \cdot [T_2])([S_2] \cdot \alpha) + (\alpha \cdot [T_2])^2 [S_2]^2 \equiv \alpha \cdot \alpha \equiv 1 \bmod 2$$

and $\sigma \cdot [T_2] = \alpha \cdot [T_2] - \alpha \cdot [T_2] = 0$. Let $\Sigma \subset M$ be an embedded surface representing the homology class $\sigma \in H_2(M; \mathbb{Z})$. The algebraic intersection of Σ and T_2 is zero. We eliminate the geometric intersection by tubing Σ in oppositely signed intersection points, obtaining a new surface Σ' in $M \setminus \nu(T_2)$ which has odd self-intersection. $\square$

We are now ready to prove the main result of this section.

Proposition 19. *For $G = F_g$, consider the family* (3-30). *There exists an infinite subset $\mathcal{K}' \subset \mathcal{K}$ of knots $K \subset S^3$ parametrizing a subfamily*

$$\{\Gamma_K \mid K \in \mathcal{K}'\} \tag{3-36}$$

of pairwise Brunnianly exotic 2-links.

Proof. Let $\Gamma_K = S_{K,1} \sqcup S_{K,2} \sqcup \cdots \sqcup S_{K,g}$ be a g-component 2-link from the family (3-30). Thanks to Proposition 16, in order to prove the theorem it is enough to show that removing the first component from each Γ_K yields a family of 2-links that is no longer exotic. By construction, we have the following diffeomorphism of pairs:

$$(M \# n(S^2 \times S^2), \Gamma_K) \approx (M_K \#_{T^2_K} (N^* \#_{T^2} \cdots \#_{T^2} N^*), \Gamma). \tag{3-37}$$

Here the notation $\#_{T^2_K}$ is used to point out the fact that the framing of that generalized fiber sum depends on the knot K, while $\Gamma = S_1 \sqcup \cdots \sqcup S_g$ is the 2-link made up by the belt spheres of the surgery (3-21) on N_g. Such diffeomorphism can be chosen to send the component $S_{K,1}$ to S_1 for each knot K. In particular, each one of the components S_i is contained in a $N^* \approx (T^2 \times S^2) \# (S^2 \times S^2)$ block, which is therefore disjoint from the $(n-1)$-component 2-link $\Gamma \setminus S_i$. In the following, we will use the copy of $S^2 \times S^2$ that intersects S_1 to stabilize $\Gamma \setminus S_1$, which is contained in the remaining $N^* \#_{T^2} \cdots \#_{T^2} N^* \approx N^*_{g-1}$. In particular, we have the following diffeomorphism of pairs:

$$(M \# g(S^2 \times S^2), \Gamma \setminus S_{K,1}) \approx ((M_K \#_{T^2_K} N^*) \#_{T^2} N^*_{g-1}, \Gamma \setminus S_1).$$

To conclude it is enough to show that there exists an infinite subfamily $\mathcal{K}' \subseteq \mathcal{K}$ such that for any two knots $K, K' \in \mathcal{K}'$ there exists a diffeomorphism between $M_K \#_{T^2_K} N^*$ and $M_{K'} \#_{T^2_{K'}} N^*$ relative to $\nu(T')$, where T' is a parallel push-off of T in N^*. In particular, we will see that there are at most four diffeomorphism types of pairs in the family $\{(M_K \#_{T^2_K} N^*, T') \mid K \subset \mathcal{K}\}$ where T' is framed. By the pigeonhole principle, we can obtain $\mathcal{K}'$.

We have, by (3-13), the identification

$$M_K \#_{T^2_K} N^* \approx (M_K \# (S^2 \times S^2)) \#_{T^2_K} (T^2 \times S^2),$$

which sends the framed 2-torus T' to the canonically framed 2-torus $T^2 \times \{p\} \subset T^2 \times S^2$.

If M is nonspin, then M_K is nonspin and, by Lemma 18, $M \setminus \nu(T_2)$ is also nonspin, and they are all simply connected. Since M_K is the generalized fiber sum between M and $S^1 \times Y_K$, where Y_K is the result of a Dehn surgery on S^3 along K, we can apply Lemma 17 to obtain a diffeomorphism ψ relative to the boundary between $(M_K \setminus \nu(T_2)) \# (S^2 \times S^2)$ and the manifold P obtained by performing two loop surgeries on $(M \setminus \nu(T_2)) \# (S^1 \times Y_K)$. It can be shown that P is diffeomorphic to $(M \setminus \nu(T_2)) \# (S^2 \times S^2)$ (modulo changing the initial framing of T_1). We extend ψ by the identity on $(T^2 \times S^2) \setminus \nu(T^2 \times \{p\})$ and obtain a diffeomorphism

$$\tilde{\psi} : (M_K \# (S^2 \times S^2)) \#_{T^2_K} (T^2 \times S^2) \to (M \# (S^2 \times S^2)) \#_{T^2_K} (T^2 \times S^2).$$

We can apply Lemma 17 a second time as follows. Consider the generalized fiber sum $\mathcal{F}_K$ between M and $(T^2 \times S^2) \setminus \nu(T^2 \times \{p\})$ along T_2, with framing depending on K, and $T^2 \times \{q\}$. Lemma 17 implies that $\mathcal{F}_K \# (S^2 \times S^2)$ is diffeomorphic to $M \# \big((T^2 \times S^2) \setminus \nu(T^2 \times \{p\})\big)^*$ by a diffeomorphism φ that keeps the boundary $\partial\nu(T^2 \times \{p\})$ fixed. Here we use the notation X^* to denote the result of two loop surgeries on a 4-manifold X along the loops described by Lemma 17. Hence, we can extend φ via the identity to a map

$$\tilde{\varphi} : (M \# (S^2 \times S^2)) \#_{T^2_K} (T^2 \times S^2) \to M \# (T^2 \times S^2)^*.$$

We do not need to determine the result of the loop surgeries $(T^2 \times S^2)^*$. Since the loops do not depend on the knot K, but their framings might, there are four possibilities at most for the framed pair $((T^2 \times S^2)^*, \tilde{\varphi}(T^2 \times \{p\})$. □

Remark 20. By taking the connected sum of the ambient manifold with a copy of $S^2 \times S^2$, the same proof of Proposition 19 shows that the 2-links of the family (3-36) (instead of their sublinks) become smoothly equivalent in the ambient 4-manifold $(M \# g(S^2 \times S^2)) \# (S^2 \times S^2)$.

Remark 21. It is possible to prove that $\mathcal{K}'$ is equal to $\mathcal{K}$. In particular, the use of the pigeonhole principle in the proof of Proposition 19 can be avoided by taking a closer look at the framing of the loops and at the framing τ_K for the 2-torus T_2 inside M_K.

Remark 22. From a pair of exotic 2-spheres in a 4-manifold, one easily obtains exotic 2-links with a prescribed number of components and the free group with g generators as 2-link group by adding unknotted 2-spheres. The results in this section prove that this is not the case for the 2-links constructed in Theorem A with 2-link group F_g.

In the case of $G = \pi_1(\Sigma_g)$ and using similar arguments, one can show a weaker property after pairing up the 2-link components, in which disregarding a pair of components from the n-component 2-link yields smoothly equivalent $(n-2)$-component 2-links.

4. Proofs

4.1. ***Proof of Theorem A.*** We collect the results of previous sections into a proof of Theorem A. The infinite set (1-2) of 2-links $\{\Gamma_K \mid K \in \mathcal{K}\}$ smoothly embedded in $M \# n(S^2 \times S^2)$ was constructed in Proposition 14. The 2-link group of Γ_K is isomorphic to the fundamental group of the manifold obtained from $M \# n(S^2 \times S^2)$ by doing surgery on every component of Γ_K. This manifold is Z_K by Proposition 14 and by Proposition 11 we know that its fundamental group $\pi_1(Z_K)$ is isomorphic to G. This settles the first clause of the theorem. The second clause of the theorem is that the collection of 2-links $\{\Gamma_K \mid K \in \mathcal{K}\}$ is an exotic family. By Proposition 14, elements of this family are pairwise topologically isotopic and by Proposition 15 they are pairwise smoothly inequivalent. The 2-links are componentwise topologically unknotted by Proposition 14 and this establishes the third clause. The infinite set of pairwise nondiffeomorphic closed 4-manifolds of the fourth clause has been described in detail in Section 3.4. □

4.2. ***Proof of Theorem B.*** The 2-links of Theorem A with 2-link group F_g are symmetric by Proposition 16. The existence of the infinite subfamily of pairwise Brunnianly exotic 2-links that stabilize after one connected sum with $S^2 \times S^2$ follows from Proposition 19 and Remark 20. □

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References

[1] S. Akbulut, "On 2-dimensional homology classes of 4-manifolds", *Math. Proc. Cambridge Philos. Soc.* **82**:1 (1977), 99–106. MR Zbl

[2] S. Akbulut, "Variations on Fintushel–Stern knot surgery on 4-manifolds", *Turkish J. Math.* **26**:1 (2002), 81–92. MR Zbl

[3] S. Akbulut, *4-manifolds*, Oxford Graduate Texts in Mathematics **25**, Oxford Univ. Press, 2016. MR Zbl

[4] D. Auckly, "Families of four-dimensional manifolds that become mutually diffeomorphic after one stabilization", *Topology Appl.* **127**:1-2 (2003), 277–298. MR Zbl

[5] D. Auckly, H. J. Kim, P. Melvin, and D. Ruberman, "Stable isotopy in four dimensions", *J. Lond. Math. Soc.* (2) **91**:2 (2015), 439–463. MR Zbl

[6] D. Auroux, S. K. Donaldson, and L. Katzarkov, "Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves", *Math. Ann.* **326**:1 (2003), 185–203. MR Zbl

[7] S. Baldridge and P. Kirk, "Symplectic 4-manifolds with arbitrary fundamental group near the Bogomolov–Miyaoka–Yau line", *J. Symplectic Geom.* **4**:1 (2006), 63–70. MR Zbl

[8] S. Baldridge and P. Kirk, "Constructions of small symplectic 4-manifolds using Luttinger surgery", *J. Differential Geom.* **82**:2 (2009), 317–361. MR Zbl

[9] R. İ. Baykur, "Dissolving knot surgered 4-manifolds by classical cobordism arguments", *J. Knot Theory Ramifications* **27**:5 (2018), art. id. 1871001. MR Zbl

[10] H. Brunn, "Über verkettung", *Bayer. Akad. Wiss. Math. Naturwiss. Abt.* **22** (1892), 77–99. Zbl

[11] A. V. Chernavskii, "Local contractibility of the group of homeomorphisms of a manifold", *Mat. Sb.* (*N.S.*) **79(121)** (1969), 307–356. In Russian; translated in *Math. USSR-Sb.* **8**:3 (1969), 287–333. MR Zbl

[12] R. Fintushel, "Knot surgery revisited", pp. 195–224 in *Floer homology*, *gauge theory*, *and low-dimensional topology*, edited by D. A. Ellwood et al., Clay Math. Proc. **5**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[13] R. Fintushel and R. J. Stern, "A fake 4-manifold with $\pi_1 = \mathbb{Z}$ and $b^+ = 4$", *Turkish J. Math.* **18**:1 (1994), 1–6. MR Zbl

[14] R. Fintushel and R. J. Stern, "Surfaces in 4-manifolds", *Math. Res. Lett.* **4**:6 (1997), 907–914. MR Zbl

[15] R. Fintushel and R. J. Stern, "Knots, links, and 4-manifolds", *Invent. Math.* **134**:2 (1998), 363–400. MR Zbl

[16] R. Fintushel and R. J. Stern, "Six lectures on four 4-manifolds", pp. 265–315 in *Low dimensional topology*, edited by T. S. Mrowka and P. S. Ozsváth, IAS/Park City Math. Ser. **15**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[17] R. Fintushel and R. Stern, "Surgery on nullhomologous tori", pp. 61–81 in *Proceedings of the Freedman Fest*, edited by R. Kirby et al., Geom. Topol. Monogr. **18**, Geom. Topol. Publ., Coventry, UK, 2012. MR Zbl

[18] M. H. Freedman, "The topology of four-dimensional manifolds", *J. Differential Geometry* **17**:3 (1982), 357–453. MR Zbl

[19] R. E. Gompf, "Sums of elliptic surfaces", *J. Differential Geom.* **34**:1 (1991), 93–114. MR Zbl

[20] R. E. Gompf, "A new construction of symplectic manifolds", *Ann. of Math.* (2) **142**:3 (1995), 527–595. MR Zbl

[21] R. E. Gompf and A. I. Stipsicz, 4-*manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[22] M. J. D. Hamilton, *On symplectic 4-manifolds and contact 5-manifolds*, Ph.D. thesis, Ludwig-Maximilians-Universität München, 2008. Zbl

[23] A. Hatcher and N. Wahl, "Stabilization for mapping class groups of 3-manifolds", *Duke Math. J.* **155**:2 (2010), 205–269. MR Zbl

[24] K. Hayden, "Exotically knotted disks and complex curves", preprint, 2021. arXiv 2003.13681v2

[25] K. Hayden, A. Kjuchukova, S. Krishna, M. Miller, M. Powell, and N. Sunukjian, "Brunnian exotic surface links in the 4-ball", *Michigan Math. J.* (online publication June 2024). DOI: 10.1307/mmj/20226312.

[26] C.-I. Ho and T.-J. Li, "Luttinger surgery and Kodaira dimension", *Asian J. Math.* **16**:2 (2012), 299–318. MR Zbl

[27] R. Mandelbaum, "Decomposing analytic surfaces", pp. 147–217 in *Geometric topology* (Athens, GA, 1977)), edited by J. C. Cantrell, Academic Press, New York, 1979. MR Zbl

[28] J. D. McCarthy and J. G. Wolfson, "Symplectic normal connect sum", *Topology* **33**:4 (1994), 729–764. MR Zbl

[29] J. Milnor, "A procedure for killing homotopy groups of differentiable manifolds", pp. 39–55 in *Proc. Sympos. Pure Math.*, *III*, Amer. Math. Soc., Providence, RI, 1961. MR Zbl

[30] B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Mathematics **603**, Springer, 1977. MR Zbl

[31] J. Munkres, "Obstructions to the smoothing of piecewise-differentiable homeomorphisms", *Ann. of Math.* (2) **72** (1960), 521–554. MR Zbl

[32] B. D. Park, "A gluing formula for the Seiberg–Witten invariant along T^3", *Michigan Math. J.* **50**:3 (2002), 593–611. MR Zbl

[33] B. Perron, "Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique", *Topology* **25**:4 (1986), 381–397. MR Zbl

[34] F. Quinn, "Isotopy of 4-manifolds", *J. Differential Geom.* **24**:3 (1986), 343–372. MR Zbl

[35] A. Ray and D. Ruberman, "Four-dimensional analogues of Dehn's lemma", *J. Lond. Math. Soc.* (2) **96**:1 (2017), 111–132. MR Zbl

[36] E. César de Sá, *Automorphisms of 3-manifolds and representations of 4-manifolds*, Ph.D. thesis, University of Warwick, 1977, available at http://webcat.warwick.ac.uk/record=b1748642~S15.

[37] E. César de Sá, "A link calculus for 4-manifolds", pp. 16–30 in *Topology of low-dimensional manifolds* (Chelwood Gate, 1977), edited by R. A. Fenn, Lecture Notes in Math. **722**, Springer, 1979. MR Zbl

[38] N. S. Sunukjian, "A note on knot surgery", *J. Knot Theory Ramifications* **24**:9 (2015), art. id. 1520003. MR Zbl

[39] N. S. Sunukjian, "Surfaces in 4-manifolds: concordance, isotopy, and surgery", *Int. Math. Res. Not.* **2015**:17 (2015), 7950–7978. MR Zbl

[40] C. H. Taubes, "The Seiberg-Witten invariants and symplectic forms", *Math. Res. Lett.* **1**:6 (1994), 809–822. MR Zbl

[41] C. H. Taubes, "The Seiberg–Witten invariants and 4-manifolds with essential tori", *Geom. Topol.* **5** (2001), 441–519. MR Zbl

[42] R. Torres, "Smoothly knotted and topologically unknotted nullhomologous surfaces in 4-manifolds", *Ann. Inst. Fourier* (online publication February 2025). DOI: 10.5802/aif.3685.

[43] C. T. C. Wall, “Diffeomorphisms of 4-manifolds”, *J. London Math. Soc.* **39** (1964), 131–140. MR Zbl

[44] A. H. Wallace, “Modifications and cobounding manifolds”, *Canadian J. Math.* **12** (1960), 503–528. MR

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