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# CLASSIFICATION OF MÖBIUS HOMOGENEOUS SUPERCONFORMAL SURFACES IN $\mathbb{S}^5$

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**Superconformal surfaces in space forms are characterized by the property that the curvature ellipse is a circle at every point, or equivalently, the normal vector bundle valued Hopf differential is isotropic. This paper focuses on the study of these surfaces within the framework of Möbius geometry. We construct new examples, with a particular emphasis on the case of codimension 3. We also obtain a complete classification of Möbius homogeneous superconformal surfaces in $\mathbb{S}^5$.**

## 1. Introduction

The investigation of superconformal surfaces in space forms is a classical and enduring topic in differential geometry. This can be traced back to over a century ago, such as in the work of Kommerell [1905] and the work of Wilson and Moore [1916], who showed that for a surface in the Euclidean space $\mathbb{R}^n$ with $n \geq 4$, at every point, a configuration consisting of the mean curvature vector $\vec{H}$ and an ellipse (called the curvature ellipse) lying in the normal space can determine the second-order scalar invariants. Here the *curvature ellipse* is defined to be the image of the unit tangent circle under the second fundamental form, II, i.e.,

$$\{\mathrm{II}(X, X) \mid X \text{ belongs to the unit tangent circle}\}.$$

A surface in space forms of codimension no less than 2 is called *superconformal* if the curvature ellipse is a circle everywhere on this surface. It can also be characterized by the isotropic property of the normal bundle valued Hopf differential. In the case of codimension 2, such surfaces which are also minimal (called superminimal surfaces in the literature) have been studied extensively, such as in the works of Borůvka [1928], Wong [1946], Bryant [1982a] and Chern and Wolfson [1983].

From the time of the works of Wintgen [1979] and Guadalupe and Rodrigues [1983], general superconformal surfaces have attracted the attention of geometers. For surfaces in the space forms $N^n(c)$, a lower bound of the Willmore functional

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was found in terms of the topology of the first normal bundle, and this bound can be achieved if and only if the surface is superconformal. To prove this global result, Wintgen, Guadalupe, and Rodrigues established the following pointwise inequality:

$$K_N \leq |\vec{H}|^2 - K + c, \tag{1}$$

where $K_N$ is the normal curvature, $K$ is the Gaussian curvature, and $c$ is the curvature of ambient space. Moreover, it was shown that equality holds in (1) if and only if the curvature ellipse is a circle. We point out that the above pointwise results had also appeared in singularity theory, due to the work of Little [1969].

For superconformal surfaces in space forms, perhaps the most notable property is that they are conformal (Möbius) invariant. Therefore, the ambient space can be chosen to be the sphere $\mathbb{S}^n$. In the case of codimension 2, the condition of superconformality is rather strong within the framework of Möbius geometry. It was shown by Burstall, Ferus, Leschke, Pedit, and Pinkall [Burstall et al. 2002] that superconformal surfaces in $\mathbb{S}^4$ arise from projections of complex curves in the twistor space $\mathbb{C}P^3$ of $\mathbb{S}^4$. Moreover, based on the work of Rouxel [1992] about the mean curvature sphere congruence of superconformal surfaces, a parametric representation of all superconformal surfaces in $\mathbb{R}^4$ is obtained by Dajczer and Tojeiro [2009] in terms of minimal surfaces.

Unlike the extensively studied case of codimension 2, superconformal surfaces of higher codimension have not been thoroughly investigated. Simple examples of such surfaces are ones Möbius equivalent to minimal and superconformal surfaces in space-forms, including 1-isotropic minimal surfaces in $\mathbb{R}^n$; minimal 2-spheres in $\mathbb{S}^n$, studied in [Barbosa 1975; Calabi 1967]; holomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$ [Bryant 1982b]; and harmonic superconformal maps from $\mathbb{H}^2$ in $\mathbb{H}^n$, studied by Hulett [2002]. The second class of examples consists of $S$-Willmore surfaces, introduced by Ejiri [1988]; see also [Ma 2005] and [Dajczer and Vlachos 2015b], where the duality of these superconformal surfaces is studied. Dajczer and Vlachos [2015a] constructed new examples by considering the pedal surface to a 2-isotropic surface in $\mathbb{R}^n$ with $n \geq 6$. Note that these new examples are of codimension no less than 4 in general. In this paper, we will construct new superconformal surfaces of codimension 3 in Section 3.

Due to the work of De Smet, Dillen, Verstraelen, and Vrancken [De Smet et al. 1999], Ge and Tang [2008] and Lu [2011], the pointwise inequality (1) has been generalized to higher-dimensional submanifolds in space forms. This is known as the DDVV inequality. Submanifolds attaining equality in the DDVV inequality everywhere are called Wintgen ideal submanifolds. Over the past fifteen years, there has been much research dedicated to the investigation of such kinds of submanifolds, mainly focusing on the construction of examples and partial classification; see [Onti et al. 2025; Xie et al. 2021] and a recent survey of Chen [2021], where more references can be found. We point out that the results and methods in most of these works do not apply to 2-dimensional Wintgen ideal submanifolds, i.e., the superconformal surfaces discussed above. An essential distinction between dimension 2 and higher dimensions can be seen from the viewpoint of Möbius geometry. For higher dimensions, the Möbius form, which serves as a fundamental Möbius invariant [Lin and Guo 2012; Hu et al. 2023; Liu et al. 2001], is nonvanishing only on the first Möbius normal bundle. However, it is important to note that this property does not hold true in the case of dimension 2.

In [Xie et al. 2021], we have classified all Möbius homogeneous Wintgen ideal submanifolds by proving that the Möbius form vanishes identically under the Möbius homogeneous assumption. This paper is devoted to the classification of Möbius homogeneous superconformal surfaces in $\mathbb{S}^5$, of which examples with nonvanishing Möbius form are found.

**Classification Theorem.** *Let $x : M^2 \to \mathbb{S}^5$ be a superconformal surface. If $x$ is Möbius homogeneous, then it is Möbius equivalent to one of the following surfaces:*

(1) *a round* 2*-sphere;*

(2) *the Veronese surface in $\mathbb{S}^4$;*

(3) *a* 2*-parameter family of surfaces described in Theorem 3.7.*

We point out that in the above classification, round 2-spheres are totally umbilic, the Veronese surface in $\mathbb{S}^4$ has positive Möbius curvature (the Gaussian curvature of the Möbius metric), and those surfaces in item (3) have vanishing Möbius curvature.

The classification is conducted within the framework of Möbius geometry, focusing on categorizing surfaces into three types based on the sign of Möbius curvature. The Möbius geometry of superconformal surfaces in $\mathbb{S}^5$ is introduced in Section 2, where an adaptive moving frame and several Möbius invariants are constructed. These tools facilitate the analysis of surfaces with positive Möbius curvature. Section 3 is dedicated to the classification of Möbius homogeneous superconformal surfaces with vanishing Möbius curvature. For this type, the moving frame introduced in Section 2 can be adjusted to a canonical form, resulting in structure equations that become a linear partial differential system with constant coefficients. From these coefficients, two Lorentz skew-symmetric matrices are derived, which can be interpreted as elements of Lie algebras corresponding to Möbius transformation groups that act on such surfaces. By determining the normal form of these Lorentz skew-symmetric matrices, we solve all examples of the structure equations, completing the classification for this type. The classification process becomes particularly intricate when considering surfaces with negative Möbius curvature. In the final section, through careful adjustments in the choice of the moving frame and the construction of additional Möbius invariants, we demonstrate that Möbius homogeneous superconformal surfaces with negative Möbius curvature do not exist.

We would like to highlight new phenomena that arise in our classification, distinguishing them from the following two categories. Firstly, unlike Möbius homogeneous Wintgen ideal submanifolds of dimension no less than 3, which are indeed isometrically homogeneous up to certain Möbius transformations, there exist Möbius homogeneous superconformal surfaces that are not isometrically homogeneous. For their distribution in the moduli space, see Figure 1. Secondly, it follows from our previous work [Wang and Xie 2014] that there are no superconformal surfaces of vanishing Möbius curvature in $\mathbb{S}^4$, which implies a distinction between superconformal surfaces of codimension 2 and those of codimension 3 within the category of Möbius homogeneous surfaces.

Inspired by the nonexistence of Möbius homogeneous superconformal surfaces with negative Möbius curvature in both $\mathbb{S}^4$ and $\mathbb{S}^5$, we would also like to propose the following problem.

**Problem.** *Do there exist Möbius homogeneous superconformal surfaces of negative Möbius curvature in* $\mathbb{S}^n$?

See [Sulanke 1988; Li et al. 2024; Lubbes 2022] for related work focusing on the classification of Möbius homogeneous submanifolds in $\mathbb{S}^n$.

## 2. Möbius geometry of superconformal surfaces in $\mathbb{S}^5$

**2.1.** ***Möbius geometry of surfaces in*** $\mathbb{S}^n$**.** In this subsection, we briefly review the theory of surfaces in Möbius geometry. For details, see [Li et al. 2001; Wang 1998].

Let $\mathbb{R}^{n+2}_1$ be the Lorentzian space of dimension $n+2$, with inner product

$$\langle y, z\rangle = -y_0z_0 + y_1z_1 + \cdots + y_{n+1}z_{n+1}$$

for $y = (y_0, y_1, \ldots, y_{n+1})$ and $z = (z_0, z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+2}_1$. In the classical light-cone model, points in the round sphere $\mathbb{S}^n$ are associated with the light-like directions $\mathbb{R}^{n+2}_1$ as below:

$$\mathbb{S}^n \to Q^n, \quad x \mapsto [(1, x)],$$

where $Q^n \triangleq \{[y] \mid y \in \mathbb{R}^{n+2}_1, \langle y, y\rangle = 0\}$ is the projectivized light-cone of $\mathbb{R}^{n+2}_1$. Under this correspondence, the Möbius (conformal) transformation group of $\mathbb{S}^n$ is isomorphic to the Lorentz orthogonal group $O^+(n+1, 1)$ preserving the time direction.

Let $x : M^2 \to \mathbb{S}^n$ be an immersion of a connected Riemann surface. For brevity, we also refer to $x$ as a surface. We assume that $x$ is umbilic-free, and denote by $\mathfrak{k}$ the norm of the trace-free part of the second fundamental form of $x$. Then

$$Y \triangleq \sqrt{2}\mathfrak{k}(1, x) : M^2 \to \mathbb{R}^{n+2}_1 \tag{2}$$

is referred to as a *canonical lift* of $x$, in the sense that its induced metric $g \triangleq \langle dY, dY\rangle$

is a Möbius invariant, known as the *Möbius metric*. An important property is that two umbilic-free surfaces $x$ and $\tilde{x}$ are Möbius equivalent if and only if their canonical lifts, $Y$ and $\tilde{Y}$, differ by a Lorentz transformation $T \in O^+(n+1, 1)$. The curvature of the Möbius metric $g$ is called the *Möbius curvature*, and we will denote it by $K$. We denote by $\Delta$ the Laplacian associated with $g$. It was proved in [Wang 1998, Theorem 1.2 and (2.1)] that $\langle \Delta Y, \Delta Y \rangle = 1 + 4K$. Define $N : M^2 \to \mathbb{R}^{n+2}_1$ by

$$N = -\tfrac{1}{2}\Delta Y - \tfrac{1}{8}\langle \Delta Y, \Delta Y \rangle Y.$$

It is easy to verify that

$$\langle Y, Y \rangle = \langle N, N \rangle = 0, \quad \langle Y, N \rangle = 1, \quad \langle N, dY \rangle = 0.$$

Now let $z = u + iv$ be a local complex coordinate on $M^2$ such that the Möbius metric $g = e^{2\lambda}|dz|^2$ for some real function $\lambda$. The orthogonal complement of $\{Y, N, Y_z, Y_{\bar{z}}\}$ in $\mathbb{R}^{n+2}_1$ is referred to as the Möbius normal bundle.

We note that dilating $Y$ by a constant results in another canonical lift of $x$, a property we will use in this paper.

A surface in $\mathbb{S}^n$ is called *Möbius homogeneous* if, for any two points $p$ and $q$ on the surface, there exists a Möbius transformation that maps $p$ to $q$ while preserving the surface. Such a surface can be regarded as an orbit of some subgroup of the Möbius transformation group of $\mathbb{S}^n$ which is isometric to the Lorentz orthogonal group $O^+(n+1, 1)$. Note that on a Möbius homogeneous surface, every globally well-defined Möbius invariant function is constant.

We will use the following lemma. It is essentially a restatement of [Wang and Xie 2014, Lemmas 3.7 and 3.8].

**Lemma 2.1.** *Let $x : M^2 \to S^n$ be a Möbius homogeneous surface, with $K$ as its Möbius curvature.*

(1) *If $K > 0$, then any Möbius invariant complex form vanishes.*

(2) *If $K = 0$, then there exists a local complex coordinate $z$ on $M^2$ such that the Möbius metric $g$ can be written as $g = |dz|^2$, and any Möbius invariant complex $m$-form $\Omega$ on $M^2$ can be expressed as $\Omega = c\, dz^m$ for some constant $c \in \mathbb{C}$.*

**2.2.** ***The structure equations and integrability conditions.*** Let $x : M^2 \to \mathbb{S}^5$ be an umbilic-free superconformal surface, and $z = u + iv$ be a local complex (isothermal) coordinate on $M^2$. With respect to this coordinate, we express the Möbius metric $g$ as $g = e^{2\lambda}|dz|^2$ for some real function $\lambda$. Let $Y : M^2 \to \mathbb{R}^7_1$ be a canonical lift of $x$. It follows that

$$\langle Y_z, Y_z \rangle = 0, \quad \langle Y_z, Y_{\bar{z}} \rangle = \tfrac{1}{2}e^{2\lambda}, \quad \langle x_{zz}, x_{zz} \rangle = 0, \quad \langle Y_{zz}, Y_{zz} \rangle = 0,$$

where the superconformal assumption has been used. Choose $\{\xi_1, \xi_2, \eta\}$ as an orthonormal frame of the Möbius normal bundle such that the projection of $Y_{zz}$ on this bundle lies in Span$\{\xi_1, \xi_2\}$. Set $\xi \triangleq \xi_1 + i\,\xi_2$. It is obvious that $\eta$ is a globally well-defined vector field along $Y$, and the choice of $\xi$ only allows a rotation $e^{i\theta}$ for some real function $\theta$.

With respect to the moving frame

$$\{Y, N, Y_z, Y_{\bar z}, \xi, \eta\},$$

the structure equations of $Y$ can be calculated as below:

$$Y_{z\bar z} = -Ae^{2\lambda}Y - \tfrac{1}{2}e^{2\lambda}N, \tag{3}$$

$$Y_{zz} = -BY - \tfrac{1}{2}\mu\xi + 2\lambda_z Y_z, \tag{4}$$

$$\xi_z = \alpha\eta - \rho\xi, \tag{5}$$

$$\xi_{\bar z} = \bar\rho\xi - 2\bar\phi Y + \beta\eta + 2e^{-2\lambda}\bar\mu Y_z, \tag{6}$$

$$\eta_z = -\tfrac{1}{2}\alpha\bar\xi - \tfrac{1}{2}\bar\beta\xi - \psi Y, \tag{7}$$

$$N_z = 2e^{-2\lambda}BY_{\bar z} + 2AY_z + \psi\eta + \phi\xi, \tag{8}$$

where $A = \frac{1}{16}(1+4K)$, and $B$, $\mu$, $\alpha$, $\beta$, $\rho$, $\phi$, $\psi$ are smooth functions defined locally on $M^2$. We point out that $e^{-4\lambda}|\mu|^2$ is exactly the squared norm of the Möbius fundamental form; hence it is a constant (see [Li et al. 2001; Wang 1998]). Moreover, $\phi$ and $\psi$ are related to the Möbius form.

The integrability conditions of $Y$ is constituted by the following eight equations:

$$B_{\bar z} - \mu\bar\phi - e^{2\lambda}A_z = 0, \tag{9}$$

$$2e^{-4\lambda}|\mu|^2 - 4A + K = 0, \tag{10}$$

$$e^{2\lambda}\phi = \mu\bar\rho + \mu_{\bar z}, \tag{11}$$

$$e^{2\lambda}\psi = \mu\beta, \tag{12}$$

$$\alpha\bar\psi - 2\rho\bar\phi - 2\bar\phi_z - \beta\psi - 2e^{-2\lambda}B\bar\mu = 0, \tag{13}$$

$$\tfrac{1}{2}\alpha\bar\alpha - \tfrac{1}{2}\beta\bar\beta + \rho_{\bar z} + \bar\rho_z - e^{-2\lambda}|\mu|^2 = 0, \tag{14}$$

$$\alpha\bar\rho - \alpha_{\bar z} + \beta\rho + \beta_z = 0, \tag{15}$$

$$\bar\beta\bar\phi - \beta\phi - \psi_{\bar z} + \bar\psi_z = 0. \tag{16}$$

**Remark 2.2.** We point out that the following Möbius invariant complex forms are globally well defined:

$$B\,dz^2 = -\langle Y_{zz}, N\rangle\,dz^2, \qquad \alpha\mu\,dz^3 = 2\langle Y_{zz}, \eta_z\rangle\,dz^3,$$

$$e^{-2\lambda}\mu\beta\,dz = 2\frac{\langle Y_{zz}, \eta_{\bar z}\rangle\,dz\,|dz|^2}{g}, \quad \alpha\phi\,dz^2 = -\langle N_z, \eta_z\rangle\,dz^2,$$

$$\phi\beta e^{-2\lambda} = -\frac{\langle N_z, \eta_{\bar z}\rangle\,|dz|^2}{g}, \quad \alpha\bar\beta\,dz^2 = \langle \eta_z, \eta_z\rangle\,dz^2, \quad \psi\,dz = \langle N_z, \eta\rangle\,dz.$$

Now we assume that $x$ is an umbilical-free Möbius homogeneous superconformal surface with positive Möbius curvature in $\mathbb{S}^5$. It follows from Lemma 2.1 (see also [Wang and Xie 2014, Lemma 3.7]) that all globally well-defined Möbius invariant complex forms vanish. Therefore, we have

$$\alpha\mu = 0, \quad \psi = 0, \quad e^{-2\lambda}\mu\beta = 0,$$

i.e.,

$$\alpha = 0, \quad \psi = 0, \quad \beta = 0,$$

which implies $\eta$ is constant along $Y$. Hence the surface $x$ is not linearly full in $\mathbb{S}^5$, namely, it is a Möbius homogeneous surface in $\mathbb{S}^4$. By the classification results in [Wang and Xie 2014], we obtain the following theorem.

**Theorem 2.3.** *Let $x : M^2 \to \mathbb{S}^5$ be an umbilical-free Möbius homogeneous superconformal surface. If $x$ has positive Möbius curvature, then $x$ is Möbius equivalent to the Veronese surface in $\mathbb{S}^4$.*

## 3. Möbius homogeneous superconformal surfaces with vanishing Möbius curvature in $\mathbb{S}^5$

In this section, we assume that $x : M^2 \to \mathbb{S}^5$ is an umbilical-free Möbius homogeneous superconformal surface with Möbius curvature $K = 0$. According to Lemma 2.1 (see also [Wang and Xie 2014, Lemma 3.8]), for $M^2$, there exist local complex coordinate charts $\{(U, z)\}$ such that on $U \subset M^2$, the Möbius metric can be expressed as $g = |dz|^2$, and the coefficients of all globally well-defined Möbius invariant complex forms are constant. Therefore, by Remark 2.2,

$$A,\ B,\ \alpha\mu,\ |\mu|^2,\ \beta\mu,\ \alpha\phi,\ \psi,\ \beta\phi,\ \alpha\bar{\beta}$$

are all constant on $U$. Write $\alpha\mu = |\alpha\mu|e^{it_0}$. From $\alpha\mu\, dz^3 = |\alpha\mu|\, d(e^{i(t_0/3)}z)^3$, it follows that by rechoosing $e^{i(t_0/3)}z$ as the new complex coordinate on $U$ we can assume that $\alpha\mu$ is real. This choice of complex coordinates ensures that $dz$ becomes a globally well-defined complex form. Moreover, note that on $U$, with respect to such choice of $z$, $\xi$ can be selected canonically such that $\mu = -\langle Y_{zz}, \bar{\xi}\rangle$ is nonnegative. Selecting such complex coordinates and $\xi$, we observe that $A$, $B$, $\psi$ and

$$\alpha = \frac{\langle \xi_z, \eta\rangle\, dz}{dz}, \quad \mu = \frac{-\langle Y_{zz}, \bar{\xi}\rangle\, dz^2}{dz^2}, \quad \beta = \frac{\langle \xi_{\bar{z}}, \eta\rangle\, d\bar{z}}{d\bar{z}}, \quad \phi = \frac{\langle N_z, \bar{\xi}\rangle\, dz}{2\,dz},$$

are all globally well-defined functions. Consequently, they are all constant.

It follows form the integrability conditions (9)–(16) that

$$\phi = 0, \quad 2A = \mu^2, \quad \rho = 0, \quad \psi = \mu\beta, \quad 2B = \alpha\bar{\beta} - \beta^2, \quad 2\mu^2 = |\alpha|^2 - |\beta|^2.$$

Substituting these formulas into the structure equations (3)–(8), we derive that

$$\frac{\partial}{\partial z}\begin{pmatrix} Y \\ N \\ Y_z \\ Y_{\bar z} \\ \xi \\ \bar\xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^2 & \alpha\bar\beta-\beta^2 & 0 & 0 & \beta\mu \\ \frac{1}{2}(\beta^2-\alpha\bar\beta) & 0 & 0 & 0 & -\frac{1}{2}\mu & 0 & 0 \\ -\frac{1}{2}\mu^2 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 2\mu & 0 & 0 & \bar\beta \\ -\beta\mu & 0 & 0 & 0 & -\frac{1}{2}\bar\beta & -\frac{1}{2}\alpha & 0 \end{pmatrix}\begin{pmatrix} Y \\ N \\ Y_z \\ Y_{\bar z} \\ \xi \\ \bar\xi \\ \eta \end{pmatrix}. \tag{17}$$

Since $x$ is umbilic-free, we have $\mu \neq 0$. It follows that $\alpha$ cannot equal zero. We choose the lift $[Y]$ of $x$ such that $\alpha = 1$. We denote by $b_1$ and $b_2$ the real and imaginary parts of $\beta$, respectively. They satisfy

$$b_1^2 + b_2^2 = 1 - 2\mu^2 < 1.$$

Using (17), it is easy to verify that

$$d\begin{pmatrix} \frac{1}{\sqrt{2}}(Y-N) \\ \frac{1}{\sqrt{2}}(Y+N) \\ Y_u \\ Y_v \\ \xi_1 \\ \xi_2 \\ \eta \end{pmatrix} = (B_1\,du + B_2\,dv)\begin{pmatrix} \frac{1}{\sqrt{2}}(Y-N) \\ \frac{1}{\sqrt{2}}(Y+N) \\ Y_u \\ Y_v \\ \xi_1 \\ \xi_2 \\ \eta \end{pmatrix}, \tag{18}$$

where $B_1$ is equal to

$$\begin{pmatrix} 0 & 0 & \frac{3b_1^2-2b_1-b_2^2+1}{4\sqrt{2}} & -\frac{(2b_1+1)b_2}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2}b_1\sqrt{1-b_1^2-b_2^2} \\ 0 & 0 & \frac{-3b_1^2+2b_1+b_2^2+3}{4\sqrt{2}} & \frac{(2b_1+1)b_2}{2\sqrt{2}} & 0 & 0 & \frac{1}{2}b_1\sqrt{1-b_1^2-b_2^2} \\ \frac{3b_1^2-2b_1-b_2^2+1}{4\sqrt{2}} & \frac{3b_1^2-2b_1-b_2^2-3}{4\sqrt{2}} & 0 & 0 & -\frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 & 0 \\ -\frac{(2b_1+1)b_2}{2\sqrt{2}} & -\frac{(2b_1+1)b_2}{2\sqrt{2}} & 0 & 0 & 0 & \frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 & 0 & 0 & \frac{1}{2}(b_1+1) \\ 0 & 0 & 0 & -\frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 & 0 & \frac{b_2}{2} \\ -\frac{1}{2}b_1\sqrt{1-b_1^2-b_2^2} & -\frac{1}{2}b_1\sqrt{1-b_1^2-b_2^2} & 0 & 0 & \frac{1}{2}(-b_1-1) & -\frac{b_2}{2} & 0 \end{pmatrix}, \tag{19}$$

and $B_2$ is equal to

$$\begin{pmatrix} 0 & 0 & \frac{(2b_1+1)b_2}{2\sqrt{2}} & \frac{b_1^2-2b_1-3b_2^2-1}{4\sqrt{2}} & 0 & 0 & -\frac{1}{2}b_2\sqrt{1-b_1^2-b_2^2} \\ 0 & 0 & -\frac{(2b_1+1)b_2}{2\sqrt{2}} & -\frac{b_1^2-2b_1-3b_2^2+3}{4\sqrt{2}} & 0 & 0 & \frac{1}{2}b_2\sqrt{1-b_1^2-b_2^2} \\ \frac{(2b_1+1)b_2}{2\sqrt{2}} & \frac{(2b_1+1)b_2}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 \\ \frac{b_1^2-2b_1-3b_2^2-1}{4\sqrt{2}} & \frac{b_1^2-2b_1-3b_2^2+3}{4\sqrt{2}} & 0 & 0 & -\frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 & 0 & -\frac{b_2}{2} \\ 0 & 0 & \frac{\sqrt{1-b_1^2-b_2^2}}{2\sqrt{2}} & 0 & 0 & 0 & \frac{1}{2}(b_1-1) \\ -\frac{1}{2}b_2\sqrt{1-b_1^2-b_2^2} & -\frac{1}{2}b_2\sqrt{1-b_1^2-b_2^2} & 0 & 0 & \frac{b_2}{2} & \frac{1}{2}(1-b_1) & 0 \end{pmatrix}. \tag{20}$$

Note that both $B_1$ and $B_2$ are Lorentz skew-symmetric, and they can be viewed as elements of the Lie algebra corresponding to the subgroup of $O(6,1)$ acting on $Y$. Differentiating in (18) we obtain that $B_1B_2 = B_2B_1$. To solve (18), we first seek the normal forms of $B_1$ and $B_2$.

It is straightforward to calculate that the characteristic polynomials of $B_1$ and $B_2$ are given by

$$xf_1(x^2) \quad \text{and} \quad xf_2(x^2), \tag{21}$$

respectively, where

$$\begin{aligned} f_1(t) = 64t^3 &+ (16b_2^2 + 64b_1 - 48b_1^2 + 48)t^2 \\ &+ (9b_1^4 - 16b_1^3 - 6b_2^2b_1^2 - 18b_1^2 + 16b_1 + b_2^4 + 2b_2^2 + 9)t \\ &- \tfrac{1}{2}(b_1+1)^2\big((b_1^2+b_2^2+2)^2 - 4b_1(b_1^2-3b_2^2) - 5\big), \\ f_2(t) = 64t^3 &+ (16b_1^2 - 64b_1 - 48b_2^2 + 48)t^2 \\ &+ (b_1^4 - 8b_1^3 - 6b_2^2b_1^2 + 22b_1^2 + 8b_2^2b_1 - 24b_1 + 9b_2^4 - 38b_2^2 + 9)t \\ &- \tfrac{1}{2}b_2^2\big((b_1^2+b_2^2+2)^2 - 4b_1(b_1^2-3b_2^2) - 5\big). \end{aligned}$$

The normal form of a single Lorentz skew-symmetric matrix has been studied in [Singer and Steinberg 1994]. We give a straightforward discussion here pertaining to our special situation, i.e., whether two commutative Lorentz skew-symmetric matrices can be transformed to the normal form simultaneously.

**Lemma 3.1.** *Let $A$ be a Lorentz skew-symmetric matrix of 7 by 7, and $\lambda$ be an eigenvalue of $A$. We have*:

(1) *$-\lambda$ is also an eigenvalue of $A$.*

(2) *If $\lambda = 0$, then the algebraic multiplicity of $\lambda$ is odd.*

(3) *If $\lambda \in \mathbb{R} \setminus \{0\}$, then the eigenspace $V_\lambda$ of $A$ corresponding to $\lambda$ is a light line.*

(4) *If $\lambda \notin \mathbb{R}$, then $\lambda$ is purely imaginary, and the realification of its eigenspace $V_\lambda$ is space-like.*

(5) *If $\lambda \neq 0$, then the algebraic multiplicity of $\lambda$ coincides with its geometric multiplicity.*

(6) *Counting multiplicities, $A$ has at most two nonzero real eigenvalues.*

*Proof.* The first two conclusions follow from

$$\det(A - \lambda I) = \det(A^t - \lambda I) = -\det(A + \lambda I),$$

and the sum of the algebraic multiplicities of eigenvalues equals 7.

Assuming $\lambda \neq 0$, if $X$ and $Y$ are two eigenvectors of $\lambda$, then using

$$\lambda\langle X, Y\rangle = \langle AX, Y\rangle = -\langle X, AY\rangle = -\lambda\langle X, Y\rangle,$$

we have $\langle X, Y\rangle = 0$. In particular, by choosing $X = Y$, we can obtain that $\langle X, X\rangle = \langle Y, Y\rangle = 0$. If $\lambda$ is real, then as real vectors, both $X$ and $Y$ are light-like; hence they must be parallel with each other, form which the third conclusion follows. If $\lambda \notin \mathbb{R}$, then the real and imaginary parts of $X$ are both space-like with the same norm and orthogonal with each other. Moreover, using

$$\lambda\langle X, \overline{X}\rangle = \langle AX, \overline{X}\rangle = -\langle X, \overline{AX}\rangle = -\bar{\lambda}\langle X, \overline{X}\rangle,$$

we can derive that $\lambda$ is purely imaginary and $\overline{X}$ is the eigenvector corresponding to $-\lambda$. This gives us the proof of the fourth conclusion.

When $\lambda \notin \mathbb{R}$, the restriction of $A$ on the realification of $V_\lambda$ is exactly Euclidean skew-symmetric, so the algebraic multiplicity of $\lambda$ equals its geometric multiplicity.

When $\lambda$ is a nonzero real eigenvalue, suppose that the sixth conclusion is not correct. Then, by the theory of Jordan forms, there exists a vector $Z$ such that

$$AZ = \lambda Z + X,$$

where $X$ is an eigenvector corresponding to $\lambda$. It follows from

$$-\lambda\langle Z, X\rangle = \langle AZ, X\rangle = \lambda\langle Z, X\rangle$$

that $\langle Z, X\rangle = 0$. Note that $\langle AZ, Z\rangle = -\langle Z, AZ\rangle$, which implies $\langle AZ, Z\rangle = 0$. Thus,

$$0 = \langle AZ, Z\rangle = \lambda\langle Z, Z\rangle.$$

Since $Z$ cannot be light-like, this gives us a contradiction, and then the fifth and sixth conclusions are proved. □

**Lemma 3.2.** *If* $0$ *is the unique real eigenvalue of* $A$, *then the root subspace* $W_0$ *corresponding to* $0$ *is Lorentzian, and there exists a Lorentzian orthonormal basis such that*

$$A|_{W_0} = \begin{pmatrix} 0 & 0 & \delta & \\ 0 & 0 & -\delta & \\ \delta & \delta & 0 & \\ & & & \vec{\mathbf{0}} \end{pmatrix},$$

*where* $\delta = 1$ *or* $0$. *Moreover, if* $\dim W_0 = 3$ *and* $\delta = 1$, *then for the Lorentz skew-symmetric matrix* $\tilde{A}$ *commutative with* $A$, *under the same Lorentzian orthonormal basis*, $\tilde{A}$ *also has the same form as above.*

*Proof.* Note that the orthogonal complement $W_0^{\perp}$ of $W_0$ is space-like since it can be spanned by the real and imaginary parts of the eigenvectors corresponding to those purely imaginary eigenvalues. As a consequence, $W_0$ is a Lorentzian subspace.

If all light-like vectors in $W_0$ are eigenvectors of $A$, then $A|_{W_0} = 0$ since $W_0$ can be spanned by light-like vectors. Otherwise, there exists a light-like vector $\nu$ such that $A\nu \neq 0$. Since $A|_{W_0}$ is nilpotent, there exists an integer $k > 1$ such that $A^k\nu = 0$ but $A^{k-1}\nu \neq 0$.

We claim that $k=3$. In fact, if $k>3$, then it follows from $\langle A^{k-1}\nu, A^{k-1}\nu\rangle = \langle A^{k-2}\nu, A^{k}\nu\rangle = 0$ and

$$\langle A^{k-2}\nu, A^{k-2}\nu\rangle = \langle A^{k-4}\nu, A^{k}\nu\rangle = 0, \quad \langle A^{k-2}\nu, A^{k-1}\nu\rangle = \langle A^{k-3}\nu, A^{k}\nu\rangle = 0$$

that as light-like vectors, $A^{k-1}\nu$ and $A^{k-2}\nu$ are parallel with each other, which is impossible. If $k=2$, a similar contradiction that $\upsilon$ and $A\nu$ are two light-like vectors parallel to each other can be obtained.

Note that it must hold that $\langle A\nu, A\nu\rangle > 0$; otherwise $\nu - \frac{1}{2}A^2\nu$ is another time-like vector perpendicular to the time-like vector $A\nu$, which is impossible. Up to dilation, we assume $\langle A\nu, A\nu\rangle = 1$. Set

$$e_1 \triangleq \nu + \tfrac{1}{2}A^2\nu, \quad e_2 \triangleq \nu - \tfrac{1}{2}A^2\nu, \quad e_3 \triangleq A\nu.$$

Then $\{e_1, e_2, e_3\}$ is a Lorentzian orthonormal basis of $V \triangleq \mathrm{Span}\{\nu, A\nu, \dots, A^{k-1}\nu\}$, and

$$A|_V(e_1, e_2, e_3) = (e_1, e_2, e_3)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since the orthogonal complement of $V$ in $W_0$ is space-like, and restricted on it $A$ can be diagonalizable, we have finished the proof of the first part.

For the second part, by assumption, $A^2\nu$ is the unique eigenvector of $A$ corresponding to 0 up to dilation. Note that $W_0$ is an invariant subspace of $\tilde{A}$. It follows from $A(\tilde{A}(A^2\nu)) = \tilde{A}(A^3\nu) = 0$ that $\tilde{A}(A^2\nu)$ is also an eigenvector corresponding to 0. So there exists a real number $\lambda_1$ such that $\tilde{A}(A^2\nu) = \lambda_1 A^2\nu$, which implies that $A(\tilde{A}(A\nu) - \lambda_1 A\nu) = 0$. Hence there exists $\lambda_2 \in \mathbb{R}$ such that $\tilde{A}(A\nu) = \lambda_1 A\nu + \lambda_2 A^2\nu$. Taking the inner product with $A\nu$ on both sides of this equation, we conclude that $\lambda_1 = 0$. As a consequence, $A(\tilde{A}\nu - \lambda_2 A\nu) = 0$, which implies there exists $\lambda_3 \in \mathbb{R}$ such that $\tilde{A}\nu = \lambda_2 A\nu + \lambda_3 A^2\nu$. Taking the inner product with $\nu$ on both sides of this equation, we conclude $\lambda_3 = 0$, i.e., $\tilde{A}\nu = \lambda_2 A\nu$, from which we finish the proof. □

Now we apply the above two lemmas to analyze the normal forms of $B_1$ and $B_2$.

**Lemma 3.3.** *In the unit disk* $\{(b_1, b_2) \mid b_1^2 + b_2^2 < 1\}$:

(1) *If*

$$(b_1^2 + b_2^2 + 2)^2 - 4b_1(b_1^2 - 3b_2^2) - 5 > 0,$$

*then* $f_1(0) \neq 0$ *and* $f_1(t)$ *has only one positive root*; *the same conclusion holds for* $f_2(t)$ *if* $b_2 \neq 0$.

(2) *If*

$$(b_1^2 + b_2^2 + 2)^2 - 4b_1(b_1^2 - 3b_2^2) - 5 < 0,$$

*then* $f_1(t)$ *has no positive root*; *the same conclusion holds for* $f_2(t)$ *if* $b_2 \neq 0$.

(3) *If*

$$(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5=0,$$

*then both* $f_1(t)$ *and* $f_2(t)$ *take zero as a simple root.*

*Proof.* It follows from item (4) of Lemma 3.1 that both $f_1(t)$ and $f_2(t)$ have at most one positive root (the multiplicity is counted here); otherwise $B_1$ and $B_2$ will have at least four nonzero eigenvalues.

If $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5>0$, then $f_1(0)<0$. It follows from $\lim_{t\to+\infty} f_1(t)=+\infty$ that $f_1(t)$ has a positive root. A similar discussion applies to $f_2(t)$.

If $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5<0$, then $f_1(0)>0$. Suppose there exists $t_0>0$ such that $f_1(t_0)=0$. Using $\lim_{t\to+\infty} f_1(t)=+\infty$ again, we conclude that $f_1(t)$ has at least two positive roots (counting the multiplicity), which is impossible. A similar discussion applies to $f_2(t)$.

If $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5=0$, then $f_1(0)=0$ and $f_2(0)=0$. It is straightforward to verify that neither

$$f_1'(0)=9b_1^4-16b_1^3-6b_2^2b_1^2-18b_1^2+16b_1+b_2^4+2b_2^2+9$$

nor

$$f_2'(0)=b_1^4-8b_1^3-6b_2^2b_1^2+22b_1^2+8b_2^2b_1-24b_1+9b_2^4-38b_2^2+9$$

can equal zero in this case, so both $f_1(t)$ and $f_2(t)$ take zero as a simple root. □

**Lemma 3.4.** *If* $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5>0$, *then there exists a matrix* $P\in O(6,1)$ *such that the Lorentz skew-symmetric matrices* $B_1$ *and* $B_2$, *as in* (19) *and* (20), *can be transformed into the normal forms*

$$P^{-1}B_1P=\begin{pmatrix}0&\delta_1&0&0&0&0&0\\ \delta_1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&\delta_2&0&0\\ 0&0&0&-\delta_2&0&0&0\\ 0&0&0&0&0&0&\delta_3\\ 0&0&0&0&0&-\delta_3&0\end{pmatrix},\quad P^{-1}B_2P=\begin{pmatrix}0&\epsilon_1&0&0&0&0&0\\ \epsilon_1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&\epsilon_2&0&0\\ 0&0&0&-\epsilon_2&0&0&0\\ 0&0&0&0&0&0&\epsilon_3\\ 0&0&0&0&0&-\epsilon_3&0\end{pmatrix},$$

*where* $\delta_j,\epsilon_j\in\mathbb{R},\ j=1,2,3.$

*Proof.* If $b_2\neq 0$, it follows from Lemma 3.3 that both $B_1$ and $B_2$ have two nonzero real eigenvalues, which are denoted by $\pm\delta_1$ and $\pm\epsilon_1$, respectively. Moreover, they both take 0 as one simple eigenvalue. According to Lemma 3.1, the remaining eigenvalues of both $B_1$ and $B_2$ are purely imaginary. We denote all the eigenvalues of $B_1$ and $B_2$ by

(22) $\{\delta_1,-\delta_1,0,i\delta_2,-i\delta_2,i\delta_3,-i\delta_3\}$ and $\{\epsilon_1,-\epsilon_1,0,i\epsilon_2,-i\epsilon_2,i\epsilon_3,-i\epsilon_3\}$

respectively, where $\delta_j, \epsilon_j \in \mathbb{R},\ j = 1, 2, 3$. Since $B_1 B_2 = B_2 B_1$, and both of them can be diagonalizable, they share the same eigenvectors, which are assumed to be

$$\{p_1,\ p_2,\ p_3,\ p_4 + i p_5,\ p_4 - i p_5,\ p_6 + i p_7,\ p_6 - i p_7\}.$$

It follows from Lemma 3.1 that $\{p_1, p_2\}$ are a pair of light-like vectors, whose inner product can be assume to be 1, and $\{p_3, p_4, p_5, p_5, p_6\}$ are space-like and orthogonal to each other, and their norms of are all assumed to be 1. Set

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}(p_1 - p_2) & \frac{1}{\sqrt{2}}(p_1 + p_2) & p_3 & p_4 & p_5 & p_6 & p_7 \end{pmatrix}.$$

Then $P \in O(6, 1)$, and this gives us the normal form of $B_1$ and $B_2$.

If $b_2 = 0$, then by a straightforward calculation, one can verify that the minimal polynomial of $B_2$ is $x(8x^2 + b_1^2 - 4b_1 + 3)$, which implies that as an eigenvalue of $B_2$, the algebraic multiplicity of 0 is equal to its geometric multiplicity. By taking $\epsilon_1 = 0$ in (22), all the discussion above applies to this case. □

**Lemma 3.5.** *If* $(b_1^2 + b_2^2 + 2)^2 - 4b_1(b_1^2 - 3b_2^2) - 5 < 0$, *then there exists a matrix* $P \in O(6, 1)$ *such that the Lorentz skew-symmetric matrices* $B_1$ *and* $B_2$, *as in* (19) *and* (20), *can be transformed into the normal forms*

$$P^{-1}B_1P = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&\delta_1&0&0&0&0\\ 0&-\delta_1&0&0&0&0&0\\ 0&0&0&0&\delta_2&0&0\\ 0&0&0&-\delta_2&0&0&0\\ 0&0&0&0&0&0&\delta_3\\ 0&0&0&0&0&-\delta_3&0 \end{pmatrix}, \quad P^{-1}B_2P = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&\epsilon_1&0&0&0&0\\ 0&-\epsilon_1&0&0&0&0&0\\ 0&0&0&0&\epsilon_2&0&0\\ 0&0&0&-\epsilon_2&0&0&0\\ 0&0&0&0&0&0&\epsilon_3\\ 0&0&0&0&0&-\epsilon_3&0 \end{pmatrix},$$

*where* $\delta_j, \epsilon_j \in \mathbb{R},\ j = 1, 2, 3$.

*Proof.* It follows from Lemma 3.3 that both $B_1$ and $B_2$ lack nonzero real eigenvalues; 0 is the simple eigenvalue of $B_1$, whereas for $B_2$, the algebraic multiplicity of 0 may exceed 1 when $b_2 = 0$, yet it still equals the geometric multiplicity. We assume that all the eigenvalues of $B_1$ and $B_2$ are

$$\{0, i\delta_1, -i\delta_1, i\delta_2, -i\delta_2, i\delta_3, -i\delta_3\} \quad \text{and} \quad \{0, i\epsilon_1, -i\epsilon_1, i\epsilon_2, -i\epsilon_2, i\epsilon_3, -i\epsilon_3\},$$

respectively, where $\delta_j, \epsilon_j \in \mathbb{R},\ j = 1, 2, 3$. Since $B_1 B_2 = B_2 B_1$, and both of them can be diagonalizable, they share the same eigenvectors, which are assumed to be

$$\{p_1,\ p_2 + i p_3,\ p_2 - i p_3,\ p_4 + i p_5,\ p_4 - i p_5,\ p_6 + i p_7,\ p_6 - i p_7\}.$$

From Lemma 3.1, it follows that the vectors $\{p_2, p_3, p_4, p_5, p_6, p_7\}$ are space-like, each with an assumed norm of 1, and are mutually orthogonal. Consequently, the vector $p_1$ is time-like, with its norm assumed to be $-1$. Set

$$P = \big(p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5\ \ p_6\ \ p_7\big)\,.$$

Then $P \in O(6, 1)$, and this gives us the normal form of $B_1$ and $B_2$. □

**Lemma 3.6.** *If* $(b_1^2 + b_2^2 + 2)^2 - 4b_1(b_1^2 - 3b_2^2) - 5 = 0$, *there exists a matrix* $P \in O(6, 1)$ *such that the Lorentz skew-symmetric matrices* $B_1$ *and* $B_2$, *as in* (19) *and* (20), *can be transformed into the normal forms*

$$P^{-1}B_1P = \begin{pmatrix} 0 & 0 & \delta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_1 & 0 & 0 & 0 & 0 \\ \delta_1 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_2 & 0 & 0 \\ 0 & 0 & 0 & -\delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_3 \\ 0 & 0 & 0 & 0 & 0 & -\delta_3 & 0 \end{pmatrix}, \quad P^{-1}B_2P = \begin{pmatrix} 0 & 0 & \epsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_1 & 0 & 0 & 0 & 0 \\ \epsilon_1 & \epsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_3 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon_3 & 0 \end{pmatrix},$$

*where* $\delta_j, \epsilon_j \in \mathbb{R},\ j = 1, 2, 3.$

*Proof.* It follows from Lemma 3.3 that both $B_1$ and $B_2$ have no nonzero real eigenvalues, and take 0 as one eigenvalue of algebraic multiplicity 3. If the geometric multiplicity is also equal to 3 for both of them, then they can be diagonalizable simultaneously, and the conclusion follows from similar discussion as above.

If one of them, assumed to be $B_1$ without loss of generality, cannot be diagonalizable on its root space $W_0$ corresponding to 0, then by using Lemma 3.2 we can derive that there exists a Lorentzian orthonormal basis of $W_0$ such that both $B_1$ and $B_2$ have the form of Lemma 3.2. Note also that the orthogonal complement $W_0^{\perp}$ of $W_0$ is space-like, where $B_1$ and $B_2$ can be diagonalized simultaneously, and the conclusion follows. □

With the help of Lemmas 3.4–3.6, the structure equations in (18) can be solved, and we obtain the following theorem.

**Theorem 3.7.** *Suppose that* $x : M^2 \to \mathbb{S}^5$ *is an umbilic-free, Möbius homogeneous, superconformal surface with the Möbius curvature* $K$ *vanishing identically. Let* $b_1$ *and* $b_2$ *be real parameters such that* $b_1^2 + b_2^2 < 1$. *Consider matrices* $B_1$ *and* $B_2$ *determined by* $b_1$ *and* $b_2$ *as in* (19) *and* (20). *Let* $\{\delta_1, \delta_2, \delta_3\}$ *and* $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ *be real numbers correspond to the eigenvalues of the matrices* $B_1$ *and* $B_2$, *respectively, as in the above three lemmas.*

(1) *If* $(b_1^2 + b_2^2 + 2)^2 - 4b_1(b_1^2 - 3b_2^2) - 5 > 0$, *then* $x$ *is Möbius equivalent to*

$$\frac{1}{\cosh\phi_1(u, v)}\big(\sinh\phi_1(u, v), c_1\cos\phi_2(u, v), c_1\sin\phi_2(u, v), c_2\cos\phi_3(u, v), c_2\sin\phi_3(u, v), c_3\big),$$

*where*

$$\phi_1(u,v)=\delta_1 u+\epsilon_1 v,\quad \phi_2(u,v)=\delta_2 u+\epsilon_2 v,\quad \phi_3(u,v)=\delta_3 u+\epsilon_3 v,$$

*and* $c_1,\ c_2,\ c_3$ *are real numbers associated with the eigenvectors of* $B_1$ *and* $B_2$, *uniquely determined by the parameters* $b_1$ *and* $b_2$, *satisfying* $c_1^2+c_2^2+c_3^2=1$.

(2) *If* $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5<0$, *then* $x$ *is Möbius equivalent to*

$$\big(c_1\cos\phi_1(u,v), c_1\sin\phi_1(u,v), c_2\cos\phi_2(u,v), c_2\sin\phi_2(u,v), c_3\cos\phi_3(u,v), c_3\sin\phi_3(u,v)\big),$$

*where*

$$\phi_1(u,v)=\delta_1 u+\epsilon_1 v,\quad \phi_2(u,v)=\delta_2 u+\epsilon_2 v,\quad \phi_3(u,v)=\delta_3 u+\epsilon_3 v,$$

*and* $c_1,\ c_2,\ c_3$ *are real numbers associated with the eigenvectors of* $B_1$ *and* $B_2$, *uniquely determined by the parameters* $b_1$ *and* $b_2$, *satisfying* $c_1^2+c_2^2+c_3^2=1$.

(3) *If* $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5=0$, *then* $x$ *is Möbius equivalent to*

$$\frac{1}{\phi_1(u,v)^2+c_1}\big(\phi_1(u,v)^2+c_1-2, 2\phi_1(u,v), c_2\cos\phi_2(u,v), c_2\sin\phi_2(u,v), c_3\cos\phi_3(u,v), c_3\sin\phi_3(u,v)\big),$$

*where*

$$\phi_1(u,v)=\delta_1 u+\epsilon_1 v,\quad \phi_2(u,v)=\delta_2 u+\epsilon_2 v,\quad \phi_3(u,v)=\delta_3 u+\epsilon_3 v,$$

*and* $c_1, c_2, c_3$ *are real numbers associated with the eigenvectors of* $B_1$ *and* $B_2$, *uniquely determined by the parameters* $b_1$ *and* $b_2$, *satisfying* $c_1^2+c_2^2+c_3^2=1$.

*Proof.* Based on the normal forms of $B_1$ and $B_2$ as described in Lemmas 3.4–3.6, and using diag$\{-1,1,1,1,1,1,1\}$ as the initial value, which differs by a Lorentz orthogonal transformation, we can solve for

$$P^{-1}\left(\tfrac{1}{\sqrt{2}}(Y^t-N^t)\ \ \tfrac{1}{\sqrt{2}}(Y^t+N^t)\ \ Y_u^t\ \ Y_v^t\ \ \xi_1^t\ \ \xi_2^t\ \ \eta^t\right)^t \tag{23}$$

from (18). Here, $P$ is the Lorentz orthogonal matrix specified in the respective lemma, determined by the eigenvectors of $B_1$ and $B_2$. Denote the matrix described in (23) by $S$. Then $Y$ can be expressed as

$$Y=\big(\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}},0,0,0,0,0\big)PS=\tfrac{1}{\sqrt{2}}(q_1+q_2)S,$$

where $q_1$ and $q_2$ are the first two row vectors of $P$.

Here, we will provide a detailed proof for item (3) only, as the proofs for the other items are similar.

In this case, where the equation $(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5=0$ holds, Lemma 3.6 can be applied. The matrix $S$ described in (23) can be expressed as

$$\begin{pmatrix} \frac{1}{2}(\delta_1u+\epsilon_1v)^2+1 & \frac{1}{2}(\delta_1u+\epsilon_1v)^2 & \delta_1u+\epsilon_1v & 0 & 0 & 0 & 0 \\ -\frac{1}{2}(\delta_1u+\epsilon_1v)^2 & 1-\frac{1}{2}(\delta_1u+\epsilon_1v)^2 & -\delta_1u-\epsilon_1v & 0 & 0 & 0 & 0 \\ \delta_1u+\epsilon_1v & \delta_1u+\epsilon_1v & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\delta_2u+v\epsilon_2) & \sin(\delta_2u+v\epsilon_2) & 0 & 0 \\ 0 & 0 & 0 & -\sin(\delta_2u+v\epsilon_2) & \cos(\delta_2u+v\epsilon_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos(\delta_3u+v\epsilon_3) & \sin(\delta_3u+v\epsilon_3) \\ 0 & 0 & 0 & 0 & 0 & -\sin(\delta_3u+v\epsilon_3) & \cos(\delta_3u+v\epsilon_3) \end{pmatrix},$$

where $\{\delta_1,\delta_2,\delta_3\}$ and $\{\epsilon_1,\epsilon_2,\epsilon_3\}$ correspond to the eigenvalues of matrices $B_1$ and $B_2$, as in Lemma 3.6, respectively. Note that $P$ is not uniquely determined. It allows for right multiplication by matrices in $G\times SO(2)\times SO(2)$, where $G$ is a matrix group generated by $3\times 3$ matrices of the forms

$$\begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1+\frac{1}{2}t^2 & \frac{1}{2}t^2 & t \\ -\frac{1}{2}t^2 & 1-\frac{1}{2}t^2 & -t \\ t & t & 1 \end{pmatrix}.$$

Using this freedom in selection, we choose $P$ such that

$$\frac{1}{\sqrt{2}}(q_1+q_2)=(c_1,c_1-2,0,c_2,0,c_3,0).$$

Then, up to a Lorentz orthogonal transformation, the canonical lift $Y$ of $x$, as defined in (2), is given by

$$\begin{aligned} Y &= \frac{1}{\sqrt{2}}(q_1+q_2)S \\ &= \big((\delta_1u+\epsilon_1v)^2+c_1,\ (\delta_1u+\epsilon_1v)^2+c_1-2,\ 2(\delta_1u+\epsilon_1v),\ c_2\cos(\delta_2u+v\epsilon_2), \\ &\qquad c_2\sin(\delta_2u+v\epsilon_2),\ c_3\cos(\delta_3u+v\epsilon_3),\ c_3\sin(\delta_3u+v\epsilon_3)\big). \end{aligned}$$

Observe that $(1,x)$ is obtained by dividing $Y$ by its first coordinate. This completes the proof of item (3). □

**Remark 3.8.** Based on (19) and (20), each pair $(b_1,b_2)$ with $b_1^2+b_2^2<1$ can fully determine the structure equation (18) and ensure that the integrability conditions (9)–(16) are satisfied. Consequently, each pair defines a Möbius homogeneous superconformal surface in $\mathbb{S}^5$. Note that different pairs lead to distinct Möbius invariants $\beta$, which means the surfaces they define are not Möbius equivalent. Therefore, the unit open disc serves as a parameter space for the set of non-Möbius equivalent superconformal surfaces with vanishing Möbius curvature in $\mathbb{S}^5$.

Figure 1 shows the respective regions determined by the sign of

$$(b_1^2+b_2^2+2)^2-4b_1(b_1^2-3b_2^2)-5$$

in the unit disc, with the negative part shaded dark gray, and characterizes the moduli space of Möbius flat, superconformal surfaces that exhibit isometric homogeneity.

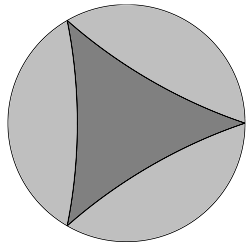

**Figure 1.** Space of Möbius flat homogeneous superconformal surfaces in $\mathbb{S}^5$.

Moreover, except for the one corresponding to $\beta = 0$ (i.e., $b_1 = b_2 = 0$), the superconformal surfaces in the above theorem are not S-Willmore surfaces.

**Remark 3.9.** The method of solving the structure equations developed in this section can be extended to the construction of superconformal surfaces of higher codimensions.

## 4. Möbius homogeneous superconformal surfaces with negative Möbius curvature in $\mathbb{S}^5$

In this section, we handle the negatively Möbius curved case. It is shown that superconformal surfaces belonging to this category do not exist.

Note that the canonical lift $Y$ of $x$ permits a constant scaling, enabling us to assume that $e^{-4\lambda}|\mu|^2 = 1$. Set

$$E \triangleq Y_z - \tau Y, \quad \hat{Y} \triangleq N + 2e^{-2\lambda}(\tau Y_{\bar{z}} + \bar{\tau} Y_z) - 2e^{-2\lambda}|\tau|^2 Y,$$

where $\tau = e^{2\lambda}\bar{\phi}/\bar{\mu}$. We point out that the 1-form

$$\tau\, dz = \frac{\bar{\phi}\, d\bar{z}\, e^{2\lambda}|dz|^2}{\bar{\mu}\, d\bar{z}^2} = \frac{g\,\langle N_{\bar{z}}, \xi\rangle\, d\bar{z}}{\langle Y_{\bar{z}\bar{z}}, \xi\rangle\, d\bar{z}^2}$$

is well defined; so $E\,dz$ and $\hat{Y}$ are both globally well defined. It is easy to check that

$$\{Y, \hat{Y}, E, \bar{E}, \xi, \bar{\xi}, \eta\}$$

forms a moving frame along $Y$ and satisfies

$$\langle \hat{Y}, \hat{Y}\rangle = 0, \quad \langle Y, \hat{Y}\rangle = 1, \quad \langle \hat{Y}, E\rangle = 0, \quad \langle \hat{Y}, \xi\rangle = 0, \quad \langle \hat{Y}, \eta\rangle = 0,$$
$$\langle E, E\rangle = 0, \quad \langle E, \xi\rangle = 0, \quad \langle E, \bar{\xi}\rangle = 0, \quad \langle E, \eta\rangle = 0.$$

With respect to this moving frame, the structure equation can be written as below:

$$Y_z = \tau Y + E, \tag{24}$$

$$E_{\bar{z}} = e^{2\lambda} h Y - \tfrac{1}{2} e^{2\lambda}\hat{Y} + \bar{\tau} E, \tag{25}$$

$$E_z = lY + (2\lambda_z - \tau)E - \tfrac{1}{2}\mu\xi, \tag{26}$$

$$\xi_z = \alpha\eta - \rho\xi, \tag{27}$$

$$\xi_{\bar z} = 2e^{-2\lambda}\bar\mu E + \bar\rho\xi + \beta\eta, \tag{28}$$

$$\eta_z = -\psi Y - \tfrac{1}{2}\alpha\bar\xi - \tfrac{1}{2}\bar\beta\xi, \tag{29}$$

$$\hat Y_z = -\tau\hat Y - 2\bar h E - 2e^{-2\lambda} l\bar E + \psi\eta. \tag{30}$$

By direct calculation, we obtain the following integrability equations:

$$l_{\bar z} = e^{2\lambda}(2h\tau + h_z), \tag{31}$$

$$e^{2\lambda}(\bar h + h) + e^{-2\lambda}|\mu|^2 - 2\lambda_{z\bar z} + (\tau_{\bar z} + \bar\tau_z) = 0, \tag{32}$$

$$\mu_{\bar z} = \mu(\bar\tau - \bar\rho), \tag{33}$$

$$e^{2\lambda}\psi = \mu\beta, \tag{34}$$

$$2e^{-2\lambda} l\bar\mu = \beta\psi - \alpha\bar\psi, \tag{35}$$

$$e^{2\lambda}(h - \bar h) = \bar\tau_z - \tau_{\bar z}, \tag{36}$$

$$\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + \rho_{\bar z} + \bar\rho_z - e^{-2\lambda}|\mu|^2 = 0, \tag{37}$$

$$\alpha\bar\rho - \alpha_{\bar z} + \beta\rho + \beta_z = 0, \tag{38}$$

$$\psi_{\bar z} + \bar\tau\psi = \bar\psi_z + \tau\bar\psi. \tag{39}$$

**Remark 4.1.** In addition to $\tau\, dz$, it is easy to verify that

$$\psi\, dz,\ \ l\, dz^2,\ \ h,\ \ e^{-2\lambda}|\alpha|^2,\ \ e^{-2\lambda}|\beta|^2$$

are all globally well-defined Möbius invariant forms. Moreover, $e^{-2\lambda}\tau_{\bar z}$ and $e^{-2\lambda}|\tau|^2$ are also globally well-defined. Hence under the assumption of Möbius homogeneity, they are constant. We assume that

$$e^{-2\lambda}|\tau|^2 = a_1, \quad e^{-2\lambda}\tau_{\bar z} = a_2. \tag{40}$$

By rotating the complex coordinate $z$, we further assume that $a_2 \in \mathbb{R}$. Then we can deduce that $h \in \mathbb{R}$ by (36) . It follows from $e^{4\lambda} = |\mu|^2$, (33) and (37) that

$$4\lambda_{\bar z z} = (\ln\mu)_{\bar z z} + (\ln\bar\mu)_{z\bar z} = \tau_{\bar z} + \bar\tau_z - e^{2\lambda}\big(1 + e^{-2\lambda}\tfrac{1}{2}(|\beta|^2 - |\alpha|^2)\big).$$

Hence the Möbius curvature is

$$K = -4e^{-2\lambda}\lambda_{z\bar z} = 1 - 2a_2 + e^{-2\lambda}\tfrac{1}{2}(|\beta|^2 - |\alpha|^2). \tag{41}$$

**Lemma 4.2.** *Let $x : M^2 \to \mathbb{S}^5$ be a Möbius homogeneous superconformal surface. If its Möbius curvature is negative, then $\alpha\beta \neq 0$, and $\psi \neq 0$.*

*Proof.* By the assumption of homogeneity, either $\alpha \equiv 0$ or $\alpha \neq 0$ everywhere, and the same holds for $\beta$.

If both $\alpha$ and $\beta$ equal zero identically, then it follows from (34) that $\psi \equiv 0$. Therefore, $\eta$ is constant along $Y$, which implies $x$ lies in $\mathbb{S}^4$ up to a Möbius transformation. But by [Wang and Xie 2014], there exists no Möbius homogeneous superconformal surface of negative Möbius curvature in $\mathbb{S}^4$.

Suppose $\alpha \equiv 0$ but $\beta \neq 0$. Then it follows from (34) and (35) that $2l\bar{\mu} = \beta^2\mu$, which implies

$$\ln l = 2\ln\beta + \ln\mu - \ln\bar{\mu} - \ln 2.$$

Note that $e^{-4\lambda}|l|^2$ is a constant, so we have

$$4\lambda_{z\bar{z}} = 2(\ln\beta)_{z\bar{z}} + 2(\ln\bar{\beta})_{z\bar{z}}.$$

Combining this with (38) and (37), we can derive that $K = e^{-2\lambda}|\beta|^2 + 2$, which contradicts the assumption that $K < 0$.

Suppose $\beta \equiv 0$ but $\alpha \neq 0$. Then it follows from (34) and (35) that $\psi \equiv 0$ and $l \equiv 0$. Using (31) and the fact that $h$ is a constant, we have

$$h\tau \equiv 0.$$

Using the fact that $e^{-2\lambda}|\alpha|^2$ is a constant and (37), we have

$$K = -4e^{-2\lambda}\lambda_{z\bar{z}} = -2e^{-2\lambda}((\ln\alpha)_{z\bar{z}} + (\ln\bar{\alpha})_{z\bar{z}}) = -2 + e^{-2\lambda}|\alpha|^2. \tag{42}$$

If $h \equiv 0$, we derive from (32) that

$$2a_2 = -\tfrac{1}{2}K - 1.$$

Substituting this into (41), we have $K = 4 - e^{-2\lambda}|\alpha|^2$. Then it follows from (42) that $K = 1$, which contradicts the assumption. If $\tau \equiv 0$, we derive from (41) that $K = 1 - \frac{1}{2}e^{-2\lambda}|\alpha|^2$, combining which with (42), we find that $K = 0$. This also gives us a contradiction.

In conclusion, we have proved that $\alpha\beta \neq 0$ everywhere. It follows from (34) that $\psi \neq 0$ everywhere. □

**Lemma 4.3.** *Let $x : M^2 \to \mathbb{S}^5$ be a Möbius homogeneous superconformal surface. If its Möbius curvature is negative, then $\tau \neq 0$.*

*Proof.* If $\tau \equiv 0$, it follows from (31) that $l_z = 0$.

If $l \equiv 0$, then from (35) we can derive that $|\alpha| = |\beta|$ since $\psi \neq 0$ by Lemma 4.2. Then (41) implies $K = 1$, which contradicts the assumption.

If $l \neq 0$, then, since $e^{-4\lambda}|l|^2$ is a constant, we can obtain a contradiction that

$$K = -4e^{-2\lambda}\lambda_{z\bar{z}} = -e^{-2\lambda}((\ln l)_{z\bar{z}} + (\ln\bar{l})_{z\bar{z}}) = 0. \qquad \square$$

In the following, we assume that $\alpha\beta \neq 0$ everywhere. Consider

$$\frac{\alpha}{\bar{\beta}} = \frac{\langle\eta_z, \xi\rangle\, dz}{\langle\eta_z, \bar{\xi}\rangle\, dz},$$

whose norm is a globally well-defined Möbius invariant complex function. It is obvious that $\xi$ can be chosen canonically such that $\alpha/\bar{\beta} \in \mathbb{R}^+$ is a constant. Then

$$\alpha\,dz,\ \ \bar{\beta}\,dz,\ \ \mu\,dz^2,\ \ \rho\,dz$$

are also globally well-defined Möbius invariant complex forms. So there exists constants $c_1, c_2, \ldots, c_6$ such that

$$\alpha = c_1\tau, \quad \bar{\beta} = c_2\tau, \quad \psi = c_3\tau, \quad l = c_4\tau^2, \quad \mu = c_5\tau^2, \quad \rho = c_6\tau,$$

with

$$c_1c_2c_3c_5 \neq 0, \quad \frac{c_1}{c_2} \in \mathbb{R}^+.$$

**Theorem 4.4.** *There does not exist Möbius homogeneous superconformal surface in $\mathbb{S}^5$ with negative Möbius curvature.*

*Proof.* We prove this theorem by contradiction and, to that end, suppose that $x : M^2 \to \mathbb{S}^5$ is a such kind of surface, with those Möbius invariants expressed as above.

Using (40), we can deduce that

$$2e^{-2\lambda}\lambda_{z\bar{z}} = e^{-2\lambda}\left(\left(\frac{\tau_{\bar{z}}}{\tau}\right)_z + \left(\frac{\bar{\tau}_z}{\bar{\tau}}\right)_{\bar{z}}\right) = 2e^{-2\lambda}\left(\frac{a_2\bar{\tau}}{a_1}\right)_z = 2\frac{a_2^2}{a_1}.$$

It follows from (31) that $h = a_2c_4$. Substituting these expressions into (32), we have

$$\frac{1}{a_1} - 2\left(\frac{a_2}{a_1}\right)^2 + 2\frac{a_2}{a_1} + 2\frac{a_2}{a_1}c_4 = 0. \tag{43}$$

It follows from (33) that

$$a_1(\bar{c}_6 - 1) + 2a_2 = 0, \tag{44}$$

which implies $c_6 \in \mathbb{R}$. Then using (34), (35) and (37)–(39), we can obtain that

$$c_3 = a_1c_5\bar{c}_2, \tag{45}$$

$$2a_1c_4\bar{c}_5 + c_1\bar{c}_3 - c_3\bar{c}_2 = 0, \tag{46}$$

$$a_1(|c_1|^2 - |c_2|^2) + 4a_2c_6 = 2, \tag{47}$$

$$\bar{c}_2(a_1c_6 + a_2) + c_1(a_1c_6 - a_2) = 0, \tag{48}$$

$$(a_1 + a_2)(\bar{c}_3 - c_3) = 0. \tag{49}$$

From (49), we see that either $a_2 = -a_1$ or $c_3 \in \mathbb{R}$. By (44), (47) and (48), we can find

$$\frac{a_2}{a_1} = \frac{c_1 + \bar{c}_2}{3c_1 + \bar{c}_2}, \quad c_6 = \frac{c_1 - \bar{c}_2}{3c_1 + \bar{c}_2}, \quad \frac{1}{a_1} = \frac{2(c_1^2 - \bar{c}_2{}^2)}{(\bar{c}_2 + 3c_1)^2} + \frac{|c_1|^2 - |c_2|^2}{2}. \tag{50}$$

If $a_2 = -a_1$, (44) implies $c_6 = 3$. Combining with (48), we can derive that $\bar{c}_2 + 2c_1 = 0$, which implies $|c_1|^2 - |c_2|^2 < 0$. This means the left-hand side of (47) is negative since $a_1 > 0$, which is impossible.

If $c_3 \in \mathbb{R}$, then $c_5\bar{c}_2 \in \mathbb{R}$. Using (45) and (46), we can find

$$c_4 = \frac{c_5\bar{c}_2(c_1 - \bar{c}_2)}{2\bar{c}_5} = \frac{(c_1 - \bar{c}_2)c_2}{2}.$$

From this and (50), we can express the left-hand side of (43) as

$$\frac{c_2\bar{c}_2{}^3 + (c_1\bar{c}_1 - 4)\bar{c}_2{}^2 + c_1\big(c_1(6\bar{c}_1 - 11c_2) + 8\big)\bar{c}_2 + 3c_1^2\big(c_1(3\bar{c}_1 - 2c_2) + 4\big)}{2(\bar{c}_2 + 3c_1)^2}. \tag{51}$$

It follows from (48) that $\bar{c}_2/c_1 \in \mathbb{R}$, so we have either $c_1, c_2 \in \mathbb{R}$ or $c_1, c_2 \in i\mathbb{R}$, since $c_1/c_2 \in \mathbb{R}^+$.

When $c_1, c_2 \in \mathbb{R}$, using the expression of $1/a_1$ in (50), we can obtain that $c_1^2 > c_2^2$. Now (51) equals

$$\frac{4(3c_1 - c_2)(c_1 + c_2) + (c_1^2 - c_2^2)(9c_1^2 - c_2^2)}{2(3c_1 + c_2)^2},$$

which can be verified easily to be positive. This gives us a contradiction with (43).

When $c_1, c_2 \in i\mathbb{R}$, we write $c_1 = ir_1$ and $c_2 = ir_2$. Then it follows from the expression of $1/a_1$ in (50) that $r_1^2 > r_2^2$. Now (51) is equal to

$$\frac{4(r_1 - r_2)(3r_1 + r_2) + (r_1^2 - r_2^2)(9r_1^2 - r_2^2)}{2(3r_1 - r_2)^2},$$

which is also positive, and we obtain a contradiction again. □

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